Chapter 3
The Air Flow Model/Boundary Fluid Structure Interaction/The Aeroelastic Problem

3.1 Introduction

In this chapter we make a precise mathematical statement of the aeroelastic problem that we wish to solve. Having described the structure model, we turn to the air flow model simplifying it to the most used case where we neglect viscosity and consider "nonviscous flow" but more importantly assume that the entropy is constant. This makes the flow vortex free so that the flow can be described in terms of the potential. Our concern is again more the structure response in air flow—"aeroelasticity"—and hence the fluid–structure boundary conditions play the dominant role in determining the aerodynamic loading on the wing structure.

Starting with the three basic conservation laws, we derive the fundamental field equation describing the air flow, The Eulerfull potential equation with the Kutta–Joukowsky boundary conditions. We present a complete statement of the aeroelastic problem at the end of the chapter for nonviscous flow and nonlinear structure models, including the simplification to Strip theory, the typical section theory.

3.2 Notation/Physical Constants

We begin with the notation we use for the basic parameters necessary to describe the flow generally throughout the book from now on. We also list the relevant physical constants we need in the process:

ρ Density
q Fluid velocity vector
q_∞ Far field velocity
U_∞ $= |q_\infty|$ This is a free parameter, the far field air speed
a_∞ $=$ Speed of sound
p Pressure
μ Viscosity

A V Balakrishnan, *Aeroelasticity: The Continuum Theory*,
DOI 10.1007/978-1-4614-3609-6_3,

S Entropy
T Temperature
e Energy per unit volume
Subscript ∞ denotes far field values

Perfect Gas Law

$p = \rho RT$
$R = c_p - c_v$
$\gamma = c_p/c_v$ Ratio of specific heats

Thermodynamic Relation:

$p\rho^{\gamma} \mathrm{e}^{-s/c_v} = Const.$
h enthalpy per unit mass $= c_p T$
E internal energy per unit mass $= c_v T$
All are functions of time t and the spatial coordinates $x, \ y, \ z$:
$f = f(t, \ x, \ y, \ z)$
The scalar functions are all positive.

Physical Constants:

All at standard air 59 F

ρ 0.00238 slug/ft^3
1.23 kg/m^3
μ 0.372×10^{-6}slug/ftsec
17.8×10^{-6} kg/msec
p 2, 116, lb/ft^2
1.0312×10^5N/m^2
a_∞ 336 m/sec
R_e Reynolds Number $U1\rho/\mu \sim\sim 10^6$
R Gas Constant 287 kg/msecunits
c_v 717
c_p 1,004
γ 1. 4
k Diffusivity $= \mu c_p$
17.8×10^{-3}
Prandtl no. $\sim\sim$ 0.7 (here taken as 1)

$$\frac{k}{R\rho_\infty} = \mu \quad \frac{c_p}{R\rho_\infty} = \frac{\gamma}{\gamma - 1}\frac{\mu}{\rho_\infty}$$

3.3 Nonviscous Flow: The Euler Full Potential Equation

Throughout this chapter we set $\mu = 0$.

The field equation of fluid flow in 3D space (R^3, orthogonal coordinates x, y, z) is derived from three basic laws of conservation which we state here in differential form (as opposed to the integral form) with t representing the time co-ordinate.

We begin with the premordial.

Conservation Laws

1. Conservation of mass: Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho q) = 0. \tag{3.1}$$

2. Conservation of momentum. The Euler momentum equation (nonviscous flow, no heat conduction)

$$\rho \frac{\partial q}{\partial t} + \rho(q \cdot \nabla)q + \nabla p = 0 \tag{3.2}$$

in the usual notation [4, 12], where $(q \cdot \nabla)q$ is the vector:

$$iq \cdot \nabla(q \cdot i) + jq \cdot \nabla(q \cdot j) + kq \cdot \nabla(q \cdot k) \quad q = iq_1 + jq_2 + kq_3,$$

where i, i, k are the orthogonal unit vectors.

Note that second spatial derivatives of q are not involved, unlike the viscous case (see Chap. 7).

