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ORDINARY DIFFERENTIAL  
EQUATIONS  
Student Solution Manual

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# Chapter 1

## Solutions

### SECTION 1.1

1. The rate of change in the population  $P(t)$  is the derivative  $P'(t)$ . The Malthusian Growth Law states that the rate of change in the population is proportional to  $P(t)$ . Thus  $P'(t) = kP(t)$ , where  $k$  is the proportionality constant. Without reference to the  $t$  variable, the differential equation becomes  $P' = kP$ .
3. Torricelli's law states that the change in height,  $h'(t)$  is proportional to the square root of the height,  $\sqrt{h(t)}$ . Thus  $h'(t) = \lambda\sqrt{h(t)}$ , where  $\lambda$  is the proportionality constant.
5. The highest order derivative is  $y''$  so the order is 2. The standard form is  $y'' = t^3/y'$ .
7. The highest order derivative is  $y''$  so the order is 2. The standard form is  $y'' = -(3y + ty')/t^2$ .
9. The highest order derivative is  $y^{(4)}$  so the order is 4. Solving for  $y^{(4)}$  gives the standard form:  $y^{(4)} = \sqrt[3]{(1 - (y''')^4)}/t$ .
11. The highest order derivative is  $y'''$  so the order is 3. Solving for  $y'''$  gives the standard form:  $y''' = 2y'' - 3y' + y$ .
13. The following table summarizes the needed calculations:

Function	$ty'(t)$	$y(t)$
$y_1(t) = 0$	$ty'_1(t) = 0$	$y_1(t) = 0$
$y_2(t) = 3t$	$ty'_2(t) = 3t$	$y_2(t) = 3t$
$y_3(t) = -5t$	$ty'_3(t) = -5t$	$y_3(t) = -5t$
$y_4(t) = t^3$	$ty'_4(t) = 3t^3$	$y_4(t) = t^3$

To be a solution, the entries in the second and third columns need to be the same. Thus  $y_1$ ,  $y_2$ , and  $y_3$  are solutions.

15. The following table summarizes the needed calculations:

Function	$y'(t)$	$2y(t)(y(t) - 1)$
$y_1(t) = 0$	$y'_1(t) = 0$	$2y_1(t)(y_1(t) - 1) = 2 \cdot 0 \cdot (-1) = 0$
$y_2(t) = 1$	$y'_2(t) = 0$	$2y_2(t)(y_2(t) - 1) = 2 \cdot 1 \cdot 0 = 0$
$y_3(t) = 2$	$y'_3(t) = 0$	$2y_3(t)(y_3(t) - 1) = 2 \cdot 2 \cdot 1 = 4$
$y_4(t) = \frac{1}{1-e^{2t}}$	$y'_4(t) = \frac{2e^{2t}}{(1-e^{2t})^2}$	$2y_4(t)(y_4(t) - 1) = 2 \frac{1}{1-e^{2t}} \left( \frac{1}{1-e^{2t}} - 1 \right)$ $= 2 \frac{1}{1-e^{2t}} \frac{e^{2t}}{1-e^{2t}} = \frac{2e^{2t}}{(1-e^{2t})^2}$

Thus  $y_1$ ,  $y_2$ , and  $y_4$  are solutions.

17. The following table summarizes the needed calculations:

Function	$2y(t)y'(t)$	$y^2 + t - 1$
$y_1(t) = \sqrt{-t}$	$2\sqrt{-t} \frac{-1}{2\sqrt{-t}} = -1$	$(\sqrt{-t})^2 + t - 1 = -1$
$y_2(t) = -\sqrt{e^t - t}$	$-2\sqrt{e^t - t} \frac{-(e^t - 1)}{2\sqrt{e^t - t}} = e^t - 1$	$(-\sqrt{e^t - t})^2 + t - 1 = e^t - 1$
$y_3(t) = \sqrt{t}$	$2\sqrt{t} \frac{1}{2\sqrt{t}} = 1$	$(\sqrt{t})^2 + t - 1 = 2t - 1$
$y_4(t) = -\sqrt{-t}$	$2(-\sqrt{-t}) \frac{1}{2\sqrt{-t}} = -1$	$(-\sqrt{-t})^2 + y - 1 = -1$

Thus  $y_1$ ,  $y_2$ , and  $y_4$  are solutions.

- 19.

$$\begin{aligned} y'(t) &= 3ce^{3t} \\ 3y + 12 &= 3(ce^{3t} - 4) + 12 = 3ce^{3t} - 12 + 12 = 3ce^{3t}. \end{aligned}$$

Note that  $y(t)$  is defined for all  $t \in \mathbb{R}$ .

- 21.

$$y'(t) = \frac{ce^t}{(1-ce^t)^2}$$

$$y^2(t) - y(t) = \frac{1}{(1-ce^t)^2} - \frac{1}{1-ce^t} = \frac{1-(1-ce^t)}{(1-ce^t)^2} = \frac{ce^t}{(1-ce^t)^2}.$$

If  $c \leq 0$  then the denominator  $1 - ce^t > 0$  and  $y(t)$  has domain  $\mathbb{R}$ . If  $c > 0$  then  $1 - ce^t = 0$  if  $t = \ln \frac{1}{c} = -\ln c$ . Thus  $y(t)$  is defined either on the interval  $(-\infty, -\ln c)$  or  $(-\ln c, \infty)$ .

**23.**

$$y'(t) = \frac{-ce^t}{ce^t - 1}$$

$$-e^y - 1 = -e^{-\ln(ce^t - 1)} - 1 = \frac{-1}{ce^t - 1} - 1 = \frac{-ce^t}{ce^t - 1}.$$

Since  $c > 0$  then  $y(t)$  is defined if and only if  $ce^t - 1 > 0$ . This occurs if  $e^t > \frac{1}{c}$  which is true if  $t > \ln \frac{1}{c} = -\ln c$ . Thus  $y(t)$  is defined on the interval  $(-\ln c, \infty)$ .

**25.**

$$y'(t) = -(c-t)^{-2}(-1) = \frac{1}{(c-t)^2}$$

$$y^2(t) = \frac{1}{(c-t)^2}.$$

The denominator of  $y(t)$  is 0 when  $t = c$ . Thus the two intervals where  $y(t)$  is defined are  $(-\infty, c)$  and  $(c, \infty)$ .

**27.** Integration gives  $y(t) = \frac{e^{2t}}{2} - t + c$ .

**29.** Observe that  $\frac{t+1}{t} = 1 + \frac{1}{t}$ . Integration gives  $y(t) = t + \ln |t| + c$ .

**31.** We integrate two times. First,  $y'(t) = -2 \cos 3t + c_1$ . Second,  $y(t) = -\frac{2}{3} \sin 3t + c_1 t + c_2$ .

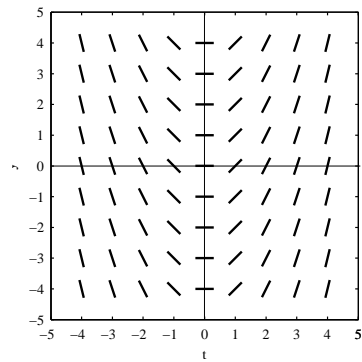
**33.** From Problem 20 the general solution is  $y(t) = ce^{-t} + 3t - 3$ . At  $t = 0$  we get  $0 = y(0) = ce^0 + 3(0) - 3 = c - 3$ . It follows that  $c = 3$  and  $y(t) = 3e^{-t} + 3t - 3$ .

**35.** From Problem 24 the general solution is  $y(t) = c(t+1)^{-1}$ . At  $t = 1$  we get  $-9 = y(1) = c(1+1)^{-1} = c/2$ . It follows that  $c = -18$  and  $y(t) = -18(t+1)^{-1}$ .

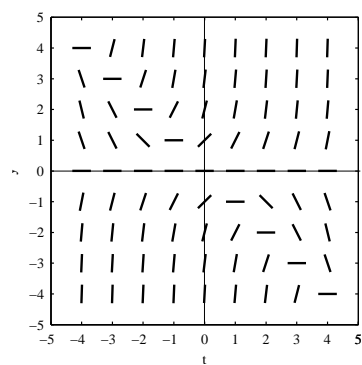
**37.** From Problem 28 the general solution is  $y(t) = -te^{-t} - e^{-t} + c$ . Evaluation at  $t = 0$  gives  $-1 = y(0) = -1 + c$  so  $c = 0$ . Hence  $y(t) = -te^{-t} - e^{-t}$ .

## SECTION 1.2

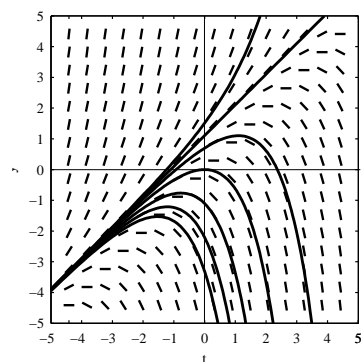
1.  $y' = t$

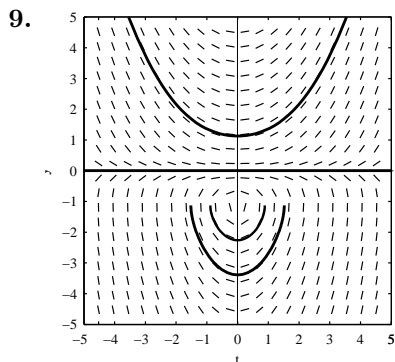
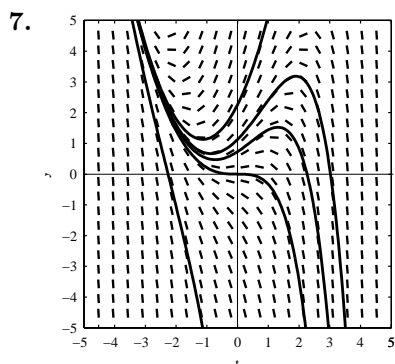


3.  $y' = y(y + t)$



5.





11. We set  $y(y + t) = 0$ . We look for constant solutions to  $y(y + t) = 0$ , and we see that  $y = 0$  is the only constant (= equilibrium) solution.
13. The equation  $1 - y^2 = 0$  has two constant solutions:  $y = 1$  and  $y = -1$
15. We substitute  $y = at + b$  into  $y' = \cos(t + y)$  to get  $a = \cos((a + 1)t + b)$ . Equality for all  $t$  means that  $\cos((a + 1)t + b)$  must be a constant function, which can occur only if the coefficient of  $t$  is 0. This forces  $a = -1$  leaving us with the equation  $-1 = \cos b$ . This implies  $b = (2n + 1)\pi$ , where  $n$  is an integer. Hence  $y = -t + (2n + 1)\pi$ ,  $n \in \mathbb{Z}$  is a family of linear solutions.
17. Implicit differentiation with respect to  $t$  gives  $2yy' - 2t - 3t^2 = 0$ .
19. Differentiation gives  $y' = 3ct^2 + 2t$ . However, from the given function we have  $ct^3 = y - t^2$  and hence  $ct^2 = \frac{y - t^2}{t}$ . Substitution gives  $y' = 3\frac{y - t^2}{t} + 2t = \frac{3y}{t} - t$ .

## SECTION 1.3

1. separable;  $h(t) = 1$  and  $g(y) = 2y(5 - y)$
3. First write in standard form:  $y' = \frac{1-2ty}{t^2}$ . We cannot write  $\frac{1-2ty}{t^2}$  as a product of a function of  $t$  and a function of  $y$ . It is not separable.
5. Write in standard form to get:  $y' = \frac{y-2yt}{t}$ . Here we can write  $\frac{y-2yt}{t} = y\frac{1-2t}{t}$ . It is separable;  $h(t) = \frac{1-2t}{t}$  and  $g(y) = y$ .
7. In standard form we get  $y' = \frac{-2ty}{t^2+3y^2}$ . We cannot write  $y' = \frac{-2ty}{t^2+3y^2}$  as a product of a function of  $t$  and a function of  $y$ . It is not separable
9. In standard form we get:  $y' = e^{-t}(y^3 - y)$  It is separable;  $h(t) = e^{-t}$  and  $g(y) = y^3 - y$
11. In standard form we get  $y' = \frac{1-y^2}{ty}$ . Clearly,  $y = \pm 1$  are equilibrium solutions. Separating the variables gives

$$\frac{y}{1-y^2}dy = \frac{1}{t}dt.$$

Integrating both sides of this equation (using the substitution  $u = 1 - y^2$ ,  $du = -2y dy$  for the integral on the left) gives

$$-\frac{1}{2} \ln |1 - y^2| = \ln |t| + c.$$

Multiplying by  $-2$ , taking the exponential of both sides, and removing the absolute values gives  $1 - y^2 = kt^{-2}$  where  $k$  is a nonzero constant. However, when  $k = 0$  the equation becomes  $1 - y^2 = 0$  and hence  $y = \pm 1$ . By considering an arbitrary constant (which we will call  $c$ ), the implicit equation  $t^2(1 - y^2) = c$  includes the two equilibrium solutions for  $c = 0$ .

13. The variables are already separated, so integrate both sides to get  $y^5/5 = t^2/2 + 2t + c$ ,  $c$  a real constant. Simplifying gives  $y^5 = \frac{5}{2}t^2 + 10t + c$ . We leave the answer in implicit form
15. In standard form we get  $y' = (1 - y) \tan t$  so  $y = 1$  is a solution. Separating variables gives  $\frac{dy}{1-y} = \tan t dt$ . The function  $\tan t$  is continuous on the interval  $(-\pi/2, \pi/2)$  and so has an antiderivative. Integration gives  $-\ln |1 - y| = -\ln |\cos t| + k_1$ . Multiplying by  $-1$  and exponentiating gives  $|1 - y| = k_2 |\cos t|$  where  $k_2$  is a positive constant. Removing the absolute value signs gives  $1 - y = k_3 \cos t$ , with  $k_3 \neq 0$ . If we allow  $k_3 = 0$  we get the equilibrium solution  $y = 1$ . Thus the solution can be written  $y = 1 - c \cos t$ ,  $c$  any real constant.



- 17.** There are two equilibrium solutions;  $y = 0$  and  $y = 4$ . Separating variables and using partial fractions gives  $\frac{1}{4} \left( \frac{1}{y} + \frac{1}{4-y} \right) dy = dt$ . Integrating and simplifying gives  $\ln \left| \frac{y}{4-y} \right| = 4t + k_1$  which is equivalent to  $\frac{y}{4-y} = ce^{4t}$ ,  $c$  a nonzero constant. Solving for  $y$  gives  $y = \frac{4ce^{4t}}{1+ce^{4t}}$ . When  $c = 0$  we get the equilibrium solution  $y = 0$ . However, there is no  $c$  which gives the other equilibrium solution  $y = 4$ .
- 19.** Separating variables gives  $\frac{dy}{y^2+1} = dt$  and integrating gives  $\tan^{-1} y = t + c$ . Thus  $y = \tan(t + c)$ ,  $c$  a real constant.
- 21.** In standard form we get  $y' = \frac{-(y+1)}{y-1} \frac{1}{1+t^2}$  from which we see that  $y = -1$  is an equilibrium solution. Separating variables and simplifying gives  $\left( \frac{2}{y+1} - 1 \right) dy = \frac{dt}{t^2+1}$ . Integrating and simplifying gives  $\ln(y+1)^2 - y = \tan^{-1} t + c$ .
- 23.** The equilibrium solution is  $y = 0$ . Separating variables gives  $y^{-2} dy = \frac{dt}{1-t}$ . Integrating and simplifying gives  $y = \frac{1}{\ln|1-t|+c}$ ,  $c$  real constant.
- 25.**  $y = 0$  is the only equilibrium solution. The equilibrium solution  $y(t) = 0$  satisfies the initial condition  $y(1) = 0$  so  $y(t) = 0$  is the required solution.
- 27.** In standard form we get  $y' = -2ty$  so  $y = 0$  is a solution. Separating variables and integrating gives  $\ln|y| = -t^2 + k$ . Solving for  $y$  gives  $y = ce^{-t^2}$  and allowing  $c = 0$  gives the equilibrium solution. The initial condition implies  $4 = y(0) = ce^0 = c$ . Thus  $y = 4e^{-t^2}$ .
- 29.** Separating variables gives  $\frac{dy}{y} = \frac{u}{u^2+1} du$  and integrating gives  $\ln|y| = \ln\sqrt{u^2+1} + k$ . Solving for  $y$  gives  $y = c\sqrt{u^2+1}$ , for  $c \neq 0$ . The initial condition gives  $2 = y(0) = c$ . So  $y = 2\sqrt{u^2+1}$ .
- 31.** Since  $y^2 + 1 \geq 1$  there are no equilibrium solutions. Separating the variables gives

$$\frac{dy}{y^2+1} = \frac{dt}{t^2},$$

and integration of both sides gives  $\tan^{-1} y = -\frac{1}{t} + c$ . Solve for  $y$  by applying the tangent function to both sides of the equation. Since  $\tan(\tan^{-1} y) = y$ , we get

$$y(t) = \tan\left(-\frac{1}{t} + c\right).$$

To find  $c$  observe that  $\sqrt{3} = y(1) = \tan(-1 + c)$ , which implies that  $c - 1 = \pi/3$ , so  $c = 1 + \pi/3$ . Hence

$$y(t) = \tan\left(-\frac{1}{t} + 1 + \frac{\pi}{3}\right).$$

To determine the maximum domain on which this solution is defined, note that the tangent function is defined on the interval  $(-\pi/2, \pi/2)$ , so that  $y(t)$  is defined for all  $t$  satisfying

$$-\frac{\pi}{2} < -\frac{1}{t} + 1 + \frac{\pi}{3} < \frac{\pi}{2}.$$

The first inequality is solved to give  $t > 6/(6 + 5\pi)$ . The second equality is solved to give  $t < 6/(6 - \pi)$ . Thus the maximum domain for the solution  $y(t)$  is the interval  $(a, b) = (6/(6 + 5\pi), 6/(6 - \pi))$ .  $\lim_{t \rightarrow b^-} y(t) = \lim_{t \rightarrow b^-} \tan\left(-\frac{1}{t} + 1 + \frac{\pi}{3}\right) = \lim_{x \rightarrow \pi/2^-} \tan x = \infty$ .

- 33.** Let  $m$  denote the number of Argon-40 atoms in the sample. Then  $8m$  is the number of Potassium-40 atoms. Let  $t$  be the age of the rock. Then  $t$  years ago there were  $m + 8m = 9m$  atoms of Potassium-40. Hence  $N(0) = 9m$ . On the other hand,  $8m = N(t) = N(0)e^{-\lambda t} = 9me^{-\lambda t}$ . This implies that  $\frac{8}{9} = e^{-\lambda t}$  and hence  $t = \frac{-\ln \frac{8}{9}}{\lambda} = \frac{-\tau}{\ln 2} \ln \frac{8}{9} \approx 212$  million years old.

- 35.** The ambient temperature is  $32^\circ$  F, the temperature of the ice water. From Equation (13) we get  $T(t) = 32 + ke^{rt}$ . At  $t = 0$  we get  $70 = 32 + k$ , so  $k = 38$  and  $T(t) = 32 + 38e^{rt}$ . After 30 minutes we have  $55 = T(30) = 32 + 38e^{30r}$  and solving for  $r$  gives  $r = \frac{1}{30} \ln \frac{23}{38}$ . To find the time  $t$  when  $T(t) = 45$  we solve  $45 = 32 + 38e^{rt}$ , with  $r$  as above. We get  $t = 30 \frac{\ln 13 - \ln 38}{\ln 23 - \ln 38} \approx 64$  minutes.

- 37.** The ambient temperature is  $T_a = 65^\circ$ . Equation (13) gives  $T(t) = 65 + ke^{rt}$  for the temperature at time  $t$ . Since the initial temperature of the thermometer is  $T(0) = 90$  we get  $90 = T(0) = 65 + k$ . Thus  $k = 25$ . The constant  $r$  is determined from the temperature at a second time:  $85 = T(2) = 65 + 25e^{2r}$  so  $r = \frac{1}{2} \ln \frac{4}{5}$ . Thus  $T(t) = 65 + 25e^{rt}$ , with  $r = \frac{1}{2} \ln \frac{4}{5}$ . To answer the first question we solve the equation  $75 = T(t) = 65 + 25e^{rt}$  for  $t$ . We get  $t = 2 \frac{\ln 2 - \ln 5}{\ln 4 - \ln 5} \approx 8.2$  minutes. The temperature at  $t = 20$  is  $T(20) = 65 + 25 \left(\frac{4}{5}\right)^{10} \approx 67.7^\circ$ .

- 39.** The ambient temperature is  $T_a = 70^\circ$ . Equation (13) gives  $T(t) = 70 + ke^{rt}$  for the temperature of the coffee at time  $t$ . We are asked to determine the initial temperature of the coffee so  $T(0)$  is unknown. However, we have the equations

$$\begin{aligned} 150 = T(5) &= 70 + ke^{5r} \\ 142 = T(6) &= 70 + ke^{6r} \end{aligned}$$

or

$$\begin{aligned} 80 &= ke^{5r} \\ 72 &= ke^{4r}. \end{aligned}$$

Dividing the second equation by the first gives  $\frac{72}{80} = e^r$  so  $r = \ln 0.9$ . From the first equation we get  $k = 80e^{-5r} \approx 135.5$ . We now calculate  $T(0) = 70 + k \approx 205.5^\circ$

41. Let us start time  $t = 0$  at 1980. Then  $P(0) = 290$ . The Malthusian growth model gives  $P(t) = 290e^{rt}$ . At  $t = 10$  (1990) we have  $370 = 290e^{10r}$  and hence  $r = \frac{1}{10} \ln \frac{37}{29}$ . At  $t = 30$  (2010) we have  $P(30) = 290e^{30r} = 290 \left(\frac{37}{29}\right)^3 \approx 602$ .
43. We have  $3P(0) = P(5) = P(0)e^{3r}$ . So  $r = \frac{\ln 3}{5}$ . Now we solve the equation  $2P(0) = P(t) = P(0)e^{rt}$  for  $t$ . We get  $t = \frac{\ln 2}{r} = \frac{5 \ln 2}{\ln 3} \approx 3.15$  years.
45. In the logistics equation  $m = 5000$  and  $P_0 = 2000$ . Thus  $P(t) = \frac{10,000,000}{2,000 + 3,000e^{-rt}} = \frac{10,000}{2 + 3e^{-rt}}$ . Since  $P(2) = 3000$  we get  $3000 = \frac{10,000}{2 + 3e^{-2r}}$ . Solving this equation for  $r$  gives  $r = \ln \frac{3}{2}$ . Now  $P(4) = \frac{10,000}{2 + 3e^{-4r}} = \frac{10,000}{2 + 3(\frac{2}{3})^4} \approx 3857$
47. We have  $P(0) = P_0 = 400$ ,  $P(3) = P_1 = 700$ , and  $P(6) = P_2 = 1000$ . Using the result of the previous problem we get  $m = \frac{700(700(400+1000)-2 \cdot 400 \cdot 1000)}{(700)^2 - 400 \cdot 1000} = 1,400$

## SECTION 1.4

1. This equation is already in standard form with  $p(t) = 3$ . An antiderivative of  $p(t)$  is  $P(t) = \int 3 dt = 3t$  so the integrating factor is  $\mu(t) = e^{3t}$ . If we multiply the differential equation  $y' + 3y = e^t$  by  $\mu(t)$ , we get the equation

$$e^{3t}y' + 3e^{3t}y = e^{4t},$$

and the left hand side of this equation is a perfect derivative, namely,  $(e^{3t}y)'$ . Thus,  $(e^{3t}y)' = e^{4t}$ . Now take antiderivatives of both sides and multiply by  $e^{-3t}$ . This gives

$$y = \frac{1}{4}e^t + ce^{-3t}$$

for the general solution of the equation. To find the constant  $c$  to satisfy the initial condition  $y(0) = -2$ , substitute  $t = 0$  into the general solution

to get  $-2 = y(0) = \frac{1}{4} + c$ . Hence  $c = -\frac{9}{4}$ , and the solution of the initial value problem is

$$y = \frac{1}{4}e^t - \frac{9}{4}e^{-3t}.$$

3. This equation is already in standard form. In this case  $p(t) = -2$ , an antiderivative is  $P(t) = -2t$ , and the integrating factor is  $\mu(t) = e^{-2t}$ . Now multiply by the integrating factor to get

$$e^{-2t}y' - 2e^{-2t}y = 1,$$

the left hand side of which is a perfect derivative  $((e^{-2t})y)'$ . Thus  $((e^{-2t})y)' = 1$  and taking antiderivatives of both sides gives

$$(e^{-2t})y = t + c,$$

where  $c \in \mathbb{R}$  is a constant. Now multiply by  $e^{2t}$  to get  $y = te^{2t} + ce^{2t}$  for the general solution. Letting  $t = 0$  gives  $4 = y(0) = c$  so

$$y = te^{2t} + 4e^{2t}.$$

5. The general solution from Problem 4 is  $y = \frac{e^t}{t} + \frac{c}{t}$ . Now let  $t = 1$  to get  $0 = e + c$ . So  $c = -e$  and  $y = \frac{e^t}{t} - \frac{e}{t}$ .

7. We first put the equation in standard form and get

$$y' + \frac{1}{t}y = \cos(t^2).$$

In this case  $p(t) = \frac{1}{t}$ , an antiderivative is  $P(t) = \ln t$ , and the integrating factor is  $\mu(t) = t$ . Now multiply by the integrating factor to get

$$ty' + y = t \cos(t^2),$$

the left hand side of which is a perfect derivative  $(ty)'$ . Thus  $(ty)' = t \cos(t^2)$  and taking antiderivatives of both sides gives  $ty = \frac{1}{2} \sin(t^2) + c$  where  $c \in \mathbb{R}$  is a constant. Now divide by  $t$  to get  $y = \frac{\sin(t^2)}{2t} + \frac{c}{t}$  for the general solution.

9. In this case  $p(t) = -3$  and the integrating factor is  $e^{\int -3 dt} = e^{-3t}$ . Now multiply to get  $e^{-3t}y' - 3e^{-3t}y = 25e^{-3t} \cos 4t$ , which simplifies to  $(e^{-3t}y)' = 25e^{-3t} \cos 4t$ . Now integrate both sides to get  $e^{-3t}y = (4 \sin 4t - 3 \cos 4t)e^{-3t} + c$ , where we computed  $\int 25e^{-3t} \cos 4t$  by parts twice. Dividing by  $e^{-3t}$  gives  $y = 4 \sin 4t - 3 \cos 4t + ce^{3t}$ .
11. In standard form we get  $z' - 2tz = -2t^3$ . An integrating factor is  $e^{\int -2t dt} = e^{-t^2}$ . Thus  $(e^{-t^2}z)' = -2t^3 e^{-t^2}$ . Integrating both sides gives

$e^{-t^2} z = (t^2 + 1)e^{-t^2} + c$ , where the integral of the right hand side is done by parts. Now divide by the integrating factor  $e^{-t^2}$  to get  $z = t^2 + 1 + ce^{t^2}$ .

13. The given equation is in standard form,  $p(t) = \cos t$ , an antiderivative is  $P(t) = -\sin t$ , and the integrating factor is  $\mu(t) = e^{-\sin t}$ . Now multiply by the integrating factor to get

$$e^{-\sin t} y' + (\cos t)e^{-\sin t} y = (\cos t)e^{-\sin t},$$

the left hand side of which is a perfect derivative  $((e^{-\sin t})y)'$ . Thus

$$((e^{-\sin t})y)' = (\cos t)e^{-\sin t}$$

and taking antiderivatives of both sides gives  $(e^{-\sin t})y = e^{-\sin t} + c$  where  $c \in \mathbb{R}$  is a constant. Now multiply by  $e^{\sin t}$  to get  $y = 1 + ce^{\sin t}$  for the general solution. To satisfy the initial condition,  $0 = y(0) = 1 + ce^{\sin 0} = 1 + c$ , so  $c = -1$ . Thus, the solution of the initial value problem is  $y = 1 - e^{\sin t}$

15. The given linear differential equation is in standard form,  $p(t) = \frac{-2}{t}$ , an antiderivative is  $P(t) = -2 \ln t = \ln t^{-2}$ , and the integrating factor is  $\mu(t) = t^{-2}$ . Now multiply by the integrating factor to get

$$t^{-2}y' - \frac{2}{t^3}y = \frac{t+1}{t^3} = t^{-2} + t^{-3},$$

the left hand side of which is a perfect derivative  $(t^{-2}y)'$ . Thus

$$(t^{-2}y)' = t^{-2} + t^{-3}$$

and taking antiderivatives of both sides gives  $(t^{-2})y = -t^{-1} - \frac{t^{-2}}{2} + c$  where  $c \in \mathbb{R}$  is a constant. Now multiply by  $t^2$  to and we get  $y = -t - \frac{1}{2} + ct^{-2}$  for the general solution. Letting  $t = 1$  gives  $-3 = y(1) = \frac{-3}{2} + c$  so  $c = \frac{-3}{2}$  and

$$y(t) = -t - \frac{1}{2} - \frac{3}{2}t^{-2}.$$

17. The given equation is in standard form,  $p(t) = a$ ,  $p(t) = a$ , an antiderivative is  $P(t) = at$ , and the integrating factor is  $\mu(t) = e^{at}$ . Now multiply by the integrating factor to get  $e^{at}y' + ae^{at}y = e^{(a+b)t}$ , the left hand side of which is a perfect derivative  $(e^{at}y)'$ . Thus  $(e^{at}y)' = e^{(a+b)t}$  and taking antiderivatives of both sides gives

$$(e^{at})y = \frac{1}{a+b}e^{(a+b)t} + c$$

where  $c \in \mathbb{R}$  is a constant. Now multiply by  $e^{-at}$  to get

$$y = \frac{1}{a+b}e^{bt} + ce^{-at}$$

for the general solution.

- 19.** In standard form we get  $y' - (\tan t)y = \sec t$ . In this case  $p(t) = -\tan t$ , an antiderivative is  $P(t) = \ln \cos t$ , and the integrating factor is  $\mu(t) = e^{P(t)} = \cos t$ . Now multiply by the integrating factor to get  $(\cos t)y' - (\sin t)y = 1$ , the left hand side of which is a perfect derivative  $((\cos t)y)'$ . Thus  $((\cos t)y)' = 1$  and taking antiderivatives of both sides gives  $(\cos t)y = t + c$  where  $c \in \mathbb{R}$  is a constant. Now multiply by  $1/\cos t = \sec t$  and we get  $y = (t + c)\sec t$  for the general solution.
- 21.** The given differential equation is in standard form,  $p(t) = -n/t$ , an antiderivative is  $P(t) = -n \ln t = \ln(t^{-n})$ , and the integrating factor is  $\mu(t) = t^{-n}$ . Now multiply by the integrating factor to get  $t^{-n}y' - nt^{-n-1}y = e^t$ , the left hand side of which is a perfect derivative  $(t^{-n}y)'$ . Thus  $(t^{-n}y)' = e^t$  and taking antiderivatives of both sides gives  $(t^{-n}y) = e^t + c$  where  $c \in \mathbb{R}$  is a constant. Now multiply by  $t^n$  to and we get  $y = t^n e^t + ct^n$  for the general solution.
- 23.** Divide by  $t$  to put the equation in the standard form

$$y' + \frac{3}{t}y = t.$$

In this case  $p(t) = 3/t$ , an antiderivative is  $P(t) = 3 \ln t = \ln(t^3)$ , and the integrating factor is  $\mu(t) = t^3$ . Now multiply the standard form equation by the integrating factor to get  $t^3y' + 3t^2y = t^4$ , the left hand side of which is a perfect derivative  $(t^3y)'$ . Thus  $(t^3y)' = t^4$  and taking antiderivatives of both sides gives  $t^3y = \frac{1}{5}t^5 + c$  where  $c \in \mathbb{R}$  is a constant. Now multiply by  $t^{-3}$  and we get  $y = \frac{1}{5}t^2 + ct^{-3}$  for the general solution. Letting  $t = -1$  gives  $2 = y(-1) = \frac{1}{5} - c$  so  $c = \frac{-9}{5}$  and

$$y = \frac{1}{5}t^2 - \frac{9}{5}t^{-3}.$$

- 25.** Divide by  $t^2$  to put the equation in the standard form

$$y' + \frac{2}{t}y = t^{-2}.$$

In this case  $p(t) = 2/t$ , an antiderivative is  $P(t) = 2 \ln t = \ln t^2$ , and the integrating factor is  $\mu(t) = t^2$ . Now multiply by the integrating factor to get  $t^2y' + 2ty = 1$ , the left hand side of which is a perfect derivative  $(t^2y)'$ . Thus  $(t^2y)' = 1$  and taking antiderivatives of both sides gives  $t^2y = t + c$  where  $c \in \mathbb{R}$  is a constant. Now multiply by  $t^{-2}$  to get  $y = \frac{1}{t} + ct^{-2}$  for the general solution. Letting  $t = 2$  gives  $a = y(2) = \frac{1}{2} + \frac{c}{4}$  so  $c = 4a - 2$

and

$$y = \frac{1}{t} + (4a - 2)t^{-2}.$$

- 27.** Let  $V(t)$  denote the volume of fluid in the tank at time  $t$ . Initially, there are 10 gal of brine. For each minute that passes there is a net decrease of  $4 - 3 = 1$  gal of brine. Thus  $V(t) = 10 - t$  gal.

**input rate:** input rate  $= 3 \frac{\text{gal}}{\text{min}} \times 1 \frac{\text{lbs}}{\text{gal}} = 3 \frac{\text{lbs}}{\text{min}}.$

**output rate:** output rate  $= 4 \frac{\text{gal}}{\text{min}} \times \frac{y(t)}{V(t)} \frac{\text{lbs}}{\text{gal}} = \frac{4y(t)}{10 - t} \frac{\text{lbs}}{\text{min}}.$

Since  $y' = \text{input rate} - \text{output rate}$ , it follows that  $y(t)$  satisfies the initial value problem

$$y' = 3 - \frac{4}{10 - t}y(t), \quad y(0) = 2.$$

Put in standard form, this equation becomes

$$y' + \frac{4}{10 - t}y = 3.$$

The coefficient function is  $p(t) = \frac{4}{10 - t}$ ,  $P(t) = \int p(t) dt = -4 \ln(10 - t) = \ln(10 - t)^{-4}$ , and the integrating factor is  $\mu(t) = (10 - t)^{-4}$ . Multiplying the standard form equation by the integrating factor gives

$$((10 - t)^{-4}y)' = 3(10 - t)^{-4}.$$

Integrating and simplifying gives  $y = (10 - t) + c(10 - t)^4$ . The initial condition  $y(0) = 2$  implies  $2 = y(0) = 10 + c10^4$  and hence  $c = -8/10^4$  so

$$y = (10 - t) - \frac{8}{10^4}(10 - t)^4.$$

Of course, this formula is valid for  $0 \leq t \leq 10$ . After 10 minutes there is no fluid and hence no salt in the tank.

- 29.** Let  $V(t)$  denote the volume of fluid in the container at time  $t$ . Initially, there are 10 L. For each minute that passes there is a net gain of  $4 - 2 = 2$  L of fluid. So  $V(t) = 10 + 2t$ . The container overflows when  $V(t) = 10 + 2t = 30$  or  $t = 10$  minutes.

**input rate:** input rate  $= 4 \frac{\text{L}}{\text{min}} \times 20 \frac{\text{g}}{\text{L}} = 80 \frac{\text{g}}{\text{min}}.$

**output rate:** output rate  $= 2 \frac{\text{L}}{\text{min}} \times \frac{y(t)}{10 + 2t} \frac{\text{g}}{\text{L}} = \frac{2y(t)}{10 + 2t} \frac{\text{g}}{\text{min}}.$

Since  $y' = \text{input rate} - \text{output rate}$ , it follows that  $y(t)$  satisfies the initial value problem

$$y' = 80 - \frac{2y}{10 + 2t}, \quad y(0) = 0.$$

Simplifying and putting in standard form gives the equation

$$y' + \frac{1}{5+t}y = 80.$$

The coefficient function is  $p(t) = \frac{1}{5+t}$ ,  $P(t) = \int p(t) dt = \ln(5+t)$ , and the integrating factor is  $\mu(t) = 5+t$ . Multiplying the standard form equation by the integrating factor gives  $((5+t)y)' = 80(5+t)$ . Integrating and simplifying gives  $y = 40(5+t) + c(5+t)^{-1}$ , where  $c$  is a constant. The initial condition  $y(0) = 0$  implies  $c = -1000$  so  $y = 40(5+t) - 1000(5+t)^{-1}$ . At the time the container overflows  $t = 10$  we have  $y(10) = 600 - \frac{1000}{15} \approx 533.33$  g of salt.

**31. input rate:** input rate =  $rc$

**output rate:** output rate =  $r \frac{P(t)}{V}$

Let  $P_0$  denote the amount of pollutant at time  $t = 0$ . Since  $P' =$  input rate  $-$  output rate it follows that  $P(t)$  is a solution of the initial value problem

$$P' = rc - \frac{rP(t)}{V}, \quad P(0) = P_0.$$

Rewriting this equation in standard form gives the differential equation  $P' + \frac{r}{V}P = rc$ . The coefficient function is  $p(t) = r/V$  and the integrating factor is  $\mu(t) = e^{rt/V}$ . Thus  $(e^{\frac{rt}{V}}P)' = rce^{\frac{rt}{V}}$ . Integrating and simplifying gives  $P(t) = cV + ke^{\frac{-rt}{V}}$ , where  $k$  is the constant of integration. The initial condition  $P(0) = P_0$  implies  $c = P_0 - cV$  so  $P(t) = cV + (P_0 - cV)e^{\frac{-rt}{V}}$ .

(a)  $\lim_{t \rightarrow \infty} P(t) = cV$ .

(b) When the river is cleaned up at  $t = 0$  we assume the input concentration is  $c = 0$ . The amount of pollutant is therefore given by  $P(t) = P_0 e^{\frac{-rt}{V}}$ . This will reduce by 1/2 when  $P(t) = \frac{1}{2}P_0$ . We solve the equation  $\frac{1}{2}P_0 = P_0 e^{\frac{-rt}{V}}$  for  $t$  and get  $t_{1/2} = V \frac{\ln 2}{r}$ . Similarly, the pollutant will reduce by 1/10 when  $t_{1/10} = V \frac{\ln 10}{r}$ .

(c) Letting  $V$  and  $r$  be given as stated for each lake gives:

Lake Erie:  $t_{1/2} = 1.82$  years,  $t_{1/10} = 6.05$  years.

Lake Ontario:  $t_{1/2} = 5.43$  years,  $t_{1/10} = 18.06$  years

**33.** Let  $y_1(t)$  and  $y_2(t)$  denote the amount of salt in Tank 1 and Tank 2, respectively, at time  $t$ . The volume of fluid at time  $t$  in Tank 1 is  $V_1(t) = 10 + 2t$  and Tank 2 is  $V_2(t) = 5 + t$ .

**input rate for Tank 1:** input rate =  $4 \frac{\text{L}}{\text{min}} \times 10 \frac{\text{g}}{\text{L}} = 40 \frac{\text{g}}{\text{min}}$ .



**output rate for Tank 1:** output rate =  $2 \frac{\text{L}}{\text{min}} \times \frac{y_1(t)}{10+2t} \frac{\text{g}}{\text{L}} = \frac{2y(t)}{10+2t} \frac{\text{g}}{\text{min}}$ . The initial value problem for Tank 1 is thus

$$y_1' = 40 - \frac{2}{10+2t}y_1, \quad y_1(0) = 0.$$

Simplifying this equation and putting it in standard form gives

$$y_1' + \frac{1}{5+t}y_1 = 40.$$

The integrating factor is  $\mu(t) = 5+t$ . Thus  $((5+t)y_1)' = 40(5+t)$ . Integrating and simplifying gives  $y_1(t) = 20(5+t) + c/(5+t)$ . The initial condition  $y(0) = 0$  implies  $c = -500$  so  $y_1 = 20(5+t) - 500/(5+t)$ .

**input rate for Tank 2:** input rate =  $2 \frac{\text{L}}{\text{min}} \times \frac{y_1(t)}{10+2t} \frac{\text{g}}{\text{L}} = 20 - \frac{500}{(5+t)^2} \frac{\text{g}}{\text{min}}$ .

**output rate for Tank 2:** output rate =  $1 \frac{\text{L}}{\text{min}} \times \frac{y_2(t)}{5+t} \frac{\text{g}}{\text{L}} = \frac{y_2(t)}{5+t} \frac{\text{g}}{\text{min}}$ . The initial value problem for Tank 2 is thus

$$y_2' = 20 - 500/(5+t)^2 - \frac{1}{(5+t)}y_2, \quad y_2(0) = 0.$$

When this equation is put in standard form we get

$$y_2' + \frac{1}{(5+t)}y_2 = 20 - \frac{500}{(5+t)^2}.$$

The integrating factor is  $\mu(t) = 5+t$ . Thus

$$((5+t)y_2)' = 20(5+t) - \frac{500}{5+t}.$$

Integrating and simplifying gives

$$y_2(t) = 10(5+t) - \frac{500 \ln(5+t)}{5+t} + \frac{c}{5+t}.$$

The initial condition  $y_2(0) = 0$  implies  $c = 500 \ln 5 - 250$  so

$$y_2(t) = 10(5+t) - \frac{500 \ln(5+t)}{5+t} + \frac{500 \ln 5 - 250}{5+t}.$$

## SECTION 1.5

1. In standard form we get  $y' = \frac{y^2 + yt + t^2}{t^2}$  which is homogeneous since the degrees of the numerator and denominator are each two. Let  $y = tv$ . Then  $v + tv' = v^2 + v + 1$  and so  $tv' = v^2 + 1$ . Separating variables gives  $\frac{dv}{v^2 + 1} = \frac{dt}{t}$ . Integrating gives  $\tan^{-1} v = \ln |t| + c$ . So  $v = \tan(\ln |t| + c)$ . Substituting  $v = y/t$  gives  $y = t \tan(\ln |t| + c)$ . The initial condition implies  $1 = y(1) = 1 \cdot \tan c = \tan c$  and hence  $c = \pi/4$ . Therefore  $y(t) = t \tan(\ln |t| + \pi/4)$ .
  
3. Since the numerator and denominator are homogeneous of degree 2 the quotient is homogeneous. Let  $y = tv$ . Then  $v + tv' = v^2 - 4v + 6$ . So  $tv' = v^2 - 5v + 6 = (v - 2)(v - 3)$ . There are two equilibrium solutions  $v = 2, 3$ . Separating the variables and using partial fractions gives  $\left( \frac{1}{v - 3} - \frac{1}{v - 2} \right) dv = \frac{dt}{t}$ . Integrating and simplifying gives  $\ln \left| \frac{v - 3}{v - 2} \right| = \ln |t| + c$ . Solving for  $v$  gives  $v = \frac{3 - 2kt}{1 - kt}$ , for  $k \neq 0$ , and so  $y = \frac{3t - 2kt^2}{1 - kt}$ , for  $k \neq 0$ . When  $k = 0$  we get  $v = 3$  or  $y = 3t$ , which is the same as the equilibrium solution  $v = 3$ . The equilibrium solution  $v = 2$  gives  $y = 2t$ . Thus we can write the solutions as  $y = \frac{3t - 2kt^2}{1 - kt}$ ,  $k \in \mathbb{R}$  and  $y = 2t$ . The initial condition  $y(2) = 4$  is satisfied for the linear equation  $y = 2t$  but has no solution for the family  $y = \frac{3t - 2kt^2}{1 - kt}$ . Thus  $y = 2t$  is the only solution.
  
5. Since the numerator and denominator are homogeneous of degree 2 the quotient is homogeneous. Let  $y = tv$ . Then  $v + tv' = \frac{3v^2 - 1}{2v}$ . Subtract  $v$  from both sides to get  $tv' = \frac{v^2 - 1}{2v}$ . The equilibrium solutions are  $v = \pm 1$ . Separating variables gives  $\frac{2v dv}{v^2 - 1} = \frac{dt}{t}$  and integrating gives  $\ln |v^2 - 1| = \ln |t| + c$ . Exponentiating gives  $v^2 - 1 = kt$  and by simplifying we get  $v = \pm \sqrt{1 + kt}$ . Now  $v = y/t$  so  $y = \pm t \sqrt{1 + kt}$ . The equilibrium solutions  $v = \pm 1$  become  $y = \pm t$ . These occur when  $k = 0$ , so are already included in the general formula.
  
7. In standard form we get  $y' = \frac{y + \sqrt{t^2 - y^2}}{t}$ . Since  $\sqrt{(\alpha t)^2 - (\alpha y)^2} = \sqrt{\alpha^2(t^2 - y^2)} = \alpha \sqrt{t^2 - y^2}$  for  $\alpha > 0$  it is easy to see that  $y' = \frac{y + \sqrt{t^2 - y^2}}{t}$  is homogeneous. Let  $y = tv$ . Then  $v + tv' = v + \sqrt{1 - v^2}$ .

Simplifying gives  $tv' = \sqrt{1-v^2}$ . Clearly  $v = \pm 1$  are equilibrium solution. Separating variables gives  $\frac{dv}{\sqrt{1-v^2}} = \frac{dt}{t}$ . Integrating gives  $\sin^{-1} v = \ln|t| + c$  and so  $v = \sin(\ln|t| + c)$ . Now substitute  $v = y/t$  to get  $y = t \sin(\ln|t| + c)$ . The equilibrium solutions imply  $y = \pm t$  are also solutions.

9. Note that although  $y = 0$  is part of the general solution it does not satisfy the initial value. Divide both sides by  $y^2$  to get  $y^{-2}y' - y^{-1} = t$ . Let  $z = y^{-1}$ . Then  $z' = -y^{-2}y'$ . Substituting gives  $-z' - z = t$  or  $z' + z = -t$ . An integrating factor is  $e^t$ . So  $(e^t z) = -te^t$ . Integrating both sides gives  $e^t z = -te^t + e^t + c$ , where we have used integration by parts to compute  $\int -te^t dt$ . Solving for  $z$  gives  $z = -t + 1 + ce^{-t}$ . Now substitute  $z = y^{-1}$  and solve for  $y$  to get  $y = \frac{1}{-t + 1 + ce^{-t}}$ . The initial condition implies  $1 = \frac{1}{1+c}$  and so  $c = 0$ . The solution is thus  $y = \frac{1}{1-t}$ .
11. Note that  $y = 0$  is a solution. First divide both sides by  $y^3$  to get  $y^{-3}y' + ty^{-2} = t$ . Let  $z = y^{-2}$ . Then  $z' = -2y^{-3}y'$ , so  $\frac{z'}{-2} = y^{-3}y'$ . Substituting gives  $\frac{z'}{-2} + tz = t$ , which in standard form is  $z' - 2tz = -2t$ . An integrating factor is  $e^{\int -2t dt} = e^{-t^2}$ , so that  $(e^{-t^2}z)' = -2te^{-t^2}$ . Integrating both sides gives  $e^{-t^2}z = e^{-t^2} + c$ , where the integral of the right hand side is done by the substitution  $u = -t^2$ . Solving for  $z$  gives  $z = 1 + ce^{t^2}$ . Since  $z = y^{-2}$  we find  $y = \pm \frac{1}{\sqrt{1+ce^{t^2}}}$ .
13. Note that  $y = 0$  is a solution. Divide by  $y^2$  and  $(1-t^2)$  to get  $y^{-2}y' - \frac{t}{1-t^2}y^{-1} = \frac{5t}{1-t^2}$ . Let  $z = y^{-1}$ . Then  $z' = -y^{-2}y'$  and substituting gives  $-z' - \frac{t}{1-t^2}z = \frac{5t}{(1-t^2)}$ . In standard form we get  $z' + \frac{t}{1-t^2}z = \frac{-5t}{1-t^2}$ . Multiplying by the integrating factor
- $$\mu(t) = e^{\int \frac{t}{1-t^2} dt} = e^{-\frac{1}{2} \ln(1-t^2)} = (1-t^2)^{-1/2}$$
- gives  $(z(1-t^2)^{-1/2})' = -5t(1-t^2)^{-3/2}$ . Integrating gives  $z(1-t^2)^{-1/2} = -5(1-t^2)^{-1/2} + c$  and hence  $z = -5 + c\sqrt{1-t^2}$ . Since  $z = y^{-1}$  we have
- $$y = \frac{1}{-5 + c\sqrt{1-t^2}}.$$
15. If we divide by  $y$  we get  $y' + ty = ty^{-1}$  which is a Bernoulli equation with  $n = -1$ . Note that since  $n < 0$ ,  $y = 0$  is *not* a solution. Dividing by  $y^{-1}$  gets us back to  $yy' + ty^2 = t$ . Let  $z = y^2$ . Then  $z' = 2yy'$  so  $\frac{z'}{2} + tz = t$

and in standard form we get  $z' + 2tz = 2t$ . An integrating factor is  $e^{t^2}$  so  $(e^{t^2}z)' = 2te^{t^2}$ . Integration gives  $e^{t^2}z = e^{t^2} + c$  so  $z = 1 + ce^{-t^2}$ . Since  $z = y^2$  we get  $y = \pm\sqrt{1 + ce^{-t^2}}$ . The initial condition implies  $-2 = y(0) = -\sqrt{1 + c}$  so  $c = 3$ . Therefore  $y = -\sqrt{1 + 3e^{-t^2}}$ .

- 17.** Note that  $y = 0$  is a solution. First divide both sides by  $y^3$  to get  $y^{-3}y' + y^{-2} = t$ . Let  $z = y^{-2}$ . Then  $z' = -2y^{-3}y'$ . So  $\frac{z'}{-2} + z = t$ .

In standard form we get  $z' - 2z = -2t$ . An integrating factor is  $e^{\int -2dt} = e^{-2t}$  and hence  $(e^{-2t}z)' = -2te^{-2t}$ . Integration by parts gives  $e^{-2t}z = (t + \frac{1}{2})e^{-2t} + c$  and hence  $z = t + \frac{1}{2} + ce^{2t}$ . Since  $z = y^{-2}$  we get  $y = \pm \frac{1}{\sqrt{t + \frac{1}{2} + ce^{2t}}}$ .

- 19.** Let  $z = 2t - 2y + 1$ . Then  $z' = 2 - 2y'$  and so  $y' = \frac{2 - z'}{2}$ . Substituting we get  $\frac{2 - z'}{2} = z^{-1}$  and in standard form we get  $z' = 2 - 2z^{-1}$ , a separable differential equation. Clearly,  $z = 1$  is an equilibrium solution. Assume for now that  $z \neq 1$ . Then separating variables and simplifying using  $1/(1 - z^{-1}) = \frac{z}{z-1} = 1 + \frac{1}{z-1}$  gives  $(1 + \frac{1}{z-1}) dz = 2 dt$ . Integrating we get  $z + \ln|z - 1| = 2t + c$ . Now substitute  $z = 2t - 2y + 1$  and simplify to get  $-2y + \ln|2t - 2y| = c$ ,  $c \in \mathbb{R}$ . (We absorb the constant 1 in  $c$ .) The equilibrium solution  $z = 1$  becomes  $y = t$ .

- 21.** Let  $z = t + y$ . Then  $z' = 1 + y'$  and substituting we get  $z' - 1 = z^{-2}$ . In standard form we get  $z' = \frac{1 + z^2}{z^2}$ . Separating variables and simplifying we get  $(1 - \frac{1}{1 + z^2}) dz = dt$ . Integrating we get  $z - \tan^{-1}z = t + c$ . Now let  $z = t + y$  and simplify to get  $y - \tan^{-1}(t + y) = c$ ,  $c \in \mathbb{R}$ .

- 23.** This is the same as Exercise 16 where the Bernoulli equation technique there used the substitution  $z = y^2$ . Here use the given substitution to get  $z' = 2yy' + 1$ . Substituting we get  $z' - 1 = z$  and in standard form  $z' = 1 + z$ . Clearly,  $z = -1$  is an equilibrium solution. Separating variables gives  $\frac{dz}{1 + z} = dt$  and integrating gives  $\ln|1 + z| = t + c$ ,  $c \in \mathbb{R}$ . Solving for  $z$  we get  $z = ke^t - 1$ , where  $k \neq 0$ . Since  $z = y^2 + t - 1$  we get  $y^2 + t - 1 = ke^t - 1$  and solving for  $y$  gives  $y = \pm\sqrt{ke^t - t}$ . The case  $k = 0$  gives the equilibrium solutions  $y = \pm\sqrt{-t}$ .

- 25.** If  $z = \ln y$  then  $z' = \frac{y'}{y}$ . Divide the given differential equation by  $y$ . Then  $\frac{y'}{y} + \ln y = t$  and substitution gives  $z' + z = t$ . An integrating factor

is  $e^t$  so  $(e^t z)' = te^t$ . Integration (by parts) gives  $e^t z = (t-1)e^t + c$  and so  $z = t-1 + ce^{-t}$ . Finally, solving for  $y$  we get  $y = e^{t-1+ce^{-t}}$ ,  $c \in \mathbb{R}$ .

## SECTION 1.6

1. This can be written in the form  $M(t, y) + N(t, y)y' = 0$  where  $M(t, y) = y^2 + 2t$  and  $N(t, y) = 2ty$ . Since  $\partial M/\partial y = 2y = \partial N/\partial t$ , the equation is exact, and the general solution is given implicitly by  $V(t, y) = c$  where the function  $V(t, y)$  is determined by the solution method for exact equations. Thus  $V(t, y) = \int (y^2 + 2t) dt + \phi(y) = y^2 t + t^2 + \phi(y)$ . The function  $\phi(y)$  satisfies

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(y^2 t + t^2) + \frac{d\phi}{dy} = 2ty + \frac{d\phi}{dy} = N(t, y) = 2ty,$$

so that  $d\phi/dy = 0$ . Thus,  $V(t, y) = y^2 t + t^2$  and the solutions to the differential equation are given implicitly by  $t^2 + ty^2 = c$ .

3. In this equation  $M = 2t^2 - y$  and  $N = t + y^2$ . Since  $\partial M/\partial y = -1$ , while  $\partial N/\partial t = 1$ , the equation is not exact.
5. In this equation  $M = 3y - 5t$  and  $N = 2y - t$ . Since  $\partial M/\partial y = 3$ , while  $\partial N/\partial t = -1$ , the equation is not exact.
7. This can be written in the form  $M(t, y) + N(t, y)y' = 0$  where  $M(t, y) = 2ty + 2t^3$  and  $N(t, y) = t^2 - y$ . Since  $\partial M/\partial y = 2t = \partial N/\partial t$ , the equation is exact, and the general solution is given implicitly by  $V(t, y) = c$  where the function  $V(t, y)$  is determined by the solution method for exact equations. Thus  $V(t, y) = \int (2ty + 2t^3) dt + \phi(y) = t^2 y + t^4/2 + \phi(y)$ . The function  $\phi(y)$  satisfies

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(t^2 y + t^4/2) + \frac{d\phi}{dy} = t^2 + \frac{d\phi}{dy} = N(t, y) = t^2 - y,$$

so that  $d\phi/dy = -y$ . Hence  $\phi(y) = -y^2/2$  so that  $V(t, y) = t^2 y + t^4/2 - y^2/2$  and the solutions to the differential equation are given implicitly by  $t^2 y + t^4/2 - y^2/2 = c$ . Multiplying by 2 and completing the square (and replacing the constant  $2c$  by  $c$ ) gives  $(y - t^2)^2 - 2t^4 = c$ .

