Chapter 2
The Pseudo-Riemannian Space–Time Manifold M_4

2.1 Review of the Special Theory of Relativity

We shall provide here a brief review of the special theory of relativity. Since the emphasis of this book is general relativity, we will state some of the results in this section without proof, for future application. For more details, the reader is referred to the references provided within this chapter.

The arena of this theory is *the four-dimensional flat pseudo-Riemannian space–time manifold* M_4 endowed with a Lorentz metric. (See [55,189,242].) Let us choose a global pseudo-Cartesian or *Minkowskian chart* for the flat manifold M_4. A typical tangent vector is depicted in Fig. 2.1.

In Fig. 2.1, the vector $\vec{\mathbf{v}}_{x_0}$ is drawn on a sheet of paper with *inherent Euclidean geometry*. However, in the pseudo-Riemannian manifold M_4 with Lorentz metric, we obtain from Examples 1.2.1 and 1.3.2 (also from (1.85i–iv)) that

$$
\begin{aligned}
\vec{\mathbf{v}}_{x_0} &= v^i \frac{\partial}{\partial x^i}\Big|_{x_0}, \\
\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{v}}_{x_0}, \vec{\mathbf{v}}_{x_0}\right) &= d_{ij} v^i v^j := \left(v^1\right)^2 + \left(v^2\right)^2 + \left(v^3\right)^2 - \left(v^4\right)^2, \\
\sigma\left(\vec{\mathbf{v}}_{x_0}\right) &= \sqrt{|d_{ij} v^i v^j|}. \qquad (2.1)
\end{aligned}
$$

Note that the components v^i of a tangent vector $\vec{\mathbf{v}}_{x_0}$ are *the same either for a positive-definite metric or for a Lorentz metric*.

The length $\sqrt{(v^1)^2 + (v^2)^2 + (v^3)^2 + (v^4)^2}$ of the vector $\vec{\mathbf{v}}_{x_0}$, as drawn in Fig. 2.1, is *different from the separation* $\sigma\left(\vec{\mathbf{v}}_{x_0}\right)$ in (2.1). Moreover, from the discussions after (1.95), it is clear that the angular inclination of the vector $\vec{\mathbf{v}}_{x_0}$ in Fig. 2.1 is *not meaningful*. Therefore, it is judicious to *exercise caution in interpreting lengths and angles of space–time vectors drawn on a piece of paper!*

A. Das and A. DeBenedictis, *The General Theory of Relativity: A Mathematical Exposition*, DOI 10.1007/978-1-4614-3658-4_2,

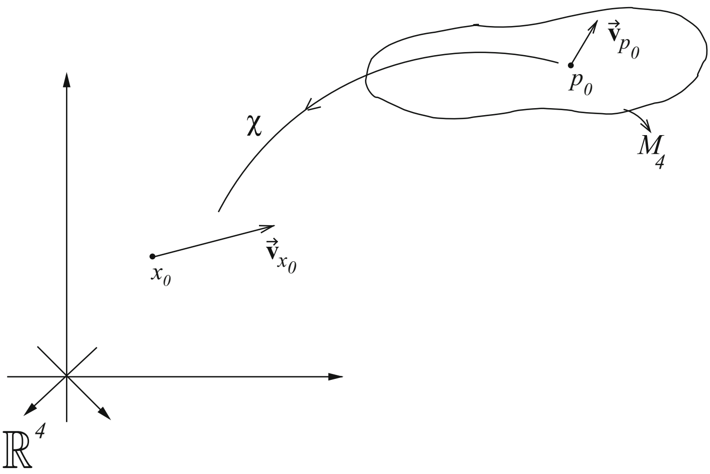


Fig. 2.1 A tangent vector $\vec{\mathbf{v}}_{p_0}$ in M_4 and its image $\vec{\mathbf{v}}_{x_0}$ in $\mathbb{R}^4$

The tangent vector space $T_{x_0}(\mathbb{R}^4)$ splits into *three distinct proper subsets*. The subset of vectors

$$W_S(x_0) := \left\{\vec{\mathbf{s}}_{x_0} : \mathbf{g}_{..}(x_0)\left(\vec{\mathbf{s}}_{x_0}, \vec{\mathbf{s}}_{x_0}\right) > 0\right\} \tag{2.2i}$$

is called the subset of *spacelike vectors*. The subset

$$W_T(x_0) := \left\{\vec{\mathbf{t}}_{x_0} : \mathbf{g}_{..}(x_0)\left(\vec{\mathbf{t}}_{x_0}, \vec{\mathbf{t}}_{x_0}\right) < 0\right\} \tag{2.2ii}$$

is called the subset of *timelike vectors*. Finally, the subset

$$W_N(x_0) := \left\{\vec{\mathbf{n}}_{x_0} : \mathbf{g}_{..}(x_0)\left(\vec{\mathbf{n}}_{x_0}, \vec{\mathbf{n}}_{x_0}\right) = 0\right\} \tag{2.2iii}$$

is the subset of *null vectors*.

Example 2.1.1. Let $\left\{\vec{\mathbf{e}}_{(a)x_0}\right\}_1^4$ be an orthonormal basis set or *tetrad*. (See (1.87) and (1.88).) Therefore,

$$\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{e}}_{(a)x_0}, \vec{\mathbf{e}}_{(b)x_0}\right) = d_{(a)(b)}. \tag{2.3}$$

Let a tangent vector be given by the equation $\vec{\mathbf{s}}_{x_0} := 2\,\vec{\mathbf{e}}_{(1)x_0}$. Therefore, $\mathbf{g}_{..}(x_0)$ $\left(\vec{\mathbf{s}}_{x_0}, \vec{\mathbf{s}}_{x_0}\right) = 4$ and $\vec{\mathbf{s}}_{x_0}$ are spacelike. Let another vector be furnished by $\vec{\mathbf{t}}_{x_0} := \sqrt{3}\,\vec{\mathbf{e}}_{(4)x_0}$. In this case, $\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{t}}_{x_0}, \vec{\mathbf{t}}_{x_0}\right) = -3$. Thus, $\vec{\mathbf{t}}_{x_0}$ is timelike. Finally, let $\vec{\mathbf{n}}_{x_0} := \vec{\mathbf{e}}_{(2)x_0} - \vec{\mathbf{e}}_{(4)x_0}$. Thus, $\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{n}}_{x_0}, \vec{\mathbf{n}}_{x_0}\right) = 0$. Therefore, $\vec{\mathbf{n}}_{x_0}$ is *a nonzero null vector*. □

Now, we shall state several theorems and a corollary regarding various tangent vectors.

Theorem 2.1.2. *Let* $\vec{\mathbf{t}}_{x_0}$ *and* $\widehat{\vec{\mathbf{t}}}_{x_0}$ *be two timelike vectors in* $T_{x_0}(\mathbb{R}^4)$. *Then* $\mathbf{g}_{..}(x_0)$ $(\vec{\mathbf{t}}_{x_0}, \widehat{\vec{\mathbf{t}}}_{x_0}) \neq 0$.

Corollary 2.1.3. *Let two timelike tangent vectors* $\vec{\mathbf{t}}_{x_0}$ *and* $\widehat{\vec{\mathbf{t}}}_{x_0}$ *have for their components* $t^4 > 0$ *and* $\hat{t}^4 > 0$. *Then,* $\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{t}}_{x_0}, \widehat{\vec{\mathbf{t}}}_{x_0}\right) < 0$.

Theorem 2.1.4. *Let* $\vec{\mathbf{t}}_{x_0}$ *be a timelike vector and* $\vec{\mathbf{n}}_{x_0}$ *be a nonzero null vector in* $T_{x_0}(\mathbb{R}^4)$. *Then,* $\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{t}}_{x_0}, \vec{\mathbf{n}}_{x_0}\right) \neq 0$.

Next, we state a very counterintuitive theorem.

Theorem 2.1.5. *Two nonzero null vectors* $\vec{\mathbf{n}}_{x_0}$ *and* $\widehat{\vec{\mathbf{n}}}_{x_0}$ *are orthogonal* (*i.e.*, $\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{n}}_{x_0}, \widehat{\vec{\mathbf{n}}}_{x_0}\right) = 0$), *if and only if they are scalar multiples of each other.*

Proofs of above theorems and corollary are available in books [55, 243].

Now, we shall briefly discuss the physical significance of the flat manifold M_4 in special relativity. Firstly, we choose physical units so that *the speed of light* $c = 1$. (In the popular c.g.s. units, $c = 2.998 \times 10^{10}$ cm.s^{-1}) *Roman indices* take from $\{1, 2, 3, 4\}$, whereas *Greek indices* take from $\{1, 2, 3\}$. The summation convention is followed for *both sets of indices*. An element $p \in M_4$ represents *an idealized point event* in the space–time continuum. In a global Minkowskian coordinate chart, $x = \chi(p) \equiv (x^1, x^2, x^3, x^4)$ stands for the coordinates of an event. The coordinates x^1, x^2, x^3 represent the usual Cartesian coordinates of *the spatial point* associated with the event. The fourth coordinate x^4 stands for the time coordinate of the event. (*Remark*: The time coordinate x^4 *need not provide the actual temporal separation!*) In units where the speed of light is not set to unity, x^4 is the time coordinate multiplied by c. Therefore, in the Minkowskian chart, the time coordinate x^4 possesses *the same units as the spatial coordinates*. (On the other hand, a spatial coordinate x^α divided by c possesses the same unit as time.)

The three-dimensional hypersurface given by

$$\mathcal{N}_{x_0} := \left\{x : d_{ij}\left(x^i - x_0^i\right)\left(x^j - x_0^j\right) = 0\right\} \tag{2.4}$$

is called the *null cone* (or “light cone”) with vertex at x_0. (See Fig. 2.2.)

The points inside and on the upper half of the null cone represent *future events* relative to x_0. Similarly, points inside and on the lower half of the cone represent *past events* relative to x_0. The events outside the cone are the (relativistic) *present events* relative to x_0.

The *causal cone* relative to x_0 is represented by the proper subset

$$\mathcal{C}_{x_0} := \left\{x : d_{ij}\left(x^i - x_0^i\right)\left(x^j - x_0^j\right) \leq 0\right\}. \tag{2.5}$$

Events in $\mathcal{C}_{x_0}$ can either causally affect or be affected by the event in x_0. This statement incorporates one of the physical postulates of the special relativity, namely, *physical actions cannot propagate faster than light*. We shall motivate this statement shortly.

The global Minkowskian charts are very useful in the special theory of relativity. The coordinate transformation (see Fig. 1.2) from one such chart to another is furnished by the equations

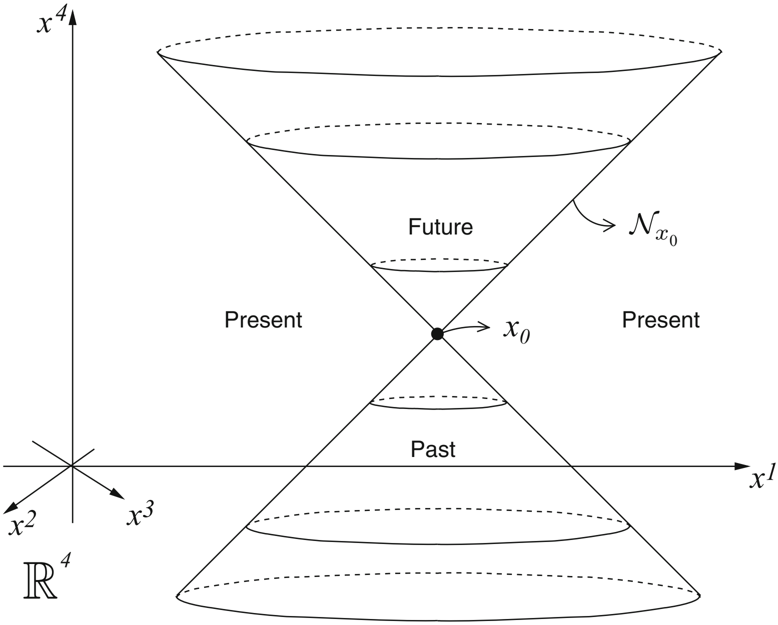


Fig. 2.2 Null cone $\mathcal{N}_{x_0}$ with vertex at x_0 (*circles* represent suppressed spheres)

$$\hat{x}^i = \widehat{X}^i(x) = c^i + l^i{}_j\, x^j, \tag{2.6i}$$

$$l^k{}_i\, d_{km}\, l^m{}_j = d_{ij}, \tag{2.6ii}$$

$$x^i = a^i{}_j\, \left(\hat{x}^j - c^j\right), \tag{2.6iii}$$

$$l^m{}_i\, a^i{}_k = \delta^m{}_k. \tag{2.6iv}$$

Here, c^i and $l^i{}_j$ are ten independent constants, or, parameters. The set of all transformations in (2.6i–iv) constitutes a Lie group denoted by $\mathcal{I}O(3,1;\mathbb{R})$. It is usually called the *Poincaré group*. The subgroup characterized by $c^1 = c^2 = c^3 = c^4 = 0$, is denoted by $O(3,1;\mathbb{R})$. It is usually known as the six-parameter *Lorentz group* [55, 114].

