Chapter 2
Line Integrals

In studying the motion of a particle along an *arc* it is convenient to consider the arc as the image of a vector-valued mapping $\gamma : [a,b] \to \mathbb{R}^3$ defined on an interval of the real line and realize $\gamma(t)$ as the position of the particle at time t. This viewpoint is also convenient in analyzing the behavior of a vector field along an arc and is the main motivation for the definitions that follow.

A. Galbis and M. Maestre, *Vector Analysis Versus Vector Calculus*, Universitext,
DOI 10.1007/978-1-4614-2200-6_2, © Springer Science+Business Media, LLC 2012

2.1 Paths

Definition 2.1.1. A *path* is a continuous mapping $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$. We call $\boldsymbol{\gamma}(a)$ the *initial point* and $\boldsymbol{\gamma}(b)$ the *final point*. The image of the path, $\boldsymbol{\gamma}([a,b])$, is called the *arc*[1] of $\boldsymbol{\gamma}$. If $\boldsymbol{\gamma}([a,b]) \subset \Omega$, we say that $\boldsymbol{\gamma}$ is a path in Ω.

Example 2.1.1. The line segment joining two points $\boldsymbol{x}, \mathbf{y} \in \mathbb{R}^n$ is the arc $[\boldsymbol{x}, \mathbf{y}] := \boldsymbol{\gamma}([0,1])$, where $\boldsymbol{\gamma} : [0,1] \to \mathbb{R}^n$ denotes the path

$$\boldsymbol{\gamma}(t) = \boldsymbol{x} + t(\boldsymbol{y} - \boldsymbol{x}) = t\boldsymbol{y} + (1-t)\boldsymbol{x}.$$

Example 2.1.2. Let $\boldsymbol{\gamma}_j : [0,2\pi] \to \mathbb{R}^2$ be given by $\boldsymbol{\gamma}_j(t) := (\cos(jt), \sin(jt))$. Then for every $j \in \mathbb{Z} \setminus \{0\}$, the arc $\boldsymbol{\gamma}_j([0,2\pi])$ is the unit circle $x^2 + y^2 = 1$ in $\mathbb{R}^2$. As the parameter t increases from 0 to 2π, the point $\boldsymbol{\gamma}_j(t)$ travels around the unit circle $|j|$ times (clockwise when j is negative and counterclockwise when j is positive).

We put

$$\boldsymbol{\gamma}'(t) := \lim_{h \to t} {}_{h \in [a,b]} \frac{\boldsymbol{\gamma}(h) - \boldsymbol{\gamma}(t)}{h - t},$$

if the limit exists. We observe that if $t \in (a,b)$, then $\boldsymbol{\gamma}'(t)$ exists if and only if $\boldsymbol{\gamma}$ is a differentiable mapping at the point t. In this case, $\boldsymbol{\gamma}'(t)$ is the $n \times 1$ column matrix of the differential at t, which is naturally viewed as a vector in $\mathbb{R}^n$. In particular, we may consider $\boldsymbol{\gamma}' : [a,b] \to \mathbb{R}^n$, and where the appropriate limits exist, repeat to find higher-order derivatives of $\boldsymbol{\gamma}$. This prompts the following definitions.

Definition 2.1.2. A path $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ is said to be a function of class C^q on $[a,b]$ if the qth derivative $\boldsymbol{\gamma}^{(q)}(t)$ exists for every $t \in [a,b]$ and $\boldsymbol{\gamma}^{(q)}$ is continuous on $[a,b]$. The mapping $\boldsymbol{\gamma}$ is said to be a *piecewise* C^q function if there exists a partition $a = t_1 < \cdots < t_k = b$ such that $\boldsymbol{\gamma}_{|[t_i,t_{i+1}]}$ is of class C^q on $[t_i, t_{i+1}]$ for every $1 \le i \le k-1$.

Definition 2.1.3. A path $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ is said to be *smooth* if $\boldsymbol{\gamma}$ is a C^1 function and $\boldsymbol{\gamma}'(t) \neq 0$ for every $t \in [a,b]$.

Unfortunately, the notation is not standard in the literature. We follow Cartan [7, 3.1, p. 48] or Edwards [9, V.1. p. 287]. However Fleming [10, 6.2, pp. 247–249] refers to *curves* as equivalence classes and a path is interpreted as a parametric representation of a curve. Marsden–Tromba [14, 3.1, p. 190] uses the term *trajectory* instead of *path* and imposes no continuity condition, while Do Carmo [5, 1-2. parameterized curves, p. 2] defines *parametric curves* (with values in $\mathbb{R}^3$) and uses the expression *regular curve* instead of smooth path.

The next example, which is a reformulation of Edwards [9, V.1, example 1, p. 287], shows that a path of class C^∞ can have corners. We first need a lemma.

---

[1] Also called the *track* or *trace* of $\boldsymbol{\gamma}$.

Lemma 2.1.1. *Let c be a real number. The function* $f:\mathbb{R}\to\mathbb{R}$,

$$f(t) := \begin{cases} c\,\mathrm{e}^{-\frac{1}{t^2}}, & t<0,\\ 0, & t=0,\\ \mathrm{e}^{-\frac{1}{t^2}}, & t>0,\end{cases}$$

is of class C^∞ *on the real line.*

Proof. We show that for each $n\in\mathbb{N}\cup\{0\}$ there is a polynomial P_n such that

$$f^{(n)}(t) := \begin{cases} c\,\frac{P_n(t)}{t^{3n}}\mathrm{e}^{-\frac{1}{t^2}}, & t<0,\\ 0, & t=0,\\ \frac{P_n(t)}{t^{3n}}\mathrm{e}^{-\frac{1}{t^2}}, & t>0.\end{cases}$$

In fact, this is obvious for $n=0$. To apply induction, we assume that the result is true for $n=k\in\mathbb{N}\cup\{0\}$. To deduce that the result is also true for $n=k+1$, it is enough to check that $f^{(k+1)}(0)=0$. We are going to use that e^{y^2} diverges to infinity faster than any polynomial in y as y tends to $+\infty$. Indeed, if $P_n(y)=a_ny^n+\cdots+a_1y+a_0$ with $a_n\neq 0$, then

$$\frac{P_n(y)}{\mathrm{e}^{y^2}} = \frac{P_n(y)}{\mathrm{e}^{y}}\cdot\frac{1}{\mathrm{e}^{y^2-y}}.$$

The function e^{-y^2+y} clearly converges to zero. On the other hand, by applying L'Hôpital's rule n times, we obtain

$$\lim_{y\to+\infty}\frac{P_n(y)}{\mathrm{e}^{y}} = \lim_{y\to+\infty}\frac{n!a_n}{\mathrm{e}^{y}} = 0.$$

Now by taking $y=\frac{1}{t}$ when $t>0$ or $y=-\frac{1}{t}$ if $t<0$ we obtain that

$$f^{(k+1)}(0) = \lim_{t\to 0}\frac{f^{(k)}(t)-f^{(k)}(0)}{t} = \lim_{t\to 0}\frac{f^{(k)}(t)}{t} = 0.$$

□

Example 2.1.3. Let $\boldsymbol{\gamma}:[-1,1]\to\mathbb{R}^2$ be defined by

$$\boldsymbol{\gamma}(t) := \begin{cases} \left(-\mathrm{e}^{-\frac{1}{t^2}+1}, \mathrm{e}^{-\frac{1}{t^2}+1}\right), & -1\le t<0,\\ (0,0), & t=0,\\ \left(\mathrm{e}^{-\frac{1}{t^2}+1}, \mathrm{e}^{-\frac{1}{t^2}+1}\right), & 0<t\le 1.\end{cases}$$

It follows from the above lemma that γ is a path of class C^∞. However, γ is not smooth, since $\gamma'(0) = (0,0)$. We note that $\gamma([-1,1]) = \{(x,|x|) \, : \, x \in [-1,1]\}$ *has a corner* at the point $(0,0)$. Several more examples can be found in Marsden–Tromba [14, 3.2, p. 205].

Our next task is to define and, if possible, to evaluate the length of a path $\gamma : [a,b] \to \mathbb{R}^n$. The basic idea consists in approximating the path by means of line segments whose endpoints are determined by a partition of the interval $[a,b]$. A path with finite length is said to be *rectifiable* or of *bounded variation*. We will show that every piecewise C^1 path is rectifiable and will obtain a formula to evaluate its length.

Definition 2.1.4. Let $\gamma : [a,b] \to \mathbb{R}^n$ be a path and let $P := \{a = t_1 < \cdots < t_k = b\}$ be a partition of $[a,b]$. The *polygonal arc* associated with P is the union of the line segments $[\gamma(t_i), \gamma(t_{i+1})]$, $1 \leq i \leq k-1$. The length of this polygonal arc is

$$L(\gamma, P) := \sum_{i=1}^{k-1} ||\gamma(t_{i+1}) - \gamma(t_i)||.$$

Lemma 2.1.2. *Let $\gamma : [a,b] \to \mathbb{R}^n$ be a path and let $P := \{a = t_1 < \cdots < t_k = b\}$ be a partition of $[a,b]$. If Q is another partition of $[a,b]$ and $P \subset Q$, then $L(\gamma, P) \leq L(\gamma, Q)$.*

Proof. We can assume without any loss of generality that Q is obtained from P by adding a single point. Thus, we assume $Q = P \cup \{s\}$, where $t_j < s < t_{j+1}$. Then

$$L(\gamma, Q) := \sum_{i \neq j} ||\gamma(t_{i+1}) - \gamma(t_i)|| + \| \gamma(s) - \gamma(t_j) \| + \| \gamma(t_{j+1}) - \gamma(s) \| .$$

Since

$$||\gamma(t_{j+1}) - \gamma(t_j)|| \leq \| \gamma(s) - \gamma(t_j) \| + \| \gamma(t_{j+1}) - \gamma(s) \|,$$

the conclusion follows. □

Definition 2.1.5. A path γ is said to be *rectifiable* or of *bounded variation* if

$$\sup\{L(\gamma, P) : P \text{ a partition of } [a,b]\} < +\infty.$$

If γ is rectifiable, we will refer to this supremum as the *length* of γ, and we will denote it by $L(\gamma)$.

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a path and $a < c < b$. Then γ is rectifiable if and only if $\gamma|_{[a,c]}$ and $\gamma|_{[c,b]}$ are. Moreover, in that case,

$$L(\gamma) = L(\gamma|_{[a,c]}) + L(\gamma|_{[c,b]}).$$

We leave the proof to the interested reader.

Definition 2.1.6. The *norm of a partition* $P := \{a = t_1 < \cdots < t_k = b\}$ is the length of the largest subinterval defined by that partition, that is,

$$\|P\| = \max\{|t_{i+1} - t_i| \ : \ i = 1, \ldots, k-1\}.$$

We are going to show that any path of class C^1 is rectifiable, but before we do so, we need to remind the reader of the topological concept of compactness and of some properties enjoyed by compact sets.

Definition 2.1.7. A subset K of $\mathbb{R}^n$ is called *compact* if for each family $\mathscr{F}$ of open subsets of $\mathbb{R}^n$ that cover K, in the sense that

$$K \subset \bigcup_{G \in \mathscr{F}} G,$$

there exists a finite subfamily $G_1, \ldots, G_m$ in $\mathscr{F}$ such that $K \subset \cup_{j=1}^m G_j$.

A subset M of $\mathbb{R}^n$ is called *bounded* if it is contained in an open ball centered at the origin. If K is a compact set, then it is bounded. This is an easy exercise, but far more can be said. The proof of the following characterizations of compact subsets in $\mathbb{R}^n$ can be found in almost any book on calculus of several variables.

Theorem 2.1.1. *Let K be a subset of $\mathbb{R}^n$. The following conditions are equivalent:*

1. *K is compact.*
2. *For each sequence $(x_j)_{j=1}^\infty \subset K$ there exists a subsequence $(x_{j_k})_{k=1}^\infty$ convergent to a point $x_0 \in K$.*
3. *(Heine–Borel–Lebesgue theorem) K is bounded and closed in $\mathbb{R}^n$.*

We also need the concept of *uniform* continuity.

Definition 2.1.8. Let M be a subset of $\mathbb{R}^n$. A mapping $\boldsymbol{f} : M \subset \mathbb{R}^n \to \mathbb{R}^m$ is *uniformly continuous* on M if given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\boldsymbol{x}, \boldsymbol{y} \in M$ with $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta$, we have

$$\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| < \varepsilon.$$

Uniform continuity of course implies continuity, but is, in fact, a stronger property. Nevertheless, both concepts coincide if the set M is a compact set, a result that we state here without proof.

Theorem 2.1.2 (Heine–Cantor theorem). *Every continuous mapping $\boldsymbol{f} : K \subset \mathbb{R}^n \to \mathbb{R}^m$ on a compact set K is uniformly continuous on K.*

We return to our study of paths.

Theorem 2.1.3. *Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ be a path of class C^1. Then $\boldsymbol{\gamma}$ is rectifiable and*

$$L(\boldsymbol{\gamma}) = \int_a^b \|\boldsymbol{\gamma}'(t)\| \mathrm{d}t.$$

Proof. We define $F : [a,b]^n \to \mathbb{R}$ by

$$F(s_1,\dots,s_n) := \sqrt{\sum_{j=1}^{n} |\boldsymbol{\gamma}_j'(s_j)|^2}.$$

Then $F(t,\dots,t) = ||\boldsymbol{\gamma}'(t)||$. Moreover, given a partition

$$P := \{a = t_1 < \cdots < t_k = b\},$$

we can apply the mean value theorem to deduce that for every $1 \le i \le k-1$ and $1 \le j \le n$, there exists $s_{ji} \in [t_i, t_{i+1}]$ such that

$$\begin{aligned} L(\boldsymbol{\gamma},P) = \sum_{i=1}^{k-1} \|\boldsymbol{\gamma}(t_{i+1}) - \boldsymbol{\gamma}(t_i)\| &= \sum_{i=1}^{k-1} \sqrt{\sum_{j=1}^{n} \left(\boldsymbol{\gamma}_j(t_{i+1}) - \boldsymbol{\gamma}_j(t_i)\right)^2} \\ &= \sum_{i=1}^{k-1} F(s_{1i},\dots,s_{ni})(t_{i+1}-t_i) \\ &= \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} F(s_{1i},\dots,s_{ni})\mathrm{d}t. \end{aligned}$$

Since F is a continuous function on the compact set $[a,b]^n$, it follows by the Heine–Cantor theorem that F is in fact uniformly continuous on $[a,b]^n$. That is, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\boldsymbol{x},\boldsymbol{y} \in [a,b]^n$ and $\|\boldsymbol{x}-\boldsymbol{y}\| < \delta$ imply

$$|F(\boldsymbol{x}) - F(\boldsymbol{y})| < \varepsilon.$$

Let us now assume that the previous partition P satisfies $\|P\| < \frac{\delta}{\sqrt{n}}$. Then

$$||(t,\dots,t) - (s_{1i},\dots,s_{ni})|| < \delta,$$

whenever $t \in [t_i, t_{i+1}]$. Hence

$$\begin{aligned} \left| L(\boldsymbol{\gamma},P) - \int_a^b ||\boldsymbol{\gamma}'(t)||\mathrm{d}t \right| &\le \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} |F(s_{1i},\dots,s_{ni}) - F(t,\dots,t)|\mathrm{d}t \\ &\le \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} \varepsilon \,\mathrm{d}t = \varepsilon \sum_{i=1}^{k-1} (t_{i+1}-t_i) = \varepsilon(b-a). \end{aligned}$$

To finish the proof, we fix a partition P_0 with norm less than $\frac{\delta}{\sqrt{n}}$. For an arbitrary partition P of $[a,b]$ we then have

$$L(\boldsymbol{\gamma},P) \le L(\boldsymbol{\gamma},P \cup P_0) \le \varepsilon(b-a) + \int_a^b ||\boldsymbol{\gamma}'(t)||\mathrm{d}t,$$

which shows that $\boldsymbol{\gamma}$ is rectifiable and $L(\boldsymbol{\gamma}) \leq \varepsilon(b-a) + \int_a^b ||\boldsymbol{\gamma}'(t)||\mathrm{d}t$. On the other hand,

$$L(\boldsymbol{\gamma}) \geq L(\boldsymbol{\gamma}, P_0) \geq \int_a^b ||\boldsymbol{\gamma}'(t)||\mathrm{d}t - \varepsilon(b-a).$$

Taking limits as ε tends to zero, we reach the desired conclusion. □

It is worth mentioning that many texts, for instance Do Carmo [5, 1-2, p. 6] or Marsden–Tromba [14, 3.2, p. 201], define the length of a path of class C^1 as $L(\boldsymbol{\gamma}) = \int_a^b \|\boldsymbol{\gamma}'(t)\|\ \mathrm{d}t$. While this point of view is very efficient, it can be puzzling to the student as to why this should be the "correct" definition. In our opinion, the polygonal arc approach to path length is far more intuitive, and as shown by the theorem above, entirely consistent.