9. This can be written in the form  $M(t, y) + N(t, y)y' = 0$  where  $M(t, y) = -y$  and  $N(t, y) = y^3 - t$ . Since  $\partial M/\partial y = -1 = \partial N/\partial t$ , the equation is exact, and the general solution is given implicitly by  $V(t, y) = c$  where the function  $V(t, y)$  is determined by the solution method for exact equations. Thus  $V(t, y) = \int (-y) dt + \phi(y) = -yt + \phi(y)$ . The function  $\phi(y)$  satisfies

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(-yt) + \frac{d\phi}{dy} = -t + \frac{d\phi}{dy} = N(t, y) = y^3 - t,$$

so that  $d\phi/dy = y^3$ . Hence,  $\phi(y) = y^4/4$  so that  $V(t, y) = y^4/4 - yt$  and the solutions to the differential equation are given implicitly by  $y^4/4 - yt = c$ .

## SECTION 1.7

1. We first change the variable  $t$  to  $u$  and write  $y'(u) = uy(u)$ . Now integrate both sides from 1 to  $t$  to get  $\int_1^t y'(u) du = \int_1^t uy(u) du$ . Now the left side is  $\int_1^t y'(u) du = y(t) - y(1) = y(t) - 1$ . Thus  $y(t) = 1 + \int_1^t uy(u) du$ .
3. Change the variable  $t$  to  $u$  and write  $y'(u) = \frac{u - y(u)}{u + y(u)}$ . Now integrate both sides from 0 to  $t$  to get  $\int_0^t y'(u) du = \int_0^t \frac{u - y(u)}{u + y(u)} du$ . The left side is  $y(t) - 1$  so  $y(t) = 1 + \int_0^t \frac{u - y(u)}{u + y(u)} du$ .
5. The corresponding integral equation is  $y(t) = 1 + \int_1^t uy(u) du$ . We then have

$$\begin{aligned} y_0(t) &= 1 \\ y_1(t) &= 1 + \int_1^t u \cdot 1 du = 1 + \left( \frac{u^2}{2} \right) \Big|_1^t = 1 + \frac{t^2}{2} - \frac{1}{2} = \frac{1+t^2}{2} \\ y_2(t) &= 1 + \int_1^t u \left( \frac{1+u^2}{2} \right) du = 1 + \left( \frac{u^2}{4} + \frac{u^4}{8} \right) \Big|_1^t = \frac{5}{8} + \frac{t^2}{4} + \frac{t^4}{8} \\ y_3(t) &= 1 + \int_1^t \left( \frac{5u}{8} + \frac{u^3}{4} + \frac{u^5}{8} \right) du = 1 + \left( \frac{5u^2}{16} + \frac{u^4}{16} + \frac{u^6}{48} \right) \Big|_1^t \\ &= \frac{29}{48} + \frac{5t^2}{16} + \frac{t^4}{16} + \frac{t^6}{48}. \end{aligned}$$

7. The corresponding integral equation is  $y(t) = \int_0^t (u + y^2(u)) du$ . We then have

$$\begin{aligned}
y_0(t) &= 0 \\
y_1(t) &= \int_0^t (u+0) du = \frac{t^2}{2} \\
y_2(t) &= \int_0^t \left( u + \left( \frac{u^2}{2} \right)^2 \right) du = \int_0^t \left( u + \frac{u^4}{4} \right) du = \frac{t^2}{2} + \frac{t^5}{20} \\
y_3(t) &= \int_0^t \left( u + \left( \frac{u^2}{2} + \frac{u^5}{20} \right)^2 \right) du = \int_0^t \left( u + \frac{u^4}{4} + \frac{u^7}{20} + \frac{u^{10}}{400} \right) du \\
&= \frac{t^2}{2} + \frac{t^5}{20} + \frac{t^8}{160} + \frac{t^{11}}{4400}.
\end{aligned}$$

9. The corresponding integral equation is

$$y(t) = \int_0^t (1 + (u - y(u))^2) du.$$

We then have

$$\begin{aligned}
y_0(t) &= 0 \\
y_1(t) &= \int_0^t (1 + (u - 0)^2) du = \left( u + \frac{u^3}{3} \right) \Big|_0^t = t + \frac{t^3}{3} \\
y_2(t) &= \int_0^t \left( 1 + \left( u - \left( u + \frac{u^3}{3} \right) \right)^2 \right) du = \int_0^t \left( 1 + \frac{u^6}{9} \right) du \\
&= \left( u + \frac{u^7}{63} \right) \Big|_0^t = t + \frac{t^7}{7 \cdot 3^2} \\
y_3(t) &= \int_0^t \left( 1 + \frac{u^{14}}{7^2 \cdot 3^4} \right) du = t + \frac{t^{15}}{15 \cdot 7^2 \cdot 3^4} \\
y_4(t) &= \int_0^t \left( 1 + \frac{u^{30}}{15^2 \cdot 7^4 \cdot 3^8} \right) du = t + \frac{t^{31}}{31 \cdot 15^2 \cdot 7^4 \cdot 3^8} \\
y_5(t) &= \int_0^t \left( 1 + \frac{u^{62}}{31^2 \cdot 15^4 \cdot 7^8 \cdot 3^{16}} \right) du = t + \frac{t^{63}}{63 \cdot 31^2 \cdot 15^4 \cdot 7^8 \cdot 3^{16}}
\end{aligned}$$

11. The right hand side is  $F(t, y) = \sqrt{y}$ . If  $\mathcal{R}$  is any rectangle about  $(1, 0)$  then there are  $y$ -coordinates that are negative. Hence  $F$  is not defined on  $\mathcal{R}$  and Picards' theorem does not apply.

13. The right hand side is  $F(t, y) = \frac{t-y}{t+y}$ . Then  $F_y(t, y) = \frac{-2t}{(t+y)^2}$ . Choose a rectangle  $\mathcal{R}$  about  $(0, -1)$  that contains no points on the line  $t+y=0$ . Then both  $F$  and  $F_y$  are continuous on  $\mathcal{R}$ . Picard's theorem applies and we can conclude there is a unique solution on an interval about 0.

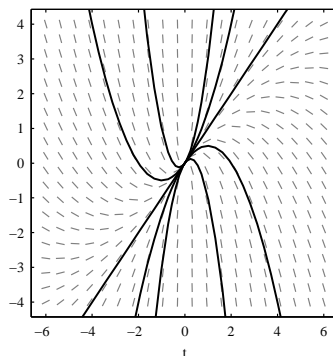
15. The corresponding integral equation is  $y(t) = 1 + \int_0^t ay(u) du$ . We thus have

$$\begin{aligned} y_0(t) &= 1 \\ y_1(t) &= 1 + \int_0^t a du = 1 + at \\ y_2(t) &= 1 + \int_0^t a(1 + au) du = 1 + \int_0^t (a + a^2u) du = 1 + at + \frac{a^2t^2}{2} \\ y_3(t) &= 1 + \int_0^t a \left( 1 + au + \frac{a^2u^2}{2} \right) du = 1 + at + \frac{a^2t^2}{2} + \frac{a^3t^3}{3!} \\ &\vdots \\ y_n(t) &= 1 + at + \frac{a^2t^2}{2} + \cdots + \frac{a^nt^n}{n!}. \end{aligned}$$

We can write  $y_n(t) = \sum_{k=0}^n \frac{a^k t^k}{k!}$ . We recognize this sum as the first  $n$  terms of the Taylor series expansion for  $e^{at}$ . Thus the limiting function is  $y(t) = \lim_{n \rightarrow \infty} y_n(t) = e^{at}$ . It is straightforward to verify that it is a solution. If  $F(t, y) = ay$  then  $F_y(t, y) = a$ . Both  $F$  and  $F_y$  are continuous on the whole  $(t, y)$ -plane. By Picard's theorem, Theorem 5,  $y(t) = e^{at}$  is the only solution to the given initial value problem.

17. Let  $F(t, y) = \cos(t + y)$ . Then  $F_y(t, y) = -\sin(t + y)$ . Let  $y_1$  and  $y_2$  be arbitrary real numbers. Then by the mean value theorem there is a number  $y_0$  in between  $y_1$  and  $y_2$  such that  $|F(t, y_1) - F(t, y_2)| = |\sin(t + y_0)| |y_1 - y_2| \leq |y_1 - y_2|$ . It follows that  $F(t, y)$  is Lipschitz on any strip. Theorem 10 implies there is a unique solution on all of  $\mathbb{R}$ .
19. 1. First assume that  $t \neq 0$ . Then  $ty' = 2y - t$  is linear and in standard form becomes  $y' - 2y/t = -1$ . An integrating factor is  $\mu(t) = e^{\int (-2/t) dt} = t^{-2}$  and multiplying both sides by  $\mu$  gives  $t^{-2}y' - 2t^{-3}y = -t^{-2}$ . This simplifies to  $(t^{-2}y)' = -t^{-2}$ . Now integrate to get  $t^{-2}y = t^{-1} + c$  or  $y(t) = t + ct^2$ . We observe that this solution is also valid for  $t = 0$ . Graphs are given below for various values of  $c$ .



Graph of  $y(t) = t + ct^3$  for various  $c$ 

2. Every solution satisfies  $y(0) = 0$ . There is no contradiction to Theorem 5 since, in standard form, the equation is  $y' = \frac{2}{t}y - 1 = F(t, y)$  and  $F(t, y)$  is not continuous for  $t = 0$ .
21. No. Both  $y_1(t)$  and  $y_2(t)$  would be solutions to the initial value problem  $y' = F(t, y)$ ,  $y(0) = 0$ . If  $F(t, y)$  and  $F_y(t, y)$  are both continuous near  $(0, 0)$ , then the initial value problem would have a unique solution by Theorem 5.
23. For  $t < 0$  we have  $y_1'(t) = 0$  and for  $t > 0$  we have  $y_1'(t) = 3t^2$ . For  $t = 0$  we calculate  $y_1'(0) = \lim_{h \rightarrow 0} \frac{y_1(h) - y_1(0)}{h - 0} = \lim_{h \rightarrow 0} \frac{y_1(h)}{h}$ . To compute this limit we show the left hand and right hand limits agree. We get

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{y_1(h)}{h} &= \lim_{h \rightarrow 0^+} \frac{h^3}{h} = \lim_{h \rightarrow 0^+} h^2 = 0 \\ \lim_{h \rightarrow 0^-} \frac{y_1(h)}{h} &= \lim_{h \rightarrow 0^+} \frac{0}{h} = 0 \end{aligned}$$

It follows that  $y_1'(t) = \begin{cases} 0, & \text{for } t < 0 \\ 3t^2 & \text{for } t \geq 0 \end{cases}$  and so

$$ty_1'(t) = \begin{cases} 0, & \text{for } t < 0 \\ 3t^3 & \text{for } t \geq 0 \end{cases}$$

On the other hand,

$$3y_1(t) = \begin{cases} 0, & \text{for } t < 0 \\ 3t^3 & \text{for } t \geq 0 \end{cases}$$

It follows that  $y_1$  is a solution. It is trivial to see that  $y_2(t)$  is a solution. There is no contraction to Theorem 5 since, in standard form  $y' = \frac{3}{t}y = F(t, y)$  has a discontinuous  $F(t, y)$  near  $(0, 0)$ . So Picard's theorem does not even apply.

## SECTION 2.1

1. Apply the Laplace transform to both sides of the equation. For the left hand side we get  $sY(s) - 2 - 4Y(s)$ , while the right hand side is 0. Solve for  $Y(s)$  to get  $Y(s) = \frac{2}{s-4}$ . From this we see that  $y(t) = 2e^{4t}$ .
3. Apply the Laplace transform to both sides of the equation. For the left hand side we get  $sY(s) - 4Y(s)$ , while the right hand side is  $1/(s-4)$ . Solve for  $Y(s)$  to get  $Y(s) = \frac{1}{(s-4)^2}$ . Therefore,  $y(t) = te^{4t}$ .
5. Apply the Laplace transform to both sides of the equation. For the left hand side we get  $sY(s) - 2 + 2Y(s)$ , while the right hand side is  $3/(s-1)$ . Solve for  $Y(s)$  to get

$$Y(s) = \frac{2}{s+2} + \frac{3}{(s-1)(s+2)} = \frac{1}{s+2} + \frac{1}{s-1}.$$

Thus  $y(t) = e^{-2t} + e^t$ .

7. Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned}\mathcal{L}\{y'' + 3y' + 2y\}(s) &= \mathcal{L}\{y''\}(s) + 3\mathcal{L}\{y'\}(s) + 2\mathcal{L}\{y\}(s) \\ &= s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) \\ &= (s^2 + 3s + 2)Y(s) - 3s - 3.\end{aligned}$$

Since the Laplace transform of 0 is 0, we now get

$$(s^2 + 3s + 2)Y(s) - 3s - 3 = 0.$$

Hence,

$$Y(s) = \frac{3s+3}{s^2+3s+2} = \frac{3(s+1)}{(s+1)(s+2)} = \frac{3}{s+2},$$

and therefore,  $y(t) = 3e^{-2t}$ .

9. Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned}
\mathcal{L}\{y'' + 25y\}(s) &= \mathcal{L}\{y''\}(s) + 25\mathcal{L}\{y\}(s) \\
&= s^2Y(s) - sy(0) - y'(0) + 25Y(s) \\
&= (s^2 + 25)Y(s) - s + 1.
\end{aligned}$$

We now get

$$(s^2 + 25)Y(s) - s + 1 = 0.$$

Hence,

$$Y(s) = \frac{s-1}{s^2+25} = \frac{s}{s^2+25} - \frac{1}{5} \frac{5}{s^2+25},$$

and therefore,  $y(t) = \cos 5t - \frac{1}{5} \sin 5t$ .

**11.** Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned}
\mathcal{L}\{y'' + 8y' + 16y\}(s) &= \mathcal{L}\{y''\}(s) + 8\mathcal{L}\{y'\}(s) + 16\mathcal{L}\{y\}(s) \\
&= s^2Y(s) - sy(0) - y'(0) + 8(sY(s) - y(0)) + 16Y(s) \\
&= (s^2 + 8s + 16)Y(s) - s - 4.
\end{aligned}$$

We now get

$$(s+4)^2Y(s) - (s+4) = 0.$$

Hence,

$$Y(s) = \frac{s+4}{(s+4)^2} = \frac{1}{s+4}$$

and therefore  $y(t) = e^{-4t}$

**13.** Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned}
\mathcal{L}\{y'' + 4y' + 4y\}(s) &= \mathcal{L}\{y''\}(s) + 4\mathcal{L}\{y'\}(s) + 4\mathcal{L}\{y\}(s) \\
&= s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 4Y(s) \\
&= (s^2 + 4s + 4)Y(s) - 1.
\end{aligned}$$

Since  $\mathcal{L}\{e^{-2t}\} = 1/(s+2)$  we get the algebraic equation

$$(s+2)^2Y(s) - 1 = \frac{1}{s+2}.$$

Hence,

$$Y(s) = \frac{1}{(s+2)^2} + \frac{1}{(s+2)^3} = \frac{1}{(s+2)^2} + \frac{1}{2} \frac{2}{(s+2)^3}$$

and therefore  $y(t) = te^{-2t} + \frac{1}{2}t^2e^{-2t}$

## SECTION 2.2

1.

$$\begin{aligned}
& \mathcal{L}\{3t+1\}(s) \\
&= \int_0^\infty (3t+1)e^{-st} dt \\
&= 3 \int_0^\infty te^{-st} dt + \int_0^\infty e^{-st} dt \\
&= 3 \left( \frac{t}{-s} e^{-st} \Big|_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt \right) + \frac{-1}{s} e^{-st} \Big|_0^\infty \\
&= 3 \left( \left( \frac{1}{s} \right) \left( \frac{-1}{s} \right) e^{-st} \Big|_0^\infty \right) + \frac{1}{s} \\
&= \frac{3}{s^2} + \frac{1}{s}.
\end{aligned}$$

3.

$$\begin{aligned}
& \mathcal{L}\{e^{2t} - 3e^{-t}\}(s) \\
&= \int_0^\infty e^{-st}(e^{2t} - 3e^{-t}) dt \\
&= \int_0^\infty e^{-st}e^{2t} dt - 3 \int_0^\infty e^{-st}e^{-t} dt \\
&= \int_0^\infty e^{-(s-2)t} dt - 3 \int_0^\infty e^{-(s+1)t} dt \\
&= \frac{1}{s-2} - \frac{3}{s+1}.
\end{aligned}$$

$$5. \mathcal{L}\{5e^{2t}\} = 5\mathcal{L}\{e^{2t}\} = \frac{5}{s-2}$$

$$7. \mathcal{L}\{t^2 - 5t + 4\} = \mathcal{L}\{t^2\} - 5\mathcal{L}\{t\} + 4\mathcal{L}\{1\} = \frac{2}{s^3} - \frac{5}{s^2} + \frac{4}{s}$$

$$9. \mathcal{L}\{e^{-3t} + 7te^{-4t}\} = \mathcal{L}\{e^{-3t}\} + 7\mathcal{L}\{te^{-4t}\} = \frac{1}{s+3} + \frac{7}{(s+4)^2}$$

$$11. \mathcal{L}\{\cos 2t + \sin 2t\} = \mathcal{L}\{\cos 2t\} + \mathcal{L}\{\sin 2t\} = \frac{s}{s^2+2^2} + \frac{2}{s^2+2^2} = \frac{s+2}{s^2+4}$$

$$13. \mathcal{L}\{(te^{-2t})^2\}(s) = \mathcal{L}\{t^2e^{-4t}\}(s) = \frac{2}{(s+4)^3}$$

$$\begin{aligned} 15. \quad \mathcal{L}\{(t + e^{2t})^2\}(s) &= \mathcal{L}\{t^2 + 2te^{2t} + e^{4t}\}(s) = \mathcal{L}\{t^2\}(s) + 2\mathcal{L}\{te^{2t}\}(s) + \\ &\quad \mathcal{L}\{e^{4t}\}(s) = \frac{2}{s^3} + \frac{2}{(s-2)^2} + \frac{1}{s-4} \end{aligned}$$

$$17. \quad \mathcal{L}\left\{\frac{t^4}{e^{4t}}\right\}(s) = \mathcal{L}\{t^4 e^{-4t}\}(s) = \frac{4!}{(s+4)^5} = \frac{24}{(s+4)^5}$$

$$19. \quad \mathcal{L}\{te^{3t}\}(s) = -(\mathcal{L}\{e^{3t}\})'(s) = -\left(\frac{1}{s-3}\right)' = \frac{1}{(s-3)^2}$$

$$\begin{aligned} 21. \quad &\text{Here we use the transform derivative principle twice to get } \mathcal{L}\{t^2 \sin 2t\}(s) = \\ &(\mathcal{L}\{\sin 2t\})'' = \left(\frac{2}{s^2+4}\right)'' = \left(\frac{-4s}{(s^2+4)^2}\right)' = \frac{12s^2-16}{(s^2+4)^3} \end{aligned}$$

$$23. \quad \mathcal{L}\{tf(t)\}(s) = -\mathcal{L}\{f(t)\}'(s) = -\left(\ln\left(\frac{s^2}{s^2+1}\right)\right)' = \frac{2s}{s^2+1} - \frac{2}{s}$$

$$25. \quad \mathcal{L}\{\text{Ei}(6t)\}(s) = \frac{1}{6} \mathcal{L}\{\text{Ei}(t)\}(s)|_{s \rightarrow s/6} = \frac{1}{6} \frac{\ln((s/6)+1)}{s/6} = \frac{\ln(s+6) - \ln 6}{s}$$

$$\begin{aligned} 27. \quad &\text{We use the identity } \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta). \quad \mathcal{L}\{\sin^2 bt\}(s) = \frac{1}{2} \mathcal{L}\{1 - \cos 2bt\}(s) = \\ &\frac{1}{2} \left( \frac{1}{s} - \frac{s}{s^2+4b^2} \right) = \frac{2b^2}{s(s^2+4b^2)}; \end{aligned}$$

$$29. \quad \text{We use the identity } \sin at \cos bt = \frac{1}{2}(\sin(a+b)t + \sin(a-b)t).$$

$$\begin{aligned} \mathcal{L}\{\sin at \cos bt\} &= \frac{1}{2}(\mathcal{L}\{\sin(a+b)t\} + \mathcal{L}\{\sin(a-b)t\}) \\ &= \frac{1}{2} \left( \frac{a-b}{s^2+(a-b)^2} + \frac{a+b}{s^2+(a+b)^2} \right). \end{aligned}$$

$$31. \quad \mathcal{L}\{\sinh bt\} = \frac{1}{2}(\mathcal{L}\{e^{bt} - e^{-bt}\}) = \frac{1}{2} \left( \frac{1}{s+b} - \frac{1}{s-b} \right) = \frac{b}{s^2-b^2}$$

$$\begin{aligned} 33. \quad &\text{Let } f(t) = \sinh bt. \text{ Then } f'(t) = b \cosh t \text{ and } f''(t) = b^2 \sinh t. \text{ Fur-} \\ &\text{ther, } f(t)|_{t=0} = 0 \text{ and } f'(t)|_{t=0} = b. \text{ Thus } b^2 \mathcal{L}\{\sinh bt\} = \mathcal{L}\{f''(t)\} = \\ &s^2 \mathcal{L}\{f(t)\} - sf(0) - f'(0) = s^2 \mathcal{L}\{f(t)\} - b. \text{ Solving for } \mathcal{L}\{f(t)\} \text{ gives} \\ &\mathcal{L}\{\sinh bt\} = \frac{b}{s^2-b^2}. \end{aligned}$$

$$35. \quad \text{Let } g(t) = \int_0^t f(u) du \text{ and note that } g'(t) = f(t) \text{ and } g(0) = \int_0^0 f(u) du = 0. \text{ Now apply the input derivative formula to } g(t), \text{ to get}$$

$$F(s) = \mathcal{L}\{f(t)\}(s) = \mathcal{L}\{g'(t)\}(s) = s\mathcal{L}\{g(t)\}(s) - g(0) = sG(s).$$

Solving for  $G(s)$  gives  $G(s) = F(s)/s$ .

37. Suppose  $f$  is of exponential type of order  $a$  and  $g$  is of exponential type of order  $b$ . Then there are numbers  $K$  and  $L$  so that  $|f(t)| \leq Ke^{at}$  and  $|g(t)| \leq Le^{bt}$ . Now  $|f(t)g(t)| \leq Ke^{at}Le^{bt} = KLe^{(a+b)t}$ . It follows that  $f + g$  is of exponential type of order  $a + b$ .
39. Suppose  $a$  and  $K$  are real and  $|y(t)| \leq Ke^{at}$ . Then  $y(t)e^{-at}$  is bounded by  $K$ . But

$$\begin{aligned} e^{t^2}e^{-at} &= e^{t^2-at+\frac{a^2}{4}}e^{-\frac{a^2}{4}} \\ &= e^{(t-\frac{a}{2})^2}e^{-\frac{a^2}{4}} \\ &= e^{u^2}e^{-\frac{a^2}{4}}, \end{aligned}$$

where  $u = t - \frac{a}{2}$ . As  $t$  approaches infinity so does  $u$ . Since  $\lim_{u \rightarrow \infty} e^{u^2} = \infty$  it is clear that  $\lim_{t \rightarrow \infty} e^{t^2}e^{-at} = \infty$ , for all  $a \in \mathbb{R}$ , and hence  $y(t)e^{-at}$  is not bounded. It follows that  $y(t)$  is not of exponential type.

41.  $y(t)$  is of exponential type because it is continuous and bounded. On the other hand,  $y'(t) = \cos(e^{t^2})e^{t^2}(2t)$ . Suppose there is a  $K$  and  $a$  so that  $|y'(t)| \leq Ke^{at}$  for all  $t \geq 0$ . We need only show that there are some  $t$  for which this inequality does not hold. Since  $\cos e^{t^2}$  oscillates between  $-1$  and  $1$  let's focus on those  $t$  for which  $\cos e^{t^2} = 1$ . This happens when  $e^{t^2}$  is a multiple of  $2\pi$ , i.e.  $e^{t^2} = 2\pi n$  for some  $n$ . Thus  $t = t_n = \sqrt{\ln(2\pi n)}$ . If the inequality  $|y'(t)| \leq Ke^{at}$  is valid for all  $t \geq 0$  it is valid for  $t_n$  for all  $n > 0$ . We then get the inequality  $2t_n e^{t_n^2} \leq Ke^{at_n}$ . Now divide by  $e^{at_n}$ , combine, complete the square, and simplify to get the inequality  $2t_n e^{(t_n - a/2)^2} \leq Ke^{a^2/4}$ . Choose  $n$  so that  $t_n > K$  and  $t_n > a$ . Then this last inequality is not satisfied. It follows that  $y'(t)$  is not of exponential type. Now consider the definite integral  $\int_0^M e^{-st}y'(t) dt$  and compute by parts: We get

$$\int_0^M e^{-st}y'(t) dt = y(t)e^{-st}\Big|_0^M + s \int_0^M e^{-st}y(t) dt.$$

Since  $y(t) = \sin(e^{t^2})$  is bounded and  $y(0) = 0$  it follows that

$$\lim_{M \rightarrow \infty} y(t)e^{-st}\Big|_0^M = 0.$$

Taking limits as  $M \rightarrow \infty$  in the equation above gives  $\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\}$ . The righthand side exists because  $y(t)$  is bounded.

- (a) Show that  $\Gamma(1) = 1$ .

- (b) Show that  $\Gamma$  satisfies the recursion formula  $\Gamma(\beta + 1) = \beta\Gamma(\beta)$ .  
 (*Hint:* Integrate by parts.)
- (c) Show that  $\Gamma(n + 1) = n!$  when  $n$  is a nonnegative integer.
- 43.** Using polar coordinates  $x = r \cos \theta$ ,  $y = r \sin \theta$ . Then  $dx dy = r dr d\theta$  and the domain of integration is the first quadrant of the plane, which in polar coordinates is given by  $0 \leq \theta \leq \pi/2$ ,  $0 \leq r < \infty$ . Thus

$$\begin{aligned}
 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy &= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta \\
 &= \frac{\pi}{2} \int_0^\infty e^{-r^2} r dr \\
 &= \left. \frac{\pi}{2} - \frac{e^{-r^2}}{2} \right|_0^\infty = \frac{\pi}{4}.
 \end{aligned}$$

Hence,  $I = \sqrt{\pi}/2$ .

### SECTION 2.3

	<b>The <math>s - 1</math> -chain</b>	
1.	$\frac{5s+10}{(s-1)(s+4)}$	$\frac{3}{s-1}$
	$\frac{2}{s+4}$	

	<b>The <math>s - 5</math> -chain</b>	
3.	$\frac{1}{(s+2)(s-5)}$	$\frac{1/7}{(s-5)}$
	$\frac{-1/7}{(s+2)}$	

5.

The $s - 1$ -chain	
$\frac{3s + 1}{(s - 1)(s^2 + 1)}$	$\frac{2}{s - 1}$
$\frac{-2s + 1}{s^2 + 1}$	

7.

The $s + 3$ -chain	
$\frac{s^2 + s - 3}{(s + 3)^3}$	$\frac{3}{(s + 3)^3}$
$\frac{s - 2}{(s + 3)^2}$	$\frac{-5}{(s + 3)^2}$
$\frac{1}{s + 3}$	$\frac{1}{s + 3}$
0	

9.

The $s + 1$ -chain	
$\frac{s}{(s + 2)^2(s + 1)^2}$	$\frac{-1}{(s + 1)^2}$
$\frac{s + 4}{(s + 2)^2(s + 1)}$	$\frac{3}{s + 1}$
$\frac{-3s - 8}{(s + 2)^2}$	



The $s - 5$ -chain	
$\frac{1}{(s-5)^5(s-6)}$	$\frac{-1}{(s-5)^5}$
$\frac{1}{(s-5)^4(s-6)}$	$\frac{-1}{(s-5)^4}$
$\frac{1}{(s-5)^3(s-6)}$	$\frac{-1}{(s-5)^3}$
$\frac{1}{(s-5)^2(s-6)}$	$\frac{-1}{(s-5)^2}$
$\frac{1}{(s-5)(s-6)}$	$\frac{-1}{s-5}$
$\frac{1}{s-6}$	

11.

13. Use the technique of distinct linear factors (Example 5).

$$\frac{13/8}{s-5} - \frac{5/8}{s+3}$$

$$15. \frac{23}{12(s-5)} + \frac{37}{12(s+7)}$$

$$17. \frac{25}{8(s-7)} - \frac{9}{8(s+1)}$$

$$19. \frac{1}{2(s+5)} - \frac{1}{2(s-1)} + \frac{1}{s-2}$$

$$21. \frac{7}{(s+4)^4}$$

23. Use Theorem 1 to write

$$\frac{s^2 + s - 3}{(s+3)^3} = \frac{A_1}{(s+3)^3} + \frac{p_1(s)}{(s+3)^2}$$

where  $A_1 = \left. \frac{s^2 + s - 3}{1} \right|_{s=-3} = 3$

and  $p_1(s) = \frac{1}{s+3}(s^2 + s - 3 - (3)(1)) = \frac{1}{s+3}(s^2 + s - 6) = s - 2$

Continuing gives

$$\frac{s-2}{(s+3)^2} = \frac{A_2}{(s+3)^2} + \frac{p_2(s)}{s+3}$$

where  $A_2 = \left. \frac{s-2}{1} \right|_{s=-3} = -5$

and  $p_2(s) = \frac{1}{s+3}(s-2 - (-5)(1)) = \frac{1}{s+3}(s+3) = 1$

Thus  $\frac{s^2+s-3}{(s+3)^3} = \frac{3}{(s+3)^3} - \frac{5}{(s+3)^2} + \frac{1}{s+3}$

*Alternate Solution:* Write  $s = (s+3) - 3$  so that

$$\begin{aligned} \frac{s^2+s-3}{(s+3)^3} &= \frac{((s+3)-3)^2 + ((s+3)-3) - 3}{(s+3)^3} \\ &= \frac{(s+3)^2 - 5(s+3) + 3}{(s+3)^3} \\ &= \frac{3}{(s+3)^3} - \frac{5}{(s+3)^2} + \frac{1}{s+3}. \end{aligned}$$

25.  $\frac{s^2-6s+7}{(s^2-4s-5)^2} = \frac{s^2-6s+7}{(s+1)^2(s-5)^2}$ , so use Theorem 1 to compute the  $(s+1)$ -chain:

$$\frac{s^2-6s+7}{(s+1)^2(s-5)^2} = \frac{A_1}{(s+1)^2} + \frac{p_1(s)}{(s+1)(s-5)^2}$$

where  $A_1 = \left. \frac{s^2-6s+7}{(s-5)^2} \right|_{s=-1} = \frac{7}{18}$

and  $p_1(s) = \frac{1}{s+1}(s^2-6s+7 - (7/18)(s-5)^2)$

$$= \frac{1}{s+1}(11s^2 - 38s - 49)/18 = \frac{1}{18}(11s - 49)$$

Continuing gives

$$\frac{1}{18} \frac{11s-49}{(s+1)(s-5)^2} = \frac{A_2}{s+1} + \frac{p_2(s)}{(s-5)^2}$$

where  $A_2 = \left. \frac{1}{18} \frac{11s-49}{(s-5)^2} \right|_{s=-1} = -5/54$

and  $p_2(s) = \frac{1}{s+1}((11s-49)/18 - (-5/54)(s-5)^2) = (5s-22)/54$

Thus  $\frac{s^2 - 6s + 7}{(s+1)^2(s-5)^2} = \frac{1/18}{(s+1)^2} - \frac{5/54}{s+1} + \frac{(5s-22)/54}{(s-5)^2}$  Now either continue with Theorem 1 or replace  $s$  with  $s = (s-5)+5$  in the numerator of the last fraction to finish the calculation and get  $\frac{s^2 - 6s + 7}{(s+1)^2(s-5)^2} = \frac{1}{54} \left( \frac{5}{s-5} + \frac{21}{(s+1)^2} + \frac{3}{(s-5)^2} - \frac{5}{s+1} \right)$

**27.** Use Theorem 1 to compute the  $(s+2)$ -chain:

$$\begin{aligned} \frac{s}{(s+2)^2(s+1)^2} &= \frac{A_1}{(s+2)^2} + \frac{p_1(s)}{(s+2)(s+1)^2} \\ \text{where } A_1 &= \left. \frac{s}{(s+1)^2} \right|_{s=-2} = -2 \\ \text{and } p_1(s) &= \frac{1}{s+2}(s - (-2)(s+1)^2) \\ &= \frac{2s^2 + 5s + 2}{s+2} = \frac{(2s+1)(s+1)}{s+2} = 2s+1 \end{aligned}$$

Continuing gives

$$\begin{aligned} \frac{2s+1}{(s+2)(s+1)^2} &= \frac{A_2}{s+2} + \frac{p_2(s)}{(s+1)^2} \\ \text{where } A_2 &= \left. \frac{2s+1}{(s+1)^2} \right|_{s=-2} = -3 \\ \text{and } p_2(s) &= \frac{1}{s+2}(2s+1 - (-3)(s+1)^2) = 3s+2 \end{aligned}$$

Thus  $\frac{s}{(s+2)^2(s+1)^2} = \frac{-2}{(s+2)^2} - \frac{3}{s+2} + \frac{3s+2}{(s+1)^2}$ . Now continue using Theorem 1 or replace  $s$  by  $(s+1)-1$  in the numerator of the last fraction to get  $\frac{s}{(s+2)^2(s+1)^2} = \frac{-2}{(s+2)^2} - \frac{3}{s+2} - \frac{1}{(s+1)^2} + \frac{3}{s+1}$

**29.** Use Theorem 1 to compute the  $(s-3)$ -chain:

$$\frac{8s}{(s-1)(s-2)(s-3)^3} = \frac{A_1}{(s-3)^3} + \frac{p_1(s)}{(s-1)(s-2)(s-3)^2}$$

where  $A_1 = \left. \frac{8s}{(s-1)(s-2)} \right|_{s=3} = 12$

and  $p_1(s) = \frac{1}{s-3}(8s - (12)(s-1)(s-2))$

$$= \frac{-12s^2 + 44s - 24}{s-3} = \frac{(-12s+8)(s-3)}{s-3} = -12s+8$$

For the second step in the  $(s-3)$ -chain:

$$\frac{-12s+8}{(s-1)(s-2)(s-3)^2} = \frac{A_2}{(s-3)^2} + \frac{p_2(s)}{(s-1)(s-2)(s-3)}$$

where  $A_2 = \left. \frac{-12s+8}{(s-1)(s-2)} \right|_{s=3} = -14$

and  $p_2(s) = \frac{1}{s-3}(-12s+8 - (-14)(s-1)(s-2))$

$$= \frac{14s^2 - 54s + 36}{s-3} = \frac{(14s-12)(s-3)}{s-3} = 14s-12$$

Continuing gives

$$\frac{14s-12}{(s-1)(s-2)(s-3)^2} = \frac{A_3}{(s-3)^2} + \frac{p_3(s)}{(s-1)(s-2)}$$

where  $A_3 = \left. \frac{14s-12}{(s-1)(s-2)} \right|_{s=3} = 15$

and  $p_3(s) = \frac{1}{s-3}(14s-12 - (15)(s-1)(s-2)) = -15s+14$

Thus  $\frac{8s}{(s-1)(s-2)(s-3)^3} = \frac{12}{(s-3)^3} - \frac{14}{(s-3)^2} + \frac{15}{s-3} + \frac{-15s+14}{(s-1)(s-2)}$ .

The last fraction has a denominator with distinct linear factors so we get

$$\frac{8s}{(s-1)(s-2)(s-3)^3} = \frac{12}{(s-3)^3} + \frac{-14}{(s-3)^2} + \frac{15}{s-3} + \frac{-16}{s-2} + \frac{1}{s-1}$$

**31.** Use Theorem 1 to compute the  $(s-2)$ -chain:

$$\frac{s}{(s-2)^2(s-3)^2} = \frac{A_1}{(s-2)^2} + \frac{p_1(s)}{(s-2)(s-3)^2}$$

where  $A_1 = \left. \frac{s}{(s-3)^2} \right|_{s=2} = 2$

and  $p_1(s) = \frac{1}{s-2}(s-(2))(s-3)^2$

$$= \frac{-2s^2 + 13s - 18}{s-2} = \frac{(-2s+9)(s-2)}{s-2} = -2s+9$$

Continuing gives

$$\frac{-2s+9}{(s-2)(s-3)^2} = \frac{A_2}{s-3} + \frac{p_2(s)}{(s-3)^2}$$

where  $A_2 = \left. \frac{-2s+9}{(s-3)^2} \right|_{s=2} = 5$

and  $p_2(s) = \frac{1}{s-2}(-2s+9-(5)(s-3)^2) = -5s+18$

Thus  $\frac{s}{(s-2)^2(s-3)^2} = \frac{2}{(s-2)^2} + \frac{5}{s-2} + \frac{-5s+18}{(s-3)^2}$ . Now continue using Theorem 1 or replace  $s$  by  $(s-3)+3$  in the numerator of the last fraction to get  $\frac{s}{(s-2)^2(s-3)^2} = \frac{2}{(s-2)^2} + \frac{5}{s-2} + \frac{3}{(s-3)^2} - \frac{5}{s-3}$

**33.** Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned} \mathcal{L}\{y'' + 2y' + y\} &= \mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} \\ &= s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) \\ &= (s^2 + 2s + 1)Y(s). \end{aligned}$$

Since  $\mathcal{L}\{9e^{2t}\} = \frac{9}{s-2}$  we get

$$Y(s) = \frac{9}{(s+1)^2(s-2)}.$$

A partial fraction decomposition gives

The $(s+1)$ -chain	
$\frac{9}{(s+1)^2(s-2)}$	$\frac{-3}{(s+1)^2}$
$\frac{3}{(s+1)(s-2)}$	$\frac{-1}{s+1}$
$\frac{1}{(s-2)}$	

It follows that

$$Y(s) = \frac{-3}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s-2}$$

and

$$y(t) = -3te^{-t} - e^{-t} + e^{2t}.$$

**35.** Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned} \mathcal{L}\{y'' - 4y' - 5y\} &= \mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} \\ &= s^2Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) - 5Y(s) \\ &= (s^2 - 4s - 5)Y(s) + s - 5. \end{aligned}$$

Since  $\mathcal{L}\{150t\} = 150/s^2$  we get the algebraic equation

$$(s^2 - 4s - 5)Y(s) + s - 5 = \frac{150}{s^2}.$$

Hence,

$$\begin{aligned} Y(s) &= \frac{-s+5}{(s+1)(s-5)} + \frac{150}{s^2(s+1)(s-5)} \\ &= \frac{-1}{s+1} + \frac{150}{s^2(s+1)(s-5)}. \end{aligned}$$

For the second term we start with the  $s$ -chain to get the following partial fraction decomposition

The $s$ -chain	
$\frac{150}{s^2(s+1)(s-5)}$	$\frac{-30}{s^2}$
$\frac{30(s-4)}{s(s+1)(s-5)}$	$\frac{24}{s}$
$\frac{-244s+124}{(s+1)(s-5)}$	$\frac{-25}{s+1}$
$\frac{1}{s-5}$	

Thus

$$Y(s) = \frac{-30}{s^2} + \frac{24}{s} - \frac{26}{s+1} + \frac{1}{s-5}$$

and Table 2.2 gives  $y(t) = -30t + 24 - 26e^{-t} + e^{5t}$ .

**37.** Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned} \mathcal{L}\{y'' - 3y' + 2y\} &= \mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} \\ &= s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) \\ &= (s^2 - 3s + 2)Y(s) - 2s + 3. \end{aligned}$$

Since  $\mathcal{L}\{4\} = 4/s$  we get the algebraic equation

$$(s-1)(s-2)Y(s) - 2s + 3 = \frac{4}{s}.$$

Hence,

$$Y(s) = \frac{2s-3}{(s-1)(s-2)} + \frac{4}{s(s-1)(s-2)}.$$

Each term has denominator a product of distinct linear terms. It is easy to see that

$$\frac{2s-3}{(s-1)(s-2)} = \frac{1}{s-1} + \frac{1}{s-2}$$

and

$$\frac{4}{s(s-1)(s-2)} = \frac{2}{s} - \frac{4}{s-1} + \frac{2}{s-2}.$$

Thus

$$Y(s) = \frac{2}{s} + \frac{3}{s-2} - \frac{3}{s-1}$$

and Table 2.2 gives  $y(t) = 2 + 3e^{2t} - 3e^t$ .

## SECTION 2.4

1. Note that  $s = i$  is a root of  $s^2 + 1$ . Applying Theorem 1 gives

$$\frac{1}{(s^2 + 1)^2(s^2 + 2)} = \frac{B_1s + C_1}{(s^2 + 1)^2} + \frac{p_1(s)}{(s^2 + 1)(s^2 + 2)}$$

$$\begin{aligned} \text{where } B_1i + C_1 &= \left. \frac{1}{(s^2 + 2)} \right|_{s=i} = \frac{1}{i^2 + 2} = 1 \\ &\Rightarrow B_1 = 0 \quad \text{and } C_1 = 1 \\ \text{and } p_1(s) &= \frac{1}{s^2 + 1}(1 - (1)(s^2 + 2)) \\ &= \frac{-s^2 - 1}{s^2 + 1} = -1. \end{aligned}$$

We now apply Theorem 1 on the remainder term  $\frac{-1}{(s^2 + 1)(s^2 + 2)}$ .

$$\frac{-1}{(s^2 + 1)(s^2 + 2)} = \frac{B_2s + C_2}{(s^2 + 1)} + \frac{p_2(s)}{(s^2 + 2)}$$

$$\begin{aligned} \text{where } B_2i + C_2 &= \left. \frac{-1}{(s^2 + 2)} \right|_{s=i} = -1 \\ &\Rightarrow B_2 = 0 \quad \text{and } C_2 = -1 \\ \text{and } p_2(s) &= \frac{1}{s^2 + 1}(-1 - (-1)(s^2 + 2)) \\ &= \frac{s^2 + 1}{s^2 + 1} = 1. \end{aligned}$$

Thus the  $(s^2 + 1)$ -chain is

The $s^2 + 1$ -chain	
$\frac{1}{(s^2 + 1)^2(s^2 + 2)}$	$\frac{1}{(s^2 + 1)^2}$
$\frac{-1}{(s^2 + 1)(s^2 + 2)}$	$\frac{-1}{(s^2 + 1)}$
$\frac{1}{s^2 + 2}$	

3. Note that  $s = \sqrt{3}i$  is a root of  $s^2 + 3$ . Applying Theorem 1 gives



$$\begin{aligned}
\frac{8s + 8s^2}{(s^2 + 3)^3(s^2 + 1)} &= \frac{B_1s + C_1}{(s^2 + 3)^3} + \frac{p_1(s)}{(s^2 + 3)^2(s^2 + 1)} \\
\text{where } B_1\sqrt{3}i + C_1 &= \left. \frac{8s + 8s^2}{(s^2 + 1)} \right|_{s=\sqrt{3}i} = \frac{8\sqrt{3}i + 8(\sqrt{3}i)^2}{(\sqrt{3}i)^2 + 1} \\
&= -4\sqrt{3}i + 12 \\
\Rightarrow B_1 &= -4 \quad \text{and } C_1 = 12 \\
\text{and } p_1(s) &= \frac{1}{s^2 + 3}(8s + 8s^2 - (-4s + 12)(s^2 + 1)) \\
&= \frac{4s^3 - 4s^2 + 12s - 12}{s^2 + 3} = 4(s - 1).
\end{aligned}$$

Apply Theorem 1 a second time on the remainder term  $\frac{4s - 4}{(s^2 + 3)^2(s^2 + 1)}$ .

$$\begin{aligned}
\frac{4s - 4}{(s^2 + 3)^2(s^2 + 1)} &= \frac{B_2s + C_2}{(s^2 + 3)^2} + \frac{p_2(s)}{(s^2 + 3)(s^2 + 1)} \\
\text{where } B_2\sqrt{3}i + C_2 &= \left. \frac{4s - 4}{(s^2 + 1)} \right|_{s=\sqrt{3}i} = -2\sqrt{3}i + 2 \\
\Rightarrow B_2 &= -2 \quad \text{and } C_2 = 2 \\
\text{and } p_2(s) &= \frac{1}{s^2 + 3}(4s - 4 - (-2s + 2)(s^2 + 1)) \\
&= \frac{2s^3 - 2s^2 + 6s - 6}{s^2 + 3} = 2s - 2.
\end{aligned}$$

A third application of Theorem 1 on the remainder term  $\frac{2s - 2}{(s^2 + 3)(s^2 + 1)}$  gives

$$\begin{aligned}
\frac{2s - 2}{(s^2 + 3)(s^2 + 1)} &= \frac{B_3s + C_3}{(s^2 + 3)} + \frac{p_3(s)}{(s^2 + 1)} \\
\text{where } B_3\sqrt{3}i + C_3 &= \left. \frac{2s - 2}{(s^2 + 1)} \right|_{s=\sqrt{3}i} = -\sqrt{3}i + 1 \\
\Rightarrow B_3 &= -1 \quad \text{and } C_3 = 1 \\
\text{and } p_3(s) &= \frac{1}{s^2 + 3}(2s - 2 - (-s + 1)(s^2 + 1)) \\
&= \frac{s^3 - s^2 + 3s - 3}{s^2 + 3} = s - 1.
\end{aligned}$$

Thus the  $(s^2 + 3)$ -chain is

The $s^2 + 3$ -chain	
$\frac{8s + 8s^2}{(s+3)^3(s^2 + 1)}$	$\frac{12 - 4s}{(s^2 + 3)^3}$
$\frac{4(s - 1)}{(s^2 + 3)^2(s^2 + 1)}$	$\frac{2 - 2s}{(s^2 + 3)^2}$
$\frac{2(s - 1)}{(s^2 + 3)(s^2 + 1)}$	$\frac{1 - s}{s^2 + 3}$
$\frac{s - 1}{s^2 + 1}$	

5. Note that  $s^2 + 2s + 2 = (s + 1)^2 + 1$  so  $s = -1 \pm i$  are the roots of  $s^2 + 2s + 2$ . We will use the root  $s = -1 + i$  for the partial fraction algorithm. Applying Theorem 1 gives

$$\frac{1}{(s^2 + 2s + 2)^2(s^2 + 2s + 3)^2} = \frac{B_1s + C_1}{(s^2 + 2s + 2)^2} + \frac{p_1(s)}{(s^2 + 2s + 2)(s^2 + 2s + 3)^2}$$

$$\text{where } B_1(-1 + i) + C_1 = \left. \frac{1}{(s^2 + 2s + 3)^2} \right|_{s=-1+i}$$

$$= \frac{1}{((-1 + i)^2 + 2)^2} = 1$$

$$\Rightarrow B_1 = 0 \quad \text{and } C_1 = 1$$

$$\begin{aligned} \text{and } p_1(s) &= \frac{1 - (1)(s^2 + 2s + 3)^2}{s^2 + 2s + 2} \\ &= \frac{-(s^2 + 2s + 4)(s^2 + 2s + 2)}{s^2 + 2s + 2} \\ &= -(s^2 + 2s + 4). \end{aligned}$$

Now apply Theorem 1 to the remainder term  $\frac{-(s^2 + 2s + 4)}{(s^2 + 2s + 2)(s^2 + 2s + 3)^2}$ .

$$\frac{-(s^2 + 2s + 4)}{(s^2 + 2s + 2)(s^2 + 2s + 3)^2} = \frac{B_2s + C_2}{(s^2 + 2s + 2)} + \frac{p_2(s)}{(s^2 + 2s + 3)^2}$$

where  $B_2(-1 + i) + C_2 = \frac{-(s^2 + 2s + 4)}{(s^2 + 2s + 3)} \Big|_{s=-1+i} = -2$

$$\Rightarrow B_2 = 0 \text{ and } C_2 = -2$$

and  $p_2(s) = \frac{-(s^2 + 2s + 4) - (-2)(s^2 + 2s + 3)^2}{s^2 + 2s + 2}$

$$= \frac{(2(s+1)^2 + 5)((s+1)^2 + 1)}{s^2 + 2s + 2}$$

$$= 2s^2 + 4s + 7.$$

Thus the  $(s^2 + 2s + 2)$ -chain is

The $s^2 + 2s + 2$ -chain	
$\frac{1}{(s^2 + 2s + 2)^2(s^2 + 2s + 3)^2}$	$\frac{1}{(s^2 + 2s + 2)^2}$
$\frac{-(s^2 + 2s + 4)}{(s^2 + 2s + 2)(s^2 + 2s + 3)^2}$	$\frac{-2}{s^2 + 2s + 2}$
$\frac{2s^2 + 4s + 7}{(s^2 + 2s + 3)^2}$	

7. Use Theorem 1 of Section 2.3 to compute the  $(s - 3)$ -chain:

$$\frac{s}{(s^2 + 1)(s - 3)} = \frac{A_1}{s - 3} + \frac{p_1(s)}{s^2 + 1}$$

where  $A_1 = \frac{s}{s^2 + 1} \Big|_{s=3} = \frac{3}{10}$

and  $p_1(s) = \frac{1}{s - 3}(s - (3/10)(s^2 + 1)) = \frac{1}{10(s - 3)}(-3s^2 + 10s - 3)$

$$= \frac{-3s + 1}{10}$$

Since the remainder term  $\frac{-3s + 1}{10(s^2 + 1)}$  is already a simple partial fraction, we conclude

$$\frac{s}{(s^2 + 1)(s - 3)} = \frac{1}{10} \left( \frac{3}{s - 3} + \frac{1 - 3s}{s^2 + 1} \right)$$

9. We compute the  $(s^2 + 4)$ -chain:

$$\frac{9s^2}{(s^2+4)^2(s^2+1)} = \frac{B_1s+C_1}{(s^2+4)^2} + \frac{p_1(s)}{(s^2+4)(s^2+1)}$$

$$\begin{aligned} \text{where } B_1(2i)+C_1 &= \left. \frac{9s^2}{s^2+1} \right|_{s=2i} = 12 \\ &\Rightarrow B_1 = 0 \quad \text{and } C_1 = 12 \\ \text{and } p_1(s) &= \frac{1}{s^2+4}(9s^2-12(s^2+1)) = \frac{-3(s^2+4)}{s^2+4} \\ &= -3 \end{aligned}$$

Now compute the next term in  $s^2+4$ -chain.

$$\begin{aligned} \frac{-3}{(s^2+4)(s^2+1)} &= \frac{B_2s+C_2}{s^2+4} + \frac{p_2(s)}{s^2+1} \\ \text{where } B_2(2i)+C_2 &= \left. \frac{-3}{s^2+1} \right|_{s=2i} = 1 \\ &\Rightarrow B_2 = 0 \quad \text{and } C_2 = 1 \\ \text{and } p_2(s) &= \frac{1}{s^2+4}(-3-(s^2+1)) \\ &= \frac{-(s^2+1)}{s^2+1} = -1. \end{aligned}$$

Since the remainder term  $\frac{-1}{s^2+1}$  is a simple partial fraction, we conclude that the complete partial fraction decomposition is

$$\frac{9s^2}{(s^2+4)^2(s^2+1)} = \frac{12}{(s^2+4)^2} + \frac{1}{s^2+4} - \frac{1}{s^2+1}$$

**11.** Use Theorem of Section 2.3 1 to compute the  $(s-3)$ -chain:

$$\begin{aligned} \frac{2}{(s^2-6s+10)(s-3)} &= \frac{A_1}{s-3} + \frac{p_1(s)}{(s^2-6s+10)} \\ \text{where } A_1 &= \left. \frac{2}{(s^2-6s+10)} \right|_{s=3} = 2 \\ \text{and } p_1(s) &= \frac{1}{s-3}(2-(2)(s^2-6s+10)) = \frac{-2s^2+12s-18}{s-3} \\ &= -2s+6 \end{aligned}$$

Since the remainder term  $\frac{-2s+6}{s^2-6s+10}$  is a simple partial fraction, we conclude that the complete partial fraction decomposition is  $\frac{2}{(s^2-6s+10)(s-3)} = \frac{2}{s-3} + \frac{6-2s}{(s-3)^2+1}$

- 13.** Note that  $s^2-4s+8 = (s-2)^2+2$  so  $s = 2 \pm 2i$  are the roots of  $s^2-4s+8$ . We will use the root  $s = 2+2i$  to compute the  $(s^2-4s+8)$ -chain. Applying Theorem 1 gives

$$\begin{aligned} \frac{25}{(s^2-4s+8)^2(s-1)} &= \frac{B_1s+C_1}{(s^2-4s+8)^2} \\ &\quad + \frac{p_1(s)}{(s^2-4s+8)(s-1)} \end{aligned}$$

$$\begin{aligned} \text{where } B_1(2+2i)+C_1 &= \left. \frac{25}{s-1} \right|_{s=2+2i} \\ &= \frac{25}{2i+1} = 5-10i \\ \Rightarrow B_1 &= -5 \quad \text{and } C_1 = 15 \\ \text{and } p_1(s) &= \frac{25 - (-5s+15)(s-1)}{s^2-4s+8} \\ &= \frac{(5)(s^2-4s+8)}{s^2-4s+8} \\ &= 5. \end{aligned}$$

Now apply Theorem 1 to the remainder term  $\frac{5}{(s^2-4s+8)(s-1)}$ .

$$\frac{5}{(s^2-4s+8)(s-1)} = \frac{B_2s+C_2}{(s^2-4s+8)} + \frac{p_2(s)}{s-1}$$

$$\begin{aligned} \text{where } B_2(2+2i)+C_2 &= \left. \frac{5}{s-1} \right|_{s=2+2i} = 1-2i \\ \Rightarrow B_2 &= -1 \quad \text{and } C_2 = 3 \\ \text{and } p_2(s) &= \frac{5 - (3-s)(s-1)}{s^2-4s+8} \\ &= \frac{(1)(s^2-4s+8)}{s^2-4s+8} \\ &= 1. \end{aligned}$$

Thus the partial fraction expansion is  $\frac{25}{(s^2 - 4s + 8)^2(s - 1)} = \frac{-5s + 15}{(s^2 - 4s + 8)^2} + \frac{-s + 3}{s^2 - 4s + 8} + \frac{1}{s - 1}$