3. Conservation of energy: First law of thermodynamics in Eulerian form for perfect gas [14]

$$E = \frac{1}{\gamma - 1}\frac{p}{\rho},$$

$$\rho \frac{\mathrm{D}E}{\mathrm{D}t} + p\nabla \cdot q = 0. \tag{3.3}$$

Or

$$\rho\left(\frac{\partial E}{\partial t} + q \cdot \nabla E\right) + p \cdot \nabla q = 0. \tag{3.4}$$

Note that we need three equations: conservation of mass, conservation of momentum, and conservation of energy. All are expressed in differential form rather than in integral form. This makes it possible to state the dynamic boundary conditions crucial for aeroelasticity.

This equation (3.4) implies [12, 14] that the total derivative of the entropy S is zero:

$$\frac{\mathrm{D}S}{\mathrm{D}t} = 0, \tag{3.5}$$

or the flow is homentropic [14].

Note that:

$$p \ \rho \ T \ S$$

are thermodynamic state variables any one of which is determined by the other three. Thus, under our assumption that the specific heats are constants, we have [14]:

$$p\rho^{-\gamma} = \text{constant} \times \mathrm{e}^{s/c_v}. \tag{3.6}$$

Isentropic Flow

A flow is said to be isentropic if

$$S = \text{constant in } t, x, y, z.$$

Homentropy does not imply isentropy. But we now assume that the flow is isentropic. In particular then the energy equation is satisfied. And from (3.6) we have the important conclusion that the pressure is a function of density. More specifically:

$$p = A\rho^{\gamma}, \tag{3.7}$$

where A is a constant determined from

$$p_\infty \rho_\infty^{-\gamma} = A.$$

Next

$$\begin{aligned}
\frac{\nabla p}{\rho} &= \frac{A}{\rho} \nabla \rho^{\gamma} \\
&= A\rho^{\gamma-1} \nabla \log \rho^{\gamma} \\
&= A\gamma\rho^{\gamma-1} \nabla \log \rho \\
&= A\frac{\gamma}{\gamma-1} \rho^{\gamma-1} \nabla \log \rho^{\gamma-1} \\
&= A\frac{\gamma}{\gamma-1} \nabla \rho^{\gamma-1}.
\end{aligned} \tag{3.8}$$

Hence the Euler equation can be purged of the pressure variable p to yield:

$$\frac{\partial q}{\partial t} + (q \cdot \nabla)q + A\frac{\gamma}{\gamma - 1}\nabla\rho^{\gamma-1} = 0. \tag{3.9}$$

This is what enables us to deduce that the flow is curl or vortex free. Thus let

$$\Omega = \nabla \times q.$$

Then taking the curl on both sides of (3.9), we obtain:

$$\nabla \times \frac{\partial q}{\partial t} + \nabla \times (q \cdot \nabla)q = 0$$

because $\nabla \times \nabla\rho^{\gamma-1} =$ the curl of a gradient is zero.

Hence

$$\frac{\partial \Omega}{\partial t} + \nabla \times (q \cdot \nabla)q = 0.$$

But

$$(q \cdot \nabla)q = \frac{1}{2}\nabla(q \,.\, q) - q \times \Omega. \tag{3.10}$$

Hence finally:

$$\frac{\partial \Omega}{\partial t} + \nabla \times (q \times \Omega) = 0. \tag{3.11}$$

Consistent with our assumption that the far field is q_∞ is that

$$q(0, x, y, z) = q_\infty = \nabla\phi_\infty$$

and hence

$$\Omega(0,\ x,\ y,\ z) = 0.$$

Now for any (smooth enough) solution $q(t, x, y, z)$, we may consider $\Omega(\cdot)$ as a solution of (3.9) which is a linear equation with zero initial condition and has the identically zero solution

$$\Omega(t, x, y, z) = 0 = \nabla \times q(t, x, y, z).$$

Or the velocity q $(\cdot)$ is curl-free. Hence it is expressible as

$$q = \nabla\phi.$$

And $\phi(t, x, y, z)$ is the velocity potential such that the far field potential

$$\begin{aligned}\phi(t, x, y, z) \text{ where } r = \sqrt{z^2 + y^2 + x^2} \to \infty,\\ = x(q_\infty \cdot i) + y(q_\infty \cdot j) + z(q_\infty \cdot k)\\ = \phi_\infty(0, x, y, z).\end{aligned}$$

