

PREFACE

It was forty years ago that my “Introduction to the arithmetic theory of automorphic functions” appeared. At present the terminology “modular form” can be counted among those most frequently heard in the conversations of mathematicians, and indeed, there are many textbooks on this topic. However, almost all of them are at the elementary level, and not so interesting from the viewpoint of the reader who already knows the basics. So, my intention in the present book is to offer something new that may satisfy the desire of such a reader. Therefore we naturally assume that the reader has at least rudimentary knowledge of modular forms of integral weight with respect to congruence subgroups of $SL_2(\mathbf{Z})$, though we state every definition and some basic theorems on such forms.

One of the principal new features of this book is the theory of modular forms of half-integral weight, another the discussion of theta functions and Eisenstein series of holomorphic and nonholomorphic types. Thus we have written the book so that the reader can learn such theories systematically. However, we present them with the following two themes as the ultimate aims:

(I) The correspondence between the forms of half-integral weight and those of integral weight.

(II) The arithmeticity of various Dirichlet series associated with modular forms of integral or half-integral weight.

The correspondence of (I) associates a cusp form of weight k with a modular form of weight $2k - 1$, where k is half an odd positive integer. I gave such a correspondence in my papers in 1973. In the present book I prove a stronger, perhaps the best possible, result with different methods.

As for (II), a typical example is a Dirichlet series

$$\mathcal{D}(s; f, g) = L(2s + 2, \omega) \sum_{n=1}^{\infty} a_n b_n n^{-s-(k+\ell)/2}$$

obtained from a cusp form $f(z) = \sum_{n=1}^{\infty} a_n \exp(2\pi i n z)$ of weight k and another form $g(z) = \sum_{n=0}^{\infty} b_n \exp(2\pi i n z)$ of weight ℓ , where $L(s, \omega)$ is the L -function of a Dirichlet character ω determined by f and g . In the crudest form, our main results show that there exists a constant $A(f)$ that depends on f, k, ℓ, ω , and an integer κ such that $D(\kappa; f, g)/A(f)$ is algebraic if a_n and b_n are algebraic, for infinitely many different g . We can of course consider $D(s; f, \chi) = \sum_{n=1}^{\infty} \chi(n) a_n n^{-s}$ with a Dirichlet character χ and ask about the nature of $D(m; f, \chi)$ for certain integers m .

Though we eventually restrict our modular forms to functions of one complex variable, some of our earlier sections provide an easy introduction to the theory of Siegel modular forms, since that gives a good perspective and makes our proofs of various facts more transparent. Also, since our second theme concerns the arithmeticity, we naturally discuss the rationality of the Fourier coefficients of a modular form, and how the form behaves under the action of an automorphism of the field to which the coefficients belong. This is a delicate problem, particularly when it is combined with the group action. Therefore, a considerable number of pages are spent on this problem. Another essential aspect of our theory is the involvement of the class of functions which we call nearly holomorphic modular forms, especially nonholomorphic Eisenstein series.

As for $D(m; f, \chi)$, we only state the results without proof, and cite two of my papers published in 1976 and 1977. My original plan was to make the book self-contained even in this respect by including the proof, but an unexpected accident made me abandon the idea. Possibly I may be excused by saying that once the reader acquires some elementary results in earlier sections of the present book, those two papers will be easy to read, and so the exclusion of the proof is not a great loss. Also, I allowed myself to quote some standard facts discussed in my books of 1971 and 2007 without proof, since I thought it awkward to reproduce the proof of every quoted fact.

It is my great pleasure to express my heartfelt thanks to my friends Koji Doi, Tomokazu Kashio, Kamal Khuri-Makdisi, Kaoru Okada, and Hiroyuki Yoshida, who read my manuscript and helped me eliminate many misprints and improve the exposition.

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