15. Note that  $s^2 + 4s + 5 = (s + 2)^2 + 1$  so  $s = -2 \pm i$  are the roots of  $s^2 + 4s + 5$ . We will use the root  $s = -2 + i$  to compute the  $(s^2 + 4s + 5)$ -chain. Applying Theorem 1 gives

$$\begin{aligned} \frac{s + 1}{(s^2 + 4s + 5)^2(s^2 + 4s + 6)^2} &= \frac{B_1s + C_1}{(s^2 + 4s + 5)^2} \\ &\quad + \frac{p_1(s)}{(s^2 + 4s + 5)(s^2 + 4s + 6)^2} \end{aligned}$$

$$\begin{aligned} \text{where } B_1(-2 + i) + C_1 &= \left. \frac{s + 1}{(s^2 + 4s + 6)^2} \right|_{s=-2+i} \\ &= -1 + i \end{aligned}$$

$$\Rightarrow B_1 = 1 \quad \text{and } C_1 = 1$$

$$\begin{aligned} \text{and } p_1(s) &= \frac{s + 1 - (s + 1)(s^2 + 4s + 6)^2}{s^2 + 4s + 5} \\ &= \frac{-(s + 1)((s^2 + 4s + 6)^2 - 1)}{s^2 + 4s + 5} \\ &= \frac{-(s + 1)(s^2 + 4s + 7)(s^2 + 4s + 5)}{s^2 + 4s + 5} \\ &= -(s + 1)(s^2 + 4s + 7). \end{aligned}$$

Now apply Theorem 1 to the remainder term  $\frac{-(s + 1)(s^2 + 4s + 7)}{(s^2 + 4s + 5)(s^2 + 4s + 6)^2}$ .

$$\frac{-(s + 1)(s^2 + 4s + 7)}{(s^2 + 4s + 5)(s^2 + 4s + 6)^2} = \frac{B_2s + C_2}{(s^2 + 4s + 5)} + \frac{p_2(s)}{(s^2 + 4s + 6)^2}$$

$$\text{where } B_2(-2 + i) + C_2 = \left. \frac{-(s + 1)(s^2 + 4s + 7)}{(s^2 + 4s + 6)^2} \right|_{s=-2+i} = 2 - 2i$$

$$\Rightarrow B_2 = -2 \text{ and } C_2 = -2$$

$$\begin{aligned} \text{and } p_2(s) &= \frac{-(s + 1)(s^2 + 4s + 7) - (-2s - 2)(s^2 + 4s + 6)^2}{s^2 + 4s + 5} \\ &= \frac{(s + 1)(2(s^2 + 4s + 6) + 1)(s^2 + 4s + 5)}{s^2 + 4s + 5} \\ &= (s + 1)(2(s^2 + 4s + 6) + 1). \end{aligned}$$

The remainder term is

$$\frac{(s+1)(2(s^2+4s+6)+1)}{(s^2+4s+6)^2} = \frac{s+1}{(s^2+4s+6)^2} + \frac{2s+2}{s^2+4s+6}$$

so the partial fraction expansion of the entire rational function is

$$\begin{aligned} \frac{s+1}{(s^2+4s+5)^2(s^2+4s+6)^2} &= \frac{s+1}{(s^2+4s+6)^2} + \frac{2s+2}{s^2+4s+6} \\ &\quad + \frac{s+1}{(s^2+4s+5)^2} - \frac{2s+2}{s^2+4s+5} \end{aligned}$$

**17.** Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned} \mathcal{L}\{y'' + 4y' + 4y\} &= \mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} \\ &= s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 4Y(s) \\ &= (s^2 + 4s + 4)Y(s) - 1. \end{aligned}$$

Since  $\mathcal{L}\{4\cos 2t\} = 4s/(s^2 + 4)$  we get the algebraic equation

$$(s+4)^2Y(s) - 1 = \frac{4s}{s^2+4}.$$

Hence,

$$Y(s) = \frac{1}{(s+2)^2} + \frac{4s}{(s^2+4)(s+2)^2}.$$

The  $(s^2 + 4)$ -chain for the second term is

The $(s^2 + 4)$ -chain	
$\frac{4s}{(s^2+4)(s+2)^2}$	$\frac{1}{s^2+4}$
$\frac{-1}{(s+2)^2}$	

Thus

$$Y(s) = \frac{1}{s^2+4}$$

and Table 2.2 gives  $y(t) = \frac{1}{2} \sin 2t$

**19.** Apply the Laplace transform to both sides. For the left hand side we get

$$\begin{aligned}
\mathcal{L}\{y'' + 4y\} &= \mathcal{L}\{y''\} + 4\mathcal{L}\{y\} \\
&= s^2Y(s) - sy(0) - y'(0) + 4Y(s) \\
&= (s^2 + 4)Y(s) - 1.
\end{aligned}$$

Since  $\mathcal{L}\{\sin 3t\} = 3/(s^2 + 9)$  we get the algebraic equation

$$(s^2 + 4)Y(s) - 1 = \frac{3}{s^2 + 9}.$$

Hence,

$$Y(s) = \frac{1}{s^2 + 4} + \frac{3}{(s^2 + 9)(s^2 + 4)}.$$

Using quadratic partial fraction recursion we obtain the  $(s^2 + 9)$ -chain

The $(s^2 + 9)$ -chain	
$\frac{3}{(s^2 + 9)(s^2 + 4)}$	$\frac{-3/5}{s^2 + 9}$
$\frac{3/5}{s^2 + 4}$	

Thus

$$Y(s) = \frac{8}{5} \frac{1}{s^2 + 4} - \frac{3}{5} \frac{1}{s^2 + 9} = \frac{4}{5} \frac{2}{s^2 + 4} - \frac{1}{5} \frac{3}{s^2 + 9}$$

and Table 2.2 gives  $y(t) = \frac{4}{5} \sin 2t - \frac{1}{5} \sin 3t$

## SECTION 2.5

1.  $\mathcal{L}^{-1}\{-5/s\} = -5\mathcal{L}^{-1}\{1/s\} = -5$
3.  $\mathcal{L}^{-1}\left\{\frac{3}{s^2} - \frac{4}{s^3}\right\} = 3\mathcal{L}^{-1}\{1/s^2\} - 2\mathcal{L}^{-1}\{2/s^3\} = 3t - 2t^2$
5.  $\mathcal{L}^{-1}\left\{\frac{3s}{s^2 + 4}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 2^2}\right\} = 3 \cos 2t$



7. First, we have  $s^2 + 6s + 9 = (s+3)^2$ . Partial fractions gives  $\frac{2s-5}{s^2+6s+9} = \frac{-11}{(s+3)^2} + \frac{2}{s+3}$ . So  $\mathcal{L}^{-1} \left\{ \frac{2s-5}{(s+3)^2} \right\} = -11te^{-3t} + 2e^{-3t}$
9.  $\frac{6}{s^2+2s-8} = \frac{6}{(s-2)(s+4)} = \frac{-1}{s+4} + \frac{1}{s-2}$ . So  $\mathcal{L}^{-1} \left\{ \frac{6}{s^2+2s-8} \right\} = \frac{-1}{e^{2t}} + \frac{1}{e^{-4t}}$
11.  $\frac{2s^2-5s+1}{(s-2)^4} = \frac{-1}{(s-2)^4} + \frac{3}{(s-2)^3} + \frac{2}{(s-2)^2}$ . So  $\mathcal{L}^{-1} \left\{ \frac{2s^2-5s+1}{(s-2)^4} \right\} = \frac{-1}{6}t^3e^{2t} + \frac{3}{2}t^2e^{2t} + 2te^{2t}$
13.  $\frac{4s^2}{(s-1)^2(s+1)^2} = \frac{1}{(s-1)^2} + \frac{1}{s-1} + \frac{1}{(s+1)^2} - \frac{1}{s+1}$ . So  $\mathcal{L}^{-1} \left\{ \frac{4s^2}{(s-1)^2(s+1)^2} \right\} = te^t + e^t + te^{-t} - e^{-t}$
15.  $\frac{8s+16}{(s^2+4)(s-2)^2} = \frac{4}{(s-2)^2} - \frac{1}{s-2} + \frac{s}{s^2+4} - \frac{2}{s^2+4}$ . So  $\mathcal{L}^{-1} \left\{ \frac{8s+16}{(s^2+4)(s-2)^2} \right\} = 4te^{2t} - e^{2t} + \cos 2t - \sin 2t$
17.  $\frac{12}{s^2(s+1)(s-2)} = \frac{-6}{s^2} + \frac{3}{s} - \frac{4}{s+1} + \frac{1}{s-2}$ . So  $\mathcal{L}^{-1} \left\{ \frac{12}{s^2(s+1)(s-2)} \right\} = 3 - 6t + e^{2t} - 4e^{-t}$
19. First we have  $s^2 + 2s + 5 = (s+1)^2 + 4$ . So  $\frac{2s}{s^2+2s+5} = \frac{2s}{(s+1)^2+4} = \frac{2(s+1)-2}{(s+1)^2+4} = 2\frac{s+1}{(s+1)^2+4} - \frac{2}{(s+1)^2+4}$ . The First Translation principle gives  $\mathcal{L}^{-1} \left\{ \frac{2s}{s^2+2s+5} \right\} = 2\mathcal{L}^{-1} \left\{ \frac{s+1}{(s+1)^2+4} \right\} - \mathcal{L}^{-1} \left\{ \frac{2}{(s+1)^2+4} \right\} = 2e^{-t} \cos 2t - e^{-t} \sin 2t$
21.  $\frac{s-1}{s^2-8s+17} = \frac{s-4}{(s-4)^2+1} + 3\frac{1}{(s-4)^2+1}$ . Thus  $\mathcal{L}^{-1} \left\{ \frac{s-1}{s^2-8s+17} \right\} = e^{4t} \cos t + 3e^{4t} \sin t$
23.  $\frac{s-1}{s^2-2s+10} = \frac{s-1}{(s-1)^2+3^2}$ . Thus  $\mathcal{L}^{-1} \left\{ \frac{s-1}{s^2-2s+10} \right\} = e^t \cos 3t$
25.  $\mathcal{L}^{-1} \left\{ \frac{8s}{(s^2+4)^2} \right\} = 8\mathcal{L}^{-1} \left\{ \frac{s}{(s^2+2^2)^2} \right\} = \frac{8}{2 \cdot 2^2} (2t \sin 2t) = 2t \sin 2t$
27. We first complete the square  $s^2 + 4s + 5 = (s+2)^2 + 1$ . By the translation principle we get  $\mathcal{L}^{-1} \left\{ \frac{2s}{(s^2+4s+5)^2} \right\} = 2\mathcal{L}^{-1} \left\{ \frac{(s+2)-2}{((s+2)^2+1)^2} \right\} =$

$$2e^{-2t} \left( \mathcal{L}^{-1} \left\{ \frac{s}{(s^2+1)^2} \right\} - 2\mathcal{L}^{-1} \left\{ \frac{1}{(s^2+1)^2} \right\} \right) = 2e^{-2t} \left( \frac{1}{2}t \sin t - 2\left(\frac{1}{2}(\sin t - t \cos t)\right) \right) = 2te^{-2t} \cos t + (t-2)e^{-2t} \sin t$$

**29.** We first complete the square  $s^2+8s+17 = (s+4)^2+1$ . By the translation

$$\begin{aligned} \text{principle we get } \mathcal{L}^{-1} \left\{ \frac{2s}{(s^2+8s+17)^2} \right\} &= 2\mathcal{L}^{-1} \left\{ \frac{(s+4)-4}{((s+4)^2+1)^2} \right\} = \\ 2e^{-4t} \left( \mathcal{L}^{-1} \left\{ \frac{s}{(s^2+1)^2} \right\} - 4\mathcal{L}^{-1} \left\{ \frac{1}{(s^2+1)^2} \right\} \right) &= 2e^{-4t} \left( \frac{1}{2}t \sin t - 4\left(\frac{1}{2}(\sin t - t \cos t)\right) \right) = \\ 4te^{-4t} \cos t + (t-4)e^{-4t} \sin t \end{aligned}$$

**31.** We first complete the square  $s^2-2s+5 = (s-1)^2+2^2$ . By the translation

$$\begin{aligned} \text{principle we get } \mathcal{L}^{-1} \left\{ \frac{1}{(s^2-2s+5)^3} \right\} &= \mathcal{L}^{-1} \left\{ \frac{1}{((s-1)^2+2^2)^3} \right\} = \\ e^t \left( \mathcal{L}^{-1} \left\{ \frac{1}{(s^2+2^2)^3} \right\} \right) &= e^t \frac{1}{2} \left( \mathcal{L}^{-1} \left\{ \frac{2}{(s^2+2^2)^3} \right\} \right) \\ = e^t \frac{1}{2 \cdot 8 \cdot 2^4} ((3-(2t)^2) \sin 2t - 6t \cos 2t) &= \\ = \frac{1}{256} ((3-4t^2)e^t \sin 2t - 6te^t \cos 2t) \end{aligned}$$

**33.** We first complete the square  $s^2-8s+17 = (s-4)^2+1$ . By the translation

$$\begin{aligned} \text{principle we get } \mathcal{L}^{-1} \left\{ \frac{s-4}{(s^2-8s+17)^4} \right\} &= \mathcal{L}^{-1} \left\{ \frac{s-4}{((s-4)^2+1)^4} \right\} = \\ e^{4t} \left( \mathcal{L}^{-1} \left\{ \frac{s}{(s^2+1)^4} \right\} \right) &= e^{4t} \frac{1}{48} ((3t-t^3) \sin t - 3t^2 \cos t) \\ = \frac{1}{48} ((-t^3+3t)e^{4t} \cos t - 3t^2e^{4t} \sin t) \end{aligned}$$

**35.** Apply the Laplace transform to get

$$s^2Y(s) - s + 1 + Y(s) = \frac{4}{s^2+1}.$$

Solving for  $Y(s)$  we get

$$Y(s) = \frac{s-1}{s^2+1} + \frac{4}{(s^2+1)^2}.$$

We use Table 2.5 to get

$$y(t) = \cos t - \sin t + 2(\sin t - t \cos t) = \cos t + \sin t - 2t \cos t.$$

**37.** Apply the Laplace transform to get

$$\begin{aligned}
(s^2 - 3)Y(s) &= 4 \left( \frac{s}{s^2 + 1} \right)'' \\
&= 4 \left( \frac{1 - s^2}{(s^2 + 1)^2} \right)' \\
&= \frac{8s(s^2 - 3)}{(s^2 + 1)^3}
\end{aligned}$$

It follows that  $Y(s) = \frac{8s}{(s^2 + 1)^3}$ . Table 2.5 now gives

$$y(t) = t \sin t - t^2 \cos t.$$

- 39.** Compute the partial fraction  $\frac{1}{(s-a)(s-b)} = \frac{1/(a-b)}{s-a} + \frac{1/(b-a)}{s-b}$ .

Then

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s-a)(s-b)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1/(a-b)}{s-a} + \frac{1/(b-a)}{s-b} \right\} = \frac{e^{at}}{a-b} + \frac{e^{bt}}{b-a}.$$

- 41.** Apply the inverse Laplace transform to the partial fraction expansion

$$\frac{1}{(s-a)(s-b)(s-c)} = \frac{1}{(a-b)(a-c)} \frac{1}{s-a} + \frac{1}{(b-a)(b-c)} \frac{1}{s-b} + \frac{1}{(c-a)(c-b)} \frac{1}{s-c}.$$

- 43.** Apply the inverse Laplace transform to the partial fraction expansion

$$\frac{s^2}{(s-a)(s-b)(s-c)} = \frac{a^2}{(a-b)(a-c)} \frac{1}{s-a} + \frac{b^2}{(b-a)(b-c)} \frac{1}{s-b} + \frac{c^2}{(c-a)(c-b)} \frac{1}{s-c}.$$

- 45.** This is directly from Table 2.4.

- 47.** This is directly from Table 2.4.

- 49.** Apply the inverse Laplace transform to the partial fraction expansion

$$\frac{s^2}{(s-a)^3} = \frac{((s-a)+a)^2}{(s-a)^3} = \frac{1}{s-a} + \frac{2a}{(s-a)^2} + \frac{a^2}{(s-a)^3}.$$

## SECTION 2.6

- 1.** The root of  $q(s)$  is 4 with multiplicity 1. Thus  $\mathcal{B}_q = \{e^{4t}\}$

3.  $q(s) = s^2 + 5s = s(s+5)$ , hence the roots of  $q(s)$  are 0 and  $-5$  each with multiplicity 1. Thus  $\mathcal{B}_q = \{1, e^{-5t}\}$
5.  $q(s) = s^2 - 6s + 9 = (s-3)^2$ , hence the root of  $q(s)$  is 3 with multiplicity 2. Thus  $\mathcal{B}_q = \{e^{3t}, te^{3t}\}$
7.  $q(s) = s^2 - s - 6 = (s-3)(s+2)$ , hence the roots of  $q(s)$  are 3 and  $-2$  each with multiplicity 1. Thus  $\mathcal{B}_q = \{e^{3t}, e^{-2t}\}$
9.  $q(s) = 6s^2 - 11s + 4 = (3s-4)(2s-1)$  so the roots are  $4/3$  and  $1/2$ , each with multiplicity 1. Hence  $\mathcal{B}_q = \{e^{4t/3}, e^{t/2}\}$
11. The quadratic formula gives roots  $\frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}$ . Hence  $\mathcal{B}_q = \{e^{(2+\sqrt{3})t}, e^{(2-\sqrt{3})t}\}$
13.  $q(s) = 4s^2 + 12s + 9 = (2s+3)^2$ ; so the root is  $-3/2$  with multiplicity 2 and hence  $\mathcal{B}_q = \{e^{-3t/2}, te^{-3t/2}\}$
15.  $q(s) = 4s^2 + 25 = 4(s^2 + (5/2)^2)$ . Therefore  $q(s)$  has complex roots  $\pm \frac{5}{2}i$ . Hence  $\mathcal{B}_q = \{\cos(5t/2), \sin(5t/2)\}$
17.  $q(s) = s^2 - 2s + 5 = s^2 - 2s + 1 + 4 = (s-1)^2 + 2^2$ . Therefore  $q(s)$  has complex roots  $1 \pm 2i$ . Hence  $\mathcal{B}_q = \{e^t \cos 2t, e^t \sin 2t\}$
19.  $q(s)$  has root  $-3$  with multiplicity 4. Hence  $\mathcal{B}_q = \{e^{-3t}, te^{-3t}, t^2e^{-3t}, t^3e^{-3t}\}$ .
21.  $q(s) = (s-1)^3$  has root 1 with multiplicity 3. Hence  $\mathcal{B}_q = \{e^t, te^t, t^2e^t\}$ .
23. We complete the square to get  $q(s) = ((s+2)^2 + 1)^2$ . Thus  $q(s)$  has complex root  $-2 \pm i$  with multiplicity 2. It follows that  $\mathcal{B}_q = \{e^{-2t} \cos t, e^{-2t} \sin t, te^{-2t} \cos t, te^{-2t} \sin t\}$ .
25. The complex roots of  $q(s)$  are  $\pm i$  with multiplicity 4. Thus  $\mathcal{B}_q = \{\cos t, \sin t, t \cos t, t \sin t, t^2 \cos t, t^2 \sin t, t^3 \cos t, t^3 \sin t\}$

## SECTION 2.7

1. Yes.
3. Yes;  $\frac{t}{e^t} = te^{-t}$ .
5. Yes;  $t \sin(4t - \frac{\pi}{4}) = t(\frac{\sqrt{2}}{2} \sin 4t - \frac{\sqrt{2}}{2} \cos 4t)$ .

7. No.
9. No;  $t^{\frac{1}{2}}$  is not a polynomial.
11. No.  $\frac{1}{\sin 2t}$  is not in  $\mathcal{E}$ .
13.  $s^4 - 1 = (s^2 - 1)(s^2 + 1) = (s - 1)(s + 1)(s^2 + 1)$ ; The roots are 1,  $-1$ , and  $\pm i$  each with multiplicity 1. Hence  $\mathcal{B}_q = \{e^t, e^{-t}, \cos t, \sin t\}$ .
15. The roots are 1 with multiplicity 3 and  $-7$  with multiplicity 2. Hence  $\mathcal{B}_q = \{e^t, te^t, t^2e^t, e^{-7t}, te^{-7t}\}$ .
17. The roots are  $-2$  with multiplicity 3 and  $\pm 2i$  with multiplicity 2. Hence  $\mathcal{B}_q = \{e^{-2t}, te^{-2t}, t^2e^{-2t}, \cos 2t, \sin 2t, t \cos 2t, t \sin 2t\}$ .
19. We must gather the roots together to get the correct multiplicity. Thus  $q(s) = (s - 2)^2(s + 3)^3$ . The roots are 2 with multiplicity 2 and  $-3$  with multiplicity 3. Hence  $\mathcal{B}_q = \{e^{2t}, te^{2t}, e^{-3t}, te^{-3t}, t^2e^{-3t}\}$ .
21. By completing the square we may write  $q(s) = (s + 4)^2((s + 3)^2 + 4)^2$ . The roots are  $-4$  with multiplicity 2 and  $-3 \pm 2i$  with multiplicity 2. Hence  $\mathcal{B}_q = \{e^{-4t}, te^{-4t}, e^{-3t} \cos 2t, e^{-3t} \sin 2t, te^{-3t} \cos 2t, te^{-3t} \sin 2t\}$ .
23. First observe that  $s^2 + 2s + 10 = (s + 1)^2 + 3^2$  and hence  $q(s) = (s - 3)^3((s + 1)^2 + 3^2)^2$ . The roots are 3 with multiplicity 3 and  $-1 \pm 3i$  with multiplicity 2. Thus  $\mathcal{B}_q = \{e^{3t}, te^{3t}, t^2e^{3t}, e^{-t} \cos 3t, e^{-t} \sin 3t, te^{-t} \cos 3t, te^{-t} \sin 3t\}$ .
25.  $2s^3 - 5s^2 + 4s - 1 = (2s - 1)(s - 1)^2$ ; hence  $\mathcal{B}_q = \{e^{t/2}, e^t, te^t\}$ .
27.  $s^4 + 5s^2 + 6 = (s^2 + 3)(s^2 + 2)$ ; hence  $\mathcal{B}_q = \{\cos \sqrt{3}t, \sin \sqrt{3}t, \cos \sqrt{2}t, \sin \sqrt{2}t\}$ .
29.  $r_1(s) = \frac{p_1(s)}{q_1(s)}$  with  $\deg p_1(s) < \deg q_1(s)$  and  $r_2(s) = \frac{p_2(s)}{q_2(s)}$  with  $\deg p_2(s) < \deg q_2(s)$ . Thus,  $r_1(s)r_2(s) = \frac{p_1(s)p_2(s)}{q_1(s)q_2(s)}$  and  $\deg(p_1(s)p_2(s)) = \deg p_1(s) + \deg p_2(s) < \deg q_1(s) + \deg q_2(s) = \deg(q_1(s)q_2(s))$ .
31. If  $r(s) \in \mathcal{R}$  then  $r(s) = \frac{p(s)}{q(s)}$  where  $\deg p(s) = m < n = \deg q(s)$ . Then

$$r'(s) = \frac{q(s)p'(s) - q'(s)p(s)}{(q(s))^2},$$

and  $\deg(q(s)p'(s) - q'(s)p(s)) \leq \max(\deg(q(s)q'(s)), \deg(q'(s)p(s))) = \max(n + (m - 1), (n - 1) + m) = n + m - 1 < 2n = \deg(q(s))^2$ . Hence  $r'(s)$  is a proper rational function.

- 33.** By exercise 32 this is true for  $n = 1$ . Now apply induction. If  $n$  is given and we assume the result is true for derivatives of order  $n - 1$ , then  $r^{(n-1)} \in \mathcal{R}_{q^n}$  but not in  $\mathcal{R}_{q^{n-1}}$ . Another application of exercise 32 then shows that  $r^{(n)} = (r^{(n-1)})' \in \mathcal{R}_{q^{n+1}}$  but not in  $\mathcal{R}_{q^n}$ .
- 35.** Observe that  $e^{t-t_0} = e^{-t_0}e^t$ . So the translate of an exponential function is a multiple of an exponential function. Also, if  $f(t)$  is a polynomial so is  $f(t-t_0)$ . By the addition rule for cos we have  $\cos b(t-t_0) = \cos bt \cos bt_0 - \sin bt \sin bt_0$  and similarly for sin. It follows that all these translates remain in  $\mathcal{E}$ . By Exercise 34 the result follows.
- 37.** By linearity of integration it is enough to show this result for  $f_n(t) = t^n e^{at}(c_1 \cos bt + c_2 \sin bt)$ , where  $c_1$  and  $c_2$  are scalars. Let  $I_n(t) = \int f_n(t) dt$ . First assume  $n = 0$ . Then a standard trick using integration by parts twice gives

$$\begin{aligned} I_0(t) : &= \int e^{at}(c_1 \cos bt + c_2 \sin bt) dt \\ &= \frac{1}{a^2 + b^2}((c_1 a - c_2 b) \cos bt + (c_1 b + c_2 a) \sin bt)e^{at}. \end{aligned}$$

Clearly,  $I_0$  is an exponential polynomial. Observe that  $I_0$  is of the same form as  $f_0$ . Now assume  $n > 0$ . Using integration by parts with  $u = t^n$  and  $dv = (c_1 \cos bt + c_2 \sin bt) dt$  we have  $I_n(t) = \int t^n (c_1 \cos bt + c_2 \sin bt) dt = t^n I_0(t) - n \int t^{n-1} I_0(t) dt$ . Since  $I_0 \in \mathcal{E}$  so are  $t^n I_0$  and  $t^{n-1} I_0$ . By induction we have  $\int t^{n-1} I_0(t) dt \in \mathcal{E}$ . It now follows that  $I_n \in \mathcal{E}$ .

- 39.** It is enough to show this for each  $f \in \mathcal{B}_q$  since differentiation is linear and  $\mathcal{E}_q = \text{Span } \mathcal{B}_q$ . Suppose  $f(t) = t^n e^{at} \cos bt$ . Then

$$f'(t) = nt^{n-1}e^{at} \cos bt - bt^n e^{at} \sin bt + at^n e^{at} \cos bt.$$

The derivative  $f'(t)$  is a linear combination of the simple exponential polynomials  $t^{n-1}e^{at} \cos bt$ ,  $t^n e^{at} \sin bt$ , and  $t^n e^{at} \cos bt$  each of which are in  $\mathcal{B}_q$ . Hence  $f'(t) \in \mathcal{E}_q$ . A similar argument applies to  $t^n e^{at} \sin bt$ .

- 41.** Observe that  $e^{t-t_0} = e^{-t_0}e^t$ . So the translate of an exponential function is a multiple of an exponential function. Also, if  $p(t)$  is a polynomial of degree  $n$  the binomial theorem implies that  $p(t-t_0)$  is a polynomial of degree  $n$ . By the addition rule for cos we have  $\cos b(t-t_0) = \cos bt \cos bt_0 - \sin bt \sin bt_0$  and similarly for sin. Thus if  $f(t) = t^n e^{at} \cos bt \in \mathcal{B}_q$  then  $f(t-t_0)$  is a linear combination of terms in  $\mathcal{B}_q$ . Since  $\mathcal{E}_q = \text{Span } \mathcal{B}_q$  it follows that all translates remain in  $\mathcal{E}_q$ .

**SECTION 2.8****1.**

$$\begin{aligned}
t * t &= \int_0^t x(t-x) dx \\
&= \int_0^t (tx - x^2) dx \\
&= \left( t \frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_{x=0}^{x=t} \\
&= t \frac{t^2}{2} - \frac{t^3}{3} = \frac{t^3}{6}
\end{aligned}$$

**3.**

$$\begin{aligned}
3 * \sin t = \sin t * 3 &= \int_0^t (\sin x)(3) dx \\
&= -3 \cos x \Big|_{x=0}^{x=t} \\
&= -3(\cos t - \cos 0) \\
&= -3 \cos t + 3
\end{aligned}$$

**5.** From the Convolution table we get

$$\begin{aligned}
\sin 2t * e^{3t} &= \frac{1}{3^2 + 2^2} (2e^{3t} - 2 \cos 2t - 3 \sin 2t) \\
&= \frac{1}{13} (2e^{3t} - 2 \cos 2t - 3 \sin 2t).
\end{aligned}$$

**7.** From the Convolution table we get

$$\begin{aligned}
t^2 * e^{-6t} &= \frac{2}{(-6)^3} (e^{-6t} - (-6 - 6t + (36t^2)/2)) \\
&= \frac{1}{108} (18t^2 - 6t - 6 + e^{-6t}).
\end{aligned}$$

**9.** From the Convolution table we get

$$\begin{aligned}
e^{2t} * e^{-4t} &= \frac{e^{2t} - e^{-4t}}{2 - (-4)} \\
&= \frac{1}{6} (e^{2t} - e^{-4t}).
\end{aligned}$$

11.

$$\begin{aligned}
 \mathcal{L}\{e^{at} * \sin bt\}(s) &= \frac{1}{s-a} \frac{b}{s^2+b^2} \\
 &= \frac{1}{s^2+b^2} \left( \frac{b}{s-a} - \frac{bs+ba}{s^2+b^2} \right) \\
 &= \frac{1}{s^2+b^2} (b\mathcal{L}\{e^{at}\} - (b\mathcal{L}\{\cos bt\} + a\mathcal{L}\{\sin bt\}))
 \end{aligned}$$

Thus

$$e^{at} * \sin bt = \frac{1}{a^2+b^2}(be^{at} - b\cos bt - a\sin bt).$$

13. First assume  $a \neq b$ . Then

$$\begin{aligned}
 \mathcal{L}\{\sin at * \sin bt\} &= \frac{a}{s^2+a^2} \frac{b}{s^2+b^2} \\
 &= \frac{1}{b^2-a^2} \left( \frac{ab}{s^2+a^2} - \frac{ab}{s^2+b^2} \right)
 \end{aligned}$$

From this it follows that

$$\sin at * \sin bt = \frac{1}{b^2-a^2}(b\sin at - a\sin bt).$$

Now assume  $a = b$ . Then

$$\mathcal{L}\{\sin at * \sin at\} = \frac{a^2}{(s^2+a^2)^2}$$

By Table 2.5 in Section 2.5 we get

$$\mathcal{L}\{\sin at * \sin at\} = \frac{1}{2a}(\sin at - at \cos at).$$

15. First assume  $a \neq b$ . Then

$$\begin{aligned}
 \mathcal{L}\{\cos at * \cos bt\} &= \frac{s}{s^2+a^2} \frac{s}{s^2+b^2} \\
 &= \frac{1}{b^2-a^2} \left( \frac{-a^2}{s^2+a^2} + \frac{b^2}{s^2+b^2} \right)
 \end{aligned}$$

From this it follows that

$$\cos at * \cos bt = \frac{1}{b^2-a^2}(-a\sin at + b\sin bt).$$



Now assume  $a = b$ . Then

$$\begin{aligned}\mathcal{L}\{\cos at * \cos at\} &= \frac{s^2}{(s^2 + a^2)^2} \\ &= \frac{1}{s^2 + a^2} - \frac{a^2}{(s^2 + a^2)^2}.\end{aligned}$$

By Table 2.5 we get

$$\mathcal{L}\{\cos at * \cos at\} = \frac{1}{a} \sin at - \frac{1}{2a}(\sin at - at \cos at) = \frac{1}{2a}(\sin at + at \cos at).$$

**17.**  $f(t) = t^2 * \sin 2t$  so  $F(s) = \frac{2}{s^3} \frac{2}{s^2 + 4} = \frac{4}{s^3(s^2 + 4)}.$

**19.**  $f(t) = t^3 * e^{-3t}$  so  $F(s) = \frac{6}{s^4} \frac{1}{s + 3} = \frac{6}{s^4(s + 3)}$

**21.**  $f(t) = \sin 2t * \sin 2t$  so  $F(s) = \frac{2}{s^2 + 2^2} \frac{2}{s^2 + 2^2} = \frac{4}{(s^2 + 4)^2}$

**23.**

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 6s + 5}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-5)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} \\ &= e^t * e^{5t} \\ &= \frac{e^t - e^{5t}}{1-5} \\ &= \frac{1}{4}(-e^t + e^{5t})\end{aligned}$$

**25.**

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{s}{(s^2 + 1)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} * \mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\} \\ &= \sin t * \cos t \\ &= \frac{1}{2}t \sin t\end{aligned}$$

**27.**

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{2}{(s-3)(s^2+4)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} * \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\
&= e^{3t} * \sin 2t \\
&= \frac{1}{13}(2e^{3t} - 2\cos 2t - 3\sin 2t)
\end{aligned}$$

29.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s-a)(s-b)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-b}\right\} \\
&= e^{at} * e^{bt} \\
&= \frac{e^{at} - e^{bt}}{a-b}.
\end{aligned}$$

31.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{G(s)}{s^2+2}\right\} &= \mathcal{L}^{-1}\{G(s)\} * \mathcal{L}^{-1}\left\{\frac{s}{s^2+\sqrt{2}}\right\} \\
&= g(t) * \cos(\sqrt{2})t \\
&= \int_0^t g(x) \cos \sqrt{2}(t-x) dx
\end{aligned}$$

33. We apply the input integral principle twice:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} &= \int_0^t \sin x dx \\
&= -\cos x \Big|_0^t \\
&= -\cos t + 1
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} &= \int_0^t 1 - \cos x dx \\
&= t - \sin t
\end{aligned}$$

35. We apply the input integral principle three times:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s(s+3)}\right\} &= \int_0^t e^{-3x} dx \\
&= \frac{e^{-3x}}{-3} \Big|_0^t \\
&= \frac{1}{3}(1 - e^{-3t}).
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+3)}\right\} &= \frac{1}{3}\int_0^t 1 - e^{-3x} dx \\
&= \frac{1}{3}\left(t - \frac{1 - e^{-3t}}{3}\right) \\
&= \frac{1}{9}(3t - 1 + e^{-3t}).
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s^3(s+3)}\right\} &= \frac{1}{9}\int_0^t 3x - 1 + e^{-3x} dx \\
&= \frac{1}{9}\left(3\frac{t^2}{2} - t - \frac{e^{-3t} - 1}{3}\right) \\
&= \frac{1}{54}(2 - 6t + 9t^2 - 2e^{-3t})
\end{aligned}$$

**37.** First,  $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+9)^2}\right\} = \frac{1}{54}(\sin 3t - 3t \cos 3t)$ . Thus

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)^2}\right\} &= \frac{1}{54}\int_0^t (\sin 3x - 3x \cos 3x) dx \\
&= \frac{1}{54}\left(-\frac{\cos 3x}{3} - \left(x \sin 3x + \frac{\cos 3x}{3}\right)\right)\bigg|_0^t \\
&= \frac{1}{54}\left(-\frac{2 \cos 3t}{3} - t \sin 3t + \frac{2}{3}\right) \\
&= \frac{1}{162}(-2 \cos 3t - 3t \sin 3t + 2).
\end{aligned}$$

### SECTION 3.1

1. no, not linear.
3. no, third order.
5. no, not constant coefficient.
7. yes;  $(D^2 - 7D + 10)(y) = 0$ ,  $q(s) = s^2 - 7s + 10$ , homogeneous
9. yes;  $D^2(y) = -2 + \cos t$ ,  $q(s) = s^2$ , nonhomogeneous
11. (a)  $Le^t = e^t + 3e^t + 2e^t = 6e^t$   
 (b)  $Le^{-t} = e^{-t} + 3(-e^{-t}) + 2e^{-t} = 0$   
 (c)  $L \sin t = -\sin t + 3(\cos t) + 2 \sin t = \sin t - 3 \cos t$

13. (a)  $L(-4 \sin t) = 4 \sin t + -4 \sin t = 0$   
 (b)  $L(3 \cos t) = 3(-\cos t) + 3 \cos t = 0$   
 (c)  $L1 = 0 + 1 = 1$
15.  $e^t$  and  $e^{4t}$  are homogeneous solution so  $y_h = c_1 e^t + c_2 e^{4t}$  are homogeneous solutions for all scalars  $c_1$  and  $c_2$ . A particular solution is  $y_p = \cos 2t$ . Thus  $y(t) = y_p(t) + y_h(t) = \cos 2t + c_1 e^t + c_2 e^{4t}$  where  $c_1, c_2$  are arbitrary constants.
17. From Exercise 15 we have  $y(t) = \cos 2t + c_1 e^t + c_2 e^{4t}$ . Since  $y' = -2 \sin 2t + c_1 e^t + 4c_2 e^{4t}$  we have

$$\begin{aligned} 1 = y(0) &= 1 + c_1 + c_2 \\ -3 = y'(0) &= c_1 + 4c_2, \end{aligned}$$

from which follows that  $c_1 = 1$  and  $c_2 = -1$ . Thus  $y(t) = \cos 2t + e^t - e^{4t}$ .

19.  $L(e^{rt}) = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = ar^2 e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)e^{rt}$
21. Let  $t = a$  be the point  $\phi_1$  and  $\phi_2$  are tangent. Then  $\phi_1(a) = \phi_2(a)$  and  $\phi_1'(a) = \phi_2'(a)$ . By the existence and uniqueness theorem  $\phi_1 = \phi_2$ .

## SECTION 3.2

- Suppose  $c_1 t + c_2 t^2 = 0$ . Evaluating at  $t = 1$  and  $t = 2$  gives  $c_1 + c_2 = 0$  and  $2c_1 + 4c_2 = 0$ . The simultaneous solution is  $c_1 = c_2 = 0$ . It follows that  $\{t, t^2\}$  is linearly independent.
- Since  $e^{t+2} = e^t e^2$  is a multiple of  $e^t$  it follows that  $\{e^t, e^{t+2}\}$  is linearly dependent.
- Since  $\ln t^2 = 2 \ln t$  and  $\ln t^5 = 5 \ln t$  they are multiples of each other and hence linearly dependent.
- Suppose  $c_1 t + c_2(1/t) = 0$ . Evaluating at  $t = 1$  and  $t = 2$  gives  $c_1 + c_2 = 0$  and  $2c_1 + c_2/2 = 0$ . The simultaneous solution is  $c_1 = c_2 = 0$ . It follows that  $\{t, 1/t\}$  is linearly independent.
- Suppose  $c_1 + c_2(1/t) + c_3(1/t^2) = 0$ . Evaluating at  $t = 1$ ,  $t = 1/2$ , and  $t = 1/3$  gives the same system as in the solution to Exercise 8 and hence  $c_1, c_2$  and  $c_3$  are zero. It follows that  $\{1, 1/t, 1/t^2\}$  on  $I = (0, \infty)$  is linearly independent.
- Let  $q(s) = s(s-1)(s+1)$ . Then  $\mathcal{B}_q = \{1, e^t, e^{-t}\}$  which is linearly independent.

- 13.** Let  $q(s) = (s-1)^5$ . Then  $\mathcal{B}_q = \{e^t, te^t, t^2e^t, t^3e^t\}$ . Linear independence follows since  $\{t^2e^t, t^3e^t, t^4e^t\} \subset \mathcal{B}_q$ .

**15.**

$$w(t, t \ln t) = \det \begin{bmatrix} t & t \ln t \\ 1 & \ln t + 1 \end{bmatrix} = t \ln t + t - t \ln t = t$$

**17.**

$$w(t^{10}, t^{20}) = \det \begin{bmatrix} t^{10} & t^{20} \\ 10t^9 & 20t^{19} \end{bmatrix} = 20t^{29} - 10t^{29} = 10t^{29}$$

**19.**

$$\begin{aligned} & w(e^{r_1 t}, e^{r_2 t}, e^{r_3 t}) \\ &= \det \begin{bmatrix} e^{r_1 t} & e^{r_2 t} & e^{r_3 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} & r_3 e^{r_3 t} \\ r_1^2 e^{r_1 t} & r_2^2 e^{r_2 t} & r_3^2 e^{r_3 t} \end{bmatrix} \\ &= e^{(r_1 + r_2 + r_3)t} \det \begin{bmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{bmatrix} \\ &= e^{(r_1 + r_2 + r_3)t} ((r_2 r_3^2 - r_3 r_2^2) - (r_1 r_3^2 - r_3 r_1^2) + (r_1 r_2^2 - r_2 r_1^2)) \\ &= e^{(r_1 + r_2 + r_3)t} (r_3 - r_1)(r_3 - r_2)(r_2 - r_1). \end{aligned}$$

The last line requires a little algebra.

**21.**

$$w(1, t, t^2, t^3) = \det \begin{bmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \\ 0 & 0 & 2 & 6t \\ 0 & 0 & 0 & 6 \end{bmatrix} = 12$$

- 23.** Let  $q(s) = (s-2)^2$ . Then  $\mathcal{B}_q = \{e^{2t}, te^{2t}\}$  is linearly independent. We can equate coefficients to get

$$\begin{aligned} 25c_1 + 10c_2 &= 0 \\ 25c_2 &= 25 \end{aligned}$$

We thus get  $c_2 = 1$  and then  $c_1 = -10/25 = -2/5$ .

- 25.** If  $q(s) = s^3$  then  $\mathcal{B}_q = \{1, t, t^2\}$  is linearly independent. Thus we can equate coefficients to get

$$\begin{aligned} a_1 &= a_2 \\ 3 &= a_1 \\ -a_2 &= -3 \end{aligned}$$

It follows that  $a_1 = 3$  and  $a_2 = 3$  is the solution.

27. Observe that  $y_1'(t) = 3t^2$  and  $y_2'(t) = \begin{cases} -3t^2 & \text{if } t < 0 \\ 3t^2 & \text{if } t \geq 0. \end{cases}$  If  $t < 0$  then  $w(y_1, y_2)(t) = \begin{pmatrix} t^3 & -t^3 \\ 3t^2 & -3t^2 \end{pmatrix} = 0$ . If  $t \geq 0$  then  $w(y_1, y_2)(t) = \begin{pmatrix} t^3 & t^3 \\ 3t^2 & 3t^2 \end{pmatrix} = 0$ . It follows that the Wronskian is zero for all  $t \in (-\infty, \infty)$ .

### SECTION 3.3

1. The characteristic polynomial is  $q(s) = s^2 - s - 2 = (s - 2)(s + 1)$  so  $\mathcal{B}_q = \{e^{2t}, e^{-t}\}$  and the general solution takes the form  $y(t) = c_1 e^{2t} + c_2 e^{-t}$ ,  $c_1, c_2 \in \mathbb{R}$
3. The characteristic polynomial is  $q(s) = s^2 + 10s + 24 = (s + 6)(s + 4)$  so  $\mathcal{B}_q = \{e^{-6t}, e^{-4t}\}$  and the general solution takes the form  $y(t) = c_1 e^{-6t} + c_2 e^{-4t}$ ,  $c_1, c_2 \in \mathbb{R}$
5. The characteristic polynomial is  $q(s) = s^2 + 8s + 16 = (s + 4)^2$  so  $\mathcal{B}_q = \{e^{-4t}, te^{-4t}\}$  and the general solution takes the form  $y(t) = c_1 e^{-4t} + c_2 te^{-4t}$ ,  $c_1, c_2 \in \mathbb{R}$
7. The characteristic polynomial is  $q(s) = s^2 + 2s + 5 = (s + 1)^2 + 4$  so  $\mathcal{B}_q = \{e^{-t} \cos 2t, e^{-t} \sin 2t\}$  and the general solution takes the form  $y(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$ ,  $c_1, c_2 \in \mathbb{R}$
9. The characteristic polynomial is  $q(s) = s^2 + 13s + 36 = (s + 9)(s + 4)$  so  $\mathcal{B}_q = \{e^{-9t}, e^{-4t}\}$  and the general solution takes the form  $y(t) = c_1 e^{-9t} + c_2 e^{-4t}$ ,  $c_1, c_2 \in \mathbb{R}$
11. The characteristic polynomial is  $q(s) = s^2 + 10s + 25 = (s + 5)^2$  so  $\mathcal{B}_q = \{e^{-5t}, te^{-5t}\}$  and the general solution takes the form  $y(t) = c_1 e^{-5t} + c_2 te^{-5t}$ ,  $c_1, c_2 \in \mathbb{R}$
13. The characteristic polynomial is  $q(s) = s^2 - 1 = (s - 1)(s + 1)$  so  $\mathcal{B}_q = \{e^t, e^{-t}\}$  and the general solution takes the form  $y(t) = c_1 e^{2t} + c_2 e^{-t}$ . The initial conditions imply that  $c_1 + c_2 = 0$  and  $c_1 - c_2 = 1$ . Solving gives  $c_1 = 1/2$  and  $c_2 = -1/2$ . Thus  $y = \frac{e^t - e^{-t}}{2}$
15. The characteristic polynomial is  $q(s) = s^2 - 10s + 25 = (s - 5)^2$  so  $\mathcal{B}_q = \{e^{5t}, te^{5t}\}$  and the general solution takes the form  $y(t) = c_1 e^{5t} + c_2 te^{5t}$ . The initial conditions imply that  $c_1 = 0$  and  $5c_1 + c_2 = 1$ . Solving gives  $c_1 = 0$  and  $c_2 = 1$ . Thus  $y = te^{5t}$

17. Let  $q(s) = (s - 3)(s + 7) = s^2 + 4s - 21$ . Then  $\mathcal{B}_q = \{e^{3t}, e^{-7t}\}$ .  
 $w(e^{3t}, e^{-7t}) = \det \begin{bmatrix} e^{3t} & e^{-7t} \\ 3e^{3t} & -7e^{-7t} \end{bmatrix} = -10e^{-4t}$ . So  $K = -10$ .

19. Let  $q(s) = (s - 3)^2 = s^2 - 6s + 9$ . Then  $\mathcal{B}_q = \{e^{3t}, te^{3t}\}$ .  $w(e^{3t}, te^{3t}) = \det \begin{bmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & (1 + 3t)e^{3t} \end{bmatrix} = (1 + 3t)e^{6t} - 3te^{6t} = e^{6t}$ . So  $K = 1$ .

21. Let  $q(s) = (s - 1)^2 + 2^2 = s^2 - 2s + 5$ . Then  $\mathcal{B}_q = \{e^t \cos 2t, e^t \sin 2t\}$ .

$$\begin{aligned} w(e^t \cos 2t, e^t \sin 2t) &= \det \begin{bmatrix} e^t \cos 2t & e^t \sin 2t \\ e^t(\cos 2t - 2 \sin 2t) & e^t(\sin 2t + 2 \cos 2t) \end{bmatrix} \\ &= e^{2t}(\sin 2t \cos 2t + 2 \cos^2 2t) \\ &\quad - e^{2t}(\cos 2t \sin 2t - 2 \sin^2 2t) \\ &= 2e^{2t}. \end{aligned}$$

So  $K = 2$ .

## SECTION 3.4

1.  $q(s)v(s) = (s + 1)(s - 2)(s - 3)$  so  $\mathcal{B}_{qv} = \{e^{-t}, e^{2t}, e^{3t}\}$  while  $\mathcal{B}_q = \{e^{-t}, e^{2t}\}$ . Since  $e^{3t}$  is the only function in the first set but not in the second  $y_p(t) = a_1 e^{3t}$ .
3.  $q(s)v(s) = (s - 2)^2(s - 3)$  so  $\mathcal{B}_{qv} = \{e^{2t}, te^{2t}, e^{3t}\}$  while  $\mathcal{B}_q = \{e^{2t}, e^{3t}\}$ . Since  $te^{2t}$  is the only function in the first set but not in the second  $y_p(t) = a_1 te^{2t}$ .
5.  $q(s)v(s) = (s - 5)^2(s^2 + 25)$  so  $\mathcal{B}_{qv} = \{e^{5t}, te^{5t}, \cos 5t, \sin 5t\}$  while  $\mathcal{B}_q = \{e^{5t}, te^{5t}\}$ . Since  $\cos 5t$  and  $\sin 5t$  are the only functions in the first set that are not in the second  $y_p(t) = a_1 \cos 5t + a_2 \sin 5t$ .
7.  $q(s)v(s) = (s^2 + 4)^2$  so  $\mathcal{B}_{qv} = \{\cos 2t, \sin 2t, t \cos 2t, t \sin 2t\}$  while  $\mathcal{B}_q = \{\cos 2t, \sin 2t\}$ . Since  $t \cos 2t$  and  $t \sin 2t$  are the only functions in the first set that are not in the second  $y_p(t) = a_1 t \cos 2t + a_2 t \sin 2t$ .
9.  $q(s)v(s) = (s^2 + 4s + 5)(s - 1)^2$  so  $\mathcal{B}_{qv} = \{e^t, te^t, e^{-2t} \cos t, e^{-2t} \sin t\}$  while  $\mathcal{B}_q = \{e^t, te^t\}$ . Since  $e^{-2t} \cos t$  and  $e^{-2t} \sin t$  are the only functions in the first set that are not in the second  $y_p(t) = a_1 e^{-2t} \cos t + a_2 e^{-2t} \sin t$ .
11. The characteristic polynomial is  $q(s) = s^2 - 3s - 10 = (s - 5)(s + 2)$ . Since  $\mathcal{L}\{7e^{-2t}\} = 7/(s + 2)$ , we set  $v(s) = s + 2$ . Then  $q(s)v(s) = (s - 5)(s + 2)^2$ . Since  $\mathcal{B}_{qv} = \{e^{5t}, e^{-2t}, te^{-2t}\}$  and  $\mathcal{B}_q = \{e^{5t}, e^{-2t}\}$  we have  $y_p = a_1 te^{-2t}$ , a test function. Substituting  $y_p$  into the differential

equation gives  $-a_1 e^{-2t} = 7e^{-t}$ . It follows that  $a_1 = -1$ . The general solution is  $y = -te^{-2t} + c_1 e^{-2t} + c_2 e^{5t}$ .

13. The characteristic polynomial is  $q(s) = s^2 + 2s + 1 = (s + 1)^2$ . Since  $\mathcal{L}\{e^{-t}\} = 1/(s + 1)$ , we set  $v(s) = s + 1$ . Then  $q(s)v(s) = (s + 1)^3$ . Since  $\mathcal{B}_{qv} = \{e^{-t}, te^{-t}, t^2 e^{-t}\}$  and  $\mathcal{B}_q = \{e^{-t}, te^{-t}\}$  we have  $y_p = a_1 t^2 e^{-t}$ , a test function. Substituting  $y_p$  into the differential equation gives  $2a_1 e^{-t} = e^{-t}$ . It follows that  $a_1 = 1/2$ . The general solution is  $y = \frac{1}{2}t^2 e^{-t} + c_1 e^{-t} + c_2 t e^{-t}$ .

15. The characteristic polynomial is  $q(s) = s^2 + 4s + 5 = (s + 2)^2 + 1$ , an irreducible quadratic. Since  $\mathcal{L}\{e^{-3t}\} = 1/(s + 3)$ , we set  $v(s) = s + 3$ . Then  $q(s)v(s) = ((s + 2)^2 + 1)(s + 3)$ . Since  $\mathcal{B}_{qv} = \{e^{-2t} \cos t, e^{-2t} \sin t, e^{-3t}\}$  and  $\mathcal{B}_q = \{e^{-2t} \cos t, e^{-2t} \sin t\}$  we have  $y_p = a_1 e^{-3t}$ , a test function. Substituting  $y_p$  into the differential equation gives  $2a_1 e^{-3t} = e^{-3t}$ . It follows that  $a_1 = 1/2$ . The general solution is  $y = \frac{1}{2}e^{-3t} + c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t$ .

17. The characteristic polynomial is  $q(s) = s^2 - 1 = (s - 1)(s + 1)$ . Since  $\mathcal{L}\{t^2\} = 2/s^3$ , we set  $v(s) = s^3$ . Then  $q(s)v(s) = (s - 1)(s + 1)s^3$ . Since  $\mathcal{B}_{qv} = \{e^t, e^{-t}, 1, t, t^2\}$  and  $\mathcal{B}_q = \{e^t, e^{-t}\}$  we have  $y_p = a_1 + a_2 t + a_3 t^2$ , a test function. Substituting  $y_p$  into the differential equation gives  $2a_3 - a_1 - a_2 t - a_3 t^2 = t^2$ . Using linear independence we equate the coefficients to get

$$\begin{aligned} 2a_3 - a_1 &= 0 \\ -a_2 &= 0 \\ -a_3 &= 1 \end{aligned}$$

It follows that  $a_3 = -1$ ,  $a_2 = 0$ , and  $a_1 = -2$ . The general solution is  $y = -t^2 - 2 + c_1 e^t + c_2 e^{-t}$ .

19. The characteristic polynomial is  $q(s) = s^2 - 4s + 4 = (s - 2)^2$ . Since  $\mathcal{L}\{e^{2t}\} = 1/(s - 2)$ , we set  $v(s) = s - 2$ . Then  $q(s)v(s) = (s - 2)^3$ . Since  $\mathcal{B}_{qv} = \{e^{2t}, te^{2t}, t^2 e^{2t}\}$  and  $\mathcal{B}_q = \{e^{2t}, te^{2t}\}$  we have  $y_p = a_1 t^2 e^{2t}$ , a test function. Substituting  $y_p$  into the differential equation gives  $2a_1 e^{2t} = e^{2t}$ . It follows that  $a_1 = 1/2$ . The general solution is  $y = \frac{1}{2}t^2 e^{2t} + c_1 e^{2t} + c_2 t e^{2t}$ .

21. The characteristic polynomial is  $q(s) = s^2 + 6s + 9 = (s + 3)^2$ . Since  $\mathcal{L}\{25te^{2t}\} = 25/(s - 2)^2$ , we set  $v(s) = (s - 2)^2$ . Then  $q(s)v(s) = (s + 3)^2(s - 2)^2$ . Since  $\mathcal{B}_{qv} = \{e^{-3t}, te^{-3t}, e^{2t}, te^{2t}\}$  and  $\mathcal{B}_q = \{e^{-3t}, te^{-3t}\}$  we have  $y_p = a_1 e^{2t} + a_2 t e^{2t}$ , a test function. Substituting  $y_p$  into the differential equation gives  $(25a_1 + 10a_2)e^{2t} + 25a_2 t e^{2t} = 25te^{2t}$ . Linear independence implies  $25a_1 + 10a_2 = 0$  and  $25a_2 = 25$ . We get  $a_2 = 1$  and  $a_1 = -2/5$ . The general solution is  $y = te^{2t} - \frac{2}{5}e^{2t} + c_1 e^{-3t} + c_2 t e^{-3t}$ .

23. The characteristic polynomial is  $q(s) = s^2 + 6s + 13 = (s + 3)^2 + 4$ , an irreducible quadratic. Since  $\mathcal{L}\{e^{-3t} \cos 2t\} = (s + 3)/((s + 3)^2 + 4)$



4), we set  $v(s) = (s + 3)^2 + 4$ . Then  $q(s)v(s) = ((s + 3)^2 + 4)^2$ . Since  $\mathcal{B}_{qv} = \{e^{-3t} \cos 2t, e^{-3t} \sin 2t, te^{-3t} \cos 2t, te^{-3t} \sin 2t\}$  and  $\mathcal{B}_q = \{e^{-3t} \cos 2t, e^{-3t} \sin 2t\}$  we have  $y_p = a_1 te^{-3t} \cos 2t + a_2 te^{-3t} \sin 2t$ , a test function. Substituting  $y_p$  into the differential equation gives (after a long calculation)  $-4a_1 e^{-3t} \sin 2t + 4a_2 e^{-3t} \cos 2t = e^{-3t} \cos 2t$ . It follows that  $-4a_1 = 0$  and  $4a_2 = 1$ . Thus  $a_1 = 0$  and  $a_2 = 1/4$ . The general solution is  $y = \frac{1}{4} te^{-3t} \sin(2t) + c_1 e^{-3t} \cos(2t) + c_2 e^{-3t} \sin(2t)$ .