Corollary 2.1.1. *Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ be a piecewise C^1 path. Then $\boldsymbol{\gamma}$ is rectifiable and $L(\boldsymbol{\gamma}) = \int_a^b \|\boldsymbol{\gamma}'(t)\|\mathrm{d}t$.*

Proof. Let $\{a = t_1 < t_2 < \cdots < t_k = b\}$ be a partition of $[a,b]$ such that $\boldsymbol{\gamma}_j := \boldsymbol{\gamma}_{|[t_j,t_{j+1}]}$ is of class C^1 for every $j = 1,\ldots,k-1$. By Theorem 2.1.3, every $\boldsymbol{\gamma}_j$ is a rectifiable path on $[t_j,t_{j+1}]$ and

$$L(\boldsymbol{\gamma}_j) = \int_{t_j}^{t_{j+1}} \|\boldsymbol{\gamma}'(t)\|\mathrm{d}t.$$

Consequently, $\boldsymbol{\gamma}$ is also rectifiable and

$$L(\boldsymbol{\gamma}) = \sum_{j=1}^{k-1} L(\boldsymbol{\gamma}_j) = \sum_{j=1}^{k-1} \int_{t_j}^{t_{j+1}} \| \boldsymbol{\gamma}'(t) \| \ \mathrm{d}t.$$

The function $t \mapsto \|\boldsymbol{\gamma}'(t)\|$ is well defined at each point of the interval $[a,b]$ except for a finite set. Moreover, its restriction to each interval $[t_j,t_{j+1}]$ is continuous, and hence Riemann integrable on $[t_j,t_{j+1}]$ for every $j = 1,\ldots,k-1$. It follows that the function is Riemann integrable on $[a,b]$. We can therefore deduce, from the properties of Riemann integrable functions, that

$$L(\boldsymbol{\gamma}) = \int_a^b \| \boldsymbol{\gamma}'(t) \| \ \mathrm{d}t.$$

□

Let $\boldsymbol{\gamma} : [0,1] \to \mathbb{R}^2$ be given by $\boldsymbol{\gamma}(0) := (0,0)$ and $\boldsymbol{\gamma}(t) := (t, t\cos\frac{\pi}{t})$ when $0 < t \leq 1$. We leave it to the reader to prove that $\boldsymbol{\gamma}$ is a continuous path but is not rectifiable.

Although we focus attention on piecewise C^1 paths throughout the text, we have found it natural to deal with the more general class of rectifiable paths for our treatment of path length. The generalization is also of benefit in the next section, where we consider work done by a vector field.

2.2 Integration of Vector Fields

The line integral was originally motivated by problems involving fluid motion and electromagnetic or other force fields.[2] Let us assume, for instance, that $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^3$ is a smooth path contained in the open set $U \subset \mathbb{R}^3$ and that there is present a force field $\boldsymbol{F} : U \to \mathbb{R}^3$. We want to evaluate the work done by the force on an object moving along the arc $\boldsymbol{\gamma}([a,b])$ from $\boldsymbol{\gamma}(a)$ to $\boldsymbol{\gamma}(b)$. We will have to take into account two basic principles:

1. The work depends only on the component of the force that is acting in the same direction as that in which the object is moving (that is, the tangent direction to the path at each point).
2. The work done by a constant field $\boldsymbol{F}_0$ to move the object through a line segment, in the same direction as $\boldsymbol{F}_0$, is the product of $\|\boldsymbol{F}_0\|$ and the length of that segment.

We recall that

$$\boldsymbol{T}(t) := \frac{\boldsymbol{\gamma}'(t)}{\|\boldsymbol{\gamma}'(t)\|}$$

is a unit vector that is tangent to the path at $\boldsymbol{\gamma}(t)$, and so the component of $\boldsymbol{F}$ that acts in the tangent direction to the path at $\boldsymbol{\gamma}(t_j)$ is

$$\left\langle \boldsymbol{F}(\boldsymbol{\gamma}(t_j)), \boldsymbol{T}(t_j) \right\rangle .$$

If we consider a short interval $[t_j, t_{j+1}]$, then the length of $\boldsymbol{\gamma}|_{[t_j,t_{j+1}]}$ is approximated by

$$\left\|\boldsymbol{\gamma}'(t_j)\right\| \cdot (t_{j+1} - t_j).$$

Hence, using the basic principles above on a very fine partition

$$P := \{a = t_1 < \cdots < t_k = b\}$$

of the interval, we see that a good approximation to the work done in moving the particle along $\boldsymbol{\gamma}$ is

$$\sum_{j=1}^{k-1} \left\langle \boldsymbol{F}(\boldsymbol{\gamma}(t_j)), \boldsymbol{\gamma}'(t_j) \right\rangle \cdot (t_{j+1} - t_j).$$

The following result tells us the limiting value of this quantity as we take finer and finer partitions (Fig. 2.1).

---

[2]A force field is, mathematically speaking, the same thing as a vector field. The term is often used when the field has a physical interpretation.

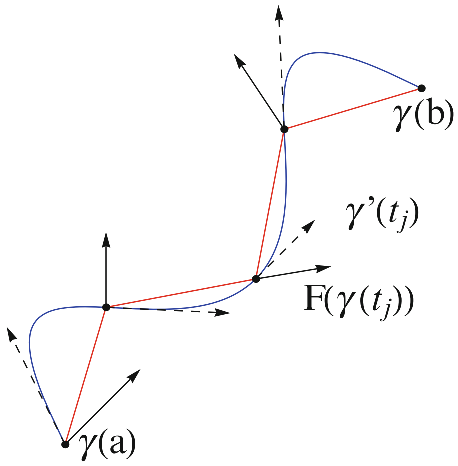


Fig. 2.1 Vector field along a path

Lemma 2.2.1. *Let* $P := \{a = t_1 < t_2 < \cdots < t_k = b\}$ *be a partition of the interval* $[a,b]$. *For any continuous function* $f : [a,b] \to \mathbb{R}$ *and for each selection* $u_j \in [t_j, t_{j+1}]$ *we have*

$$\lim_{\|P\|\to 0} \sum_{j=1}^{k-1} (t_{j+1} - t_j) \cdot f(u_j) \;=\; \int_a^b f(t)\,\mathrm{d}t.$$

Proof. By the Heine–Cantor theorem, f is uniformly continuous, which means that for every $\varepsilon > 0$ one can find $\delta > 0$ such that

$$|f(s) - f(t)| \leq \varepsilon$$

whenever $s, t \in [a,b]$ and $|s - t| \leq \delta$. Consider now a partition

$$P = \{a = t_1 < t_2 < \cdots < t_k = b\}$$

with $0 <\| P \|< \delta$. Then

$$\begin{aligned} \left| \sum_{j=1}^{k-1} (t_{j+1} - t_j) \cdot f(u_j) - \int_a^b f(t) \right| &= \left| \sum_{j=1}^{k-1} \int_{t_j}^{t_{j+1}} \Big(f(u_j) - f(t) \Big)\, \mathrm{d}t \right| \\ &\leq \sum_{j=1}^{k-1} \int_{t_j}^{t_{j+1}} \big| f(u_j) - f(t) \big| \, \mathrm{d}t \\ &\leq \varepsilon \sum_{j=1}^{k-1} \big(t_{j+1} - t_j \big) = \varepsilon (b - a). \end{aligned}$$

Since ε is arbitrarily small, the result follows. □

Lemma 2.2.1 and its preceding discussion show that the work done by the force field $\boldsymbol{F}$ on a particle as it moves along $\boldsymbol{\gamma}$ is

$$\int_a^b \left\langle \boldsymbol{F}(\boldsymbol{\gamma}(t)), \boldsymbol{\gamma}'(t) \right\rangle \mathrm{d}t.$$

This type of integral, known as a line integral, appears often in the sequel, and so we make a formal definition.

Definition 2.2.1. Let $\boldsymbol{F} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field and let $\boldsymbol{\gamma}$ be a piecewise C^1 path in U. The line integral of $\boldsymbol{F}$ along $\boldsymbol{\gamma}$ is given by

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} := \int_a^b \left\langle \boldsymbol{F}(\boldsymbol{\gamma}(t)), \boldsymbol{\gamma}'(t) \right\rangle \mathrm{d}t.$$

2.3 Integration of Differential Forms

In order to facilitate later calculations, and also to provide a better framework for dealing with the general Stokes's theorem, it is convenient at this point to introduce the notion of differential form. A key observation is that any vector $\boldsymbol{v} \in \mathbb{R}^n$ defines a linear $\mathbb{R}$-valued mapping, that is, a *linear form*,

$$\varphi_v : \mathbb{R}^n \to \mathbb{R},$$

by

$$\varphi_v(\boldsymbol{h}) := \langle \boldsymbol{v}, \boldsymbol{h} \rangle .$$

Conversely, any linear form L on $\mathbb{R}^n$ coincides with φ_v for a unique vector $\boldsymbol{v} \in \mathbb{R}^n$. Indeed, if $L : \mathbb{R}^n \to \mathbb{R}$, the mapping

$$\boldsymbol{\varphi} : \mathbb{R}^n \to (\mathbb{R}^n)^*, \quad \boldsymbol{v} \mapsto \boldsymbol{\varphi}_v,$$

maps the vector $(L(e_1), \ldots, L(e_n))$ to the linear form L. Thus $\boldsymbol{\varphi}$ is a linear isomorphism from the vectors in $\mathbb{R}^n$ to the linear forms on $\mathbb{R}^n$.

We denote by $\mathrm{d}x_j$ the linear form associated with the vector $\boldsymbol{e}_j$ of the canonical basis of $\mathbb{R}^n$. That is,

$$\mathrm{d}x_j(\boldsymbol{h}) = \left\langle \boldsymbol{e}_j, \boldsymbol{h} \right\rangle = h_j.$$

It easily follows that if $\boldsymbol{v} = (v_1, \ldots, v_n)$, then φ_v is the linear form

$$\sum_{j=1}^{n} v_j \cdot \mathrm{d}x_j.$$

After identifying vectors with linear forms, it is quite natural to identify a vector field on a set U with a mapping that associates to each point of U a linear form.

Definition 2.3.1. Let $U \subset \mathbb{R}^n$ be an open set. A *differential form of degree* 1 on U, or simply a 1-*form*, is a mapping

$$\boldsymbol{\omega} : U \subset \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^n, \mathbb{R}) = (\mathbb{R}^n)^*.$$

Given a 1-form $\boldsymbol{\omega}$, a vector $\boldsymbol{x} \in U$, and an integer $j \in \{1, \ldots, n\}$, we will denote the scalar $\boldsymbol{\omega}(\mathbf{x})(e_j) \in \mathbb{R}$ by $f_j(\mathbf{x})$. Evidently, each f_j is a function from U to $\mathbb{R}$, and by linearity of $\boldsymbol{\omega}(x) \in (\mathbb{R}^n)^*$, for any $\boldsymbol{h} = (h_1, \ldots, h_n) \in \mathbb{R}^n$ we can write

$$\boldsymbol{\omega}(\boldsymbol{x})(\boldsymbol{h}) = \sum_{j=1}^{n} \boldsymbol{\omega}(\mathbf{x})(e_j).h_j = \sum_{j=1}^{n} f_j(\boldsymbol{x}) h_j = \left(\sum_{j=1}^{n} f_j(\boldsymbol{x}) \mathrm{d}x_j \right)(\boldsymbol{h}).$$

Hence $\boldsymbol{\omega}(\boldsymbol{x}) = \sum_{j=1}^{n} f_j(\boldsymbol{x})\mathrm{d}x_j$ for all $\boldsymbol{x} \in U$, which we will abbreviate to

$$\boldsymbol{\omega} = \sum_{j=1}^{n} f_j \cdot \mathrm{d}x_j$$

and call f_j the *component* functions of $\boldsymbol{\omega}$.

The notion of differential form of degree 1 is, as we shall highlight below, a generalization of the concept of differential of a function. It is a very powerful algebraic tool for studying integration on curves or surfaces.

A 1-form $\boldsymbol{\omega}$ is continuous or of class C^q if its component functions f_j are continuous or, respectively, of class C^q. From now on *every differential form of degree* 1 *is assumed to be continuous*.

Observe that $\boldsymbol{\omega}(\boldsymbol{x})$ is the linear form associated with the vector

$$\boldsymbol{F}(\boldsymbol{x}) := (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x})),$$

where, as usual, the f_j denote the component functions of $\boldsymbol{\omega}$. Consequently, the study of the 1-form $\boldsymbol{\omega}$ is essentially equivalent to the study of the vector field $\boldsymbol{F}$, and we may interpret differential forms of degree 1 and vector fields as two different ways of visualizing the same mathematical object. Vector fields provide the most appropriate point of view for formulating problems from physics or engineering, but in order to solve these problems mathematically, it is often more convenient to express them in terms of differential forms.

What we have just seen is another important example of changing the meaning, or interpretation, of a mathematical concept by renaming it. We have moved from the realm of linear algebra into that of differential geometry by viewing a linear mapping, namely the projection $\boldsymbol{h} \to h_j$, as a differential form, $\mathrm{d}x_j$.

Example 2.3.1. Let $g : U \subset \mathbb{R}^n \to \mathbb{R}$ be a function of class C^1 on the open set U. The differential of g at point $\boldsymbol{x} \in U$ is the linear mapping $\mathrm{d}g(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ given by

$$(\mathrm{d}g)(\boldsymbol{x})(\boldsymbol{h}) = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j}(\boldsymbol{x})h_j = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j}(\boldsymbol{x})\mathrm{d}x_j(\boldsymbol{h}).$$

Hence

$$\mathrm{d}g = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j}\mathrm{d}x_j.$$

Since each partial derivative $\frac{\partial g}{\partial x_j}$ is a continuous function, we conclude that the mapping $U \to \mathscr{L}(\mathbb{R}^n, \mathbb{R})$, $\boldsymbol{x} \to (\mathrm{d}g)(\boldsymbol{x})$, is a (continuous) differential form of degree 1. We represent it by $\boldsymbol{\omega} = \mathrm{d}g$ (the differential of g).

Definition 2.3.2. Let $\boldsymbol{\omega}$ be a continuous 1-form on U and let $\boldsymbol{\gamma} : [a,b] \to U$ be a piecewise C^1 path. Then

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} := \int_a^b \boldsymbol{\omega}(\boldsymbol{\gamma}(t))\big(\boldsymbol{\gamma}'(t)\big)\,\mathrm{d}t.$$

We observe that the previous function is well defined and continuous except on a finite set. Since it is bounded, it is integrable in the sense of Riemann and Lebesgue. Moreover, expressing the 1-form $\boldsymbol{\omega}$ in terms of its component functions, $\boldsymbol{\omega} = \sum_{j=1}^n f_j \mathrm{d}x_j$, we have

$$\begin{aligned}\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = \int_a^b \left(\sum_{j=1}^n f_j(\boldsymbol{\gamma}(t))\mathrm{d}x_j(\boldsymbol{\gamma}'(t))\right)\mathrm{d}t &= \int_a^b \left(\sum_{j=1}^n f_j(\boldsymbol{\gamma}(t))\boldsymbol{\gamma}_j'(t)\right)\mathrm{d}t \\ &= \int_a^b \big\langle \boldsymbol{F}(\boldsymbol{\gamma}(t)),\ \boldsymbol{\gamma}'(t)\big\rangle\,\mathrm{d}t,\end{aligned}$$

where $\boldsymbol{F} := (f_1,\dots,f_n) : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is the vector field associated with $\boldsymbol{\omega}$. This simple calculation highlights an important fact: when $\boldsymbol{F}$ is the vector field associated with the differential form $\boldsymbol{\omega}$ of degree 1, we have

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = \int_{\boldsymbol{\gamma}} \boldsymbol{F}.$$

2.4 Parameter Changes

The line integral $\int_{\boldsymbol{\gamma}} \boldsymbol{F}$ depends on the vector field $\boldsymbol{F}$ and also on the path $\boldsymbol{\gamma}$. In this section, we plan to analyze what happens on replacing $\boldsymbol{\gamma}$ by some other path with the same trace. Further results in this direction will be obtained in the optional Sect. 2.7.