Example 2.1.6. Assuming $|v| < 1$, a *Lorentz transformation* is provided by

$$\begin{aligned} \hat{x}^1 &= x^1, \quad \hat{x}^2 = x^2, \\ \hat{x}^3 &= \frac{x^3 - vx^4}{\sqrt{1-(v)^2}}, \\ \hat{x}^4 &= \frac{-vx^3 + x^4}{\sqrt{1-(v)^2}}. \end{aligned} \tag{2.7}$$

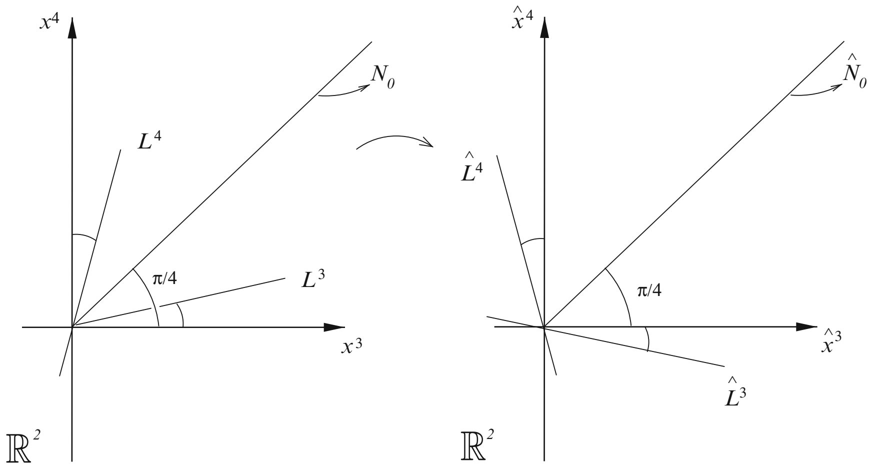


Fig. 2.3 A Lorentz transformation inducing a mapping between two coordinate planes

Suppressing two dimensions characterized by the x^1-axis and x^2-axis, we shall exhibit qualitatively the transformation (2.7) in Fig. 2.3. (We assume $0 < v < 1$.)

In Fig. 2.3, the x^3-axis and x^4-axis are mapped into two straight lines $\hat{L}^3$ and $\hat{L}^4$ respectively. The line N_0 is mapped into $\widehat{N}_0$. Thus, N_0 remains *invariant* under the mapping. (This line physically represents the trajectory of a photon, or other massless particle, in space–time!) The preimages of $\hat{x}^3$-axis and $\hat{x}^4$-axis are lines L^3 and L^4, respectively. In the inherent Euclidean geometry of the plane of Fig. 2.3, two lines L^3 and L^4 do *not intersect orthogonally!* However, in the relativistic Lorentz metric, two lines L^3 and L^4 *do* intersect orthogonally! The Lorentz transformation in (2.7) mediates, physically speaking, the transformation from one (inertial) observer to another (inertial) observer moving with velocity v along the x^3-axis. (A popular name for the Lorentz transformation in (2.7) is a *boost*.) This mapping induces the phenomena of length contraction and time dilation between observers in relative motion [55, 89]. □

Now, we shall discuss (Minkowskian) tensor fields in the flat space–time manifold. We use a global Minkowskian coordinate chart so that

$$\begin{aligned}
\mathbf{g}_{..}(x) &= d_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2 + \mathrm{d}x^3 \otimes \mathrm{d}x^3 - \mathrm{d}x^4 \otimes \mathrm{d}x^4 \\
&= \delta_{\alpha\beta}\, \mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta - \mathrm{d}x^4 \otimes \mathrm{d}x^4, \\
\mathbf{g}^{..}(x) &= d^{ij}\, \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}\,, \\
\operatorname{sgn}(\mathbf{g}_{..}(x)) &= +2. \qquad (2.8)
\end{aligned}$$

The natural orthonormal basis set or tetrad is provided by

$$\vec{\mathbf{e}}_{(a)}(x) = \lambda^i_{(a)}(x)\,\frac{\partial}{\partial x^i} := \delta^i_{(a)}\,\frac{\partial}{\partial x^i}\,. \tag{2.9}$$

(See (1.104).)

A tensor field ${}^r_s\mathbf{T}(x)$ and its Minkowskian components satisfy

$${}^r_s\mathbf{T}(x) = T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x)\,\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s}. \tag{2.10}$$

(Compare the above with (1.30).)

The transformations of tensor field components under a Poincaré transformation in (2.6i–iv) are provided by

$$\widehat{T}^{k_1,\ldots,k_r}{}_{l_1,\ldots,l_s}(\hat{x}) = l^{k_1}{}_{i_1},\ldots,\, l^{k_r}{}_{i_r}\, a^{j_1}{}_{l_1},\ldots,\, a^{j_s}{}_{l_s}\, T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x). \tag{2.11}$$

(Compare the above with (1.37).)

It should be noted that covariant derivatives in Minkowskian charts are *equivalent to partial derivatives*, as the Christoffel symbols are identically zero in Minkowskian charts in flat space–time.

Example 2.1.7. Consider the Lorentz metric tensor in (2.8). Under a Poincaré transformation, using (2.11), we deduce that

$$\hat{d}_{ij} = a^k{}_i\, a^l{}_j\, d_{kl} = d_{ij}\,.$$

(For the derivation of the last step, we have used (2.6ii, iv).) Therefore, the Lorentz metric tensor behaves as a *numerical tensor* under Poincaré transformations. (Consult [56] for discussions on numerical tensors.) □

One of the mathematical axioms of the special relativity is that *every natural law must be expressible as a tensor field equation*

$$\begin{aligned} {}^r_s\mathbf{T}(x) &= {}^r_s\mathbf{O}(x), \\ \text{or}\quad T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(x) &= 0. \end{aligned} \tag{2.12}$$

(Sometimes, such tensor equations are restricted on a curve.)

By a Poincaré transformation, (2.12) simply imply that

$$\widehat{T}^{k_1,\ldots,k_r}{}_{l_1,\ldots,l_s}(\hat{x}) = 0. \tag{2.13}$$

Therefore, a natural law, as formulated by one inertial (i.e., experiencing no net force) observer, must be *exactly similar* to that of another moving (inertial) observer. This is the principle of special relativistic "covariance." (Later, other types of objects, *spinor fields*, had to be added on.)

Now, we shall deal with differentiable, parametric curves in space–time. We shall consider a nondegenerate parametric curve $\mathcal{X}$ of class C^2. (See Fig. 1.6.) We reiterate (1.19) and (1.20) as

$$\begin{aligned} x &= \mathcal{X}(t), \\ x^i &= \mathcal{X}^i(t), \\ \sum_{j=1}^{4}&\left[\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\right]^2 > 0, \\ t &\in [a,b] \subset \mathbb{R}. \end{aligned} \tag{2.14}$$

Now, we explore the invariant values of

$$\mathbf{g}_{..}(\mathcal{X}(t))\left[\vec{\boldsymbol{\mathcal{X}}}'(t), \vec{\boldsymbol{\mathcal{X}}}'(t)\right] = d_{ij}\,\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}. \tag{2.15}$$

A *spacelike curve* satisfies

$$d_{ij}\,\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t} > 0. \tag{2.16i}$$

A *timelike curve* is characterized by

$$d_{ij}\,\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t} < 0. \tag{2.16ii}$$

Moreover, a (nondegenerate) *null curve* satisfies

$$d_{ij}\,\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t} = 0. \tag{2.16iii}$$

Qualitative pictures, together with an upper null cone, are exhibited in Fig. 2.4. Note that the slope of timelike curves is everywhere steeper than that of the null cone, the slope of null curves is coincident with the slope of the null cone, and the slope of spacelike curves is shallower than the null cone.

From relativistic physics, it is known that (1) an (idealized) massive point particle travels along a timelike curve; (2) a massless point particle (like a photon) traverses along a null curve; (3) and moreover, hypothetical superluminal particles called tachyons possess trajectories which are represented by spacelike curves.

Now, we shall introduce the arc separation along *a timelike curve*. (See (1.193).) It is furnished by

$$s = \mathcal{S}(t) := \int_a^t \sqrt{-d_{ij}\,\frac{\mathrm{d}\mathcal{X}^i(u)}{\mathrm{d}u}\,\frac{\mathrm{d}\mathcal{X}^j(u)}{\mathrm{d}u}}\cdot \mathrm{d}u. \tag{2.17}$$

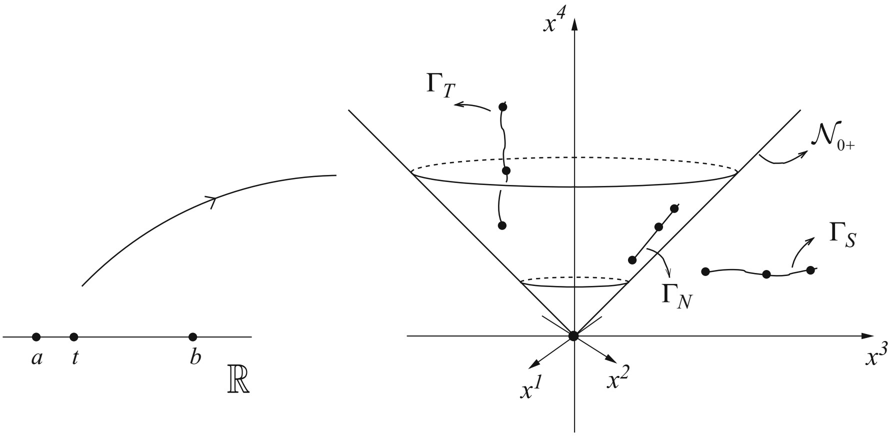


Fig. 2.4 Images Γ_S, Γ_T, and Γ_N of a spacelike, timelike, and a null curve

The above arc separation s (sometimes popularly denoted by τ also) is called the *proper time* along the timelike curve. It is *assumed*[1] to provide *the actual time separation* between the events $\mathcal{X}(a)$ and $\mathcal{X}(t)$ as an idealized, standard point clock traverses along the timelike curve $\mathcal{X}$ (or, the *world line*). The arc separation along a null curve is *exactly zero*.

An *inertial observer* in special relativity theory traverses a timelike, straight line satisfying

$$\frac{d^2\mathcal{X}^i(t)}{dt^2} = 0. \tag{2.18}$$

The above equation is that of *a geodesic* in flat space–time with a Minkowskian coordinate chart. (Compare with (1.183).)

An inertial observer is an idealized, massive point being carrying a (point) standard clock and four orthogonal directions inherent in a tetrad. He or she is also capable of sending photons outward and to receive them at later times after reflections from other specular point objects travelling on timelike world lines. In this process, the observer can assign *operational Minkowskian coordinates* to the events around him/her. Such a method of measurements is called *Minkowskian chronometry*. (Consult the book by Synge [242].) *A non-inertial, idealized observer* follows a timelike world line with $\frac{d^2\mathcal{X}^i(t)}{dt^2} \neq 0$.

Remark: It should be mentioned that the world line of an inertial (or else non-inertial) observer violates the uncertainty principle of quantum mechanics, which prohibits the simultaneous existence of exact position and velocity. (The arguments here are for classical idealized objects where quantum corrections are negligible for these purposes.)

[1] There are strong physical arguments for this assumption which are beyond the scope of this brief review of special relativity. The interested reader is referred to [55].

Now, we shall study the equations of motion of a massive particle of *constant mass*. In prerelativistic physics, Newton's equations of motion are furnished by

$$m\frac{\mathrm{d}^2\mathcal{X}^\alpha(t)}{\mathrm{d}t^2} =: m\frac{\mathrm{d}V^\alpha(t)}{\mathrm{d}t} = f^\alpha(\mathbf{x}, t, \mathbf{v})_{|x^\alpha=\mathcal{X}^\alpha(t)}, \; V^\alpha(t) = \frac{\mathrm{d}\mathcal{X}^\alpha(t)}{\mathrm{d}t}. \tag{2.19i}$$

Here, $m > 0$ is the mass parameter and t is the (absolute) time. Moreover, $\mathbf{x} = (x^1, x^2, x^3)$ are Cartesian coordinates; $V^\alpha(t)$ and $f^\alpha(\mathbf{x}, t, \mathbf{v})_{|..}$ are the Cartesian components of *instantaneous velocity* and the net external force vector, respectively. One consequence of (2.19i) is that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(m/2)\delta_{\alpha\beta}V^\alpha(t)V^\beta(t)\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[k + (m/2)\delta_{\alpha\beta}V^\alpha(t)V^\beta(t)\right]$$

$$= \delta_{\alpha\beta}V^\alpha(t)\left[f^\beta(\cdot\cdot)_{|..}\right]. \tag{2.19ii}$$

The above equation can be physically interpreted as "the rate of increase of the kinetic energy plus an undetermined energy constant k is equal to the rate of work performed by the external force."