The Euler Full Potential Equation

Assuming entropic flow, we are now ready to derive the field equation for the velocity potential $\phi(\cdot)$ using the continuity equation. First though we rewrite the Euler equation using (3.7) and (3.8) as:

$$\frac{\partial \nabla\phi}{\partial t} + \frac{1}{2}\nabla[\nabla\phi, \nabla\phi] + \frac{A\gamma}{\gamma - 1}\nabla\rho^{\gamma-1} = 0. \tag{3.12}$$

Hence

$$\begin{aligned} \frac{\partial\phi}{\partial t} + \frac{1}{2}[\nabla\phi,\ \nabla\phi] + A\frac{\gamma}{\gamma-1}\rho^{\gamma-1} &= \text{constant} = \text{far field values} \\ &= \frac{1}{2}[q_\infty,\ q_\infty] + \frac{\gamma}{\gamma-1}A\rho_\infty^{\gamma-1}. \end{aligned}$$

Now because p is a function of ρ,

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} = \gamma A\rho^{\gamma-1} = a^2,$$

where a is the local speed of sound with the far field value a_∞, so that

$$a_\infty^2 = \gamma A\ \rho_\infty^{\gamma-1},$$

or

$$\gamma A = \frac{a_\infty^2}{\rho_\infty^{\gamma-1}}. \tag{3.13}$$

Hence we have

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(U^2 - U_\infty^2\right) = \frac{a_\infty^2}{\gamma-1}\left(1 - \frac{\rho^{\gamma-1}}{\rho_\infty^{\gamma-1}}\right),$$

where we use U for the flow speed. Hence we have for the density:

$$\rho^{\gamma-1} = \rho_\infty^{\gamma-1}\left(1 - \left(\frac{\gamma-1}{a_\infty^2}\right)\left(\frac{1}{2}\left(U^2 - U_\infty^2\right) + \partial_t\phi\right)\right). \tag{3.14}$$

Now using the continuity equation:

$$\begin{aligned} \frac{\partial\rho^{\gamma-1}}{\partial t} &= (\gamma-1)\rho^{\gamma-2}\frac{\partial\rho}{\partial t} \\ &= -(\gamma-1)\rho^{\gamma-2}\nabla\cdot(\rho\nabla\phi) \\ &= -(\gamma-1)\rho^{\gamma-2}(\rho\Delta\phi + \nabla\phi\cdot\nabla\rho) \\ &= -(\gamma-1)\rho^{\gamma-1}\Delta\phi - \nabla\phi\cdot\nabla\rho^{\gamma-1} \end{aligned}$$

using

$$(\gamma - 1)\frac{\nabla \rho}{\rho} = \frac{\nabla \rho^{\gamma-1}}{\rho^{\gamma-1}}.$$

From (3.14) we have:

$$\frac{\partial \rho^{\gamma-1}}{\partial t} = -\rho_\infty^{\gamma-1}\left(\frac{\gamma-1}{a_\infty^2}\right)\frac{\partial^2 \phi}{\partial t^2} + \nabla\phi \cdot \partial t \nabla \phi)$$
$$-\nabla \rho^{\gamma-1} \cdot \nabla\phi = -\rho_\infty^{\gamma-1}\left(\frac{\gamma-1}{a_\infty^2}\right)\left(\frac{1}{2}\nabla\left|\nabla\phi\right|^2 + \partial t \nabla \phi\right)\cdot\nabla\phi.$$

Hence

$$\rho_\infty^{\gamma-1}\left(\frac{\gamma-1}{a_\infty^2}\right)\left(\frac{\partial^2\phi}{\partial t^2} + \nabla\phi\cdot\frac{\partial\nabla\phi}{\partial t}\right)$$
$$= (\gamma-1)\rho_\infty^{\gamma-1}\left(1 - \left(\frac{\gamma-1}{a_\infty^2}\right)\left(\frac{1}{2}\left(U_\infty^2 - U^2\right) + \frac{\partial\phi}{\partial t}\right)\right)\Delta\phi$$
$$-\rho_\infty^{\gamma-1}\left(\frac{\gamma-1}{a_\infty^2}\right)\left(\frac{1}{2}\nabla||\nabla\phi||^2 + \frac{\partial\nabla\phi}{\partial t}\right)\cdot\nabla\phi.$$