Definition 2.4.1. Let $\boldsymbol{\alpha} : [a,b] \to \mathbb{R}^n$ and $\boldsymbol{\beta} : [c,d] \to \mathbb{R}^n$ be two paths. We say that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are *equivalent*, and write $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$, if there is a mapping $\varphi : [a,b] \to [c,d]$, of class C^1, such that $\varphi([a,b]) = [c,d]$, $\varphi'(t) > 0$ for every $t \in [a,b]$ and $\boldsymbol{\alpha} = \boldsymbol{\beta} \circ \varphi$ (Fig. 2.2).

By the mean value theorem, the conditions on φ in this definition ensure that φ is strictly increasing, hence bijective, and that $c = \varphi(a)$ and $d = \varphi(b)$. It then follows that $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$ implies $\boldsymbol{\beta} \sim \boldsymbol{\alpha}$. Indeed, $\boldsymbol{\beta} = \boldsymbol{\alpha} \circ \varphi^{-1}$, and φ^{-1} has the necessary properties for equivalence.

The next result is usually referred to as the *chain rule* or the *composite function theorem*. Its proof can be found in any text on differential calculus of several variables; for example [9, Theorem 3.1, p. 76] or [10, 4.4, p. 134].

Theorem 2.4.1. *Let U and V be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. If the mappings $\boldsymbol{F} : U \longrightarrow V$ and $\boldsymbol{G} : V \longrightarrow \mathbb{R}^p$ are differentiable at $\boldsymbol{a} \in U$ and $\boldsymbol{F}(\boldsymbol{a}) \in V$ respectively, then their composition $\boldsymbol{H} = \boldsymbol{G} \circ \boldsymbol{F}$ is differentiable at $\boldsymbol{a}$, and*

$$\mathrm{d}\boldsymbol{H}(\boldsymbol{a}) = \mathrm{d}\boldsymbol{G}(\boldsymbol{F}(\boldsymbol{a})) \circ \mathrm{d}\boldsymbol{F}(\boldsymbol{a}),$$

or, in terms of their associated matrices,

$$\boldsymbol{H}'(\boldsymbol{a}) = \boldsymbol{G}'(\boldsymbol{F}(\boldsymbol{a}))\boldsymbol{F}'(\boldsymbol{a}).$$

Furthermore, if $\boldsymbol{F}$ and $\boldsymbol{G}$ are of class C^q ($1 \le q \le \infty$) on their respective domains of definition, then $\boldsymbol{H}$ is also C^q on U.

Observe that the chain rule hypothesis involves functions defined on open sets. Nevertheless, when the first function is defined on a closed interval $[a,b]$, a variant of the result is also true.

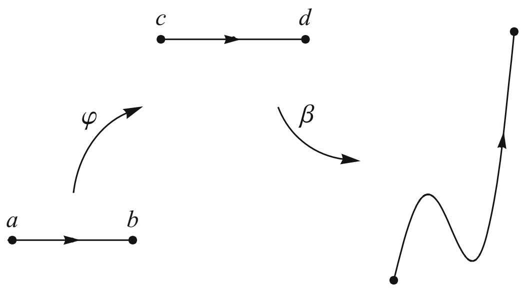


Fig. 2.2 $\boldsymbol{\beta}$ and $\boldsymbol{\beta} \circ \varphi$ are equivalent paths

Corollary 2.4.1. *Let V be an open subset of $\mathbb{R}^m$, $\boldsymbol{\gamma} : [a,b] \longrightarrow V$ a path that has a derivative at each point of $[a,b]$, and suppose $\boldsymbol{G} : V \longrightarrow \mathbb{R}^m$ is differentiable at each point of $\boldsymbol{\gamma}([a,b])$. Then $\boldsymbol{G} \circ \boldsymbol{\gamma} : [a,b] \longrightarrow \mathbb{R}^m$ has a derivative at each point of $[a,b]$ and*

$$(\boldsymbol{G} \circ \boldsymbol{\gamma})'(t) = \boldsymbol{G}'(\boldsymbol{\gamma}(t)) \cdot \boldsymbol{\gamma}'(t),$$

for every $t \in [a,b]$. Here, the derivatives at $t = a$ and $t = b$ are understood to be derivatives from the right and from the left respectively. Furthermore, if $\boldsymbol{\gamma}$ is C^q on $[a,b]$ and $\boldsymbol{G}$ is C^q on the open set V, then $\boldsymbol{G} \circ \boldsymbol{\gamma}$ is also C^q on $[a,b]$.

Proof. We write $\boldsymbol{\gamma}(t) = (\gamma_1(t), \ldots, \gamma_m(t))$, where $\gamma_j : [a,b] \longrightarrow \mathbb{R}$ has a derivative on $[a,b]$ for each $j = 1, \ldots, n$ (or C^q on $[a,b]$ respectively).

It is known that γ_j can be extended to $\tilde{\gamma}_j : \mathbb{R} \to \mathbb{R}$ with a derivative at each point of $\mathbb{R}$ (or C^q on $\mathbb{R}$ respectively). We take $\tilde{\boldsymbol{\gamma}} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m) : \mathbb{R} \longrightarrow \mathbb{R}^m$. Now we define $U = (\tilde{\boldsymbol{\gamma}})^{-1}(V)$. Since $\tilde{\boldsymbol{\gamma}}$ is continuous on $\mathbb{R}$, U is an open subset of $\mathbb{R}$ containing $[a,b]$. Moreover, we have $\tilde{\boldsymbol{\gamma}} : U \longrightarrow V$ and $\boldsymbol{G} : V \longrightarrow \mathbb{R}^m$ with $\tilde{\boldsymbol{\gamma}}$ and $\boldsymbol{G}$ differentiable (respectively of class C^q) on their respective domains. Now the chain rule gives the conclusion. $\square$

The same argument clearly proves the following further variation of the chain rule.

Corollary 2.4.2. *Let $\alpha : [a,b] \longrightarrow \mathbb{R}$ be differentiable at each point of $[a,b]$ and let $\boldsymbol{\beta} : [c,d] \longrightarrow \mathbb{R}^m$ be a path that has a derivative at each point of $[c,d]$ with $\alpha([a,b]) \subset [c,d]$. Then $\boldsymbol{\beta} \circ \alpha : [a,b] \longrightarrow \mathbb{R}^m$ has a derivative at each point of $[a,b]$ and*

$$(\boldsymbol{\beta} \circ \alpha)'(t) = \boldsymbol{\beta}'(\alpha(t))\alpha'(t),$$

for every $t \in [a,b]$. Furthermore, if α is C^q on $[a,b]$ and $\boldsymbol{\beta}$ is C^q on $[c,d]$, then $\boldsymbol{\beta} \circ \alpha$ is also C^q on $[a,b]$.

Proposition 2.4.1. *If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are two equivalent paths and one of them, say $\boldsymbol{\beta}$, is piecewise C^1, then $\boldsymbol{\alpha}$ is piecewise C^1 and $L(\boldsymbol{\alpha}) = L(\boldsymbol{\beta})$.*

Proof. Write $\boldsymbol{\alpha} = \boldsymbol{\beta} \circ \varphi$, where φ is as in Definition 2.4.1 and let $Q = \{c = u_1 < \cdots < u_k = d\}$ be a partition of $[c,d]$ such that $\boldsymbol{\beta}_{|[u_j,u_{j+1}]}$ is C^1 on $[u_j, u_{j+1}]$. If we take $t_j := \varphi^{-1}(u_j)$, then $P = \{a = t_1 < \cdots < t_k = b\}$ is a partition of $[a,b]$.

Since $\boldsymbol{\alpha}|_{[t_j,t_{j+1}]} = (\boldsymbol{\beta} \circ \varphi)|_{[t_j,t_{j+1}]} = (\boldsymbol{\beta}|_{[u_j,u_{j+1}]}) \circ \varphi|_{[t_j,t_{j+1}]}$ is the composition of two C^1 mappings (on closed intervals), it is itself C^1 on $[t_j, t_{j+1}]$, and thus $\boldsymbol{\alpha}$ is piecewise C^1. Now,

$$L(\boldsymbol{\beta}) = \int_{c=\varphi(a)}^{d=\varphi(b)} ||\boldsymbol{\beta}'(u)|| \mathrm{d}u = \sum_{j=1}^{k-1} \int_{u_j}^{u_{j+1}} ||\boldsymbol{\beta}'(u)|| \mathrm{d}u.$$

For each $1 \leq j \leq k-1$ we can apply the change of variable $u = \varphi(t)$ to obtain

$$\begin{aligned}\int_{u_j}^{u_{j+1}} ||\boldsymbol{\beta}'(u)||\mathrm{d}u &= \int_{t_j}^{t_{j+1}} ||\boldsymbol{\beta}'(\varphi(t))||\varphi'(t)\mathrm{d}t \\ &= \int_{t_j}^{t_{j+1}} ||\boldsymbol{\beta}'(\varphi(t))\varphi'(t)||\mathrm{d}t = \int_{t_j}^{t_{j+1}} ||(\boldsymbol{\beta}\circ\varphi)'(t)||\mathrm{d}t \\ &= \int_{t_j}^{t_{j+1}} ||\boldsymbol{\alpha}'(t)||\mathrm{d}t.\end{aligned}$$

Summing for all values of $1 \le j \le k-1$, we conclude that

$$L(\boldsymbol{\beta}) = \int_a^b ||\boldsymbol{\alpha}'(t)||\mathrm{d}t = L(\boldsymbol{\alpha}).$$

□

Proposition 2.4.2. *Let $\boldsymbol{\omega}$ be a continuous 1-form on U, and let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two piecewise C^1 paths in U with $\boldsymbol{\alpha} \sim \boldsymbol{\beta}$. Then*

$$\int_{\boldsymbol{\alpha}} \boldsymbol{\omega} = \int_{\boldsymbol{\beta}} \boldsymbol{\omega}.$$

Proof. Let us first assume that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are paths of class C^1. Then, employing the same notation as the previous proof, we have

$$\begin{aligned}\int_{\boldsymbol{\beta}} \boldsymbol{\omega} &= \int_{\varphi(a)}^{\varphi(b)} \boldsymbol{\omega}(\boldsymbol{\beta}(u))\big(\boldsymbol{\beta}'(u)\big)\,\mathrm{d}u \\ &= \int_a^b \boldsymbol{\omega}(\boldsymbol{\beta}(\varphi(t)))\big(\boldsymbol{\beta}'(\varphi(t))\big)\,\varphi'(t)\mathrm{d}t \\ &= \int_a^b \boldsymbol{\omega}(\boldsymbol{\beta}(\varphi(t)))\big((\boldsymbol{\beta}\circ\varphi)'(t)\big)\,\mathrm{d}t \\ &= \int_a^b \boldsymbol{\omega}(\boldsymbol{\alpha}(t))(\boldsymbol{\alpha}'(t))\mathrm{d}t = \int_{\boldsymbol{\alpha}} \boldsymbol{\omega}.\end{aligned}$$

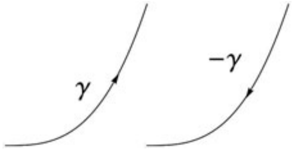
Again we are using the chain rule theorem for extensions of $\boldsymbol{\beta}$ and φ to the whole real line. In the general case, we consider a partition

$$P := \{a = t_1 < t_2 < \cdots < t_k = b\}$$

of the interval $[a,b]$ for which $\boldsymbol{\beta}$ is of class C^1 on each subinterval $[u_j, u_{j+1}] = [\varphi(t_j), \varphi(t_{j+1})]$. Then $\boldsymbol{\alpha} = \boldsymbol{\beta}\circ\varphi$ is also of class C^1 on each subinterval $[t_j, t_{j+1}]$. We therefore have from above that

$$\int_{\boldsymbol{\beta}_j} \boldsymbol{\omega} = \int_{\boldsymbol{\alpha}_j} \boldsymbol{\omega},$$

Fig. 2.3 $\boldsymbol{\gamma}$ and $-\boldsymbol{\gamma}$ are opposite paths



where $\boldsymbol{\beta}_j = \boldsymbol{\beta}_{|[u_j,u_{j+1}]}$ and $\boldsymbol{\alpha}_j = \boldsymbol{\alpha}_{|[t_j,t_{j+1}]}$. Summing this identity for all values of j from 1 to $k-1$, we get

$$\int_{\boldsymbol{\beta}} \boldsymbol{\omega} = \int_{\boldsymbol{\alpha}} \boldsymbol{\omega}.$$

□

Notice that in the above proof neither $\varphi'(t_j)$ nor $(\boldsymbol{\beta}\circ\varphi)'(t_j)$ is necessarily defined for $j = 1,\dots,k-1$.

Definition 2.4.2. Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ be a piecewise C^1 path. The *opposite* path (Fig. 2.3) is defined by $(-\boldsymbol{\gamma}) : [-b,-a] \to \mathbb{R}^n$, $(-\boldsymbol{\gamma})(t) := \boldsymbol{\gamma}(-t)$.

The opposite path $-\boldsymbol{\gamma}$ moves along $\boldsymbol{\gamma}([a,b])$ in the opposite direction to that given by the path $\boldsymbol{\gamma}$. The initial point of $-\boldsymbol{\gamma}$ is the final point of $\boldsymbol{\gamma}$ and vice versa. The reader will note the difference between $(-\boldsymbol{\gamma})(t)$ and $-(\boldsymbol{\gamma}(t))$.

Definition 2.4.3. Let $\boldsymbol{\alpha} : [a,b] \to \mathbb{R}^n$ and $\boldsymbol{\beta} : [c,d] \to \mathbb{R}^n$ be two piecewise C^1 paths such that $\boldsymbol{\alpha}(b) = \boldsymbol{\beta}(c)$. By a *union of the paths* $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, denoted by $\boldsymbol{\alpha}\cup\boldsymbol{\beta}$, we mean any piecewise C^1 path $\boldsymbol{\xi} : [e,f] \to \mathbb{R}^n$ with the property that for some $e < r < f$,

$$\boldsymbol{\xi}_{|[e,r]} \sim \boldsymbol{\alpha} \quad \text{and} \quad \boldsymbol{\xi}_{|[r,f]} \sim \boldsymbol{\beta}.$$

It is obvious that[3] the trace of $\boldsymbol{\alpha}\cup\boldsymbol{\beta}$ is $\boldsymbol{\alpha}([a,b]) \cup \boldsymbol{\beta}([c,d])$ and $\boldsymbol{\alpha}\cup\boldsymbol{\beta}$ consists of tracing over $\boldsymbol{\alpha}$ first and then over $\boldsymbol{\beta}$. Such a union of paths always exists, a concrete example being given by $\boldsymbol{\xi} : [0,1] \to \mathbb{R}^n$, where

$$\boldsymbol{\xi}(t) = \begin{cases} \boldsymbol{\alpha}\big(2t(b-a)+a\big), & 0 \le t \le \frac{1}{2}, \\ \boldsymbol{\beta}\big((2t-1)(d-c)+c\big), & \frac{1}{2} \le t \le 1. \end{cases}$$

Proposition 2.4.3. *Let $\boldsymbol{\omega}$ be a continuous 1-form on an open subset U of $\mathbb{R}^n$ and let $\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\gamma}$ be three piecewise C^1 paths in U with the final point of $\boldsymbol{\alpha}$ coinciding with the initial point of $\boldsymbol{\beta}$. Then*

(1) $\displaystyle\int_{-\boldsymbol{\gamma}} \boldsymbol{\omega} = -\int_{\boldsymbol{\gamma}} \boldsymbol{\omega}$,

(2) $\displaystyle\int_{\boldsymbol{\alpha}\cup\boldsymbol{\beta}} \boldsymbol{\omega} = \int_{\boldsymbol{\alpha}} \boldsymbol{\omega} + \int_{\boldsymbol{\beta}} \boldsymbol{\omega}$.

[3] The right-hand side here is a normal set union. Note that unlike set union, path union is not a symmetric operation.

