Chapter 2

Quadratic Operators and Quadratic Functional Equation

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Abstract In the first part of this paper, we consider some quadratic difference operators (e.g., Lobaczewski difference operators) and quadratic-linear difference operators (d'Alembert difference operators and quadratic difference operators) in some special function spaces X_λ. We present results about boundedness and find the norms of such operators. We also present new results about the quadratic functional equation. The second part is devoted to the so-called double quadratic difference property in the class of differentiable functions. As an application we prove the stability result in the sense of Ulam–Hyers–Rassias for the quadratic functional equation in a special class of differentiable functions.

Key words Quadratic, d'Alembert, and Lobaczewski difference operators · X_λ spaces · Quadratic functional equation · Stability

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2.1 The X_λ and X_λ^2 Spaces

We shall introduce the spaces X_λ and X_λ^2 (see [7]). A. Bielecki also studied similar spaces in [4] and applied them in the theory of differential equations.

Definition 2.1 Let X and Y be two normed vector spaces and $\lambda \geq 0$. Define

$$X_\lambda := \left\{ f : X \to Y : \|f(x)\| \leq M_f e^{\lambda \|x\|},\ x \in X \right\},$$

Dedicated to Professor Themistocles M. Rassias on the occasion of his 60th birthday.

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where M_f is a real constant depending on f. Moreover, for $f \in X_\lambda$

$$\|f\| := \sup_{x\in X}\left\{e^{-\lambda\|x\|}\left\|f(x)\right\|\right\}. \tag{2.1}$$

Let us note that the space X_λ with the norm (2.1) was considered by S. Czerwik and K. Dłutek in [10]. It is easy to prove the following

Lemma 2.1 *The space* $(X_\lambda, \|\cdot\|)$, *where* $\|\cdot\|$ *is defined by* (2.1), *is a linear normed space*.

Definition 2.2 Let X and Y be two normed vector spaces and $\lambda \geq 0$. Define

$$X_\lambda^2 := \left\{g\colon X\times X \to Y : \left\|g(x,y)\right\| \leq M_g e^{\lambda(\|x\|+\|y\|)},\ x,y\in X\right\},$$

where M_g is a real constant depending on g. Moreover, for $g \in X_\lambda^2$

$$\|g\| := \sup_{x,y\in X}\left\{e^{-\lambda(\|x\|+\|y\|)}\left\|g(x,y)\right\|\right\}. \tag{2.2}$$

We have

Lemma 2.2 *The space* $(X_\lambda^2, \|\cdot\|)$, *where* $\|\cdot\|$ *is defined by* (2.2), *is a linear normed space*.

2.2 Quadratic Difference Operator in X_λ Spaces

We define the quadratic difference operator $Q(f)$ by

$$Q(f)(x,y) := f(x+y) + f(x-y) - 2f(x) - 2f(y) \tag{2.3}$$

for $x, y \in X$, $f \in X_\lambda$.

Then we have

Theorem 2.1 *The quadratic difference operator* $Q\colon X_\lambda \to X_\lambda^2$, *given by the formula* (2.3), *is a linear bounded operator satisfying the inequality*

$$\left\|Q(f)\right\| \leq 6\|f\|, \quad f \in X_\lambda. \tag{2.4}$$

Proof First, we shall verify that if $f \in X_\lambda$, then $Q(f) \in X_\lambda^2$. We have

$$\begin{aligned}\left\|Q(f)(x,y)\right\| &= \left\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\right\| \\ &\leq M_f e^{\lambda(\|x+y\|)} + M_f e^{\lambda(\|x-y\|)} + 2M_f e^{\lambda(\|x\|)} + 2M_f e^{\lambda(\|y\|)} \\ &\leq 6M_f e^{\lambda(\|x\|+\|y\|)},\end{aligned}$$

thus $Q(f) \in X_\lambda^2$ as claimed.

Clearly, Q is linear. For $f \in X_\lambda$, we obtain

$$\begin{aligned}\|Q(f)\| &= \sup_{x,y\in X} e^{-\lambda(\|x\|+\|y\|)}\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \\ &\le \sup_{x,y\in X} e^{-\lambda(\|x\|+\|y\|)}\|f(x+y)\| + \sup_{x,y\in X} e^{-\lambda(\|x\|+\|y\|)}\|f(x-y)\| \\ &\quad + 2\sup_{x\in X} e^{-\lambda(\|x\|+\|y\|)}\|f(x)\| + 2\sup_{y\in X} e^{-\lambda(\|x\|+\|y\|)}\|f(y)\| \\ &= \|f\| + \|f\| + 2\|f\| + 2\|f\| = 6\|f\|.\end{aligned}$$

Therefore,

$$\|Q(f)\| \le 6\|f\|, \quad f \in X_\lambda,$$

which concludes the proof. □

Under some additional assumptions, we can prove some further results. In fact, we have

Theorem 2.2 *Let* $\mathbb{R} \subset X$, $\mathbb{R} \subset Y$ *and* $\|x\| = |x|$ *for* $x \in \mathbb{R}$. *Then*

$$\|Q\| = 6. \tag{2.5}$$

Proof Let $\{x_n\}$ be a strictly decreasing sequence of positive numbers such that

$$\lim_{n\to\infty} x_n = 0.$$

Let us define for $n \in \mathbb{N}$ a function f_n by

$$f_n(x) := \begin{cases} -e^{2\lambda x_n}, & x = x_n, \\ e^{2\lambda x_n}, & x = 2x_n, \\ e^{2\lambda x_n}, & x = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, we have

$$\|f_n(x)\| \le e^{2\lambda x_n} e^{\lambda\|x\|}, \quad x \in X,$$

so $f_n \in X_\lambda$ for all $n \in \mathbb{N}$. Moreover,

$$e^{-\lambda\|x\|}\|f_n(x)\| = \begin{cases} e^{2\lambda x_n}, & x = 0, \\ e^{2\lambda x_n}, & x = x_n, \\ 1, & x = 2x_n, \\ 0, & \text{otherwise}. \end{cases}$$

Because the sequence $\{x_n\}$ is a decreasing sequence of positive numbers convergent to zero, we obtain that $\|f_n\| = e^{2\lambda x_n}$ for all $n \in \mathbb{N}$. We also have

$$\begin{aligned}\big\|Q(f_n)\big\| &= \sup_{x,y\in X}\big\{e^{-\lambda(\|x\|+\|y\|)}\big\|f(x+y)+f(x-y)-2f(x)-2f(y)\big\|\big\}\\ &\geq e^{-\lambda x_n}\big\|f_n(2x_n)+f_n(0)-4f_n(x_n)\big\| = e^{-\lambda x_n}\cdot 6e^{\lambda x_n} = 6.\end{aligned}$$

Thus $\|Q(f_n)\| \geq 6$ for $n \in \mathbb{N}$. We also know from (2.4) that $\|Q\| \leq 6$. Suppose on the contrary that $\|Q\| < 6$. Then there exists $\varepsilon > 0$ such that

$$\big\|Q(f_n)\big\| \leq (6-\varepsilon)\|f_n\|, \quad f_n \in X_\lambda.$$

On the other hand, we have for $f_n \in X_\lambda$

$$6 \leq \big\|Q(f_n)\big\| \leq (6-\varepsilon)e^{2\lambda x_n}.$$

Taking into account that $x_n \to 0$ as $n \to \infty$, we get $6 \leq 6-\varepsilon$, where $\varepsilon > 0$, which is impossible. Thus we obtain eventually that $\|Q\| = 6$, and the proof is complete. □

For further information on new results concerning the quadratic difference operator on other spaces, see also the papers [9, 11, 12].

2.3 D'Alembert and Lobaczewski Difference Operators in X_λ Spaces

In this section, we shall recall the definition of the quadratic bounded operator. The Lobaczewski difference operator is an interesting example of a quadratic operator.

Here we shall present the ideas and main results obtained by S. Czerwik and K. Król in [13].

Let X and Y be linear spaces over a field $\mathbb{K}$.