In special relativity, using a Minkowskian chart, we represent the motion curve by

$$\begin{aligned} x^\alpha &= \mathcal{X}^{\#\alpha}(s), \\ t \equiv x^4 &= \mathcal{X}^{\#4}(s). \end{aligned} \tag{2.20}$$

Here, $s \in [0, s_1]$ and $\mathcal{X}^{\#k}$ are C^2-functions. Furthermore, we have chosen s to be the arc separation parameter or proper time. (See (1.193).) It is also assumed that the motion curve is timelike and future pointing, that is,

$$d_{kl}\,\frac{\mathrm{d}\mathcal{X}^{\#k}(s)}{\mathrm{d}s}\,\frac{\mathrm{d}\mathcal{X}^{\#l}(s)}{\mathrm{d}s} \equiv -1, \tag{2.21i}$$

$$\frac{\mathrm{d}\mathcal{X}^{\#4}(s)}{\mathrm{d}s} > 0, \tag{2.21ii}$$

$$d_{kl}\,\frac{\mathrm{d}\mathcal{X}^{\#k}(s)}{\mathrm{d}s}\,\frac{\mathrm{d}^2\mathcal{X}^{\#l}(s)}{\mathrm{d}s^2} \equiv 0. \tag{2.21iii}$$

We denote the *four-velocity* components by

$$u^i = \mathcal{U}^i(s) := \frac{\mathrm{d}\mathcal{X}^{\#i}(s)}{\mathrm{d}s}\,, \tag{2.22i}$$

$$d_{ij}\,u^i u^j = -1, \tag{2.22ii}$$

$$|u^4| \geq 1. \tag{2.22iii}$$

In the four-dimensional space of four-velocity components, the quadratic constraint (2.22ii) yields a three-dimensional hyperhyperboloid of two disconnected subsets. (See Fig. 2.5.)

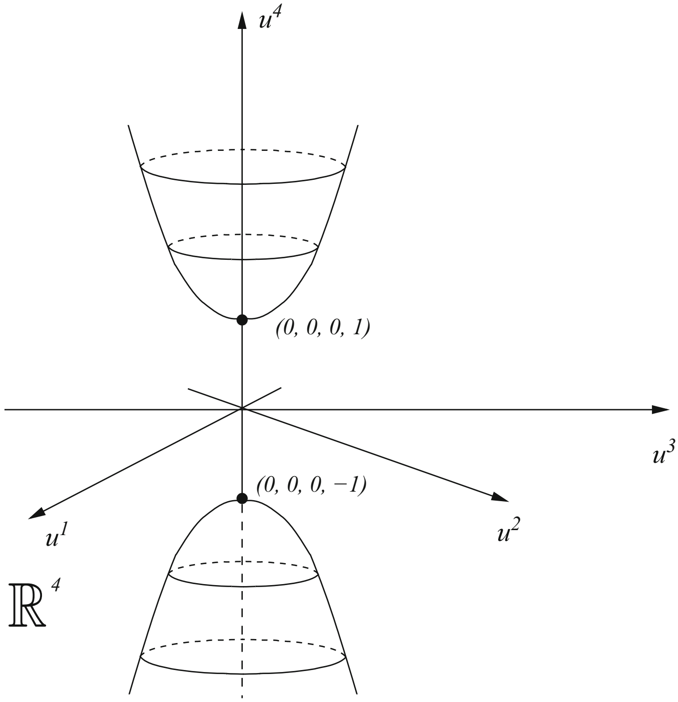


Fig. 2.5 The three-dimensional hyperhyperboloid representing the 4-velocity constraint

To compare relativistic velocity components with nonrelativistic velocity components, we need to reparametrize the motion curve by (2.17). Thus, we obtain

$$\begin{aligned} x^{\alpha} &= \mathcal{X}^{\#\alpha}(\mathcal{S}(t)) = \mathcal{X}^{\alpha}(t), \\ x^{4} &= \mathcal{X}^{\#4}(s) = \mathcal{X}^{4}(t) = t, \\ V^{\alpha}(t) &:= \frac{\mathrm{d}\mathcal{X}^{\alpha}(t)}{\mathrm{d}t}, \\ \|\vec{\mathbf{V}}(t)\| &:= +\sqrt{\delta_{\mu\nu}V^{\mu}(t)V^{\nu}(t)}\,. \end{aligned} \tag{2.23}$$

(Note that $V^{\alpha}(t)$ are *not spatial components of a relativistic 4-vector*.)

Using (1.24), (2.17), (2.22i), and (2.23), we derive that

$$\begin{aligned} \mathcal{U}^{\alpha}(s) &= \frac{\mathrm{d}\mathcal{X}^{\#\alpha}(s)}{\mathrm{d}s} = \left[\frac{\mathrm{d}\mathcal{S}(t)}{\mathrm{d}t}\right]^{-1}\frac{\mathrm{d}\mathcal{X}^{\alpha}(t)}{\mathrm{d}t} = \frac{V^{\alpha}(t)}{\sqrt{1-\|\vec{\mathbf{V}}(t)\|^2}}, \\ \mathcal{U}^{4}(s) &= \frac{\mathrm{d}\mathcal{X}^{\#4}(s)}{\mathrm{d}s} = \left[\frac{\mathrm{d}\mathcal{S}(t)}{\mathrm{d}t}\right]^{-1}\frac{\mathrm{d}\mathcal{X}^{4}(t)}{\mathrm{d}t} = \frac{1}{\sqrt{1-\|\vec{\mathbf{V}}(t)\|^2}}. \end{aligned} \tag{2.24}$$

The above equations are mathematically valid *only for* $\| \vec{\mathbf{V}}(t) \| < 1$. Or, in other words, the nonrelativistic speed of a massive material particle *must be strictly less than the speed of light*.

Now, we shall generalize Newton's equations of motion (2.19i, ii) to the appropriate relativistic equations. For a particle with *constant mass* $m > 0$, we *assume that the relativistic equations are furnished by*:

$$m \frac{\mathrm{d}^2 \mathcal{X}^{\#i}(s)}{\mathrm{d}s^2} = F^i(x, u)\Big|_{x=\mathcal{X}^{\#}(s),\, u=\frac{\mathrm{d}\mathcal{X}^{\#}(s)}{\mathrm{d}s}}. \tag{2.25}$$

Here, the *4-force vector* components $F^i(\cdot\cdot)$ are continuous functions of *eight variables*. (See [55] and [243].) Note that from (2.21iii), we obtain

$$\begin{aligned} & d_{ij} \frac{\mathrm{d}\mathcal{X}^{\#i}(s)}{\mathrm{d}s} F^j(\cdot\cdot)_{|..} \equiv 0, \\ & \frac{\mathrm{d}\mathcal{X}^{\#4}(s)}{\mathrm{d}s} F^4(\cdot\cdot)_{|..} = \delta_{\alpha\beta} \frac{\mathrm{d}\mathcal{X}^{\#\alpha}(s)}{\mathrm{d}s} F^\beta(\cdot\cdot)_{|..}. \end{aligned} \tag{2.26}$$

Thus, by Theorems 2.1.2 and 2.1.4, *a nonzero 4-force vector must be spacelike*.

Relativistic equations (2.25) yield, using (2.23) and (2.24), the equations

$$m \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{V^\alpha(t)}{\sqrt{1- \| \vec{\mathbf{V}}(t) \|^2}}\right] = \sqrt{1- \| \vec{\mathbf{V}}(t) \|^2} \cdot F^\alpha(\cdot\cdot)_{|x=\mathcal{X}(t)}, \tag{2.27i}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m}{\sqrt{1- \| \vec{\mathbf{V}}(t) \|^2}}\right] = \sqrt{1- \| \vec{\mathbf{V}}(t) \|^2} \cdot F^4(\cdot\cdot)_{|..}. \tag{2.27ii}$$

In the low speed $\left(\| \vec{\mathbf{V}}(t) \| \ll 1\right)$ regime, the equations in (2.27i) reduce to Newton's equations of motion (2.19i), provided we equate

$$f^\alpha(\cdot\cdot)_{|..} = \sqrt{1- \| \vec{\mathbf{V}}(t) \|^2} \cdot F^\alpha(\cdot\cdot)_{|..}. \tag{2.28}$$

From (2.28) and (2.26), we derive that

$$\sqrt{1- \| \vec{\mathbf{V}}(t) \|^2} \cdot F^4(\cdot\cdot)_{|..} = \delta_{\alpha\beta} V^\alpha(t) f^\beta(\cdot\cdot)_{|..}. \tag{2.29}$$

Substituting (2.29) into (2.27ii), we deduce that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m}{\sqrt{1- \| \vec{\mathbf{V}}(t) \|^2}}\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[m + (m/2)\delta_{\alpha\beta} V^\alpha(t) V^\beta(t) + 0\left(\| \vec{\mathbf{V}}(t) \|^4\right)\right] \\ &= \delta_{\alpha\beta} V^\alpha(t) f^\beta(\cdot\cdot)_{|..}. \end{aligned} \tag{2.30}$$

Comparing (2.30) with (2.19ii), we are prompted to express the energy of the particle as

$$E\left(\parallel \vec{\mathbf{v}} \parallel\right) := \frac{m}{\sqrt{1 - \parallel \vec{\mathbf{v}} \parallel^2}}, \tag{2.31i}$$

$$\lim_{\|\vec{\mathbf{v}}\| \to 0_+} E\left(\parallel \vec{\mathbf{v}} \parallel\right) = E(0_+) = m. \tag{2.31ii}$$

Explicitly reinstating (temporarily) the speed of light c, (2.31ii) reveals the rest energy as $E(0_+) = mc^2$. (*This is arguably the most celebrated equation* in modern science! [87].) Since energy $E(\cdot\cdot)$ was derived from the expression $m\frac{d\mathcal{X}^{\#4}(s)}{ds}$, it is logical to define the (kinetic) *4-momentum* of a massive particle as

$$\begin{aligned} p^i &= \mathcal{P}^i(s) := m\frac{d\mathcal{X}^{\#i}(s)}{ds}, \\ p_i &= \mathcal{P}_i(s) = d_{ij}\,\mathcal{P}^j(s), \\ p_\alpha &= p^\alpha, \quad p_4 = -p^4. \end{aligned} \tag{2.32}$$

The constraint (2.22ii) yields a similar constraint on 4-momentum components in (2.32), and it is furnished by

$$d_{ij}\,p^i p^j = d^{ij}\,p_i p_j = -(m)^2. \tag{2.33}$$

The above yields a three-dimensional hyperhyperboloid in the four-dimensional momentum space. (Compare with the corresponding Fig. 2.5.) This hypersurface is known as the *mass shell* in relativistic physics.

Example 2.1.8. Let us investigate the timelike motion of a constant mass particle under a nonzero, constant-valued 4-force vector. By (2.25), we write

$$m\frac{d^2\mathcal{X}^{\#i}(s)}{ds^2} = K^i = \text{const.}, \tag{2.34i}$$

$$\frac{d^2\mathcal{X}^{\#i}(s)}{ds^2} = \frac{K^i}{m} =: C^i = \text{const.} \tag{2.34ii}$$

By the Theorems 2.1.2 and 2.1.4, we conclude that

$$d_{ij}\,C^i C^j > 0. \tag{2.35}$$

Solving differential equations (2.34ii) under the initial values $\mathcal{X}^{\#i}(0) = x_0^i$, $\frac{d\mathcal{X}^{\#i}(s)}{ds}_{|s=0} = u_0^i$, we obtain the solution as

$$\mathcal{X}^{\#i}(s) = x_0^i + u_0^i \cdot s + (1/2)C^i \cdot (s)^2. \tag{2.36}$$

Using the identity (2.21i), we deduce from (2.36) that

$$0 \equiv \frac{\mathrm{d}}{\mathrm{d}s}\left[d_{ij}\,\frac{\mathrm{d}\mathcal{X}^{\# i}(s)}{\mathrm{d}s}\,\frac{\mathrm{d}^2\mathcal{X}^{\# j}(s)}{\mathrm{d}s^2}\right] = d_{ij}C^iC^j. \tag{2.37}$$

Equation (2.37) yielding $d_{ij}C^iC^j = 0$ obviously *violates the strict inequality (2.35)*. Therefore, we conclude that such *a particle motion with constant 4-acceleration is impossible*.[2] □

Now, we shall investigate the motions of particles inside *an extended body*. It is the theme of the usual nonrelativistic *continuum mechanics*. The scope of this branch of applied mathematics encompasses fluids, elastic materials, visco-elastic materials, plasmas, and so on. (Usually, the relativistic effects, which are dominant at high velocities, are neglected in these disciplines.)

Let us consider a nonrelativistic fluid motion in Euler's approach. There is a three-velocity field $\mathcal{V}^\alpha(\mathbf{x}, t)$ which is a measure of the instantaneous fluid velocity of a particle at the spatial point $\mathbf{x} = (x^1, x^2, x^3)$ and time t. The nonlinear equations of motion of streamlines are governed by

$$\frac{\partial \mathcal{V}^\alpha(\cdot\cdot)}{\partial t} + \mathcal{V}^\beta(\mathbf{x}, t)\,\partial_\beta \mathcal{V}^\alpha(\cdot\cdot) = [\rho(\mathbf{x}, t)]^{-1}\,\partial_\beta\sigma^{\alpha\beta} + \phi^\alpha_{(\mathrm{ext})}(\mathbf{x}, t). \tag{2.38}$$

Here, $\rho(\mathbf{x}, t) > 0$ is the *mass density*, $\sigma^{\beta\alpha}(\mathbf{x}, t) \equiv \sigma^{\alpha\beta}(\mathbf{x}, t)$ is the *stress density*, and $\phi^\alpha_{(\mathrm{ext})}(\mathbf{x}, t)$ is the force density due to external influences. (See [244].) The streamlines are furnished by solutions or integral curves of the differential equations

$$\frac{\mathrm{d}\mathcal{X}^\alpha(t)}{\mathrm{d}t} = \mathcal{V}^\alpha(\mathbf{x}, t)_{|..}, \tag{2.39}$$

where $\mathcal{V}^\alpha(\mathbf{x},\ t)$ satisfy (2.38). (Compare with (1.75).)