Proof. Since $(-\gamma)'(t) = -\gamma'(-t)$, the substitution $u = -t$ gives

$$\begin{aligned}\int_{-\boldsymbol{\gamma}} \boldsymbol{\omega} &= \int_{-b}^{-a} \boldsymbol{\omega}(\boldsymbol{\gamma}(-t))\left(-\boldsymbol{\gamma}'(-t)\right)\mathrm{d}t \\ &= \int_{a}^{b} -\boldsymbol{\omega}(\boldsymbol{\gamma}(u))\left(\boldsymbol{\gamma}'(u)\right)\mathrm{d}u = -\int_{\boldsymbol{\gamma}} \boldsymbol{\omega}.\end{aligned}$$

Also

$$\begin{aligned}\int_{\boldsymbol{\alpha}\cup\boldsymbol{\beta}} \boldsymbol{\omega} &= \int_{\mathrm{e}}^{r} \boldsymbol{\omega}(\boldsymbol{\xi}(t))\left(\boldsymbol{\xi}'(t)\right)\mathrm{d}t + \int_{r}^{f} \boldsymbol{\omega}(\boldsymbol{\xi}(t))\left(\boldsymbol{\xi}'(t)\right)\mathrm{d}t \\ &= \int_{\boldsymbol{\xi}_{|[e,r]}} \boldsymbol{\omega} + \int_{\boldsymbol{\xi}_{|[r,f]}} \boldsymbol{\omega} = \int_{\boldsymbol{\alpha}} \boldsymbol{\omega} + \int_{\boldsymbol{\beta}} \boldsymbol{\omega}.\end{aligned}$$

$\square$

2.5 Conservative Fields: Exact Differential Forms

We begin this section with an example of a vector field for which the line integral over two different paths from $(0,0)$ to $(1,1)$ is the same.

Example 2.5.1. Let $\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by $\boldsymbol{F}(x,y) = (x,y)$ and let us consider the paths (Fig. 2.4)

$$\boldsymbol{\gamma}_1 : [0,1] \to \mathbb{R}^2 \text{ and } \boldsymbol{\gamma}_2 : [0,1] \to \mathbb{R}^2$$

defined by $\boldsymbol{\gamma}_1(t) = (t,t)$ and $\boldsymbol{\gamma}_2(t) = (t,t^2)$. Then

$$\int_{\boldsymbol{\gamma}_1} \boldsymbol{F} = \int_0^1 \langle (t,t),(1,1)\rangle \; \mathrm{d}t = \int_0^1 (t+t)\; \mathrm{d}t = 1.$$

Also

$$\int_{\boldsymbol{\gamma}_2} \boldsymbol{F} = \int_0^1 \left\langle (t,t^2),(1,2t)\right\rangle \; \mathrm{d}t = \int_0^1 (t+2t^3)\; \mathrm{d}t = 1.$$

The fact that the two line integrals here take the same value is not surprising if one remembers that a line integral represents the work done by a force field in moving a particle along a path, while it is well known from physics that under the action of a gravitational field,[4] the work done to move an object between two different points depends only on the difference of the potential energies at these points, and is therefore independent of the path taken (see Example 2.5.4). Our next example, a vector field for which the line integral over two different paths from $(0,0)$ to $(1,1)$ is different, shows that this behavior is not universal.

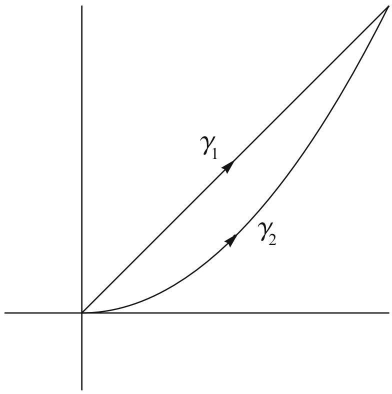


Fig. 2.4 The paths of Example 2.5.1

[4] Although the vector field in our example is not the gravitational field of Example 1.2.2, it is similar enough, in a sense yet to be defined, that the physical argument remains valid.

Example 2.5.2. Let $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ be the paths given in Example 2.5.1 but let us consider instead the vector field

$$\boldsymbol{F}(x,y) = \left(-y+\frac{3}{8}, x-\frac{1}{2}\right).$$

Then

$$\int_{\boldsymbol{\gamma}_1} \boldsymbol{F} = \int_0^1 \left\langle \left(-t+\frac{3}{8}, t-\frac{1}{2}\right), (1,1) \right\rangle \,\mathrm{d}t = \int_0^1 -\frac{1}{8}\mathrm{d}t = -\frac{1}{8},$$

while

$$\int_{\boldsymbol{\gamma}_2} \boldsymbol{F} = \int_0^1 \left\langle \left(-t^2+\frac{3}{8}, t-\frac{1}{2}\right), (1,2t) \right\rangle \,\mathrm{d}t = \int_0^1 \left(t^2-t+\frac{3}{8}\right)\mathrm{d}t = \frac{1}{3}-\frac{1}{8}.$$

Definition 2.5.1. Let $\boldsymbol{F} = (f_1,\ldots,f_n)$ be a vector field on an open subset U of $\mathbb{R}^n$ with associated 1-form $\boldsymbol{\omega} = f_1 \cdot \mathrm{d}x_1 + \cdots + f_n \cdot \mathrm{d}x_n$. If there is a function of class C^1, $f : U \subset \mathbb{R}^n \to \mathbb{R}$, such that $\nabla f = \boldsymbol{F}$ (or equivalently, $\mathrm{d}f = \boldsymbol{\omega}$) on U, then the vector field $\boldsymbol{F}$ is said to be *conservative* and the 1-form $\boldsymbol{\omega}$ is said to be *exact*. The scalar function f is called a *potential* of the conservative vector field $\boldsymbol{F}$.

Conservative fields have a similar behavior to that of Example 2.5.1 in that their line integrals are independent of path. This is an immediate consequence of the following result, which is a generalization of the fundamental theorem of calculus.

Theorem 2.5.1. *Let $g : U \subset \mathbb{R}^n \to \mathbb{R}$ be a function of class C^1 on an open set U and $\boldsymbol{\gamma} : [a,b] \to U$ a piecewise C^1 path. Then*

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\nabla} g = \int_{\boldsymbol{\gamma}} \mathrm{d}g = g(\boldsymbol{\gamma}(b)) - g(\boldsymbol{\gamma}(a)).$$

Proof. Since $\mathrm{d}g$ is the 1-form associated to the vector field $\boldsymbol{\nabla} g$, the first equation follows from our observation immediately preceding Sect. 2.4. For the second equation, we take the usual expansion of the 1-form $\mathrm{d}g$,

$$\mathrm{d}g = \sum_{j=1}^{n} \frac{\partial g}{\partial x_j} \cdot \mathrm{d}x_j$$

and apply the chain rule,

$$\sum_{j=1}^{n} \frac{\partial g}{\partial x_j}(\boldsymbol{\gamma}(t))\boldsymbol{\gamma}_j'(t) = (g \circ \boldsymbol{\gamma})'(t),$$

which is valid for all but a finite set of t in $[a,b]$, to obtain

$$\int_{\boldsymbol{\gamma}} \mathrm{d}g = \int_a^b \left(\sum_{j=1}^{n} \frac{\partial g}{\partial x_j}(\boldsymbol{\gamma}(t))\boldsymbol{\gamma}_j'(t) \right) \mathrm{d}t = \int_a^b (g \circ \boldsymbol{\gamma})'(t)\mathrm{d}t.$$

Let $P := \{a = t_1 < \cdots < t_k = b\}$ be a partition such that $\boldsymbol{\gamma}|_{[t_i,t_{i+1}]}$ is of class C^1 for every $1 \le i \le k-1$. The fundamental theorem of calculus gives

$$\begin{aligned}\int_{\boldsymbol{\gamma}} \mathrm{d}g &= \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} (g \circ \boldsymbol{\gamma})'(t)\mathrm{d}t \\ &= \sum_{i=1}^{k-1} \big((g \circ \boldsymbol{\gamma})(t_{i+1}) - (g \circ \boldsymbol{\gamma})(t_i)\big) \\ &= g(\boldsymbol{\gamma}(b)) - g(\boldsymbol{\gamma}(a)).\end{aligned}$$

□

For our next result we recall the concepts of connected set and path-connected set.

Definition 2.5.2. Let C be a subset of $\mathbb{R}^n$.

1. C is said to be *connected* if there do not exist two open sets V and W in $\mathbb{R}^n$ such that

   a. $C \subset V \cup W$;
   b. $C \cap V \neq \varnothing$ and $C \cap W \neq \varnothing$;
   c. $C \cap V \cap W = \varnothing$.

   In other words, a set C is connected if and only if with the topology induced on C by $\mathbb{R}^n$, the only subsets of C that are both open and closed are C itself and the empty set $\varnothing$.
2. C is said to be *path connected* if given $\boldsymbol{x}, \boldsymbol{y} \in C$, there exists a path $\boldsymbol{\alpha} : [a,b] \to C$ such that $\boldsymbol{\alpha}(a) = \boldsymbol{x}$ and $\boldsymbol{\alpha}(b) = \boldsymbol{y}$. If the connecting path can be chosen to be polygonal, then the set is called *polygonally connected*.

The continuous image of a connected set is connected, from which it follows that every path-connected set is connected. For an open subset of $\mathbb{R}^n$, the converse also holds, and the two concepts coincide. This result can be found, for example, in [1, Theorem 4.43].

Theorem 2.5.2. *Let U be a connected open subset of $\mathbb{R}^n$. Then U is polygonally connected.*

Theorem 2.5.3. *Let $\boldsymbol{F} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field on an open set U. The following conditions are equivalent:*

(1) $\boldsymbol{F}$ is a conservative vector field.
(2) If $\boldsymbol{\gamma} : [a,b] \to U \subset \mathbb{R}^n$ is a piecewise C^1 path that is closed ($\boldsymbol{\gamma}(a) = \boldsymbol{\gamma}(b)$), then

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} = 0.$$

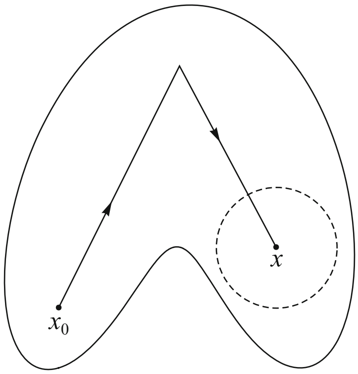


Fig. 2.5 A polygonal path in U from $\mathbf{x_0}$ to $\boldsymbol{x}$

(3) If $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are two piecewise C^1 paths with the same initial and final points, then

$$\int_{\boldsymbol{\gamma}_1} \boldsymbol{F} = \int_{\boldsymbol{\gamma}_1} \boldsymbol{F}.$$

Proof. $(1) \Rightarrow (2)$. By hypothesis, there is a C^1 function f on U,

$$f : U \subset \mathbb{R}^n \to \mathbb{R},$$

with $\boldsymbol{\nabla} f = \boldsymbol{F}$. Let $\boldsymbol{\gamma} : [a,b] \to U \subset \mathbb{R}^n$ be a closed piecewise C^1 path. According to Theorem 2.5.1,

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} = f(\boldsymbol{\gamma}(b)) - f(\boldsymbol{\gamma}(a)) = 0.$$

$(2) \Rightarrow (3)$. If $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ have the same initial point and also the same final point, then

$$\boldsymbol{\gamma} := \boldsymbol{\gamma}_1 \cup (-\boldsymbol{\gamma}_2)$$

is a closed piecewise C^1 path in U. Consequently,

$$\int_{\boldsymbol{\gamma}_1} \boldsymbol{F} - \int_{\boldsymbol{\gamma}_2} \boldsymbol{F} = \int_{\boldsymbol{\gamma}} \boldsymbol{F} = 0.$$

$(3) \Rightarrow (1)$. Let us first assume that U is an open and connected set in $\mathbb{R}^n$. Then U is also path connected and polygonally connected. We fix a point $\boldsymbol{x}_0 \in U$ and define the potential function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ as follows (Fig. 2.5): For every $\boldsymbol{x} \in U$ let $\boldsymbol{\gamma}_x$ be any polygonal path in U from $\mathbf{x}_0$ to $\boldsymbol{x}$ and let

$$f(\boldsymbol{x}) := \int_{\boldsymbol{\gamma}_x} \boldsymbol{F}.$$

The hypothesis of (3) implies that $f(\boldsymbol{x})$ does not depend on the particular polygonal path $\boldsymbol{\gamma}_x$ chosen, and thus the definition is unambiguous. To prove the result, we will show that f is a function of class C^1 whose gradient coincides with the vector field $\boldsymbol{F}$.

Given $\boldsymbol{x} \in U$, we can find $\delta > 0$ such that the open ball $B(\boldsymbol{x}, \delta)$ centered at $\boldsymbol{x}$ and with radius δ is contained in U. Observe that for every $1 \leq j \leq n$ and $0 < |t| < \delta$, if $\boldsymbol{\gamma}_x$ is a polygonal path in U from point $\mathbf{x}_0$ to point $\boldsymbol{x}$, then $\boldsymbol{\gamma}_x \cup [\boldsymbol{x}, \boldsymbol{x} + t\mathbf{e}_j]$ is a polygonal path in U from $\mathbf{x}_0$ to $\boldsymbol{x} + t\mathbf{e}_j$. Hence

$$f(\boldsymbol{x}+t\mathbf{e}_j) - f(\boldsymbol{x}) = \int_{\boldsymbol{\gamma}_x \cup [\boldsymbol{x}, \boldsymbol{x}+t\mathbf{e}_j]} \boldsymbol{F} - \int_{\boldsymbol{\gamma}_x} \boldsymbol{F} = \int_{[\boldsymbol{x}, \boldsymbol{x}+t\mathbf{e}_j]} \boldsymbol{F}.$$

Let us assume for simplicity that $t > 0$ and parameterize the line segment by $\boldsymbol{\gamma}(s) = \boldsymbol{x} + s\mathbf{e}_j$, $0 \leq s \leq t$. We also write $\boldsymbol{F}$ in terms of component functions, $\boldsymbol{F} = (f_1, \ldots, f_n)$. Then for each $1 \leq j \leq n$,

$$f(\boldsymbol{x}+t\mathbf{e}_j) - f(\boldsymbol{x}) = \int_0^t \langle \boldsymbol{F}(\boldsymbol{x}+s\mathbf{e}_j), \mathbf{e}_j \rangle \, \mathrm{d}s = \int_0^t f_j(\boldsymbol{x}+s\mathbf{e}_j) \, \mathrm{d}s.$$

It follows that

$$\begin{aligned}\left| \frac{f(\boldsymbol{x}+t\mathbf{e}_j) - f(\boldsymbol{x})}{t} - f_j(\mathbf{x}) \right| &= \left| \frac{1}{t} \int_0^t (f_j(\boldsymbol{x}+s\mathbf{e}_j) - f_j(\boldsymbol{x})) \, \mathrm{d}s \right| \\ &\leq \frac{1}{t} \int_0^t |f_j(\boldsymbol{x}+s\mathbf{e}_j) - f_j(\boldsymbol{x})| \mathrm{d}s \\ &\leq \max_{0 \leq s \leq t} |f_j(\boldsymbol{x}+s\mathbf{e}_j) - f_j(\boldsymbol{x})|.\end{aligned}$$

Since each f_j is a continuous function, the above expression tends to zero as t tends to zero. As a consequence,

$$\frac{\partial f}{\partial x_j}(\boldsymbol{x}) = f_j(\boldsymbol{x}).$$

In particular, f is a function of class C^1 and $\boldsymbol{\nabla} f = \boldsymbol{F}$ on U. In the general case in which U is not connected, the above argument can be applied to each connected component of U in order to construct a suitable potential function (see, for instance, Burkill [4, p. 60]). □

Definition 2.5.3. An open set $U \subset \mathbb{R}^n$ is said to be *starlike* with respect to the point $\boldsymbol{a} \in U$ if the segment $[\boldsymbol{a}, \boldsymbol{x}]$ is contained in U for every $\boldsymbol{x} \in U$.

We recall that the triangle with vertices $\boldsymbol{a}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ is the set

$$\{\alpha\boldsymbol{a} + \beta\boldsymbol{x} + \gamma\boldsymbol{y} \ : \ 0 \leq \alpha, \beta, \gamma \text{ and } \alpha + \beta + \gamma = 1\}.$$

By its boundary we understand the closed polygonal arc $[\boldsymbol{a}, \boldsymbol{x}] \cup [\boldsymbol{x}, \boldsymbol{y}] \cup [\boldsymbol{y}, \boldsymbol{a}]$.