25. The characteristic polynomial is  $q(s) = s^2 - 5s - 6 = (s - 6)(s + 1)$ . Since  $\mathcal{L}\{e^{3t}\} = 1/(s - 3)$ , we set  $v(s) = s - 3$ . Then  $q(s)v(s) = (s - 6)(s + 1)(s - 3)$ . Since  $\mathcal{B}_{qv} = \{e^{6t}, e^{-t}, e^{3t}\}$  and  $\mathcal{B}_q = \{e^{6t}, e^{-t}\}$  we have  $y_p = a_1 e^{3t}$ , a test function. Substituting  $y_p$  into the differential equation gives  $-12a_1 e^{3t} = e^{3t}$ . It follows that  $a_1 = -1/12$ . The general solution is  $y = \frac{-1}{12} e^{3t} + c_1 e^{6t} + c_2 e^{-t}$ . Since  $y' = \frac{-1}{4} e^{3t} + 6c_1 e^{6t} - c_2 e^{-t}$  the initial condition imply

$$\begin{aligned} \frac{-1}{12} + c_1 + c_2 &= 2 \\ \frac{-1}{4} + 6c_1 - c_2 &= 1 \end{aligned}$$

It is easy to calculate that  $c_1 = 10/21$  and  $c_2 = 135/84$ . Thus  $y = \frac{-1}{12} e^{3t} + \frac{10}{21} e^{6t} + \frac{135}{84} e^{-t}$ .

27. The characteristic polynomial is  $q(s) = s^2 + 1$ . Since  $\mathcal{L}\{10e^{2t}\} = 10/(s - 2)$ , we set  $v(s) = s - 2$ . Then  $q(s)v(s) = (s^2 + 1)(s - 2)$ . Since  $\mathcal{B}_{qv} = \{\cos t, \sin t, e^{2t}\}$  and  $\mathcal{B}_q = \{\cos t, \sin t\}$  we have  $y_p = a_1 e^{2t}$ , a test function. Substituting  $y_p$  into the differential equation gives  $5a_1 e^{2t} = 10e^{2t}$  and hence  $a_1 = 2$ . The general solution is  $y = 2e^{2t} + c_1 \cos t + c_2 \sin t$ . Since  $y' = 4e^{2t} - c_1 \sin t + c_2 \cos t$  the initial conditions imply

$$\begin{aligned} 2 + c_1 &= 0 \\ 4 + c_2 &= 0 \end{aligned}$$

and so  $c_1 = -2$  and  $c_2 = -4$ . Thus  $y = 2e^{2t} - 2 \cos t - 4 \sin t$ .

## SECTION 3.5

1. The characteristic polynomial is  $q(s) = s^2 - 4 = (s - 2)(s + 2)$  and  $\mathcal{L}\{e^{-6t}\} = 1/(s + 6)$ . Thus

$$\mathcal{L}\{y\} = \frac{1}{(s - 2)(s + 2)(s + 6)} = \frac{\frac{1}{32}}{s + 6} + \frac{p(s)}{(s - 2)(s + 2)}.$$

A particular solution is  $y_p = \frac{1}{32} e^{-6t}$  and the general solution is  $y = \frac{1}{32} e^{-6t} + c_1 e^{2t} + c_2 e^{-2t}$

3. The characteristic polynomial is  $q(s) = s^2 + 5s + 6 = (s + 2)(s + 3)$  and  $\mathcal{L}\{e^{-2t}\} = 1/(s + 2)$ . Thus

$$\mathcal{L}\{y\} = \frac{1}{(s + 2)^2(s + 3)} = \frac{1}{(s + 2)^2} + \frac{p(s)}{(s + 2)(s + 3)}.$$

A particular solution is  $y_p = te^{-2t}$  and the general solution is  $y = te^{-2t} + c_1e^{-2t} + c_2e^{-3t}$

5. The characteristic polynomial is  $q(s) = s^2 + 2s - 8 = (s - 2)(s + 4)$  and  $\mathcal{L}\{6e^{-4t}\} = 6/(s + 4)$ . Thus

$$\mathcal{L}\{y\} = \frac{6}{(s - 2)(s + 4)^2} = \frac{-1}{(s + 4)^2} + \frac{p(s)}{(s - 2)(s + 4)}.$$

A particular solution is  $y_p = -te^{-4t}$  and the general solution is  $y = -te^{-4t} + c_1e^{2t} + c_2e^{-4t}$

7. The characteristic polynomial is  $q(s) = s^2 + 6s + 9 = (s + 3)^2$  and  $\mathcal{L}\{25e^{2t}\} = 25/((s - 2)^2)$ . Thus

$$\mathcal{L}\{y\} = \frac{25}{(s - 2)^2(s + 3)^2} = \frac{1}{(s - 2)^2} - \frac{2}{5} \frac{1}{s - 2} + \frac{p(s)}{(s + 3)^2}.$$

A particular solution is  $y_p = te^{2t} - \frac{2}{5}e^{2t}$  and the general solution is  $y = te^{2t} - \frac{2}{5}e^{2t} + c_1e^{-3t} + c_2te^{-3t}$

9. The characteristic polynomial is  $q(s) = s^2 - 8s + 25 = (s - 4)^2 + 9$  and  $\mathcal{L}\{36te^{4t} \sin 3t\} = 216(s - 4)/((s - 4)^2 + 9)^2$ . Thus

$$\mathcal{L}\{y\} = \frac{216(s - 4)}{((s - 4)^2 + 9)^3}.$$

This is a partial fraction. Table 2.5 gives  $y = -3t^2e^{4t} \cos 3t + te^{4t} \sin 3t$ . A particular solution is  $y_p = -3t^2e^{4t} \cos 3t + te^{4t} \sin 3t$  and the general solution is  $y = -3t^2e^{4t} \cos 3t + te^{4t} \sin 3t + c_1e^{4t} \cos 3t + c_2e^{4t} \sin 3t$

11. The characteristic polynomial is  $q(s) = s^2 + 2s + 1 = (s + 1)^2$  and  $\mathcal{L}\{\cos t\} = s/(s^2 + 1)$ . Thus

$$\mathcal{L}\{y\} = \frac{s}{(s + 1)^2(s^2 + 1)} = \frac{1}{2} \frac{1}{s^2 + 1} + \frac{p(s)}{(s + 1)^2}.$$

A particular solution is  $y_p = \frac{1}{2} \sin t$  and the general solution is  $y = \frac{1}{2} \sin t + c_1e^{-t} + c_2te^{-t}$

## SECTION 3.6

1. The force is 16 lbs. A length of 6 inches is  $1/2$  ft. The spring constant is  $k = 16/(1/2) = 32$  lbs/ft.
3. The force exerted by the mass is  $40 \cdot 9.8 = 392$  N. Thus  $k = 392/.8 = 490$  N/m.
5. The force is 4 lbs and the velocity is  $1/2$  ft per second. So  $\mu = \text{Force/velocity} = \frac{4}{1/2} = 8$  lbs s/ft.
7. Let  $x$  be the force. Then  $100 = x/4$  so  $x = 400$  lbs.
9. The mass is  $m = 6$ . The spring constant is given by  $k = 2/.1 = 20$ . The damping constant is  $\mu = 0$ . Since no external force is mentioned we may assume it is zero. The initial conditions are  $y(0) = .1$  m and  $y'(0) = 0$ . The following equation

$$6y'' + 20y = 0, \quad y(0) = .1, \quad y'(0) = 0$$

represents the model for the motion of the body. The characteristic polynomial is  $q(s) = 6s^2 + 20 = 6(s^2 + \sqrt{10/3})^2$ . Thus  $y = c_1 \cos \sqrt{10/3}t + c_2 \sin \sqrt{10/3}t$ . The initial conditions imply  $c_1 = 1/10$  and  $c_2 = 0$ . Thus

$$y = \frac{1}{10} \cos \sqrt{10/3}t.$$

The motion is undamped free or simple harmonic motion. Since  $y$  is written in the form  $y = A \cos \omega t + \phi$  we can read off the amplitude, frequency, and phase shift; they are  $A = 1/10$ ,  $\beta = \sqrt{10/3}$ , and  $\phi = 0$ .

11. The mass is  $m = 16/32 = 1/2$  slugs. The spring constant  $k$  is given by  $k = 16/(6/12) = 32$ . The damping constant is given by  $\mu = 4/2 = 2$ . Since no external force is mentioned we may assume it is zero. The initial conditions are  $y(0) = 1$  and  $y'(0) = 1$ . The following equation  $\frac{1}{2}y'' + 2y' + 32y = 0$ ,  $y(0) = 1$ ,  $y'(0) = 1$  models the motion of the body. The characteristic polynomial is  $q(s) = \frac{1}{2}s^2 + 2s + 32 = \frac{1}{2}(s^2 + 4s + 64) = \frac{1}{2}((s + 2)^2 + \sqrt{60})^2$ . Thus

$$y = c_1 e^{-2t} \cos \sqrt{60}t + c_2 e^{-2t} \sin \sqrt{60}t.$$

The initial conditions imply  $c_1 = 1$  and  $c_2 = 3/\sqrt{60}$ . Thus

$$y = e^{-2t} \cos \sqrt{60}t + \frac{3}{\sqrt{60}} e^{-2t} \sin \sqrt{60}t.$$

The discriminant of the characteristic equation is  $D = 2^2 - 4 \cdot (1/2) \cdot 32 = -60 < 0$  so the motion is underdamped free motion. Let  $A = \sqrt{1 + \left(\frac{3}{\sqrt{60}}\right)^2} = \sqrt{23/20}$ . If  $\tan \phi = 3/\sqrt{60} = \sqrt{60}/20$  then  $\phi \approx .3695$ . We can write

$$y = \sqrt{\frac{23}{20}} e^{-2t} \cos(\sqrt{60}t + \phi).$$

13. The mass is  $m = 2/32 = 1/16$  slug. The spring constant  $k$  is given by  $k = 2/(4/12) = 6$  and the damping constant is  $\mu = 0$ . The initial conditions are  $y(0) = 0$  and  $y'(0) = 8/12 = 2/3$ . The equation  $\frac{1}{16}y'' + 6y = 0$  or equivalently  $y'' + 96y = 0$  with initial conditions  $y(0) = 0$ ,  $y'(0) = 2/3$  models the motion of the body. The characteristic polynomial is  $q(s) = s^2 + 96$  so  $y = c_1 \cos \sqrt{96}t + c_2 \sin \sqrt{96}t$ . The initial conditions imply  $c_1 = 0$  and  $c_2 = \frac{\sqrt{6}}{36}$ . Thus  $y = \frac{\sqrt{6}}{36} \sin \sqrt{96}t = \frac{\sqrt{6}}{36} \cos(\sqrt{96}t - \frac{\pi}{2})$ . The motion is undamped free or simple harmonic motion so the mass crosses equilibrium.

15. By the quadratic formula the roots of  $q(s) = ms^2 + \mu s + k$  are

$$s = \frac{-\mu \pm \sqrt{\mu^2 - 4mk}}{2m} = \frac{-\mu}{2m} \pm \sqrt{\left(\frac{\mu}{2m}\right)^2 - \frac{k}{m}}.$$

If the discriminant  $D = \mu^2 - 4mk$  is negative then the roots are complex and the real part is  $\frac{-\mu}{2m}$  which is negative. If the discriminant is zero then  $\frac{-\mu}{2m}$  is a negative double root. If the discriminant is positive then both roots are real and distinct. It is enough to show that the larger of the two,  $r = \frac{-\mu}{2m} + \sqrt{\left(\frac{\mu}{2m}\right)^2 - \frac{k}{m}}$  is negative. Let  $p = \frac{\mu}{2m} + \sqrt{\left(\frac{\mu}{2m}\right)^2 - \frac{k}{m}}$  and observe that it is positive. Further,

$$\begin{aligned} rp &= \left( \frac{-\mu}{2m} + \sqrt{\left(\frac{\mu}{2m}\right)^2 - \frac{k}{m}} \right) \left( \frac{\mu}{2m} + \sqrt{\left(\frac{\mu}{2m}\right)^2 - \frac{k}{m}} \right) \\ &= -\frac{\mu^2}{4m^2} + \frac{\mu^2}{4m^2} - \frac{k}{m} \\ &= -\frac{k}{m} < 0 \end{aligned}$$

Since  $p > 0$  it follows the  $r < 0$ .

It follows that a solution to  $my'' + \mu y' + ky = 0$  is of the following form

1.  $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$  where  $r_1$  and  $r_2$  are negative.
2.  $y = (c_1 + c_2 t) e^{rt}$  where  $r$  is negative
3.  $y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$  where  $\alpha$  is negative

In each case  $\lim_{t \rightarrow \infty} y(t) = 0$ .

## SECTION 4.1

1. yes;  $(D^3 - 3D)y = e^t$ , order 3,  $q(s) = s^3 - 3s$ , nonhomogeneous
3. no, because of the presence of  $y^4$
5. (a)  $L(e^{2t}) = 8e^{2t} - 4(2e^{2t}) = 0$   
 (b)  $L(3^{-2t}) = -8e^{-2t} - 4(-2e^{-2t}) = 0$   
 (c)  $L(2) = 0 - 4(0) = 0$
7. (a)  $Le^{-t} = e^{-t} + 5e^{-t} + 4e^{-t} = 10e^{-t}$   
 (b)  $L \cos t = \cos t + 5(-\cos t) + 4 \cos t = 0$   
 (c)  $L \sin 2t = 16 \sin 2t + 5(-4 \sin 2t) + 4 \sin 2t = 0$
9.  $e^{2t}$ ,  $e^{-2t}$ , and 1 are homogeneous solution so  $y_h = c_1 e^{2t} + c_2 e^{-2t} + c_3$  are homogeneous solutions for all scalars  $c_1$ ,  $c_2$ , and  $c_3$ . A particular solution is  $y_p = te^{2t}$ . Thus  $y(t) = y_p(t) + y_h(t) = te^{2t} + c_1 e^{2t} + c_2 e^{-2t} + c_3$  where  $c_1$ ,  $c_2$ , and  $c_3$  are arbitrary constants
11. From Exercise 9 we have  $y(t) = y_p(t) + y_h(t) = te^{2t} + c_1 e^{2t} + c_2 e^{-2t} + c_3$ . Since

$$\begin{aligned} y &= te^{2t} + c_1 e^{2t} + c_2 e^{-2t} + c_3 \\ y' &= (1 + 2t)e^{2t} + 2c_1 e^{2t} - 2c_2 e^{-2t} \\ y'' &= (4 + 4t)e^{2t} + 4c_1 e^{2t} + 4c_2 e^{-2t} \end{aligned}$$

we have

$$\begin{aligned} 2 = y(0) &= c_1 + c_2 + c_3 \\ -1 = y'(0) &= 1 + 2c_1 - 2c_2 \\ 16 = y''(0) &= 4 + 4c_1 + 4c_2, \end{aligned}$$

from which follows that  $c_1 = 1$ ,  $c_2 = 2$ , and  $c_3 = -1$ . Thus  $y(t) = te^{2t} + e^{2t} + 2e^{-2t} - 1$

## SECTION 4.2

1. The characteristic polynomial is  $q(s) = s^3 - 1 = (s - 1)(s^2 + s + 1) = (s - 1)((s + 1/2)^2 + 3/4)$ . Thus  $\mathcal{B}_q = \left\{ e^t, e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t, e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t \right\}$ . It follows that  $y(t) = c_1 e^{-t} + c_2 e^{-\frac{1}{2}t} \cos \frac{\sqrt{3}}{2}t + c_3 e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2}t$ .
3. The characteristic polynomial is  $q(s) = s^4 - 1 = (s^2 - 1)(s^2 + 1) = (s - 1)(s + 1)(s^2 + 1)$ . Thus  $\mathcal{B}_q = \{e^t, e^{-t}, \cos t, \sin t\}$ . It follows that  $y(t) = c_1 e^t + c_2 e^{-t} + c_3 \sin t + c_4 \cos t$ .
5. The characteristic polynomial is  $q(s) = s^4 - 5s^2 + 4 = (s^2 - 1)(s^2 - 4) = (s - 1)(s + 1)(s - 2)(s + 2)$ . Thus  $\mathcal{B}_q = \{e^t, e^{-t}, e^{2t}, e^{-2t}\}$ . It follows that  $y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-2t}$ .
7. The characteristic polynomial is  $q(s) = (s + 2)(s^2 + 25)$ . Thus  $\mathcal{B}_q = \{e^{-2t}, \cos 5t, \sin 5t\}$ . It follows that  $y(t) = c_1 e^{-2t} + c_2 \cos 5t + c_3 \sin 5t$ .
9. The characteristic polynomial is  $q(s) = (s + 3)(s - 1)(s + 3)^2 = (s - 1)(s + 3)^3$ . Thus  $\mathcal{B}_q = \{e^t, e^{-3t}, t e^{-3t}, t^2 e^{-3t}\}$ . It follows that  $y(t) = c_1 e^t + c_2 e^{-3t} + c_3 t e^{-3t} + c_4 t^2 e^{-3t}$ .
11. The characteristic polynomial is  $q(s) = s^4 - 1 = (s^2 - 1)(s^2 + 1) = (s - 1)(s + 1)(s^2 + 1)$ . Thus  $\mathcal{B}_q = \{e^t, e^{-t}, \cos t, \sin t\}$ . It follows that  $y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$ . Since

$$\begin{aligned}
 y(t) &= c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t \\
 y'(t) &= c_1 e^t - c_2 e^{-t} - c_3 \sin t + c_4 \cos t \\
 y''(t) &= c_1 e^t + c_2 e^{-t} - c_3 \cos t - c_4 \sin t \\
 y'''(t) &= c_1 e^t - c_2 e^{-t} + c_3 \sin t - c_4 \cos t
 \end{aligned}$$

we have

$$\begin{aligned}
 -1 = y(0) &= c_1 + c_2 + c_3 \\
 6 = y'(0) &= c_1 - c_2 + c_4 \\
 -3 = y''(0) &= c_1 + c_2 - c_3 \\
 2 = y'''(0) &= c_1 - c_2 - c_4
 \end{aligned}$$

from which we get  $c_1 = 1$ ,  $c_2 = -3$ ,  $c_3 = 1$ , and  $c_4 = 2$ . Hence  $y = e^t - 3e^{-t} + \cos t + 2 \sin t$

## SECTION 4.3

1. Since  $q(s) = s^3 - s = s(s-1)(s+1)$  we have  $\mathcal{B}_q = \{1, e^t, e^{-t}\}$  and since  $q(s)v(s) = s(s-1)(s+1)^2$  we have  $\mathcal{B}_{qv} = \{1, e^t, e^{-t}, te^{-t}\}$ . Thus  $\mathcal{B}_{qv} \setminus \mathcal{B}_q = \{te^{-t}\}$  and  $y = cte^{-t}$  is the test function.
3.  $q(s) = s(s-1)(s+1)$  we have  $\mathcal{B}_q = \{1, e^t, e^{-t}\}$  and since  $q(s)v(s) = s(s-1)(s+1)(s-2)$  we have  $\mathcal{B}_{qv} = \{1, e^t, e^{-t}, e^{2t}\}$ . Thus  $\mathcal{B}_{qv} \setminus \mathcal{B}_q = \{e^{2t}\}$  and  $y = ce^{2t}$  is the test function.
5. We have  $q(s) = s^3 - s = s(s-1)(s+1)$  and  $\mathcal{L}\{e^t\} = \frac{1}{s-1}$ . Let  $v(s) = s-1$ . Then  $q(s)v(s) = s(s-1)^2(s+1)$ ,  $\mathcal{B}_q = \{1, e^t, e^{-t}\}$  and  $\mathcal{B}_{qv} = \{1, e^t, te^t, e^{-t}\}$  and  $\mathcal{B}_{qv} \setminus \mathcal{B}_q = \{te^t\}$ . It follows that  $y = cte^t$  is the test function. Since

$$\begin{aligned} y &= cte^t \\ y' &= c(1+t)e^t \\ y'' &= c(2+t)e^t \\ y''' &= c(3+t)e^t \end{aligned}$$

we have  $c(3+t)e^t - c(1+t)e^t = e^t$ . Simplifying we get  $2ce^t = e^t$  which implies  $c = 1/2$ . It follows that  $y = \frac{1}{2}te^t + c_1e^{-t} + c_2e^t + c_3$

7. We have  $q(s) = s^4 - 5s^2 + 4 = (s^2-1)(s^2-4) = (s-1)(s+1)(s-2)(s+2)$  and  $\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$ . Let  $v(s) = s-2$ . Then  $q(s)v(s) = (s-1)(s+1)(s-2)^2(s+2)$ ,  $\mathcal{B}_q = \{e^t, e^{-t}, e^{2t}, e^{-2t}\}$  and  $\mathcal{B}_{qv} = \{e^t, e^{-t}, e^{2t}, te^{2t}, e^{-2t}\}$ . Thus  $\mathcal{B}_{qv} \setminus \mathcal{B}_q = \{te^{2t}\}$ . It follows that  $y = cte^{2t}$  is the test function and

$$\begin{aligned} y &= cte^{2t} \\ y' &= c(1+2t)e^{2t} \\ y'' &= c(4+4t)e^{2t} \\ y''' &= c(12+8t)e^{2t} \\ y^{(4)} &= c(32+16t)e^{2t}. \end{aligned}$$

Substituting into the differential equation and simplifying gives  $12ce^{2t} = e^{2t}$ . We thus get  $c = 1/12$ . It follows that  $y = \frac{1}{12}te^{2t} + c_1e^t + c_2e^{-t} + c_3e^{2t} + c_4e^{-2t}$

9. We have  $q(s) = s^3 - s = s(s-1)(s+1)$  and  $\mathcal{L}\{e^t\} = \frac{1}{s-1}$ . Thus  $Y(s) = \frac{1}{s(s+1)(s-1)^2}$ . One iteration of the partial fraction decomposition algorithm gives

Incomplete $(s-1)$ -chain	
$\frac{1}{s(s+1)(s-1)^2}$	$\frac{1/2}{(s-1)^2}$
$\frac{p(s)}{s(s+1)(s-1)}$	

It follows that  $y_p = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = \frac{1}{2}te^t$  and the general solution is  $y = \frac{1}{2}te^t + c_1e^{-t} + c_2e^t + c_3$

11. We have  $q(s) = s(s^2 + 4)$  and  $\mathcal{L}\{t\} = \frac{1}{s^2}$ . Thus  $Y(s) = \frac{1}{s^3(s^2+4)}$ . The partial fraction decomposition algorithm gives

Incomplete $s$ -chain	
$\frac{1}{s^3(s^2+4)}$	$\frac{1/4}{s^3}$
$\frac{-s/4}{s^2(s^2+4)}$	0
$\frac{-1/4}{s(s^2+4)}$	

It follows that  $y_p = \frac{t^2}{8}$  and the general solution is  $y = \frac{t^2}{8} + c_1 + c_2 \cos 2t + c_3 \sin 2t$ .

13. We have  $q(s) = s^3 - s^2 + s - 1 = (s-1)(s^2+1)$  and  $\mathcal{L}\{4\cos t\} = \frac{4s}{s^2+1}$ . Thus  $Y(s) = \frac{4s}{(s-1)(s^2+1)^2}$ . The partial fraction decomposition algorithm gives

Incomplete $s^2+1$ -chain	
$\frac{4s}{(s-1)(s^2+1)^2}$	$\frac{-2s+2}{(s^2+1)^2}$
$\frac{p(s)}{(s-1)(s^2+1)}$	



It follows from Table 2.9 that  $y_p = (-t \sin t + \sin t - t \cos t)$ . But since  $\sin t$  is a homogeneous solution we can write the general solution as  $y = -t(\sin t + \cos t) + c_1 e^t + c_2 \cos t + c_2 \sin t$ .

## SECTION 4.4

1. Here  $L_1 = D - 6$  and  $L_2 = D$ . It is easy to see that  $L = q(D)$ , where  $q(s) = s^2 - 6s + 8 = (s - 2)(s - 4)$ . Therefore  $y_1$  and  $y_2$  are linear combinations of  $\mathcal{B}_q = \{e^{2t}, e^{4t}\}$ . Next we recursively extend the initial values to derivatives of order 1 to get

$$\begin{aligned} y_1(0) &= 2 & y_2(0) &= -1 \\ y_1'(0) &= 16 & y_2'(0) &= 4 \end{aligned}$$

If  $y = c_1 e^{2t} + c_2 e^{4t}$  then

$$\begin{aligned} c_1 + c_2 &= y(0) \\ 2c_1 + 4c_2 &= y'(0) \end{aligned}$$

For  $y_1$  we get

$$\begin{aligned} c_1 + c_2 &= 2 \\ 2c_1 + 4c_2 &= 16 \end{aligned}$$

which gives  $c_1 = -4$  and  $c_2 = 6$ . Thus  $y_1(t) = -4e^{2t} + 6e^{4t}$ . For  $y_2$  we get

$$\begin{aligned} c_1 + c_2 &= -1 \\ 2c_1 + 4c_2 &= 4 \end{aligned}$$

which gives  $c_1 = -4$  and  $c_2 = 3$ . Thus  $y_2(t) = -4e^{2t} + 3e^{4t}$ .

3. Here  $L_1 = D$  and  $L_2 = D$ . It is easy to see that  $L = q(D)$ , where  $q(s) = s^2 + 4$ . Therefore  $y_1$  and  $y_2$  are linear combinations of  $\mathcal{B}_q = \{\cos 2t, \sin 2t\}$ . Next we recursively extend the initial values to derivatives of order 1 to get

$$\begin{aligned} y_1(0) &= 1 & y_2(0) &= -1 \\ y_1'(0) &= -2 & y_2'(0) &= 2 \end{aligned}$$

If  $y = c_1 \cos 2t + c_2 \sin 2t$  then

$$\begin{aligned} c_1 &= y(0) \\ 2c_2 &= y'(0) \end{aligned}$$

For  $y_1$  we get

$$\begin{aligned} c_1 &= 1 \\ 2c_2 &= -2 \end{aligned}$$

which gives  $c_1 = 1$  and  $c_2 = -1$ . Thus  $y_1(t) = \cos 2t - \sin 2t$ . For  $y_2$  we get

$$\begin{aligned} c_1 &= -1 \\ 2c_2 &= 2 \end{aligned}$$

which gives  $c_1 = -1$  and  $c_2 = 1$ . Thus  $y_2(t) = -\cos 2t + \sin 2t$ .

5. Here  $L_1 = D + 4$  and  $L_2 = D^2 - 6D + 23$ . It is easy to see that  $L = q(D)$ , where  $q(s) = (s+4)(s^2-6s+23)-90 = (s^2-1)(s-2) = (s+1)(s-1)(s-2)$ . Therefore  $y_1$  and  $y_2$  are linear combinations of  $\mathcal{B}_q = \{e^{-t}, e^t, e^{2t}\}$ . Next we recursively extend the initial values to derivatives of order 2 to get

$$\begin{aligned} y_1(0) &= 0 & y_2(0) &= 2 \\ y_1'(0) &= 20 & y_2'(0) &= 2 \\ y_1''(0) &= -60 & y_2''(0) &= -34. \end{aligned}$$

If  $y = c_1 e^{-t} + c_2 e^t + c_3 e^{2t}$  then

$$\begin{aligned} c_1 + c_2 + c_3 &= y(0) \\ -c_1 + c_2 + 2c_3 &= y'(0) \\ c_1 + c_2 + 4c_3 &= y''(0). \end{aligned}$$

For  $y_1$  we get

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ -c_1 + c_2 + 2c_3 &= 20 \\ c_1 + c_2 + 4c_3 &= -60. \end{aligned}$$

which gives  $c_1 = -20$ ,  $c_2 = 40$ , and  $c_3 = -20$ . Thus  $y_1(t) = -20e^{-t} + 40e^t - 20e^{2t}$ . For  $y_2$  we get

$$\begin{aligned} c_1 + c_2 + c_3 &= 2 \\ -c_1 + c_2 + 2c_3 &= 2 \\ c_1 + c_2 + 4c_3 &= -34. \end{aligned}$$

which gives  $c_1 = -6$ ,  $c_2 = 20$ , and  $c_3 = -12$ . Thus  $y_2(t) = -6e^{-t} + 20e^t - 12e^{2t}$ .

7. Here  $L_1 = D^2 + 2D + 6$  and  $L_2 = D^2 - 2D + 6$ . It is easy to see that  $L = q(D)$ , where  $q(s) = (s^2 + 2s + 6)(s^2 - 2s + 6) - 45 = s^4 + 8s^2 - 9 = (s^2 - 1)(s^2 + 9) = (s - 1)(s + 1)(s^2 + 9)$ . Therefore  $y_1$  and  $y_2$  are linear combinations of  $\mathcal{B}_q = \{e^t, e^{-t}, \cos 3t, \sin 3t\}$ . Next we recursively extend the initial values to derivatives of order 3 to get

$$\begin{aligned} y_1(0) &= 0 & y_2(0) &= 6 \\ y_1'(0) &= 0 & y_2'(0) &= 6 \\ y_1''(0) &= 30 & y_2''(0) &= -24 \\ y_1'''(0) &= -30 & y_2'''(0) &= -84 \end{aligned}$$

If  $y = c_1 e^t + c_2 e^{-t} + c_3 \cos 3t + c_4 \sin 3t$  then

$$\begin{aligned} c_1 + c_2 + c_3 &= y(0) \\ c_1 - c_2 + 3c_4 &= y'(0) \\ c_1 + c_2 - 9c_3 &= y''(0) \\ c_1 - c_2 - 27c_4 &= y'''(0) \end{aligned}$$

For  $y_1$  we get

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_1 - c_2 + 3c_4 &= 0 \\ c_1 + c_2 - 9c_3 &= 30 \\ c_1 - c_2 - 27c_4 &= -30 \end{aligned}$$

which gives  $c_1 = 0$ ,  $c_2 = 3$ ,  $c_3 = -3$ , and  $c_4 = 1$ . Thus  $y_1(t) = 3e^{-t} - 3 \cos 3t + \sin 3t$ . For  $y_2$  we get

$$\begin{aligned} c_1 + c_2 + c_3 &= 6 \\ c_1 - c_2 + 3c_4 &= 6 \\ c_1 + c_2 - 9c_3 &= -24 \\ c_1 - c_2 - 27c_4 &= -84 \end{aligned}$$

which gives  $c_1 = 0$ ,  $c_2 = 3$ ,  $c_3 = 3$  and  $c_4 = 3$ . Thus and  $y_2(t) = 3e^{-t} + 3 \cos 3t + 3 \sin 3t$ .

9. Here  $a = 2$ ,  $b = 1$ , and  $c = 2$  and the coupled system that describes the motion is given by

$$\begin{aligned} y_1'' + 3y_1 &= y_2 \\ y_2'' + 2y_2 &= 2y_1. \end{aligned}$$

Let  $\mathbf{L}_1 = \mathbf{D}^2 + 3$  and  $\mathbf{L}_2 = \mathbf{D}^2 + 2$ . Then  $y_1$  and  $y_2$  a solutions to  $q(\mathbf{D})y = 0$ , where  $q(s) = (s^2+3)(s^2+2)-2 = s^4+5s^2+4 = (s^2+1)(s^2+4)$ . Thus  $y_1$  and  $y_2$  a linear combinations of  $\mathcal{B}_q = \{\cos t, \sin t, \cos 2t, \sin 2t\}$ . Next we recursively extend the initial values to derivatives of order 3 to get

$$\begin{aligned} y_1(0) &= 3 & y_2(0) &= 0 \\ y_1'(0) &= 3 & y_2'(0) &= 0 \\ y_1''(0) &= -9 & y_2''(0) &= 6 \\ y_1'''(0) &= -9 & y_2'''(0) &= 6 \end{aligned}$$

If  $y = c_1 \cos t + c_2 \sin t + c_3 \cos 2t + c_4 \sin 2t$  then

$$\begin{aligned} c_1 + c_3 &= y(0) \\ c_2 + 2c_4 &= y'(0) \\ -c_1 - 4c_3 &= y''(0) \\ -c_2 - 8c_4 &= y'''(0) \end{aligned}$$

For  $y_1$  we get

$$\begin{array}{rccccccc} c_1 & & & + & c_3 & & = & 3 \\ & c_2 & & & & + & 2c_4 & = & 3 \\ -c_1 & & & - & 4c_3 & & = & -9 \\ & -c_2 & & & & - & 8c_4 & = & -9 \end{array}$$

which gives  $c_1 = 1$ ,  $c_2 = 1$ ,  $c_3 = 2$ , and  $c_4 = 1$ . Thus  $y_1(t) = \cos t + \sin t + 2 \cos 2t + \sin 2t$ . For  $y_2$  we get

$$\begin{array}{rccccccc} c_1 & & & + & c_3 & & = & 0 \\ & c_2 & & & & + & 2c_4 & = & 0 \\ -c_1 & & & - & 4c_3 & & = & 6 \\ & -c_2 & & & & - & 8c_4 & = & 6 \end{array}$$

which gives  $c_1 = 2$ ,  $c_2 = 2$ ,  $c_3 = -2$  and  $c_4 = -1$ . Thus  $y_2(t) = 2 \cos t + 2 \sin t - 2 \cos 2t - \sin 2t$ .

11. 1. We begin by taking the Laplace transform of each equation above to get

$$\begin{aligned} q_1(s)Y_1(s) - p_1(s) &= \lambda_1 Y_2(s) \\ q_2(s)Y_2(s) - p_2(s) &= \lambda_2 Y_1(s) \end{aligned}$$

which can be rewritten:

$$\begin{aligned} q_1(s)Y_1(s) - \lambda_1 Y_2(s) &= p_1(s) \\ q_2(s)Y_2(s) - \lambda_2 Y_1(s) &= p_2(s). \end{aligned}$$

In matrix form this becomes

$$\begin{pmatrix} q_1(s) & -\lambda_1 \\ -\lambda_2 & q_2(s) \end{pmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} p_1(s) \\ p_2(s) \end{pmatrix}$$

2. The inverse of the coefficient matrix is

$$\begin{pmatrix} q_1(s) & -\lambda_1 \\ -\lambda_2 & q_2(s) \end{pmatrix}^{-1} = \frac{1}{q_1(s)q_2(s) - \lambda_1\lambda_2} \begin{pmatrix} q_2(s) & \lambda_1 \\ \lambda_2 & q_1(s) \end{pmatrix}$$

and therefore

$$\begin{aligned} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} &= \frac{1}{q_1(s)q_2(s) - \lambda_1\lambda_2} \begin{pmatrix} q_2(s) & \lambda_1 \\ \lambda_2 & q_1(s) \end{pmatrix} \begin{pmatrix} p_1(s) \\ p_2(s) \end{pmatrix} \\ &= \frac{1}{q_1(s)q_2(s) - \lambda_1\lambda_2} \begin{pmatrix} p_1(s)q_2(s) + \lambda_1 p_2(s) \\ p_2(s)q_1(s) + \lambda_2 p_1(s) \end{pmatrix}. \end{aligned}$$

13. We first take the Laplace transform of each equation to get

$$\begin{aligned} sY_1(s) - 2 - Y_1 &= -2Y_2(s) \\ sY_2(s) - (-2) - Y_2(s) &= 2Y_1(s). \end{aligned}$$

We associate the  $Y_1$  and  $Y_2$ . In matrix form we get

$$\begin{pmatrix} s-1 & 2 \\ -2 & s-1 \end{pmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

By matrix inversion we get

$$\begin{aligned} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} &= \begin{pmatrix} s-1 & 2 \\ -2 & s-1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ -2 \end{pmatrix} \\ &= \frac{1}{(s-1)^2 + 4} \begin{pmatrix} s-1 & -2 \\ 2 & s-1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \end{pmatrix} \\ &= \frac{1}{(s-1)^2 + 4} \begin{pmatrix} 2s+2 \\ -2s+6 \end{pmatrix} = \begin{pmatrix} \frac{2(s-1)+4}{(s-1)^2+2^2} \\ \frac{-2(s-1)+4}{(s-1)^2+2^2} \end{pmatrix} \end{aligned}$$

We now get by Laplace inversion

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 2e^t \cos 2t + 2e^t \sin 2t \\ -2e^t \cos 2t + 2e^t \sin 2t \end{pmatrix}.$$

**15.** We first take the Laplace transform of each equation to get

$$\begin{aligned} sY_1(s) - 1 + 2Y_1 &= 5Y_2(s) \\ s^2Y_2(s) - 3 - 2(sY_2(s)) + 5Y_2(s) &= 2Y_1(s). \end{aligned}$$

We associate the  $Y_1$  and  $Y_2$ . In matrix form we get

$$\begin{pmatrix} s+2 & -5 \\ -2 & s^2-2s+5 \end{pmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

By matrix inversion we get

$$\begin{aligned} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} &= \begin{pmatrix} s+2 & -5 \\ -2 & s^2-2s+5 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ &= \frac{1}{s^3+s} \begin{pmatrix} s^2-2s+5 & 5 \\ 2 & s+2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ &= \frac{1}{s(s^2+1)} \begin{pmatrix} s^2-2s+20 \\ 3s+8 \end{pmatrix} = \begin{pmatrix} \frac{20}{s} - 19\frac{s}{s^2+1} - 2\frac{1}{s^2+1} \\ \frac{8}{s} - 8\frac{s}{s^2+1} + 3\frac{1}{s^2+1} \end{pmatrix} \end{aligned}$$

We now get by Laplace inversion

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 20 - 19 \cos t - 2 \sin t \\ 8 - 8 \cos t + 3 \sin t \end{pmatrix}.$$

### SECTION 4.5

1. The only characteristic mode is  $e^{-5t}$ . Thus the zero-input response is  $y(t) = ce^{-5t}$ . The initial condition  $a = y(0) = 10$  implies  $c = 10$ . Thus  $y(t) = 10e^{-5t}$ . The characteristic value is  $-5$ , to the left of the imaginary axis. Hence the system is stable.
3. The characteristic polynomial is  $q(s) = s^2 - 4s + 3 = (s - 3)(s - 1)$ . The characteristic modes are  $\{e^t, e^{3t}\}$ . Thus  $y(t) = c_1e^t + c_2e^{3t}$ . The initial condition  $\mathbf{a} = (2, 4) = (y(0), y'(0))$  implies  $c_1 = 1$  and  $c_2 = 1$ . Thus the zero-input response is  $y(t) = e^t + e^{3t}$ . The characteristic values are  $1, 3$  and both lie to the right of the imaginary axis. Hence the system is unstable.
5. The characteristic polynomial is  $q(s) = s^2 + 4s + 5 = (s + 2)^2 + 1$ . The characteristic modes are  $\{e^{-2t} \cos t, e^{-2t} \sin t\}$ . Thus  $y(t) = c_1e^{-2t} \cos t + c_2e^{-2t} \sin t$ . The initial condition  $\mathbf{a} = (0, 1) = (y(0), y'(0))$  implies  $c_1 = 0$  and  $c_2 = 1$ . Thus the zero-input response is  $y(t) = e^{-2t} \sin t$ . The characteristic values are  $-2 + i, -2 - i$  and both lie to the left of the imaginary axis. Hence the system is stable.
7. The characteristic polynomial is  $q(s) = s^2 + 6s + 9 = (s + 3)^2$ . The characteristic modes are  $\{e^{-3t}, te^{-3t}\}$ . Thus  $y(t) = c_1e^{-3t} + c_2te^{-3t}$ . The initial condition  $\mathbf{a} = (1, 1) = (y(0), y'(0))$  implies  $c_1 = 1$  and  $c_2 = 4$ . Thus the zero-input response is  $y(t) = e^{-3t} + 4te^{-3t}$ . The characteristic value is  $-3$  with multiplicity 2 lies to the left of the imaginary axis. Hence the system is stable.
9. The characteristic polynomial is  $q(s) = s^2 - 2s + 2 = (s - 1)^2 + 1$ . The characteristic modes are  $\{e^t \cos t, e^t \sin t\}$ . Thus  $y(t) = c_1e^t \cos t + c_2e^t \sin t$ . The initial condition  $\mathbf{a} = (1, 2) = (y(0), y'(0))$  implies  $c_1 = 1$  and  $c_2 = 1$ . Thus the zero-input response is  $y(t) = e^t \cos t + e^t \sin t$ . The characteristic values are  $1 + i, 1 - i$  and lie to the right of the imaginary axis. Hence the system is unstable.
11. The characteristic polynomial is  $q(s) = (s + 1)(s^2 + 1)$ . The characteristic modes are  $\{e^{-t}, \cos t, \sin t\}$ . Thus  $y(t) = c_1e^{-t} + c_2 \cos t + c_3 \sin t$ . The initial condition  $\mathbf{a} = (1, -1, 1) = (y(0), y'(0), y''(0))$  implies  $c_1 = 1$ ,  $c_2 = 0$ , and  $c_3 = 0$ . Thus the zero-input response is  $y(t) = e^{-t}$ . The characteristic values are  $-1, i$ , and  $-i$ . The system is then marginally stable.

13. The characteristic mode is  $e^{-t}$  so  $y(t) = ce^{-t}$ . For the unit impulse we have  $y(0) = 1$  and this implies  $c = 1$ . Thus  $y(t) = e^{-t}$ .
15. The characteristic polynomial is  $q(s) = s^2 - 4 = (s - 2)(s + 2)$  and hence the characteristic modes are  $\{e^{2t}, e^{-2t}\}$ . Hence,  $y(t) = c_1 e^{2t} + c_2 e^{-2t}$ . For the unit impulse we have  $y(0) = 0$  and  $y'(0) = 1$  and this implies  $c_1 = 1/4$  and  $c_2 = -1/4$ . Thus  $y(t) = \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}$ .
17. The characteristic polynomial is  $q(s) = s^3 + s = s(s^2 + 1)$ . The characteristic modes are  $\{1, \cos 2t, \sin 2t\}$ . For the unit impulse we have  $y(0) = 0$ ,  $y'(0) = 0$ , and  $y''(0) = 1$  and this implies  $c_1 = 1$ ,  $c_2 = -1$  and  $c_3 = 0$ . Thus  $y(t) = 1 - \cos(t)$ .
19. Since  $f$  is bounded there is an  $M$  such that  $|f(t)| \leq M$  for all  $t \geq 0$ . We then have

$$\begin{aligned} |t^k e^{\alpha t} \cos \beta t * f(t)| &= \left| \int_0^t x^k e^{\lambda x} \cos \beta x f(t-x) dx \right| \\ &\leq \int_0^t x^k e^{\alpha x} |f(t-x)| dx \\ &\leq M \int_0^t x^k e^{\alpha x} dx \\ &= M(C + p(t)e^{\alpha t}), \end{aligned}$$

where  $C$  and  $p(t)$  are as in Exercise 18, which also implies that  $t^k e^{\alpha t} \cos \beta t * f$  is bounded. The argument for  $t^k e^{\alpha t} \sin \beta t * f$  is the same.

## SECTION 5.1

1. No, it is not linear because of the presence of the product  $y'y$ .
3. yes, nonhomogeneous, yes
5. yes, nonhomogeneous, no
7. yes, nonhomogeneous, no
9. No, it is not linear because of the presence of  $\sin y$ .
11. yes, homogeneous, no
13.
  1.  $L(\frac{1}{t}) = t^2(2t^{-3}) + t(-t^{-2}) - t^{-1} = (2 - 1 - 1)t^{-1} = 0$
  2.  $L(1) = t^2(0) + t(0) - 1 = -1$
  3.  $L(t) = t^2(0) + t(1) - t = 0$
  4.  $L(t^r) = t^2 r(r-1)t^{r-2} + t(rt^{r-1}) - t^r = (r^2 - 1)t^r$

15.  $y'_p = C(2t - t^2)e^{-t}$  and  $y''_p = C(t^2 - 4t + 2)e^{-t}$ . Thus

$$\begin{aligned} t^2 y''_p + t y'_p - y_p &= C(t^3 - 4t^2 + 2t)e^{-t} + C(-t^3 + 3t^2 - 2t)e^{-t} + C(-t^2)e^{-t} \\ &= C(-2t^2)e^{-t} \end{aligned}$$

The equation  $C(-2t^2)e^{-t} = t^2 e^{-t}$  implies  $C = \frac{-1}{2}$ .

17. If  $y = e^{-t}$  then  $y' = -e^{-t}$  and  $y'' = e^{-t}$  so that  $\mathbf{L}y = (t-1)e^{-t} - t(-e^{-t}) + e^{-t} = 2te^{-t}$ . Parts (1) follows. If  $y = e^t$  then  $\mathbf{L}y = (t-1)(e^t) - t(e^t) + (e^t) = 0$ . It follows that  $y = e^t$  is a solution to  $\mathbf{L}y = 0$ . If  $y = t$  then  $y' = 1$  and  $y'' = 0$ . Thus  $\mathbf{L}y = (t-1)(0) - t(1) + t = 0$ . Part (2) now follows. By linearity every function of the form  $y(t) = e^{-t} + c_1 e^t + c_2 t$  is a solution to  $\mathbf{L}y = 2te^{-t}$ , where  $c_1$  and  $c_2$  are constants. If we want a solution to  $\mathbf{L}(y) = 2te^{-t}$  with  $y(0) = a$  and  $y'(0) = b$ , then we need to solve for  $c_1$  and  $c_2$ : Since  $y(t) = e^{-t} + c_1 e^t + c_2 t$  we have  $y'(t) = -e^{-t} + c_1 e^t + c_2$ . Hence,

$$\begin{aligned} a &= y(0) = 1 + c_1 \\ b &= y'(0) = -1 + c_1 + c_2. \end{aligned}$$

These equations give  $c_1 = a - 1$  and  $c_2 = b - a + 2$ . Particular choices of  $a$  and  $b$  give the answers for Part (3).

- (3)a.  $y(t) = e^{-t} - e^t + 2t$
- (3)b.  $y(t) = e^{-t} + (0)e^t + (1)t = e^{-t} + t$
- (3)c.  $y(t) = e^{-t} + (-e^t + 3t)$
- (3)d.  $y(t) = e^{-t} + (a-1)e^t + (b-a+2)t$

19. Write the equation in the standard form:

$$y'' + \frac{3}{t}y' - \frac{1}{t^2}y = t^2.$$

The forcing function is continuous on  $\mathbb{R}$  while the coefficient functions,  $\frac{3}{t}$  and  $-\frac{1}{t^2}$ , are continuous except at  $t = 0$ . Thus the largest intervals of common continuity are  $(0, \infty)$  and  $(-\infty, 0)$ . Since the initial conditions are given at  $t_0 = -1$  it follows from Theorem 6 that the interval  $(-\infty, 0)$  is the largest interval with a unique solution.

21. Write the equation in the standard form:

$$y'' + \frac{y}{\sin t} = \frac{\cos t}{\sin t}.$$



The intervals of continuity are of the form  $(k\pi, (k+1)\pi)$ ,  $k \in \mathbb{Z}$ . Since  $t_0 = \frac{\pi}{2}$  it follows that the maximal interval for a unique solution is  $(0, \pi)$ .

- 23.** The common interval of continuity of the coefficient functions is  $(3, \infty)$  and  $t_0 = 10$  is in this interval.
- 25.** The initial condition occurs at  $t = 0$  which is precisely where  $a_2(t) = t^2$  has a zero. Theorem 6 does not apply.
- 27.** The assumptions say that  $y_1(t_0) = y_2(t_0)$  and  $y'_1(t_0) = y'_2(t_0)$ . Both  $y_1$  and  $y_2$  therefore satisfies the same initial conditions. By the uniqueness part of Theorem 6  $y_1 = y_2$ .

## SECTION 5.2

1. dependent;  $2t$  and  $5t$  are multiples of each other.
3. independent; If  $c_1 \ln t + c_2 t \ln t = 0$  then evaluating at  $t = e$  and  $t = e^2$  gives  $c_1 + ec_2 = 0$  and  $2c_1 + 2e^2c_2 = 0$ . These equations imply that  $c_1$  and  $c_2$  are both zero so  $\{\ln t, t \ln t\}$  is linearly independent.
5. independent, If  $c_1 \ln 2t + c_2 \ln 5t = 0$  then evaluating at  $t = 1$  and  $t = e$  gives  $(\ln 2)c_1 + (\ln 5)c_2 = 0$  and  $(1 + \ln 2)c_1 + (\ln 5 + 1)c_2 = 0$ . These equations imply that  $c_1$  and  $c_2$  are both zero so  $\{\ln t, t \ln t\}$  is linearly independent.
7.  $f'_1(t) = e^t - 1$  and  $f''_2(t) = e^t$ . Thus  $(t-1)f''_1 - tf'_1 + f_1 = (t-1)(e^t) - t(e^t - 1) + e^t - t = 0$ . Similarly,  $f'_2(t) = 1$  and  $f''_2(t) = 0$ . Thus  $(t-1)f''_2 - tf'_2 + f_2 = -t(1) + t = 0$ . Now,

$$w(t) = \begin{vmatrix} e^t - t & t \\ e^t - 1 & 1 \end{vmatrix} = e^t - t - (e^t - 1)t = (1 - t)e^t.$$

On the other hand the coefficient function of  $y'$  in the standard form of the differential equation is  $a_1(t) = -\frac{t}{t-1} = -1 - \frac{1}{t-1}$ . Integrating gives  $\int_0^t -1 - \frac{1}{x-1} dx = -x - \ln|x-1| \Big|_0^t = -t + \ln(1-t)$ , (since  $x-1 < 0$ ) and  $e^{-\int_0^t a_1(x) dx} = e^t(1-t)$ . At  $t = 0$  we have  $w(1) = 1$  so Abel's formula is verified. It follows from Proposition 4 that  $f_1$  and  $f_2$  are linearly independent. By Theorem 2 the solution set is  $\{c_1(e^t - t) + c_2t : c_1, c_2 \in \mathbb{R}\}$

9.  $f'_1(t) = \frac{-2\sin(2\ln t)}{t}$ ,  $f''_1(t) = \frac{2\sin(2\ln t) - 4\cos(2\ln t)}{t^2}$ ,  $f'_2(t) = \frac{2\cos(2\ln t)}{t}$ , and  $f''_2(t) = \frac{-4\sin(2\ln t) - 2\cos(2\ln t)}{t^2}$ . Thus  $t^2 f''_1 + t f'_1 + 4f_1 = 2\sin(2\ln t) - 4\cos(2\ln t) - 2\sin(2\ln t) + 4\cos(2\ln t) = 0$ . Similarly,  $t^2 f''_2 + t f'_2 + 4f_2 = -4\sin(2\ln t) - 2\cos(2\ln t) + 2\cos(2\ln t) + 4\sin(2\ln t) = 0$ . Now,

$$w(t) = \left| \begin{array}{cc} \cos(2 \ln t) & \sin(2 \ln t) \\ \frac{-2 \sin(2 \ln t)}{t} & \frac{2 \cos(2 \ln t)}{t} \end{array} \right| = \frac{4}{t}.$$

On the other hand the coefficient function of  $y'$  in the standard form of the differential equation is  $a_1(t) = \frac{1}{t}$ . Integrating gives  $\int_1^t \frac{1}{x} dx = \ln t$  and  $e^{-\int_0^t a_1(x) dx} = 1/t$ . At  $t = 1$  we have  $w(1) = 1$  so Abel's formula is verified. It follows from Proposition 4 that  $f_1$  and  $f_2$  are linearly independent. By Theorem 2 the solution set is  $\{c_1 \cos(2 \ln t) + c_2 \sin(2 \ln t) : c_1, c_2 \in \mathbb{R}\}$

11. 1. Suppose  $at^3 + b|t^3| = 0$  on  $(-\infty, \infty)$ . Then for  $t = 1$  and  $t = -1$  we get

$$\begin{aligned} a + b &= 0 \\ -a + b &= 0. \end{aligned}$$

These equations imply  $a = b = 0$ . So  $y_1$  and  $y_2$  are linearly independent.

2. Observe that  $y_1'(t) = 3t^2$  and  $y_2'(t) = \begin{cases} -3t^2 & \text{if } t < 0 \\ 3t^2 & \text{if } t \geq 0. \end{cases}$  If  $t < 0$

then  $w(y_1, y_2)(t) = \begin{pmatrix} t^3 & -t^3 \\ 3t^2 & -3t^2 \end{pmatrix} = 0$ . If  $t \geq 0$  then  $w(y_1, y_2)(t) = \begin{pmatrix} t^3 & t^3 \\ 3t^2 & 3t^2 \end{pmatrix} = 0$ . It follows that the Wronskian is zero for all  $t \in (-\infty, \infty)$ .

3. The condition that the coefficient function  $a_2(t)$  be nonzero in Theorem 2 and Proposition 4 is essential. Here the coefficient function,  $t^2$ , of  $y''$  is zero at  $t = 0$ , so Proposition 4 does not apply on  $(-\infty, \infty)$ . The largest open intervals on which  $t^2$  is nonzero are  $(-\infty, 0)$  and  $(0, \infty)$ . On each of these intervals  $y_1$  and  $y_2$  are linearly dependent.
4. Consider the cases  $t < 0$  and  $t \geq 0$ . The verification is then straightforward.
5. Again the condition that the coefficient function  $a_2(t)$  be nonzero is essential. The Uniqueness and Existence theorem does not apply.

### SECTION 5.3

1. The indicial polynomial is  $Q(s) = s^2 + s - 2 = (s + 2)(s - 1)$ . There are two distinct roots 1 and  $-2$ . The fundamental set is  $\{t, t^{-2}\}$ . The general solution is  $y(t) = c_1 t + c_2 t^{-2}$ .