Restricting the vector field equations (2.38) on a streamline $x^\alpha = \mathcal{X}^\alpha(t)$, we obtain

$$\rho(\mathbf{x}, t)_{|..} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\,[\mathcal{V}^\alpha(\cdot\cdot)]_{|..} = \left[\partial_\beta\sigma^{\alpha\beta} + \rho\phi^\alpha_{(\mathrm{ext})}(\cdot\cdot)\right]_{|..}. \tag{2.40}$$

It is obvious that Newton's equations of motion (2.19i) have motivated (2.40). *In the absence of an external force*, (2.38) reduce to

$$\rho(\cdot\cdot)\left[\frac{\partial \mathcal{V}^\alpha(\cdot\cdot)}{\partial t} + \mathcal{V}^\beta(\cdot\cdot)\,\partial_\beta\mathcal{V}^\alpha(\cdot\cdot)\right] - \partial_\beta\sigma^{\alpha\beta} = 0. \tag{2.41}$$

Now, in the relativistic theory, we need to visualize a (bounded) compact fluid body as a *world tube* in space–time. (See the Fig. 2.6.)

---

[2] Often, the term "constant acceleration" in the relativity literature refers to the condition $d_{ij}\,\frac{\mathrm{d}\mathcal{U}^i(s)}{\mathrm{d}s}\,\frac{\mathrm{d}\mathcal{U}^j(s)}{\mathrm{d}s} = \text{const.}$, which may be satisfied with nonconstant $\frac{\mathrm{d}\mathcal{U}^i(s)}{\mathrm{d}s} = \frac{\mathrm{d}^2\mathcal{X}^{\# i}(s)}{\mathrm{d}s^2}$.

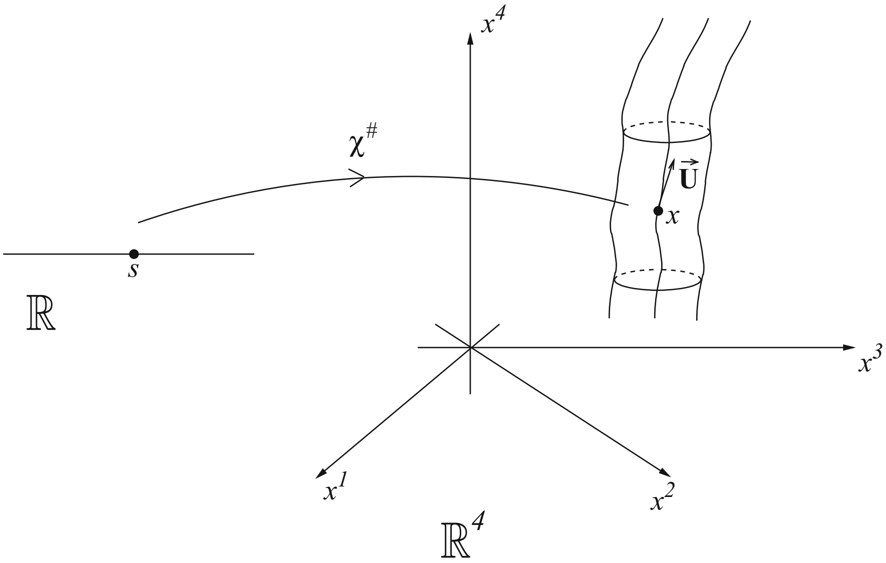


Fig. 2.6 A world tube and a curve representing a fluid streamline

The timelike, future-pointing 4-velocity vector field $\vec{\mathbf{U}}(x)$ satisfies

$$U_i(x)U^i(x) = d_{ij}U^i(x)U^j(x) \equiv -1,$$
$$U^4(x) \geq 1. \tag{2.42}$$

A fluid streamline or curve of class C^2 is characterized by integral curves of the differential equations

$$\frac{\mathrm{d}\mathcal{X}^{\#i}(s)}{\mathrm{d}s} = U^i(x)_{|x=\mathcal{X}^{\#}(s)}. \tag{2.43}$$

(Compare with (2.39).)

The appropriate relativistic generalizations of the equations of motion in (2.41) are furnished by

$$\rho(x)\left[U^j(x)\,\partial_j U^i\right] - \left[\delta^i{}_k + U^i(x)U_k(x)\right]\partial_j s^{kj} = 0. \tag{2.44}$$

Here, $\rho(x) > 0$ is the proper energy density (including mass energy density). Moreover, $U^i(x)$ are components of the 4-velocity field of streamlines, and $s^{ji}(x) \equiv s^{ij}(x)$ are components of *the relativistic symmetric stress tensor field* of differentiability class C^1.

Now, we are in a position to define the relativistic, symmetric *energy–momentum–stress tensor* field by

$$
\begin{aligned}
\mathbf{T}^{\cdot\cdot}(x) &:= T^{ij}(x)\,\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j},\\
T^{ij}(x) &:= \rho(x)U^i(x)U^j(x) - s^{ij}(x)\\
&\equiv T^{ji}(x).
\end{aligned}
\tag{2.45}
$$

Remarks: (i) Some authors call the above tensor the stress–momentum–energy tensor, or stress–energy tensor, or simply the stress tensor.

(ii) In relativistic quantum field theory, $T^{ij}(x)$ sometimes represents components of *the canonical energy–momentum–stress tensor* for which $T^{ji}(x) \not\equiv T^{ij}(x)$. In this text, *we will not be utilizing the canonical variant of this tensor*.

Note that $T^{ij}(x)$ in (2.45) possesses ten linearly independent components.

Now, we shall state and prove a theorem about the energy–momentum–stress tensor field.

Theorem 2.1.9. *Let a symmetric energy–momentum–stress tensor field* $\mathbf{T}^{\cdot\cdot}(x)$ *be defined by* (2.45) *in* $D \subset \mathbb{R}^4$ *representing a fluid world tube. Furthermore, let the tensor field components* $T^{ij}(x)$ *be of differentiability class* C^1. *Then, the four partial differential equations*

$$
\partial_j T^{ij} = 0 \tag{2.46}
$$

imply the relativistic equations of streamline motion in (2.44). *Moreover,* (2.46) *implies one additional differential equation*

$$
\partial_j\left[\rho U^j\right] + U_i(x)\,\partial_j s^{ij} = 0. \tag{2.47}
$$

Proof. By (2.45) and (2.46), we obtain

$$
\partial_j\left[\rho U^i U^j - s^{ij}\right] = \rho(x)\,U^j(x)\,\partial_j U^i + U^i(x)\,\partial_j\left[\rho U^j\right] - \partial_j s^{ij} = 0. \tag{2.48}
$$

Multiplying the above by $U_i(x)$ and contracting, we derive that

$$
(1/2)\rho U^j\partial_j\left[U_iU^i\right] + \left[U_iU^i\right]\partial_j\left[\rho U^j\right] - U_i\partial_j s^{ij} = 0.
$$

Now, substituting $U_i(x)U^i(x) \equiv -1$ from (2.42) into the preceding equation, we deduce (2.47). Putting (2.47) into (2.48), we derive the equations of motion for the streamlines in (2.44). ■

Remark: The equation $\partial_j T^{ij} = 0$ in (2.46) is called the *differential conservation of energy–momentum*.

Now, we shall try to understand physically the energy–momentum–stress tensor in (2.45). Suppose that an idealized inertial observer passes through the fluid and intersects a streamline at the event x in Fig. 2.6. Assume that *this test observer does not disturb any of the streamlines*. He carries an orthonormal basis $\left\{\vec{\mathbf{e}}_{(a)}(x)\right\}_1^4$ or a tetrad. Imagine that this observer manages to measure numbers associated with

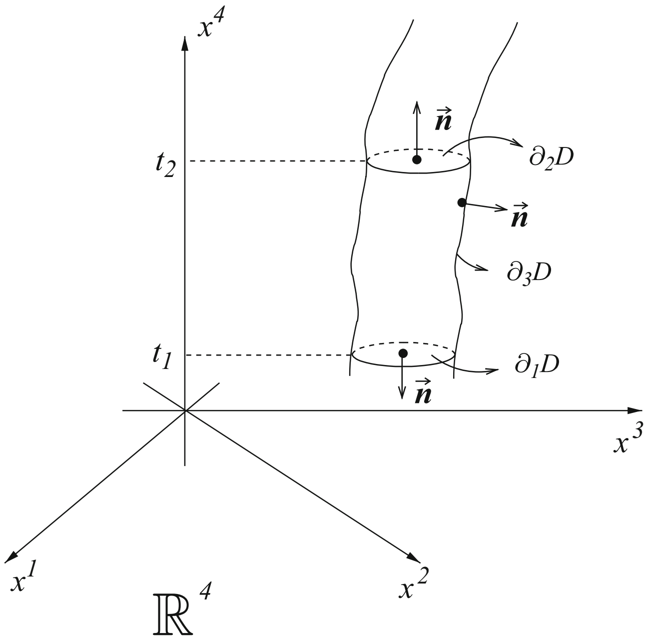


Fig. 2.7 A doubly sliced world tube of an extended body

the orthonormal components (or *physical components*) $T^{(a)(b)}(x)$ of the energy–momentum–stress tensor $\mathbf{T}^{\cdot\cdot}(x)$. In such a scenario, physical interpretations can be furnished. $T^{(4)(4)}(x)$ is the sum of the material energy density $\rho(x)\left[U^4(x)\right]^2$ and the internal energy density $-s^{(4)(4)}(x)$. $T^{(\alpha)(4)}(x)$ are the sums of kinetic momentum density $\rho(x)U^\alpha(x)U^4(x)$ and internal momentum density $-s^{(\alpha)(4)}(x)$. ($T^{(\alpha)(4)}(x)$ is often called the *energy flux* or *energy flux vector*, although it is not truly a vector.) Finally, $T^{(\alpha)(\beta)}(x)$ are sums of material stress components $\rho(x)U^\alpha(x)U^\beta(x)$ and the *negative of the usual internal stress components* $-s^{(\alpha)(\beta)}(x)$. A thorough discussion of the physical interpretation of the energy–momentum–stress tensor may be found in [230, 243].

Now, we are in a position to investigate the *integral conservation* or *total conservation of energy–momentum* of a fluid or other extended bodies. We provide another figure of a material world tube with two "horizontal caps" in Fig. 2.7.

Now, we shall state and prove the integral conservation laws. (See Fig. 2.7.)

Theorem 2.1.10. *Let* $\mathbf{T}^{\cdot\cdot}(x) \not\equiv \mathbf{0}^{\cdot\cdot}(x)$ *and be of class* C^1 *inside a material world tube. Furthermore, let* $\mathbf{T}^{\cdot\cdot}(x) \equiv \mathbf{0}^{\cdot\cdot}(x)$ *outside the world tube. Let* $x^4 = t_1$ *and* $x^4 = t_2$ *yield three-dimensional cross sections of the world tube denoted by* $\partial_1 D$ *and* $\partial_2 D$*, respectively. Moreover, let* $\partial_3 D$ *be the three-dimensional boundary "wall" around the domain* D *of the tube. Furthermore, let boundary conditions* $T^{ij}(x)n_j(x)_{|\partial_3 D} = 0$*, where* n^i *denotes components of unit outer* *normal, be satisfied. Let the overall boundary* $\partial D := \partial_1 D \cup \partial_2 D \cup \partial_3 D$ *be non-null, continuous, piecewise differentiable, and orientable. Then, the differential conservation equations* (2.46) *imply the integral conservation equations*

$$\begin{aligned} P^i &:= \int\limits_{\partial_1 D} T^{4i}(x^1, x^2, x^3, t_1)\, dx^1 dx^2 dx^3 \\ &= \int\limits_{\partial_2 D} T^{4i}(x^1, x^2, x^3, t_2)\, dx^1 dx^2 dx^3 \\ &= \int\limits_{\partial D_{(t)}} T^{4i}(x^1, x^2, x^3, t)\, dx^1 dx^2 dx^3 \\ &= \text{const.} \end{aligned} \tag{2.49}$$

Proof. Applying Gauss' Theorem 1.3.27 and differential conservation equations (2.46), we obtain

$$\begin{aligned} 0 = & \int\limits_{D} \left[\partial_j T^{ij}\right] dx^1 dx^2 dx^3 dx^4 \\ = & \int\limits_{\partial_1 D} T^{ij}\, n_j\, dx^1 dx^2 dx^3 + \int\limits_{\partial_2 D} T^{ij}\, n_j\, dx^1 dx^2 dx^3 + \int\limits_{\partial_3 D} T^{ij}\, n_j\, d^3 v \\ = & -\int\limits_{\partial_1 D} T^{i4}\, dx^1 dx^2 dx^3 + \int\limits_{\partial_2 D} T^{i4} dx^1 dx^2 dx^3 + 0. \end{aligned}$$

Thus, (2.49) is validated. ■

The constant-valued components P^i represent the conserved *total 4-momentum* of the extended body.