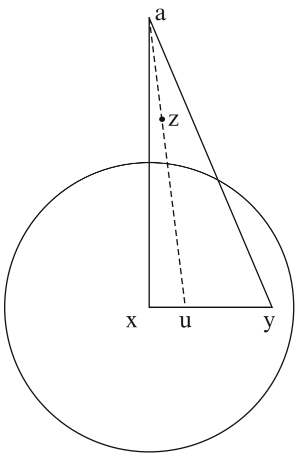


Fig. 2.6 Triangle of Lemma 2.5.1

Lemma 2.5.1. *Let U be starlike with respect to the point $\boldsymbol{a} \in U$. If $\boldsymbol{y} \in B(\boldsymbol{x}, R) \subset U$, then the triangle (Fig. 2.6) with vertex $\{\boldsymbol{a}, \boldsymbol{x}, \boldsymbol{y}\}$ is contained in U.*

Proof. Let $\mathbf{z} = \alpha \boldsymbol{a} + \beta \boldsymbol{x} + \gamma \boldsymbol{y}$ be given, where $\alpha, \beta, \gamma \geq 0$ and $\alpha + \beta + \gamma = 1$. If $\alpha = 1$, then $z = a \in U$. In other case, we put

$$\boldsymbol{u} = \frac{\beta}{1-\alpha}\boldsymbol{x} + \frac{\gamma}{1-\alpha}\boldsymbol{y}.$$

Since

$$\frac{\beta}{1-\alpha} + \frac{\gamma}{1-\alpha} = 1,$$

we obtain that $\boldsymbol{u} \in [\boldsymbol{x}, \boldsymbol{y}] \subset B(\boldsymbol{x}, R) \subset U$. Finally, from $\boldsymbol{z} = \alpha \boldsymbol{a} + (1-\alpha)\boldsymbol{u}$ we conclude that

$$\boldsymbol{z} \in [\boldsymbol{a}, \boldsymbol{u}] \subset U.$$

□

Theorem 2.5.4. *If the open set U is starlike with respect to the point $\boldsymbol{a} \in U$, then the conditions of Theorem 2.5.3 are equivalent to the following:*

(4) *If $\boldsymbol{\gamma}$ is the boundary of a triangle contained in U, then*

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = 0.$$

Proof. Since condition (2) clearly implies (4), it is enough to prove that (4) implies (1). We do this by arguing that the function f defined by

$$f(\boldsymbol{x}) := \int_{[\boldsymbol{a}, \boldsymbol{x}]} \boldsymbol{F}$$

is, in fact, a potential for the vector field $\boldsymbol{F}$. For every $\boldsymbol{x} \in U$ we choose $R_{\boldsymbol{x}} > 0$ with $B(\boldsymbol{x}, R_{\boldsymbol{x}}) \subset U$. According to the previous lemma, for each $1 \leq j \leq n$ and for all $t \in \mathbb{R}$ with $|t| < R_{\boldsymbol{x}}$, the triangle with vertices $\boldsymbol{a}$, $\boldsymbol{x}$, and $\boldsymbol{x} + t\mathbf{e}_j$ is contained in U. Hence, we deduce from condition (4) that

$$f(\boldsymbol{x} + t\mathbf{e}_j) - f(\boldsymbol{x}) = \int_{[\boldsymbol{a}, \boldsymbol{x}+t\mathbf{e}_j]} \boldsymbol{F} - \int_{[\boldsymbol{a}, \boldsymbol{x}]} \boldsymbol{F} = \int_{[\boldsymbol{x}, \boldsymbol{x}+t\mathbf{e}_j]} \boldsymbol{F}.$$

The argument now proceeds as in the proof of (3) $\Rightarrow$ (1) in Theorem 2.5.3. $\square$

Of course, Theorem 2.5.3 can also be interpreted as a characterization of those differential forms of degree 1 that are exact.

Example 2.5.3. Let

$$\boldsymbol{\omega} = -\frac{y}{x^2 + y^2} \cdot \mathrm{d}x + \frac{x}{x^2 + y^2} \cdot \mathrm{d}y,$$

which is a continuous 1-form in $U := \mathbb{R}^2 \setminus \{(0,0)\}$. Then $\boldsymbol{\omega}$ is not exact, since

$$\boldsymbol{\gamma} : [0, 2\pi] \to \mathbb{R}^2, \ \boldsymbol{\gamma}(t) := (\cos(t), \sin(t)),$$

defines a closed path contained in U for which $\displaystyle\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = 2\pi \neq 0.$

However, if we consider $V := \mathbb{R}^2 \setminus \{(0, y) : y \in \mathbb{R}\}$ and $g : V \to (-\frac{\pi}{2}, \frac{\pi}{2})$ defined by

$$g(x, y) := \arctan \frac{y}{x},$$

then $\boldsymbol{\omega} = \mathrm{d}g$ on V.

Thus, the fact that a vector field is conservative depends not only on the expression of the field but also on the region U we are dealing with. That is, a vector field admitting a potential function on a given open set may not be conservative on some larger set.

Example 2.5.4. The gravitational field is conservative.

Let us consider a particle of mass M located at the origin. The force of attraction exerted on a particle of unit mass located at point $(x, y, z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ is

$$\boldsymbol{F}(x, y, z) = -\frac{GM}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \, (x, y, z),$$

where G is the gravitational constant. Since the magnitude of the force is the same at all points equidistant from the origin, it seems reasonable to expect that this is also true for the potential function.

Consequently, we look for a function of $r = \sqrt{x^2+y^2+z^2}$ whose derivative is $-\frac{GM}{r^2}$. An example of such a function is $\frac{GM}{r}$, and it is easy to check that

$$f(x,y,z) := \frac{GM}{\sqrt{x^2+y^2+z^2}}$$

satisfies $\boldsymbol{\nabla} f = \boldsymbol{F}$. That is, f is a potential function for the gravitational field.

One should remark that what is called the gravitational potential in physics is the function $V := -f$. Hence, the work done by the field to move a particle from point $\boldsymbol{A}$ to point $\boldsymbol{B}$ is independent of the trajectory of the particle, and its value is the difference of potentials $V(\boldsymbol{A}) - V(\boldsymbol{B})$.

We now obtain a necessary condition for a vector field to be conservative.

Theorem 2.5.5. *Let $\boldsymbol{F} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a conservative vector field of class C^1 on an open set U and write $\boldsymbol{F} := (f_1, f_2, \ldots, f_n)$. Then*

$$\frac{\partial f_j}{\partial x_k}(\boldsymbol{x}) = \frac{\partial f_k}{\partial x_j}(\boldsymbol{x})$$

for every $j,k \in \{1,\ldots,n\}$ and every $\boldsymbol{x} \in U$.

Proof. By hypothesis, there is a function $g : U \subset \mathbb{R}^n \to \mathbb{R}$ such that $\boldsymbol{\nabla} g = \boldsymbol{F}$. Therefore $f_j = \frac{\partial g}{\partial x_j}$, and we can write

$$\frac{\partial f_j}{\partial x_k}(\boldsymbol{x}) = \frac{\partial^2 g}{\partial x_k\, \partial x_j}(\boldsymbol{x})$$

and similarly

$$\frac{\partial f_k}{\partial x_j}(\boldsymbol{x}) = \frac{\partial^2 g}{\partial x_j\, \partial x_k}(\boldsymbol{x})$$

for all $\boldsymbol{x} \in U$. Since $\boldsymbol{F}$ is a function of class C^1 on U, it follows that g is of class C^2, and Schwarz's theorem concerning the symmetry of second-order partial derivatives gives our result. $\square$

In particular, if $\boldsymbol{F} = (P,Q)$ is a conservative vector field on $U \subset \mathbb{R}^2$ of class C^1, then for every $(x,y) \in U$,

$$\frac{\partial Q}{\partial x}(x,y) = \frac{\partial P}{\partial y}(x,y).$$

It also follows from Definition 1.2.11 and Theorem 2.5.5 that every conservative vector field $\boldsymbol{F} = (f_1, f_2, f_3)$ on the open set $U \subset \mathbb{R}^3$ of class C^1 satisfies

$$\begin{aligned} \operatorname{Curl} \boldsymbol{F}(x,y,z) &= \left(\tfrac{\partial f_3}{\partial y} - \tfrac{\partial f_2}{\partial z}, \tfrac{\partial f_1}{\partial z} - \tfrac{\partial f_3}{\partial x}, \tfrac{\partial f_2}{\partial x} - \tfrac{\partial f_1}{\partial y}\right) \\ &= 0 \end{aligned}$$

for all $(x,y,z) \in U$.

In Example 2.5.3 we proved that the vector field

$$\boldsymbol{F} = (P,Q) : U \subset \mathbb{R}^2 \to \mathbb{R}^2$$

defined on $U = \mathbb{R}^2 \setminus \{(0,0)\}$ by

$$P(x,y) = -\frac{y}{x^2+y^2},\ Q(x,y) = \frac{x}{x^2+y^2}$$

is not conservative. However,

$$\frac{\partial Q}{\partial x}(x,y) = \frac{\partial P}{\partial y}(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Thus, the condition in Theorem 2.5.5 is not sufficient, in general, for a vector field to be conservative. However, if we impose certain geometric conditions on the domain U, then the condition of Theorem 2.5.5 is sufficient, as the next theorem shows. Before stating the result, we need to recall the following fact about the derivative of an integral (also called a parametric derivative).

Proposition 2.5.1. *Suppose $f : [a,b] \times [c,d] \to \mathbb{R}$ is a function of class C^1 and $g : [a,b] \to \mathbb{R}$ is defined by*

$$g(x) = \int_c^d f(x,t)\mathrm{d}t.$$

Then g is of class C^1 on $[a,b]$ and

$$g'(x) = \int_c^d \frac{\partial f}{\partial x}(x,t)\mathrm{d}t$$

for all $x \in [a,b]$.

The proof of this statement follows the proof of $(3) \Rightarrow (1)$ in Theorem 2.5.3, but the interested reader can find stronger results concerning the derivative of an integral, for example in [15, Theorem 9.42, p. 236].

Theorem 2.5.6 (Poincaré's lemma). *Let $\boldsymbol{F} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field of class C^1, $\boldsymbol{F} := (f_1, f_2, \ldots, f_n)$, on an open set U that is starlike with respect to the point $\boldsymbol{a}$. If*

$$\frac{\partial f_j}{\partial x_k}(\boldsymbol{x}) = \frac{\partial f_k}{\partial x_j}(\boldsymbol{x})$$

for every choice of the indices $1 \le j,k \le n$ and for all $\boldsymbol{x} \in U$, then $\boldsymbol{F}$ is conservative.

Proof. We define $g : U \subset \mathbb{R}^n \to \mathbb{R}$ by

$$g(\boldsymbol{x}) := \int_0^1 \sum_{j=1}^n f_j(\boldsymbol{a} + t(\boldsymbol{x} - \boldsymbol{a})) \cdot (x_j - a_j)\ \mathrm{d}t.$$

It follows that g is a function of class C^1 on U and that

$$
\begin{aligned}
\frac{\partial g}{\partial x_k}(\boldsymbol{x}) &= \sum_{j\neq k}(x_j-a_j)\cdot\int_0^1 \frac{\partial f_j}{\partial x_k}(\boldsymbol{a}+t(\boldsymbol{x}-\boldsymbol{a}))t\;\mathrm{d}t \\
&\quad+\int_0^1\left\{\frac{\partial f_k}{\partial x_k}(\boldsymbol{a}+t(\boldsymbol{x}-\boldsymbol{a}))t(x_k-a_k)+f_k(\boldsymbol{a}+t(\boldsymbol{x}-\boldsymbol{a}))\right\}\;\mathrm{d}t \\
&= \int_0^1\left[\sum_{j=1}^{n}(x_j-a_j)\frac{\partial f_k}{\partial x_j}(\boldsymbol{a}+t(\boldsymbol{x}-\boldsymbol{a}))t+f_k(\boldsymbol{a}+t(\boldsymbol{x}-\boldsymbol{a}))\right]\;\mathrm{d}t \\
&= \int_0^1\frac{d}{\mathrm{d}t}\left(f_k(\boldsymbol{a}+t(\boldsymbol{x}-\boldsymbol{a}))t\right)\;\mathrm{d}t = f_k(\boldsymbol{x}),
\end{aligned}
$$

for every $\boldsymbol{x}\in U$. This proves that $\boldsymbol{\nabla} g=\boldsymbol{F}$ and $\boldsymbol{F}$ is a conservative vector field on U. $\square$

Corollary 2.5.1. *Let $\boldsymbol{F}: U\subset\mathbb{R}^3\to\mathbb{R}^3$ be a vector field of class C^1 on an open starlike set U. Then $\boldsymbol{F}$ is conservative on U if and only if* $\operatorname{Curl}\boldsymbol{F}=0$.

2.6 Green's Theorem

This section constitutes a first approach to studying the result discovered in 1828 by George Green. This result is now known as Green's theorem and can be viewed as a generalization of the fundamental theorem of calculus. It states that the value of a double integral over the region bounded by a path is determined by the value of a line integral over that path. We will encounter Green's theorem again in Chap. 9, where it will appear as a particular case of the general Stokes's theorem. The results of Chap. 9 will allow us to apply Green's theorem to more general regions than those considered in this section, where we restrict attention to regions of type I (vertically simple) and type II (horizontally simple) (see Marsden–Tromba [14, 8.1, p. 494]).

For each $\varepsilon > 0$ let us consider the paths

$$\boldsymbol{\gamma}_\varepsilon^j : \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right] \longrightarrow \mathbb{R}^2$$

defined by

$$\boldsymbol{\gamma}_\varepsilon^1(t) := \left(t, -\frac{\varepsilon}{2}\right), \; \boldsymbol{\gamma}_\varepsilon^2(t) := \left(\frac{\varepsilon}{2}, t\right), \; \boldsymbol{\gamma}_\varepsilon^3(t) := \left(t, \frac{\varepsilon}{2}\right), \; \boldsymbol{\gamma}_\varepsilon^4(t) := \left(-\frac{\varepsilon}{2}, t\right).$$

Then

$$\boldsymbol{\gamma}_\varepsilon = \boldsymbol{\gamma}_\varepsilon^1 \cup \boldsymbol{\gamma}_\varepsilon^2 \cup (-\boldsymbol{\gamma}_\varepsilon^3) \cup (-\boldsymbol{\gamma}_\varepsilon^4)$$

represents the boundary of a square oriented counterclockwise (Fig. 2.7).

With this notation we have the following.

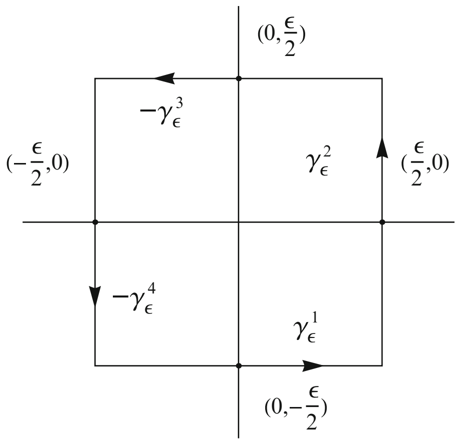


Fig. 2.7 The paths $\boldsymbol{\gamma}_\varepsilon^j$

Proposition 2.6.1. *Let $\boldsymbol{F}: U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field of class C^1 on the open set U with components $\boldsymbol{F} = (P, Q)$. Then for every $(x_0, y_0) \in U$ we have*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{(x_0,y_0)+\boldsymbol{\gamma}_\varepsilon} \boldsymbol{F} \quad = \quad \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)(x_0, y_0).$$

Proof. The line integral $\int_{(x_0,y_0)+\boldsymbol{\gamma}_\varepsilon} \boldsymbol{F}$ can be written as the difference between

$$\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \left(Q\left(x_0 + \frac{\varepsilon}{2}, y_0 + t \right) - Q\left(x_0 - \frac{\varepsilon}{2}, y_0 + t \right) \right) \mathrm{d}t$$

and

$$\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \left(P\left(x_0 + t, y_0 + \frac{\varepsilon}{2} \right) - P\left(x_0 + t, y_0 - \frac{\varepsilon}{2} \right) \right) \mathrm{d}t.$$

For every $t \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, we apply the mean value theorem to the functions $Q(\cdot, y_0 + t)$ and $P(x_0 + t, \cdot)$ to obtain points x_ε, y_ε (which depend on ε and also on t) such that $|x_0 - x_\varepsilon| < \frac{\varepsilon}{2}$, $|y_0 - y_\varepsilon| < \frac{\varepsilon}{2}$, and

$$\frac{1}{\varepsilon^2} \int_{(x_0,y_0)+\boldsymbol{\gamma}_\varepsilon} \boldsymbol{F} = \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \left(\frac{\partial Q}{\partial x}(x_\varepsilon, y_0 + t) - \frac{\partial P}{\partial y}(x_0 + t, y_\varepsilon) \right) \mathrm{d}t.$$

Since $\frac{\partial Q}{\partial x}$ is continuous at (x_0, y_0), for every $r > 0$ there exist $\delta > 0$ such that $(x, y) \in U$ and

$$\left| \frac{\partial Q}{\partial x}(x, y) - \frac{\partial Q}{\partial x}(x_0, y_0) \right| \leq r,$$

whenever $|x - x_0| < \delta$ and $|y - y_0| < \delta$. Then for each $0 < \varepsilon < \delta$ and $t \in (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$, we obtain

$$\left| \frac{\partial Q}{\partial x}(x_\varepsilon, y_0 + t) - \frac{\partial Q}{\partial x}(x_0, y_0) \right| \leq r.$$

Now we can write

$$\begin{aligned} &\left| \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \frac{\partial Q}{\partial x}(x_\varepsilon, y_0 + t)\, \mathrm{d}t - \frac{\partial Q}{\partial x}(x_0, y_0) \right| \\ &\quad = \left| \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \left(\frac{\partial Q}{\partial x}(x_\varepsilon, y_0 + t) - \frac{\partial Q}{\partial x}(x_0, y_0) \right) \mathrm{d}t \right| \leq \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} r\, \mathrm{d}t = r. \end{aligned}$$

We have used here the fact that $\left|\int_a^b f(t)\ \mathrm{d}t\right| \le \int_a^b |f(t)|\ \mathrm{d}t$. This proves that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \frac{\partial Q}{\partial x}(x_\varepsilon, y_0+t)\ \mathrm{d}t = \frac{\partial Q}{\partial x}(x_0,y_0).$$

A similar argument gives

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \frac{\partial P}{\partial y}(x_0+t, y_\varepsilon)\ \mathrm{d}t = \frac{\partial P}{\partial y}(x_0,y_0),$$

from which the conclusion follows. □

In some texts, the line integral $\int_{\boldsymbol{\alpha}} \boldsymbol{F}$ is called the *circulation* of the vector field $\boldsymbol{F}$ along the path $\boldsymbol{\alpha}$, and the limit

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\int_{(x_0,y_0)+\boldsymbol{\gamma}_\varepsilon} \boldsymbol{F}$$

is called the *rate of circulation* of the vector field $\boldsymbol{F}$ at the point (x_0,y_0). We refer to the comments after Corollary 9.4.1 for the physical interpretation of this expression.