Definition 2.3 An operator $Q\colon X \to Y$ is called quadratic if it satisfies the following equations

$$Q(x+y)+Q(x-y) = 2Q(x)+2Q(y), \quad x,y \in X, \tag{2.6}$$

$$Q(kx) = k^2Q(x), \quad x \in X,\ k \in \mathbb{K}. \tag{2.7}$$

Definition 2.4 A quadratic operator $Q\colon X \to Y$, where X, Y are linear normed spaces over $\mathbb{K}$, is called bounded if there exists an $M \geq 0$ such that

$$\big\|Q(x)\big\| \leq M\|x\|^2, \quad x \in X. \tag{2.8}$$

A norm of a quadratic bounded operator $Q\colon X \to Y$ is defined by

$$\|Q\| := \sup_{x\in X}\big\{\big\|Q(x)\big\| : \|x\| \leq 1\big\}. \tag{2.9}$$

By $B_Q(X,Y)$ we denote the space of all bounded quadratic operators. It is easy to prove that $B_Q(X,Y)$ with the norm given by (2.9) is a linear normed space.

Let $\mathbb{C}$ denote the set of all complex numbers. For a set X, a symbol $\mathbb{C}^X$ denotes the set of all functions $f\colon X \to \mathbb{C}$.

Definition 2.5 For a linear space X, the Lobaczewski difference operator $L\colon \mathbb{C}^X \to \mathbb{C}^{X^2}$ is defined by

$$L(f)(x,y) := f^2\left(\frac{x+y}{2}\right) - f(x)f(y), \quad x,y \in X. \tag{2.10}$$

One can verify that we have

Remark 2.1 The Lobaczewski difference operator $L\colon \mathbb{C}^X \to \mathbb{C}^{X^2}$ defined by (2.10) is a quadratic operator.

We can also prove that the Lobaczewski operator $L\colon X_\lambda \to X_\lambda^2$, where $Y = \mathbb{C}$, is a quadratic bounded operator. We have even more (see [13]).

Theorem 2.3 *Let $Y = \mathbb{C}$. The Lobaczewski difference operator defined by* (2.10) *belongs to $B_Q(X_\lambda, X_\lambda^2)$, and for all $f \in X_\lambda$ we have*

$$\big\|L(f)\big\| \le 2\|f\|^2. \tag{2.11}$$

Under some additional assumptions, we can find the norm of L. In fact, the following is true.

Theorem 2.4 *Let $\mathbb{R}_+ \subset X$, $Y = \mathbb{C}$, and $\|x\| = |x|$ for all $x \in \mathbb{R}_+$. Then*

$$\|L\| = 2, \tag{2.12}$$

where L is given by (2.10).

The proof, similar to the proof of Theorem 2.2, can be found in [13].

Now we shall present results about the d'Alembert difference operator.

Definition 2.6 We denote by B_{LQ} the space

$$\begin{aligned} B_{LQ}(X,Y) := \big\{T \in Y^X : \exists L \in B(X,Y) \text{ and } \exists Q \in B_Q(X,Y) \\ \text{such that } T = L + Q\big\}. \end{aligned}$$

Here, of course, $B(X,Y)$ stands for the space of linear bounded operators from X to Y.

For $T = L + Q \in B_{LQ}(X,Y)$, we define

$$\|T\| := \|L\| + \|Q\|.$$

We say that such an operator T is a bounded linear-quadratic operator.

Definition 2.7 Let X be a linear space. The d'Alembert difference operator $A : \mathbb{C}^X \to \mathbb{C}^{X^2}$ is defined by

$$A(f)(x, y) := f(x + y) + f(x - y) - 2f(x)f(y), \quad x, y \in X. \tag{2.13}$$

In the sequel, we present the following

Theorem 2.5 ([13]) *Let $Y = \mathbb{C}$ and X be a normed space. The d'Alembert difference operator $A\colon X_\lambda \to X^2_\lambda$ defined by (2.13) belongs to $B_{LQ}(X_\lambda, X^2_\lambda)$, and for all $f \in X_\lambda$ we have*

$$\|A(f)\| \le 2\|f\| + 2\|f\|^2.$$

Proof On account of (2.13), we get $A = L_A + Q_A$, where the linear operator $L_A : X_\lambda \to X^2_\lambda$ and the quadratic operator $Q_A : X_\lambda \to X^2_\lambda$ are given by

$$L_A(f)(x, y) := f(x + y) + f(x - y),$$
$$Q_A(f)(x, y) := -2f(x)f(y).$$

Now, for any $f \in X_\lambda$ we obtain successively

$$\begin{aligned}\|L_A(f)\| &= \sup_{x,y\in X} \left\{e^{-\lambda(\|x\|+\|y\|)}\left|f(x + y) + f(x - y)\right|\right\}\\ &\le \sup_{x,y\in X} \left\{e^{-\lambda(\|x\|+\|y\|)}\left|f(x + y)\right|\right\} + \sup_{x,y\in X} \left\{e^{-\lambda(\|x\|+\|y\|)}\left|f(x - y)\right|\right\}\\ &\le \sup_{x,y\in X} \left\{e^{-\lambda\|x+y\|}\left|f(x + y)\right|\right\} + \sup_{x,y\in X} \left\{e^{-\lambda\|x+y\|}\left|f(x - y)\right|\right\} = 2\|f\|.\end{aligned}$$

Therefore, $L_A \in B(X_\lambda, X^2_\lambda)$. We shall now prove that Q_A is bounded and $\|Q_A\| = 2$. Indeed, for $f \in X_\lambda$ we get

$$\begin{aligned}\|Q_A(f)\| &= \sup_{x,y\in X} \left\{e^{-\lambda\|x+y\|}\left|2f(x)f(y)\right|\right\}\\ &= 2\sup_{x\in X}\left\{e^{-\lambda\|x\|}\left|f(x)\right|\right\} \cdot \sup_{y\in X}\left\{e^{-\lambda\|y\|}\left|f(y)\right|\right\} = 2\|f\|^2.\end{aligned}$$

Thus $Q_A \in B_Q(X_\lambda, X^2_\lambda)$ and $\|Q_A\| = 2$. Since $A = L_A + Q_A$, we get that $A \in B_{LQ}(X_\lambda, X^2_\lambda)$ and

$$\|A(f)\| = \|L_A(f) + Q_A(f)\| \le \|L_A(f)\| + \|Q_A(f)\| \le 2\|f\| + 2\|f\|^2,$$

as claimed. □

Under additional assumptions, one can compute the norm of A. Namely, we have

Theorem 2.6 *Let X be a linear normed space, $\mathbb{R}_+ \subset X$, $Y = \mathbb{C}$, and $\|x\| = |x|$ for $x \in \mathbb{R}_+$. Then*

$$\|A\| = 4.$$

The proof, similar to the proof of Theorem 2.2, can be found in [13].

2.4 Quadratic Functional Equation and Functional Equations for Quadratic Differences

At first, we shall give the formula for the general solution of the generalized quadratic functional equation on a group. The result is due to K. Dłutek (see [7]).

Theorem 2.7 *Let G_1 and G_2 be groups with division by two. Let $A, B, C, D\colon G_1 \to G_2$ satisfy the equation*

$$A(x) + B(y) = C(x + y) + D(x - y), \quad x, y \in G_1.$$

Then there exist a quadratic function $K : G_1 \to G_2$ (i.e., a function satisfying the equation (2.6)), additive functions $E, F : G_1 \to G_2$ and constants $S_1, S_2, S_3, S_4 \in G_2$ such that

$$\begin{aligned} A(x) &= 2K(x) + E(x) + F(x) + S_1, \\ B(x) &= 2K(x) + E(x) - F(x) + S_2, \\ C(x) &= K(x) + E(x) + S_3, \\ D(x) &= K(x) + F(x) + S_4 \end{aligned}$$

for all $x \in G_1$ and $S_1 + S_2 = S_3 + S_4$.

Now we shall state the result concerning the properties of the quadratic difference operator Q on L^P-spaces; for more details, see [11].