3. The indicial polynomial is  $Q(s) = 9s^2 - 6s + 1 = (3s - 1)^2$ . There is one root,  $1/3$ , with multiplicity 2. The fundamental set is  $\{t^{\frac{1}{3}}, t^{\frac{1}{3}} \ln t\}$ . The general solution is  $y(t) = c_1 t^{\frac{1}{3}} + c_2 t^{\frac{1}{3}} \ln t$ .
5. The indicial polynomial is  $Q(s) = 4s^2 - 4s + 1 = (2s - 1)^2$ . The root is  $\frac{1}{2}$  with multiplicity 2. The fundamental set is  $\{t^{\frac{1}{2}}, t^{\frac{1}{2}} \ln t\}$ . The general solution is  $y(t) = c_1 t^{\frac{1}{2}} + c_2 t^{\frac{1}{2}} \ln t$ .
7. The indicial polynomial is  $Q(s) = s^2 + 6s + 9 = (s + 3)^2$ . The root is  $-3$  with multiplicity 2. The fundamental set is  $\{t^{-3}, t^{-3} \ln t\}$ . The general solution is  $y(t) = c_1 t^{-3} + c_2 t^{-3} \ln t$ .
9. The indicial polynomial is  $Q(s) = s^2 - 4 = (s - 2)(s + 2)$ . There are two distinct roots, 2 and  $-2$ . The fundamental set is  $\{t^2, t^{-2}\}$ . The general solution is  $y(t) = c_1 t^2 + c_2 t^{-2}$ .
11. The indicial polynomial is  $Q(s) = s^2 - 4s + 13 = (s - 2)^2 + 9$ . There are two complex roots,  $2 + 3i$  and  $2 - 3i$ . The fundamental set is  $\{t^2 \cos(3 \ln t), t^2 \sin(3 \ln t)\}$ . The general solution is  $y(t) = c_1 t^2 \cos(3 \ln t) + c_2 t^2 \sin(3 \ln t)$ .
13. The indicial polynomial is  $Q(s) = 4s^2 - 4s + 1 = (2s - 1)(2s - 1)$ . There is a double root,  $r = \frac{1}{2}$ . The fundamental set is  $\{t^{\frac{1}{2}}, t^{\frac{1}{2}} \ln t\}$ . The general solution is  $y(t) = c_1 t^{\frac{1}{2}} + c_2 t^{\frac{1}{2}} \ln t$ . The initial conditions imply

$$\begin{aligned} c_1 &= 2 \\ \frac{1}{2}c_1 + c_2 &= 0. \end{aligned}$$

Thus  $c_1 = 2$  and  $c_2 = -1$ . Hence  $y = 2t^{1/2} - t^{1/2} \ln t$

15. The coefficient functions for the given equation in standard form are  $a_1(t) = -4/t$  and  $a_2(t) = 6/t^2$  both of which are not defined at the initial condition  $t_0 = 0$ . Thus the uniqueness and existence theorem does not guarantee a solution. In fact, the condition that  $y'(0)$  exist presupposes that  $y$  is defined near  $t = 0$ . For  $t$  positive the indicial polynomial is  $Q(s) = s^2 - 5s + 6 = (s - 6)(s + 1)$  and therefore  $y(t) = c_1 t^6 + c_2 t^{-1}$ . The only way that  $y$  can be extended to  $t = 0$  is that  $c_2 = 0$ . In this case  $y(t) = c_1 t^6$  cannot satisfy the given initial conditions. Thus, no solution is possible.

## SECTION 5.4

1. By L'Hospital's rule  $\lim_{t \rightarrow 0} \frac{e^{bt} - e^{at}}{t} = b - a$ . So Theorem 4 applies and gives

$$\begin{aligned}\mathcal{L}\left\{\frac{e^{bt} - e^{at}}{t}\right\}(s) &= \int_s^\infty \frac{1}{\sigma - b} - \frac{1}{\sigma - a} d\sigma \\ &= \lim_{M \rightarrow \infty} \ln\left(\frac{M - b}{M - a}\right) - \ln\left(\frac{s - b}{s - a}\right) \\ &= \ln\left(\frac{s - a}{s - b}\right)\end{aligned}$$

3. Apply L'Hospital's rule twice to get  $\lim_{t \rightarrow \infty} 2 \frac{\cos bt - \cos at}{t^2} = a^2 - b^2$ . Now use Exercise 2 to get

$$\begin{aligned}\mathcal{L}\left\{2 \frac{\cos bt - \cos at}{t^2}\right\}(s) &= \int_s^\infty \ln\left(\frac{\sigma^2 + a^2}{\sigma^2 + b^2}\right) d\sigma \\ &= \lim_{M \rightarrow \infty} \left( \int_s^M \ln(\sigma^2 + a^2) d\sigma - \int_s^M \ln(\sigma^2 + b^2) d\sigma \right).\end{aligned}$$

We now use two facts from calculus:

1.  $\int \ln(x^2 + a^2) dx = x \ln(x^2 + a^2) - 2x + 2a \tan^{-1}(x/a) + C$
2.  $\lim_{x \rightarrow \infty} x \ln\left(\frac{x^2 + a^2}{x^2 + b^2}\right) = 0$

The first fact is shown by integration by parts and the second fact is shown by L'Hospital's rule. We now get (after some simplifications)

$$\mathcal{L}\left\{2 \frac{\cos bt - \cos at}{t^2}\right\}(s) = s \ln\left(\frac{s^2 + b^2}{s^2 + a^2}\right) + 2a \tan^{-1}\left(\frac{a}{s}\right) - 2b \tan^{-1}\left(\frac{b}{s}\right)$$

5. Applying the Laplace transform we get

$$Y' + \frac{3s + 2}{s^2 + s} Y = \frac{2y_0}{s^2 + s}.$$

The integrating factor is  $I = s^2(s + 1)$ ; we get  $Y(s) = \frac{y_0}{s+1} + \frac{C}{s^2(s+1)}$ . Laplace inversion gives

$$\begin{aligned}y(t) &= y_0 e^{-t} + C(t - 1 + e^{-t}) \\ &= (y_0 + C)e^{-t} + C(t - 1).\end{aligned}$$

Let  $c_1 = C$  and  $c_2 = y_0 + C$  to get  $y(t) = c_1 e^{-t} + c_2(t - 1)$ .

7. Applying the Laplace transform we get  $Y'(s) = \frac{-y_0}{(s+2)^2}$  and therefore  $Y(s) = \frac{y_0}{s+2} + C$ . However, since  $\lim_{s \rightarrow \infty} Y(s) = 0$  we must have  $C = 0$ . Hence  $y(t) = y_0 e^{-2t}$ .

9. Applying the Laplace transform we get  $Y'(s) + \frac{6s}{s^2+1}Y(s) = 0$ . An integrating factor is  $I = (s^2+1)^3$ . We then get  $Y(s) = \frac{C}{(s^2+1)^3}$ , and  $y(t) = (C/8)((3-t^2)\sin t - 3t\cos t)$

11. Apply the Laplace transform to get  $Y'(s) = \frac{-y_0}{s(s-1)} = y_0 \left( \frac{1}{s} - \frac{1}{s-1} \right)$ . Then  $Y(s) = y_0 \ln \left( \frac{s}{s-1} \right) + C$ . Take  $C = 0$  since  $\lim_{s \rightarrow \infty} Y(s) = 0$ . Hence  $y(t) = y_0 \frac{e^t - 1}{t}$ .

13. Apply the Laplace transform, simplify, and get  $Y'(s) = \frac{-y_0}{(s^2-5s+6)} = y_0 \left( \frac{1}{s-2} - \frac{1}{s-3} \right)$ . Then  $Y(s) = y_0 \ln \left( \frac{s-2}{s-3} \right) + C$ . Take  $C = 0$ . Then  $y(t) = y_0 \left( \frac{e^{3t} - e^{2t}}{t} \right)$ .

15. Apply the Laplace transform, simplify, and get  $Y'(s) = \frac{-sy_0}{s(s^2+1)} - \frac{2y_1}{s(s^2+1)} = -y_0 \frac{1}{s^2+1} - 2y_1 \left( \frac{1}{s} - \frac{s}{s^2+1} \right)$ . Integrating gives  $Y(s) = -y_0 \tan^{-1}(s) + y_1 \ln \left( \frac{s^2+1}{s^2} \right) + C$ . Since  $\lim_{s \rightarrow \infty} Y(s) = 0$  we must have  $C = y_0 \frac{\pi}{2}$  and hence  $Y(s) = y_0 \tan^{-1} \left( \frac{1}{s} \right) + y_1 \ln \left( \frac{s^2+1}{s^2} \right)$ . Therefore  $y(t) = y_0 \frac{\sin t}{t} + 2y_1 \frac{1 - \cos t}{t}$

17. We use the formula

$$\frac{d^n}{dt^n}(f(t)g(t)) = \sum_{k=0}^n \binom{n}{k} \frac{d^k}{dt^k} f(t) \cdot \frac{d^{n-k}}{dt^{n-k}} g(t).$$

Observe that

$$\frac{d^k}{dt^k} e^{-t} = (-1)^k e^{-t}$$

and

$$\frac{d^{n-k}}{dt^{n-k}} t^n = n(n-1) \cdots (k+1) t^k.$$

It now follows that

$$\begin{aligned}
 & \frac{1}{n!} e^t \frac{d^n}{dt^n} (e^{-t} t^n) \\
 = & \frac{1}{n!} e^t \sum_{k=0}^n \binom{n}{k} \frac{d^k}{dt^k} e^{-t} \frac{d^{n-k}}{dt^{n-k}} t^n \\
 = & e^t \sum_{k=0}^n \binom{n}{k} (-1)^k e^{-t} \frac{n(n-1) \cdots (k+1)}{n!} t^k \\
 = & \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{t^k}{k!} \\
 = & \ell_n(t).
 \end{aligned}$$

- 21.** Hint: Take the Laplace transform of each side. Use the previous exercise and the binomial theorem.
- 23.** We compute the Laplace transform of both sides. We'll do a piece at a time.

$$\begin{aligned}
 & \mathcal{L} \{ (2n+1) \ell_n \} (s) \\
 = & (2n+1) \frac{(s-1)^n}{s^{n+1}} \\
 = & \frac{(s-1)^{n-1}}{s^{n+2}} (2n+1)(s(s-1)).
 \end{aligned}$$

$$\begin{aligned}
 & \mathcal{L} \{ -t \ell_n \} (s) \\
 = & \left( \frac{(s-1)^n}{s^{n+1}} \right)' \\
 = & \frac{(s-1)^{n-1}}{s^{n+2}} (n+1-s).
 \end{aligned}$$

$$\begin{aligned}
 & -n \mathcal{L} \{ \ell_{n-1} \} (s) \\
 = & -n \frac{(s-1)^{n-1}}{s^n} \\
 = & \frac{(s-1)^{n-1}}{s^{n+2}} (-ns^2).
 \end{aligned}$$

We have written each so that the common factor is  $\frac{(s-1)^{n-1}}{s^{n+2}}$ . The coefficients are

$$\begin{aligned}
& n+1-s+(2n+1)(s(s-1))-ns^2 \\
&= (n+1)(s^2-2s+1) \\
&= (n+1)(s-1)^2
\end{aligned}$$

The right hand side is now

$$\begin{aligned}
& \frac{1}{n+1} \left( (n+1)(s-1)^2 \frac{(s-1)^{n-1}}{s^{n+2}} \right) \\
&= \frac{(s-1)^{n+1}}{s^{n+2}} \\
&= \mathcal{L}\{\ell_{n+1}\}(s).
\end{aligned}$$

Taking the inverse Laplace transform completes the verification.

**25.** First of all  $\int_0^\infty e^{-t}\ell_n(t) dt = \mathcal{L}\{\ell_n\}(1) = 0$ . Thus

$$\begin{aligned}
& \int_t^\infty e^{-x}\ell_n(x) dx \\
&= -\int_0^t e^{-x}\ell_n(x) dx \\
&= -e^{-t} \int_0^\infty e^{t-x}\ell_n(x) dx \\
&= -e^{-t}(e^t * \ell_n(t)).
\end{aligned}$$

By the convolution theorem

$$\begin{aligned}
& \mathcal{L}\{e^t * \ell_n\}(s) \\
&= \frac{1}{s-1} \frac{(s-1)^n}{s^{n+1}} \\
&= \frac{(s-1)^{n-1}}{s^{n+1}} \\
&= \frac{(s-1)^{n-1}}{s^n} \left(1 - \frac{s-1}{s}\right) \\
&= \frac{(s-1)^{n-1}}{s^n} - \frac{(s-1)^n}{s^{n+1}} \\
&= \mathcal{L}^{-1}\{\ell_{n-1}(t)\} - \mathcal{L}^{-1}\{\ell_n(t)\}.
\end{aligned}$$

It follows by inversion that  $e^t * \ell_n = \ell_{n-1} - \ell_n$  and substituting this formula into the previous calculation gives the needed result.

## SECTION 5.5

1. Let  $y_2(t) = t^2 u(t)$ . Then  $t^4 u'' + t^3 u' = 0$ , which gives  $u' = t^{-1}$  and  $u(t) = \ln t$ . Substituting gives  $y_2(t) = t^2 \ln t$ . The general solution can be written  $y(t) = c_1 t^2 + c_2 t^2 \ln t$ .
3. Let  $y_2(t) = t^{\frac{1}{2}} u(t)$ . Then  $4t^{\frac{5}{2}} u'' + 4t^{\frac{3}{2}} u' = 0$  leads to  $u' = 1/t$  and hence  $u(t) = \ln t$ . Thus  $y_2(t) = \sqrt{t} \ln t$ . The general solution can be written  $y(t) = c_1 \sqrt{t} + c_2 \sqrt{t} \ln t$ .
5. Let  $y_2(t) = tu(t)$ . Then  $u$  satisfies  $t^3 u'' - t^3 u' = 0$ . Thus  $u' = e^t$  and  $u = e^t$ . It follows that  $y_2(t) = te^t$  is a second independent solution. The general solution can be written  $y(t) = c_1 t + c_2 te^t$ .
7. Let  $y_2(t) = u(t) \sin t^2$ . Then  $u(t)$  satisfies  $t \sin t^2 u'' + (4t^2 \cos t^2 - \sin t^2) u' = 0$  and hence  $\frac{u''}{u'} = \frac{1}{t} - 4t \frac{\cos t^2}{\sin t^2}$ . It follows that  $u' = t \csc^2 t^2$  and therefore  $u(t) = \frac{-1}{2} \cot t^2$ . We now get  $y_2(t) = \frac{-1}{2} \cos t^2$ . The general solution can be written  $y(t) = c_1 \sin t^2 + c_2 \cos t^2$ .
9. Let  $y_2(t) = u(t) \tan t$ . Then  $u'' \tan t + 2u' \sec^2 t = 0$  which gives  $u' = \cot^2 t = \csc^2 t - 1$ . Hence  $u = -\cot t - t$  and  $y_2(t) = -1 - t \tan t$ . The general solution can be written  $y(t) = c_1 \tan t + c_2(1 + t \tan t)$ .
11. The functions  $\tan t$  and  $\sec t$  are continuous except at points of the form  $\frac{\pi}{2} + 2n\pi$ ,  $n \in \mathbb{Z}$ . We will work in the interval  $(-\pi/2, \pi/2)$ . Let  $y_2(t) = u(t) \tan t$ . Then  $u'' \tan t + u'(\tan^2 t + 2) = 0$  and hence  $\frac{u''}{u'} = -\tan t - 2 \cot t$ . It follows that  $\ln |u'| = \ln |\cos t| - 2 \ln |\sin t|$  and thus  $u' = \cos t \sin^{-2} t$ . Further  $u(t) = \frac{-1}{\sin t}$  and we have  $y_2(t) = -\sec t$ . The general solution can be written  $y(t) = c_1 \tan t + c_2 \sec t$ .
13. Let  $y_2 = u \frac{\sin 2t}{1+\cos 2t}$ . Then  $u(t)$  satisfies  $u'' \sin 2t + 4u' = 0$  and hence  $\frac{u''}{u'} = -4 \csc 2t$ . We now get  $\ln u' = 2 \ln |\csc 2t + \cot 2t|$ . Thus  $u' = (\csc 2t + \cot 2t)^2 = \csc^2 2t + 2 \csc 2t \cot 2t + \cot^2 2t = 2 \csc^2 2t + 2 \csc 2t \cot 2t - 1$ . By integrating we get  $u = -\cot 2t - \csc 2t - t = -\frac{1+\cos 2t}{\sin 2t} - t$ . It now follows that  $y_2 = -1 - \frac{t \sin 2t}{1+\cos 2t}$ . The general solution can be written  $y(t) = c_1 \frac{\sin 2t}{1+\cos 2t} + c_2 \left(1 + \frac{t \sin 2t}{1+\cos 2t}\right)$ .
15. Let  $y_2(t) = (1-t^2)u(t)$ . Substitution gives  $(1-t^2)^2 u'' - 4t(1-t^2)u' = 0$  and hence  $\frac{u''}{u'} = -2 \frac{-2t}{1-t^2}$ . From this we get  $u' = \frac{1}{(1-t^2)^2}$ . Integrating  $u'$  by partial fractions give  $u = \frac{1}{2} \frac{t}{1-t^2} + \frac{1}{4} \ln \left( \frac{1+t}{1-t} \right)$  and hence

$$y_2(t) = \frac{1}{2}t + \frac{1}{4}(1-t^2) \ln \left( \frac{1+t}{1-t} \right).$$



The general solution can be written

$$y = c_1(1 - t^2) + c_2 \left( \frac{1}{2}t + \frac{1}{4}(1 - t^2) \ln \left( \frac{1+t}{1-t} \right) \right).$$

## SECTION 5.6

1.  $\sin t$  and  $\cos t$  form a fundamental set for the homogeneous solutions. Let  $y_p(t) = u_1 \cos t + u_2 \sin t$ . Then the matrix equation
 
$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \sin t \end{pmatrix}$$
 implies  $u_1'(t) = -\sin^2 t = \frac{1}{2}(\cos 2t - 1)$  and  $u_2'(t) = \cos t \sin t = \frac{1}{2}(\sin 2t)$ . Integration give  $u_1(t) = \frac{1}{4}(\sin(2t) - 2t) = \frac{1}{2}(\sin t \cos t - t)$  and  $u_2(t) = \frac{-1}{4} \cos 2t = \frac{-1}{4}(2 \cos^2 t - 1)$ . This implies  $y_p(t) = \frac{1}{4} \sin t - \frac{1}{2}t \cos t$ . Since  $\frac{1}{4} \sin t$  is a homogeneous solution we can write the general solution in the form  $y(t) = \frac{-1}{2}t \cos t + c_1 \cos t + c_2 \sin t$ . We observe that a particular solution is the imaginary part of a solution to  $y'' + y = e^{it}$ . We use the incomplete partial fraction method and get  $Y(s) = \frac{1}{(s-i)^2(s+i)}$ . This can be written  $Y(s) = \frac{1}{2i} \frac{1}{(s-i)^2} + \frac{p(s)}{(s-i)(s+i)}$ . From this we get  $y_p(t) = \text{Im} \left( \frac{1}{2i} \mathcal{L}^{-1} \left\{ \frac{1}{(s-i)^2} \right\} \right) = \text{Im} \frac{-i}{2} t e^{it} = \frac{-1}{2} t \cos t$ . The general solution is  $y(t) = \frac{-1}{2} t \cos t + c_1 \cos t + c_2 \sin t$ .
3. The functions  $e^t \cos 2t$  and  $e^t \sin 2t$  form a fundamental set. Let  $y_p(t) = c_1 e^t \cos 2t + c_2 e^t \sin 2t$ . Then the matrix equation
 
$$W(e^t \cos 2t, e^t \sin 2t) \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ e^t \end{pmatrix}$$
 implies that  $u_1'(t) = \frac{-1}{2} \sin 2t$  and  $u_2'(t) = \frac{1}{2} \cos 2t$ . Hence,  $u_1(t) = \frac{1}{4} \cos 2t$  and  $u_2(t) = \frac{1}{4} \sin 2t$ . From this we get  $y_p(t) = \frac{1}{4} e^t \cos^2 2t + \frac{1}{4} e^t \sin^2 2t = \frac{1}{4} e^t$ . On the other hand, the method of undetermined coefficients implies that a particular solution is of the form  $y_p(t) = C e^t$ . Substitution gives  $4C e^t = e^t$  and hence  $C = \frac{1}{4}$ . It follows that  $y_p(t) = \frac{1}{4} e^t$ . Furthermore, the general solution is  $y(t) = \frac{1}{4} e^t + c_1 e^t \cos 2t + c_2 e^t \sin 2t$ .
5. A fundamental set is  $\{e^t, e^{2t}\}$ . The matrix equation
 
$$\begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ e^{3t} \end{pmatrix}$$
 implies  $u_1'(t) = -e^{2t}$  and  $u_2'(t) = e^t$ . Hence  $u_1(t) = \frac{-1}{2} e^{2t}$ ,  $u_2(t) = e^t$ , and  $y_p(t) = \frac{-1}{2} e^{2t} e^t + e^t e^{2t} = \frac{1}{2} e^{3t}$ . The general solution is  $y(t) = \frac{1}{2} e^{3t} + c_1 e^t + c_2 e^{2t}$ . The method of undetermined coefficients implies that a particular solution is of the form  $y_p = C e^{3t}$ . Substitution gives  $2C e^{3t} = 3e^{3t}$  and hence  $C = \frac{1}{2}$ . The general solution is as above.

7. A fundamental set is  $\{e^t, te^t\}$ . The matrix equation

$\begin{pmatrix} e^t & te^t \\ e^t & e^t + te^t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{e^t}{t} \end{pmatrix}$  implies  $u_1'(t) = -1$  and  $u_2'(t) = \frac{1}{t}$ . Hence,  $u_1(t) = -t$ ,  $u_2(t) = \ln t$ , and  $y_p(t) = -te^t + t \ln te^t$ . Since  $-te^t$  is a homogeneous solution we can write the general solution as  $y(t) = t \ln te^t + c_1 e^t + c_2 te^t$ .

9. The associated homogeneous equation is Cauchy-Euler with indicial equation  $s^2 - 3s + 2 = (s-2)(s-1)$ . It follows that  $\{t, t^2\}$  forms a fundamental set. We put the given equation in standard form to get  $y'' - \frac{2}{t}y' + \frac{2}{t^2}y = t^2$ . Thus  $f(t) = t^2$ . The matrix equation

$\begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ t^2 \end{pmatrix}$  implies  $u_1'(t) = -t^2$  and  $u_2'(t) = t$ . Hence  $u_1(t) = -\frac{t^3}{3}$ ,  $u_2(t) = \frac{t^2}{2}$ , and  $y_p(t) = -\frac{t^3}{3}t + \frac{t^2}{2}t^2 = \frac{t^4}{6}$ . It follows that the general solution is  $y(t) = \frac{t^4}{6} + c_1 t + c_2 t^2$ .

11. The homogeneous equation is Cauchy-Euler with indicial equation  $s^2 - 2s + 1 = (s-1)^2$ . It follows that  $\{t, t \ln t\}$  is a fundamental set. After writing in standard form we see the forcing function  $f(t)$  is  $\frac{1}{t}$ . The matrix equation

$\begin{pmatrix} t & t \ln t \\ 1 & \ln t + 1 \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{t} \end{pmatrix}$  implies  $u_1'(t) = \frac{-\ln t}{t}$  and  $u_2'(t) = \frac{1}{t}$ . Hence  $u_1(t) = \frac{-\ln^2 t}{2}$ ,  $u_2(t) = \ln t$ , and  $y_p(t) = \frac{-t}{2} \ln^2 t + t \ln^2 t = \frac{t}{2} \ln^2 t$ . The general solution is  $y(t) = \frac{t}{2} \ln^2 t + c_1 t + c_2 t \ln t$ .

13. The matrix equation

$\begin{pmatrix} \tan t & \sec t \\ \sec^2 t & \sec t \tan t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix}$  implies  $u_1'(t) = t$  and  $u_2'(t) = -t \sin t$ . Hence  $u_1(t) = \frac{t^2}{2}$ ,  $u_2(t) = t \cos t - \sin t$ , and  $y_p(t) = \frac{t^2}{2} \tan t + (t \cos t - \sin t) \sec t = \frac{t^2}{2} \tan t + t - \tan t$ . Since  $\tan t$  is a homogeneous solution we can write the general solution as  $y(t) = \frac{t^2}{2} \tan t + t + c_1 \tan t + c_2 \sec t$ .

15. After put in standard form the forcing function  $f$  is  $4t^4$ . The matrix equation

$\begin{pmatrix} \cos t^2 & \sin t^2 \\ -2t \sin t^2 & 2t \cos 2t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 4t^4 \end{pmatrix}$  implies  $u_1'(t) = -2t^3 \sin t^2$  and  $u_2'(t) = 2t^3 \cos t^2$ . Integration by parts gives  $u_1(t) = t^2 \cos t^2 - \sin t^2$  and  $u_2(t) = t^2 \sin t^2 + \cos t^2$ . Hence  $y_p(t) = t^2 \cos^2 t^2 - \cos t^2 \sin t^2 + t^2 \sin^2 t^2 + \cos t^2 \sin t^2 = t^2$ . The general solution is  $y(t) = t^2 + c_1 \cos t^2 + c_2 \sin t^2$ .

17. Let  $a$  and  $t$  be in the interval  $I$ . Let  $z_1$  and  $z_2$  be the definite integrals defined as follows:

$$z_1(t) = \int_a^t \frac{-y_2(x)f(x)}{w(y_1, y_2)(x)} dx$$

$$z_2(t) = \int_a^t \frac{y_1(x)f(x)}{w(y_1, y_2)(x)} dx.$$

These definite integrals determine the constant of integration in Theorem 1 so that  $z_1(a) = z_2(a) = 0$ . It follows that

$$\begin{aligned} y_p(t) &= z_1(t)y_1(t) + z_2(t)y_2(t) \\ &= \int_a^t \frac{-y_2(x)y_1(t)f(x)}{w(y_1, y_2)(x)} dx + \int_a^t \frac{y_1(x)y_2(t)f(x)}{w(y_1, y_2)(x)} dx \\ &= \int_a^t \frac{(y_1(x)y_2(t) - y_2(x)y_1(t))}{w(y_1, y_2)(x)} f(x) dx \\ &= \int_a^t \frac{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1(t) & y_2(t) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} f(x) dx. \end{aligned}$$

19. Let  $y_1(t) = e^{-at}$  and  $y_2(t) = e^{at}$ . Then  $\{y_1, y_2\}$  is a fundamental set. We have

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1(t) & y_2(t) \end{vmatrix} = e^{-ax}e^{at} - e^{ax}e^{-at} = e^{a(t-x)} - e^{-a(t-x)} = 2 \sinh(a(t-x))$$

and

$$w(y_1, y_2)(x) = \begin{vmatrix} e^{-at} & e^{at} \\ -ae^{-at} & ae^{at} \end{vmatrix} = 2a.$$

Thus

$$\begin{aligned} y_p(t) &= \int_0^t \frac{2 \sinh a(t-x)}{2a} f(x) dx \\ &= \frac{1}{a} f(t) * \sinh at. \end{aligned}$$

Applying the Laplace transform to  $y'' - a^2y = f$ , with initial conditions  $y(0) = y'(0) = 0$ , gives  $s^2Y(s) - a^2Y(s) = F(s)$ . Solving for  $Y(s)$  we get

$$Y(s) = \frac{F(s)}{s^2 - a^2} = \frac{1}{a} \frac{a}{s^2 - a^2} F(s).$$

The convolution theorem gives a particular solution

$$y_p(t) = \frac{1}{a} \sinh at * f(t).$$

- 21.** Let  $y_1(t) = e^{at}$  and  $y_2(t) = e^{bt}$ . Then  $\{y_1, y_2\}$  is a fundamental set. We have

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1(t) & y_2(t) \end{vmatrix} = e^{ax}e^{bt} - e^{bx}e^{at} = e^{ax+bt} - e^{bx+at}$$

and

$$w(y_1, y_2)(x) = \begin{vmatrix} e^{ax} & e^{bx} \\ ae^{ax} & be^{bx} \end{vmatrix} = (b-a)e^{(a+b)x}.$$

Thus

$$\begin{aligned} y_p(t) &= \int_0^t \frac{e^{ax+bt} - e^{bx+at}}{(b-a)e^{(a+b)x}} f(x) dx \\ &= \frac{1}{b-a} \int_0^t (e^{b(t-x)} - e^{a(t-x)}) f(x) dx \\ &= \frac{1}{b-a} f(t) * (e^{bt} - e^{at}). \end{aligned}$$

Applying the Laplace transform to  $y'' - (a+b)y' + aby = f$ , with initial conditions  $y(0) = y'(0) = 0$ , gives  $s^2Y(s) - (a+b)sY(s) + abY(s) = F(s)$ . Solving for  $Y(s)$  we get

$$Y(s) = \frac{F(s)}{(s-a)(s-b)} = \frac{1}{a-b} \left( \frac{1}{s-a} - \frac{1}{s-b} \right) F(s).$$

The convolution theorem gives a particular solution

$$y_p(t) = \frac{1}{a-b} (e^{at} - e^{bt}) * f(t).$$

## SECTION 6.1

1. Graph (c)
3. Graph (e)
5. Graph (f)
7. Graph (h)
9.  $\int_0^5 f(t) dt = \int_0^2 (t^2 - 4) dt + \int_2^3 0 dt + \int_3^5 (-t + 3) dt = (t^3/3 - 4t)|_0^2 + 0 + (-t^2/2 + 3t)|_3^5 = (8/3 - 8) + (-25/2 + 15) - (-9/2 + 9) = -22/3.$
11.  $\int_0^{2\pi} |\sin x| dx = \int_0^\pi \sin x dx + \int_\pi^{2\pi} -\sin x dx = -\cos x|_0^\pi + \cos x|_\pi^{2\pi} = 4.$

13.  $\int_2^5 f(t) dt = \int_2^3 (3-t) dt + \int_3^4 2(t-3) dt + \int_4^6 2 dt = 1/2 + 1 + 4 = 11/2$
15.  $\int_0^6 f(u) du = \int_0^1 u du + \int_1^2 (2-u) du + \int_2^6 1 du = 1/2 + 1/2 + 4 = 5.$
17. A is true since  $y(t)$  satisfies the differential equation on each subinterval. B is true since the left and right limits agree at  $t = 2$ . C is not true since  $y(0) = 1 \neq 2$ .
19. A is true since  $y(t)$  satisfies the differential equation on each subinterval. B is false since  $\lim_{t \rightarrow 2^-} y(t) = 1 + e^{-8}$  while  $\lim_{t \rightarrow 2^+} y(t) = 1$ . C is false since B is false.
21. A is true since  $y(t)$  satisfies the differential equation on each subinterval. B is true since  $\lim_{t \rightarrow 1^-} y(t) = -2e + e^2 = \lim_{t \rightarrow 1^+} y(t)$ . C is false since  $\lim_{t \rightarrow 1^-} y'(t) = -3e + 2e^2$  while  $\lim_{t \rightarrow 1^+} y'(t) = 3e^2 - 2e$ . D is false since C is false.
23. A is true since  $y(t)$  satisfies the differential equation on each subinterval. B is true since  $\lim_{t \rightarrow 1^-} y(t) = -2e + e^2 = \lim_{t \rightarrow 1^+} y(t)$ . C is true since  $\lim_{t \rightarrow 1^-} y'(t) = -3e + 2e^2 = \lim_{t \rightarrow 1^+} y'(t)$ . D is true since  $y(0) = y'(0) = 0$ .
25. The general solution of  $y' - y = 1$  on the interval  $[0, 2)$  is found by using the integrating factor  $e^{-t}$ . The general solution is  $y(t) = -1 + ce^t$  and the initial condition  $y(0) = 0$  gives  $c = 1$ , so that  $y(t) = -1 + e^t$  for  $t \in [0, 2)$ . Continuity of  $y(t)$  at  $t = 2$  will then give  $y(2) = \lim_{t \rightarrow 2^-} y(t) = -1 + e^2$ , which will provide the initial condition for the next interval  $[2, 4)$ . The general solution of  $y' - y = -1$  on  $[2, 4)$  is  $y(t) = 1 + ke^t$ . Thus  $-1 + e^2 = y(2) = 1 + ke^2$  and solve for  $k$  to get  $k = -2e^{-2} + 1$ , so that  $y(t) = 1 + (-2e^{-2} + 1)e^t$  for  $t \in [2, 4)$ . Continuity will then give  $y(4) = 1 + (-2e^{-2} + 1)e^4$ , which will provide the initial condition for the next interval  $[4, \infty)$ . The general solution to  $y' - y = 0$  on  $[4, \infty)$  is  $y(t) = be^t$  and the constant  $b$  is obtained from the initial condition  $be^4 = y(4) = 1 + (-2e^{-2} + 1)e^4$ , which gives  $b = e^{-4} - 2e^{-2} + 1$ , so that  $y(t) = (e^{-4} - 2e^{-2} + 1)e^t$  for  $t \in [4, \infty)$ . Putting these three pieces together, we find that the solution is

$$y(t) = \begin{cases} -1 + e^t & \text{if } 0 \leq t < 2, \\ 1 - 2e^{t-2} + e^t & \text{if } 2 \leq t < 4 \\ e^{t-4} - 2e^{t-2} + e^t & \text{if } 4 \leq t < \infty. \end{cases}$$

27. The general solution of  $y' - y = f(t)$  on any interval is found by using the integrating factor  $e^{-t}$ . The general solution on the interval  $[0, 1)$  is  $y(t) = ae^t$  and since the initial condition is  $y(0) = 0$ , the solution on  $[0, 1)$  is  $y(t) = 0$ . Continuity then gives  $y(1) = 0$ , which will be the initial

condition for the interval  $[1, 2)$ . The general solution of  $y' - y = t - 1$  on the interval  $[1, 2)$  is  $y(t) = -t + be^t$  and the initial condition  $y(1) = 0$  gives  $0 = -1 + be^1$  so that  $b = e^{-1}$ . Thus  $y(t) = -t + e^{-1}e^t = -t + e^{t-1}$  for  $t \in [0, 2)$ . Continuity of  $y(t)$  at  $t = 2$  will then give  $y(2) = \lim_{t \rightarrow 2^-} y(t) = -2 + e^1$ , which will provide the initial condition for the next interval  $[2, 3)$ . The general solution of  $y' - y = 3 - t$  on  $[2, 3)$  is  $y(t) = t - 2 + ce^t$ . Thus  $-2 + e^1 = y(2) = ce^2$  and solve for  $c$  to get  $c = -2e^{-2} + e^{-1}$ , so that  $y(t) = t - 2 + (-2e^{-2} + e^{-1})e^t = t - 2 - 2e^{t-2} + e^{t-1}$  for  $t \in [2, 3)$ . Continuity will then give  $y(3) = 1 - 2e^1 + e^2$ , which will provide the initial condition for the next interval  $[3, \infty)$ . The general solution to  $y' - y = 0$  on  $[4, \infty)$  is  $y(t) = ke^t$  and the constant  $k$  is obtained from the initial condition  $ke^3 = y(3) = 1 - 2e^1 + e^2$ , which gives  $c = e^{-3} - 2e^{-2} + e^{-1}$ , so that  $y(t) = (e^{-3} - 2e^{-2} + e^{-1})e^t = e^{t-3} - 2e^{t-2} + e^{t-1}$  for  $t \in [3, \infty)$ . Putting these three pieces together, we find that the solution is

$$y(t) = \begin{cases} 0 & \text{if } 0 \leq t < 1, \\ -t + e^{t-1} & \text{if } 1 \leq t < 2, \\ t - 2 - 2e^{t-2} + e^{t-1} & \text{if } 2 \leq t < 3 \\ e^{t-3} - 2e^{t-2} + e^{t-1} & \text{if } 3 \leq t < \infty. \end{cases}$$

- 29.** The characteristic polynomial of the equation  $y'' - y = f(t)$  is  $s^2 - 1 = (s-1)(s+1)$  so the homogeneous equation has the solution  $y_h(t) = ae^t + be^{-t}$  for constants  $a$  and  $b$ . On the interval  $[0, 1]$  the equation  $y'' - y = t$  has a particular solution  $y_p(t) = -t$  so the general solution has the form  $y(t) = -t + ae^t + be^{-t}$ . The initial conditions give  $0 = y(0) = a + b$  and  $1 = y'(0) = -1 + a - b$ . Solving gives  $a = 1$ ,  $b = -1$  so  $y(t) = -t + e^t - e^{-t}$  on  $[0, 1)$ . By continuity it follows that  $y(1) = -1 + e^1 - e^{-1}$  and  $y'(1) = -1 + e^1 + e^{-1}$  and these constitute the initial values for the equation  $y'' - y = 0$  on the interval  $[1, \infty)$ . The general solution on this interval is  $y(t) = ae^t + be^{-t}$  and at  $t = 1$  we get  $y(1) = ae^1 + be^{-1} = -1 + e^1 - e^{-1}$  and  $y'(1) = ae^1 - be^{-1} = -1 + e^1 + e^{-1}$ . Solving for  $a$  and  $b$  gives  $a = 1 - e^{-1}$  and  $b = -1$  so that  $y(t) = (1 - e^{-1})e^t - e^{-t} = e^t - e^{t-1} - e^{-1}$ . Putting the two pieces together gives

$$y(t) = \begin{cases} -t + e^t - e^{-t} & \text{if } 0 \leq t < 1, \\ e^t - e^{t-1} - e^{-1} & 1 \leq t < \infty. \end{cases}$$

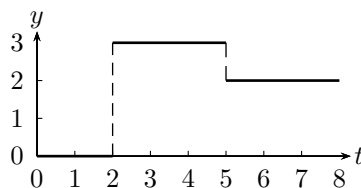
- 33.** 1.  $|f(t)| = |\sin(1/t)| \leq 1$  for all  $t \neq 0$ , while  $|f(0)| = |0| = 0 \leq 1$ .  
 2. It is enough to observe that  $\lim_{t \rightarrow 0^+}$  does not exist. But letting  $t_n = \frac{1}{n\pi}$  gives  $f(t_n) = \sin n\pi = 0$  for all positive integers  $n$ , while letting  $t_n = \frac{2}{(4n+1)\pi}$  gives  $f(t_n) = \sin(1/t_n) = \sin((4n+1)\pi/2) = \sin(2n\pi +$

$\frac{\pi}{2}) = 1$  so there is one sequence  $t_n \rightarrow 0$  with  $f(t_n) \rightarrow 0$  while another sequence  $t_n \rightarrow 0$  with  $f(t_n) \rightarrow 1$  so  $f(t)$  cannot be continuous at 0.

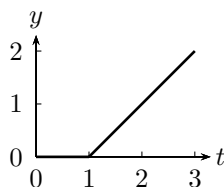
3. To be piecewise continuous,  $f(t)$  would have to have a limit at  $t$  approaches 0 from above, and this is not true as shown in part 2.

## SECTION 6.2

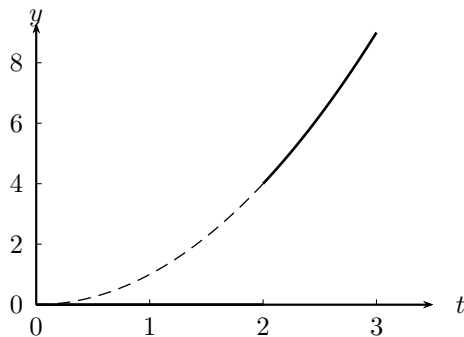
1.  $f(t) = 3h(t-2) - h(t-5) = \begin{cases} 0 & \text{if } t < 2, \\ 3 & \text{if } 2 \leq t < 5, \\ 2 & \text{if } t \geq 5. \end{cases}$  Thus, the graph is



3. This function is  $g(t-1)h(t-1)$  where  $g(t) = t$ , so the graph of  $f(t)$  is the graph of  $g(t) = t$  translated 1 unit to the right and then truncated at  $t = 1$ , with the graph before  $t = 1$  replaced by the line  $y = 0$ . Thus the graph is

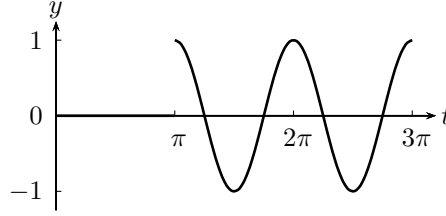


5. This function is just  $t^2$  truncated at  $t = 2$ , with the graph before  $t = 2$  replaced by the line  $y = 0$ . Thus the graph is



where the dashed line is the part of the  $t^2$  graph that has been truncated. It is only shown for emphasis and it is not part of the graph.

7. This is the function  $\cos 2t$  shifted  $\pi$  units to the right and then truncated at  $t = \pi$ . The graph is



9. (a)  $(t-2)\chi_{[2,\infty)}(t)$ ; (b)  $(t-2)h(t-2)$ ;  
 (c)  $\mathcal{L}\{(t-2)h(t-2)\} = e^{-2s}\mathcal{L}\{t\} = e^{-2s}/s^2$ .
11. (a)  $(t+2)\chi_{[2,\infty)}(t)$ ; (b)  $(t+2)h(t-2)$ ;  
 (c)  $\mathcal{L}\{(t+2)h(t-2)\} = e^{-2s}\mathcal{L}\{(t+2)+2\} = e^{-2s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$ .
13. (a)  $t^2\chi_{[4,\infty)}(t)$ ; (b)  $t^2h(t-4)$ ;  
 (c)  $\mathcal{L}\{t^2h(t-4)\} = e^{-4s}\mathcal{L}\{(t+4)^2\} = e^{-4s}\mathcal{L}\{t^2 + 8t + 16\}$   
 $= e^{-4s}\left(\frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}\right)$ .
15. (a)  $(t-4)^2\chi_{[2,\infty)}(t)$ ; (b)  $(t-4)^2h(t-2)$ ;  
 (c)  $\mathcal{L}\{(t-4)^2h(t-2)\} = e^{-2s}\mathcal{L}\{((t+2)-4)^2\} = e^{-2s}\mathcal{L}\{t^2 - 4t + 4\}$   
 $= e^{-2s}\left(\frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s}\right)$ .
17. (a)  $e^t\chi_{[4,\infty)}(t)$ ; (b)  $e^th(t-4)$ ;  
 (c)  $\mathcal{L}\{e^th(t-4)\} = e^{-4s}\mathcal{L}\{e^{t+4}\} = e^{-4s}e^4\mathcal{L}\{e^t\}$   
 $= e^{-4(s-1)}\frac{1}{s-1}$ .
19. (a)  $te^t\chi_{[4,\infty)}(t)$ ; (b)  $te^th(t-4)$ ;  
 (c)  $\mathcal{L}\{te^th(t-4)\} = e^{-4s}\mathcal{L}\{(t+4)e^{t+4}\} = e^{-4s}e^4\mathcal{L}\{te^t + 4e^t\}$   
 $= e^{-4(s-1)}\left(\frac{1}{(s-1)^2} + \frac{4}{s-1}\right)$ .
21. (a)  $t\chi_{[0,1)}(t) + (2-t)\chi_{[1,\infty)}(t)$ ; (b)  $t + (2-2t)h(t-1)$ ;  
 (c)  $\mathcal{L}\{t + (2-2t)h(t-1)\} = \mathcal{L}\{t\} + e^{-s}\mathcal{L}\{(2-2(t+1))\}$   
 $= \mathcal{L}\{t\} + e^{-s}\mathcal{L}\{-2t\} = \frac{1}{s^2} - \frac{2e^{-s}}{s^2}$ .
23. (a)  $t^2\chi_{[0,2)}(t) + 4\chi_{[2,3)}(t) + (7-t)\chi_{[3,\infty)}(t)$ ;  
 (b)  $t^2 + (4-t^2)h(t-2) + (3-t)h(t-3)$ ;  
 (c)  $\mathcal{L}\{t^2 + (4-t^2)h(t-2) + (3-t)h(t-3)\}$



$$\begin{aligned}
&= \mathcal{L}\{t^2\} + e^{-2s} \mathcal{L}\{4 - (t+2)^2\} + e^{-3s} \mathcal{L}\{3 - (t+3)\} \\
&= \frac{2}{s^3} - e^{-2s} \left( \frac{2}{s^3} + \frac{4}{s^2} \right) - \frac{e^{-3s}}{s^2}.
\end{aligned}$$

$$\begin{aligned}
25. \quad & \text{(a) } \sum_{n=0}^{\infty} (t-n) \chi_{[n, n+1)}(t); \\
& \text{(b) } t - \sum_{n=1}^{\infty} h(t-n); \\
& \text{(c) } \mathcal{L}\{t - \sum_{n=1}^{\infty} h(t-n)\} = \mathcal{L}\{t\} - \sum_{n=1}^{\infty} \mathcal{L}\{h(t-n)\} \\
&= \frac{1}{s^2} - \sum_{n=1}^{\infty} \frac{e^{-ns}}{s} = \frac{1}{s^2} - \frac{1}{s} \sum_{n=1}^{\infty} (e^{-s})^n \\
&= \frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}.
\end{aligned}$$

$$\begin{aligned}
27. \quad & \text{(a) } \sum_{n=0}^{\infty} (2n+1-t) \chi_{[2n, 2n+2)}(t); \text{ (b) } -(t+1) + 2 \sum_{n=0}^{\infty} h(t-2n); \\
& \text{(c) } -\frac{1}{s^2} - \frac{1}{s} + \frac{2}{s(1-e^{-2s})}.
\end{aligned}$$

$$\begin{aligned}
29. \quad & \mathcal{L}^{-1} \left\{ \frac{e^{-3s}}{s^2} \right\} = h(t-3) \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \right\} \Big|_{t \rightarrow t-3} \\
&= h(t-3) (t)|_{t \rightarrow t-3} = (t-3)h(t-3) = \begin{cases} 0 & \text{if } 0 \leq t < 3, \\ t-3 & \text{if } t \geq 3. \end{cases}
\end{aligned}$$

$$\begin{aligned}
31. \quad & \mathcal{L}^{-1} \left\{ \frac{e^{-\pi s}}{s^2+1} \right\} = h(t-\pi) \mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} \right\} \Big|_{t \rightarrow t-\pi} \\
&= h(t-\pi) (\sin t)|_{t \rightarrow t-\pi} = h(t-\pi) \sin(t-\pi) \\
&= \begin{cases} 0 & \text{if } 0 \leq t < \pi, \\ \sin(t-\pi) & \text{if } t \geq \pi \end{cases} = \begin{cases} 0 & \text{if } 0 \leq t < \pi, \\ -\sin t & \text{if } t \geq \pi. \end{cases}
\end{aligned}$$

$$\begin{aligned}
33. \quad & \mathcal{L}^{-1} \left\{ \frac{e^{-\pi s}}{s^2+2s+5} \right\} = h(t-\pi) \mathcal{L}^{-1} \left\{ \frac{1}{s^2+2s+5} \right\} \Big|_{t \rightarrow t-\pi} \\
&= h(t-\pi) \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2+2^2} \right\} \Big|_{t \rightarrow t-\pi} = h(t-\pi) \left( \frac{1}{2} e^{-t} \sin 2t \right) \Big|_{t \rightarrow t-\pi} \\
&= \frac{1}{2} e^{-(t-\pi)} \sin 2(t-\pi) h(t-\pi) = \begin{cases} 0 & \text{if } 0 \leq t < \pi, \\ \frac{1}{2} e^{-(t-\pi)} \sin 2t & \text{if } t \geq \pi. \end{cases}
\end{aligned}$$

$$\begin{aligned}
35. \quad & \mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s^2+4} \right\} = h(t-2) \mathcal{L}^{-1} \left\{ \frac{1}{s^2+4} \right\} \Big|_{t \rightarrow t-2} \\
&= h(t-2) \left( \frac{1}{2} \sin 2t \right) \Big|_{t \rightarrow t-2} = \frac{1}{2} h(t-2) \sin 2(t-2) \\
&= \begin{cases} 0 & \text{if } 0 \leq t < 2, \\ \frac{1}{2} \sin 2(t-2) & \text{if } t \geq 2. \end{cases}
\end{aligned}$$

$$\begin{aligned}
37. \quad \mathcal{L}^{-1} \left\{ \frac{se^{-4s}}{s^2 + 3s + 2} \right\} &= h(t-4) \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 3s + 2} \right\} \Big|_{t \rightarrow t-4} \\
&= h(t-4) \mathcal{L}^{-1} \left\{ \frac{2}{s+2} - \frac{1}{s+1} \right\} \Big|_{t \rightarrow t-4} \\
&= h(t-4) (2e^{-2t} - e^{-t}) \Big|_{t \rightarrow t-4} = h(t-4) (2e^{-2(t-4)} - e^{-(t-4)}) \\
&= \begin{cases} 0 & \text{if } 0 \leq t < 4, \\ 2e^{-2(t-4)} - e^{-(t-4)} & \text{if } t \geq 4. \end{cases}
\end{aligned}$$

$$\begin{aligned}
39. \quad \mathcal{L}^{-1} \left\{ \frac{1 - e^{-5s}}{s^2} \right\} &= \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \right\} - h(t-5) \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \right\} \Big|_{t \rightarrow t-5} \\
&= t - h(t-5) (t) \Big|_{t \rightarrow t-5} = t - (t-5)h(t-5) = \begin{cases} t & \text{if } 0 \leq t < 5, \\ 5 & \text{if } t \geq 5. \end{cases}
\end{aligned}$$

$$\begin{aligned}
41. \quad \mathcal{L}^{-1} \left\{ e^{-\pi s} \frac{2s+1}{s^2 + 6s + 13} \right\} &= h(t-\pi) \mathcal{L}^{-1} \left\{ \frac{2s+1}{s^2 + 6s + 13} \right\} \Big|_{t \rightarrow t-\pi} \\
&= h(t-\pi) \mathcal{L}^{-1} \left\{ \frac{2(s+3) - 5}{(s+3)^2 + 2^2} \right\} \Big|_{t \rightarrow t-\pi} \\
&= h(t-\pi) \mathcal{L}^{-1} \left\{ \frac{2(s+3)}{(s+3)^2 + 2^2} \right\} \Big|_{t \rightarrow t-\pi} \\
&\quad + h(t-\pi) \mathcal{L}^{-1} \left\{ \frac{-5}{(s+3)^2 + 2^2} \right\} \Big|_{t \rightarrow t-\pi} \\
&= h(t-\pi) (2e^{-3t} \cos 2t - \frac{5}{2}e^{-3t} \sin 2t) \Big|_{t \rightarrow t-\pi} \\
&= h(t-\pi) e^{-3(t-\pi)} (2 \cos 2(t-\pi) - \frac{5}{2} \sin 2(t-\pi)) \\
&= \begin{cases} 0 & \text{if } 0 \leq t < \pi, \\ e^{-3(t-\pi)} (2 \cos 2t - \frac{5}{2} \sin 2t) & \text{if } t \geq \pi. \end{cases}
\end{aligned}$$

43. Let  $b > 0$ . Since  $f_1$  and  $f_2$  are piecewise continuous on  $[0, \infty)$  they only have finitely many jump discontinuities on  $[0, b)$ . It follows that  $f_1 + cf_2$  have only finitely many jump on  $[0, b)$ . Thus  $f_1 + cf_2$  is piecewise continuous on  $[0, \infty)$ .