If $f(x,y)$ denotes the density (mass per unit area) of a planar object U, then one can evaluate its total mass as

$$\iint_U f(x,y)\ \mathrm{d}(x,y).$$

By analogy, it seems reasonable to expect that the circulation of the vector field $\boldsymbol{F}$ along a path bounding a region U can be obtained as a double integral over U of the rate of circulation, that is, of the function

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Green's theorem shows that this intuition is correct in some cases.

Up to this point, we have dealt with the integral only of continuous functions on an interval $[a,b]$, and so we have required only the elementary properties of the Riemann integral. Henceforth, however, we will need tools from the theory of integration in several variables. In that setting, one has to choose between two possibilities: the Riemann integral and the Lebesgue integral. The Riemann integral on an n-rectangle in $\mathbb{R}^n$, and its extension to bounded subsets with Jordan content, usually called Jordan measurable sets, is by far the more intuitive. On the other hand, the Lebesgue integral on $\mathbb{R}^n$, or specifically, on a measurable subset of $\mathbb{R}^n$, is the more powerful theory, even if less intuitive. We have decided to use the Lebesgue integral. Why? A key point is that any open or any closed subset of $\mathbb{R}^n$

is Lebesgue measurable but, in general, not Jordan measurable. Moreover, every continuous function on a compact set is Lebesgue integrable, and two of the most important theorems of integral calculus, namely Fubini's theorem and the change of variable theorem, hold in far more generality for the Lebesgue than for Riemann integral.

Notwithstanding the differences between the two theories of integration, and our preference for the theory of Lebesgue, the reader should be aware that for our purposes, and with enough hard toil, it is possible to show that most of the situations we encounter in the text actually fit into the framework of the Riemann integral and Jordan content, and so the reader may continue to think and visualize in terms of the Riemann integral. But we are not going to bridge that gap explicitly here. A book in which both theories appear and can be compared is that of Apostol [1].

Definition 2.6.1. (i) A set $R = \Pi_{j=1}^{n} I_j$ is called an *n-rectangle* in $\mathbb{R}^n$ if each I_j is a bounded interval in $\mathbb{R}$, i.e., if there exist $a_j \leq b_j$ real numbers such that $(a_j, b_j) \subseteq I_j \subseteq [a_j, b_j]$ for every $j = 1, \ldots, n$. In that case, the Lebesgue measure of R is defined to be

$$m(R) = \Pi_{j=1}^{n}(b_j - a_j).$$

(ii) A subset N of $\mathbb{R}^n$ is called a *null set* if given $\varepsilon > 0$, there exists a sequence of n-rectangles $(R_k)_{k=1}^{\infty}$ such that $N \subset \cup_{k=1}^{\infty} R_k$ and

$$\sum_{k=1}^{\infty} m(R_k) < \varepsilon.$$

(iii) A property $P(x)$, where $x \in \mathbb{R}^n$, is said to be true *almost everywhere* if there exists a null set $N \subset \mathbb{R}^n$ such that the property $P(x)$ holds for every $x \in \mathbb{R}^n \setminus N$.

(iv) Given a subset K of $\mathbb{R}^n$, the *characteristic function* of K, denoted by χ_K, is defined by

$$\chi_K(\boldsymbol{x}) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{if } x \notin K. \end{cases}$$

(v) A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is called a *step function* if φ is a linear combination of characteristic functions of n-rectangles, i.e., if

$$\varphi = \sum_{k=1}^{m} c_j \chi_{R_j},$$

where $c_j \in \mathbb{R}$ and R_j is an n-rectangle.

(vi) A subset $M \subset \mathbb{R}^n$ is said to be *measurable* if there exists a sequence $(\varphi_h)_{h=1}^{\infty}$ of step functions and a null subset N of $\mathbb{R}^n$ such that the sequence of real numbers $(\varphi_h(x))$ converges to $\chi_M(x)$ for every $x \in \mathbb{R}^n \setminus N$, i.e., if the sequence $(\varphi_h)_{h=1}^{\infty}$ converges pointwise almost everywhere to χ_M.

It can be proved that for every $\varepsilon > 0$, a null set can be covered by a sequence (R_k) of n-cubes, that is, n-rectangles with all sides of equal length, such that $\sum_{k=1}^{\infty} m(R_k) < \varepsilon$.

The family $\mathscr{M}$ of Lebesgue measurable sets in $\mathbb{R}^n$ is going to be very important throughout, but we need only the following very basic facts about that family:

1. Every open subset and every closed subset of $\mathbb{R}^n$ is measurable.
2. If M is a measurable set, then $\mathbb{R}^n \setminus M$ is also measurable.
3. If $(M_k)_{k=1}^{\infty}$ is a (countable) family of measurable sets, then $\cap_{k=1}^{\infty} M_k$ and $\cup_{k=1}^{\infty} M_k$ are measurable sets too.
4. Every null set is Lebesgue measurable.

We will need to use Fubini's theorem, and so we state it. We follow the notation of Stromberg [18, Theorem 6.121, p. 352] and refer to that book for the definition of the Lebesgue integral in $\mathbb{R}^n$. The space of Lebesgue integrable functions on $\mathbb{R}^n$ is denoted by $L(\mathbb{R}^n)$. If A is a measurable subset of $\mathbb{R}^n$, we will say that the function $f : A \to \mathbb{R}$ is Lebesgue integrable on A if extended as 0 outside A and denoting that extension by $f\chi_A$, then the function $f\chi_A$ is Lebesgue integrable on $\mathbb{R}^n$. In that case, by definition,

$$\int_A f := \int_A f(\boldsymbol{x})\mathrm{d}\boldsymbol{x} := \int_{\mathbb{R}^n} f\chi_A(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.$$

If $A = [a,b]$, we keep the classical notation and write $\int_{[a,b]} f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$. In the case that f has two or three variables, we will write

$$\iint_A f(x,y)\ \mathrm{d}(x,y) \text{ or } \iiint_A f(x,y,z)\ \mathrm{d}(x,y,z).$$

We prefer to write $\mathrm{d}(x,y)$ instead of $\mathrm{d}x\,\mathrm{d}y$ to avoid confusion with the exterior product $\mathrm{d}x \wedge \mathrm{d}y$ (see Chap. 6).

Theorem 2.6.1 (Fubini's theorem). *Let $f \in L(\mathbb{R}^{n+p})$. Then*

(i) $f_{\boldsymbol{x}}(\boldsymbol{y}) = f(\boldsymbol{x},\boldsymbol{y})$ *belongs to* $L(\mathbb{R}^p)$ *almost everywhere in* $\mathbb{R}^n$.
(ii) $f_{\boldsymbol{y}}(\boldsymbol{x}) = f(\boldsymbol{x},\boldsymbol{y})$ *belongs to* $L(\mathbb{R}^n)$ *almost everywhere in* $\mathbb{R}^p$.
(iii) $F(\boldsymbol{x}) := \int_{\mathbb{R}^p} f_{\boldsymbol{x}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}$ *is defined (that is, the integral exists) almost everywhere in* $\mathbb{R}^n$ *and is Lebesgue integrable in* $\mathbb{R}^n$. *Also* $G(\boldsymbol{y}) := \int_{\mathbb{R}^n} f_{\boldsymbol{y}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$ *is defined almost everywhere in* $\mathbb{R}^p$, *and moreover,* G *belongs to* $L(\mathbb{R}^p)$.
(iv)

$$\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^p} f(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{y}\Big)\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^{n+p}} f(\boldsymbol{x},\boldsymbol{y})\mathrm{d}(\boldsymbol{x},\boldsymbol{y}) = \int_{\mathbb{R}^p}\Big(\int_{\mathbb{R}^n} f(\boldsymbol{x},\boldsymbol{y})\mathrm{d}\boldsymbol{x}\Big)\mathrm{d}\boldsymbol{y}.$$

A variant of Fubini's theorem particularly useful to us is the following. Let $A \subset [a,b] \times [c,d]$ be a measurable set and $f : A \to \mathbb{R}$ a Lebesgue integrable function on A. For $x \in [a,b]$, let $A_x = \{y \in \mathbb{R} : (x,y) \in A\}$. Then

$$\iint_A f(x,y)\mathrm{d}(x,y) = \int_a^b \Big(\int_{A_x} f(x,y)\mathrm{d}y\Big)\mathrm{d}x.$$

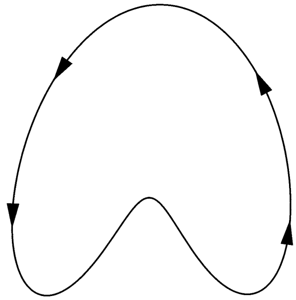


Fig. 2.8 A compact set with positively oriented boundary

The following concept will be discussed more precisely in Sect. 8.5.

Definition 2.6.2. Let $K \subset \mathbb{R}^2$ be a compact set whose boundary ∂K is equal to $\boldsymbol{\gamma}([a,b])$, where $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^2$ is a closed piecewise C^1 path. We say that $\boldsymbol{\gamma}$ is positively oriented if γ traverses ∂K once in such a way that the region K always lies to the left (Fig. 2.8). To be rigorous one must to assume that the compact set k satisfies condition (8.10), see for instance Cartan [7, 4.4.1]

Theorem 2.6.2 (Green's theorem). *Let $K \subset \mathbb{R}^2$ be a compact set with positively oriented boundary parameterized by γ (as in Definition 2.6.2). Let $\boldsymbol{\omega} = P\mathrm{d}x + Q\mathrm{d}y$ be a 1-form of class C^1 on an open set $U \subset \mathbb{R}^2$ such that $K \subset U$. Then*

$$\int_{\boldsymbol{\gamma}} P\mathrm{d}x + Q\mathrm{d}y = \iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}(x,y).$$

Using the notation of vector fields, this can be written as

$$\int_{\boldsymbol{\gamma}} (P,Q) = \iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}(x,y).$$

The reader may feel ill at ease with Definition 2.6.2 and Theorem 2.6.2. This is to be expected and welcomed. We have encountered for the first time a certain dichotomy that is often overlooked or brushed aside, but which we must specifically address in this book. Definition 2.6.2 and Theorem 2.6.2 have both an *intuitive* and a *mathematically rigorous* formulation, and these are not easily reconciled. If we look at Green's theorem from an intuitive point of view (typically, this is the case within the context of vector calculus), then there is generally no real difficulty. For example, if we want to apply it to situations arising in physical problems, then it is obvious what is meant by left and right, and we do not expect any trouble in deciding whether a region K lies to our left. However, from a purely mathematical point of view (i.e., within the context of vector analysis), the interpretation of "left" and "right" is not clear at all, and will demand from us much effort in developing adequate machinery to deal with the concept. Later, in Chaps. 5 and 9, we will present a

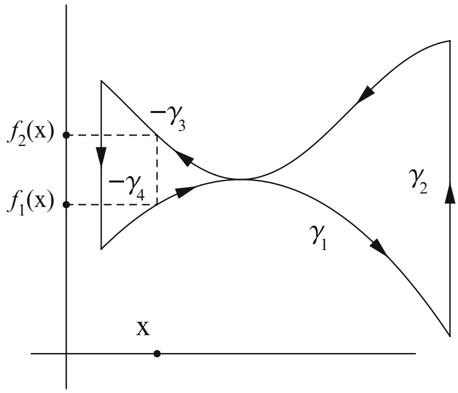


Fig. 2.9 A region of type I and its positively oriented boundary

rigorous definition of orientation and a general version of Green's theorem. For now, we content ourselves with working toward a proof of the theorem in some particular regions K for which the orientation can be very easily defined, and for which that definition of orientation coincides with our physical intuition.

Definition 2.6.3. A compact set $K \subset \mathbb{R}^2$ is said to be a region of type I (Fig. 2.9) if it can be described as

$$K = \{(x,y) \in \mathbb{R}^2 \ : \ a \le x \le b \ , \ f_1(x) \le y \le f_2(x)\},$$

where $f_1, f_2 : [a,b] \to \mathbb{R}$ are piecewise C^1 functions, $f_1 \le f_2$.

The positively oriented boundary of K is the path union

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}_1 \cup \boldsymbol{\gamma}_2 \cup (-\boldsymbol{\gamma}_3) \cup (-\boldsymbol{\gamma}_4),$$

where

$$\boldsymbol{\gamma}_1 : [a,b] \to \mathbb{R}^2 \ , \ \boldsymbol{\gamma}_1(t) := (t, f_1(t));$$

$$\boldsymbol{\gamma}_2 : [f_1(b), f_2(b)] \to \mathbb{R}^2 \ , \ \boldsymbol{\gamma}_2(t) := (b,t);$$

$$\boldsymbol{\gamma}_3 : [a,b] \to \mathbb{R}^2 \ , \ \boldsymbol{\gamma}_3(t) := (t, f_2(t));$$

$$\boldsymbol{\gamma}_4 : [f_1(a), f_2(a)] \to \mathbb{R}^2 \ , \ \boldsymbol{\gamma}_4(t) := (a,t).$$

Lemma 2.6.1. *Let K be a compact set that is a region of type I and let P be a continuous function on K admitting continuous partial derivative $\frac{\partial P}{\partial y}$ on a neighborhood of K. Then*

$$\int_{\boldsymbol{\gamma}} P \mathrm{d}x = -\iint_K \frac{\partial P}{\partial y} \mathrm{d}(x,y).$$

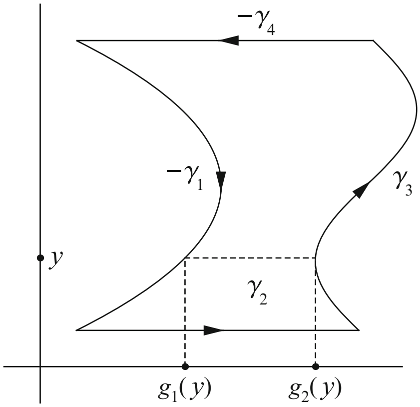


Fig. 2.10 A region of type II and its positively oriented boundary

Proof. Since $\boldsymbol{\gamma}_2'(t) = \boldsymbol{\gamma}_4'(t) = (0,1), \boldsymbol{\gamma}_1'(t) = (1, f_1'(t))$, and $\boldsymbol{\gamma}_3'(t) = (1, f_2'(t))$ for all t, we have

$$\int_{\boldsymbol{\gamma}} P\mathrm{d}x = \int_{\boldsymbol{\gamma}_1} P\mathrm{d}x - \int_{\boldsymbol{\gamma}_3} P\mathrm{d}x = \int_a^b P(t, f_1(t))\mathrm{d}t - \int_a^b P(t, f_2(t))\mathrm{d}t.$$

On the other hand, every continuous function on a compact set K is integrable, and according to Fubini's theorem,

$$\iint_K \frac{\partial P}{\partial y}\mathrm{d}(x,y) = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} \frac{\partial P}{\partial y}(x,y)\mathrm{d}y \right) \mathrm{d}x = \int_a^b \left(P(x, f_2(x)) - P(x, f_1(x)) \right) \mathrm{d}x.$$

Comparing these two equations, we have

$$\int_{\boldsymbol{\gamma}} P\mathrm{d}x = -\iint_K \frac{\partial P}{\partial y}\,\mathrm{d}(x,y).$$

□

Definition 2.6.4. The compact set K is said to be a region of type II (Fig. 2.10) if it can be analytically described as

$$K = \{(x,y) \ : \ c \le y \le d; g_1(y) \le x \le g_2(y)\},$$

where $g_1, g_2 : [c,d] \to \mathbb{R}$ are piecewise C^1 functions, $g_1 \le g_2$.