Hence finally:

$$\frac{\partial^2\phi}{\partial t^2} + \nabla\phi\cdot\frac{\partial\nabla\phi}{\partial t} = a_\infty^2\Delta\phi\left(1 - \frac{\gamma-1}{a_\infty^2}\left(\frac{1}{2}\left(U^2 - U_\infty^2\right) + \frac{\partial\phi}{\partial t}\right)\right)$$
$$-\frac{1}{2}\nabla\phi\cdot\nabla(||\nabla\phi||^2) - \frac{\partial\nabla\phi}{\partial t}\cdot\nabla\phi,$$

which we can rewrite in the form:

$$\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}||\nabla\phi||^2 = a_\infty^2\left(1 + \frac{\gamma-1}{a_\infty^2}\left(\frac{U_\infty^2}{2} - \frac{||\nabla\phi||^2}{2} - \frac{\partial\phi}{\partial t}\right)\right)\Delta\phi$$
$$-\nabla\phi\cdot\nabla\left(\frac{||\nabla\phi||^2}{2}\right). \tag{3.15}$$

This is the 3D Euler full potential equation valid except on the boundary specified later.

Acceleration Potential and Pressure

We can now derive an explicit expression for the pressure in terms of the fluid velocity. The acceleration is defined by the total derivative of the velocity:

$$a(t) = \frac{\mathrm{D}q}{\mathrm{D}t} = \frac{\partial q}{\partial t} + (q \cdot \nabla)q$$

and for potential flow by (3.10)

$$(q \cdot \nabla)q = \frac{1}{2}\nabla(q \cdot q).$$

Hence

$$a(t) = \nabla\left(\partial t\phi + \frac{1}{2}|\nabla\phi|^2\right) = \nabla\psi(t),$$

where the acceleration potential

$$\psi(t) = \partial t\phi + \frac{|\nabla\phi|^2}{2} \tag{3.16}$$

and the far field value

$$\psi_\infty = \frac{U_\infty^2}{2}.$$

The flow being isentropic

$$p = A\rho^\gamma$$

and by (3.14)

$$\rho = \rho_\infty\left(1 - (\gamma - 1)\left(\frac{\psi - \psi_\infty}{a_\infty^2}\right)\right)^{1/(\gamma-1)} \tag{3.17}$$

and hence

$$p = A\rho_\infty^\gamma\left(1 - (\gamma - 1)\frac{(\psi - \psi_\infty)}{a_\infty^2}\right)^{\gamma/(\gamma-1)}. \tag{3.18}$$

And a very reasonable (and universal) approximation here is to take:

$$p = A\rho_\infty^\gamma\left(1 - \gamma\frac{(\psi - \psi_\infty)}{a_\infty^2}\right) \tag{3.19}$$

consistent with

$$0 < M = \frac{U_\infty}{a_\infty} < 1,$$

where M is the Mach number, and that the "perturbation" of the flow by the airplane is small compared to the far field speed. In any event we use (3.19) for p throughout for isentropic flow.

The main relations we have are thus (3.19) and (3.15).

What distinguishes aeroelasticity from aerodynamics is of course the interaction with the structure dynamics on the boundary, a singular boundary that complicates the problem further.

Boundary Conditions

The far field condition

$$\phi(t, \infty) = U_\infty(x \cos\alpha + y \sin\alpha),$$

where α is the specified "angle of attack."

As with any field equation, the conditions on the boundary determine the solution. Here it is further complicated by the fact we have taken into account the structure dynamics as well.

Flow-Structure Interaction:
Hence we first need to specify the structure model.

The Simplest Wing Structure Model
The simplest wing model is a slender "thin" plate whose thickness is then taken to be zero, rectangular in shape (unswept wing) uniform, with ℓ denoting the half wing span and $2b$ the width or chord length, so that b is the halfchord.

We only consider wings of high-aspect ratio:

$$\ell \gg b,$$

which would justify the flexible model. However, we do consider a finite rectangular plane (Finite plane) as the boundary for the aerodynamic equations. Thus we have 3D aerodynamics and a 2D wing boundary.