Theorem 2.8 *Let (G, Σ, μ) be a complete measurable Abelian group, $\mu(G) < \infty$ and let $(E, \|\cdot\|)$ be a Banach space. If $1 \le p \le \infty$, then the quadratic difference operator*

$$Q\colon L^P_\mu(G, E) \to L^P_{\mu\times\mu}(G \times G, E)$$

given by (2.3) is linear, continuous, and invertible. Moreover, the inverse operator Q^{-1} defined for $h \in Q[L^P_\mu(G, E)]$ is continuous and has the form

$$Q^{-1}h(\cdot) = \big(2\mu(G)\big)^{-1} \int_G h(x, \cdot)\, d\mu(x).$$

For some problems, particularly for the problem of Ulam–Hyers–Rassias stability of functional equations, functional equations for quadratic differences are very useful (see [3, 7, 8]). Let us present a few such equations, which we will need in the proof of Theorem 2.11.

Theorem 2.9 *Let X, Y be Abelian groups and $f: X \to Y$ be a function. Then $Q(f)$ given by formula (2.3) satisfies the following functional equations*

$$\begin{aligned} &Q(f)(x+y, s+t) + Q(f)(x-y, s-t) + 2Q(f)(x,y) + 2Q(f)(s,t) \\ &\quad = Q(f)(x+s, y+t) + Q(f)(x-s, y-t) + 2Q(f)(x,s) \\ &\qquad + 2Q(f)(y,t), \end{aligned} \tag{2.14}$$

$$\begin{aligned} &Q(f)(x+y, s) + Q(f)(x-y, s) + 2Q(f)(x,y) \\ &\quad = Q(f)(x+s, y) + Q(f)(x-s, y) + 2Q(f)(x,s), \end{aligned} \tag{2.15}$$

$$\begin{aligned} &Q(f)(x+y, t) + Q(f)(x-y, t) + 2Q(f)(x,y) \\ &\quad = Q(f)(x, y+t) + Q(f)(x, y-t) + 2Q(f)(y,t) \end{aligned} \tag{2.16}$$

for all $x, y, s, t \in X$.

There are also interesting partial differential equations for quadratic differences (see [3, 7, 8]). Let X and Y be normed spaces. The space of all functions $f: X \to Y$ that are n-times differentiable will be denoted by $D^n(X,Y)$. By $\partial_k^n f$, $k = 1, 2$, we denote, as usual, the nth partial derivative of $f: X \times X \to Y$ with respect to the kth variable.

Theorem 2.10 *Let $f: X \to Y$ be a function such that $Q(f) \in D^2(X \times X, Y)$. Then we have*

$$\begin{aligned} &\partial_2^2\big(Q(f)\big)(x+y, 0) + \partial_2^2\big(Q(f)\big)(x-y, 0) \\ &\quad = 2\partial_1^2\big(Q(f)\big)(x,y) + 2\partial_2^2\big(Q(f)\big)(x,0), \end{aligned} \tag{2.17}$$

$$\begin{aligned} &\partial_2^2\big(Q(f)\big)(x+y, 0) + \partial_2^2\big(Q(f)\big)(x-y, 0) \\ &\quad = 2\partial_2^2\big(Q(f)\big)(x,y) + 2\partial_2^2\big(Q(f)\big)(y,0), \end{aligned} \tag{2.18}$$

$$2\partial_{12}^2\big(Q(f)\big)(x,y) = \partial_2^2\big(Q(f)\big)(x+y,0) - \partial_2^2\big(Q(f)\big)(x-y,0) \tag{2.19}$$

for all $x, y \in X$.

From Theorems 2.9 and 2.10, we easily obtain the following corollary.

Corollary 2.1 *Let $f: X \to Y$ be a function such that $Q(f) \in D^2(X \times X, Y)$. Then we have*

$$\partial_1\big(Q(f)\big)(0,0) = 0, \tag{2.20}$$

$$\partial_{12}^2\big(Q(f)\big)(0,0)=0,$$
$$\partial_2^2\big(Q(f)\big)(0,0)=0. \tag{2.21}$$

Moreover, for all $x \in X$ we have

$$\partial_2\big(Q(f)\big)(x,0)=\partial_2\big(Q(f)\big)(0,0). \tag{2.22}$$

2.5 Double Quadratic Difference Property

In 1940, S.M. Ulam posed the following problem (cf. [29]):

We are given a group $(X,+)$ and a metric group $(Y,+,d)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that if $f\colon X\to Y$ satisfies the inequality

$$d\big[f(x+y),f(x)+f(y)\big]<\delta \quad \text{for all } x,y\in X,$$

then a homomorphism $A\colon X\to Y$ exists with

$$d\big[f(x),A(x)\big]<\varepsilon \quad \text{for all } x\in X?$$

One can ask a similar question for other important functional equations. The first partial solution of this problem was given by D.H. Hyers [16] under the assumption that X and Y are Banach spaces. In 1978, Themistocles M. Rassias extended the theorem of Hyers by considering an unbounded Cauchy difference (see [23]). During the last decades, the stability problems of various functional equations have been extensively investigated by many authors (see, e.g., [1, 2, 7, 8, 14, 15, 17, 18, 24–27]).

Assume that X and Y are normed spaces. For a function $f\colon X\to Y$, we put

$$\|f\|_{\sup}:=\sup_{x\in X}\big\|f(x)\big\|.$$

For the quadratic difference, the stability problem can be reformulated as follows. Let $\varepsilon>0$ be given. Does there exist a $\delta>0$ such that if $f\colon X\to Y$ satisfies

$$\big\|Q(f)\big\|_{\sup}<\delta,$$

then there exists a quadratic function $K\colon X\to Y$ with

$$\|f-K\|_{\sup}<\varepsilon?$$

We can consider Ulam's problem for different norms. In this paper, we are going to prove the stability of the quadratic functional equation in the class of differentiable functions. The same problem for the Cauchy type functional equations was solved by J. Tabor and J. Tabor in [28].

Let X and Y be a real normed space and a real Banach space, respectively. By $\mathbb{N}_0$, $\mathbb{N}$, $\mathbb{R}$ we denote the sets of all nonnegative integers, positive integers, and real numbers, respectively. Let $f\colon X \to Y$ be an n-times Fréchet differentiable function. By $D^n f$, $n \in \mathbb{N}$, we denote the nth derivative of f, and $D^0 f$ stands for f. By $C^n(X, Y)$ we denote the space of n-times continuously differentiable functions and by $BC^n(X, Y)$ the subspace of $C^n(X, Y)$ consisting of bounded functions. Moreover, $C^0(X, Y)$ and $C^\infty(X, Y)$ stand for the space of continuous functions and the space of infinitely many times continuously differentiable functions, respectively.

Following an idea of J. Tabor and J. Tabor [28], we assume that we are given a norm in $X \times X$ such that $\|(x_1, x_2)\|$ is a function of $\|x_1\|$ and $\|x_2\|$, and the following condition is satisfied

$$\big\|(x, 0)\big\| = \big\|(0, x)\big\| = \|x\|, \quad x \in X.$$

Let $i_1\colon X \to X \times X$, $i_2\colon X \to X \times X$ be injections defined by

$$\begin{aligned} i_1(x) &:= (x, 0), \quad x \in X, \\ i_2(y) &:= (0, y), \quad y \in X. \end{aligned}$$

Let $L\colon X \times X \to X$ be a bounded linear mapping. It follows directly from the assumed conditions on the norm in $X \times X$ that

$$\begin{aligned} \|L \circ i_1\| &\le \|L\|\|i_1\| = \|L\|, \\ \|L \circ i_2\| &\le \|L\|\|i_2\| = \|L\|. \end{aligned}$$

Therefore, if $F\colon X \times X \to Y$ is n-times differentiable for $n \in \mathbb{N}$, then

$$\left.\begin{aligned} \|\partial_1 F(x, y)\| &= \|DF(x, y) \circ i_1\| \le \|DF(x, y)\|, \\ \|\partial_2 F(x, y)\| &= \|DF(x, y) \circ i_2\| \le \|DF(x, y)\| \end{aligned}\right\} \tag{2.23}$$

and

$$\left.\begin{aligned} \|\partial_1^{i-2}\partial_2^2 F(x, y)\| &\le \|D^i F(x, y)\|, \\ \|\partial_1^2\partial_2^{i-2} F(x, y)\| &\le \|D^i F(x, y)\| \end{aligned}\right\} \tag{2.24}$$

for all $x, y \in X$ and $i = 2, 3, \dots, n$.