The positively oriented boundary of K is the path union

$$\boldsymbol{\gamma} = (-\boldsymbol{\gamma}_1) \cup \boldsymbol{\gamma}_2 \cup \boldsymbol{\gamma}_3 \cup (-\boldsymbol{\gamma}_4),$$

where

$$\boldsymbol{\gamma}_1 : [c,d] \to \mathbb{R}^2\ ,\ \boldsymbol{\gamma}_1(t) := (g_1(t), t);$$

$$\boldsymbol{\gamma}_2 : [g_1(c), g_2(c)] \to \mathbb{R}^2 \ , \ \boldsymbol{\gamma}_2(t) := (t, c);$$

$$\boldsymbol{\gamma}_3 : [c, d] \to \mathbb{R}^2 \ , \ \boldsymbol{\gamma}_3(t) := (g_2(t), t);$$

$$\boldsymbol{\gamma}_4 : [g_1(d), g_2(d)] \to \mathbb{R}^2 \ , \ \boldsymbol{\gamma}_4(t) := (t, d).$$

We have the following analogue of Lemma 2.6.1.

Lemma 2.6.2. *Let K be a compact set that is a region of type II and let Q be a continuous function on K admitting continuous partial derivative $\dfrac{\partial Q}{\partial x}$ on a neighborhood of K. Then*

$$\int_{\boldsymbol{\gamma}} Q\,\mathrm{d}y = \iint_K \frac{\partial Q}{\partial x}\,\mathrm{d}(x,y).$$

Theorem 2.6.3. *Let K be a region of type I or a region of type II and let $\boldsymbol{\omega} = P\mathrm{d}x + Q\mathrm{d}y$ be a 1-form of class C^1 on some open rectangle containing K. Then*

$$\int_{\boldsymbol{\gamma}} P\mathrm{d}x + Q\mathrm{d}y = \iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}(x,y).$$

Proof. We choose a, b, c, and d such that K is contained in the rectangle $R = [a,b] \times [c,d]$ and we assume that the 1-form $\boldsymbol{\omega}$ is defined and is of class C^1 on the open rectangle T containing R. We will present the proof in the case that K is a region of type I. In the case that K is a region of type II, a similar argument does the job.

We have already proved that

$$\int_{\boldsymbol{\gamma}} P\mathrm{d}x = -\iint_K \frac{\partial P}{\partial y}\mathrm{d}(x,y),$$

and so we need only to obtain

$$\int_{\boldsymbol{\gamma}} Q\mathrm{d}y = \iint_K \frac{\partial Q}{\partial x}\mathrm{d}(x,y).$$

Notice that we cannot apply the previous lemma, since K is not necessarily a region of type II. We proceed as follows.

For $(x,y) \in T$, we define $V(x,y) := \int_c^y Q(x,t)\mathrm{d}t$. The resultant function V has the property that

$$\mathrm{d}V(x,y) = F(x,y)\ \mathrm{d}x + Q(x,y)\ \mathrm{d}y,$$

where

$$F(x,y) = \int_c^y \frac{\partial Q}{\partial x}(x,t)\mathrm{d}t.$$

The expression for F is obtained after applying Proposition 2.5.1 (see also, Apostol [1, Theorem 10.39]). Since by Theorem 2.5.1,

$$\int_{\boldsymbol{\gamma}} \mathrm{d}V = 0,$$

we have

$$\int_{\boldsymbol{\gamma}} Q(x,y)\ \mathrm{d}y = -\int_{\boldsymbol{\gamma}} F(x,y)\ \mathrm{d}x.$$

Moreover, K is a region of type I, and on a neighborhood of K,

$$\frac{\partial F}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y),$$

because for each fixed x the function $y \mapsto F(x,y)$ is a primitive of $y \mapsto \frac{\partial Q}{\partial x}(x,y)$. An application of Lemma 2.6.1 to $F(x,y)\ \mathrm{d}x$ gives

$$-\int_{\boldsymbol{\gamma}} F(x,y)\ \mathrm{d}x = \iint_K \frac{\partial F}{\partial y}\ \mathrm{d}(x,y).$$

Finally,

$$\int_{\boldsymbol{\gamma}} Q(x,y)\ \mathrm{d}y = \iint_K \frac{\partial Q}{\partial x}(x,y)\ \mathrm{d}(x,y)$$

and

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = \iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ \mathrm{d}(x,y).$$

$\square$

Definition 2.6.5. A compact set K is said to be a *simple region* if K is a region of type I and also a region of type II.

The hypothesis for Green's theorem on simple regions can be relaxed somewhat, since it is enough that the 1-form be defined on a neighborhood of K. Before providing the proof, let us establish a notational convention that is valid by the following fact (see Proposition 2.7.2). Suppose K is a simple region and $\boldsymbol{\gamma}_1 : [a,b] \to \mathbb{R}^2$, $\boldsymbol{\gamma}_2 : [c,d] \to \mathbb{R}^2$ are two simple, closed, and piecewise smooth paths with

$$\boldsymbol{\gamma}_1([a,b]) = \boldsymbol{\gamma}_2([c,d]) = \partial K$$

and such that $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ *induce the same orientation* on the boundary ∂K. Then according to Proposition 2.7.2,

$$\int_{\boldsymbol{\gamma}_1} \boldsymbol{\omega} = \int_{\boldsymbol{\gamma}_2} \boldsymbol{\omega}.$$

Consequently, it makes sense to write

$$\int_{\partial K} \boldsymbol{\omega}$$

if we interpret the integration as being done along any *simple*, (Definition 2.7.1) closed, and piecewise C^1 path that gives a parameterization of ∂K such that the region K always lies to the left.

Theorem 2.6.4. *Let $K \subset \mathbb{R}^2$ be a simple region and let $\boldsymbol{\omega} = P\mathrm{d}x + Q\mathrm{d}y$ be a 1-form of class C^1 on an open neighborhood of K. Then*

$$\int_{\partial K} \boldsymbol{\omega} = \iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}(x,y).$$

Proof. Since K is a region of type I, we have

$$\int_{\partial K} P\mathrm{d}x = -\iint_K \frac{\partial P}{\partial y} \mathrm{d}(x,y),$$

and since it is also a region of type II, we obtain

$$\int_{\partial K} Q\mathrm{d}y = \iint_K \frac{\partial Q}{\partial x} \mathrm{d}(x,y).$$

It is enough to sum these two identities. □

Green's Formula[5] allows us to give a direct proof of the sufficiency condition for a vector field to be conservative (Theorem 2.5.6) on an open and starlike set in the plane.

Theorem 2.6.5 (Poincaré's lemma). *Let $\boldsymbol{F} = (P,Q)$ be a vector field of class C^1 on the open and starlike set $U \in \mathbb{R}^2$ and suppose that*

$$\frac{\partial Q}{\partial x}(x,y) = \frac{\partial P}{\partial y}(x,y)$$

for every $(x,y) \in U$. Then $\boldsymbol{F}$ is conservative.

Proof. Let $\boldsymbol{\omega} = P\mathrm{d}x + Q\mathrm{d}y$ be the 1-form associated with the vector field $\boldsymbol{F}$. It is enough to prove that

[5] Green's theorem is essentially a number of different results, which hold in various domains with different hypotheses, that establish a common integral identity. We shall refer to that integral identity as Green's formula.

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = 0$$

whenever $\boldsymbol{\gamma}$ is the boundary of some triangle K contained in U. Since K is a simple region, it follows from Green's theorem that

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = \iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}(x,y) = 0.$$

□

A general version of Poincaré's lemma on $\mathbb{R}^n$ will be obtained in Chap. 6 (see Sect. 6.6) after we have completed our study of differential forms.

2.7 Appendix: Comments on Parameterization

If we ask a student to evaluate the integral of a vector field along the unit circle oriented counterclockwise, and we don't specify a parameterization of the circle, the student will probably use the "natural" parameterization $\boldsymbol{\gamma} : [0, 2\pi] \to \mathbb{R}^2$ given by $\boldsymbol{\gamma}(t) = \big(\cos(t),\ \sin(t)\big)$. However, this is not the only parameterization available, and it is quite natural to ask how this freedom in the choice of parameterization might affect the solution to the problem posed. That is, suppose that we have another path $\boldsymbol{\alpha}$ whose trajectory is also the unit circle and with the property that we move along the circle counterclockwise, but without any other obvious relation to the original path $\boldsymbol{\gamma}$. Can we be sure that

$$\int_{\boldsymbol{\alpha}} \boldsymbol{F} = \int_{\boldsymbol{\gamma}} \boldsymbol{F}\ ?$$

In this section, given from the point of view of vector analysis, we give a rigorous answer to this question (Proposition 2.7.2), thereby showing why so much freedom is typically allowed in parameterizing a curve.

Lemma 2.7.1. *Let $\boldsymbol{\varphi} : [c,d] \to \mathbb{R}^n$ and $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ be two injective paths of class C^1 such that*

$$\boldsymbol{\gamma}([a,b]) \subset \boldsymbol{\varphi}([c,d]).$$

Let us assume that there is $1 \le k \le n$ with $\varphi_k'(s) \neq 0$ for all $s \in [c,d]$. Then

$$\boldsymbol{\varphi}^{-1} \circ \boldsymbol{\gamma} : [a,b] \to [c,d]$$

is a function of class C^1.

Proof. Since $\boldsymbol{\varphi}$ is C^1, the intermediate value theorem implies that $\varphi_k'(s) > 0$ for all $s \in [c,d]$ or $\varphi_k'(s) < 0$ for all $s \in [c,d]$. Without loss of generality we will assume the former. Since φ_k can be extended to a function of class C^1 on the whole real line and $\varphi_k'(s) > 0$ for all $s \in [c,d]$, we have that φ_k is a strictly increasing bijection between $[c,d]$ and $[e,f] := \varphi_k([c,d])$ and also that its inverse $h := \varphi_k^{-1}$ is a function of class C^1 on $[e,f]$. Hence $\boldsymbol{\varphi}$ is bijective onto its image. Let

$$\pi_k : \mathbb{R}^n \to \mathbb{R}$$

denote the projection onto the kth coordinate. From the fact that the mapping $\varphi_k \circ \boldsymbol{\varphi}^{-1}$ associates to each point in the range of $\boldsymbol{\varphi}$ its kth coordinate and $\boldsymbol{\gamma}([a,b]) \subset \boldsymbol{\varphi}([c,d])$ we can deduce $\varphi_k \circ \boldsymbol{\varphi}^{-1} \circ \boldsymbol{\gamma} = \pi_k \circ \boldsymbol{\gamma}$. That is,

$$\boldsymbol{\varphi}^{-1} \circ \boldsymbol{\gamma} = h \circ \pi_k \circ \boldsymbol{\gamma}.$$

Now the conclusion follows by the chain rule. □

Proposition 2.7.1. *Let $\boldsymbol{\varphi} : [c,d] \to \mathbb{R}^n$ and $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ be two injective smooth paths. If $\boldsymbol{\gamma}([a,b]) = \boldsymbol{\varphi}([c,d])$ and both paths define the same "orientation" (that is, $\boldsymbol{\varphi}(s) = \boldsymbol{\gamma}(t)$ implies that $\boldsymbol{\varphi}'(s)$ is a positive multiple of $\boldsymbol{\gamma}'(t)$), then $\boldsymbol{\varphi}$ and $\boldsymbol{\gamma}$ are equivalent paths.*

Proof. The mapping $\boldsymbol{\varphi} : [c,d] \to \boldsymbol{\varphi}\big([c,d]\big)$ is a homeomorphism because it is a continuous bijection between two compact sets. Hence, the mapping

$$\theta := \boldsymbol{\varphi}^{-1} \circ \boldsymbol{\gamma} : [a,b] \to [c,d]$$

is a continuous bijection and $\boldsymbol{\varphi} \circ \theta = \boldsymbol{\gamma}$. We have to prove that θ is a mapping of class C^1 and $\theta'(t) > 0$ for all $t \in [a,b]$. To do this, we fix $t_0 \in (a,b)$ and we observe that $\boldsymbol{\varphi}'(\theta(t_0)) \neq 0$. Since $\boldsymbol{\varphi}' \circ \theta$ is a continuous function, we deduce that there exist $1 \leq k \leq n$ and $a < \alpha < t_0 < \beta < b$ such that

$$\varphi_k'(\theta(t)) \neq 0$$

for every $t \in [\alpha,\beta]$. We define

$$[\xi,\eta] := \theta\big([\alpha,\beta]\big).$$

Then the restriction of $\boldsymbol{\gamma}$ to $[\alpha,\beta]$ has the same trajectory as the restriction of $\boldsymbol{\varphi}$ to $[\xi,\eta]$. According to the previous lemma,

$$\theta : [\alpha,\beta] \to [\xi,\eta]$$

is a function of class C^1. Consequently,

$$\boldsymbol{\gamma}'(t) = \boldsymbol{\varphi}'(\theta(t)) \cdot \theta'(t)$$

for every $t \in [\alpha,\beta]$. Since, by hypothesis, $\boldsymbol{\gamma}'(t)$ is a positive multiple of $\boldsymbol{\varphi}'(\theta(t))$, we deduce that $\theta'(t) > 0$ for every $t \in [\alpha,\beta]$. In particular, θ is of class C^1 on a neighborhood of t_0 and $\theta'(t_0) > 0$.

In the case $t_0 = a$ we find $a < \beta < b$ such that

$$\varphi_k'(\theta(t)) \neq 0$$

for all $t \in [a,\beta]$. We now proceed as before but with $\alpha = a$ in order to finally deduce that θ is of class C^1 on $[a,\beta]$ and $\theta' > 0$ at every point of $[a,\beta]$. A similar argument covers the case $t_0 = b$. □

Definition 2.7.1. A closed path $\boldsymbol{\varphi} : [c,d] \to \mathbb{R}^n$ is said to be *simple* if $\boldsymbol{\varphi}$ is piecewise C^1 and $\boldsymbol{\varphi}|_{[c,d)}$ is injective.

Lemma 2.7.2. *Let $\boldsymbol{\varphi} : [c,d] \to \mathbb{R}^n$ be a simple path. Then*

$$\boldsymbol{\varphi} : (c,d) \to \boldsymbol{\varphi}\big((c,d)\big)$$

is a homeomorphism.

Proof. Let $\{t_n\}$ be a sequence in (c,d) such that

$$\lim_{n\to\infty} \boldsymbol{\varphi}(t_n) = \boldsymbol{\varphi}(t_0)$$

for some $t_0 \in (c,d)$. In order to conclude that $\{t_n\}$ converges to t_0, it is enough to prove that each convergent subsequence of $\{t_n\}$ is convergent to t_0. Let us assume that $\{n_l\}$ is an increasing sequence of natural numbers and that

$$\lim_{l\to\infty} t_{n_l} = \xi.$$

Since $\xi \in [c,d]$ and $\boldsymbol{\varphi}$ is continuous, we obtain

$$\boldsymbol{\varphi}(\xi) = \lim_{l\to\infty} \boldsymbol{\varphi}(t_{n_l}) = \boldsymbol{\varphi}(t_0).$$

From the injectivity of $\boldsymbol{\varphi}|_{[c,d)}$ and $\boldsymbol{\varphi}|_{(c,d]}$ we deduce $\xi = t_0$. Consequently, $\{t_n\}$ converges to t_0 and

$$\boldsymbol{\varphi}^{-1} : \boldsymbol{\varphi}\big((c,d)\big) \to (c,d)$$

is continuous. $\square$

In saying that a path $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ is *piecewise smooth*, we mean that there is a partition

$$\{a = t_0 < t_1 < \cdots < t_l = b\},$$

such that there exists $\boldsymbol{\gamma}'(t) \neq 0$ for every $t \neq t_j$ and $\boldsymbol{\gamma}$ admits nonzero one-sided derivatives at each point t_j.