We choose the spatial co-ordinate system consistent with the aircraft rigid body dynamics. Thus the x-axis is along the airplane axis, the y-axis is the span axis and the negative z-axis is the "plunge" axis. Thus the boundary for the field equation is the rectangle in the xy-plane described by:

$$\Gamma = \{-b \leq x \leq b; \quad 0 \leq y \leq \ell\}.$$

For the structure dynamics, however, we specialize to a beam model, ignoring the dependence on the chord variable. Such a model was described by Goland, as we have seen in Chap. 2 where the structure is endowed with two degrees of freedom: plunge and pitch.

The plunge displacement denoted h (t, y) is then the displacement of the wing along the negative z-axis. It is uniform along the chord. The pitch $\vartheta(t, y)$ is the twist angle in radians about an axis parallel to the y-axis at distance ab from the x-axis and again does not depend on the chord variable x. Hence $h(t, y)$ and $\vartheta(t, y)$ are defined on:

$$0 \leq t; \quad 0 \leq y \leq \ell.$$

The wing is fixed to the fusilage. Thus we have a clamped–free (CF) model with:

$$h(t, 0) = 0 = h'(t, 0); \quad \vartheta(t, 0) = 0.$$

The wing is free at the other end, and hence

$$h'''(t,\ell) = 0 = h''(t,\ell) = 0; \quad \vartheta'(t,\ell) = 0.$$

These are then the "end" conditions to be satisfied in addition to the force/momentum balance equations.

The resulting structure dynamic equations are described in Chap. 2. Here we shall describe the air–wing interaction dynamics.

Thus we have two sets of conditions.

I. The Attached Flow Condition

The normal velocity of the air along the wing is equal to the normal velocity of the wing. We may need to distinguish between the top and bottom of the wing if we allow for discontinuity in the fluid velocity across the wing, even though the thickness is zero.

The total displacement of the wing is given by:

$$\overrightarrow{k}\, z(t,x,y) \quad \text{where} \quad z(t,x,y) = -(h(t,\ y) +\ (x-ab)\ \vartheta\ (t,y))$$

and hence the normal velocity of the wing is given by:

$$\begin{aligned}\overrightarrow{k}\,\frac{\mathrm{D}z(t)}{\mathrm{D}(t)} &= \overrightarrow{k}\left(\frac{\partial z(t)}{\partial t} + q(t,x,y,0)\cdot\nabla z(t)\right)\\ &= -\overrightarrow{k}\left(\dot{h}(t,y) + (x-a)\dot{\vartheta}(t,y) + (q(t,x,y,0)\cdot\overrightarrow{i})\vartheta(t,y)\right.\\ &\quad\left. + \left(q(t,x,y,0)\cdot\overrightarrow{j}\right)\left(\frac{\partial}{\partial y}(h(t,y)+(x-a)\vartheta(t,y))\right)\right).\end{aligned}$$

Hence allowing for discontinuity in the flow, the flow tangency conditions become:

$$\begin{aligned}\frac{\partial\phi(t,x,y,0+)}{\partial z} &= \frac{\partial\phi_\infty}{\partial z} + (-1)\left[\dot{h}(t,y) + (x-a)\dot{\vartheta}(t,y)\right.\\ &\quad + (q(t,x,y,0+)\cdot\overrightarrow{i})\vartheta(t,y) + (q(t,x,y,0+)\cdot\overrightarrow{j})\\ &\quad\left.\times\left(\frac{\partial}{\partial y}(h(t,y)+(x-a)\#(t,y))\right)\right], x, y\epsilon\Gamma, \qquad (3.20)\end{aligned}$$

$$\frac{\partial\phi(t,x,y,0-)}{\partial z} = \frac{\partial\phi_\infty}{\partial z} + (-1)\Big[\dot{h}(t,y) + (x-a)\dot{\vartheta}(t,y)(q(t,x,y,0-)\cdot\vec{i}\,)$$
$$\times\,\vartheta(t,y) + (q(t,x,y,0-)\cdot\vec{j}\,)\left(\frac{\partial}{\partial y}(h(t,y)+(x-a)\vartheta(t,y))\right)\Big],$$
$$\times\, x, y\epsilon\Gamma. \tag{3.21}$$