Let $n \in \mathbb{N}$ and let $f\colon X \to Y$ be n-times differentiable. Then $Q(f)$ is also n-times differentiable, and by (2.24) we have

$$\big\|Df(x + y) - Df(x - y) - 2Df(y)\big\| \le \big\|D\big(Q(f)\big)(x, y)\big\|, \tag{2.25}$$

$$\big\|D^2 f(x + y) + D^2 f(x - y) - 2D^2 f(y)\big\| \le \big\|D^2\big(Q(f)\big)(x, y)\big\| \tag{2.26}$$

for all $x, y \in X$. Moreover, for $n \ge 3$, we obtain from (2.24)

$$\big\|D^i f(x + y) + D^i f(x - y)\big\| \le \big\|D^i\big(Q(f)\big)(x, y)\big\| \tag{2.27}$$

for all $x, y \in X$ and $i = 3,\ 4,\ \dots,\ n$.

We will prove that the class $C^n(\mathbb{R}, Y)$ has the so-called double quadratic difference property, i.e., if $f: \mathbb{R} \to Y$ is such a function that $Q(f) \in C^n(\mathbb{R} \times \mathbb{R}, Y)$, then there exists exactly one quadratic function $K_0: \mathbb{R} \to Y$ such that $f - K_0 \in C^n(\mathbb{R}, Y)$ (see also [3]). The problem of the double difference property for the Cauchy difference $C(f)(x, y) := f(x + y) - f(x) - f(y) \in C^n(X \times X, Y)$ has been investigated in [28]. For more details about the double difference property, the reader is referred to [19].

Lemma 2.3 (See also [3]) *Let $f: X \to Y$ be a function such that $Q(f) \in C^2(X \times X, Y)$. Then $K_0: X \to Y$ given by the formula*

$$\begin{aligned} K_0(x) = f(x) - f(0) + \frac{1}{2}\partial_2\big(Q(f)\big)(0,0)(x) \\ - \frac{1}{2}\int_0^1 \int_0^t \partial_2^2\big(Q(f)\big)(ux, 0)\big(x^2\big)\, du\, dt, \quad x \in X \end{aligned} \tag{2.28}$$

is a quadratic function.

Proof Let $f_1(x) := f(x) - f(0)$ for all $x \in X$. Then $Q(f_1) = Q(f) + 2f(0) \in C^n(X \times X, Y)$ and $Q(f_1)(0,0) = 0$. Moreover, $\partial_2(Q(f_1)) = \partial_2(Q(f))$ and $\partial_2^2(Q(f_1)) = \partial_2^2(Q(f))$. Let us fix arbitrary $x, y \in X$ and consider a function

$$\varphi(t) := Q(f_1)(tx, ty), \quad t \in \mathbb{R}.$$

Obviously, $\varphi \in C^2(\mathbb{R}, Y)$. Then we have

$$D\varphi(t) = \partial_1\big(Q(f_1)\big)(tx, ty)(x) + \partial_2\big(Q(f_1)\big)(tx, ty)(y), \quad t \in \mathbb{R}.$$

Hence and from (2.20), we get

$$D\varphi(0) = \partial_2\big(Q(f_1)\big)(0,0)(y).$$

Therefore, we obtain

$$\begin{aligned} Q(f_1)(x, y) = \varphi(1) - \varphi(0) = \int_0^1 D\varphi(t)\, dt = \int_0^1 \int_0^t D^2\varphi(u)\, du\, dt + D\varphi(0) \\ = \int_0^1 \int_0^t D^2\big(Q(f_1)\big)(ux, uy)(x, y)\, du\, dt + \partial_2\big(Q(f_1)\big)(0,0)(y) \\ = \int_0^1 \int_0^t \partial_1^2\big(Q(f_1)\big)(ux, uy)\big(x^2\big)\, du\, dt \\ + 2\int_0^1 \int_0^t \partial_{12}^2\big(Q(f_1)\big)(ux, uy)(xy)\, du\, dt \\ + \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(ux, uy)\big(y^2\big)\, du\, dt + \partial_2\big(Q(f_1)\big)(0,0)(y). \end{aligned}$$

Thus

$$\begin{aligned} Q(f_1)(x,y) &= \int_0^1 \int_0^t \partial_1^2\big(Q(f_1)\big)(ux,uy)\big(x^2\big)\,du\,dt \\ &\quad + 2\int_0^1 \int_0^t \partial_{12}^2\big(Q(f_1)\big)(ux,uy)(xy)\,du\,dt \\ &\quad + \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(ux,uy)\big(y^2\big)\,du\,dt \\ &\quad + \partial_2\big(Q(f_1)\big)(0,0)(y), \quad x,y\in X. \end{aligned} \tag{2.29}$$

We define the function $K_0\colon X\to Y$ by the formula

$$\begin{aligned} K_0(x) &:= f_1(x) + \frac{1}{2}\partial_2\big(Q(f_1)\big)(0,0)(x) \\ &\qquad - \frac{1}{2}\int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(ux,0)\big(x^2\big)\,du\,dt, \quad x\in X. \end{aligned}$$

We show that K_0 is a quadratic function. By making use of (2.17), (2.18), (2.19), and (2.29), we obtain for all $x,y\in X$

$$\begin{aligned} &K_0(x+y)+K_0(x-y)-2K_0(x)-2K_0(y) \\ &\quad = Q(f_1)(x,y) - \partial_2\big(Q(f_1)\big)(0,0)(y) \\ &\qquad - \frac{1}{2}\int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(ux+uy,0)(x+y)^2\,du\,dt \\ &\qquad - \frac{1}{2}\int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(ux-uy,0)(x-y)^2\,du\,dt \\ &\qquad + \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(ux,0)\big(x^2\big)\,du\,dt \\ &\qquad + \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(uy,0)\big(y^2\big)\,du\,dt \\ &\quad = \int_0^1 \int_0^t \partial_1^2\big(Q(f_1)\big)(ux,uy)\big(x^2\big)\,du\,dt \\ &\qquad + 2\int_0^1 \int_0^t \partial_{12}^2\big(Q(f_1)\big)(ux,uy)(xy)\,du\,dt \\ &\qquad + \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(ux,uy)\big(y^2\big)\,du\,dt \\ &\qquad - \frac{1}{2}\int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(ux+uy,0)\big(x^2\big)\,du\,dt \end{aligned}$$

$$
\begin{aligned}
&- \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(ux+uy,0)(xy)\,du\,dt \\
&- \frac{1}{2}\int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(ux+uy,0)\big(y^2\big)\,du\,dt \\
&- \frac{1}{2}\int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(ux-uy,0)\big(x^2\big)\,du\,dt \\
&+ \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(ux-uy,0)(xy)\,du\,dt \\
&- \frac{1}{2}\int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(ux-uy,0)\big(y^2\big)\,du\,dt \\
&+ \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(ux,0)\big(x^2\big)\,du\,dt \\
&+ \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(uy,0)\big(y^2\big)\,du\,dt \\
= &\int_0^1 \int_0^t \bigg[\partial_1^2\big(Q(f_1)\big)(ux,uy) - \frac{1}{2}\partial_2^2\big(Q(f_1)\big)(ux+uy,0) \\
&- \frac{1}{2}\partial_2^2\big(Q(f_1)\big)(ux-uy,0) + \partial_2^2\big(Q(f_1)\big)(ux,0)\bigg]\big(x^2\big)\,du\,dt \\
&+ \int_0^1 \int_0^t \big[2\partial_{12}^2\big(Q(f_1)\big)(ux,uy) - \partial_2^2\big(Q(f_1)\big)(ux+uy,0) \\
&+ \partial_2^2\big(Q(f_1)\big)(ux-uy,0)\big](xy)\,du\,dt \\
&+ \int_0^1 \int_0^t \bigg[\partial_2^2\big(Q(f_1)\big)(ux,uy) - \frac{1}{2}\partial_2^2\big(Q(f_1)\big)(ux+uy,0) \\
&- \frac{1}{2}\partial_2^2\big(Q(f_1)\big)(ux-uy,0) + \partial_2^2\big(Q(f_1)\big)(uy,0)\bigg]\big(y^2\big)\,du\,dt = 0.
\end{aligned}
$$