Let us now assume that $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ and $\boldsymbol{\varphi} : [c,d] \to \mathbb{R}^n$ are two simple paths, which are piecewise smooth and satisfy $\boldsymbol{\gamma}([a,b]) = \boldsymbol{\varphi}([c,d])$. We say that the two paths induce the same *orientation* if whenever $\boldsymbol{\gamma}(t) = \boldsymbol{\varphi}(s)$ for t and s such that $\boldsymbol{\gamma}$ is differentiable at t and $\boldsymbol{\varphi}$ is differentiable at s, then $\boldsymbol{\gamma}'(t)$ is a positive multiple of $\boldsymbol{\varphi}'(s)$.

Proposition 2.7.2. *Let $\boldsymbol{\gamma} : [a,b] \to \mathbb{R}^n$ and $\boldsymbol{\varphi} : [c,d] \to \mathbb{R}^n$ two simple paths that are piecewise smooth and satisfy $\boldsymbol{\gamma}([a,b]) = \boldsymbol{\varphi}([c,d]) \subset U$, where $U \subset \mathbb{R}^n$ is open. If the two paths induce the same orientation, then for every differential form $\boldsymbol{\omega}$ of degree* 1 *and class C^1 on U, we have*

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = \int_{\boldsymbol{\varphi}} \boldsymbol{\omega}.$$

Proof. Let us first assume that $\boldsymbol{\gamma}$ and $\boldsymbol{\varphi}$ are two piecewise smooth paths with the same initial point (and also the same final point, since they are closed paths). We introduce partitions of the intervals $[a,b]$ and $[c,d]$ as follows. In the optimal case that $\boldsymbol{\gamma}$ and $\boldsymbol{\varphi}$ are smooth on their respective domains, we consider arbitrary partitions $\{a=a_0<a_1<a_2=b\}$ and $\{c=c_0<c_1<c_2=d\}$ with the property that $\boldsymbol{\gamma}(a_1)=\boldsymbol{\varphi}(c_1)$. In the general case that either $\boldsymbol{\varphi}$ or $\boldsymbol{\gamma}$ is not smooth, let Q be a nonempty finite set containing all the points of the form $\boldsymbol{\gamma}(t)$, where $t\in(a,b)$ is such that $\boldsymbol{\gamma}$ is not differentiable at t, and also all the points of the form $\boldsymbol{\varphi}(s)$, where $s\in(c,d)$ for which $\boldsymbol{\varphi}$ is not differentiable at s. Then Q consists of $m-1$ elements for some $m\geq 2$. We can find partitions

$$\{a=a_0<a_1<\cdots<a_m=b\},\quad \{c=c_0<c_1<\cdots<c_m=d\}$$

of the intervals $[a,b]$ and $[c,d]$ with the property that

$$\boldsymbol{\gamma}\big(\{a_1,\dots,a_{m-1}\}\big)=\boldsymbol{\varphi}\big(\{c_1,\dots,c_{m-1}\}\big)=Q.$$

Observe that our partitions have at least three points and that

$$\boldsymbol{\gamma}|_{[a_{j-1},a_j]} \text{ and } \boldsymbol{\varphi}|_{[c_{j-1},c_j]}$$

are injective and smooth paths. We now put

$$I_j:=(a_{j-1},a_j),\quad \text{and hence}\quad \overline{I_j}=[a_{j-1},a_j].$$

By Lemma 2.7.2, $\boldsymbol{\gamma}(I_j)$ is an open and connected set in $\boldsymbol{\gamma}\big((a,b)\big)=\boldsymbol{\varphi}\big((c,d)\big)$, and hence, after applying again Lemma 2.7.2,

$$J_j:=\boldsymbol{\varphi}^{-1}\big(\boldsymbol{\gamma}(I_j)\big)$$

is an open (and proper) subinterval of (c,d). It is obvious that $\boldsymbol{\gamma}(I_j)=\boldsymbol{\varphi}(J_j)$. Moreover, $\boldsymbol{\gamma}(\overline{I_j})=\boldsymbol{\varphi}(\overline{J_j})$. In fact, if $t=\lim_{n\to\infty}t_n$ for $t_n\in I_j$, then

$$\boldsymbol{\gamma}(t)=\lim_{n\to\infty}\boldsymbol{\varphi}(s_n)$$

for some sequence $\{s_n\}$ in J_j. If $\{s_{n_k}\}$ is a subsequence convergent to $s\in\overline{J_j}$, then $\boldsymbol{\gamma}(t)=\boldsymbol{\varphi}(s)\in\boldsymbol{\varphi}(\overline{J_j})$. This proves that $\boldsymbol{\gamma}(\overline{I_j})\subset\boldsymbol{\varphi}(\overline{J_j})$. A similar argument gives the reverse inclusion. Since

$$\boldsymbol{\gamma}_{|\overline{I_j}} \text{ and } \boldsymbol{\varphi}_{|\overline{J_j}}$$

are smooth and injective, we can apply Proposition 2.7.1 to conclude that these are equivalent paths. The sets $\{\overline{J}_1,\overline{J}_2,\dots,\overline{J}_{m-1}\}$ are mutually disjoint (or have at most one point in common) and form a covering of $[c,d]$, so we can finally conclude that

$$\int_{\boldsymbol{\gamma}} \boldsymbol{\omega} = \sum_{j=1}^{m} \int_{\boldsymbol{\gamma}_{|\overline{I_j}}} \boldsymbol{\omega} = \sum_{j=1}^{m} \int_{\boldsymbol{\varphi}_{|\overline{J_j}}} \boldsymbol{\omega}$$

$$= \int_{\boldsymbol{\varphi}} \boldsymbol{\omega}.$$

We now analyze the case that φ and γ do not have the same initial (and final) point, that is, $\boldsymbol{\varphi}(c) \neq \boldsymbol{\gamma}(a)$. We take $a < t_0 < b$ such that $\boldsymbol{\gamma}(t_0) = \boldsymbol{\varphi}(c)$ and we define

$$\boldsymbol{\lambda} := \boldsymbol{\gamma}_{|[t_0,b]} \cup \boldsymbol{\gamma}_{|[a,t_0]},$$

which is a simple and piecewise smooth closed path. Moreover, the trace of λ coincides with

$$\boldsymbol{\gamma}([a,b]) = \boldsymbol{\varphi}([c,d]),$$

$\boldsymbol{\lambda}$ and $\boldsymbol{\varphi}$ have the same initial (and final) point, and the two paths $\boldsymbol{\lambda}$ and $\boldsymbol{\varphi}$ induce the same orientation. Hence

$$\int_{\boldsymbol{\lambda}} \boldsymbol{\omega} = \int_{\boldsymbol{\varphi}} \boldsymbol{\omega}.$$

Finally, we obtain

$$\int_{\boldsymbol{\lambda}} \boldsymbol{\omega} = \int_{\boldsymbol{\gamma}_{|[t_0,b]}} \boldsymbol{\omega} + \int_{\boldsymbol{\gamma}_{|[a,t_0]}} \boldsymbol{\omega}$$

$$= \int_{\boldsymbol{\gamma}} \boldsymbol{\omega},$$

and the proposition is proved. □

Remark 2.7.1. Physically, one might wish to consider two particles moving along the same closed trajectory with the same direction of movement but whose initial (and hence final) points differ. In this case, the two paths $\boldsymbol{\gamma}$ and $\boldsymbol{\varphi}$ are not equivalent. In fact, it follows from Definition 2.4.1 that two equivalent paths have the same initial (and final) point. For example, the paths $\gamma\colon [0,2\pi] \to \mathbb{R}^2$, $\gamma(t) = (\cos t, \sin t)$, and $\varphi : [\pi, 3\pi] \to \mathbb{R}^2$, $\varphi(s) = (\cos s, \sin s)$, are not equivalent. Another example of two inequivalent paths that satisfy the hypothesis of Proposition 2.7.2 are γ and $\beta : [\pi, 3\pi] \to \mathbb{R}^2$, $\beta(s) = (\cos(s + \frac{\pi}{2}), \sin(s + \frac{\pi}{2}))$.

2.8 Exercises

Exercise 2.8.1. Evaluate the length of the path

$$\boldsymbol{\gamma} : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}^2, \quad \boldsymbol{\gamma}(t) = (\mathrm{e}^t \cos(t), \mathrm{e}^t \sin(t)).$$

Exercise 2.8.2. Obtain a parameterization, counterclockwise, of the ellipse

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \quad (a, b > 0).$$

Exercise 2.8.3. Integrate the vector field

$$\boldsymbol{F}(x,y) = (y^2, -2xy)$$

along the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$, oriented counterclockwise.

Exercise 2.8.4. (1) Find a path $\boldsymbol{\gamma}$ whose trajectory is the intersection of the cylinder $x^2 + y^2 = 1$ with the plane $x + y + z = 1$ and with the additional properties that the initial (and final) point is $(0, -1, 2)$ and the projection onto the xy-plane is oriented counterclockwise.

(2) Evaluate

$$\int_{\boldsymbol{\gamma}} xy \, \mathrm{d}x + yz \, \mathrm{d}y - x \, \mathrm{d}z.$$

Exercise 2.8.5. Let $\boldsymbol{\gamma}$ be a simple path whose trajectory is the intersection of the coordinate planes with the portion of the unit sphere in the first octant, oriented according to the sequence

$$(1,0,0), \ (0,1,0), \ (0,0,1), \ (1,0,0).$$

Evaluate

$$\int_{\boldsymbol{\gamma}} z \, \mathrm{d}x + x \, \mathrm{d}y + y \, \mathrm{d}z.$$

Exercise 2.8.6. Find a path whose trajectory is the intersection of the upper hemisphere of the sphere with radius $2a$,

$$x^2 + y^2 + z^2 = 4a^2,$$

with the cylinder

$$x^2 + (y - a)^2 = a^2.$$

Exercise 2.8.7. Determine whether the vector field

$$\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2, \quad \boldsymbol{F}(x,y) = \left(x^3 y,\ x\right),$$

is conservative.

Exercise 2.8.8. Find a potential for the vector field

$$\boldsymbol{F} : \mathbb{R}^2 \to \mathbb{R}^2, \quad \boldsymbol{F}(x,y) = \left(2xy,\ x^2 - y^2\right).$$

Exercise 2.8.9. For the vector field

$$\boldsymbol{F} : \mathbb{R}^3 \to \mathbb{R}^3, \quad \boldsymbol{F}(x,y,z) = \left(2xy,\ x^2 + z^2,\ 2zy\right),$$

(1) show that $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$;
(2) find a potential for $\boldsymbol{F}$.

Exercise 2.8.10. (1) Show that the vector field

$$\boldsymbol{F}(x,y) = \left(\mathrm{e}^x(\sin(x+y) + \cos(x+y)) + 1\ ,\ \mathrm{e}^x \cos(x+y)\right)$$

is conservative on $\mathbb{R}^2$ and find a potential.

(2) Evaluate the line integral

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F},$$

where

$$\boldsymbol{\gamma} : [0,\pi] \to \mathbb{R}^2\ ;\ \boldsymbol{\gamma}(t) = \left(\sin(\pi \mathrm{e}^{\sin(t)}), \cos^5(t)\right).$$

Exercise 2.8.11. Evaluate

$$\int_{\boldsymbol{\gamma}} x\, \mathrm{d}x + y\, \mathrm{d}y + z\, \mathrm{d}z,$$

where $\boldsymbol{\gamma}(t) = \left(\cos^4(t), \sin^2(t) + \cos^3(t), t\right)$, $0 \le t \le \pi$.

Exercise 2.8.12. For the vector field

$$\boldsymbol{F} = (f_1, f_2) : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$$

defined by

$$f_1(x,y) = \frac{-y}{x^2+y^2}, \quad f_2(x,y) = \frac{x}{x^2+y^2},$$

(1) Show that

$$\frac{\partial f_2}{\partial x}(x,y) = \frac{\partial f_1}{\partial y}(x,y)$$

for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.

(2) Let $\boldsymbol{\gamma}$ be the unit circle oriented counterclockwise. Show that

$$\int_{\boldsymbol{\gamma}} \boldsymbol{F} = 2\pi.$$

Is this fact a contradiction to Poincaré's lemma?

(3) Argue whether this statement is true: for every closed and piecewise C^1 path $\boldsymbol{\alpha} : [a,b] \to \mathbb{R}^2 \setminus \{(0,0)\}$ such that $\alpha_1(t) \geq 0$ for all $t \in [a,b]$,

$$\int_{\boldsymbol{\alpha}} \boldsymbol{F} = 0.$$

(4) Evaluate $\int_{\boldsymbol{\gamma}} \boldsymbol{F}$, where

$$\boldsymbol{\gamma}(t) = \left(\cos(t),\ \sin^7(t)\right), \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

Exercise 2.8.13. Let $\boldsymbol{\gamma}(t) = (\cos(t),\ \sin(t))$ be given and let us consider the paths

$$\boldsymbol{\alpha} := \boldsymbol{\gamma}_{|[-\pi,\pi]} \text{ and } \boldsymbol{\beta} := \boldsymbol{\gamma}_{|[0,2\pi]}.$$

(1) Are $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ equivalent paths?

(2) Justify why

$$\int_{\boldsymbol{\alpha}} \boldsymbol{F} = \int_{\boldsymbol{\beta}} \boldsymbol{F}$$

for any continuous vector field $\boldsymbol{F} : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined on the unit circle.

Exercise 2.8.14. Let $\boldsymbol{\gamma}$ be the path whose trajectory is the union of the graph of $y = x^3$ from $(0,0)$ to $(1,1)$ and the segment from $(1,1)$ to $(0,0)$. Using Green's theorem, evaluate

$$\int_{\boldsymbol{\gamma}} (x^2 + y^2)\ \mathrm{d}x + (2xy + x^2)\ \mathrm{d}y.$$

Exercise 2.8.15. Use Green's theorem to evaluate the line integral

$$\int_{\boldsymbol{\gamma}} (x^2 + 3x^2y^2)\ \mathrm{d}x + (2x^3y + x^2)\ \mathrm{d}y,$$

where $\boldsymbol{\gamma}$ is the boundary of the region lying between the graphs of $y = 0$ and $y = 4 - x^2$, oriented counterclockwise.

Exercise 2.8.16. Let K be a region of type I or type II and let $\boldsymbol{\gamma}$ be a path whose trajectory is the boundary of K oriented counterclockwise. Then

$$\text{area}\,(K) = \int_{\boldsymbol{\gamma}} \frac{1}{2}(-y\,\mathrm{d}x + x\,\mathrm{d}y) = \int_{\boldsymbol{\gamma}} -y\,\mathrm{d}x = \int_{\boldsymbol{\gamma}} x\,\mathrm{d}y.$$

Exercise 2.8.17. Evaluate the area bounded by the cycloid $\boldsymbol{\gamma} : [0, 2\pi] \to \mathbb{R}^2$,

$$\boldsymbol{\gamma}(t) = (at - a\sin(t), a - a\cos(t))$$

$(a > 0)$ and the x-axis.

Exercise 2.8.18. Evaluate the area bounded by the two coordinate axes and the path

$$\boldsymbol{\gamma} : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}^2, \quad \boldsymbol{\gamma}(t) = (\sin^4(t), \cos^4(t)).$$

Exercise 2.8.19. Evaluate the area limited by the circles

$$C_1 := \left\{(x,y) \in \mathbb{R}^2 \;:\; x^2 + y^2 = a^2\right\}$$

and

$$C_2 := \left\{(x,y) \in \mathbb{R}^2 \;:\; x^2 + y^2 = 2ax\right\} \;(a > 0).$$

Chapter 3
Regular *k*-Surfaces

Roughly speaking, a regular surface in $\mathbb{R}^3$ is a two-dimensional set of points, in the sense that it can be locally described by two parameters (the local coordinates) and with the property that it is smooth enough (that is, there are no vertices, edges, or self-intersections) to guarantee the existence of a tangent plane to the surface at each point.

A. Galbis and M. Maestre, *Vector Analysis Versus Vector Calculus*, Universitext,
DOI 10.1007/978-1-4614-2200-6_3,

Springer

http://www.springer.com/978-1-4614-2199-3

Vector Analysis Versus Vector Calculus
Galbis, A.; Maestre, M.
2012, XIII, 375 p. 79 illus., 59 illus. in color., Softcover
ISBN: 978-1-4614-2199-3