II. Kutta–Joukowski Conditions

These conditions are peculiar to aeroelasticity and are given in terms of the pressure which is discontinuous across the wing. We define the pressure Jump by δp:

$$\delta p(t,x,y) = p(t,x,y,0+) - p(t,x,y,0-). \tag{3.22}$$

Defining

$$\delta\psi(t,x,y) = \psi(t,x,y,0+) - \psi(t,x,y,0-),$$

we have from (3.20) that

$$\delta p = -A\rho_\gamma^\infty \frac{\gamma}{a_2^\infty}\delta\psi = -\rho_\infty\delta\psi. \tag{3.23}$$

II. The pressure jump is zero off the wing

$$\delta p(t,x,y) = 0 \quad \text{for } x, y \text{ not in } \Gamma. \tag{3.24}$$

Or

$$\delta\psi(t,x,y) = 0 \quad \text{for } x, y \text{ not in } \Gamma. \tag{3.25}$$

II. Kutta condition

$$\delta p(t,x,y) = 0 \text{ as } x \to b-,\ x, y \text{ in } \Gamma. \tag{3.26}$$

Or

$$\delta\psi(t,x,y) = 0 \text{ as } x \to b-,\ x, y \text{ in } \Gamma. \tag{3.27}$$

Structure Dynamics

We are now ready to complete the structure dynamics models begun in Chap. 2 by including the aerodynamic lift and moment which are expressed in terms of the structure state variables.

Goland Model

$$m\ddot{h}(t,y) + S\ddot{\vartheta}(t,y) + EI\ h''''(t,y) = L(t,y) = \int_{-b}^{b} \delta p(t,x,y)\mathrm{d}x,$$
$$0 < y < \ell, \qquad (3.28)$$

$$I_\vartheta \ddot{\vartheta}(t,y) + S\ddot{h}(t,y) - GJ\vartheta''(t,y) = M(t,y) = \int_{-b}^{b} (x-ab)\delta p(t,x,y)\mathrm{d}x$$
$$0 < y < \ell. \qquad (3.29)$$

Dowell–Hodges Model

$$m\ddot{h}(t,y) + EI_1 h''''(t,y) + (EI_2 - EI_1)(\vartheta(t,y)v(t,y)'')'' = mg\sin\varphi + L(t,y),$$
$$m\ddot{v}(t,y) + EI_2 v''''(t,y) + (EI_2 - EI_1)(\vartheta(t,y)h(t,y)'')'' = mg\cos\varphi,$$
$$I_\vartheta \ddot{\vartheta}(t,y) - GJ\vartheta''(t,y) + (EI_2 - EI_1)h(t,y)''v(t,y)'' = M(t,y),$$
$$0 < t;\ 0 < y < \ell.$$

3.4 Problem Statement

The Aeroelastic Problem: 3D/Nonzero Angle of Attack/2D Boundary

We can now present a complete statement of the combined 3D aeroelastic problem by the following equations.

Field equation

$$\frac{\partial^2\phi}{\partial t^2} + \partial t\ ||\nabla\phi||^2 = a_\infty^2 \left(1 + \frac{\gamma - 1}{a_\infty^2}\left(\frac{U_\infty^2}{2} - \frac{||\nabla\phi||^2}{2} - \partial t\phi\right)\right)\Delta\phi$$
$$- \nabla\phi \bullet \nabla\left(\frac{||\nabla\phi||^2}{2}\right). \qquad (3.30)$$

Far field

$$\phi(t,\infty) = xU_\infty \cos\alpha + yU_\infty \sin\alpha.$$

Boundary conditions

$$\begin{aligned}
&\Gamma = \{\{x,y\}, -b < x < b; 0 < y < \ell\} \\
&\quad \times \frac{\partial\phi(t,x,y,0+)}{\partial z} = \frac{\partial\phi_\infty}{\partial z} + (-1)\Big[\dot{h}(t,y) + (x-a)\dot{\vartheta}(t,y) \\
&\quad + (q(t,x,y,0+)\cdot\overrightarrow{i})\vartheta(t,y) + (q(t,x,y,0+)\cdot\overrightarrow{j}) \\
&\quad \times \left(\frac{\partial}{\partial y}(h(t,y) + (x-a)\vartheta(t,y))\right)\Big], \quad x,y \epsilon \Gamma. \qquad (3.31)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial\phi(t,x,y,0-)}{\partial z} = \frac{\partial\phi_\infty}{\partial z} &+ (-1)\Big[\dot{h}(t,y) + (x-a)\dot{\vartheta}(t,y) \\
&\times (q(t,x,y,0-)\cdot\overrightarrow{i})\vartheta(t,y)\,(q(t,x,y,0-)\cdot\overrightarrow{j}) \\
&\times \left(\frac{\partial}{\partial y}(h(t,y) + (x-a)\vartheta(t,y))\right)\Big], x, y \in \Gamma. \qquad (3.32)
\end{aligned}$$