We can define the *relativistic total angular momentum* of an extended body *relative to the event* x_0, by the following equations:

$$\begin{aligned} J^{ik} &:= \int\limits_{\partial D_{(t)}} \big[(x^i - x_0^i)\, T^{k4}(x^1, x^2, x^3, t) \\ &\qquad - (x^k - x_0^k)\, T^{i4}(x^1, x^2, x^3, t)\big] dx^1 dx^2 dx^3, \\ J^{ki} &= -J^{ik} = \text{const.} \end{aligned} \tag{2.50}$$

The proof for the constancies of J^{ik} will be left as Problem #7 in Exercise 2.1.

Example 2.1.11. Consider the case of *the incoherent dust* (pressureless fluid) which happens to be one of the simplest of all materials. The definition is summarized in the energy–momentum–stress tensor

$$T^{ij}(x) := \rho(x)U^i(x)U^j(x). \tag{2.51}$$

Equation (2.47) yields

$$\partial_j \left[\rho U^j\right] = 0 \tag{2.52}$$

and is known as the *relativistic continuity equation* for the dust.

The equations for streamline motion (2.44) reduce to

$$\begin{aligned} &\rho(x)\left[U^j(x)\partial_j U^i\right] = 0, \\ \text{or,} \quad &\left[U^j(x)\partial_j U^i\right]_{|x=\mathcal{X}^{\#}(s)} = \frac{\mathrm{d}^2\mathcal{X}^{\#i}(s)}{\mathrm{d}s^2} = 0. \end{aligned} \tag{2.53}$$

Therefore, it is evident that the dust particles, in the absence of external forces, move along timelike straight lines (or geodesics) in the flat space–time. (It is an expected result, as the absence of pressure and shears implies that adjacent elements of the dust cannot exert any forces on each other.) □

The special theory of relativity emerged from investigations of the speed of light, which is also the speed of electromagnetic wave propagation. Therefore, the electromagnetic field equations are of particular interest in special relativity theory.

The classical electromagnetic field is governed by *Maxwell's equations*.[3] As three-dimensional vector field equations, they are (in the absence of sources):

$$\nabla \times \vec{\mathbf{H}}(\mathbf{x}, t) = \frac{\partial \vec{\mathbf{E}}(\mathbf{x}, t)}{\partial t}, \tag{2.54i}$$

$$\nabla \cdot \vec{\mathbf{E}}(\cdot\cdot) = 0, \tag{2.54ii}$$

$$\nabla \cdot \vec{\mathbf{H}}(\cdot\cdot) = 0, \tag{2.54iii}$$

$$\nabla \times \vec{\mathbf{E}}(\cdot\cdot) = -\frac{\partial \vec{\mathbf{H}}(\cdot\cdot)}{\partial t}. \tag{2.54iv}$$

Here, t is the time variable, $\mathbf{x} = (x^1, x^2, x^3)$ represents Cartesian coordinates, and ∇ is the (spatial) gradient operator. Moreover, $\vec{\mathbf{E}}(\cdot\cdot)$ and $\vec{\mathbf{H}}(\cdot\cdot)$ are electric and magnetic fields, respectively. The above equations can be cast neatly into Minkowskian tensor

[3] We are using *Lorentz–Heaviside units*. Especially, we have set c, the speed of light, to be *unity* to be consistent with units used throughout most of the book.

field equations. To achieve this, we have to identify the electric and magnetic three-vectors with an antisymmetric, second-order Minkowskian tensor as shown below:

$$\left[F_{ij}(x)\right]_{4\times 4} := \begin{bmatrix} 0 & H_3(x) & -H_2(x) & E_1(x) \\ -H_3(x) & 0 & H_1(x) & E_2(x) \\ H_2(x) & -H_1(x) & 0 & E_3(x) \\ -E_1(x) & -E_2(x) & -E_3(x) & 0 \end{bmatrix}$$

$$\equiv -\left[F_{ji}(x)\right]. \tag{2.55}$$

Maxwell's equations (2.54i–iv) are (exactly) equivalent to the four-dimensional tensor field equations:

$$\partial_j F^{ij} = 0, \tag{2.56i}$$

$$\text{and} \quad \partial_k F_{ij} + \partial_i F_{jk} + \partial_j F_{ki} = 0. \tag{2.56ii}$$

Thus, Maxwell's equations (2.54i–iv), which were discovered approximately 40 years before the advent of the special theory of relativity, were *already relativistic!*

Example 2.1.12. We assume here the tensor field $\mathbf{F}_{..}(x)$ is of class C^2. Differentiating the mixed tensor form of (2.56ii), we obtain

$$\begin{aligned} 0 &= \partial_i \partial_k F^i_{\ j} + d^{il} \partial_i \partial_l F_{jk} - \partial_i \partial_j F^i_{\ k} \\ &= \partial_k \left[\partial_i F^i_{\ j}\right] + \Box F_{jk} - \partial_j \left[\partial_i F^i_{\ k}\right] \\ &= 0 + \Box F_{jk} - 0 \\ &= \left[\frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2} + \frac{\partial^2}{(\partial x^3)^2} - \frac{\partial^2}{(\partial x^4)^2}\right] F_{jk}(x). \end{aligned}$$

Note that these equations are wave equations with a speed of unity. Therefore, the above equations show that electromagnetic wave fields travel with the speed of light. The wave operator $\Box$ remains invariant under Poincaré transformations (2.6i), particularly under the Lorentz transformation (2.7). Therefore, *the speed of light (or a photon) remains unchanged for a moving inertial observer.* □

Equations (2.56i,ii) can be expressed succinctly in terms of differential 2-forms as introduced in (1.64). Let us repeat the 2-form here as

$$\mathbf{F}_{..}(x) = (1/2) F_{ij}(x)\, \mathrm{d}x^i \wedge \mathrm{d}x^j.$$

By (1.69), we can express Maxwell's equations (2.56ii) as

$$\mathrm{d}\,[\mathbf{F}_{..}(x)] = \mathbf{0}_{...}(x). \tag{2.57}$$

(Glance through (1.70) for the electromagnetic 4-potential $A_i(x)\,\mathrm{d}x^i$ and for gauge transformations.) We repeat the Hodge-star operation in (1.113). (See also Example 1.3.6.) Thus, we express

$$*\,\mathbf{F}_{..}(x) = (1/2)\left[\eta^{ij}{}_{kl}\,F_{ij}(x)\right]\mathrm{d}x^k \wedge \mathrm{d}x^l. \tag{2.58}$$

Therefore, Maxwell's equations (2.56i), by the use of Example 1.3.6, reduce to

$$\mathrm{d}\,[*\mathbf{F}_{..}(x)] = *\mathbf{0}_{...}. \tag{2.59}$$

In the presence of electrically charged matter, the *Maxwell–Lorentz equations* (using Lorentz–Heaviside units) are

$$\partial_k F^{ik} = J^i(x), \tag{2.60i}$$

$$\partial_k F_{ij} + \partial_i F_{jk} + \partial_j F_{ki} = 0, \tag{2.60ii}$$

$$\partial_i J^i = \partial_i \partial_k F^{ik} \equiv 0. \tag{2.60iii}$$

In physical terms, $J^4(x)$ and $J^\alpha(x)$ represent the electrical charge density and current density, respectively.

We introduce the Hodge-star operation (1.113) on the 4-*charge-current* vector $J^i(x)$ to express

$$*\,\mathbf{J}_{...}(x) = \left[\eta^{l}{}_{ijk}\,J_l(x)\right]\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k. \tag{2.61}$$

Therefore, the electromagnetic equations (2.60i–iii) reduce to

$$\mathrm{d}\,[*\mathbf{F}_{..}(x)] = \;*\,\mathbf{J}_{...}(x), \tag{2.62i}$$

$$\mathrm{d}\,[\mathbf{F}_{..}(x)] = \mathbf{0}_{...}(x), \tag{2.62ii}$$

$$\mathrm{d}\,[*\mathbf{J}_{...}(x)] = \mathrm{d}^2\,[*\mathbf{F}_{..}(x)] \equiv *\mathbf{0}_{....}(x). \tag{2.62iii}$$

(The Poincaré lemma in (1.66) is evident for (2.62iii).)

Now, we shall introduce the electromagnetic energy–momentum–stress tensor as the following:

$$\begin{aligned} {}_{\text{(em)}}T^{ij}(x) &:= F^{ik}(x)F^{j}{}_{k}(x) - (1/4)d^{ij}\,F_{kl}(x)F^{kl}(x) \\ &\equiv {}_{\text{(em)}}T^{ji}(x). \end{aligned} \tag{2.63}$$

Example 2.1.13. It will be instructive to work out the divergence $\partial_j \left[{}_{(\mathrm{em})}T^{ij}\right]$, which represents the force density experienced by the charged material due to the electromagnetic field. By (2.63), (2.60i), and (2.60ii), we derive that

$$\begin{aligned}
\partial_j\, {}_{(\mathrm{em})}T^{ij} &= F^{ik}(x)\partial_j F^{j}_{\ k} + (1/2)\left[\partial_j F^{ik} \cdot F^{j}_{\ k}(x) + \partial_j F^{ik} \cdot F^{j}_{\ k}(x)\right] \\
&\quad -(1/2) \cdot d^{ij} \cdot \partial_j F_{kl} \cdot F^{kl}(x) \\
&= -F^{ik}(x)\partial_j\ F_k^{\ j} + (1/2)\, d^{il} F^{jk}(x) \left[\partial_j F_{lk} + \partial_k F_{jl} + \partial_l F_{kj}\right] \\
&= -F^{ik}(x) J_k(x) + 0. \qquad \square
\end{aligned}$$

Now, we shall explore properties of a *charged dust* with assumptions

$$J^i(x) := \sigma(x) U^i(x), \tag{2.64i}$$

$$T^{ij}(x) = \rho(x) U^i(x) U^j(x) + {}_{(\mathrm{em})}T^{ij}(x). \tag{2.64ii}$$

Here, $\sigma(x)$ is the (proper) *electrical charge density* and the 4-velocity field components $U^i(x)$ satisfy (2.42). A pertinent consequence for a charged dust model will be furnished now.

Theorem 2.1.14. *Let the functions $\rho(x)$, $\sigma(x)$, and $U^i(x)$ be of class C^1 and $F_{ij}(x)$ be of class C^2 in a domain $D \subset \mathbb{R}^4$. Then, the differential conservation equations, $\partial_j T^{ij} = 0$, imply the equations:*

$$\rho(x) U^j(x) \partial_j U^i = \sigma(x) F^{ik}(x) U_k(x). \tag{2.65}$$

Proof. Using (2.64ii), (2.60i,ii), (2.42), and Example 2.1.13, we obtain that

$$\begin{aligned}
0 = \partial_j T^{ij} &= \partial_j \left[\rho U^i U^j\right] + \partial_j \left[{}_{(\mathrm{em})}T^{ij}\right] \\
&= \rho(x) U^j(x) \partial_j U^i + U^i(x) \partial_j \left[\rho U^j\right] - \sigma(x) F^{ik}(x) U_k(x).
\end{aligned}$$

Multiplying the above by $U_i(x)$ and using $U_i(x)U^i(x) \equiv -1$, $F^{ik}(x)U_i(x)U_k(x) \equiv 0$, we deduce the continuity equation

$$\partial_j \left[\rho U^j\right] = 0.$$

Therefore, (2.65) follows. ■

Restricting on a charged stream line, we derive the equations of motion as

$$\rho\left(\mathcal{X}^{\#}(s)\right) \frac{\mathrm{d}^2 \mathcal{X}^{\# i}(s)}{\mathrm{d}s^2} = \sigma\left(\mathcal{X}^{\#}(s)\right) \left[F^{i}_{\ k}(x)\right]_{|..} \frac{\mathrm{d}\mathcal{X}^{\# k}(s)}{\mathrm{d}s}. \tag{2.66}$$

For comparison, we cite here that the *relativistic Lorentz equations of motion* of a charged particle of mass m and charge e are known to be

$$m\frac{\mathrm{d}^2\mathcal{X}^{\#i}(s)}{\mathrm{d}s^2} = e\left[F^i_{\ k}(x)\right]_{|..}\frac{\mathrm{d}\mathcal{X}^{\#k}(s)}{\mathrm{d}s}. \tag{2.67}$$

The similarity between (2.66) and (2.67) is unmistakable.

Example 2.1.15. The three spatial components of the equations of motion (2.67) yield, by (2.24):

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{V^\alpha(t)}{\sqrt{1-\parallel \vec{\mathbf{V}}(t)\parallel^2}}\right] = e\delta^{\alpha\beta}\left[F_{\beta\gamma}(\mathbf{x},t)_{|\mathcal{X}^\alpha(t)}V^\gamma(t) + F_{\beta 4}(\cdot\cdot)_{|..}\right].$$

Using (2.54i–iv) and the antisymmetric permutation symbol in (1.54), the preceding equations imply that

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{V^\alpha(t)}{\sqrt{1-\parallel \vec{\mathbf{V}}(t)\parallel^2}}\right] = e\left[E^\alpha(\mathbf{x},t)_{|..} + \delta^{\alpha\beta}\varepsilon_{\beta\gamma\mu}V^\gamma(t)H^\mu(\cdot\cdot)_{|..}\right]. \tag{2.68}$$

Similarly, the fourth equation in (2.67) yields

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{\sqrt{1-\parallel \vec{\mathbf{V}}(t)\parallel^2}}\right] = e\delta_{\alpha\beta}V^\alpha(t)E^\beta(\cdot\cdot)_{|..}. \tag{2.69}$$

The right-hand side of the above equation represents the rate of work done by the external electric field. *The rate of work done by the magnetic field is exactly zero.* □

We shall now summarize the main theoretical accomplishments of the special theory of relativity in the following:

1. The special theory of relativity explains the puzzle about the speed of light arising out of the Michelson-Morley experiments [181], which indicated that the speed of light is invariant. As a consequence, it is established that the maximal speed of propagation of physical actions is the speed of light.
2. There is *an equivalence of all inertial observers or frames* in regards to the formulation of natural laws.
3. Length contraction of a moving rod and time dilation of a moving clock are predicted (see [55, 89, 230]).
4. The rest mass $m > 0$ of a body possesses potential energy $E = mc^2$. This enormously potent equation was first discovered by Einstein [87].