Therefore, K_0 is a quadratic function, which completes the proof. □

Theorem 2.11 *Let $n \geq 2$ be a fixed positive integer and let $f: \mathbb{R} \to Y$ be a function such that $Q(f) \in C^n(\mathbb{R}\times\mathbb{R}, Y)$. Then there exists a unique quadratic function $K_0: \mathbb{R} \to Y$ such that $f - K_0 \in C^n(\mathbb{R}, Y)$ and $D^2(f-K_0)(0) = 0$. Moreover, we have for all $x \in \mathbb{R}$*

$$
\begin{aligned}
D(f-K_0)(x) &= \frac{1}{2}\int_0^x \partial_2^2\big(Q(f)\big)(s,0)\,ds - \frac{1}{2}\partial_2\big(Q(f)\big)(0,0), \\
D^2(f-K_0)(x) &= \frac{1}{2}\partial_2^2\big(Q(f)\big)(x,0),
\end{aligned}
$$

$$\left\| D^k(f-K_0)(x) \right\| \le \frac{1}{2} \left\| D^k\big(Q(f)\big)(x,0) \right\|, \quad k \in \mathbb{N}\backslash\{1\},\ k \le n.$$

Proof Let $f_1(x) := f(x) - f(0)$ for all $x \in \mathbb{R}$. On account of Lemma 2.3, there exists a quadratic function K_0 given by (2.28). Now we prove that $f - K_0$ is a differentiable function. Fix arbitrary $x, h \in \mathbb{R}$, $h \neq 0$. Then we get

$$\begin{aligned}
&\frac{1}{h}\big[f_1(x+h) - K_0(x+h) - \big(f_1(x) - K_0(x)\big)\big] \\
&\quad = \frac{1}{h}\bigg[\frac{1}{2}\int_0^1\int_0^t \partial_2^2\big(Q(f_1)\big)\big(u(x+h),0\big)(x+h)^2\,du\,dt \\
&\qquad - \frac{1}{2}\partial_2\big(Q(f_1)\big)(0,0)(x+h) \\
&\qquad - \frac{1}{2}\int_0^1\int_0^t \partial_2^2\big(Q(f_1)\big)(ux,0)\big(x^2\big)\,du\,dt + \frac{1}{2}\partial_2\big(Q(f_1)\big)(0,0)(x)\bigg] \\
&\quad = \frac{1}{2h}\bigg[\int_0^{x+h}\int_0^v \partial_2^2\big(Q(f_1)\big)(s,0)\,ds\,dv \\
&\qquad - \int_0^x\int_0^v \partial_2^2\big(Q(f_1)\big)(s,0)\,ds\,dv - \partial_2\big(Q(f_1)\big)(0,0)(h)\bigg] \\
&\quad = \frac{1}{2h}\bigg[\int_x^{x+h}\int_0^v \partial_2^2\big(Q(f_1)\big)(s,0)\,ds\,dv - \partial_2\big(Q(f_1)\big)(0,0)(h)\bigg] \\
&\quad = \frac{1}{2h}\bigg[\int_0^1\int_0^{x+th} \partial_2^2\big(Q(f_1)\big)(s,0)(h)\,ds\,dt - \partial_2\big(Q(f_1)\big)(0,0)(h)\bigg] \\
&\quad = \frac{1}{2}\bigg[\int_0^1\int_0^{x+th} \partial_2^2\big(Q(f_1)\big)(s,0)\,ds\,dt - \partial_2\big(Q(f_1)\big)(0,0)\bigg] \\
&\quad \longrightarrow \quad \frac{1}{2}\int_0^1\int_0^x \partial_2^2\big(Q(f_1)\big)(s,0)\,ds\,dt \\
&\qquad\qquad - \frac{1}{2}\partial_2\big(Q(f_1)\big)(0,0) = \frac{1}{2}\int_0^x \partial_2^2\big(Q(f_1)\big)(s,0)\,ds - \frac{1}{2}\partial_2\big(Q(f_1)\big)(0,0)
\end{aligned}$$

for $h \to 0$. Hence the function $f - K_0 = f_1 - K_0 + f(0)$ is differentiable at every $x \in \mathbb{R}$, $\partial_2(Q(f_1)) = \partial_2(Q(f))$, $\partial_2^2(Q(f_1)) = \partial_2^2(Q(f))$ and

$$D(f-K_0)(x) = \frac{1}{2}\int_0^x \partial_2^2\big(Q(f)\big)(s,0)\,ds - \frac{1}{2}\partial_2\big(Q(f)\big)(0,0), \quad x \in \mathbb{R}. \tag{2.30}$$

Moreover, since the function $f - K_0$ is differentiable at every $x \in \mathbb{R}$, then there exists also the second difference derivative which is equal to the second derivative (see [21]). We show that

$$D^2(f - K_0)(x) = \frac{1}{2}\partial_2^2\big(Q(f)\big)(x,0), \quad x \in \mathbb{R}.$$

Fix arbitrary $x, h \in \mathbb{R}$ and let a function $\psi : \mathbb{R} \to Y$ be given by

$$\psi(t) := Q(f_1)(x, th), \quad t \in \mathbb{R}.$$

Then $\psi \in C^n(\mathbb{R}, Y)$ and we have

$$\begin{aligned} Q(f_1)(x,h) &= \psi(1) - \psi(0) \\ &= \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(x, uh)\big(h^2\big)\,du\,dt \\ &\quad + \partial_2\big(Q(f_1)\big)(x,0)(h), \quad x, h \in \mathbb{R}. \end{aligned}$$

From (2.22) we get

$$\partial_2\big(Q(f_1)\big)(x,0) = \partial_2\big(Q(f_1)\big)(0,0), \quad x \in \mathbb{R},$$

hence

$$\begin{aligned} Q(f_1)(x,h) &= \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(x, uh)\big(h^2\big)\,du\,dt \\ &\quad + \partial_2\big(Q(f_1)\big)(0,0)(h), \quad x, h \in \mathbb{R}. \end{aligned} \tag{2.31}$$

Next, from (2.18) we obtain

$$\begin{aligned} &\int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(x+uh, 0)\big(h^2\big)\,du\,dt \\ &\quad + \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(x-uh, 0)\big(h^2\big)\,du\,dt \\ &= 2\int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(x, uh)\big(h^2\big)\,du\,dt \\ &\quad + 2\int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(uh, 0)\big(h^2\big)\,du\,dt, \quad x, h \in \mathbb{R}. \end{aligned} \tag{2.32}$$

Therefore, using (2.31) and (2.32), for each fixed $x, h \in \mathbb{R}$ we have

$$\Big\| f_1(x+h) - K_0(x+h) - 2f_1(x) + 2K_0(x) + f_1(x-h) - K_0(x-h) \\ - \frac{1}{2}\partial_2^2\big(Q(f_1)\big)(x,0)\big(h^2\big)\Big\|$$

$$
\begin{aligned}
&= \left\| Q(f_1)(x,h) + 2f_1(h) - 2K_0(h) - \frac{1}{2}\partial_2^2\big(Q(f_1)\big)(x,0)\big(h^2\big) \right\| \\
&= \bigg\| \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(x,uh)\big(h^2\big)\,du\,dt + \partial_2\big(Q(f_1)\big)(0,0)(h) \\
&\quad + \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(uh,0)\big(h^2\big)\,du\,dt \\
&\quad - \partial_2\big(Q(f_1)\big)(0,0)(h) - \frac{1}{2}\partial_2^2\big(Q(f_1)\big)(x,0)\big(h^2\big) \bigg\| \\
&= \bigg\| \frac{1}{2}\int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(x+uh,0)\big(h^2\big)\,du\,dt \\
&\quad + \frac{1}{2}\int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(x-uh,0)\big(h^2\big)\,du\,dt \\
&\quad - \int_0^1 \int_0^t \partial_2^2\big(Q(f_1)\big)(x,0)\big(h^2\big)\,du\,dt \bigg\| \\
&= \bigg\| \int_0^1 \int_0^t \bigg[\frac{1}{2}\partial_2^2\big(Q(f_1)\big)(x+uh,0) + \frac{1}{2}\partial_2^2\big(Q(f_1)\big)(x-uh,0) \\
&\quad - \partial_2^2\big(Q(f_1)\big)(x,0) \bigg] \big(h^2\big)\,du\,dt \bigg\| \\
&\le \|h\|^2 \sup_{u\in[0,1]} \bigg\| \frac{1}{2}\partial_2^2\big(Q(f_1)\big)(x+uh,0) \\
&\quad + \frac{1}{2}\partial_2^2\big(Q(f_1)\big)(x-uh,0) - \partial_2^2\big(Q(f_1)\big)(x,0) \bigg\|.
\end{aligned}
$$