### SECTION 6.3

1. We write the forcing function as  $f(t) = 3h(t-1)$ . Applying the Laplace transform, partial fractions, and simplifying gives

$$Y(s) = \frac{-3}{s(s+2)} e^{-s} = \frac{-3}{2} \left( \frac{1}{s} - \frac{1}{s+2} \right) e^{-s}.$$

Laplace inversion now gives

$$y = -\frac{3}{2}h(t-1) \begin{cases} (1 - e^{-2(t-1)}) & \text{if } 0 \leq t < 1 \\ -\frac{3}{2}(1 - e^{-2(t-1)}) & \text{if } 1 \leq t < \infty \end{cases}.$$

3. We write the forcing function as  $f(t) = 2\chi_{[2,3)} = 2h(t-2) - 2h(t-3)$ . Applying the Laplace transform, partial fractions, and simplifying gives

$$\begin{aligned} Y(s) &= \frac{2}{s(s-3)} (e^{-2s} - e^{-3s}) \\ &= \frac{2}{3} \left( \frac{1}{s-3} - \frac{1}{s} \right) (e^{-2s} - e^{-3s}). \end{aligned}$$

Laplace inversion now gives

$$\begin{aligned} y &= \frac{2}{3} \left( (e^{3(t-2)} - 1) h(t-2) - (e^{3(t-3)} - 1) h(t-3) \right) \\ &= \begin{cases} 0 & \text{if } 0 \leq t < 2 \\ \frac{2}{3} (e^{3(t-2)} - 1) & \text{if } 2 \leq t < 3 \\ \frac{2}{3} (e^{3(t-2)} - e^{3(t-3)}) & \text{if } 3 \leq t < \infty \end{cases}. \end{aligned}$$

5. We write the forcing function as

$$\begin{aligned} f(t) &= 12e^t \chi_{[0,1)} + 12e \chi_{[1,\infty)} \\ &= 12e^t - 12(e^t - e)h(t-1). \end{aligned}$$

Applying the Laplace transform, partial fractions, and simplifying gives

$$\begin{aligned} Y(s) &= \frac{2}{s-4} + \frac{12}{(s-1)(s-4)} - e^{-s} \left( \frac{12e}{(s-1)(s-4)} - \frac{12e}{s(s+4)} \right) \\ &= \frac{6}{s-4} - \frac{4}{s-1} - e^{-s} e \left( \frac{-4}{s-1} + \frac{1}{s-4} + \frac{3}{s} \right). \end{aligned}$$

Laplace inversion now gives

$$\begin{aligned} y &= 6e^{4t} - 4e^t - e \left( -4e^{t-1} + e^{4(t-1)} + 3 \right) h(t-1) \\ &= 6e^{4t} - 4e^t + 4e^t h(t-1) - e^{4t-3} h(t-1) - 3eh(t-1) \\ &= \begin{cases} 6e^{4t} - 4e^t & \text{if } 0 \leq t < 1 \\ 6e^{4t} - e^{4t-3} - 3e & \text{if } 1 \leq t < \infty \end{cases}. \end{aligned}$$

7. Applying the Laplace transform, partial fractions, and simplifying gives

$$\begin{aligned}
 Y(s) &= \frac{e^{-3s}}{s(s^2 + 9)} \\
 &= \frac{1}{9} \left( \frac{1}{s} - \frac{s}{s^2 + 9} \right) e^{-3s}.
 \end{aligned}$$

Laplace inversion now gives

$$y = \frac{1}{9} (1 - \cos 3(t-3)) h(t-3) = \begin{cases} 0 & \text{if } 0 \leq t < 3 \\ \frac{1}{9} (1 - \cos 3(t-3)) & \text{if } 3 \leq t < \infty \end{cases}.$$

9. Write the forcing function as  $f(t) = 6\chi_{[1,3)} = 6h(t-1) - 6h(t-3)$ . Now apply the Laplace transform, partial fractions, and simplify to get

$$\begin{aligned}
 Y(s) &= \frac{6}{s(s+2)(s+3)} (e^{-s} - e^{-3s}) \\
 &= \left( \frac{1}{s} - \frac{3}{s+2} + \frac{2}{s+3} \right) (e^{-s} - e^{-3s})
 \end{aligned}$$

Now we take the inverse Laplace transform and simplify to get

$$\begin{aligned}
 y &= \left( 1 - 3e^{-2(t-1)} + 2e^{-3(t-1)} \right) h(t-1) \\
 &\quad - \left( 1 - 3e^{-2(t-3)} + 2e^{-3(t-3)} \right) h(t-3) \\
 &= \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ 1 - 3e^{-2(t-1)} + 2e^{-3(t-1)} & \text{if } 1 \leq t < 3 \\ 3e^{-2(t-3)} - 3e^{-2(t-1)} - 2e^{-3(t-3)} + 2e^{-3(t-1)} & \text{if } 3 \leq t < \infty \end{cases}
 \end{aligned}$$

11. Apply the Laplace transform, partial fractions, and simplify to get

$$\begin{aligned}
 Y(s) &= \frac{1}{(s+1)^2} + \frac{1}{s(s+1)^2} e^{-3s} \\
 &= \frac{1}{(s+1)^2} + \left( -\frac{1}{(s+1)^2} - \frac{1}{s+1} + \frac{1}{s} \right) e^{-3s}
 \end{aligned}$$

Laplace inversion gives

$$\begin{aligned}
 y &= te^{-t} + \left( -(t-3)e^{-(t-3)} - e^{-(t-3)} + 1 \right) h(t-3) \\
 &= te^{-t} + \left( 1 - (t-2)e^{-(t-3)} \right) h(t-3) \\
 &= \begin{cases} te^{-t} & \text{if } 0 \leq t < 3 \\ 1 + te^{-t} - (t-2)e^{-(t-3)} & \text{if } 3 \leq t < \infty \end{cases}.
 \end{aligned}$$

- 13.** For the first three minutes, source one adds salt at a rate of  $1 \frac{\text{lb}}{\text{gal}} \cdot 2 \frac{\text{gal}}{\text{min}} = 2 \frac{\text{lbs}}{\text{min}}$ . and after that source two takes over and adds salt at a rate of  $5 \frac{\text{lb}}{\text{gal}} \cdot 2 \frac{\text{gal}}{\text{min}} = 10 \frac{\text{lbs}}{\text{min}}$ . Thus the rate at which salt is being added is given by the function

$$\begin{aligned} f(t) &= \begin{cases} 2 & \text{if } 0 \leq t < 3 \\ 10 & \text{if } 3 \leq t < \infty. \end{cases} \\ &= 2\chi_{[0,3)} + 10\chi_{[3,\infty)} \\ &= 2(1 - h(t-3)) + 10(h(t-3)) \\ &= 2 + 8h(t-3). \end{aligned}$$

The output rate of salt is given by  $\frac{y(t)}{4} \cdot 2 = \frac{y(t)}{2}$  lbs/min. We are thus led to the differential equation

$$y' + \frac{1}{2}y(t) = 2 + 8h(t-3), \quad y(0) = 0.$$

We take the Laplace transform of both sides and use partial fractions to get

$$\begin{aligned} Y(s) &= \frac{2}{s(s+1/2)} + \frac{8e^{-3s}}{s(s+1/2)} \\ &= \frac{4}{s} - \frac{4}{s+1/2} + e^{-3s} \left( \frac{16}{s} - \frac{16}{s+1/2} \right). \end{aligned}$$

Laplace inversion now gives

$$\begin{aligned} y(t) &= 4 - 4e^{-\frac{t}{2}} + 16h(t-3) - 16e^{-\frac{(t-3)}{2}}h(t-3) \\ &= \begin{cases} 4 - 4e^{-\frac{t}{2}} & \text{if } 0 \leq t < 3 \\ 20 - 4e^{-\frac{t}{2}} - 16e^{-\frac{(t-3)}{2}} & \text{if } t \geq 3. \end{cases} \end{aligned}$$

- 15.** For the first two minutes, source one adds salt at a rate of  $1 \frac{\text{kg}}{\text{L}} \cdot 3 \frac{\text{L}}{\text{min}} = 3 \frac{\text{kg}}{\text{min}}$ . Thereafter source two takes over for two minutes but the input rate of salt is 0. Thereafter source one take over again and adds salt to the tank at a rate of  $3 \frac{\text{kg}}{\text{min}}$ . Thus the rate at which salt is being added is given by the function

$$\begin{aligned} f(t) &= \begin{cases} 3 & \text{if } 0 \leq t < 2 \\ 0 & \text{if } 2 \leq t < 4 \\ 3 & \text{if } 4 \leq t < \infty. \end{cases} \\ &= 3\chi_{[0,2)} + 3\chi_{[4,\infty)} \\ &= 3(1 - h(t-2) + h(t-4)). \end{aligned}$$

The output rate of salt is given by  $\frac{y(t)}{10} \cdot 3 = \frac{3}{10}y(t)$  kg/min. We are thus led to the differential equation

$$y' + \frac{3}{10}y(t) = 3(1 - h(t-2) + h(t-4)), \quad y(0) = 2.$$

We take the Laplace transform of both sides, simplify, and use partial fractions to get

$$\begin{aligned} Y(s) &= \frac{2}{s + 3/10} + \frac{3}{s(s + 3/10)}(1 - e^{-2s} + e^{-4s}) \\ &= \frac{10}{s} - \frac{8}{s + 3/10} + \left( \frac{10}{s} - \frac{10}{s + 3/10} \right) (e^{-4s} - e^{-2s}). \end{aligned}$$

Laplace inversion now gives

$$\begin{aligned} y(t) &= 10 - 8e^{-3t/10} - \left( 10 - 10e^{-3(t-2)/10} \right) h(t-2) \\ &\quad + \left( 10 - 10e^{-3(t-4)/10} \right) h(t-4) \\ &= \begin{cases} 10 - 8e^{-3t/10} & \text{if } 0 \leq t < 2 \\ 10e^{-3(t-2)/10} - 8e^{-3t/10} & \text{if } 2 \leq t < 4 \\ 10 - 8e^{-3t/10} + 10e^{-3(t-2)/10} - 10e^{-3(t-4)/10} & \text{if } 4 \leq t < \infty \end{cases}. \end{aligned}$$

## SECTION 6.4

1. Take the Laplace transform, solve for  $Y(s)$ , and simplify to get  $Y(s) = \frac{e^{-s}}{s+2}$ . Laplace inversion then gives

$$\begin{aligned} y &= e^{-2(t-1)}h(t-1) \\ &= \begin{cases} 0 & \text{if } 0 \leq t < 1 \\ e^{-2(t-1)} & \text{if } 1 \leq t < \infty \end{cases}. \end{aligned}$$

3. Take the Laplace transform, solve for  $Y(s)$ , and simplify to get  $Y(s) = \frac{2}{s-4} + \frac{e^{-4s}}{s-4}$ . Laplace inversion then gives

$$\begin{aligned} y &= 2e^{4t} + e^{4(t-4)}h(t-4) \\ &= \begin{cases} 2e^{4t} & \text{if } 0 \leq t < 4 \\ 2e^{4t} + e^{4(t-4)} & \text{if } 4 \leq t < \infty \end{cases}. \end{aligned}$$

5. We begin by taking the Laplace transform of each side and simplifying to get  $Y(s) = \frac{1}{s^2+4} + \frac{e^{-\pi s}}{s^2+4}$ . Laplace inversion then gives

$$\begin{aligned} y &= \frac{1}{2} \sin 2t + \frac{1}{2} \sin 2(t - \pi) h(t - \pi) \\ &= \frac{1}{2} \sin 2t + \frac{1}{2} \sin(2t) h(t - \pi) \\ &= \begin{cases} \frac{\sin 2t}{2} & \text{if } 0 \leq t < \pi \\ \sin 2t & \text{if } \pi \leq t < \infty \end{cases} . \end{aligned}$$

7. Apply the Laplace transform, partial fractions, and simplify to get

$$\begin{aligned} Y(s) &= \frac{s+3}{(s+1)(s+3)} + \frac{2e^{-2s}}{(s+1)(s+3)} \\ &= \frac{1}{s+1} + \left( \frac{1}{s+1} - \frac{1}{s+3} \right) e^{-2s} . \end{aligned}$$

Laplace inversion gives

$$\begin{aligned} y &= e^{-t} + \left( e^{-(t-2)} - e^{-3(t-2)} \right) h(t-2) \\ &= \begin{cases} e^{-t} & \text{if } 0 \leq t < 2 \\ e^{-t} + e^{-(t-2)} - e^{-3(t-2)} & \text{if } 2 \leq t < \infty \end{cases} . \end{aligned}$$

9. Take the Laplace transform, apply partial fractions, and simplify to get

$$\begin{aligned} Y(s) &= -\frac{s+1}{(s+2)^2} + \frac{3}{(s+2)^2} e^{-s} \\ &= \frac{1}{(s+2)^2} - \frac{1}{s+2} + \frac{3}{(s+2)^2} e^{-s} . \end{aligned}$$

Laplace inversion now gives

$$\begin{aligned} y &= te^{-2t} - e^{-2t} + 3(t-1)e^{-2(t-1)} h(t-1) \\ &= \begin{cases} te^{-2t} - e^{-2t} & \text{if } 0 \leq t < 1 \\ te^{-2t} - e^{-2t} + 3(t-1)e^{-2(t-1)} & \text{if } 1 \leq t < \infty \end{cases} . \end{aligned}$$

11. The input rate of salt is  $6 + 4\delta_3$  while the output rate is  $3\frac{y(t)}{12}$ . We thus have the differential equation  $y' + \frac{1}{4}y = 6 + 4\delta_3$ ,  $y(0) = 0$ . Take the Laplace transform, apply partial fractions, and simplify to get

$$\begin{aligned}
 Y(s) &= \frac{6}{s(s+1/4)} + \frac{4}{s+1/4}e^{-3s} \\
 &= \frac{24}{s} - \frac{24}{s+1/4} + \frac{4}{s+1/4}e^{-3s}.
 \end{aligned}$$

Laplace inversion now gives

$$\begin{aligned}
 y &= 24 - 24e^{-\frac{1}{4}t} + 4e^{-\frac{1}{4}(t-3)}h(t-3) \\
 &= \begin{cases} 24 - 24e^{-\frac{1}{4}t} & \text{if } 0 \leq t < 3 \\ 24 - 24e^{-\frac{1}{4}t} + 4e^{-\frac{1}{4}(t-3)} & \text{if } 3 \leq t < \infty \end{cases}.
 \end{aligned}$$

- 13.** Clearly,  $y(0) = 0$ . The input rate is  $\delta_0 + \delta_2 + \delta_4 + \delta_6$  while the output rate is  $y$ . We are thus led to the differential equation  $y' + y = \sum_{k=0}^3 \delta_{2k}$ ,  $y(0) = 0$ . Take the Laplace transform and solve for  $Y(s)$  to get

$$Y(s) = \sum_{k=0}^3 \frac{e^{-2ks}}{s+1}.$$

Laplace inversion gives

$$\begin{aligned}
 y &= \sum_{k=0}^3 e^{-(t-2k)}h(t-2k) \\
 &= \begin{cases} e^{-t} & \text{if } 0 \leq t < 2 \\ e^{-t} + e^{-(t-2)} & \text{if } 2 \leq t < 4 \\ e^{-t} + e^{-(t-2)} + e^{-(t-4)} & \text{if } 4 \leq t < 6 \\ e^{-t} + e^{-(t-2)} + e^{-(t-4)} + e^{-(t-6)} & \text{if } 6 \leq t < \infty \end{cases}.
 \end{aligned}$$

Using the formula  $1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r}$  we get

$$\begin{aligned}
 y(6) &= \sum_{k=0}^3 e^{-(6-2k)} = \sum_{k=0}^3 e^{-2k} = \sum_{k=0}^3 (e^{-2})^k \\
 &= \frac{1 - (e^{-2})^4}{1 - e^{-2}} = 1.156 \text{ lb.}
 \end{aligned}$$

- 15.** The mass is  $m = 2$ . The spring constant  $k$  is given by  $k = 8/1 = 8$ . The damping constant is given by  $\mu = 8/1 = 8$ . The external force is  $2\delta_4$ . The initial conditions are  $y(0) = .1$  and  $y'(0) = .05$ . The equation  $2y'' + 8y' + 8y = 2\delta_4$ ,  $y(0) = .1$ ,  $y'(0) = .05$  models the motion of the body. Divide by two to get  $y'' + 4y' + 4y = \delta_4$ . Apply the Laplace transform to get



$$Y(s) = \frac{.1}{s+2} + \frac{1}{(s+2)^2} e^{-4s}.$$

Laplace inversion now gives

$$\begin{aligned} y &= \frac{1}{10} e^{-2t} + (t-4)e^{-2(t-4)} h(t-4) \\ &= \begin{cases} \frac{1}{10} e^{-2t} & \text{if } 0 \leq t < 4 \\ \frac{1}{10} e^{-2t} + (t-4)e^{-2(t-4)} & \text{if } 4 \leq t < \infty \end{cases}. \end{aligned}$$

17. Clearly  $m = 1$ ,  $\mu = 0$ , and  $k = 1$ . The forcing function is  $\delta_0 + \delta_\pi + \cdots + \delta_{5\pi} = \sum_{k=0}^5 \delta_{\pi k}$ . The differential equation that describes the motion is

$$y'' + y = \sum_{k=0}^5 \delta_{\pi k}.$$

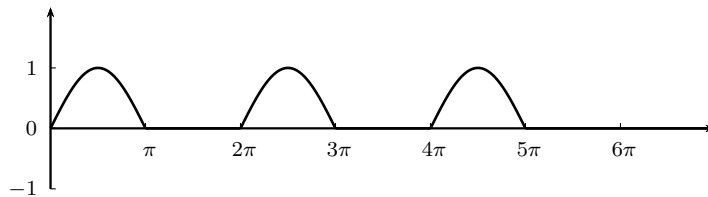
Apply the Laplace transform to get

$$Y(s) = \sum_{k=0}^5 \frac{e^{-\pi k s}}{s^2 + 1}.$$

Laplace inversion now gives

$$\begin{aligned} y &= \sum_{k=0}^5 (\sin(t - \pi k)) h(t - \pi k) \\ &= \sum_{k=0}^5 (-1)^k (\sin t) h(t - \pi k) \\ &= \begin{cases} \sin t & \text{if } 0 \leq t < \pi \\ 0 & \text{if } \pi \leq t < 2\pi \\ \sin t & \text{if } 2\pi \leq t < 3\pi \\ 0 & \text{if } 3\pi \leq t < 4\pi \\ \sin t & \text{if } 4\pi \leq t < 5\pi \\ 0 & \text{if } 5\pi \leq t < \infty \end{cases}. \end{aligned}$$

The graph is given below.



At  $t = 0$  the hammer imparts a velocity to the system causing harmonic motion. At  $t = \pi$  the hammer strikes in precisely the right way to stop the motion. Then at  $t = 2\pi$  the process repeats.

19. The value of  $y(t)$  at  $t = c$  for  $y$  given by Exercise 18 is  $y_0 e^{-ac}$ . Thus the differential equation we need to solve is  $y' + ay = 0$ ,  $y(c) = y_0 e^{-ac} + k$ , on the interval  $[c, \infty)$ . We get the general solution  $y(t) = be^{-at}$ . The initial condition implies  $be^{-ac} = y_0 e^{-ac} + k$ . Solving for  $b$  gives  $b = y_0 + ke^{ac}$ . Thus

$$y = y_0 e^{-at} + ke^{-a(t-c)},$$

on the interval  $[c, \infty)$ .

## SECTION 6.5

1.  $F(s) = \frac{1}{s-1}$  and  $G(s) = \frac{1}{s} - \frac{e^{-s}}{s}$ . Thus  $F(s)G(s) = \frac{1}{(s-1)s} - \frac{e^{-s}}{(s-1)s}$ . Partial fractions gives  $F(s)G(s) = \left(\frac{1}{s-1} - \frac{1}{s}\right) - \left(\frac{1}{s-1} - \frac{1}{s}\right)e^{-s}$  and Laplace inversion gives

$$\begin{aligned} f * g(t) &= e^t - 1 - (e^{t-1} - 1)h(t-1) \\ &= \begin{cases} e^t - 1 & \text{if } 0 \leq t < 1 \\ e^t - e^{t-1} & \text{if } 1 \leq t < \infty \end{cases}. \end{aligned}$$

3.  $F(s) = e^{-s}\mathcal{L}\{t+1\} = \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s}$  and  $G(s) = \frac{1}{s}(e^{-3s} - e^{-4s})$ . Thus

$$F(s)G(s) = \left(\frac{1}{s^3} + \frac{1}{s^2}\right)(e^{-4s} - e^{-5s})$$

Laplace inversion now gives

$$\begin{aligned} f * g &= \left(\frac{(t-4)^2}{2} + (t-4)\right)h(t-4) - \left(\frac{(t-5)^2}{2} + (t-5)\right)h(t-5) \\ &= \begin{cases} 0 & \text{if } 0 \leq t < 4 \\ \frac{(t-4)^2}{2} + (t-4) & \text{if } 4 \leq t < 5 \\ t - 7/2 & \text{if } 5 \leq t < \infty \end{cases}. \end{aligned}$$

5.  $F(s) = \frac{1}{s} - \frac{1}{s}e^{-2s}$  and  $G(s) = \frac{1}{s} - \frac{1}{s}e^{-2s}$ . Thus

$$F(s)G(s) = \frac{1}{s^2}(1 - 2e^{-2s} + e^{-4s})$$

Laplace inversion now gives

$$\begin{aligned} f * g &= t - 2(t-2)h(t-2) + (t-4)h(t-4) \\ &= \begin{cases} t & \text{if } 0 \leq t < 2 \\ -t + 4 & \text{if } 2 \leq t < 4 \\ 0 & \text{if } 4 \leq t < \infty \end{cases} . \end{aligned}$$

7.

$$\begin{aligned} \sin t * (\delta_0 + \delta_\pi) &= \sin t + \sin(t-\pi)h(t-\pi) \\ &= \sin t - (\sin t)h(t-\pi) \\ &= \begin{cases} \sin t & \text{if } 0 \leq t < \pi \\ 0 & \text{if } \pi \leq t < \infty \end{cases} . \end{aligned}$$

9. The unit impulse response function is  $\zeta(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$ . The homogeneous solution is  $y_h = 2e^{3t}$ . Observe that  $Y_p(s) = \frac{1}{s-3} \frac{e^{-2s}}{s} = \frac{1}{3} \left( \frac{1}{s-3} - \frac{1}{s} \right) e^{-2s}$ . It follows that the particular solution is

$$\begin{aligned} y_p &= \zeta * (h(t-2)) \\ &= \mathcal{L}^{-1}\{Y_p(s)\} \\ &= \frac{1}{3} \left( e^{3(t-2)} - 1 \right) h(t-2). \end{aligned}$$

and

$$\begin{aligned} y &= y_h + y_p \\ &= 2e^{3t} + \frac{1}{3} \left( e^{3(t-2)} - 1 \right) h(t-2) \\ &= \begin{cases} 2e^{3t} & \text{if } 0 \leq t < \infty \\ 2e^{3t} + \frac{1}{3} (e^{3(t-2)} - 1) & \text{if } 1 \leq t < \infty \end{cases} . \end{aligned}$$

11. The unit impulse response function is  $\zeta(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+8}\right\} = e^{-8t}$ . The homogeneous solution is  $y_h = -2e^{-8t}$ . The particular solution is  $y_p = \zeta * \chi_{[3,5)}$ . Observe that

$$\begin{aligned}
Y_p(s) &= \mathcal{L}\{y_p\} \\
&= \frac{1}{s+8} \left( \frac{e^{-3s}}{s} - \frac{e^{-5s}}{s} \right) \\
&= \frac{1}{8} \left( \frac{1}{s} - \frac{1}{s+8} \right) (e^{-3s} - e^{-5s}).
\end{aligned}$$

It follows that

$$\begin{aligned}
y_p &= \mathcal{L}^{-1}\{Y_p(s)\} \\
&= \frac{1}{8} \left( 1 - e^{-8(t-3)} \right) h(t-3) - \frac{1}{8} \left( 1 - e^{-8(t-5)} \right) h(t-5).
\end{aligned}$$

and

$$\begin{aligned}
y &= y_h + y_p \\
&= -2e^{-8t} + \frac{1}{8} \left( 1 - e^{-8(t-3)} \right) h(t-3) - \frac{1}{8} \left( 1 - e^{-8(t-5)} \right) h(t-5) \\
&= \begin{cases} -2e^{-8t} & \text{if } 0 \leq t < 3 \\ -2e^{-8t} + \frac{1}{8} (1 - e^{-8(t-3)}) & \text{if } 3 \leq t < 5 \\ -2e^{-8t} + \frac{1}{8} (e^{-8(t-5)} - e^{-8(t-3)}) & \text{if } 5 \leq t < \infty \end{cases}
\end{aligned}$$

- 13.** The unit impulse response function is  $\zeta(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3} \sin 3t$ . The homogeneous solution is  $y_h = \cos 3t$ . For the particular solution  $y_p$  we have

$$\begin{aligned}
y_p &= \frac{1}{3} \sin 3t * \chi_{[0,2\pi)} \\
&= \frac{1}{3} \int_0^t \sin(3(t-u)) \chi_{[0,2\pi)} du \\
&= \frac{1}{3} \begin{cases} \int_0^t \sin(3(t-u)) du & \text{if } 0 \leq t < 2\pi \\ \int_0^{2\pi} \sin(3(t-u)) du & \text{if } 2\pi \leq t < \infty \end{cases} \\
&= \frac{1}{9} \begin{cases} 1 - \cos 3t & \text{if } 0 \leq t < 2\pi \\ 0 & \text{if } 2\pi \leq t < \infty \end{cases}.
\end{aligned}$$

- 15.** Let  $y$  be the homogenous solution to  $q(\mathbf{D})y = 0$  with the given initial conditions. Observe that

$$\mathcal{L}\left\{\mathbf{D}^k y\right\} = \begin{cases} s^k Y(s) & \text{if } k < n \\ s^n Y(s) - 1/a_n & \text{if } k = n \end{cases}.$$

and thus

$$\mathcal{L}\{a_k \mathbf{D}^k y\} = \begin{cases} a_k s^k Y(s) & \text{if } k < n \\ a_n s^n Y(s) - 1 & \text{if } k = n \end{cases}.$$

Therefore  $\mathcal{L}\{q(\mathbf{D})y\} = q(s)Y(s) - 1 = 0$  from which we get

$$Y(s) = \frac{1}{q(s)}.$$

Hence  $y = \zeta$  is the unit impulse response function.

17. Suppose  $a_0\zeta + a_1\zeta' + \cdots + a_{n-1}\zeta^{(n-1)} = 0$ . Apply the Laplace transform and use Exercise 16 to get

$$\frac{a_0 + a_1 s + \cdots + a_{n-1} s^{n-1}}{q(s)} = 0.$$

It follows that the numerator must be identically 0 and hence the coefficients  $a_k = 0$ , for each  $k$ . Thus  $\{\zeta, \zeta', \dots, \zeta^{(n-1)}\}$  is linearly independent.

19. This follows from Exercises 17 and 18.

21. By the input derivative formula we get

$$\mathcal{L}\{\mathbf{D}^k y\} = s^k Y(s) - s^{k-1}y_0 - \cdots - y_{k-1} = s^k Y(s) - \sum_{l=0}^{k-1} s^l y_{k-1-l},$$

for  $k \geq 1$ . It follows that

$$\mathcal{L}\{q(\mathbf{D})y\} = q(s)Y(s) - \sum_{k=1}^n \sum_{l=0}^{k-1} a_k s^l y_{k-1-l}.$$

Therefore

$$Y(s) = \sum_{k=1}^n \sum_{l=0}^{k-1} a_k \frac{s^l}{q(s)} y_{k-1-l}.$$

Laplace inversion and Exercise 15 give

$$y(t) = \sum_{k=1}^n \sum_{l=0}^{k-1} a_k \zeta^{(l)} y_{k-1-l}.$$

Reversing the order of the sum gives

$$\begin{aligned}
y(t) &= \sum_{l=0}^{n-1} \left( \sum_{k=l+1}^n a_k y_{k-l-1} \right) \zeta^{(l)} \\
&= \sum_{l=0}^{n-1} \left( \sum_{k=0}^{n-l-1} a_{k+l+1} y_k \right) \zeta^{(l)},
\end{aligned}$$

where in the second line and second sum we shifted  $k$  to  $k + l + 1$ . It follows that the coefficients are given by

$$c_l = \sum_{k=0}^{n-l-1} a_{k+l+1} y_k.$$

- 23.** We have  $q(s) = s^2 - 2s + 1 = (s - 1)^2$  and the unit impulse response function is  $\zeta = \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^2} \right\} = te^t$ . To compute  $c_0$  we write

$$\begin{array}{ccc}
1 & -2 & 1 \\
& 2 & -3
\end{array}$$

and get  $c_0 = -2 \cdot 2 + 1 \cdot (-3) = -7$ . For  $c_1$  we consider

$$\begin{array}{ccc}
1 & -2 & 1 \\
& 2 & -3
\end{array}$$

and get  $c_1 = 1 \cdot 2 = 2$ . It follows from Exercise 21 that

$$\begin{aligned}
y &= c_0 \zeta + c_1 \zeta' \\
&= -7te^t + 2(e^t + te^t) \\
&= 2e^t - 5te^t.
\end{aligned}$$

- 25.** We have  $q(s) = s^3 + s = s(s^2 + 1)$ . Partial fractions give  $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$ . Thus  $\zeta = \mathcal{L}^{-1} \{q(s)\} = 1 - \cos t$ . To compute  $c_0$  we write

$$\begin{array}{cccc}
0 & 1 & 0 & 1 \\
& 1 & 0 & 4
\end{array}$$

and get  $c_0 = 1 + 4 = 5$ . For  $c_1$  we consider

$$\begin{array}{cccc}
0 & 1 & 0 & 1 \\
& 1 & 0 & 4
\end{array}$$

and get  $c_1 = 0$ . For  $c_2$  we consider

$$\begin{array}{cccc} 0 & 1 & 0 & 1 \\ & & 1 & 0 & 4 \end{array}$$

and get  $c_2 = 1$ . It follows from Exercise 21 that

$$\begin{aligned} y &= c_0\zeta + c_1\zeta' + c_2\zeta'' \\ &= 5(1 - \cos t) + 0(\sin t) + 1(\cos t) \\ &= 5 - 4\cos t. \end{aligned}$$

## SECTION 6.6

1.

$$\sum_{n=0}^{\infty} (t-n)^2 \chi_{[n, n+1)}(t) = (< t >_1)^2.$$

3.

$$\begin{aligned} \sum_{n=0}^{\infty} n^2 \chi_{[3n, 3(n+1))}(t) &= \frac{1}{9} \sum_{n=0}^{\infty} (3n)^2 \chi_{[3n, 3(n+1))}(t) \\ &= \frac{1}{9} ([t]_3)^2. \end{aligned}$$

5.

$$\begin{aligned} \sum_{n=0}^{\infty} (t+n) \chi_{[2n, 2(n+1))}(t) &= \sum_{n=0}^{\infty} (t-2n + \frac{3}{2}2n) \chi_{[2n, 2(n+1))}(t) \\ &= < t >_2 + \frac{3}{2} [t]_2. \end{aligned}$$

7.

$$\begin{aligned} \mathcal{L}\{f(< t >_3)\} &= \frac{1}{1-e^{-3s}} \mathcal{L}\{e^t - e^t h(t-3)\} \\ &= \frac{1}{1-e^{-3s}} \left( \frac{1}{s-1} - e^{-3s} \mathcal{L}\{e^{t+3}\} \right) \\ &= \frac{1}{1-e^{-3s}} \left( \frac{1}{s-1} - \frac{e^{-3s}e^3}{s-1} \right) \\ &= \frac{1-e^{-3(s-1)}}{1-e^{-3s}} \frac{1}{s-1}. \end{aligned}$$

9.

$$\begin{aligned}
\mathcal{L}\{f(< t >_{2p})\} &= \frac{1}{1 - e^{-2ps}} \left( \int_0^{2p} e^{-st} f(t) dt \right) \\
&= \frac{1}{1 - e^{-2ps}} \left( \int_0^p e^{-st} dt - \int_p^{2p} e^{-st} dt \right) \\
&= \frac{1}{1 - e^{-2ps}} \left( \frac{e^{-ps} - 1}{-s} + \frac{e^{-2ps} - e^{-ps}}{s} \right) \\
&= \frac{(1 - e^{-ps})^2}{1 - e^{-2ps}} \frac{1}{s} \\
&= \frac{1 - e^{-ps}}{1 + e^{-ps}} \frac{1}{s}.
\end{aligned}$$

11. Since  $< t >_p = t - [t]_p$  we have  $[t]_p = t - < t >_p$ . Hence

$$\begin{aligned}
\mathcal{L}\{[t]_p\} &= \mathcal{L}\{t\} - \mathcal{L}\{< t >_p\} \\
&= \frac{1}{s^2} - \frac{1}{s^2} \left( 1 - \frac{spe^{-ps}}{1 - e^{-ps}} \right) \\
&= \frac{pe^{-ps}}{s(1 - e^{-ps})} \\
&= \frac{p}{s(e^{ps} - 1)}
\end{aligned}$$

13. On the interval  $[2n, 2n+2)$  we have  $f(t) = e^{-2n}$  thus

$$f(t) = \sum_{n=0}^{\infty} e^{-2n} \chi_{[2n, 2n+2)}(t).$$

We now have



$$\begin{aligned}
\mathcal{L}\{f([t]_2)\} &= \sum_{n=0}^{\infty} e^{-2n} \left( \frac{e^{-2ns} - e^{-(2n+2)s}}{s} \right) \\
&= \sum_{n=0}^{\infty} e^{-2n} e^{-2ns} \frac{1 - e^{-2s}}{s} \\
&= \frac{1 - e^{-2s}}{s} \sum_{n=0}^{\infty} e^{-2n(s+1)} \\
&= \frac{1 - e^{-2s}}{s} \sum_{n=0}^{\infty} \left( e^{-2(s+1)} \right)^n \\
&= \frac{1 - e^{-2s}}{s} \frac{1}{1 - e^{-2(s+1)}} \\
&= \frac{1 - e^{-2s}}{1 - e^{-2(s+1)}} \frac{1}{s}
\end{aligned}$$

15.

$$\begin{aligned}
\mathcal{L}\{f([t]_p)\} &= \sum_{n=0}^{\infty} f(np) \mathcal{L}\{\chi_{[np, (n+1)p)}\} \\
&= \sum_{n=0}^{\infty} f(np) \frac{e^{-nps} - e^{-(n+1)ps}}{s} \\
&= \frac{1 - e^{-ps}}{s} \sum_{n=0}^{\infty} f(np) e^{-nps}.
\end{aligned}$$

17. Let  $F(s) = \frac{1 - e^{-4(s-2)}}{(1 - e^{-4s})(s-2)}$ . We first write

$$\begin{aligned}
F(s) &= \sum_{n=0}^{\infty} \frac{1 - e^{-4(s-2)}}{s - 2} e^{-4ns} \\
&= \sum_{n=0}^{\infty} \frac{e^{-4ns} - e^8 e^{-4(n+1)s}}{s - 2}
\end{aligned}$$

Laplace inversion now gives

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \sum_{n=0}^{\infty} e^{2(t-4n)} h(t-4n) - e^8 e^{2(t-4(n+1))} h(t-4(n+1)) \\
&= \sum_{n=0}^{\infty} e^{2(t-4n)} (h(t-4n) - h(t-4(n+1))) \\
&= e^{2t} \sum_{n=0}^{\infty} e^{-2(4n)} \chi_{[4n, 4(n+1))} \\
&= e^{2t} e^{-2[t]_4} = e^{2(t-[t]_4)} = e^{2\langle t \rangle_4}.
\end{aligned}$$

19.

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \sum_{n=0}^{\infty} (-1)^n \mathcal{L}^{-1}\left\{\frac{e^{-pns}}{s+a}\right\} \\
&= \sum_{n=0}^{\infty} (-1)^n e^{-a(t-pn)} h(t-pn) \\
&= e^{-at} \sum_{N=0}^{\infty} \sum_{n=0}^N (-1)^n e^{apn} \chi_{[Np, (N+1)p)} \\
&= e^{-at} \sum_{N=0}^{\infty} \frac{1 - (-e^{ap})^{N+1}}{1 - e^{ap}} \\
&= e^{-at} \sum_{N=0}^{\infty} \frac{1 - (-1)^{N+1} e^{a(N+1)p}}{1 + e^{ap}} \\
&= e^{-at} \begin{cases} \frac{1+e^{a(N+1)p}}{1+e^{ap}} & \text{if } t \in [Np, (N+1)p), (N \text{ even}) \\ \frac{1-e^{a(N+1)p}}{1+e^{ap}} & \text{if } t \in [Np, (N+1)p), (N \text{ odd}) \end{cases} \\
&= e^{-at} \left( \frac{1 + (-1)^{\frac{[t]_p}{p}} e^{a([t]_p+p)}}{1 + e^{ap}} \right).
\end{aligned}$$

## SECTION 6.7

1. On the interval  $[0, 2)$  the input rate is  $2 \cdot 4 = 8$  lbs salt per minute. On the interval  $[2, 4)$  the input rate is  $1 \cdot 4 = 4$  lbs salt per minute. The input function  $f(t)$  is periodic with period 4. We can thus write  $f(t) = 4 + 4 \text{sw}_2(t)$ . The output rate is  $\frac{y(t)}{10} \cdot 4$ . The resulting differential equation that models this problem is

$$y' + \frac{4}{10}y = 4 + 4\text{sw}_2(t), \quad y(0) = 0.$$

Taking the Laplace transform and simplifying gives

$$Y(s) = \frac{4}{s(s + \frac{2}{5})} + \frac{4}{s(s + \frac{2}{5})} \frac{1}{1 + e^{-2s}}.$$

Partial fractions gives

$$\frac{4}{s(s + \frac{2}{5})} = \frac{10}{s} - \frac{10}{s + \frac{2}{5}}$$

and hence

$$Y(s) = \frac{10}{s} - \frac{10}{s + \frac{2}{5}} + \frac{10}{s} \frac{1}{1 + e^{-2s}} - \frac{10}{s + \frac{2}{5}} \frac{1}{1 + e^{-2s}}.$$

Let

$$\begin{aligned} Y_1(s) &= \frac{10}{s} - \frac{10}{s + \frac{2}{5}} \\ Y_2(s) &= \frac{10}{s} \frac{1}{1 + e^{-2s}} \\ Y_3(s) &= \frac{10}{s + \frac{2}{5}} \frac{1}{1 + e^{-2s}}. \end{aligned}$$

Example 6.6.2 and Exercise 6.6.19 are useful for taking the inverse Laplace transforms of  $Y_2$  and  $Y_3$ . We get

$$\begin{aligned} y_1(t) &= 10 - 10e^{-\frac{2t}{5}} \\ y_2(t) &= 10\text{sw}_2(t) \\ y_3(t) &= \frac{10e^{-\frac{2t}{5}}}{1 + e^{\frac{4}{5}}} \left( 1 + e^{\frac{4}{5}} \begin{cases} e^{\frac{4N}{5}} & \text{if } t \in [2N, 2(N+1)), \text{ N even} \\ -e^{\frac{4N}{5}} & \text{if } t \in [2N, 2(N+1)), \text{ N odd} \end{cases} \right) \\ &= \frac{10e^{-\frac{2t}{5}}}{1 + e^{\frac{4}{5}}} \left( 1 + e^{\frac{4}{5}} (-1)^{[t/2]_1} e^{\frac{2}{5}[t]_2} \right). \end{aligned}$$

It now follows that

$$\begin{aligned} y(t) &= y_1(t) + y_2(t) - y_3(t) \\ &= 10 - 10e^{-\frac{2t}{5}} + 10\text{sw}_2(t) - \frac{10e^{-\frac{2t}{5}}}{1 + e^{\frac{4}{5}}} \left( 1 + e^{\frac{4}{5}} (-1)^{[t/2]_1} e^{\frac{2}{5}[t]_2} \right) \end{aligned}$$

When  $t = 2N$  and  $N$  is even then

$$y(2N) = 20 - 10e^{\frac{-2}{5}2N} - 10\frac{e^{\frac{-2}{5}2N}}{1 + e^{\frac{4}{5}}} \left(1 + e^{\frac{4}{5}}e^{\frac{2}{5}2N}\right).$$

Continuing  $y(2N)$  to all  $t \geq 0$  gives

$$l(t) = 20 - 10e^{\frac{-2}{5}t} - 10\frac{e^{\frac{-2}{5}t}}{1 + e^{\frac{4}{5}}} \left(1 + e^{\frac{4}{5}}e^{\frac{2}{5}t}\right),$$

a function whose graph bounds the graph of  $y$  from below. In a similar way for  $t = 2N$ ,  $N$  odd, we get

$$u(t) = 10 - 10e^{\frac{-2}{5}t} - 10\frac{e^{\frac{-2}{5}t}}{1 + e^{\frac{4}{5}}} \left(1 - e^{\frac{4}{5}}e^{\frac{2}{5}t}\right),$$

whose graph bounds the graph of  $y$  from above. Now observe that

$$\lim_{t \rightarrow \infty} l(t) = 20 - \frac{10e^{\frac{4}{5}}}{1 + e^{\frac{4}{5}}} \approx 13.10 \quad \text{and} \quad \lim_{t \rightarrow \infty} u(t) = 10 + \frac{10e^{\frac{4}{5}}}{1 + e^{\frac{4}{5}}} \approx 16.9$$

Thus the amount of salt fluctuates from 13.10 pounds to 16.90 pounds in the long term.

3. The input function is  $5 \sum_{n=1}^{\infty} \delta_{2n} = 5\delta_0(< t >_2)$ . and therefore the differential equation that models this system is

$$y' + \frac{1}{2}y = 5\delta_0(< t >_2), \quad y(0) = 0.$$

By Proposition 6.6.6 the Laplace transform gives

$$Y(s) = \frac{5}{s + \frac{1}{2}} \frac{1}{1 - e^{-2s}}.$$

By Theorem 6.6.7 Laplace inversion gives

$$\begin{aligned} y(t) &= 5 \sum_{N=0}^{\infty} \left( \sum_{n=0}^N e^{-\frac{1}{2}(t-2n)} \right) \chi_{[2N, 2(N+1))} \\ &= 5e^{-\frac{1}{2}t} \sum_{N=0}^{\infty} \frac{e^{N+1} - 1}{e - 1} \chi_{[2N, 2(N+1))} \\ &= 5e^{-\frac{1}{2}t} \frac{e^{\frac{1}{2}[t]_2+1} - 1}{e - 1}. \end{aligned}$$

The solution is sandwiched in between a lower and upper curve. The upper curve,  $u(t)$ , is obtained by setting  $t = 2m$  to be an even integer in the formula for the solution and then continuing it to all reals. We obtain

$$u(2m) = 5e^{-\frac{1}{2}2m} \frac{e^{\frac{1}{2}[2m]_2+1} - 1}{e - 1} = 5e^{-\frac{1}{2}2m} \frac{e^{\frac{1}{2}2m+1} - 1}{e - 1}.$$

Thus

$$u(t) = 5e^{-\frac{1}{2}t} \frac{e^{\frac{1}{2}t+1} - 1}{e - 1} = 5 \frac{e - e^{-\frac{1}{2}t}}{e - 1}.$$

In a similar way, the lower curve,  $l(t)$ , is obtained by setting  $t = 2(m+1)^-$  (slightly less than the even integer  $2(m+1)$ ) and continuing to all reals. We obtain

$$l(t) = 5 \frac{1 - e^{-\frac{1}{2}t}}{e - 1}.$$

An easy calculation gives

$$\lim_{t \rightarrow \infty} u(t) = 5 \frac{e}{e-1} \simeq 7.91 \quad \text{and} \quad \lim_{t \rightarrow \infty} l(t) = 5 \frac{1}{e-1} \simeq 2.91.$$

This means that the salt fluctuation in the tank varies between 2.91 and 7.91 pounds for large values of  $t$ .

5. Let  $y(t)$  be the number of alligators at time  $t$  measured in months. We assume the Malthusian growth model  $y' = ry$ . Thus  $y(t) = y(0)e^{rt} = 3000e^{rt}$ . To determine the growth rate  $r$  we know  $y(-12) = 2500$  (12 months earlier there were 2500 alligators). Thus  $2500 = 3000e^{-12r}$  and hence  $r = \frac{1}{12} \ln \frac{6}{5}$ . The elite force of Cajun alligator hunters instantaneously remove 40 alligators at the beginning of each month. This can be modeled by  $40(\delta_0 + \delta_1 + \dots) = 40\delta_0(< t >_1)$ . The mathematical model is thus

$$y' = ry - 40\delta_0(< t >_1), \quad y(0) = 3000,$$

where  $r = \frac{1}{12} \ln \frac{6}{5}$ . We apply the Laplace transform and use Proposition 6.6.6 to get

$$Y(s) = \frac{3000}{s - r} - 40 \frac{1}{s - r} \frac{1}{1 - e^{-s}}.$$

Let

$$\begin{aligned} Y_1(s) &= \frac{3000}{s - r} \\ Y_2(s) &= 40 \frac{1}{s - r} \frac{1}{1 - e^{-s}} \end{aligned}$$

Then  $Y(s) = Y_1(s) + Y_2(s)$ . We use Theorem 6.6.7 to get

$$\begin{aligned}
y_1(t) &= 3000e^{rt} \\
y_2(t) &= 40 \sum_{N=0}^{\infty} \left( \sum_{n=0}^N e^{r(t-n)} \right) \chi_{[N, N+1)} \\
&= 40e^{rt} \sum_{N=0}^{\infty} \frac{1 - e^{-r(N+1)}}{1 - e^{-r}} \chi_{[N, N+1)} \\
&= 40 \frac{e^{rt} - e^{-r([t]_1 - t + 1)}}{1 - e^{-r}}.
\end{aligned}$$

It follows now that

$$\begin{aligned}
y(t) &= y_1(t) - y_2(t) \\
&= 3000e^{rt} - 40 \frac{e^{rt} - e^{-r([t]_1 - t + 1)}}{1 - e^{-r}}.
\end{aligned}$$

To determine the population at the beginning of 5 years = 60 months we compute

$$\begin{aligned}
y(60) &= 3000e^{60r} - 40 \frac{e^{60r} - e^{-r}}{1 - e^{-r}} \\
&= 7464.96 - 3988.16 \\
&\approx 3477
\end{aligned}$$

## SECTION 6.8

1. Since  $c\beta = \sqrt{2}$  is not an odd multiple of  $\pi$  we get

$$\begin{aligned}
y(t) &= 2 \left( 2 \operatorname{sw}_1(t) - (-1)^{[t]_1} (\cos < t >_1 - \alpha \sin < t >_1) \right) \\
&\quad - 2 (\cos t + \alpha \sin t),
\end{aligned}$$

where  $\alpha = \frac{-\sin \sqrt{2}}{1 + \cos \sqrt{2}}$ . Since  $\frac{\beta c}{\pi} = \frac{2}{\pi}$  is irrational the motion is non periodic

3. Since  $c\beta = 2\pi$  is not an odd multiple of  $\pi$  we get

$$\begin{aligned}
y(t) &= \frac{1}{\pi^2} \left( 2 \operatorname{sw}_2(t) - (-1)^{[t/2]_1} \cos \pi < t >_2 - \cos \pi t \right) \\
&= \frac{1}{\pi^2} \left( 2 \operatorname{sw}_2(t) - \cos \pi t \left( (-1)^{[t/2]_1} + 1 \right) \right),
\end{aligned}$$

where we have used the identity  $\cos \pi < t >_2 = \cos \pi t$ . Since  $\frac{\beta c}{\pi} = 2$  is rational the motion is periodic.

5. Since  $c\beta = \pi$  is an odd multiple of  $\pi$  we get  $y(t) = \frac{2}{\pi^2} (\text{sw}_1(t) - [t]_1 \cos \pi t - \cos \pi t)$ . Resonance occurs.
7. Since  $c\beta = \pi$  is not a multiple of  $2\pi$  and  $\gamma = 0$  we get

$$y(t) = \sin t + \sin \langle t \rangle_\pi + (1 + (-1)^{[t/\pi]_1}) \sin t,$$

where we have used  $\sin \langle t \rangle_\pi = (-1)^{[t/\pi]_1} \sin t$ . Since  $\frac{\beta c}{2\pi} = \frac{1}{2}$  is rational the motion is periodic.

9. Since  $c\beta = 1$  is not a multiple of  $2\pi$  and  $\gamma = \frac{-\sin 1}{1 - \cos 1}$  we get

$$y(t) = \sin t + \gamma \cos t + \sin \langle t \rangle_1 - \gamma \cos \langle t \rangle_1,$$

where  $\gamma = \frac{-\sin 1}{1 - \cos 1}$ . Since  $\frac{\beta c}{2\pi} = \frac{1}{2\pi}$  is not rational the motion is non periodic.

11. Since  $\beta c = 2\pi$  we get

$$y(t) = 2(\sin t)(1 + [t/2\pi]_1).$$

Resonance occurs.

13. First we have

$$\begin{aligned} \sum_{n=0}^N (e^{i\theta})^n &= \sum_{n=0}^N e^{in\theta} \\ &= \sum_{n=0}^N \cos n\theta + i \sin n\theta \\ &= \sum_{n=0}^N \cos n\theta + i \sum_{n=0}^N \sin n\theta. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{n=0}^N (e^{i\theta})^n &= \frac{e^{i(N+1)\theta} - 1}{e^{i\theta} - 1} \\ &= \frac{\cos(N+1)\theta - 1 + i \sin(N+1)\theta}{\cos \theta - 1 + i \sin \theta} \\ &= \frac{\cos(N+1)\theta - 1 + i \sin(N+1)\theta}{\cos \theta - 1 + i \sin \theta} \cdot \frac{\cos \theta - 1 - i \sin \theta}{\cos \theta - 1 - i \sin \theta} \end{aligned}$$

The product of the denominators simplifies to  $2 - 2\cos \theta$ . The product of the numerators has a real and imaginary part. Call them  $R$  and  $I$ ,

respectively. Then

$$\begin{aligned}
 R &= (\cos((N+1)\theta) - 1)(\cos\theta - 1) + \sin((N+1)\theta)\sin\theta \\
 &= \cos((N+1)\theta)\cos\theta + \sin((N+1)\theta)\sin\theta - \cos(N+1)\theta - \cos\theta + 1 \\
 &= \cos(N\theta) - \cos(N\theta)\cos\theta + \sin(N\theta)\sin\theta - \cos\theta + 1 \\
 &= (\cos(N\theta) + 1)(1 - \cos\theta) + \sin(N\theta)\sin\theta
 \end{aligned}$$

and

$$\begin{aligned}
 I &= \sin((N+1)\theta)(\cos\theta - 1) - \sin\theta(\cos((N+1)\theta) - 1) \\
 &= \sin((N+1)\theta)\cos\theta - \sin((N+1)\theta) + \sin\theta - \cos((N+1)\theta)\sin\theta \\
 &= \sin(N\theta) - \sin(N\theta)\cos\theta - \cos(N\theta)\sin\theta + \sin\theta \\
 &= \sin(N\theta)(1 - \cos\theta) + \sin\theta(1 - \cos(N\theta)).
 \end{aligned}$$

Equating real and imaginary parts and simplifying now gives

$$\begin{aligned}
 \sum_{n=0}^N \cos n\theta &= \frac{R}{2 - 2\cos\theta} \\
 &= \frac{(\cos(N\theta) + 1)(1 - \cos\theta) + \sin(N\theta)\sin\theta}{2 - 2\cos\theta} \\
 &= \frac{1}{2}(1 + \cos N\theta + \gamma \sin N\theta)
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{n=0}^N \sin n\theta &= \frac{I}{2 - 2\cos\theta} \\
 &= \frac{\sin(N\theta)(1 - \cos\theta) + \sin\theta(1 - \cos(N\theta))}{2 - 2\cos\theta} \\
 &= \frac{1}{2}(\sin N\theta + \gamma(1 - \cos N\theta))
 \end{aligned}$$

**15.** Let

$$\begin{aligned}
 R(v) &= \sum_{n=0}^N \cos nv = \frac{1}{2}(1 + \cos Nv + \gamma \sin Nv) = \operatorname{Re} \sum_{n=0}^N e^{inv} \\
 I(v) &= \sum_{n=0}^N \sin nv = \frac{1}{2}(\sin Nv + \gamma(1 - \cos Nv)) = \operatorname{Im} \sum_{n=0}^N e^{inv},
 \end{aligned}$$

as in Exercise 13. Now



$$\begin{aligned}
\sum_{n=0}^N \cos(u + nv) &= \operatorname{Re} \sum_{n=0}^N e^{i(u+nv)} \\
&= \operatorname{Re} \left( e^{iu} \sum_{n=0}^N e^{inv} \right) \\
&= \operatorname{Re} ((\cos u + i \sin u)(R(v) + iI(v))) \\
&= (\cos u)R(v) - (\sin u)I(v) \\
&= \frac{1}{2} (\cos u + \cos u \cos Nv + \gamma \cos u \sin Nv) \\
&\quad - \frac{1}{2} (\sin u \sin Nv + \gamma(\sin u - \sin u \cos Nv)) \\
&= \frac{1}{2} (\cos u + \cos(u + Nv) + \gamma(-\sin u + \sin(u + Nv))).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{n=0}^N \sin(u + nv) &= \operatorname{Im} \sum_{n=0}^N e^{i(u+nv)} \\
&= \operatorname{Im} \left( e^{iu} \sum_{n=0}^N e^{inv} \right) \\
&= \operatorname{Im} ((\cos u + i \sin u)(R(v) + iI(v))) \\
&= (\sin u)R(v) + (\cos u)I(v) \\
&= \frac{1}{2} (\sin u + \sin u \cos Nv + \gamma \sin u \sin Nv) \\
&\quad + \frac{1}{2} (\cos u \sin Nv + \gamma(\cos u - \cos u \cos Nv)) \\
&= \frac{1}{2} (\sin u + \sin(u + Nv) + \gamma(\cos u - \cos(u + Nv))).
\end{aligned}$$

## SECTION 7.1

1. The ratio test gives  $\frac{(n+1)^2}{n^2} \rightarrow 1$ .  $R = 1$ .
3. The ratio test gives  $\frac{2^n n!}{2^{n+1}(n+1)!} = \frac{1}{2(n+1)} \rightarrow 0$ .  $R = \infty$ .
5. The ratio test gives  $\frac{(n+1)!}{n!} = n+1 \rightarrow \infty$ .  $R = 0$ .
7.  $t$  is a factor in this series which we factor out to get  $t \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}$ . Since  $t$  is a polynomial its presence will not change the radius of convergence.

Let  $u = t^2$  in the new powers series to get  $\sum_{n=0}^{\infty} \frac{(-1)^n u^n}{(2n)!}$ . The ratio test gives  $\left| \frac{(-1)^{n+1} (2n)!}{(-1)^n (2n+2)!} \right| = \frac{1}{(2n+1)(2n+2)} \rightarrow 0$ . The radius of convergence in  $u$  and hence  $t$  is  $\infty$ .

9. The expression in the denominator can be written  $\frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{1} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n+1)}{2 \cdot 4 \cdots 2n} = \frac{(2n+1)!}{2^n (1 \cdot 2 \cdot 3 \cdots n)} = \frac{(2n+1)!}{2^n n!}$  and the given power series is  $\sum_{n=0}^{\infty} \frac{(n!)^2 2^n t^n}{(2n+1)!}$ . The ratio test gives  $\frac{((n+1)!)^2 2^{n+1} (2n+1)!}{(n!)^2 2^n (2n+3)!} = \frac{(n+1)^2 2}{(2n+3)(2n+2)} \rightarrow \frac{1}{2}$ .  $R = 2$ .
11. Use the geometric series to get  $\frac{1}{t-a} = \frac{-1}{a} \frac{1}{1-\frac{t}{a}} = \frac{-1}{a} \sum_{n=0}^{\infty} \left(\frac{t}{a}\right)^n = -\sum_{n=0}^{\infty} \frac{t^n}{a^{n+1}}$ .
13.  $\frac{\sin t}{t} = \frac{1}{t} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$ .
15. Recall that  $\tan^{-1} t = \int \frac{1}{1+t^2} dt$ . Using the result of Exercise 10 we get  $\tan^{-1} t = \sum_{n=0}^{\infty} \int (-1)^n t^{2n} dt + C = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1} + C$ . Since  $\tan^{-1} 0 = 0$  it follows that  $C = 0$ . Thus  $\tan^{-1} t = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1}$ .
17. Since  $\tan t$  is odd we can write  $\tan t = \sum_{n=0}^{\infty} d_{2n+1} t^{2n+1}$  and hence  $\sin t = \cos t \sum_{n=0}^{\infty} d_{2n+1} t^{2n+1}$ . Writing out a few terms gives  $t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots = (1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots)(d_1 t + d_3 t^3 + d^5 t^5 \cdots)$ . Collecting like powers of  $t$  gives the following recursion relations

$$\begin{aligned} d_1 &= 1 \\ d_3 - \frac{d_1}{2!} &= \frac{-1}{3!} \\ d_5 - \frac{d_3}{2!} + \frac{d_1}{4!} &= \frac{1}{5!} \\ d_7 - \frac{d_5}{2!} + \frac{d_3}{4!} - \frac{d_1}{6!} &= \frac{-1}{7!}. \end{aligned}$$

Solving these equations gives

$$\begin{aligned}
 d_1 &= 1 \\
 d_3 &= \frac{1}{3} \\
 d_5 &= \frac{2}{15} \\
 d_7 &= \frac{17}{315}.
 \end{aligned}$$

Thus  $\tan t = 1 + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \dots$ .