In Table 2.1, we elaborate on some aesthetic aspects of the special theory of relativity in regards to the unification of distinct physical entities occurring in the nonrelativistic regime.

Table 2.1 Correspondence between relativistic and non-relativistic physical quantities

Physical entities in nonrelativistic physics	Corresponding unification in special relativity
Absolute space $\mathbb{E}_3$ and absolute time $\mathbb{R}$	Flat space–time manifold M_4
3-momentum p_α and energy E	4-momentum p_i with $p_i p^i = -m^2$
For electromagnetic wave, 3-wave numbers k_α and the frequency ν	4-wave numbers k_i with $k_i k^i = 0$
Electric field vector $\vec{\mathbf{E}}$ and magnetic field vector $\vec{\mathbf{H}}$	Second-order, antisymmetric electromagnetic field tensor F_{ij}
For a fluid or a deformable body: energy density ρ, momentum density ρV^α, and symmetric stress tensor $\sigma^{\alpha\beta}$	Symmetric energy–momentum–stress tensor $T^{ij} := \rho U^i U^j - s^{ij}$

So far, we have used only pseudo-Cartesian or Minkowskian coordinate charts for the flat space–time manifold M_4. Sometimes, more general coordinate charts are necessary. For example, to derive the energy levels of a hydrogen atom, the relativistic Dirac equation is investigated in the spherical polar coordinates for spatial dimensions. Let us go back to the equations pertaining to a general coordinate chart in (1.1) and Figs. 1.1 and 1.2. We shall apply such mappings to the flat space–time manifold M_4. Consider the mappings $\widehat{\chi} : \widehat{U} \to \mathbb{R}^4$ and $\widehat{\chi}^{-1} : \widehat{D} \to M_4$. If we restrict the mapping $\widehat{\chi}^{-1}$ to the two-dimensional domain $\widehat{D}_{(2)} := \{\widehat{x} \in \widehat{D} \subset \mathbb{R}^4 : \widehat{x}^1 = \widehat{x}^2 = 0\}$, then the corresponding restricted mapping $\widehat{\chi}^{-1}{}_{|\widehat{D}_2}$ can be qualitatively exhibited as in Fig. 2.8.

By (1.1) and (1.37), we conclude that

$$ {}_2\widehat{\mathbf{X}}'\,[\mathbf{g}_{..}(x)] = \widehat{\mathbf{g}}_{..}(\widehat{x}) = \left[\frac{\partial X^k(\widehat{x})}{\partial \widehat{x}^i}\,\frac{\partial X^l(\widehat{x})}{\partial x^j}\,d_{kl}\right] \mathrm{d}\widehat{x}^i \otimes \mathrm{d}\widehat{x}^j, \tag{2.70i} $$

$$ \widehat{\mathbf{e}}^{(a)}(\widehat{x}) = \delta^{(a)}{}_k\,\frac{\partial X^k(\widehat{x})}{\partial \widehat{x}^i}\,\mathrm{d}\widehat{x}^i, \tag{2.70ii} $$

$$ \widehat{\left\{\begin{matrix} i \\ j\ k \end{matrix}\right\}} \not\equiv 0, \tag{2.70iii} $$

$$ \widehat{\gamma}_{(a)(b)(c)}(\widehat{x}) \not\equiv 0. \tag{2.70iv} $$

Equations (1.163), (1.161), and (2.70i–iv) imply that

$$ \widehat{\mathbf{R}}^{\cdot}{}_{\dots}(\widehat{x}) \equiv \widehat{\mathbf{0}}^{\cdot}{}_{\dots}(\widehat{x}). \tag{2.71} $$

The vanishing of the curvature tensor in the domain $\widehat{D} \subset \mathbb{R}^4$ indicates *the flatness of the domain* $\widehat{U} \subset M_4$ *in the curvilinear coordinate chart* $(\widehat{\chi}, \widehat{U})$ (in spite of $\widehat{\left\{\begin{matrix} i \\ j\ k \end{matrix}\right\}} \not\equiv 0$ and $\widehat{\gamma}_{(a)(b)(c)}(\widehat{x}) \not\equiv 0$).

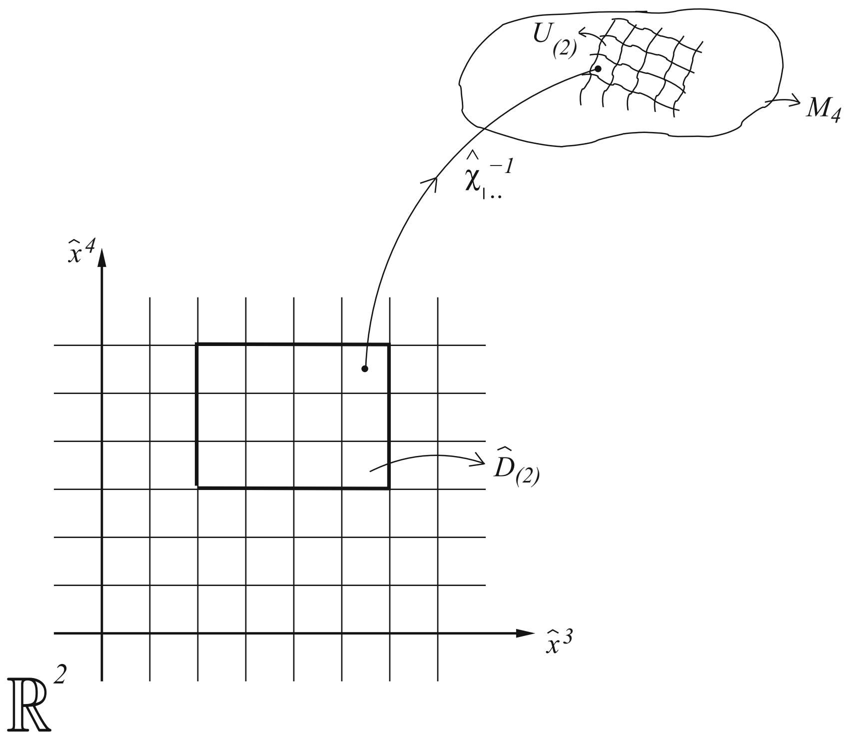


Fig. 2.8 Mapping of a rectangular coordinate grid into a curvilinear grid in the space–time manifold

Example 2.1.16. Consider the flat space–time manifold and a global Minkowskian chart (χ, M_4) given by

$$x \equiv \left(x^1, x^2, x^3, x^4\right) = \chi(p) \in D = \mathbb{R}^4.$$

Consider the coordinate transformation to spherical polar and time coordinates which is furnished by

$$\begin{aligned}
\widehat{x}^1 &= \widehat{X}^1(x) := +\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \equiv r, \\
\widehat{x}^2 &= \widehat{X}^2(x) := \text{Arccos}\left[x^3\big/\sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\right] \equiv \theta, \\
\widehat{x}^3 &= \widehat{X}^3(x) := \text{arc}(x^1, x^2) \equiv \varphi, \\
\widehat{x}^4 &= x^4 \equiv t, \\
\widehat{D}_s = \widehat{D} &:= \left\{\widehat{x} \in \mathbb{R}^4 : \widehat{x}^1 > 0,\ 0 < \widehat{x}^2 < \pi,\ -\pi < \widehat{x}^3 < \pi,\ -\infty < \widehat{x}^4 < \infty\right\}.
\end{aligned}$$

(See Problem # 1 in Exercise 1.1.) It can be noted that the above chart is *not global*.

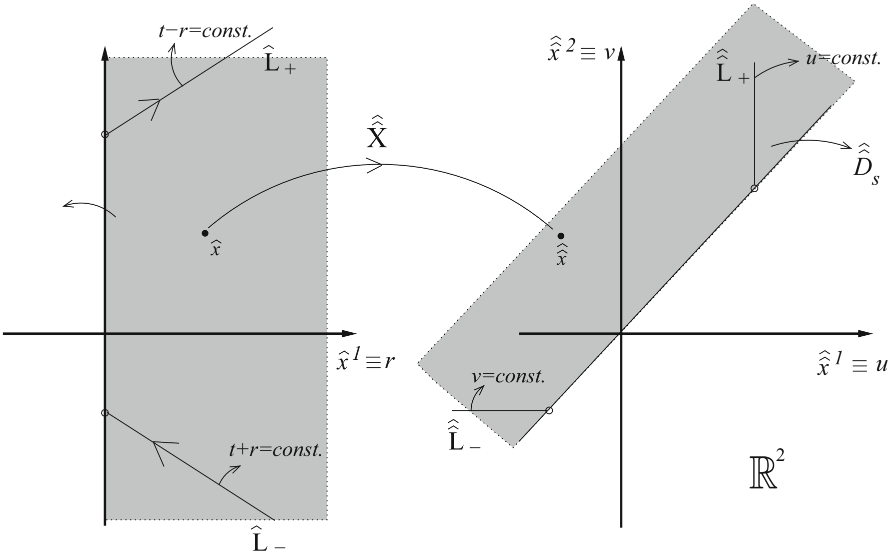


Fig. 2.9 A coordinate transformation mapping half lines $\hat{L}_+$ and $\hat{L}_-$ into half lines $\hat{\hat{L}}_+$ and $\hat{\hat{L}}_-$

The metric tensor in (2.70i) reduces to the orthogonal form

$$
\begin{aligned}
{}_2\widehat{\mathbf{X}}'\,[\mathbf{g}_{..}(x)] = \widehat{\mathbf{g}}_{..}(\widehat{x}) &= d\widehat{x}^1 \otimes d\widehat{x}^1 + (\widehat{x}^1)^2 d\widehat{x}^2 \otimes d\widehat{x}^2 + (\widehat{x}^1 \sin\widehat{x}^2)^2 d\widehat{x}^3 \otimes d\widehat{x}^3 \\
&\quad - d\widehat{x}^4 \otimes d\widehat{x}^4, \\
ds^2 &= (d\widehat{x}^1)^2 + (\widehat{x}^1)^2\left[(d\widehat{x}^2)^2 + (\sin\widehat{x}^2)^2 (d\widehat{x}^3)^2\right] - (d\widehat{x}^4)^2 \\
&\equiv dr^2 + r^2\left[(d\theta)^2 + \sin^2\theta\,(d\varphi^2)\right] - dt^2.
\end{aligned}
$$

(See (1.195).) □

Example 2.1.17. Let us make another coordinate transformation from the preceding chart in Example 2.1.16. Let it be furnished by equations

$$
\begin{aligned}
\widehat{\widehat{x}}^1 &= \widehat{x}^4 - \widehat{x}^1 \equiv u, \\
\widehat{\widehat{x}}^2 &= \widehat{x}^4 + \widehat{x}^1 \equiv v, \\
\widehat{\widehat{x}}^3 &= \widehat{x}^2 \equiv \theta, \\
\widehat{\widehat{x}}^4 &= \widehat{x}^3 \equiv \varphi, \\
\widehat{\widehat{D}}_s &:= \left\{\widehat{\widehat{x}} \in \mathbb{R}^4 : -\infty < \widehat{\widehat{x}}^2 < \infty,\ 0 < \widehat{\widehat{x}}^2 - \widehat{\widehat{x}}^1,\ 0 < \widehat{\widehat{x}}^3 < \pi,\ -\pi < \widehat{\widehat{x}}^4 < \pi\right\}.
\end{aligned}
$$

See Fig. 2.9 depicting the transformation (*suppressing angular coordinates*).

The oriented half lines $\widehat{L}_+$ and $\widehat{L}_-$ represent possible trajectories of outgoing and incoming photons for an observer pursuing the $\widehat{x}^4$-coordinate line, respectively. Since the null lines (or photon paths) are expressed by $u = \text{const.}$ and $v = \text{const.}$, these pair of coordinates are called *double-null coordinates*, as u and v coordinate lines are coincident with the paths of null (light) rays.