Since $\partial_2^2(Q(f_1)) = \partial_2^2(Q(f))$ is a continuous function,

$$
\frac{1}{2}\partial_2^2\big(Q(f_1)\big)(x+uh,0) + \frac{1}{2}\partial_2^2\big(Q(f_1)\big)(x-uh,0) - \partial_2^2\big(Q(f_1)\big)(x,0) \to 0
$$

for $h \to 0$. Hence we get

$$
D^2(f - K_0)(x) = \frac{1}{2}\partial_2^2\big(Q(f)\big)(x,0), \quad x \in \mathbb{R}. \tag{2.33}
$$

Since $\partial_2^2(Q(f))(x,0) \in C^{n-2}(\mathbb{R}\times\mathbb{R}, Y)$, one has $\partial_2^2(Q(f))(x,0) \in C^{n-2}(\mathbb{R}, Y)$. Finally, $D^2(f-K_0) \in C^{n-2}(\mathbb{R}, Y)$, i.e., $f - K_0 \in C^n(\mathbb{R}, Y)$. Moreover, from (2.21) we also have

$$
D^2(f - K_0)(0) = \frac{1}{2}\partial_2^2\big(Q(f)\big)(0,0) = 0.
$$

To prove the uniqueness of K_0, consider quadratic functions $K_1, K_2\colon \mathbb{R} \to Y$ such that $f - K_1, f - K_2 \in C^n(\mathbb{R}, Y)$, $D^2(f-K_1)(0) = D^2(f-K_2)(0) = 0$ and conditions (2.30), (2.33) hold. Then $K_1 - K_2$ is a quadratic function and $K_1 - K_2 \in C^n(\mathbb{R}, Y)$. Therefore, for every $x \in \mathbb{R}$, we have

$$D^2(K_1 - K_2)(x) = D^2\big((f - K_2) - (f - K_1)\big)(x)$$
$$= D^2(f - K_2)(x) - D^2(f - K_1)(x) = 0.$$

Since $D^2(K_1 - K_2) = 0$, $D(K_1 - K_2)(x)$ is a constant function for every $x \in \mathbb{R}$. But

$$D(K_1 - K_2)(0) = D(f - K_2)(0) - D(f - K_1)(0) = 0,$$

so $D(K_1 - K_2) = 0$. Analogously, we have that $(K_1 - K_2)(x)$ is a constant function for every $x \in \mathbb{R}$; and since $(K_1 - K_2)(0) = 0$, it yields that $K_1 = K_2$.

It remains to prove that

$$\big\| D^k(f - K_0)(x)\big\| \le \frac{1}{2}\big\| D^k\big(Q(f)\big)(x, 0)\big\|, \quad k \in \mathbb{N}\backslash\{1\},\ k \le n \tag{2.34}$$

for every $x \in \mathbb{R}$. Let $g := f - K_0$. Then $g \in C^n(\mathbb{R}, Y)$, and consequently we have $Q(g) = Q(f) \in C^n(\mathbb{R} \times \mathbb{R}, Y)$. Making use of (2.26) and the fact that $D^2 g(0) = 0$, we obtain

$$\big\| D^2 g(x)\big\| = \big\| D^2 g(x) - D^2 g(0)\big\| \le \frac{1}{2}\big\| D^2\big(Q(g)\big)(x, 0)\big\|, \quad x \in \mathbb{R},$$

which proves (2.34) for $k = 2$. For $3 \le k \le n$, $k \in \mathbb{N}$, condition (2.34) follows directly from (2.27), which completes the proof. □

Corollary 2.2 ([3]) *Under the assumptions of Theorem 2.11, we have*

$$\big\| D^k(f - K_0)(0)\big\| \le \frac{1}{2}\big\| D^k\big(Q(f)\big)(0, 0)\big\|, \quad k \in \mathbb{N}_0\backslash\{1\},\ k \le n, \tag{2.35}$$

$$\big\| D^k(f - K_0)\big\|_{\sup} \le \frac{1}{2}\big\| D^k\big(Q(f)\big)\big\|_{\sup}, \quad k \in \mathbb{N}\backslash\{1\},\ k \le n. \tag{2.36}$$

Proof The case $k = 0$ in (2.35) is trivial because obviously $f(0) = -\frac{1}{2}Q(f)(0, 0)$. From (2.34) we obtain (2.35) for $k \ge 2$ and (2.36). The proof is completed. □

Remark 2.2 Let the assumptions of Theorem 2.11 be satisfied and let $\partial_2(Q(f))(0, 0) = 0$. Then the inequality (2.34) (and consequently (2.35) and (2.36)) also holds for $k = 1$.

Proof If $\partial_2(Q(f))(0, 0) = 0$, then from (2.30) we obtain $D(f - K_0)(0) = 0$. Let $g := f - K_0$. Hence $g \in C^n(\mathbb{R}, Y)$, $Q(g) = Q(f) \in C^n(\mathbb{R} \times \mathbb{R}, Y)$, and $C(g) \in C^n(\mathbb{R} \times \mathbb{R}, Y)$. Therefore, on account of (2.23), we get

$$\big\| Dg(x + y) - Dg(y)\big\| \le \big\| D\big(C(g)\big)(x, y)\big\|, \quad x, y \in \mathbb{R}. \tag{2.37}$$

One can easily check that for any function $h\colon \mathbb{R} \to Y$ the following equality holds

$$2C(h)(x,y) + 2C(h)(x,-y) = Q(h)(x,y) + Q(h)(x,-y), \quad x, y \in \mathbb{R},$$

where $C(f)$ denotes the Cauchy difference. Then, in particular, for a function g we obtain

$$D\big(C(g)\big)(x,0) = \frac{1}{2}D\big(Q(g)\big)(x,0) = \frac{1}{2}D\big(Q(f)\big)(x,0), \quad x \in \mathbb{R}.$$

Therefore, by virtue of (2.37) with $y = 0$, from the above equality and the fact that $Dg(0) = 0$, we have

$$\big\|Dg(x)\big\| = \big\|Dg(x) - Dg(0)\big\| \le \big\|D\big(C(g)\big)(x,0)\big\| = \frac{1}{2}\big\|D\big(Q(g)\big)(x,0)\big\|,$$
$$x \in \mathbb{R},$$

which proves the inequality (2.34) for $k = 1$. □

It is still an open problem to prove that the function $f - K_0$ which occurs in Theorem 2.11 is differentiable for every $x \in X$, where X denotes a real normed space.

Corollary 2.3 ([3]) *The quadratic function $K_0\colon \mathbb{R} \to Y$ occurring in Lemma 2.3 and Theorem 2.11 can be defined by the formula*

$$K_0(x) = \frac{1}{2}\lim_{n\to\infty} n^2\left[f\left(\frac{x}{n}\right) + f\left(-\frac{x}{n}\right) - 2f(0)\right], \quad x \in \mathbb{R}. \tag{2.38}$$

Theorem 2.11 states, in particular, that the class of infinitely many times differentiable functions has the double quadratic difference property. We may show that the class of analytic functions also has this property.

Corollary 2.4 ([3]) *Let $f : \mathbb{R} \to Y$ be a function such that $Q(f)$ is analytic. Then there exists exactly one quadratic function $K : \mathbb{R} \to Y$ such that $f - K$ is analytic and $D^2(f - K)(0) = 0$.*

Now we give some auxiliary results which will be used in the sequel.