Kutta–Joukowski conditions

$$\psi(t) = \frac{\partial\phi}{\partial t} + \frac{||\nabla\phi||^2}{2}, \qquad (3.33)$$

$$\delta\psi(t,x,y) = 0 \quad \text{for } x, y \text{ not in } \Gamma, \qquad (3.34)$$

$$\delta\psi(t,x,y) = 0 \quad \text{as } x \to b-, x,\ y \text{ in } \Gamma. \qquad (3.35)$$

Structure Dynamics: Linear (Goland)

$$\ddot{h}(t,y) + S\ddot{\vartheta}(t,y) + EIh''''(t,y) = -\rho_\infty \int_{-b}^{b} \delta\psi(t,x,y)\,\mathrm{d}x, \quad 0 < y < \ell, \qquad (3.36)$$

$$I_\vartheta \ddot{\vartheta}(t,y) + S\ddot{h}(t,y) - GJ\vartheta''(t,y) = -\rho_\infty \int_{-b}^{b} (x-ab)\delta\psi(t,x,y)\mathrm{d}x \quad 0 < y < \ell, \tag{3.37}$$

plus CF end conditions:

$$h(t,0) = 0 = h'(t,0); \vartheta(t,0) = 0,$$
$$h'''(t,\ell) = 0 = h''(t,\ell) = 0; \vartheta'(t,\ell) = 0,$$

or FF end conditions:

$$h'''(t,0) = 0 = h''(t,0) = 0; \vartheta'(t,0) = 0,$$
$$h'''(t,\ell) = 0 = h''(t,1) = 0; \vartheta'(t,\ell) = 0.$$

Structure Dynamics Nonlinear

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The state variables are the same as in the Goland. For the nonlinearities see Chap. 2, Sect. 2.7. The end conditions are also the same as in the Goland model.

Dowell–Hodges

Here there is an extra state variable $v(t,\cdot)$:

$$m\ddot{h}(t,y) + EI_1 h''''(t,y) + (EI_2 - EI_1)(\vartheta(t,y)v(t,y)'')''$$
$$= mg\sin\varphi - \rho_\infty \int_{-b}^{b} \delta\psi(t,x,y)\mathrm{d}x, \quad 0 < y < \ell, \tag{3.38}$$
$$m\ddot{v}(t,y) + EI_2 v''''(t,y) + (EI_2 - EI_1)(\vartheta(t,y)h(t,y)'')'' = mg\cos\varphi, \tag{3.39}$$
$$I_\vartheta \ddot{\vartheta}(t,y) - GJ\vartheta''(t,y) + (EI_2 - EI_1)h(t,y)''v(t,y)''$$
$$= -\rho_\infty \int_{-b}^{b} (x-ab)\delta\psi(t,x,y)\mathrm{d}x 0 < t; \quad 0 < y < \ell. \tag{3.40}$$

The end conditions for $v(t,.)$ are the same as for $h(t,.)$.

Typical Section (Strip) Theory

A universally invoked simplification is to neglect the dependence on the y-co-ordinate in consequence of the assumed high-aspect ratio ℓ/b of the wing.

Problem Statement

Typical Section Theory

2D Aerodynamics 1D Structure

The field equations: with the y-co-ordinate omitted:

$$\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right)$$
$$= a_\infty^2\left(1 + \frac{\gamma-1}{a_\infty^2}\left(\frac{U_\infty^2}{2} - \frac{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2}{2} - \frac{\partial\phi}{\partial t}\right)\right)\Delta\phi$$
$$-\nabla\phi\cdot\nabla\left(\frac{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2}{2}\right) \quad -\infty < x, z < \infty. \tag{3.41}$$