The metric in the preceding Example 2.1.17 is transformed into

$$\begin{aligned} {}_2\widehat{\widehat{\mathbf{X}}}'\left[\widehat{\mathbf{g}}_{..}(\widehat{x})\right] = \widehat{\widehat{\mathbf{g}}}_{..}\left(\widehat{\widehat{x}}\right) &= -\mathrm{d}\widehat{\widehat{x}}^1 \otimes \mathrm{d}\widehat{\widehat{x}}^2 + (1/4)\left[\widehat{\widehat{x}}^2 - \widehat{\widehat{x}}^1\right]^2 \\ &\quad \times \left[\mathrm{d}\widehat{\widehat{x}}^3 \otimes \mathrm{d}\widehat{\widehat{x}}^3 + \left(\sin\widehat{\widehat{x}}^3\right)^2 \mathrm{d}\widehat{\widehat{x}}^4 \otimes \mathrm{d}\widehat{\widehat{x}}^4\right], \\ \mathrm{d}s^2 &= -\mathrm{d}u\,\mathrm{d}v + (1/4)(v-u)^2\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right]. \end{aligned}$$

Consider a three-dimensional *null hypersurface* specified by

$$\mathcal{N}_3 := \left\{\widehat{\widehat{x}} \in \mathbb{R}^4 : \widehat{\widehat{x}}^2 = k,\ 0 < k - \widehat{\widehat{x}}^1 < \infty,\ 0 < \widehat{\widehat{x}}^3 < \pi,\ -\pi < \widehat{\widehat{x}}^4 < \pi\right\}.$$

The induced metric on $\mathcal{N}_3$, by (1.224), is furnished as

$$\widehat{\widehat{\mathbf{g}}}_{..}(\widehat{\widehat{x}})_{|\mathcal{N}_3} = (1/4)\left(k - \widehat{\widehat{x}}^1\right)^2 \left[\mathrm{d}\widehat{\widehat{x}}^3 \otimes \mathrm{d}\widehat{\widehat{x}}^3 + \left(\sin\widehat{\widehat{x}}^3\right)^2 \mathrm{d}\widehat{\widehat{x}}^4 \otimes \mathrm{d}\widehat{\widehat{x}}^4\right],$$

$$\mathrm{d}s^2_{|..} = (1/4)(k-u)^2\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right].$$

(Although (1.223) was pursued later for nonnull hypersurfaces, it is still valid for a null hypersurface.) The preceding equation yields *a two-dimensional metric for a three-dimensional hypersurface* $\mathcal{N}_3$! There is a loss of one dimension because a null coordinate line has a zero separation. □

A tensor field equation

$${}^r_s\mathbf{T}(x) = {}^r_s\mathbf{0}(x), \quad x \in D_{(e)} \subset \mathbb{R}^4$$

in a Minkowskian chart implies that

$${}^r_s\widehat{\mathbf{T}}(\widehat{x}) = {}^r_s\widehat{\mathbf{0}}(\widehat{x}), \quad \widehat{x} \in \widehat{D}_{(e)} \subset \widehat{D} \subset \mathbb{R}^4 \tag{2.72}$$

in a general coordinate chart and vice-versa. Under a successive general transformation, (2.72) yields an equivalent tensor field equation

$${}^r_s\widehat{\widehat{\mathbf{T}}}\left(\widehat{\widehat{x}}\right) = {}^r_s\widehat{\widehat{\mathbf{0}}}\left(\widehat{\widehat{x}}\right), \quad \widehat{\widehat{x}} \in \widehat{\widehat{D}}_{(e)} \subset \widehat{\widehat{D}}_s \subset \widehat{\widehat{D}} \subset \mathbb{R}^4. \tag{2.73}$$

The equivalence of tensor field equations (2.72) and (2.73) is called the *general covariance* of the tensor field equations. The special theory of relativity therefore, by demanding that physically relevant formulae be expressible as tensor equations, is covariant under the group of general coordinate transformations. This reflects the fact that physical effects must be independent of the coordinate system with which they are measured.

Let us cast some of the tensor field equations in a general coordinate chart characterized by (2.70i,ii). The metric in (2.70i) and the orthonormal 1-forms in (2.70ii) will yield Christoffel symbols and Ricci rotation coefficients which are *not necessarily zero*. We can define covariant derivatives from the definitions in (1.124i), (1.124ii), (1.134), and (1.139ii).

It is useful to transform some of the special relativistic equations we have studied into general coordinate systems. Equation (2.42) implies that

$$\begin{aligned}\widehat{g}_{ij}(\widehat{x})\,\widehat{U}^i(\widehat{x})\,\widehat{U}^j(\widehat{x}) &= d_{(a)(b)}\,\widehat{U}^{(a)}(\widehat{x})\,\widehat{U}^{(b)}(\widehat{x})\\ &\equiv -1.\end{aligned} \tag{2.74}$$

The equations of motion for stream lines in (2.44) go over into

$$\widehat{\rho}(\widehat{x})\left[\widehat{U}^j(\widehat{x})\widehat{\nabla}_j\widehat{U}^i\right] - \left[\delta^i{}_k + \widehat{U}^i(\widehat{x})\,\widehat{U}_k(\widehat{x})\right]\widehat{\nabla}_j\,\widehat{s}^{kj} = 0. \tag{2.75}$$

The energy–momentum–stress tensor in (2.45) and the differential conservation equation (2.46) transform into

$$\widehat{T}^{ij}(\widehat{x}) = \widehat{\rho}(\widehat{x})\,\widehat{U}^i(\widehat{x})\,\widehat{U}^j(\widehat{x}) - \widehat{s}^{ij}(\widehat{x}) \equiv \widehat{T}^{ji}(\widehat{x}), \tag{2.76i}$$

$$\widehat{T}^{(a)(b)}(\widehat{x}) = \widehat{\rho}(\widehat{x})\,\widehat{U}^{(a)}(\widehat{x})\,\widehat{U}^{(b)}(\widehat{x}) - \widehat{s}^{(a)(b)}(\widehat{x}) \equiv \widehat{T}^{(a)(b)}(\widehat{x}), \tag{2.76ii}$$

$$\widehat{\nabla}_j\,\widehat{T}^{ij} = 0, \tag{2.76iii}$$

$$\widehat{\nabla}_{(b)}\widehat{T}^{(b)(a)} = 0. \tag{2.76iv}$$

Finally, the Maxwell–Lorentz (electromagnetic) field equations (2.60i–iii) transform into

$$\widehat{\nabla}_k\,\widehat{F}^{ik} = \left(1/\sqrt{|\widehat{g}|}\right)\frac{\partial}{\partial\widehat{x}^k}\left[\sqrt{|\widehat{g}|}\,\widehat{F}^{ik}\right] = \widehat{J}^i(\widehat{x}), \tag{2.77i}$$

$$\widehat{\partial}_k\,\widehat{F}_{ij} + \widehat{\partial}_i\,\widehat{F}_{jk} + \widehat{\partial}_j\,\widehat{F}_{ki} = 0, \tag{2.77ii}$$

$$\widehat{\nabla}_i\,\widehat{J}^i = 0, \tag{2.77iii}$$

$$\widehat{\nabla}_{(c)}\,\widehat{F}^{(a)(c)} = \widehat{J}^{(a)}(\widehat{x}), \tag{2.78i}$$

$$\widehat{\nabla}_{(c)}\,\widehat{F}_{(a)(b)} + \widehat{\nabla}_{(a)}\,\widehat{F}_{(b)(c)} + \widehat{\nabla}_{(b)}\,\widehat{F}_{(c)(a)} = 0, \tag{2.78ii}$$

$$\widehat{\nabla}_{(a)}\,\widehat{J}^{(a)} = 0. \tag{2.78iii}$$

Remarks: (i) Equations (2.75), (2.76i–iv), (2.77i–iii), and (2.78i–iii) *all possess "general covariance" under the set of general coordinate transformations*.

(ii) *Special relativistic equations, expressed as tensor equations in general coordinates, are already "general relativistic"!* In the sequel, the space–time continuum will be treated as *a curved manifold*, so that $\widehat{\mathbf{R}}^{\cdot}_{\cdots}(\widehat{x}) \not\equiv \widehat{\mathbf{0}}^{\cdot}_{\cdots}(\widehat{x})$. Fortunately, (2.75), (2.76i–iv), (2.77i–iii), (2.78i–iii), and others, pass over smoothly into the arena of curved space–time.

Exercises 2.1

1. Determine whether or not any of the subset of vectors among $W_S(x_0)$, $W_T(x_0)$, or $W_N(x_0)$ defined in (2.2i–iii) is a vector subspace.
2. Let $\vec{\mathbf{t}}_{x_0}$ and $\widehat{\vec{\mathbf{t}}}_{x_0}$ be two future-pointing timelike vectors. Prove *the reversed Schwarz inequality*

$$\sigma\left(\vec{\mathbf{t}}_{x_0}\right) \cdot \sigma\left(\widehat{\vec{\mathbf{t}}}_{x_0}\right) \leq \left|\mathbf{g}_{..}(x_0)\left(\vec{\mathbf{t}}_{x_0}, \widehat{\vec{\mathbf{t}}}_{x_0}\right)\right|.$$

3. Define a 4×4 *Lorentz matrix* by $[L] := [l^i{}_j]$, where entries $l^i{}_j$ satisfy (2.66).
   (i) Prove that (2.66) can be expressed as the matrix equation

$$[L]^{\mathrm{T}}[D]\,[L] = [D]. \quad (\text{Here, } [D] := [d_{ij}].)$$

   (ii) Deduce that $\det[L] = \pm 1$.
   (iii) Prove that under the composition rule as the matrix multiplication, the set of all (4×4) Lorentz matrices constitutes *a group*.

Remark: This group, which is called *the Lorentz group*, is denoted by $O(3, 1; \mathbb{R})$.)

4. Consider a special relativistic tensor wave equation

$$\Box\phi_{l_1,\ldots,l_s} := d^{ij}\,\partial_i\,\partial_j\,\phi_{l_1,\ldots,l_s} = 0.$$

Let $2 \times (4)^s$ functions $f_{(l_1,\ldots,l_s)}$, $g_{(l_1,\ldots,l_s)} \in C^2(D_{(e)} \subset \mathbb{R}^4; \mathbb{R})$. These functions are *otherwise arbitrary*. Prove that

$$\begin{aligned}\phi_{l_1,\ldots,l_s}(x) := {} & f_{(l_1,\ldots,l_s)}\left(k_1x^1 + k_2x^2 + k_3x^3 - \nu(\mathbf{k})\right)\\ & + g_{(l_1,\ldots,l_s)}\left(k_1x^1 + k_2x^2 + k_3x^3 + \nu(\mathbf{k})\right),\end{aligned}$$

where $\nu(\mathbf{k}) := \sqrt{\delta^{\alpha\beta}k_\alpha k_\beta}$, solves the wave equation.

5. Consider the upper branch of the three-dimensional hyperhyperbola $\sum_3$ in Fig. 2.5. A parametric representation is furnished by

$$\begin{aligned}
u^1 &= \xi^1\left(w^1, w^2, w^3\right) := \sinh w^1 \cdot \sin w^2 \cdot \cos w^3, \\
u^2 &= \xi^2\left(w^1, w^2, w^3\right) := \sinh w^1 \cdot \sin w^2 \cdot \sin w^3, \\
u^3 &= \xi^3\left(w^1, w^2, w^3\right) := \sinh w^1 \cdot \cos w^2, \\
u^4 &= \cosh w^1, \\
\mathcal{D}_3 &:= \left\{\left(w^1, w^2, w^3\right) \in \mathbb{R}^3 : -\infty < w^1 < \infty,\ 0 < w^2 < \pi,\ -\pi < w^3 < \pi\right\}.
\end{aligned}$$

Prove that the hypersurface $\sum_3$ is a three-dimensional space of constant curvature.

6. Suppose that a particle with possibly variable mass (like a radioactive particle or exhaust-emitting rocket) is moving under an external force. The relativistic equations of motion (2.25) are generalized to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[M(s)\frac{\mathrm{d}\mathcal{X}^{\#i}(s)}{\mathrm{d}s}\right] = F^i(x, u)\Big|_{x=\mathcal{X}^{\#}(s),\, u=\frac{\mathrm{d}\mathcal{X}^{\#}(s)}{\mathrm{d}s}}, \qquad M(s) > 0.$$

   (i) Using the equations above, prove that $M(s)$ is constant-valued if and only if the orthogonality $d_{ij}\, F^i(\cdot\cdot)_{|..}\, \frac{\mathrm{d}\mathcal{X}^{\#j}(s)}{\mathrm{d}s} \equiv 0$ holds.

   (ii) Prove that the separation $\sqrt{d_{ij}\, \frac{\mathrm{d}^2\mathcal{X}^{\#i}(s)}{\mathrm{d}s^2}\, \frac{\mathrm{d}^2\mathcal{X}^{\#j}(s)}{\mathrm{d}s^2}}$ of the 4-acceleration vector $\frac{\mathrm{d}^2\mathcal{X}^{\#i}(s)}{\mathrm{d}s^2}\, \frac{\partial}{\partial x^i}_{|..}$ is constant-valued if and only if $[M(s)]^{-1}\, d_{ij}\, \frac{\mathrm{d}^2\mathcal{X}^{\#i}(s)}{\mathrm{d}s^2}\, F^j(\cdot\cdot)_{|..}$ is constant-valued.