**19.**  $e^t \sin t = (1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots)(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots) = t + (1)t^2 + (\frac{-1}{3!} + \frac{1}{2!})t^3 + (\frac{-1}{3!} + \frac{1}{3!})t^4 + (\frac{1}{5!} - \frac{1}{2!}\frac{1}{3!} + \frac{1}{4!})t^5 \dots = t + t^2 + \frac{1}{3}t^3 - \frac{1}{30}t^5.$

**21.** The ratio test gives infinite radius of convergence. Let  $f(t)$  be the function defined by the given power series. Then

$$\begin{aligned}
 f(t) &= \sum_{n=0}^{\infty} (-1)^n \frac{n+1}{n!} t^n \\
 &= \sum_{n=0}^{\infty} (-1)^n \left( \frac{n}{n!} + \frac{1}{n!} \right) t^n \\
 &= \sum_{n=1}^{\infty} (-1)^n \frac{1}{(n-1)!} t^n + \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} t^n \\
 &= t \sum_{n=0}^{\infty} (-1)^{n+1} \frac{t^n}{n!} + \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \\
 &= -te^{-t} + e^{-t}
 \end{aligned}$$

**23.** It is easy to check that the interval of convergence is  $(-1, 1)$ . Let  $f(t)$  be the function defined by the given power series. Then

$$\begin{aligned}
 \int f(t) dt &= \sum_{n=0}^{\infty} (n+1) \frac{t^{n+1}}{n+1} + c \\
 &= t \sum_{n=0}^{\infty} t^n + c \\
 &= \frac{t}{1-t} + c.
 \end{aligned}$$

Differentiation gives

$$f(t) = \frac{1}{(1-t)^2}.$$

- 25.** It is not hard to see that the interval of convergence is  $(-1, 1)$ . Let  $f(t)$  be the given power series. A partial fraction decomposition gives

$$\frac{1}{(2n+1)(2n-1)} = \frac{1}{2} \left( \frac{1}{2n-1} - \frac{1}{2n+1} \right).$$

Therefore

$$f(t) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n-1} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n+1}.$$

Let  $f_1(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n-1}$  and  $f_2(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n+1}$ . Then  $f_1(t) = -t + \frac{t^2}{2} \ln \frac{1+t}{1-t}$  by Exercise 24. Observe that  $f_2(0) = 0$  and

$$\begin{aligned} f_2'(t) &= \sum_{n=0}^{\infty} t^{2n} \\ &= \frac{1}{1-t^2} = \frac{1}{2} \frac{1}{1+t} + \frac{1}{2} \frac{1}{1-t}. \end{aligned}$$

Integration and the fact that  $f_2(0) = 0$  gives  $f_2(t) = \frac{1}{2} \ln \frac{1+t}{1-t}$ . It follows now that

$$\begin{aligned} f(t) &= \frac{1}{2}(f_1(t) - f_2(t)) \\ &= \frac{1}{2} \left( -t + \frac{t^2}{2} \ln \frac{1+t}{1-t} - \frac{1}{2} \ln \frac{1+t}{1-t} \right) \\ &= \frac{-t}{2} + \frac{t^2-1}{4} \ln \frac{1+t}{1-t}. \end{aligned}$$

- 27.** The binomial theorem:  $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$ .
- 29.** The ratio  $\frac{c_{n+1}}{c_n}$  is  $\frac{1}{2}$  if  $n$  is even and 2 if  $n$  is odd. Thus  $\lim_{n \rightarrow \infty} \left| \frac{c_{n+1}}{c_n} \right|$  does not exist. The ratio test does not apply. The root test gives that  $\sqrt[n]{c_n}$  is 1 if  $n$  is odd and  $\sqrt[2]{2}$  if  $n$  is even. As  $n$  approaches  $\infty$  both even and odd terms approach 1. It follows that the radius of convergence is 1.
- 31.** Suppose  $f^{(n)}(t) = e^{\frac{-1}{t}} p_n(\frac{1}{t})$  where  $p_n$  is a polynomial. Then  $f^{(n+1)}(t) = e^{\frac{-1}{t}} \left( \frac{1}{t^2} \right) p_n(\frac{1}{t}) + p_n'(\frac{1}{t}) \left( \frac{-1}{t^2} \right) e^{\frac{-1}{t}} = e^{\frac{-1}{t}} p_{n+1}(\frac{1}{t})$ , where  $p_{n+1}(x) = x^2(p_n(x) - p_n'(x))$ . By mathematical induction it follows that  $f^{(n)}(t) = e^{\frac{-1}{t}} p_n(\frac{1}{t})$  for all  $n = 1, 2, \dots$
- 33.** Since  $f(t) = 0$  for  $t \leq 0$  clearly  $\lim_{t \rightarrow 0^-} f^{(n)}(t) = 0$ . The previous problems imply that the right hand limits are also zero. Thus  $f^{(n)}(0)$  exist and is 0.

## SECTION 7.2

1. Let  $y(t) = \sum_{n=0}^{\infty} c_n t^n$ . Then  $c_{n+2} = \frac{c_n}{(n+2)(n+1)}$ . Consider even and odd cases to get  $c_{2n} = \frac{c_0}{(2n)!}$  and  $c_{2n+1} = \frac{c_1}{(2n+1)!}$ . Thus  $y(t) = c_0 \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} = c_0 \cosh t + c_1 \sinh t$ . (see Example 7.1.7) We observe that the characteristic polynomial is  $s^2 - 1 = (s-1)(s+1)$  so  $\{e^t, e^{-t}\}$  is a fundamental set. But  $\cosh t = \frac{e^t + e^{-t}}{2}$  and  $\sinh t = \frac{e^t - e^{-t}}{2}$ ; the set  $\{\cosh t, \sinh t\}$  is also a fundamental set.
3. Let  $y(t) = \sum_{n=0}^{\infty} c_n t^n$ . Then  $c_{n+2}(n+2)(n+1) + k^2 c_n = 0$  or  $c_{n+2} = -\frac{k^2 c_n}{(n+2)(n+1)}$ . We consider first the even case.

$$\begin{array}{ll} n=0 & c_2 = -\frac{k^2 c_0}{2 \cdot 1} \\ n=2 & c_4 = -\frac{k^2 c_2}{4 \cdot 3} = \frac{k^4 c_0}{4!} \\ n=4 & c_6 = -\frac{k^6 c_0}{6!} \\ & \vdots \end{array}$$

From this it follows that  $c_{2n} = (-1)^n \frac{k^{2n} c_0}{(2n)!}$ . The odd case is similar. We get  $c_{2n+1} = (-1)^n \frac{k^{2n+1} c_1}{(2n+1)!}$ . The power series expansion becomes

$$\begin{aligned} y(t) &= \sum_{n=0}^{\infty} c_n t^n \\ &= c_0 \sum_{n=0}^{\infty} (-1)^n \frac{k^{2n} t^{2n}}{(2n)!} \\ &\quad + c_1 \sum_{n=0}^{\infty} (-1)^n \frac{k^{2n+1} t^{2n+1}}{(2n+1)!} \\ &= c_0 \cos kt + c_1 \sin kt. \end{aligned}$$

5. Let  $y(t) = \sum_{n=0}^{\infty} c_n t^n$ . Then the recurrence relation is

$$(n+2)(n+1)c_{n+2} - (n-2)(n+1)c_n = 0$$

or

$$c_{n+2} = \frac{n-2}{n+2} c_n.$$

Since there is a difference of two in the indices we consider the even and odd case. We consider first the even case.

$$\begin{aligned} n=0 & & c_2 &= -c_0 \\ n=2 & & c_4 &= \frac{0}{4}c_2 = 0 \\ n=4 & & c_6 &= \frac{2}{6}c_4 = 0 \\ & \vdots & & \vdots \end{aligned}$$

It follows that  $c_{2n} = 0$  for all  $n = 2, 3, \dots$ . Thus

$$\begin{aligned} \sum_{n=0}^{\infty} c_{2n} t^{2n} &= c_0 + c_2 t^2 + 0t^4 + \dots \\ &= c_0(1 - t^2) \end{aligned}$$

and hence  $y_0(t) = 1 - t^2$ . We now consider the odd case.

$$\begin{aligned} n=1 & & c_3 &= \frac{-1}{3}c_1 \\ n=3 & & c_5 &= \frac{1}{5}c_3 = -\frac{1}{5 \cdot 3}c_1 \\ n=5 & & c_7 &= \frac{3}{7}c_5 = -\frac{1}{7 \cdot 5}c_1 \\ n=7 & & c_9 &= \frac{5}{9}c_7 = -\frac{1}{9 \cdot 7}c_1 \\ & \vdots & & \vdots \end{aligned}$$

From this we see that  $c_{2n+1} = \frac{-c_1}{(2n+1)(2n-1)}$ . Thus

$$\sum_{n=0}^{\infty} c_{2n+1} t^{2n+1} = -c_1 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)(2n-1)}$$

and hence  $y_1(t) = - \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)(2n-1)}$ . By Exercise 7.1.25 we can write  $y_1$  as

$$y_1(t) = \frac{t}{2} - \frac{t^2 - 1}{4} \ln \left( \frac{1+t}{1-t} \right).$$

The general solution is

$$y(t) = c_0(1 - t^2) - c_1 \left( \frac{t}{2} + \frac{t^2 - 1}{4} \ln \left( \frac{1-t}{1+t} \right) \right).$$

(See also Exercise 5.5.15.)

**7.** Let  $y(t) = \sum_{n=0}^{\infty} c_n t^n$ . Then the recurrence relation is

$$c_{n+2} = \frac{2}{n+2} c_{n+1} - \frac{n-1}{(n+2)(n+1)} c_n.$$

For the first several terms we get

$$\begin{aligned}
 n=0 & \quad c_2 = 0c_1 + \frac{1}{2}c_0 = \frac{1}{2!}c_0 \\
 n=1 & \quad c_3 = \frac{1}{2}c_2 - 0 = \frac{1}{3!}c_0 \\
 n=2 & \quad c_4 = \frac{2}{4}c_3 - \frac{1}{4 \cdot 3}c_2 = \frac{1}{4!}c_0 \\
 n=3 & \quad c_5 = \frac{3}{5}c_4 - \frac{2}{5 \cdot 4}c_3 = \frac{3}{5!}c_0 - \frac{2}{5!}c_0 = \frac{1}{5!}c_0 \\
 & \quad \vdots \quad \quad \quad \vdots
 \end{aligned}$$

In general,

$$c_n = \frac{1}{n!}c_0, \quad n = 2, 3, \dots$$

We now get

$$\begin{aligned}
 y(t) &= \sum_{n=0}^{\infty} c_n t^n \\
 &= c_0 + c_1 t + \sum_{n=2}^{\infty} c_n t^n \\
 &= (c_1 - c_0)t + c_0 + c_0 t + c_0 \sum_{n=2}^{\infty} \frac{t^n}{n!} \\
 &= (c_1 - c_0)t + c_0 e^t \\
 &= c_0(e^t - t) + c_1 t.
 \end{aligned}$$

9. Let  $y(t) = \sum_{n=0}^{\infty} c_n t^n$ . Then the recurrence relation is

$$c_{n+2} = -\frac{(n-2)(n-3)}{(n+2)(n+1)}c_n.$$

The even case gives:

$$\begin{aligned}
 n=0 & \quad c_2 = -\frac{6}{2}c_0 = -3c_0 \\
 n=2 & \quad c_4 = 0 \\
 n=4 & \quad c_6 = 0 \\
 & \quad \vdots \quad \quad \quad \vdots
 \end{aligned}$$

Hence

$$\sum_{n=0}^{\infty} c_{2n} t^{2n} = c_0 + c_2 t^2 = c_0(1 - 3t^2).$$

The odd case gives

$$\begin{array}{ll}
n = 1 & c_3 = -\frac{1}{3}c_1 \\
n = 3 & c_5 = 0 \\
n = 5 & c_7 = 0 \\
\vdots & \vdots
\end{array}$$

Hence

$$\sum_{n=0}^{\infty} c_{2n+1} t^{2n+1} = c_1 t + c_3 t^3 = c_1 \left( t - \frac{t^3}{3} \right).$$

The general solution is

$$y(t) = c_0(1 - 3t^2) + c_1 \left( t - \frac{t^3}{3} \right).$$

**11.** Let  $n$  be an integer. Then  $e^{inx} = (e^{ix})^n$ . By Euler's formula this is

$$(\cos x + i \sin x)^n = \cos nx + i \sin nx.$$

**13.** By de Moivre's formula  $\sin(n+1)x$  is the imaginary part of  $(\cos x + i \sin x)^{n+1}$ . The binomial theorem gives

$$\begin{aligned}
\cos(n+1)x + i \sin(n+1)x &= (\cos x + i \sin x)^{n+1} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} \cos^{n+1-k} x i^k \sin^k x
\end{aligned}$$

Only the odd powers of  $i$  contribute to the imaginary part. It follows that

$$\begin{aligned}
\sin(n+1)x &= \operatorname{Im} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2j+1} \cos^{n+1-(2j+1)} x (i^{2j+1}) \sin^{2j+1} x \\
&= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n+1}{2j+1} \cos^{n-2j} x (1 - \cos^2 x)^j \sin x,
\end{aligned}$$

where we use the greatest integer function  $\lfloor x \rfloor$  to denote the greatest integer less than or equal to  $x$ . Now replace  $t = \cos x$  and using the definition  $\sin x U_n(\cos x) = \sin(n+1)x$  to get  $U_n(t) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n+1}{2j+1} t^{n-2j} (1 - t^2)^j$ . It follows that  $\sin nx$  is a product of  $\sin x$  and a polynomial in  $\cos x$ .

**15.** We have



$$\sin((n+2)x) = \sin((n+1)x + x) = \sin((n+1)x) \cos x + \cos((n+1)x) \sin x$$

and hence

$$(\sin x)U_{n+1}(\cos x) = (\sin x)U_n(x) \cos x + (\sin x)T_{n+1}(\cos x).$$

Now divide by  $\sin x$  and let  $t = \cos x$ . We get

$$U_{n+1}(t) = tU_n(t) + T_{n+1}(t).$$

- 17.** By using the sum and difference formula it is easy to verify the following trigonometric identity:

$$2 \sin a \cos b = \sin(a + b) + \sin(a - b).$$

Let  $a = (n+1)x$  and  $b = x$ . Then

$$2 \cos x \sin(n+1)x = \sin((n+2)x) + \sin(nx)$$

and hence

$$2(\cos x)U_n(\cos x)/\sin x = U_{n+1}(\cos x)/\sin x + U_{n-1}(\cos x)/\sin x.$$

Now cancel out  $\sin x$  and let  $t = \cos x$ .

- 19.** By using the sum and difference formula it is easy to verify the following trigonometric identity:

$$2 \sin a \sin b = \cos(b - a) - \cos(a + b).$$

Let  $a = x$  and  $b = nx$ . Then

$$2 \sin x \sin nx = \cos((n-1)x) - \cos((n+1)x)$$

and hence

$$2U_{n-1}(\cos x) = T_{n-1}(\cos x) - T_{n+1}(\cos x).$$

Now let  $t = \cos x$ , replace  $n$  by  $n+1$ , and divide by 2.

### SECTION 7.3

- 1.** The function  $\frac{t}{1-t^2}$  is analytic except at  $t = 1$  and  $t = -1$ . The function  $\frac{1}{1+t}$  is analytic except at  $t = -1$ . It follows that  $t = 1$  and  $t = -1$  are the only singular points. Observe that  $(t-1)\left(\frac{t}{1-t^2}\right) = \frac{-t}{1+t}$  is analytic

at 1 and  $(t-1)^2 \left( \frac{1}{1+t} \right)$  is analytic at  $t=1$ . It follows that 1 is a regular singular point. Also observe that  $(t+1) \left( \frac{t}{1-t^2} \right) = \frac{t}{1-t}$  is analytic at  $-1$  and  $(t+1)^2 \left( \frac{1}{1+t} \right) = (1+t)$  is analytic at  $t=-1$ . It follows that  $-1$  is a regular singular point. Thus 1 and  $-1$  are regular points.

3. Both  $3t(1-t)$  and  $\frac{1-e^t}{t}$  are analytic. There are no singular points and hence no regular points.
5. We first write it in standard form:  $y'' + \frac{1-t}{t}y' + 4y = 0$ . While the coefficient of  $y$  is analytic the coefficient of  $y'$  is  $\frac{1-t}{t}$  is analytic except at  $t=0$ . It follows that  $t=0$  is a singular point. Observe that  $t \left( \frac{1-t}{t} \right) = 1-t$  is analytic and  $4t^2$  is too. It follows that  $t=0$  is a regular point.
7. The indicial equation is  $q(s) = s(s-1) + 2s = s^2 + s = s(s+1)$  The exponents of singularity are 0 and  $-1$ . Theorem 2 guarantees one Frobenius solution but there could be two.
9. In standard form the equation is  $t^2y'' + t(1-t)y' + \lambda ty = 0$ . The indicial equation is  $q(s) = s(s-1) + s = s^2$  The exponent of singularity is 0 with multiplicity 2. Theorem 2 guarantees that there is one Frobenius solution. The other has a logarithmic term.
11.  $y_1(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!} = \frac{1}{t} \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} = \frac{1}{t} \sin t$ .  $y_2$  is done similarly.
13. Let  $y(t) = t^{-2}v(t)$ . Then  $y'(t) = -2t^{-3}v(t) + t^{-2}v'(t)$  and  $y''(t) = 6t^{-4}v(t) - 4t^{-3}v'(t) + t^{-2}v''(t)$ . From which we get

$$\begin{aligned} t^2 y'' &= 6t^{-2}v(t) - 4t^{-1}v'(t) + v''(t) \\ 5ty' &= -10t^{-2}v(t) + 5t^{-1}v'(t) \\ 4y &= 4t^{-2}v(t). \end{aligned}$$

Adding these terms and remembering that we are assuming the  $y$  is a solution we get

$$0 = t^{-1}v'(t) + v''(t).$$

From this we get  $\frac{v''}{v'} = \frac{-1}{t}$ . Integrating we get  $\ln v'(t) = -\ln t$  and hence  $v'(t) = \frac{1}{t}$ . Integrating again gives  $v(t) = \ln t$ . It follows that  $y(t) = t^{-2} \ln t$  is a second independent solution. The indicial polynomial is  $q(s) = s(s-1) + 5s + 4 = (s-2)^2$ . Case 3 of the theorem guarantees that one solution is a Frobenius solution and the other has logarithmic term.

In each case below we let  $y = t^r \sum_{n=0}^{\infty} c_n t^n$  where we assume  $c_0 \neq 0$  and  $r$  is the exponent of singularity.

15. Indicial polynomial:  $p(s) = s(s-3)$ ; exponents of singularity  $s = 0$  and  $s = 3$ .

$$\begin{aligned} n = 0 \quad c_0(r)(r-3) &= 0 \\ n = 1 \quad c_1(r-2)(r+1) &= 0 \\ n \geq 1 \quad c_n(n+r)(n+r-3) &= -c_{n-1} \end{aligned}$$

**r=3:**

$$\begin{aligned} n \text{ odd} \quad c_n &= 0 \\ n = 2m \quad c_{2m} &= 3c_0 \frac{(-1)^m (2m+2)}{(2m+3)!} \end{aligned}$$

$$y(t) = 3c_0 \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2) t^{2m+3}}{(2m+3)!} = 3c_0 (\sin t - t \cos t).$$

**r=0:** One is lead to the equation  $0c_3 = 0$  and we can take  $c_3 = 0$ . Thus

$$\begin{aligned} n \text{ odd} \quad c_n &= 0 \\ n = 2m \quad c_{2m} &= c_0 \frac{(-1)^{m+1} (2m-1)}{(2m)!} \end{aligned}$$

$$y(t) = c_0 \sum_{m=0}^{\infty} \frac{(-1)^{m+1} (2m-1) t^{2m}}{(2m)!} = c_0 (t \sin t + \cos t).$$

**General Solution:**  $y = c_1 (\sin t - t \cos t) + c_2 (t \sin t + \cos t)$ .

17. Indicial polynomial:  $p(s) = (s-1)^2$ ; exponents of singularity  $s = 1$ , multiplicity 2. There is one Frobenius solution.

**r = 1 :** Let  $y(t) = \sum_{n=0}^{\infty} c_n t^{n+1}$ . Then

$$n \geq 1 \quad n^2 c_n - n c_{n-1} = 0.$$

This is easy to solve. We get  $c_n = \frac{1}{n!} c_0$  and hence

$$y(t) = c_0 \sum_{n=0}^{\infty} \frac{1}{n!} t^{n+1} = c_0 t e^t.$$

**Logarithmic Solution:** Let  $y_1(t) = t e^t$ . The second independent solution is necessarily of the form

$$y(t) = y_1(t) \ln t + \sum_{n=0}^{\infty} c_n t^{n+1}.$$

Substitution into the differential equation leads to

$$t^2 e^t + \sum_{n=1}^{\infty} (n^2 c_n - n c_{n-1}) t^{n+1} = 0.$$

We write out the power series for  $t^2 e^t$  and add corresponding coefficients to get

$$n^2 c_n - n c_{n-1} + \frac{1}{(n-1)!}.$$

The following list is a straightforward verification:

$$\begin{aligned} n=1 & \quad c_1 = -1 \\ n=2 & \quad c_2 = \frac{-1}{2} \left(1 + \frac{1}{2}\right) \\ n=3 & \quad c_3 = \frac{-1}{3!} \left(1 + \frac{1}{2} + \frac{1}{3}\right) \\ n=4 & \quad c_4 = \frac{-1}{4!} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right). \end{aligned}$$

Let  $s_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ . Then an easy argument gives that

$$c_n = \frac{-s_n}{n!}.$$

We now have a second independent solution

$$y_2(t) = te^t \ln t - t \sum_{n=1}^{\infty} \frac{s_n t^n}{n!}.$$

**General Solution:**

$$y = c_1 te^t + c_2 \left( te^t \ln t - t \sum_{n=1}^{\infty} \frac{s_n t^n}{n!} \right).$$

19. Indicial polynomial:  $p(s) = (s-2)(s+1)$ ; exponents of singularity  $s = 2$  and  $s = -1$ .

$$\begin{aligned} n=0 & \quad c_0(r-2)(r+1) = 0 \\ n \geq 1 & \quad c_n(n+r-2)(n+r+1) = -c_{n-1}(n+r-1) \end{aligned}$$

**r=2:**

$$c_n = 6c_0 \frac{(-1)^n(n+1)}{(n+3)!}, \quad n \geq 1$$

$$y(t) = 6c_0 \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)t^n}{(n+3)!} = 6c_0 \left( \frac{(t+2)e^{-t}}{t} + \frac{t-2}{t} \right).$$

**r=-1:** The recursion relation becomes  $c_n(n-3)(n) = c_{n-1}(n-2) = 0$ . Thus

$$\begin{aligned} n=1 & \quad c_1 = -\frac{c_0}{2} \\ n=2 & \quad c_2 = 0 \\ n=3 & \quad 0c_3 = 0 \end{aligned}$$

We can take  $c_3 = 0$  and then  $c_n = 0$  for all  $n \geq 2$ . We now have  $y(t) = c_0 t^{-1} \left(1 - \frac{t}{2}\right) = \frac{c_0}{2} \left(\frac{2-t}{t}\right)$ .

**General Solution:**  $y = c_1 \frac{2-t}{t} + c_2 \frac{(t+2)e^{-t}}{t}$ .

- 21.** Indicial polynomial:  $p(s) = s(s-2)$ ; exponents of singularity  $r = 0$  and  $r = 2$ .

$$\begin{aligned} n = 0 & \quad c_0 r(r-2) = 0 \\ n \geq 1 & \quad c_n(n+r)(n+r-2) = -c_{n-1}(n+r-3) \end{aligned}$$

**r=2:** The recursion relation becomes  $c_n = -\frac{n-1}{n(n+2)}c_{n-1}$ . For  $n = 1$  we see that  $c_1 = 0$  and hence  $c_n = 0$  for all  $n \geq 1$ . It follows that  $y(t) = c_0 t^2$  is a solution.

**r=0:** The recursion relation becomes  $c_n(n)(n-2) = -c_{n-1}(n-3) = 0$ . Thus

$$\begin{aligned} n = 1 & \quad -c_1 = -\frac{c_0}{-2} \\ n = 2 & \quad 0c_2 = -2c_0 \quad \Rightarrow \Leftarrow \end{aligned}$$

The  $n = 2$  case implies an inconsistency in the recursion relation since  $c_0 \neq 0$ . Since  $y_1(t) = t^2$  is a Frobenius solution a second independent solution can be written in the form

$$y(t) = t^2 \ln t + \sum_{n=0}^{\infty} c_n t^n.$$

Substitution leads to

$$t^3 + 2t^2 + (-c_1 - 2c_0)t + \sum_{n=2}^{\infty} (c_n(n)(n-2) + c_{n-1}(n-3))t^n = 0$$

and the following relations:

$$\begin{aligned} n = 1 & \quad -c_1 - 2c_0 = 0 \\ n = 2 & \quad 2 - c_1 = 0 \\ n = 3 & \quad 1 + 3c_3 = 0 \\ n \geq 4 & \quad n(n-2)c_n + (n-3)c_{n-3} = 0. \end{aligned}$$

We now have  $c_0 = -1$ ,  $c_1 = 2$ ,  $c_3 = -1/3$ .  $c_2$  can be arbitrary so we choose  $c_2 = 0$ , and  $c_n = \frac{-(n-3)c_{n-1}}{n(n-2)}$ , for  $n \geq 4$ . A straightforward calculation gives

$$c_n = \frac{2(-1)^n}{n!(n-2)}.$$

A second independent solution is

$$y_2(t) = t^2 \ln t + \left( -1 + 2t - \frac{t^3}{3} + \sum_{n=4}^{\infty} \frac{2(-1)^n t^n}{n!(n-2)} \right).$$

**General Solution:**  $y = c_1 t^2 + c_2 y_2(t)$ .

- 23.** Indicial polynomial:  $p(s) = (s^2 + 1)$ ; exponents of singularity  $r = \pm i$ . Let  $r = i$  (the case  $r = -i$  gives equivalent results). The recursion relation that arises from  $y(t) = \sum_{n=0}^{\infty} c_n t^{n+i}$  is  $c_n((n+i)^2 + 1) - c_{n-1}((n-2+i)^2 + 1) = 0$  and hence

$$c_n = \frac{(n-2)(n-2+2i)}{n(n+2i)} c_{n-1}.$$

A straightforward calculation gives the first few terms as follows:

$$\begin{aligned} n=1 \quad c_1 &= \frac{1-2i}{1+2i} c_0 \\ n=2 \quad c_2 &= 0 \\ n=3 \quad c_3 &= 0 \end{aligned}$$

and hence  $c_n = 0$  for all  $n \geq 2$ . Therefore  $y(t) = c_0(t^i + \left(\frac{1-2i}{1+2i}\right)t^{1+i})$ . Since  $t > 0$  we can write  $t^i = e^{i \ln t} = \cos \ln t + i \sin \ln t$ , by Euler's formula. Separating the real and imaginary parts we get two independent solutions

$$\begin{aligned} y_1(t) &= -3 \cos \ln t - 4 \sin \ln t + 5t \cos \ln t \\ y_2(t) &= -3 \sin \ln t + 4 \cos \ln t + 5t \sin \ln t. \end{aligned}$$

- 25.** Indicial polynomial:  $p(s) = (s^2 + 1)$ ; exponents of singularity  $r = \pm i$ . Let  $r = i$  (the case  $r = -i$  gives equivalent results). The recursion relation that arises from  $y(t) = \sum_{n=0}^{\infty} c_n t^{n+i}$  is

$$\begin{aligned} n=1 \quad c_1 &= c_0 \\ n \geq 2 \quad c_n((n+i)^2 + 1) &+ c_{n-1}(-2n - 2i + 1) + c_{n-2} = 0 \end{aligned}$$

A straightforward calculation gives the first few terms as follows:

$$\begin{aligned} n=1 \quad c_1 &= c_0 \\ n=2 \quad c_2 &= \frac{1}{2!} c_0 \\ n=3 \quad c_3 &= \frac{1}{3!} c_0 \\ n=4 \quad c_4 &= \frac{1}{4!} c_0. \end{aligned}$$

An easy induction argument gives

$$c_n = \frac{1}{n!} c_0.$$

We now get

$$\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} c^n t^{n+i} \\
&= c_0 \sum_{n=0}^{\infty} \frac{t^{n+i}}{n!} \\
&= c_0 t^i e^t.
\end{aligned}$$

Since  $t > 0$  we can write  $t^i = e^{i \ln t} = \cos \ln t + i \sin \ln t$ , by Euler's formula. Now separating the real and imaginary parts we get two independent solutions

$$y_1(t) = e^t \cos \ln t \quad \text{and} \quad y_2(t) = e^t \sin \ln t.$$

## SECTION 7.4

1. Let

$$y(t) = t^{2k+1} \sum_{n=0}^{\infty} c_n t^n = \sum_{n=0}^{\infty} c_n t^{n+2k+1}$$

Then

$$\begin{aligned}
ty''(t) &= \sum_{n=0}^{\infty} (n+2k)(n+2k+1)c_n t^{n+2k} \\
2ity'(t) &= \sum_{n=1}^{\infty} 2i(n+2k)c_{n-1} t^{n+2k} \\
-2ky'(t) &= \sum_{n=0}^{\infty} -2k(n+2k+1)c_n t^{n+2k} \\
-2iky(t) &= \sum_{n=1}^{\infty} -2ikc_{n-1} t^{n+2k}
\end{aligned}$$

By assumption the sum of the series is zero. The  $n = 0$  terms in the first and third sum give  $(2k)(2k+1)c_0 - 2k(2k+1)c_0 = 0$ . Thus we can start all the series at  $n = 1$ . For  $n \geq 1$  we get  $n(n+2k+1)c_n + 2i(n+k)c_{n-1} = 0$  which implies

$$c_n = \frac{-2i(n+k)}{n(n+2k+1)} c_{n-1}.$$

Since  $c_0 \neq 0$  it follows from this recursion relation that  $c_n \neq 0$  for all  $n \geq 0$ . Therefore the Frobenius solution  $y(t)$  is not a polynomial.

3. Since differentiation respects the real and imaginary parts of complex-valued functions we have

$$\begin{aligned}
B_k(t) &= \operatorname{Re}(b_k(t)e^{it}) \\
B'_k(t) &= \operatorname{Re}((b_k(t)e^{it})') = \operatorname{Re}(b'_k(t) + ib_k(t))e^{it} \\
B''_k(t) &= \operatorname{Re}((b''_k(t) + 2ib'_k(t) - b_k(t))e^{it}).
\end{aligned}$$

It follows now from Proposition 4 that

$$\begin{aligned}
0 &= t^2 B''_k(t) - 2kt B'_k(t) + (t^2 + 2k) B_k(t) \\
&= t^2 \operatorname{Re}(b_k(t)e^{it})'' - 2kt \operatorname{Re}(b_k(t)e^{it})' + (t^2 + 2k) \operatorname{Re}(b_k(t)e^{it}) \\
&= \operatorname{Re}((t^2(b''_k(t) + 2ib'_k(t) - b_k(t)) - 2kt(b'_k(t) + ib_k(t)) + (t^2 + 2k)b_k(t))e^{it}) \\
&= \operatorname{Re}((t^2 b''_k(t) + 2t(it - k)b'_k(t) - 2k(it - 1)b_k(t))e^{it}).
\end{aligned}$$

Apply Lemma 6 to get

$$t^2 b''_k(t) + 2t(it - k)b'_k(t) - 2k(it - 1)b_k(t) = 0.$$

5. Let  $g(t) = \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 - 1)^k} \right\}$ . Then

$$\begin{aligned}
\mathcal{L} \{tg(t)\} &= -\frac{d}{ds} (\mathcal{L} \{g(t)\}) = -\frac{d}{ds} \left( \frac{s}{(s^2 - 1)^k} \right) \\
&= -\frac{(s^2 - 1)^k - 2ks^2(s^2 - 1)^{k-1}}{(s^2 - 1)^{2k}} \\
&= \frac{2ks^2 - (s^2 - 1)}{(s^2 - 1)^{k+1}} = \frac{(2k - 1)(s^2 - 1) + 2k}{(s^2 - 1)^{k+1}} \\
&= \frac{2k - 1}{(s^2 - 1)^k} + \frac{2k}{(s^2 - 1)^{k+1}}.
\end{aligned}$$

Divide by  $2k$ , solve for the second term in the last line, and apply the inverse Laplace transform to get

$$\begin{aligned}
\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 - 1)^{k+1}} \right\} &= \frac{t}{2k} g(t) - \frac{(2k - 1)}{2k} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 - 1)^k} \right\} \\
&= \frac{t}{2k} \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 - 1)^k} \right\} - \frac{(2k - 1)}{2k} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 - 1)^k} \right\}.
\end{aligned}$$

By the definition of  $C_k$  and  $D_k$  we get

$$\frac{1}{2^k k!} C_k(t) = \frac{t}{2^k k!} D_{k-1}(t) - \frac{2k - 1}{2^k k!} C_{k-1}(t).$$

Simplifying gives the result.



7. Multiply the equation in Exercise 5 by  $t$  and use the formula in Exercise 4 to get  $D_{k+1}(t) = t^2 D_{k-1} - (2k-1)D_k(t)$ . Now shift  $k$  and the formula follows.
9. By the Input Derivative Principle we have

$$\begin{aligned} \frac{1}{2^k k!} \mathcal{L}\{C'_k(t)\} &= \frac{1}{2^k k!} (s \mathcal{L}\{C_k(t)\} - C_k(0)) \\ &= \frac{s}{(s^2 - 1)^{k+1}} \\ &= \frac{1}{2^k k!} \mathcal{L}\{D_k(t)\}. \end{aligned}$$

Laplace inversion gives the first formula. In a similar way the Input Derivative Principle gives

$$\begin{aligned} \frac{1}{2^k k!} \mathcal{L}\{D'_k(t)\} &= \frac{1}{2^k k!} (s \mathcal{L}\{D_k(t)\} - D_k(0)) \\ &= \frac{s^2}{(s^2 - 1)^{k+1}} \\ &= \frac{s^2 - 1}{(s^2 - 1)^{k+1}} + \frac{1}{(s^2 - 1)^{k+1}} \\ &= \frac{1}{(s^2 - 1)^k} + \frac{1}{(s^2 - 1)^{k+1}} \\ &= \frac{1}{2^{k-1}(k-1)!} \mathcal{L}\{C_{k-1}(t)\} + \frac{1}{2^k k!} \mathcal{L}\{C_k(t)\}. \end{aligned}$$

Simplifying and Laplace inversion gives the result.

11. Since  $s^2 - 1 = (s-1)(s+1)$  it follows that  $\frac{1}{(s^2-1)^{k+1}}$  is an  $s-1$ -chain and an  $s+1$ -chain, each of length  $k+1$ . Hence there are constants  $\alpha_n$  and  $\beta_n$  so that

$$\frac{1}{(s^2 - 1)^{k+1}} = \sum_{n=1}^{k+1} \frac{\alpha_n}{(s-1)^n} + \frac{\beta_n}{(s+1)^n}.$$

Now replace  $s$  by  $-s$ . The left-hand side does not change so we get

$$\begin{aligned} &\frac{1}{(s^2 - 1)^{k+1}} \\ &= \sum_{n=1}^{k+1} \frac{\alpha_n}{(-s-1)^n} + \frac{\beta_n}{(-s+1)^n} \\ &= \sum_{n=1}^{k+1} \frac{\alpha_n(-1)^n}{(s+1)^n} + \frac{\beta_n(-1)^n}{(s-1)^n}. \end{aligned}$$

It follows now by the uniqueness of partial fraction decompositions that

$$\beta_n = (-1)^n \alpha_n.$$

Laplace inversion now gives

$$\begin{aligned} & \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 - 1)^{k+1}} \right\} \\ &= \sum_{n=1}^{k+1} \alpha_n \frac{t^{n-1}}{(n-1)!} e^t + \alpha_n (-1)^n \frac{t^{n-1}}{(n-1)!} e^{-t} \\ &= \sum_{n=1}^{k+1} \alpha_n \frac{t^{n-1}}{(n-1)!} e^t - \alpha_n \frac{(-t)^{n-1}}{(n-1)!} e^{-t} \end{aligned}$$

Let  $f(t) = \sum_{n=1}^{k+1} \alpha_n \frac{t^{n-1}}{(n-1)!}$ . Then

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 - 1)^{k+1}} \right\} = f(t)e^t - f(-t)e^{-t}.$$

Up to the constant  $2^k k!$ , the polynomial  $f(t)$  is  $c_k(t)$ . A similar argument gives the second part of the problem.

- 13.** 1. It is easy to see from the definition of  $c_k$  and Exercise 5 that  $c_k$  satisfies

$$c_{k+2}(t) = t^2 c_k(t) - (2k+3)c_{k+1}(t)$$

and therefore  $c_{k+2}(0) = -(2k+3)c_{k+1}(0)$ . An easy check gives  $c_1(t) = \frac{t-1}{2}$  and thus  $c_1(0) = \frac{-1}{2}$ . Recursively we get

$$\begin{aligned} c_1(0) &= \frac{-1}{2} & c_3(0) &= -5c_2(0) = \frac{-5 \cdot 3}{2} \\ c_2(0) &= -3c_1(0) = \frac{3}{2} & c_4(0) &= 7c_3(0) = \frac{7 \cdot 5 \cdot 3}{2} \end{aligned}$$

Inductively, we get

$$\begin{aligned} c_k(0) &= \frac{(-1)^k}{2} (2k-1) \cdot (2k-3) \cdot (2k-5) \cdots 1 \\ &= \frac{(-1)^k}{2} \frac{(2k)!}{2^k k!} \\ &= \frac{(-1)^k (2k)!}{2^{k+1} k!}. \end{aligned}$$

2. From Exercise 4 it is easy to see that  $d_k(t) = t c_{k-1}$  and so  $d'_k(0) = c_{k-1}(0) = \frac{(-1)^{k-1} (2(k-1))!}{2^k (k-1)!}$ .

- 15.** Merely put the previous calculations together.

**SECTION 8.1**

$$1. B + C = \begin{bmatrix} 1 & 1 \\ -1 & 7 \\ 0 & 3 \end{bmatrix}, \quad B - C = \begin{bmatrix} 1 & -3 \\ 5 & -1 \\ -2 & 1 \end{bmatrix}, \quad \text{and } 2B - 3C = \begin{bmatrix} 2 & -8 \\ 13 & -6 \\ -5 & 1 \end{bmatrix}$$

$$3. A(B + C) = AB + AC = \begin{bmatrix} 3 & 4 \\ 1 & 13 \end{bmatrix}, \quad (B + C)A = \begin{bmatrix} 3 & -1 & 7 \\ 3 & 1 & 25 \\ 5 & 0 & 12 \end{bmatrix}$$

$$5. AB = \begin{bmatrix} 6 & 4 & -1 & -8 \\ 0 & 2 & -8 & 2 \\ 2 & -1 & 9 & -5 \end{bmatrix}$$

$$7. CA = \begin{bmatrix} 8 & 0 \\ 4 & -5 \\ 8 & 14 \\ 10 & 11 \end{bmatrix}$$

$$9. ABC = \begin{bmatrix} 8 & 9 & -48 \\ 4 & 0 & -48 \\ -2 & 3 & 40 \end{bmatrix}.$$

$$15. \begin{bmatrix} 0 & 0 & 1 \\ 3 & -5 & -1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$17. (a) \text{ Choose, for example, } A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

$$(b) (A + B)^2 = A^2 + 2AB + B^2 \text{ precisely when } AB = BA.$$

$$19. B^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

$$21. (a) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = \begin{bmatrix} v_2 \\ v_1 \end{bmatrix}; \text{ the two rows of } A \text{ are switched. (b) } \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} v_1 + cv_2 \\ v_2 \end{bmatrix}; \text{ to the first row is added } c \text{ times the second row while the second row is unchanged, (c) to the second row is added } c \text{ times the first row while the first row is unchanged. (d) the first row is multiplied by } a \text{ while the second row is unchanged, (e) the second row is multiplied by } a \text{ while the first row is unchanged.}$$

**23.**

$$\begin{aligned}
& F(\theta_1)F(\theta_2) \\
&= \begin{bmatrix} \cosh \theta_1 & \sinh \theta_1 \\ \sinh \theta_1 & \cosh \theta_1 \end{bmatrix} \begin{bmatrix} \cosh \theta_2 & \sinh \theta_2 \\ \sinh \theta_2 & \cosh \theta_2 \end{bmatrix} \\
&= \begin{bmatrix} \cosh \theta_1 \cosh \theta_2 + \sinh \theta_1 \sinh \theta_2 & \cosh \theta_1 \sinh \theta_2 + \sinh \theta_1 \cosh \theta_2 \\ \sinh \theta_1 \cosh \theta_2 + \cosh \theta_1 \sinh \theta_2 & \sinh \theta_1 \sinh \theta_2 + \cosh \theta_1 \cosh \theta_2 \end{bmatrix} \\
&= \begin{bmatrix} \cosh(\theta_1 + \theta_2) & \sinh(\theta_1 + \theta_2) \\ \sinh(\theta_1 + \theta_2) & \cosh(\theta_1 + \theta_2) \end{bmatrix} \\
&= F(\theta_1 + \theta_2),
\end{aligned}$$

We used the addition formulas for sinh and cosh in the second line.

## SECTION 8.2

$$1. A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 1 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 6 \end{bmatrix}, \text{ and } [A|\mathbf{b}] = \begin{bmatrix} 1 & 4 & 3 & 2 \\ 1 & 1 & -1 & 4 \\ 2 & 0 & 1 & 1 \\ 0 & 1 & -1 & 6 \end{bmatrix}.$$

$$\begin{array}{rclcl}
& x_1 & - & & x_3 & + & 4x_4 & + & 3x_5 & = & 2 \\
3. & 5x_1 & + & 3x_2 & - & 3x_3 & - & x_4 & - & 3x_5 & = & 1 \\
& 3x_1 & - & 2x_2 & + & 8x_3 & + & 4x_4 & - & 3x_5 & = & 3 \\
& -8x_1 & + & 2x_2 & & & + & 2x_4 & + & x_5 & = & -4
\end{array}$$

5. RREF

$$7. m_2(1/2)(A) = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$9. t_{1,3}(-3)(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$11. \begin{bmatrix} 1 & 0 & 0 & -11 & -8 \\ 0 & 1 & 0 & -4 & -2 \\ 0 & 0 & 1 & 9 & 6 \end{bmatrix}$$

$$13. \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$15. \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$17. \begin{bmatrix} 1 & 4 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$$

$$19. \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix}, \alpha \in \mathbb{R}$$

$$21. \begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R}$$

$$23. \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

$$25. \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \alpha \in \mathbb{R}$$

$$27. \emptyset$$

$$29. \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$31. \left\{ \begin{bmatrix} -34 \\ -40 \\ 39 \\ 1 \end{bmatrix} \right\}$$

$$33. \text{ The equation } \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \text{ has solution } a = 2 \text{ and } b = 3. \text{ By}$$

$$\text{Proposition 7 } \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} \text{ is a solution.}$$

35. If  $\mathbf{x}_i$  is the solution set for  $A\mathbf{x} = \mathbf{b}_i$  then  $\mathbf{x}_1 = \begin{bmatrix} -7/2 \\ 7/2 \\ -3/2 \end{bmatrix}$ ,  $\mathbf{x}_2 = \begin{bmatrix} -3/2 \\ 3/2 \\ -1/2 \end{bmatrix}$ ,  
 and  $\mathbf{x}_3 = \begin{bmatrix} 7 \\ -6 \\ 3 \end{bmatrix}$ .

### SECTION 8.3

1.  $\begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$

3. not invertible

5. not invertible

7.  $\begin{bmatrix} -6 & 5 & 13 \\ 5 & -4 & -11 \\ -1 & 1 & 3 \end{bmatrix}$

9.  $\begin{bmatrix} -29 & 39/2 & -22 & 13 \\ 7 & -9/2 & 5 & -3 \\ -22 & 29/2 & -17 & 10 \\ 9 & -6 & 7 & -4 \end{bmatrix}$

11.  $\begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ -1 & -1 & 0 & 1 \end{bmatrix}$

13.  $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

15.  $\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{10} \begin{bmatrix} -2 & 4 & 4 \\ -2 & -1 & 4 \\ -6 & 2 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 16 \\ 11 \\ 18 \end{bmatrix}$

17.  $\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{2} \begin{bmatrix} -58 & 39 & -44 & 26 \\ 14 & -9 & 10 & -6 \\ -44 & 29 & -34 & 20 \\ 18 & -12 & 14 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 19 \\ -4 \\ 15 \\ -6 \end{bmatrix}$

19.  $(A^t)^{-1} = (A^{-1})^t$

21.  $F(\theta)^{-1} = F(-\theta)$

## SECTION 8.4

1. 1

3. 10

5. -21

7. 2

9. 0

$$11. \frac{1}{s^2-6s+8} \begin{bmatrix} s-3 & 1 \\ 1 & s-3 \end{bmatrix} \quad s=2, 4$$

$$13. \frac{1}{(s-1)^3} \begin{bmatrix} (s-1)^2 & 3 & s-1 \\ 0 & (s-1)^2 & 0 \\ 0 & 3(s-1) & (s-1)^2 \end{bmatrix} \quad s=1$$

$$15. \frac{1}{s^3+s^2+4s+4} \begin{bmatrix} s^2+s & 4s+4 & 0 \\ -s-1 & s^2+s & 0 \\ s-4 & 4s+4 & s^2+4 \end{bmatrix} \quad s=-1, \pm 2i$$

17. no inverse

$$19. \frac{1}{8} \begin{bmatrix} 4 & -4 & 4 \\ -1 & 3 & -1 \\ -5 & -1 & 3 \end{bmatrix}$$

$$21. \frac{1}{6} \begin{bmatrix} 2 & -98 & 9502 \\ 0 & 3 & -297 \\ 0 & 0 & 6 \end{bmatrix}$$

$$23. \frac{1}{15} \begin{bmatrix} 55 & -95 & 44 & -171 \\ 50 & -85 & 40 & -150 \\ 70 & -125 & 59 & -216 \\ 65 & -115 & 52 & -198 \end{bmatrix}$$

$$25. \det A = 1, \det A(1, \mathbf{b}) = \det \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = 5, \text{ and } \det A(2, \mathbf{b}) = \det \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} = -3. \text{ It follows that } x_1 = 5/1 = 5 \text{ and } x_2 = -3/1 = -3$$

27.  $\det A = -10$ ,  $\det A(1, \mathbf{b}) = \det \begin{bmatrix} -2 & 0 & -2 \\ 1 & -2 & 0 \\ 2 & 2 & -1 \end{bmatrix} = -16$ ,  $\det A(2, \mathbf{b}) = \det \begin{bmatrix} 1 & -2 & -2 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} = -11$ , and  $\det A(3, \mathbf{b}) = \det \begin{bmatrix} 1 & 0 & -2 \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{bmatrix} = -18$ . It follows that  $x_1 = 16/10$ ,  $x_2 = 11/10$ , and  $x_3 = 18/10$ .

## SECTION 8.5

1. The characteristic polynomial is  $c_A(s) = (s-1)(s-2)$ . The eigenvalues are thus  $s = 1, 2$ . The eigenspaces are  $E_1 = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$  and  $E_2 = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ .
3. The characteristic polynomial is  $c_A(s) = s^2 - 2s + 1 = (s-1)^2$ . The only eigenvalue is  $s = 1$ . The eigenspace is  $E_1 = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ .
5. The characteristic polynomial is  $c_A(s) = s^2 + 2s - 3 = (s+3)(s-1)$ . The eigenvalues are thus  $s = -3, 1$ . The eigenspaces are  $E_{-3} = \text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$  and  $E_1 = \text{Span} \left\{ \begin{bmatrix} -3 \\ 2 \end{bmatrix} \right\}$ .
7. The characteristic polynomial is  $c_A(s) = s^2 + 2s + 10 = (s+1)^2 + 3^2$ . The eigenvalues are thus  $s = -1 \pm 3i$ . The eigenspaces are  $E_{-1+3i} = \text{Span} \left\{ \begin{bmatrix} 7+i \\ 10 \end{bmatrix} \right\}$  and  $E_{-1-3i} = \text{Span} \left\{ \begin{bmatrix} 7-i \\ 10 \end{bmatrix} \right\}$ .
9. The eigenvalues are  $s = -2, 3$ .  $E_{-2} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ ,  $E_3 = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ .
11. The eigenvalues are  $s = 0, 2, 3$ .  $E_0 = \text{NS}(A) = \text{Span} \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ ,  $E_2 = \text{Span} \left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}$ ,  $E_3 = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ .



13. We write  $c_A(s) = (s-2)((s-2)^2 + 1)$  to see that the eigenvalues are

$$s = 2, 2 \pm i. \quad E_2 = \text{Span} \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}, \quad E_{2+i} = \text{Span} \left\{ \begin{bmatrix} -4+3i \\ 4+2i \\ 5 \end{bmatrix} \right\}, \quad E_{2-i} = \text{Span} \left\{ \begin{bmatrix} -4-3i \\ 4-2i \\ 5 \end{bmatrix} \right\}$$

## SECTION 9.2

1. nonlinear, because of the presence of the product  $y_1 y_2$ .

3. We may write the system in the form

$$\mathbf{y}' = \begin{bmatrix} \sin t & 0 \\ 1 & \cos t \end{bmatrix} \mathbf{y}.$$

It is linear and homogeneous, but not constant coefficient.

5. We write the system in the form

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix} \mathbf{y}.$$

It is linear, constant coefficient, and homogeneous.

7. First note that  $y_1(0) = 0$  and  $y_2(0) = 1$ , so the initial condition is satisfied. Then

$$\mathbf{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} e^t - 3e^{3t} \\ 2e^t - 3e^{3t} \end{bmatrix}$$

while

$$\begin{aligned} \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{y}(t) &= \begin{bmatrix} 5(e^t - e^{3t}) - 2(2e^t - e^{3t}) \\ 4(e^t - e^{3t}) - (2e^t - e^{3t}) \end{bmatrix} \\ &= \begin{bmatrix} e^t - 3e^{3t} \\ 2e^t - 3e^{3t} \end{bmatrix}. \end{aligned}$$

Thus  $\mathbf{y}'(t) = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{y}(t)$ , as required.

9. First note that  $y_1(0) = 1$  and  $y_2(0) = 3$ , so the initial condition is satisfied. Then

$$\mathbf{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -e^{-t} + e^t + te^t \\ -3e^{-t} + e^t + te^t \end{bmatrix}$$

while

$$\begin{aligned} \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} + te^t \\ 3e^{-t} + te^t \end{bmatrix} + \begin{bmatrix} e^t \\ e^t \end{bmatrix} &= \begin{bmatrix} 2e^{-t} + 2te^t - 3e^{-t} - te^t + e^t \\ 3e^{-t} + 3te^t - 6e^{-t} - 2te^t + e^t \end{bmatrix} \\ &= \begin{bmatrix} -e^{-t} + te^t + e^t \\ -3e^{-t} + te^t + e^t \end{bmatrix}. \end{aligned}$$

Thus  $\mathbf{y}'(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{y}(t) + \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ , as required.

In solutions, 11–15,  $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix}$ .

- 11.** Let  $y_1 = y$  and  $y_2 = y'$ . Then  $y_1' = y' = y_2$  and  $y_2' = y'' = -5y' - 6y + e^{2t} = -6y_1 - 5y_2 + e^{2t}$ . Letting  $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ , this can be expressed in vector form as

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

- 13.** Let  $y_1 = y$  and  $y_2 = y'$ . Then  $y_1' = y' = y_2$  and  $y_2' = y'' = k^2y + A \cos \omega t = k^2y_1 + A \cos \omega t$ . Letting  $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ , this can be expressed in vector form as

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ k^2 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ A \cos \omega t \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

- 15.** Let  $y_1 = y$  and  $y_2 = y'$ . Then  $y_1' = y' = y_2$  and  $y_2' = y'' = -\frac{2}{t}y' - \frac{1}{t^2}y = -\frac{1}{t^2}y_1 - \frac{2}{t}y_2$ . Letting  $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ , this can be expressed in vector form as

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -\frac{1}{t^2} & -\frac{2}{t} \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(1) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

**17.**  $A'(t) = \begin{bmatrix} -3e^{-3t} & 1 \\ 2t & 2e^{2t} \end{bmatrix}$

**19.**  $\mathbf{y}'(t) = \begin{bmatrix} 1 \\ 2t \\ t^{-1} \end{bmatrix}$

**21.**  $\mathbf{v}'(t) = \begin{bmatrix} -2e^{-2t} & \frac{2t}{t^2+1} & -3 \sin 3t \end{bmatrix}$

$$23. \frac{1}{4} \begin{bmatrix} e^2 - e^{-2} & e^2 + e^{-2} - 2 \\ 2 - e^2 - e^{-2} & e^2 - e^{-2} \end{bmatrix}$$

$$25. \begin{bmatrix} 4 & 8 \\ 12 & 16 \end{bmatrix}$$

$$27. \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} \\ \frac{2}{s^3} & \frac{1}{s-2} \end{bmatrix}$$

$$29. \begin{bmatrix} \frac{3!}{s^4} & \frac{2s}{(s^2+1)^2} & \frac{1}{(s+1)^2} \\ \frac{2-s}{s^3} & \frac{s-3}{s^2-6s+13} & \frac{3}{s} \end{bmatrix}$$

$$31. \frac{2}{s^2-1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$33. \begin{bmatrix} 1 & 2t & 3t^2 \end{bmatrix}$$

$$35. \text{ We have } \begin{bmatrix} \frac{2s}{s^2-1} & \frac{2}{s^2-1} \\ \frac{2}{s^2-1} & \frac{2s}{s^2-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} + \frac{1}{s-1} & \frac{-1}{s+1} + \frac{1}{s-1} \\ \frac{-1}{s+1} + \frac{1}{s-1} & \frac{1}{s+1} + \frac{1}{s-1} \end{bmatrix} \cdot \text{Laplace}$$

inversion gives  $\begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}$

### SECTION 9.3

1.

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \\ A^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \\ A^3 &= \begin{bmatrix} 1 & 0 \\ 0 & -8 \end{bmatrix} \\ &\vdots \\ A^n &= \begin{bmatrix} 1 & 0 \\ 0 & (-2)^n \end{bmatrix}. \end{aligned}$$

It follows now that

$$\begin{aligned}
e^{At} &= I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \cdots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & 0 \\ 0 & -2t \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} t^2 & 0 \\ 0 & (-2t)^2 \end{bmatrix} + \cdots + \frac{1}{n!} \begin{bmatrix} t^n & 0 \\ 0 & (-2t)^n \end{bmatrix} + \cdots \\
&= \begin{bmatrix} e^t & 0 \\ 0 & e^{-2t} \end{bmatrix}.
\end{aligned}$$

3.

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
A^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \\
A^3 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A
\end{aligned}$$

It follows now that  $A^n = I$  if  $n$  is even and  $A^n = A$  if  $n$  is odd. Thus

$$\begin{aligned}
e^{At} &= I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \cdots \\
&= I + At + I\frac{t^2}{2!} + A\frac{t^3}{3!} + I\frac{t^4}{4!} + A\frac{t^5}{5!} + \cdots \\
&= I \left( 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \right) + A \left( t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots \right) \\
&= I \cosh t + A \sinh t \\
&= \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}.
\end{aligned}$$

5.

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
A^2 &= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \\
A^3 &= \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \\
&\vdots \\
A^n &= \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}.
\end{aligned}$$

It follows now that

$$\begin{aligned}
e^{At} &= I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \cdots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t & t \\ t & t \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} 2t^2 & 2t^2 \\ 2t^2 & 2t^2 \end{bmatrix} + \cdots + \frac{1}{n!} \begin{bmatrix} 2^{n-1}t^n & 2^{n-1}t^n \\ 2^{n-1}t^n & 2^{n-1}t^n \end{bmatrix} + \cdots
\end{aligned}$$

The  $(1, 1)$  entry is

$$\begin{aligned}
1 + \sum_{n=1}^{\infty} \frac{2^{n-1}t^n}{n!} &= 1 + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2t)^n}{n!} \\
&= \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} \\
&= \frac{1}{2} + \frac{1}{2} e^{2t}
\end{aligned}$$

The  $(1, 2)$  entry is

$$\begin{aligned}
0 + \sum_{n=1}^{\infty} \frac{2^{n-1}t^n}{n!} &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2t)^n}{n!} \\
&= -\frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} \\
&= -\frac{1}{2} + \frac{1}{2} e^{2t}
\end{aligned}$$

Since the  $(1, 1)$  entry and the  $(2, 2)$  entry are equal and the  $(1, 2)$  entry and the  $(2, 1)$  entry are equal we have

$$e^{At} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{2t} & -\frac{1}{2} + \frac{1}{2}e^{2t} \\ -\frac{1}{2} + \frac{1}{2}e^{2t} & \frac{1}{2} + \frac{1}{2}e^{2t} \end{bmatrix}$$

7.