Far field potential

$$\phi(t,\infty) = xU_\infty\cos\alpha + zU_\infty\sin\alpha.$$

Boundary conditions

Γ is now just the Chord: $-b < x < b$ and the flow tangency condition becomes:

$$\text{Total Displacement} = \overrightarrow{k}\, z(t,x) \text{ where}$$

$$z(t,x) = -(h(t,y) + (x-ab)\vartheta(t,y))$$

and hence the normal velocity of the wing is given by

$$\begin{aligned}\vec{k}\frac{\mathrm{D}z(t)}{\mathrm{D}(t)} &= \vec{k}\left(\frac{\partial z(t)}{\partial t} + q(t,x,y,0)\cdot\nabla z(t)\right),\\ &= -\vec{k}\left(\dot{h}\,(t,y) + (x-ab)\dot{\vartheta}\,(t,y) + \left(q\,(t,x,y,0)\cdot\vec{i}\right)\vartheta(t,y)\right).\end{aligned}$$

Hence allowing for discontinuity in the flow, the flow tangency conditions become:

$$\begin{aligned}\frac{\partial\phi(t,x,0+)}{\partial z} = \frac{\partial\phi_\infty}{\partial z} + (-1)\Big[\dot{h}(t,y) + (x-a)\dot{\vartheta}(t,y)\\ + \left(q(t,x,0+)\cdot\vec{i}\right)\vartheta(t,y)\Big].\end{aligned} \tag{3.42}$$

And

$$\begin{aligned}\frac{\partial\phi(t,x,0-)}{\partial z} = \frac{\partial\phi_\infty}{\partial z} + (-1)\Big[\dot{h}(t,y) + (x-a)\dot{\vartheta}(t,y)\\ + \left(q(t,x,0-)\cdot\vec{i}\right)\vartheta(t,y)\Big]\,x\epsilon\Gamma.\end{aligned} \tag{3.43}$$

Kutta–Joukowski Conditions

$$\begin{aligned}\delta p(t,x) &= p(t,x,0+) - p(t,x,0-),\\ \delta\psi(t,x) &= \psi(t,x,0+) - \psi(t,x,0-),\\ \delta p &= -\rho_\infty\delta\psi.\end{aligned}$$

The pressure jump is zero off the wing.

$$\delta p(t,x) = 0 \quad \text{for } |x| > b,$$

Or

$$\delta\psi\ (t,x) = 0 \quad \text{for } |x| > b.$$

Kutta condition

$$\delta p(t,x) \to 0 \text{ as } x \to b-, \text{in}\Gamma.$$

Or,

$$\delta\psi(t,x)| \to 0\text{as}x \to b-,\ |x| < b.$$

The structure equations with the indicated aerodynamic loading remain the same.

This is then the precise statement of the aeroelastic problem continuum equations. The objective is to determine the stability of the structure state as a function of U_∞, the air speed.

To anticipate the theory that follows, the main conclusion is that for a given value of M (equivalently, speed of sound, equivalently altitude) there is a speed, called flutter speed, denoted U_{F}, for $U_\infty < U_{\mathrm{F}}$, the structure is stable (see later for precise definition of stability), and for $U > U_{\mathrm{F}}$ the structure is unstable.

Notes and Comments

It is interesting to note that none of the books on aeroelasticity, including Dowell et al. [17], Hodges and Pierce [5] or Bisplinghof, Ashley, and Halfman [6] care to make a precise statement of the aeroelastic problem as we do in this chapter. Indeed without such a statement it is not clear what it is that the computer codes used universally (see the many recent papers on aeroelasticity, for example [75, 81–83, 93]) are providing the (approximate) solution to, even omitting the cases where the solvability of the problem cannot be established.

Indeed without such a formulation it is not possible to define the Flutter Speed calculating which is a main objective of the theory.

We should note that most progress has been made for the typical section case (2D aerodynamics) and it is fortunate that flexibility is consistent with high-aspect ratio so that the typical section approximation is reasonable (without necessarily being very high).

Regarding the foundational conservation laws, following [4, 12] we have invoked three of them rather than the first two as in [4, 17], for example. The triad is essential for aeroelasticity. As Meyer [14] notes the third is the Euler version of the first law of thermodynamics.

http://www.springer.com/978-1-4614-3608-9

Aeroelasticity
The Continuum Theory
Balakrishnan, A.
2012, XII, 400 p., Hardcover
ISBN: 978-1-4614-3608-9