Lemma 2.4 ([5]) *Let $(G, +)$ be an Abelian group. If a function $f : G \to Y$ satisfies the inequality*

$$\big\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\big\| \le \varepsilon, \quad x, y \in G$$

for some $\varepsilon > 0$, then there exists a unique quadratic function $K : G \to Y$ such that

$$\big\|f(x) - K(x)\big\| \le \frac{1}{2}\varepsilon, \quad x \in G.$$

Moreover, the function K is given by the formula

$$K(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}}, \quad x \in G.$$

In [6], S. Czerwik provided a generalization of the above result and also proved that if a function $\mathbb{R} \ni t \to f(tx)$ is continuous for each fixed $x \in E$, where E denotes a real normed space, then $K(tx) = t^2 K(x)$ for all $t \in \mathbb{R}$ and $x \in E$.

The following lemma is some kind of an analogue to the Mean Value Theorem for real valued functions.

Lemma 2.5 ([20]) *Let a mapping $T : B \to Y$, $B \subset X$, where B is an open set, be two times Fréchet differentiable. Let $x, h \in B$, and let for every $0 \le \alpha \le 1$, $(x + \alpha h) \in B$. Then*

$$\left\| T(x+h) - T(x) - DT(x)h \right\| \le \frac{1}{2}\|h\|^2 \sup_{0<\alpha<1} \left\| D^2T(x+\alpha h)\right\|.$$

Similarly as for the case of Euler's Theorem for positive homogeneous functions (see [22]), one can prove the following lemma.

Lemma 2.6 *Let $T : \mathbb{R} \to Y$ be a homogeneous function of degree 2 such that $T \in C^2(\mathbb{R}, Y)$. Then*

$$T(x) = \frac{1}{2}x^2 D^2T(x), \quad x \in \mathbb{R}. \tag{2.39}$$

Proof Since T is a homogeneous function of degree 2, then

$$T(\alpha x) = \alpha^2 T(x), \quad x, \alpha \in \mathbb{R}.$$

Let us fix an arbitrary $x \in \mathbb{R}$. Differentiating both sides of the above equality with respect to α, we obtain

$$D^2T(\alpha x)x^2 = 2T(x), \quad \alpha \in \mathbb{R}.$$

Since $x \in \mathbb{R}$ was chosen arbitrarily, for $\alpha = 1$ we get the equality (2.39), which completes the proof. □

Lemma 2.7 *Let $K : \mathbb{R} \to Y$ be a quadratic function and let $f : \mathbb{R} \to Y$ be a mapping such that $f \in C^2(\mathbb{R}, Y)$. Assume also that $f - K$ is bounded and*

$$\sup_{(x,y)\in\mathbb{R}\times\mathbb{R}} \left\| D^2f(x) - D^2f(y)\right\| \le \varepsilon \tag{2.40}$$

for some $\varepsilon > 0$. Then K is differentiable and

$$\sup_{x\in\mathbb{R}} \left\| D^2K(x) - D^2f(x)\right\| \le \varepsilon. \tag{2.41}$$

Proof Since $f - K$ is bounded, then $Q(f)$ is also bounded and, on account of Lemma 2.4, we have

$$K(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^{2n}}, \quad x \in \mathbb{R}.$$

Since f is a continuous at each $x \in \mathbb{R}$, K is also continuous and it is of the form (see also [7])

$$K(x) = x^2 K(1), \quad x \in \mathbb{R}.$$

Thus K is differentiable. Let us fix an arbitrary $y \in \mathbb{R}$. Applying Lemma 2.5 to the function

$$T(x) := f(x) - \frac{1}{2}x^2 D^2 f(y), \quad x \in \mathbb{R}$$

and the inequality (2.40), we get for all $x \in \mathbb{R}$

$$\begin{aligned}\left\| f(x) - \frac{1}{2}x^2 D^2 f(y) - f(0) - xDf(0)\right\| &\le \frac{1}{2}|x|^2 \sup_{x\in\mathbb{R}}\left\| D^2 f(x) - D^2 f(y)\right\| \\ &\le \frac{1}{2}\varepsilon |x|^2.\end{aligned}$$

Replacing x by $2^n x$ and dividing both sides of the above inequality by 2^n, we have

$$\left\| \frac{f(2^n x)}{2^{2n}} - \frac{1}{2}x^2 D^2 f(y) - \frac{f(0)}{2^{2n}} - \frac{xDf(0)}{2^n}\right\| \le \frac{1}{2}\varepsilon|x|^2, \quad x \in \mathbb{R}.$$

Letting $n \to \infty$, we conclude that

$$\left\| K(x) - \frac{1}{2}x^2 D^2 f(y)\right\| \le \frac{1}{2}\varepsilon|x|^2, \quad x \in \mathbb{R}.$$

Therefore, by virtue of Lemma 2.6, we obtain

$$\left\| \frac{1}{2}x^2 D^2 K(x) - \frac{1}{2}x^2 D^2 f(y)\right\| \le \frac{1}{2}\varepsilon|x|^2, \quad x \in \mathbb{R},$$

and hence

$$\left\| D^2 K(x) - D^2 f(y)\right\| \le \varepsilon, \quad x \in \mathbb{R}.$$

Since y was arbitrary,

$$\sup_{x\in\mathbb{R}}\left\| D^2 K(x) - D^2 f(x)\right\| \le \varepsilon,$$

which completes the proof. $\square$

Theorem 2.12 *Let $n \geq 2$ be a fixed positive integer and let $f: \mathbb{R} \to Y$ be such a function that $Q(f) \in BC^n(\mathbb{R} \times \mathbb{R}, Y)$. Then there exists a unique quadratic function $K_\infty: \mathbb{R} \to Y$ such that $f - K_\infty \in BC^n(\mathbb{R}, Y)$. Moreover,*

$$\left\| D^k(f - K_\infty)(0) \right\| \leq \frac{1}{2} \left\| D^k\big(Q(f)\big)(0,0) \right\|, \quad k \in \mathbb{N}_0 \backslash \{2\},\ k \leq n, \tag{2.42}$$

$$\left\| D^k(f - K_\infty) \right\|_{\sup} \leq \frac{1}{2} \left\| D^k\big(Q(f)\big) \right\|_{\sup}, \quad k \in \mathbb{N}_0 \backslash \{1\},\ k \leq n. \tag{2.43}$$

Proof By virtue of Theorem 2.11, there exists a unique quadratic function $K_0: \mathbb{R} \to Y$ such that $f_1 := f - K_0 \in C^n(\mathbb{R}, Y)$. Then $Q(f_1) = Q(f) \in BC^n(\mathbb{R} \times \mathbb{R}, Y)$. It means, in particular, that $Q(f_1)$ is bounded. By Lemma 2.4, there exists a unique quadratic function $K_1: \mathbb{R} \to Y$ such that

$$\sup_{x \in \mathbb{R}} \left\| f_1(x) - K_1(x) \right\| \leq \frac{1}{2} \sup_{(x,y) \in \mathbb{R} \times \mathbb{R}} \left\| Q(f_1)(x,y) \right\|. \tag{2.44}$$

We put

$$K_\infty(x) := K_0(x) + K_1(x), \quad x \in \mathbb{R}.$$

Clearly, K_∞ is also a quadratic function and $f - K_\infty = f_1 - K_1 \in C^n(\mathbb{R}, Y)$. From (2.26) we obtain

$$\sup_{(x,y) \in \mathbb{R} \times \mathbb{R}} \left\| D^2 f_1(x) - D^2 f_1(y) \right\| \leq \frac{1}{2} \sup_{(x,y) \in \mathbb{R} \times \mathbb{R}} \left\| D^2\big(Q(f_1)\big)(x,y) \right\|. \tag{2.45}$$

Conditions (2.44) and (2.45) mean that the functions f_1 and K_1 satisfy the assumptions of Lemma 2.7 with

$$\varepsilon = \frac{1}{2} \sup_{(x,y) \in \mathbb{R} \times \mathbb{R}} \left\| D^2\big(Q(f_1)\big)(x,y) \right\|.$$

Thus K_1 is differentiable and, by making use of (2.41) and (2.45), we get

$$\begin{aligned} \sup_{x \in \mathbb{R}} \left\| D^2(f - K_\infty)(x) \right\| &= \sup_{x \in \mathbb{R}} \left\| D^2(f_1 - K_1)(x) \right\| \\ &\leq \frac{1}{2} \sup_{(x,y) \in \mathbb{R} \times \mathbb{R}} \left\| D^2\big(Q(f_1)\big)(x,y) \right\|. \end{aligned} \tag{2.46}$$

For $k = 0$, the inequality (2.42) is obvious; for $k = 1$, it follows from (2.25); and for $k \geq 3$, it is a trivial consequence of (2.27). Making use of (2.44), (2.46), and (2.27), we obtain (2.43), which completes the proof. □

Corollary 2.5 *The quadratic function K_∞ occurring in Theorem 2.12 can be defined by the formula*

$$K_\infty(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^{2n}}, \quad x \in \mathbb{R}.$$

Proof The formula for K_∞ is a trivial consequence of the fact that the function $f - K_\infty$ is bounded. □

Comparing the formulae for K_0 and K_∞, one can easily notice that these functions are usually different. We will see it in the following example.