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\
A^2 &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \\
A^3 &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix} \\
A^4 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix} \\
A^5 &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 32 \end{bmatrix}
\end{aligned}$$

The  $(1, 1)$  entry and the  $(2, 2)$  entry of  $e^{At}$  are equal and are

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots.$$

The  $(1, 2)$  entry and the  $(2, 1)$  entry of  $e^{At}$  have opposite signs. The  $(2, 1)$  entry is

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots.$$

The  $(3, 3)$  entry is

$$e^{2t} = 1 + 2t + \frac{(2t)^2}{2!} + \cdots.$$

All other entries are zero thus

$$e^{At} = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}$$

9. The characteristic polynomial is  $c_A(s) = s(s-3)$  and  $sI - A = \begin{bmatrix} s-1 & 1 \\ 2 & s-2 \end{bmatrix}$ . Thus  $(sI - A)^{-1} = \begin{bmatrix} \frac{s-2}{s(s-3)} & \frac{-1}{s(s-3)} \\ \frac{-2}{s(s-3)} & \frac{s-1}{s(s-3)} \end{bmatrix}$ . A partial fraction decomposition of each entry gives

$$(sI - A)^{-1} = \frac{1}{s} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} + \frac{1}{s-3} \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix}.$$

$$\text{Thus } e^{At} = \mathcal{L}^{-1}(sI - A)^{-1} = \begin{bmatrix} \frac{2}{3} + \frac{1}{3}e^{3t} & \frac{1}{3} - \frac{1}{3}e^{3t} \\ \frac{2}{3} - \frac{2}{3}e^{3t} & \frac{1}{3} + \frac{2}{3}e^{3t} \end{bmatrix}$$

11. The characteristic polynomial is  $c_A(s) = (s-3)(s+1)+5 = s^2-2s+2 = (s-1)^2+1$  and  $sI - A = \begin{bmatrix} s-3 & -5 \\ 1 & s+1 \end{bmatrix}$ . Thus the resolvent matrix is

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s+1}{(s-1)^2+1} & \frac{5}{(s-1)^2+1} \\ \frac{-1}{(s-1)^2+1} & \frac{s-3}{(s-1)^2+1} \end{bmatrix} \text{ which we write as}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s-1}{(s-1)^2+1} & \frac{5}{(s-1)^2+1} \\ \frac{-1}{(s-1)^2+1} & \frac{s-1}{(s-1)^2+1} \end{bmatrix} + \begin{bmatrix} \frac{2}{(s-1)^2+1} & 0 \\ 0 & \frac{-2}{(s-1)^2+1} \end{bmatrix}.$$

Therefore

$$e^{At} = \begin{bmatrix} e^t \cos t + 2e^t \sin t & 5e^t \sin t \\ -e^t \sin t & e^t \cos t - 2e^t \sin t \end{bmatrix}$$

13. The characteristic polynomial is  $c_A(s) = s^3$  and  $sI - A = \begin{bmatrix} s & -1 & -1 \\ 0 & s & -1 \\ 0 & 0 & s \end{bmatrix}$ .

$$\text{Thus } (sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{s+1}{s^3} \\ 0 & \frac{1}{s} & \frac{1}{s^2} \\ 0 & 0 & \frac{1}{s} \end{bmatrix} \text{ and } e^{At} = \begin{bmatrix} 1 & t & t + \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

15. Let  $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$  and  $N = 2$ . Then by Example 6  $e^{Mt} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$ .

Thus

$$\begin{aligned} e^{At} &= \begin{bmatrix} e^{Mt} & 0 \\ 0 & e^{Nt} \end{bmatrix} \\ &= \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}. \end{aligned}$$

## SECTION 9.4

1. The characteristic matrix and characteristic polynomial are

$$sI - A = \begin{bmatrix} s-2 & 1 \\ -1 & s \end{bmatrix} \quad \text{and} \quad c_A(s) = s^2 - 2s + 1 = (s-1)^2.$$

The characteristic polynomial has root 1 with multiplicity 2.

The standard basis of  $\mathcal{E}_{c_A}$  is  $\mathcal{B}_{c_A} = \{e^t, te^t\}$ . It follows that

$$e^{At} = Me^t + Nte^t.$$

Differentiating we obtain

$$\begin{aligned} Ae^{At} &= Me^t + N(e^t + te^t) \\ &= (M + N)e^t + Nte^t. \end{aligned}$$

Now, evaluate each equation at  $t = 0$  to obtain:

$$\begin{aligned} I &= M \\ A &= M + N. \end{aligned}$$

from which we get

$$\begin{aligned} M &= I \\ N &= A - I. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} &= Ie^t + (A - I)te^t \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e^t + \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} te^t \\ &= \begin{bmatrix} e^t + te^t & -te^t \\ te^t & e^t - te^t \end{bmatrix} \end{aligned}$$

- 3.**  $c_A(s) = (s - 2)(s + 2) + 4 = s^2$ . Thus  $\mathcal{B}_{c_A} = \{1, t\}$  and Fulmer's method gives

$$e^{At} = M_1 + M_2 t.$$

Differentiating and evaluating at  $t = 0$  gives

$$\begin{aligned} M_1 &= I \\ M_2 &= A. \end{aligned}$$

Thus

$$e^{At} = \begin{bmatrix} 1 + 2t & t \\ -4t & 1 - 2t \end{bmatrix}.$$

- 5.** The characteristic polynomial is  $c_A(s) = s^2 - 2s + 2 = (s - 1)^2 + 1$ . The standard basis of  $\mathcal{E}_{c_A}$  is  $\mathcal{B}_{c_A} = \{e^t \cos t, e^t \sin t\}$ . It follows that



$$e^{At} = Me^t \cos t + Ne^t \sin t.$$

Differentiating and evaluating at  $t = 0$  gives

$$\begin{aligned} I &= M \\ A &= M + N. \end{aligned}$$

from which we get

$$\begin{aligned} M &= I \\ N &= A - I. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} &= Ie^t \cos t + (A - I)e^t \sin t \\ &= \begin{bmatrix} e^t \cos t & 0 \\ 0 & e^t \cos t \end{bmatrix} + \begin{bmatrix} 3e^t \sin t & -10e^t \sin t \\ e^t \sin t & -3e^t \sin t \end{bmatrix} \\ &= \begin{bmatrix} e^t \cos t + 3e^t \sin t & -10e^t \sin t \\ e^t \sin t & e^t \cos t - 3e^t \sin t \end{bmatrix} \end{aligned}$$

7. The characteristic polynomial is  $c_A(s) = s^2 - 4$  and has roots  $-2, 2$ . The standard basis of  $\mathcal{E}_{c_A}$  is  $\mathcal{B}_{c_A} = \{e^{2t}, e^{-2t}\}$ . It follows that

$$e^{At} = Me^{2t} + Ne^{-2t}.$$

Differentiating and evaluating at  $t = 0$  gives

$$\begin{aligned} I &= M + N \\ A &= 2M - 2N. \end{aligned}$$

from which we get

$$\begin{aligned} M &= \frac{1}{4}(A + 2I) \\ N &= -\frac{1}{4}(A - 2I). \end{aligned}$$

Thus,

$$\begin{aligned}
e^{At} &= \frac{1}{4}(A + 2I)e^{2t} - \frac{1}{4}(A - 2I)e^{-2t} \\
&= \frac{1}{4} \begin{bmatrix} -7 & 11 \\ -7 & 11 \end{bmatrix} e^{2t} - \frac{1}{4} \begin{bmatrix} -11 & 11 \\ -7 & 7 \end{bmatrix} e^{-2t} \\
&= \frac{1}{4} \begin{bmatrix} -7e^{2t} + 11e^{-2t} & 11e^{2t} - 11e^{-2t} \\ -7e^{2t} + 7e^{-2t} & 11e^{2t} - 7e^{-2t} \end{bmatrix}
\end{aligned}$$

9. The characteristic polynomial is  $c_A(s) = s^2 - 4s + 13 = ((s - 2)^2 + 3^2)$ . The standard basis of  $\mathcal{E}_{c_A}$  is  $\mathcal{B}_{c_A} = \{e^{2t} \cos 3t, e^{2t} \sin 3t\}$ . It follows that

$$e^{At} = Me^{2t} \cos 3t + Ne^{2t} \sin 3t.$$

Differentiating and evaluating at  $t = 0$  gives

$$\begin{aligned}
I &= M \\
A &= 2M + 3N.
\end{aligned}$$

from which we get

$$\begin{aligned}
M &= I \\
N &= \frac{1}{3}(A - 2I) = \begin{bmatrix} 8 & 13 \\ -5 & -8 \end{bmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}
e^{At} &= Ie^{2t} \cos 3t + \frac{1}{3}(A - I)e^{2t} \sin 3t \\
&= \begin{bmatrix} e^{2t} \cos 3t & 0 \\ 0 & e^{2t} \cos 3t \end{bmatrix} + \begin{bmatrix} 8e^{2t} \sin 3t & 13e^{2t} \sin 3t \\ -5e^{2t} \sin 3t & -8e^{2t} \sin 3t \end{bmatrix} \\
&= \begin{bmatrix} e^{2t} \cos 3t + 8e^{2t} \sin 3t & 13e^{2t} \sin 3t \\ -5e^{2t} \sin 3t & e^{2t} \cos 3t - 8e^{2t} \sin 3t \end{bmatrix}
\end{aligned}$$

11. The characteristic polynomial is  $c_A(s) = (s + 2)^2$ . The standard basis is  $\mathcal{B}_{c_A} = \{e^{-2t}, te^{-2t}\}$ . It follows that

$$e^{At} = Me^{-2t} + Nte^{-2t}.$$

Differentiating and evaluating at  $t = 0$  gives

$$\begin{aligned}
I &= M \\
A &= -2M + N.
\end{aligned}$$

from which we get

$$\begin{aligned} M &= I \\ N &= A + 2I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} &= Ie^{-2t} + (A + 2I)te^{-2t} \\ &= \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-2t} \end{bmatrix} + \begin{bmatrix} -te^{-2t} & te^{-2t} \\ -te^{-2t} & te^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} e^{-2t} - te^{-2t} & te^{-2t} \\ -te^{-2t} & e^{-2t} + te^{-2t} \end{bmatrix} \end{aligned}$$

**13.** In this case  $\mathcal{B}_{c_A} = \{1, e^t, e^{-t}\}$ . It follows that

$$e^{At} = M + Ne^t + Pe^{-t}.$$

Differentiating and evaluating at  $t = 0$  gives

$$\begin{aligned} M + N + P &= I \\ N - P &= A \\ N + P &= A^2 \end{aligned}$$

from which we get

$$\begin{aligned} M &= I - A^2 \\ N &= \frac{A^2 + A}{2} \\ P &= \frac{A^2 - A}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} &= M + Ne^t + Pe^{-t} \\ &= \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ 2 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} e^t + \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix} e^{-t} \\ &= \begin{bmatrix} 2 - e^{-t} & 0 & -1 + e^{-t} \\ 0 & e^t & 0 \\ 2 - 2e^{-t} & 0 & -1 + 2e^{-t} \end{bmatrix} \end{aligned}$$

**15.** The standard basis of  $\mathcal{E}_{c_A}$  is

$$\mathcal{B}_{c_A} = \{e^t, e^t \cos t, e^t \sin t\}.$$

Therefore

$$e^{At} = Me^t + Ne^t \sin t + Pe^t \cos t.$$

Differentiating twice and simplifying we get the system:

$$\begin{aligned} e^{At} &= Me^t + Ne^t \sin t + Pe^t \cos t \\ Ae^{At} &= Me^t + (N - P)e^t \sin t + (N + P)e^t \cos t \\ A^2 e^{At} &= Me^t - 2Pe^t \sin t + 2Ne^t \cos t. \end{aligned}$$

Now evaluating at  $t = 0$  gives

$$\begin{aligned} I &= M + P \\ A &= M + N + P \\ A^2 &= M + 2N \end{aligned}$$

and solving gives

$$\begin{aligned} N &= A - I \\ M &= A^2 - 2A + 2I \\ P &= -A^2 + 2A - I. \end{aligned}$$

A straightforward calculation gives  $A^2 = \begin{bmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ 2 & 0 & -2 \\ \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}$  and

$$N = \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ 1 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \quad M = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \quad \text{and} \quad P = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

Hence,

$$\begin{aligned} e^{At} &= \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} e^t + \begin{bmatrix} 0 & -\frac{1}{2} & 0 \\ 1 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} e^t \sin t + \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} e^t \cos t \\ &= \frac{1}{2} \begin{bmatrix} e^t + e^t \cos t & -e^t \sin t & e^t - e^t \cos t \\ 2e^t \sin t & 2e^t \cos t & -2e^t \sin t \\ e^t - e^t \cos t & e^t \sin t & e^t + e^t \cos t \end{bmatrix}. \end{aligned}$$

**17.** In this case  $\mathcal{B}_{c_A} = \{e^t, \cos 2t, \sin 2t\}$ . It follows that

$$e^{At} = Me^t + N \cos 2t + P \sin 2t.$$

Differentiating twice and evaluating at  $t = 0$  gives

$$\begin{aligned}
M + N &= I \\
M + 2P &= A \\
M - 4N &= A^2
\end{aligned}$$

from which we get

$$\begin{aligned}
M &= \frac{A^2 + 4I}{5} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix} \\
N &= \frac{I - A^2}{5} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \\
P &= \frac{-A^2 + 5A - 4I}{10} = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}
e^{At} &= Me^t + N \cos 2t + P \sin 2t \\
&= \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & 2 \end{bmatrix} e^t + \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix} \cos 2t + \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \sin 2t \\
&= \begin{bmatrix} -e^t + 2 \cos 2t & -\sin 2t & e^t - \cos 2t \\ 2 \sin 2t & \cos 2t & -\sin 2t \\ -2e^t + 2 \cos 2t & -\sin 2t & 2e^t - \cos 2t \end{bmatrix}
\end{aligned}$$

**19.** In this case  $\mathcal{B}_{c_A} = \{\cos t, \sin t, t \cos t, t \sin t\}$ . It follows that

$$e^{At} = M \cos t + N \sin t + Pt \cos t + Qt \sin t.$$

Differentiating three times and evaluating at  $t = 0$  gives

$$\begin{aligned}
M &= I \\
N + P &= A \\
-M + 2Q &= A^2 \\
-N - 3P &= A^3
\end{aligned}$$

from which we get

$$\begin{aligned}
M &= & I &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
N &= \frac{A(A^2 + 3I)}{2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\
P &= \frac{-A(A^2 + I)}{2} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \\
Q &= \frac{A^2 + I}{2} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}
e^{At} &= M \cos t + N \sin t + Pt \cos t + Qt \sin t \\
&= \begin{bmatrix} \cos t - t \cos t & \sin t - t \sin t & t \cos t & t \sin t \\ -\sin t + t \sin t & \cos t - t \cos t & -t \sin t & t \cos t \\ -t \cos t & -t \sin t & \cos t + t \cos t & \sin t + t \sin t \\ t \sin t & -t \cos t & -\sin t - t \sin t & \cos t + t \cos t \end{bmatrix}.
\end{aligned}$$

- 21.** The standard basis is  $\mathcal{B}_{c_A} = \{e^{rt}, te^{rt}\}$  so that  $e^{At} = Me^{rt} + Nte^{rt}$ . Fulmer's method gives

$$\begin{aligned}
I &= M \\
A &= rM + N
\end{aligned}$$

which are easily solved to give

$$M = I \quad \text{and} \quad N = (A - rI).$$

Hence,

$$e^{At} = Ie^{rt} + (A - rI)te^{rt} = (I + (A - rI)t)e^{rt}. \quad (1)$$

## SECTION 9.5

1. It is easy to see that  $e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix}$ . Thus

$$\mathbf{y}(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -2e^{3t} \end{bmatrix}.$$

3. The characteristic polynomial is  $c_A(s) = (s-2)^2$ . Thus  $\mathcal{B}_{c_A} = \{e^{2t}, te^{2t}\}$  and

$$e^{At} = M_1 e^{2t} + M_2 t e^{2t}.$$

Differentiating and evaluating at  $t = 0$  gives

$$\begin{aligned} I &= M_1 \\ A &= 2M_1 + M_2. \end{aligned}$$

and hence  $M_1 = I$  and  $M_2 = A - 2I$ . We thus get  $e^{At} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$  and

$$\mathbf{y}(t) = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -e^{2t} + 2te^{2t} \\ 2e^{2t} \end{bmatrix}$$

5. The characteristic polynomial is  $c_A(s) = s^2 - 1 = (s+1)(s-1)$ . Thus  $\mathcal{B}_{c_A} = \{e^{-t}, e^t\}$  and

$$e^{At} = M_1 e^{-t} + M_2 e^t.$$

Differentiating and evaluating at  $t = 0$  gives

$$\begin{aligned} I &= M_1 + M_2 \\ A &= -M_1 + M_2. \end{aligned}$$

and hence  $M_1 = \frac{1}{2}(-A + I)$  and  $M_2 = \frac{1}{2}(A + I)$ . We thus get  $e^{At} = \frac{1}{2} \begin{bmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{bmatrix}$  and

$$\mathbf{y}(t) = \frac{1}{2} \begin{bmatrix} 3e^t - e^{-t} & -e^t + e^{-t} \\ 3e^t - 3e^{-t} & -e^t + 3e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}.$$

7. The characteristic polynomial is  $c_A(s) = (s-1)^2$ . Thus  $\mathcal{B}_{c_A} = \{e^t, te^t\}$  and

$$e^{At} = M_1 e^t + M_2 t e^t.$$

Differentiating and evaluating at  $t = 0$  gives

$$\begin{aligned} I &= M_1 \\ A &= M_1 + M_2. \end{aligned}$$

and hence  $M_1 = I$  and  $M_2 = A - I$ . We thus get  $e^{At} = \begin{bmatrix} e^t + 2te^t & -4te^t \\ te^t & e^t - 2te^t \end{bmatrix}$

$$\text{and } \mathbf{y}(t) = \begin{bmatrix} e^t + 2te^t & -4te^t \\ te^t & e^t - 2te^t \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^t - 2te^t \\ e^t - te^t \end{bmatrix}$$

9. The characteristic polynomial is  $c_A(s) = (s+1)(s^2+4)$ . Thus  $\mathcal{B}_{c_A} = \{e^{-t}, \cos 2t, \sin 2t\}$  and

$$e^{At} = M_1 e^{-t} + M_2 \cos 2t + M_3 \sin 2t.$$

Differentiating and evaluating at  $t = 0$  gives

$$\begin{aligned} I &= M_1 + M_2 \\ A &= -M_1 + 2M_3 \\ A^2 &= M_1 - 4M_2. \end{aligned}$$

and hence

$$\begin{aligned} M_1 &= \frac{1}{5}(A^2 + 4I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \\ M_2 &= -\frac{1}{5}(A^2 - I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ M_3 &= \frac{1}{10}(A^2 + 5A + 4I) = \begin{bmatrix} 0 & 2 & 0 \\ -1/2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}. \end{aligned}$$

$$\text{We thus get } e^{At} = \begin{bmatrix} \cos 2t & 2 \sin 2t & 0 \\ -\frac{1}{2} \sin 2t & \cos 2t & 0 \\ -e^{-t} + \cos 2t & 2 \sin 2t & e^{-t} \end{bmatrix} \text{ and hence}$$

$$\mathbf{y}(t) = \begin{bmatrix} \cos 2t & 2 \sin 2t & 0 \\ -\frac{1}{2} \sin 2t & \cos 2t & 0 \\ -e^{-t} + \cos 2t & 2 \sin 2t & e^{-t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cos 2t + 2 \sin 2t \\ \cos 2t - \sin 2t \\ 2 \cos 2t + 2 \sin 2t \end{bmatrix}$$

11. A straightforward calculation gives

$$e^{At} = e^{-t} \cos 2t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{-t} \sin 2t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

It follows that



$$\begin{aligned}
\mathbf{y}_h(t) &= e^{At} \mathbf{y}_0 \\
&= \left( e^{-t} \cos 2t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{-t} \sin 2t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} e^{-t} \cos 2t \\ -e^{-t} \sin 2t \end{bmatrix}.
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{y}_p &= e^{At} * f(t) \\
&= \left( e^{-t} \cos 2t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{-t} \sin 2t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) * \begin{bmatrix} 5 \\ 0 \end{bmatrix} \\
&= e^{-t} \cos 2t * 1 \begin{bmatrix} 5 \\ 0 \end{bmatrix} + e^{-t} \sin 2t * 1 \begin{bmatrix} 0 \\ -5 \end{bmatrix} \\
&= (1 - e^{-t} \cos 2t + 2e^{-t} \sin 2t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (2 - 2e^{-t} \cos 2t - e^{-t} \sin 2t) \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 - e^{-t} \cos 2t + 2e^{-t} \sin 2t \\ -2 + 2e^{-t} \cos 2t + e^{-t} \sin 2t \end{bmatrix}.
\end{aligned}$$

It now follows that

$$\begin{aligned}
\mathbf{y}(t) &= \mathbf{y}_h + \mathbf{y}_p \\
&= \begin{bmatrix} 1 + 2e^{-t} \sin 2t \\ -2 + 2e^{-t} \cos 2t \end{bmatrix}
\end{aligned}$$

**13.** A straightforward calculation gives

$$e^{At} = e^{-t} \cos 2t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{-t} \sin 2t \begin{bmatrix} 0 & -2 \\ 1/2 & 0 \end{bmatrix}$$

It follows that

$$\begin{aligned}
\mathbf{y}_h(t) &= e^{At} \mathbf{y}_0 \\
&= \left( e^{-t} \cos 2t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{-t} \sin 2t \begin{bmatrix} 0 & -2 \\ 1/2 & 0 \end{bmatrix} \right) \begin{bmatrix} 2 \\ -1 \end{bmatrix} \\
&= e^{-t} \cos 2t \begin{bmatrix} 2 \\ -1 \end{bmatrix} + e^{-t} \sin 2t \begin{bmatrix} 2 \\ 1 \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{y}_p &= e^{At} * f(t) \\
&= \left( e^{-t} \cos 2t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^{-t} \sin 2t \begin{bmatrix} 0 & -2 \\ 1/2 & 0 \end{bmatrix} \right) * \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\
&= ((e^{-t} \cos 2t) * 1) \begin{bmatrix} 4 \\ 1 \end{bmatrix} + ((e^{-t} \sin 2t) * 1) \begin{bmatrix} -2 \\ 2 \end{bmatrix} \\
&= \frac{1}{5} (1 - e^{-t} \cos 2t + 2e^{-t} \sin 2t) \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \frac{1}{5} (2 - 2e^{-t} \cos 2t - e^{-t} \sin 2t) \begin{bmatrix} -2 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 2e^{-t} \sin 2t \\ 1 - e^{-t} \cos 2t \end{bmatrix}.
\end{aligned}$$

It now follows that

$$\begin{aligned}
\mathbf{y}(t) &= \mathbf{y}_h + \mathbf{y}_p \\
&= e^{-t} \cos 2t \begin{bmatrix} 2 \\ -1 \end{bmatrix} + e^{-t} \sin 2t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2e^{-t} \sin 2t \\ 1 - e^{-t} \cos 2t \end{bmatrix} \\
&= \begin{bmatrix} 2e^{-t} \cos 2t + 4e^{-t} \sin 2t \\ 1 + e^{-t} \sin 2t - 2e^{-t} \cos 2t \end{bmatrix}.
\end{aligned}$$

15. A straightforward calculation gives

$$e^{At} = e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix}$$

It follows that

$$\begin{aligned}
\mathbf{y}_h(t) &= e^{At} \mathbf{y}_0 \\
&= \left( e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2te^t \\ e^t - 4te^t \end{bmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{y}_p &= e^{At} * f(t) \\
&= \left( e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + te^t \begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix} \right) * t \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= e^t * t \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= (e^t - t - 1) \begin{bmatrix} 1 \\ -2 \end{bmatrix}.
\end{aligned}$$

It now follows that

$$\begin{aligned}
\mathbf{y}(t) &= \mathbf{y}_h + \mathbf{y}_p \\
&= \begin{bmatrix} 2te^t \\ e^t - 4te^t \end{bmatrix} + (e^t - t - 1) \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 2te^t + e^t - t - 1 \\ -4te^t - e^t + 2t + 2 \end{bmatrix}
\end{aligned}$$

17. A straightforward calculation gives

$$e^{At} = e^t \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} + te^t \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

Clearly

$$\mathbf{y}_h(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

while

$$\begin{aligned}
\mathbf{y}_p(t) &= e^{At} * f(t) \\
&= \left( e^t \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} + te^t \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix} \right) * e^{2t} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\
&= (e^t * e^{2t}) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + (te^t * e^{2t}) \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} + (e^{2t} * e^{2t}) \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \\
&= (e^{2t} - e^t) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + (e^{2t} - te^t - e^t) \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} + te^{2t} \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} te^t \\ 2te^{2t} - e^{2t} + e^t \\ -2te^{2t} + te^t \end{bmatrix}
\end{aligned}$$

It now follows that

$$\mathbf{y}(t) = \mathbf{y}_h + \mathbf{y}_p = \begin{bmatrix} te^t \\ 2te^{2t} - e^{2t} + e^t \\ -2te^{2t} + te^t \end{bmatrix}$$

19.  $y_1$  and  $y_2$  are related to each other as follows:

$$\begin{aligned}y_1' &= 2 - 2y_1 \\y_2' &= 2y_1 - y_2\end{aligned}$$

with initial conditions  $y_1(0) = 4$  and  $y_2(0) = 0$ . Let  $A = \begin{bmatrix} -2 & 0 \\ 2 & -1 \end{bmatrix}$ ,  $\mathbf{f}(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ , and  $\mathbf{y}(0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ . We need to solve the system  $\mathbf{y}' = A\mathbf{y} + \mathbf{f}$ ,  $\mathbf{y}(0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ . It is easy to check that

$$e^{At} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} e^{-2t}.$$

It follows that

$$\mathbf{y}_h(t) = \left( \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} e^{-2t} \right) \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix} e^{-t} + \begin{bmatrix} 4 \\ -8 \end{bmatrix} e^{-2t}$$

and

$$\begin{aligned}\mathbf{y}_p &= e^{At} * \begin{bmatrix} 2 \\ 0 \end{bmatrix} = (e^{-t} * 1) \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + (e^{-2t} * 1) \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\&= (1 - e^{-t}) \begin{bmatrix} 0 \\ 4 \end{bmatrix} + (1 - e^{-2t}) \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\&= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -4 \end{bmatrix} e^{-t} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t}.\end{aligned}$$

It now follows that

$$\begin{aligned}\mathbf{y}(t) &= \mathbf{y}_h(t) + \mathbf{y}_p(t) \\&= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix} e^{-t} + \begin{bmatrix} 3 \\ -6 \end{bmatrix} e^{-2t}.\end{aligned}$$

The concentration of salt in Tank 2 is  $1/2$  if  $y_2(t) = 1$ . We thus solve  $y_2(t) = 1$ , i.e.  $2 + 4e^{-t} - 6e^{-2t} = 1$  for  $t$ . Let  $x = e^{-t}$ . Then  $-6x^2 + 4x + 1 = 0$ . The quadratic formula gives  $x = \frac{2+\sqrt{10}}{6}$ . Since  $x > 0$  we have  $e^{-t} = x = \frac{2+\sqrt{10}}{6}$ . Solving for  $t$  we get  $t = -\ln\left(\frac{2+\sqrt{10}}{6}\right) = 0.1504$  minutes or 9.02 seconds.

**21.**  $y_1$  and  $y_2$  are related to each other as follows:

$$\begin{aligned}y_1' &= 2 + 3y_2 - 5y_1 \\y_2' &= 2 + 5y_1 - 7y_2\end{aligned}$$

with initial conditions  $y_1(0) = 0$  and  $y_2(0) = 0$ . Let  $A = \begin{bmatrix} -5 & 3 \\ 5 & -7 \end{bmatrix}$ ,  $\mathbf{f}(t) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ , and  $\mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . We need to solve the system  $\mathbf{y}' = A\mathbf{y} + \mathbf{f}$ ,  $\mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ . It is easy to check that

$$e^{At} = \frac{e^{-2t}}{8} \begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix} + \frac{e^{-10t}}{8} \begin{bmatrix} 3 & -3 \\ -5 & 5 \end{bmatrix}.$$

Clearly  $\mathbf{y}_h = 0$  while

$$\begin{aligned} \mathbf{y}(t) = \mathbf{y}_p(t) &= e^{At} * \mathbf{f}(t) \\ &= \left( \frac{e^{-2t}}{8} \begin{bmatrix} 5 & 3 \\ 5 & 3 \end{bmatrix} + \frac{e^{-10t}}{8} \begin{bmatrix} 3 & -3 \\ -5 & 5 \end{bmatrix} \right) * 1 \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= \frac{e^{-2t} * 1}{8} \begin{bmatrix} 16 \\ 16 \end{bmatrix} \\ &= (1 - e^{-2t}) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Thus

$$\begin{aligned} y_1(t) &= 1 - e^{-2t} \\ y_2(t) &= 1 - e^{-2t}. \end{aligned}$$

## SECTION 9.6

1. The characteristic polynomial is  $c_A(s) = s^2 - 1 = (s+1)(s-1)$ . There are two distinct eigenvalues,  $\lambda_1 = -1$  and  $\lambda_2 = 1$ . An easy calculation give that  $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  is an eigenvector with eigenvalue  $-1$  and  $v_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$  is an eigenvector with eigenvalue  $1$ . Let  $P = \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix}$ . Then  $J = P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ . Since there is a distinct positive and negative real eigenvalue the critical point is a saddle.
3. The characteristic polynomial is  $c_A(s) = s^2 + 4s + 5 = (s+2)^2 + 1$  and has complex roots  $-2 \pm i$ . A calculation gives an eigenvector  $v = \begin{bmatrix} -3-i \\ 5 \end{bmatrix}$  for

$-2 - i$ . Let  $v_1 = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$  and  $v_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ . Let  $P = [v_1 \ v_2] = \begin{bmatrix} -3 & -1 \\ 5 & 0 \end{bmatrix}$ . Then  $J = P^{-1}AP = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$  and the origin is a stable spiral node.

5. In this case  $A$  is of type  $J_3$  with positive eigenvalue 2. The origin is an unstable star node.
7. The characteristic polynomial is  $c_A(s) = s^2 - 6s + 8 = (s-2)(s-4)$ . There are two distinct eigenvalues,  $\lambda_1 = 2$  and  $\lambda_2 = 4$ . An easy calculation gives that  $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  is an eigenvector with eigenvalue 2 and  $v_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$  is an eigenvector with eigenvalue 4. Let  $P = \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix}$ . Then  $J = P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ . Since both eigenvalues are positive the origin is an unstable node.
9. The characteristic polynomial is  $c_A(s) = s^2 - 2s + 5 = (s-1)^2 + 2^2$ . So  $1 \pm 2i$  are the eigenvalues. An eigenvector for  $1 - 2i$  is  $\begin{bmatrix} -1 + i \\ 4 \end{bmatrix}$ . Let  $P = \begin{bmatrix} -1 & 1 \\ 4 & 0 \end{bmatrix}$ . Then  $J = P^{-1}AP = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ . The origin is an unstable star node.
11. Let  $(x, y)$  be a point on the  $P(L)$ . Suppose  $P^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Let  $\begin{bmatrix} u \\ v \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$ . Then  $(u, v)$  is on  $L$  and so

$$\begin{aligned} 0 &= Du + Ev + F \\ &= D(ax + by) + E(cx + dy) + F \\ &= (Da + Ec)x + (Db + Ed)y + F \\ &= D'x + E'y + F, \end{aligned}$$

where  $(D', E') = (Da + Ec, Db + Ed) = (D, E)P^{-1}$ . It follows that  $(x, y)$  satisfies the equation of a line. A line goes through the origin if and only if  $F = 0$ . If the equation for  $L$  has  $F = 0$  then the above calculation shows the equation for  $P(L)$  does too.

13. Let  $C$  be the graph of a power curve in the  $(u, v)$  plane and  $P(C)$  the transform of  $C$ . Let  $(x, y)$  be a point of  $P(C)$  and  $(u, v)$  the point on  $C$  such that  $P \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ . If  $P^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  then  $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$ . Replace  $u$  and  $v$  in the equation  $Au + Bv = (Cu + Dv)^p$  by  $ax + by$  and  $cx + dy$ , respectively. We then get  $(Aa + Bc)x + (Ab + Bd)y = ((Ca + Dc)x + (Cb + Dd)y)^p$ . Thus  $P(C)$  is the graph of a power curve.

- 15.** The characteristic polynomial takes the form  $c_A(s) = s^2 - (\text{tr } A)s + \det A$ . Let  $\lambda = \text{tr } A$ . Since  $\det A = 0$  we have  $c_A(s) = s^2 - \lambda s = s(s - \lambda)$ . Now consider two cases:

$\lambda \neq 0$ : In this case  $A$  has two distinct eigenvalues, 0 and  $\lambda$ . Let  $v_1$  be an eigenvector with eigenvalue 0 and  $v_2$  an eigenvector with eigenvalue  $\lambda$ . Then  $v_1$  and  $v_2$  are linearly independent. If  $P = [v_1 \ v_2]$  then  $P$  is invertible and

$$AP = [Av_1 \ Av_2] = [0 \ \lambda v_2] = [v_1 \ v_2] \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} = PJ_1.$$

Now multiply both sides on the left by  $P^{-1}$  to get that  $P^{-1}AP = J_1$ .

$\lambda = 0$ : In this case  $c_A(s) = s^2$ . Since  $A$  is not zero there must be a vector  $v_1$  that is not an eigenvector. Let  $v_2 = Av_1$ . Then  $v_2$  is an eigenvector with eigenvalue 0 since, by the Cayley-Hamilton theorem,

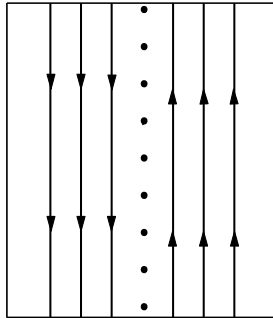
$$Av_2 = A^2v_1 = 0.$$

Now let  $P = [v_1 \ v_2]$ . Then

$$AP = A[v_1 \ v_2] = [Av_1 \ Av_2] = [v_2 \ 0] = [v_1 \ v_2] \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = P \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Now multiply both sides on the left by  $P^{-1}$  to get that  $P^{-1}AP = J_2$ .

- 17.** If  $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$  then the equation  $J_2\mathbf{c} = 0$  implies  $c_1 = 0$ . It follows that each point on the  $v$ -axis is an equilibrium point. Now assume  $c_1 \neq 0$ . The solution to  $\mathbf{w}' = J_2\mathbf{w}$ ,  $\mathbf{w}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$  is  $u(t) = c_1$  and  $v(t) = tc_1 + c_2$ . The path  $(u(t), v(t)) = (c_1, c_2) + t(0, c_1)$ ,  $t \in \mathbb{R}$ , is a vertical line that passes through the initial condition  $(c_1, c_2)$  and points upward if  $c_1 > 0$  and downward if  $c_1 < 0$ . The phase portrait is given below:



19. It is not difficult to see that  $e^{At} = I + tA$ . Let  $v_1$  be an eigenvector. By Lemma 9.5.9,  $e^{At}v_1 = v_1$ . So each eigenvector is an equilibrium point. Let  $v$  be a vector that is not an eigenvector. By the Cayley-Hamilton theorem  $A^2v = 0$  so  $Av$  is an eigenvector. Furthermore  $e^{At}v = v + tAv$ . The trajectory is a line parallel to  $Av$  going through  $v$ .
21. Assume  $c_1 > 0$  and thus  $x > 0$  (the case where  $c_1 < 0$  is similar). We have  $y' = \frac{\ln x/c_1}{\lambda} + \frac{1}{\lambda} + \frac{c_2}{c_1}$ . It follows that  $\lim_{x \rightarrow 0^+} y' = -\infty$  if  $\lambda > 0$  and  $\lim_{x \rightarrow 0^+} y' = \infty$  if  $\lambda < 0$ .
23.  $y'' = \frac{1}{\lambda x}$  and the result follows.

## SECTION 9.7

1. Observe that  $\Phi'(t) = \begin{bmatrix} -e^{-t} & 2e^{2t} \\ -e^{-t} & 8e^{2t} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} e^{-t} & e^{2t} \\ e^{-t} & 4e^{2t} \end{bmatrix} = A(t)\Phi(t)$ . Also,  $\det \Phi(t) = 4e^t - e^t = 3e^t \neq 0$ . Thus  $\Phi(t)$  is a fundamental matrix. The general solution can be written in the form  $y(t) = \Phi(t)c$ , where  $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$  is a constant vector. The initial condition implies  $y(0) = \Phi(0)c$  or

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 + 4c_2 \end{bmatrix}.$$

Solving for  $c$  we get  $c_1 = 2$  and  $c_2 = -1$ . It follows that

$$y(t) = \Phi(t)c = 2 \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix} - \begin{bmatrix} e^{2t} \\ 4e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{2t} \\ 2e^{-t} - 4e^{2t} \end{bmatrix}.$$

The standard fundamental matrix at  $t = 0$  is

$$\begin{aligned} \Psi(t) = \Phi(t)\Phi(0)^{-1} &= \begin{bmatrix} e^{-t} & e^{2t} \\ e^{-t} & 4e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}^{-1} \\ &= \frac{1}{3} \begin{bmatrix} e^{-t} & e^{2t} \\ e^{-t} & 4e^{2t} \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 4e^{-t} - e^{2t} & -e^{-t} + e^{2t} \\ 4e^{-t} - 4e^{2t} & -e^{-t} + 4e^{2t} \end{bmatrix}. \end{aligned}$$

3. Observe that



$$\begin{aligned}
\Phi'(t) &= \begin{bmatrix} t \cos(t^2/2) & -t \sin(t^2/2) \\ -t \sin(t^2/2) & -\cos(t^2/2) \end{bmatrix} \\
&= \begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix} \begin{bmatrix} \sin(t^2/2) & \cos(t^2/2) \\ \cos(t^2/2) & -\sin(t^2/2) \end{bmatrix} \\
&= A(t)\Phi(t).
\end{aligned}$$

Also,  $\det \Phi(t) = -\sin^2(t^2/2) - \cos^2(t^2/2) = -1 \neq 0$ . Thus  $\Phi(t)$  is a fundamental matrix. The general solution can be written in the form  $\mathbf{y}(t) = \Phi(t)\mathbf{c}$ , where  $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$  is a constant vector. The initial condition implies  $\mathbf{y}(0) = \Phi(0)\mathbf{c}$  or

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \end{bmatrix}.$$

Thus  $c_1 = 0$  and  $c_2 = 1$ . It follows that

$$\mathbf{y}(t) = \Phi(t)\mathbf{c} = 0 \begin{bmatrix} \sin(t^2/2) \\ \cos(t^2/2) \end{bmatrix} + \begin{bmatrix} \cos(t^2/2) \\ -\sin(t^2/2) \end{bmatrix} = \begin{bmatrix} \cos(t^2/2) \\ -\sin(t^2/2) \end{bmatrix}.$$

The standard fundamental matrix at  $t = 0$  is

$$\begin{aligned}
\Psi(t) = \Phi(t)\Phi(0)^{-1} &= \begin{bmatrix} \sin(t^2/2) & \cos(t^2/2) \\ \cos(t^2/2) & -\sin(t^2/2) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \sin(t^2/2) & \cos(t^2/2) \\ \cos(t^2/2) & -\sin(t^2/2) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \cos(t^2/2) & \sin(t^2/2) \\ -\sin(t^2/2) & \cos(t^2/2) \end{bmatrix}.
\end{aligned}$$

5. Observe that

$$\begin{aligned}
\Phi'(t) &= \begin{bmatrix} -\cos t + t \sin t & -\sin t - t \cos t \\ \sin t + t \cos t & -\cos t + t \sin t \end{bmatrix} \\
&= \begin{bmatrix} 1/t & 1 \\ -1 & 1/t \end{bmatrix} \begin{bmatrix} -t \cos t & -t \sin t \\ t \sin t & -t \cos t \end{bmatrix} \\
&= A(t)\Phi(t).
\end{aligned}$$

Also,  $\det \Phi(t) = t^2 \cos^2 t + t^2 \sin^2 t = t^2 \neq 0$ . Thus  $\Phi(t)$  is a fundamental matrix. The general solution can be written in the form  $\mathbf{y}(t) = \Phi(t)\mathbf{c}$ , where  $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$  is a constant vector. The initial condition implies  $\mathbf{y}(\pi) = \Phi(\pi)\mathbf{c}$  or

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \pi c_1 \\ \pi c_2 \end{bmatrix}.$$

Thus  $c_1 = 1/\pi$  and  $c_2 = -1/\pi$ . It follows that

$$\mathbf{y}(t) = \Phi(t)\mathbf{c} = \frac{1}{\pi} \begin{bmatrix} -t \cos t \\ t \sin t \end{bmatrix} - \frac{1}{\pi} \begin{bmatrix} -t \sin t \\ -t \cos t \end{bmatrix} = \frac{t}{\pi} \begin{bmatrix} -\cos t + \sin t \\ \cos t + \sin t \end{bmatrix}.$$

The standard fundamental matrix at  $t = \pi$  is

$$\begin{aligned} \Psi(t) = \Phi(t)\Phi(\pi)^{-1} &= \begin{bmatrix} -t \cos t & -t \sin t \\ t \sin t & -t \cos t \end{bmatrix} \begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix}^{-1} \\ &= \frac{1}{\pi} \begin{bmatrix} -t \cos t & -t \sin t \\ t \sin t & -t \cos t \end{bmatrix}. \end{aligned}$$

7. Observe that

$$\begin{aligned} \Phi'(t) &= \begin{bmatrix} 0 & te^t \\ 0 & e^t \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1/t & 1/t \end{bmatrix} \begin{bmatrix} 1 & (t-1)e^t \\ -1 & e^t \end{bmatrix} \\ &= A(t)\Phi(t). \end{aligned}$$

Also,  $\det \Phi(t) = e^t + te^t - e^t = te^t \neq 0$ . Thus  $\Phi(t)$  is a fundamental matrix. The general solution can be written in the form  $\mathbf{y}(t) = \Phi(t)\mathbf{c}$ , where  $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$  is a constant vector. The initial condition implies  $\mathbf{y}(0) = \Phi(0)\mathbf{c}$  or

$$\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & e \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ -c_1 + ec_2 \end{bmatrix}.$$

Thus  $c_1 = -3$  and  $c_2 = 1/e$ . It follows that

$$\mathbf{y}(t) = \Phi(t)\mathbf{c} = -3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{e} \begin{bmatrix} (t-1)e^t \\ e^t \end{bmatrix} = \begin{bmatrix} (t-1)e^{t-1} - 3 \\ e^{t-1} + 3 \end{bmatrix}.$$

The standard fundamental matrix at  $t = 0$  is

$$\begin{aligned} \Psi(t) = \Phi(t)\Phi(0)^{-1} &= \begin{bmatrix} 1 & (t-1)e^t \\ -1 & e^t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & e \end{bmatrix}^{-1} \\ &= \frac{1}{e} \begin{bmatrix} 1 & (t-1)e^t \\ -1 & e^t \end{bmatrix} \begin{bmatrix} e & 0 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{e} \begin{bmatrix} e + (t-1)e^t & (t-1)e^t \\ -e + e^t & e^t \end{bmatrix} \\ &= \begin{bmatrix} 1 + (t-1)e^{t-1} & (t-1)e^{t-1} \\ -1 + e^{t-1} & e^{t-1} \end{bmatrix}. \end{aligned}$$

9. Let  $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ . Then  $A(t) = \frac{1}{t}A$ . We first compute  $e^{Au}$ . The characteristic polynomial,  $c_A(s)$  is

$$c_A(s) = \det(sI - A) = \det \begin{bmatrix} s & 1 \\ -1 & s-2 \end{bmatrix} = s^2 - 2s + 1 = (s-1)^2.$$

It follows that  $\mathcal{B}_{c_A} = \{e^u, ue^u\}$ . Using Fulmer's method we have  $e^{Au} = e^u M_1 + ue^u M_2$ . Differentiating and evaluating at  $u = 0$  gives

$$\begin{aligned} I &= M_1 \\ A &= M_1 + M_2. \end{aligned}$$

It follows that  $M_1 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and  $M_2 = A - I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$ . Thus

$$e^{Au} = e^u \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ue^u \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Since  $\ln t$  is an antiderivative of  $\frac{1}{t}$  and  $\ln 1 = 0$  we have by Proposition 12

$$\begin{aligned} \Psi(t) &= e^{\ln t} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (\ln t) e^{\ln t} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} t - t \ln t & -t \ln t \\ t \ln t & t + t \ln t \end{bmatrix}, \end{aligned}$$

is the standard fundamental matrix for  $\mathbf{y}'(t) = A(t)\mathbf{y}(t)$  at  $t = 1$ . The homogeneous solution is given by

$$\begin{aligned} \mathbf{y}_h(t) &= \Psi(t)\mathbf{y}(1) \\ &= \begin{bmatrix} t - t \ln t & -t \ln t \\ t \ln t & t + t \ln t \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2t - 2t \ln t \\ 2t \ln t \end{bmatrix}. \end{aligned}$$

The particular solution is given by

$$\begin{aligned} \mathbf{y}_p(t) &= \Psi(t) \int_1^t \Psi(u)^{-1} \mathbf{f}(u) du \\ &= \begin{bmatrix} t - t \ln t & -t \ln t \\ t \ln t & t + t \ln t \end{bmatrix} \int_1^t \frac{1}{u} \begin{bmatrix} 1 + \ln u & \ln u \\ -\ln u & 1 - \ln u \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} du \\ &= \begin{bmatrix} t - t \ln t & -t \ln t \\ t \ln t & t + t \ln t \end{bmatrix} \ln t \begin{bmatrix} 1 \\ -1 \end{bmatrix} du \\ &= t \ln t \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{aligned}$$

It follows that

$$\mathbf{y}(t) = \mathbf{y}_h(t) + \mathbf{y}_p(t) = \begin{bmatrix} 2t - 2t \ln t \\ 2t \ln t \end{bmatrix} + \begin{bmatrix} t \ln t \\ -t \ln t \end{bmatrix} = \begin{bmatrix} 2t - t \ln t \\ t \ln t \end{bmatrix}.$$

11. Let  $A = \begin{bmatrix} 3 & 5 \\ -1 & -3 \end{bmatrix}$ . Then  $A(t) = \sec(t)A$ . The characteristic polynomial of  $A$  is  $c_A(s) = s^2 - 4 = (s - 2)(s + 2)$ . Hence  $\mathcal{B}_{c_A} = \{e^{2t}, e^{-2t}\}$  and

$$e^{Au} = M_1 e^{2u} + M_2 e^{-2u}.$$

Differentiating and evaluating at  $u = 0$  give the equations  $I = M_1 + M_2$  and  $A = 2M_1 - 2M_2$ . It follows that

$$M_1 = \frac{1}{4} \begin{bmatrix} 5 & 5 \\ -1 & -1 \end{bmatrix} \quad \text{and} \quad M_2 = \frac{1}{4} \begin{bmatrix} -1 & -5 \\ 1 & 5 \end{bmatrix}$$

and

$$e^{Au} = \frac{1}{4} \begin{bmatrix} 5e^{2u} - e^{-2u} & 5e^{2u} - 5e^{-2u} \\ -e^{2u} + e^{-2u} & -e^{2u} + 5e^{-2u} \end{bmatrix}.$$

If  $b(t) = \int_0^t \sec u \, du = \ln |\sec t + \tan t|$  then  $\Psi(t) = e^{Ab(t)}$ . If  $X = (\sec t + \tan t)^2$  then  $X^{-1} = (\sec t - \tan t)^2$  and

$$\begin{aligned} \Psi(t) &= \frac{1}{4} \begin{bmatrix} 5X - \frac{1}{X} & 5X - 5\frac{1}{X} \\ -X + \frac{1}{X} & -X + 5\frac{1}{X} \end{bmatrix} \\ &= \begin{bmatrix} \sec^2 t + 3 \sec t \tan t + \tan^2 t & 5 \sec t \tan t \\ -\sec t \tan t & \sec^2 t - 3 \sec t \tan t + \tan^2 t \end{bmatrix}. \end{aligned}$$

From this it follows that the homogeneous solution is

$$\begin{aligned} \mathbf{y}_h(t) &+ \begin{bmatrix} \sec^2 t + 3 \sec t \tan t + \tan^2 t & 5 \sec t \tan t \\ -\sec t \tan t & \sec^2 t - 3 \sec t \tan t + \tan^2 t \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 \sec^2 t + 11 \sec t \tan t + 2 \tan^2 t \\ \sec^2 t - 5 \sec t \tan t + \tan^2 t \end{bmatrix} \end{aligned}$$

Since the forcing function  $\mathbf{f}$  is identically zero the particular solution is zero. Hence  $\mathbf{y} = \mathbf{y}_h$ .

13. Let  $v_1(t)$  and  $v_2(t)$  denote the volume of brine in Tank 1 and Tank 2, respectively. Then  $v_1(t) = v_2(t) = 2 - t$ . The following differential equations describe the system

$$\begin{aligned}y_1'(t) &= -\frac{3}{2-t}y_1(t) + \frac{1}{2-t}y_2(t) + 6 \\y_2'(t) &= \frac{1}{2-t}y_1(t) - \frac{3}{2-t}y_2(t) + 0,\end{aligned}$$

with initial conditions  $y_1(0) = 0$  and  $y_2(0) = 20$ . In matrix form,  $\mathbf{y}'(t) = A(t)\mathbf{y}(t) + \mathbf{f}(t)$ , we have

$$A(t) = \begin{bmatrix} -3/(2-t) & 1/(2-t) \\ 1/(2-t) & -3/(2-t) \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 20 \end{bmatrix}.$$

Let  $a(t) = \frac{-1}{2-t}$ . Then we can write  $A(t) = a(t)A$  where

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

The characteristic polynomial is  $c_A(s) = s^2 - 6s + 8 = (s-2)(s-4)$ . We now have  $\mathcal{B}_{c_A} = \{e^{2t}, e^{4t}\}$ . It follows that  $e^{Au} = M_1e^{2u} + M_2e^{4u}$ . Differentiating and setting  $u = 0$  we get

$$\begin{aligned}I &= M_1 + M_2 \\A &= 2M_1 + 4M_2.\end{aligned}$$

An easy calculation gives

$$\begin{aligned}M_1 &= \frac{1}{2}(4I - A) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\M_2 &= \frac{1}{2}(A - 2I) = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\end{aligned}$$

and

$$e^{Au} = \frac{e^{2u}}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{e^{4u}}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Let  $b(t) = \int_0^t a(u) du = \int_0^t \frac{-1}{2-u} du = \ln \frac{2-t}{2}$ . Then the standard fundamental matrix is

$$\Psi(t) = e^{Au}|_{u=b(t)} = \frac{(2-t)^2}{8} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{(2-t)^4}{32} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

For the homogeneous solution we have

$$\begin{aligned}
\mathbf{y}_h(t) &= \mathbf{\Psi}(t)\mathbf{y}(0) \\
&= \frac{(2-t)^2}{8} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 20 \end{bmatrix} + \frac{(2-t)^4}{32} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 20 \end{bmatrix} \\
&= \frac{(2-t)^2}{2} \begin{bmatrix} 5 \\ 5 \end{bmatrix} + \frac{(2-t)^4}{8} \begin{bmatrix} -5 \\ 5 \end{bmatrix}.
\end{aligned}$$

For the particular solution straightforward calculations give

$$\begin{aligned}
\mathbf{\Psi}^{-1}(u) &= \frac{2}{(2-u)^4} \begin{bmatrix} (2-u)^2 + 4 & (2-u)^2 - 4 \\ (2-u)^2 - 4 & (2-u)^2 + 4 \end{bmatrix}, \\
\mathbf{\Psi}^{-1}\mathbf{f}(u) &= \frac{2}{(2-u)^4} \begin{bmatrix} (2-u)^2 + 4 & (2-u)^2 - 4 \\ (2-u)^2 - 4 & (2-u)^2 + 4 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} \\
&= \frac{12}{(2-u)^4} \begin{bmatrix} (2-u)^2 + 4 \\ (2-u)^2 - 4 \end{bmatrix} \\
&= 12 \begin{bmatrix} (2-u)^{-2} + 4(2-u)^{-4} \\ (2-u)^{-2} - 4(2-u)^{-4} \end{bmatrix},
\end{aligned}$$

and

$$\int_0^t \mathbf{\Psi}^{-1}(u)\mathbf{f}(u) du = \frac{4}{(2-t)^3} \begin{bmatrix} 3(2-t)^2 + 4 \\ 3(2-t)^2 - 4 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix}.$$

Finally, we get

$$\begin{aligned}
\mathbf{y}_p(t) &= \mathbf{\Psi}(t) \int_0^t \mathbf{\Psi}^{-1}(u)\mathbf{f}(u) du \\
&= \left( \frac{(2-t)^2}{8} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{(2-t)^4}{32} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \int_0^t \mathbf{\Psi}^{-1}(u)\mathbf{f}(u) du \\
&= (2-t) \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \frac{(2-t)^2}{2} \begin{bmatrix} 3 \\ 3 \end{bmatrix} - \frac{(2-t)^4}{8} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.
\end{aligned}$$

We now add the homogeneous and particular solutions together and simplify to get

$$\begin{aligned}
\mathbf{y}(t) &= \mathbf{y}_h + \mathbf{y}_p \\
&= (2-t) \begin{bmatrix} 4 \\ 2 \end{bmatrix} + (2-t)^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{(2-t)^4}{4} \begin{bmatrix} -3 \\ 3 \end{bmatrix}.
\end{aligned}$$

We now get

$$y_1(t) = 4(2-t) + (2-t)^2 - \frac{3}{4}(2-t)^4$$

$$y_2(t) = 2(2-t) + (2-t)^2 + \frac{3}{4}(2-t)^3.$$

The amount of fluid in each tank after 1 minute is  $v_1(1) = v_2(1) = 1$ . Thus the concentrations (grams/L) of salt in Tank 1 is  $y_1(1)/1$  and in Tank 2 is  $y_2(1)/1$ , i.e.

$$\frac{y_1(1)}{1} = \frac{17}{4} \quad \text{and} \quad \frac{y_2(1)}{1} = \frac{15}{4}.$$

Ordinary Differential Equations

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