Example Let $f : \mathbb{R} \to Y$ be a bounded function such that $f \in C^2(\mathbb{R}, Y)$. Since f is bounded, $K_\infty = 0$. One can easily check that the following equalities hold:

$$\partial_2\big(Q(f)\big)(0,0) = -2Df(0),$$
$$\partial_2^2\big(Q(f)\big)(x,0) = 2D^2 f(x) - 2D^2 f(0), \quad x \in \mathbb{R}.$$

Therefore, by applying the formula of K_0 given by (2.28), we obtain that

$$K_0(x) = \frac{1}{2} D^2 f(0) x^2, \quad x \in \mathbb{R}, \tag{2.47}$$

hence

$$D^2 K_0(x) = D^2 f(0), \quad x \in \mathbb{R}.$$

Thus $K_0 = K_\infty$ if and only if $D^2 f(0) = 0$.

Clearly, one can also obtain the formula (2.47) from (2.38).

2.6 Stability

Let $f : X \to Y$ be a function such that $f \in C^n(X, Y)$ for $n \in \mathbb{N}_0$. In subspaces of $C^n(X, Y)$, one can consider different norms defined in terms of $\|D^i f(0)\|$, $\|D^i f\|_{\sup}$ for $i \le n$. For example, the following norms

$$\begin{aligned} \|f\| &:= \sum_{i=0}^{n-1} \big\|D^i f(0)\big\| + \big\|D^n f\big\|_{\sup}, \\ \|f\| &:= \sum_{i=0}^{n} \big\|D^i f\big\|_{\sup}, \\ \|f\| &:= \max_{i=0,\ldots,n} \big\|D^i f\big\|_{\sup} \end{aligned} \tag{2.48}$$

are used very often. Obviously, several other norms can be introduced. We will prove the stability result for the quadratic difference $Q(f)$ in a possibly general setting. We will use the following convention: if $m, n \in \mathbb{N}_0$ and $m > n$, then $\sum_{i=m}^{n} a_i = 0$.

In the sequel, we will use the following assumptions introduced in [28]. Let $n \in \mathbb{N}_0 \cup \{\infty\}$ be fixed. In the set $[0,\infty]^{2n+2}$, we introduce the following order

$$(x_1,\ x_2,\ \ldots) \le (y_1,\ y_2,\ \ldots)$$

iff $x_i \le y_i$ for $i \in \mathbb{N}$, $i \le 2n + 2$.

Let $p\colon [0,\infty]^{2n+2} \to [0,\infty]$ be any function satisfying the following conditions:

(i) $p(x+y) \le p(x) + p(y)$, $x, y \in [0,\infty]^{2n+2}$,
(ii) $p(\alpha x) = \alpha p(x)$, $x, y \in [0,\infty]^{2n+2}$, $\alpha \in [0,\infty]$,
(iii) $x \le y \Longrightarrow p(x) \le p(y)$, $x, y \in [0,\infty]^{2n+2}$.

We additionally assume that $0 \cdot \infty = 0$. From (ii) we obtain that $p(0) = 0$.

We define the mapping $\Phi\colon C^n(X,Y) \to [0,\infty]^{2n+2}$ by the formula

$$\Phi(f) := \left(\left\| f(0)\right\|,\ \|f\|_{\sup},\ \left\|Df(0)\right\|,\ \|Df\|_{\sup},\ \ldots\right)$$

and put

$$S_p(X,Y) := \left\{f \in C^n(X,Y) : p\left(\Phi(f)\right) < \infty\right\}.$$

Since $p(0) = 0$, S_p contains at least the zero function. It is easy to notice that S_p is a linear space and that $p \circ \Phi|S_p$ is a seminorm. We will denote this seminorm by $\|\cdot\|_p$. The same notations we will apply for the space $C^n(X \times X, Y)$.

Now we are able to prove the main theorem of this section.

Theorem 2.13 *Let $f\colon \mathbb{R} \to Y$ be a function such that $Q(f) \in S_p(\mathbb{R} \times \mathbb{R}, Y)$ and $\partial_2(Q(f))(0,0) = 0$. We additionally assume that the function p does not depend on the second or fourth and fifth variables. Then there exists a quadratic function $K\colon \mathbb{R} \to Y$ such that $f - K \in S_p(\mathbb{R}, Y)$ and*

$$\|f - K\|_p \le \frac{1}{2}\left\|Q(f)\right\|_p.$$

Proof Assume that $Q(f) \in C^n(\mathbb{R} \times \mathbb{R}, Y)$. Suppose that p does not depend on the second variable. Then

$$p(0,\ \infty,\ 0,\ \ldots) = p(0,\ 0,\ 0,\ \ldots) = 0.$$

By Theorem 2.11, there exists exactly one quadratic function $K_0\colon \mathbb{R} \to Y$ satisfying conditions (2.35) and (2.36). Then

$$\Phi(f - K_0) \le \frac{1}{2}\left[\Phi\left(Q(f)\right) + (0,\infty,0,\ldots)\right],$$

and hence from (i), (ii), and (iii) we have

$$p\left(\Phi(f - K_0)\right) \le \frac{1}{2} p\left(\Phi\left(Q(f)\right) + (0,\infty,0,\ldots)\right) \le \frac{1}{2} p\left(\Phi\left(Q(f)\right)\right),$$

i.e.,

$$\|f - K_0\|_p \le \frac{1}{2}\left\|Q(f)\right\|_p.$$

Suppose now that p does not depend on the fourth and fifth variables. If $Q(f) \in BC^n(\mathbb{R} \times \mathbb{R}, Y)$, then by Theorem 2.12 there exists exactly one quadratic function $K_\infty \colon \mathbb{R} \to Y$ satisfying conditions (2.42) and (2.43). Hence

$$\Phi(f - K_\infty) \leq \frac{1}{2}\big[\Phi\big(Q(f)\big) + (0, 0, 0, \infty, \infty, 0, \ldots)\big],$$

and consequently from (i), (ii), and (iii) we get

$$p\big(\Phi(f - K_\infty)\big) \leq \frac{1}{2} p\big(\Phi\big(Q(f)\big) + (0, 0, 0, \infty, \infty, 0, \ldots)\big) \leq \frac{1}{2} p\big(\Phi\big(Q(f)\big)\big),$$

i.e.,

$$\|f - K_\infty\|_p \leq \frac{1}{2}\big\|Q(f)\big\|_p.$$

If $Q(f)$ is unbounded, then $\|Q(f)\|_{\sup} = \infty$. By Theorem 2.11, we can find a quadratic function such that the conditions (2.35) and (2.36) hold. Then $\Phi(f - K_0) \leq \frac{1}{2}\Phi(Q(f))$, and hence

$$\|f - K_0\|_p \leq \frac{1}{2}\big\|Q(f)\big\|_p.$$

The proof is completed. □

One can easily notice that if we defined for $n \in \mathbb{N}_0$

$$p(x_1, x_2, \ldots, x_{2n+2}) := \sum_{i=1}^{n} x_{2i-1} + x_{2n+2}, \tag{2.49}$$

then we would obtain stability of the quadratic functional equation in the norm defined by the formula (2.48).

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