7. Prove the integral conservation laws for the relativistic total angular momentum given in (2.50).

8. Consider the electrically charged dust model discussed in (2.64i,ii) and Theorem 2.1.14.

   (i) Assuming the junction conditions $\rho U^i n_{i\,|...} = 0 = \sigma U^i n_{i\,|...}$, prove the following integral conservation laws regarding the *total mass* and the *total charge*:

$$M := \int\limits_{D(t)} \rho\,(\mathbf{x}, t)\, U^4(\mathbf{x}, t)\, \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \text{const.},$$

$$Q := \int\limits_{D(t)} \sigma\,(\mathbf{x}, t)\, U^4(\mathbf{x}, t)\, \mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \text{const.}$$

(ii) Show the existence of two fixed spatial points $\mathbf{x}_{(1)}, \mathbf{x}_{(2)} \in D_{(t)}$ such that

$$\frac{Q}{M} = \frac{\sigma\,(\mathbf{x}_{(2)}, t)\, U^4(\mathbf{x}_{(2)}, t)}{\rho\,(\mathbf{x}_{(1)}, t)\, U^4(\mathbf{x}_{(1)}, t)} = \text{const.}$$

9. Consider a coordinate chart $(\widehat{\chi}, \widehat{U})$ for M_4 such that

$$\widehat{\mathbf{g}}_{..}(\widehat{x}) = \delta_{\alpha\beta}\, \mathrm{d}\hat{x}^{\alpha} \otimes \mathrm{d}\hat{x}^{\beta} - \left(\widehat{x}^1 + \widehat{x}^2 + \widehat{x}^3\right)^2 \mathrm{d}\widehat{x}^4 \otimes \mathrm{d}\widehat{x}^4,$$
$$\widehat{D} := \left\{\widehat{x} \in \mathbb{R}^4 : \widehat{x}^1 + \widehat{x}^2 + \widehat{x}^3 \geq 1,\ \widehat{x}^4 \in \mathbb{R}\right\}.$$

Prove by explicit computations that $\widehat{\mathbf{R}}^{\cdot}{}_{\dots}(\widehat{x}) \equiv \widehat{\mathbf{0}}^{\cdot}{}_{\dots}(\widehat{x})$ for $\widehat{x} \in \widehat{D} \subset \mathbb{R}^4$.

10. Consider the spherical polar coordinate chart dealt with in Example 2.1.16. In this chart, check Maxwell's equation (2.57) for the following two cases:

(i) The electromagnetic 2-form field for an electric dipole of strength $p_{(1)}$ given by

$$\widehat{\mathbf{F}}_{..}(\widehat{x}) = p_{(1)} \cdot \left[\frac{2\cos\widehat{x}^2}{(\widehat{x}^1)^3}\, \mathrm{d}\widehat{x}^1 \wedge \mathrm{d}\widehat{x}^4 + \frac{\sin\widehat{x}^2}{(\widehat{x}^1)^2}\, \mathrm{d}\widehat{x}^2 \wedge \mathrm{d}\widehat{x}^4\right]$$

(ii) The electromagnetic 2-form field for a vibrating electric dipole with frequency ν, furnished by

$$\begin{aligned}\widehat{\mathbf{F}}_{..}(\widehat{x}) = \text{Real part of }\ p_{(1)} \cdot \Bigg\{ & \mathrm{e}^{i\nu(\widehat{x}^1 - \widehat{x}^4)}\Bigg[2\cos\widehat{x}^2\left(\frac{1}{(\widehat{x}^1)^3} - \frac{i\nu}{(\widehat{x}^1)^2}\right)\mathrm{d}\widehat{x}^1 \wedge \mathrm{d}\widehat{x}^4 \\ & + \sin\widehat{x}^2\left(\frac{1}{(\widehat{x}^1)^2} - \frac{i\nu}{\widehat{x}^1} - \nu^2\right)\mathrm{d}\widehat{x}^2 \wedge \mathrm{d}\widehat{x}^4 \\ & - \sin\widehat{x}^2\left(\frac{i\nu}{\widehat{x}^1} + \nu^2\right)\mathrm{d}\widehat{x}^1 \wedge \mathrm{d}\widehat{x}^2\Bigg]\Bigg\}.\end{aligned}$$

11. Using a Minkowskian coordinate chart, solve for the Killing vector fields of flat M_4 with help of (1.171ii).

Answers and Hints to Selected Exercises

1. None of the subsets of $W_S(x_0)$, $W_T(x_0)$, or $W_N(x_0)$ is a vector subspace.
3. (ii) Taking determinants of both sides of the matrix equation, it can be shown that

$$\left\{\det[L]^{\mathrm{T}}\right\} \cdot (-1) \cdot \{\det[L]\} = -1,$$
$$\text{or}\quad \{\det[L]\}^2 = 1.$$

5. The intrinsic metric of the hypersurface $\sum_3$ is given by

$$\overline{\mathbf{g}}_{..}(w) = \mathrm{d}w^1 \otimes \mathrm{d}w^1 + (\sinh w^1)^2\, \mathrm{d}w^2 \otimes \mathrm{d}w^2 + \left[(\sinh w^1)(\sin w^2)\right]^2 \mathrm{d}w^3 \otimes \mathrm{d}w^3,$$
$$\overline{R}_{\alpha\beta\gamma\delta}(w) = (-1)\cdot\left[\overline{g}_{\alpha\gamma}(w)\cdot\overline{g}_{\beta\delta}(w) - \overline{g}_{\alpha\delta}(w)\cdot\overline{g}_{\beta\gamma}(w)\right].$$

6. (i) Multiplying equations of motion by $d_{ij}\,\frac{\mathrm{d}\mathcal{X}^{\#j}(s)}{\mathrm{d}s}$ and contracting, it can be proved that

$$-\frac{\mathrm{d}M(s)}{\mathrm{d}s} + M(s)\underbrace{\left[d_{ij}\frac{\mathrm{d}\mathcal{X}^{\#j}(s)}{\mathrm{d}s}\,\frac{\mathrm{d}^2\mathcal{X}^{\#i}(s)}{\mathrm{d}s^2}\right]}_{\equiv 0} = d_{ij}\,F^i(\cdot\cdot)_{|..}\,\frac{\mathrm{d}\mathcal{X}^{\#j}(s)}{\mathrm{d}s}.$$

7. Introduce a $(3+0)$th order tensor field by

$$\mathcal{T}^{ijk}(x) := \left(x^i - x_0^i\right)T^{jk}(x) - \left(x^j - x_0^j\right)T^{ik}(x).$$

Derive, using $T^{ji}(x) \equiv T^{ij}(x)$ and $\partial_j T^{ij} = 0$, that $\partial_k \mathcal{T}^{ijk} = 0$.

8. (ii) Use the mean value theorem of integrals. (See [32].)
9. Often, for such calculations, it is time-saving to use computational symbolic algebra programs as in Appendix 8.
10. (i)

$$\mathrm{d}\,\widehat{\mathbf{F}}_{..}(\widehat{x}) = p_{(1)}\cdot\left[-\frac{2\sin\widehat{x}^2}{(\widehat{x}^1)^3}\,\mathrm{d}\widehat{x}^2\wedge\mathrm{d}\widehat{x}^1\wedge\mathrm{d}\widehat{x}^4 - \frac{2\sin\widehat{x}^2}{(\widehat{x}^1)^3}\,\mathrm{d}\widehat{x}^1\wedge\mathrm{d}\widehat{x}^2\wedge\mathrm{d}\widehat{x}^4\right]$$
$$\equiv \widehat{\mathbf{0}}_{...}(\widehat{x}).$$

11. Ten Killing vector fields are:

$$\vec{\mathbf{K}}_{(r)}(x) := \delta^l_{\ (r)}\,\frac{\partial}{\partial x^l},$$
$$\vec{\mathbf{K}}_{(r)(s)}(x) := \left(d_{(r)i}\,\delta^j_{\ (s)} - d_{(s)i}\,\delta^j_{\ (r)}\right)x^i\,\frac{\partial}{\partial x^j} \equiv -\vec{\mathbf{K}}_{(s)(r)}(x),$$
$$r, s \in \{1, 2, 3, 4\}.$$

(*Remarks*: (i) Compare the above with the answers for Problem #7(ii) in Exercise 1.2. (ii) These Killing vectors are also called *generators for the Poincaré group* $\mathcal{I}O(3,1;\mathbb{R})$.)

2.2 Curved Space–Time and Gravitation

In the preceding section on special relativity theory, the set of all inertial observers (observers moving relative to each other with constant three-velocities) forms a privileged class of observers. In Minkowskian coordinate charts, natural laws are expressed by each of the inertial observers with *exactly similar* (Minkowskian) tensor field equations. Naturally, the next investigations focus on natural laws expressible by an observer moving with *constant three-acceleration* (i.e., $\frac{d^2\mathcal{X}^\alpha(t)}{dt^2} =$ const.) relative to an inertial observer. Our familiar experiences of objects moving with constant three-accelerations include apples (or other massive objects) falling under Earth's gravity. (We are neglecting air resistance due to Earth's atmosphere, and we are assuming the fall is taking place over distances where changes in the strength of the gravitational field can be neglected.)

Recall that Newton's equations of motion in an external gravitational field are provided by

$$m\frac{d^2\mathcal{X}^\alpha(t)}{dt^2} = f^\alpha(..)_{|..} =: -m\delta^{\alpha\beta}\frac{\partial W(\mathbf{x})}{\partial x^\beta}_{|x^\alpha=\mathcal{X}^\alpha(t)}, \tag{2.79i}$$

$$\text{or,} \quad \frac{d^2\mathcal{X}^\alpha(t)}{dt^2} = -\,\delta^{\alpha\beta}\partial_\beta W_{|..}. \tag{2.79ii}$$

(See (2.19i).) In (2.79i, ii), $W(\mathbf{x})$ represents the *Newtonian gravitational potential*. From (2.79ii), it is clear that the motion curve in a gravitational field is *independent of the mass $m > 0$ of the particle*.[4]

Very near the Earth's surface, the potential may be well approximated by $W(\mathbf{x}) = gx^3 +$ const., where the x^3-axis is coincident with the vertical direction. (Here, g denotes the constant gravitational acceleration *which must not be confused with* $\det\left[g_{ij}\right]$.) Integrating the equations of motion (2.79ii) in Earth's gravity yields

$$\begin{aligned}
\mathcal{X}^1(t) &= {}_{(i)}x^1 + {}_{(i)}v^1 t,\\
\mathcal{X}^2(t) &= {}_{(i)}x^2 + {}_{(i)}v^2 t,\\
\mathcal{X}^3(t) &= {}_{(i)}x^3 + {}_{(i)}v^3 t - \frac{1}{2}\, g\,(t)^2.
\end{aligned} \tag{2.80}$$

Here, ${}_{(i)}x^1$, ${}_{(i)}x^2$, ${}_{(i)}x^3$ represent the initial position of the particle whereas ${}_{(i)}v^1$, ${}_{(i)}v^2$, ${}_{(i)}v^3$ represent the initial three-velocity components. Let a second idealized

[4] The equivalency between *gravitational mass* and *inertial mass* is assumed. Current experimental limits place the equivalency of these two types of mass to within 10^{-12} [226], and it is widely believed that they are equivalent.

(point) observer start initially from $({}_{(0)}x^1, {}_{(0)}x^2, {}_{(0)}x^3)$ and with zero initial three-velocity. Her trajectory, falling freely under gravity, will be given by

$$\begin{aligned} \hat{\mathcal{X}}^1(t) &= {}_{(0)}x^1, \\ \hat{\mathcal{X}}^2(t) &= {}_{(0)}x^2, \\ \hat{\mathcal{X}}^3(t) &= {}_{(0)}x^3 - \frac{1}{2}g\,(t)^2. \end{aligned} \tag{2.81}$$

The relative three-velocity components, between two falling particles, are furnished by

$$\begin{aligned} \frac{\mathrm{d}\mathcal{X}^\alpha(t)}{\mathrm{d}t} - \frac{\mathrm{d}\hat{\mathcal{X}}^\alpha(t)}{\mathrm{d}t} &= {}_{(i)}v^\alpha = \text{const.}, \\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}\eta^\alpha(t) := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\mathcal{X}^\alpha(t) - \hat{\mathcal{X}}^\alpha(t)\right] &\equiv 0. \end{aligned} \tag{2.82}$$

Or, in other words, *nearby objects free-falling near the surface of the Earth move from each other with constant 3-velocities*. However, *the relative 3-acceleration* $\frac{\mathrm{d}^2\eta^\alpha(t)}{\mathrm{d}t^2} \equiv 0$.

Now, consider an idealized observer inside a freely falling elevator in Earth's gravitational field. Let him throw some massive particles inside the falling elevator. In free fall, he experiences no effects of gravity; moreover, he sees the particles move around him with constant 3-velocities (ignoring bounces off the walls). He can well conclude that gravitational forces have *disappeared*. Consider another example: Astronauts inside the international space station work under conditions similar to the observer in the elevator.[5] Therefore, these astronauts are experiencing conditions valid in the special theory of relativity in inertial frames. On the other hand, it is well known that in instants after a rocket launching, the astronauts inside the rocket experience *an apparent enhancing* of gravity. Thus, one may conclude that 3-accelerations of an observer in space can *apparently generate or annihilate gravitation* (at least in the local sense). This surprising fact is known as the *principle of equivalence*.

Let us investigate more critically the scenarios involving constant 3-accelerations on *a nonlocal scale*. Consider three massive point objects falling freely under the influence of gravity near the surface of the Earth where the acceleration is

[5] The apparent zero-gravity effects experienced by astronauts in orbit are exactly analogous to the elevator example. The "zero-gravity" effects are due to the fact that the astronauts and the orbiting vehicle are in free fall and *not* due to the fact that gravity is too weak to have any appreciable effects. A quick calculation, using (2.79ii) and the remark after Example 1.3.28, reveals that the gravitational acceleration at 320 km above the Earth's surface (low Earth orbit) is approximately 91% of its value at the surface of the Earth.

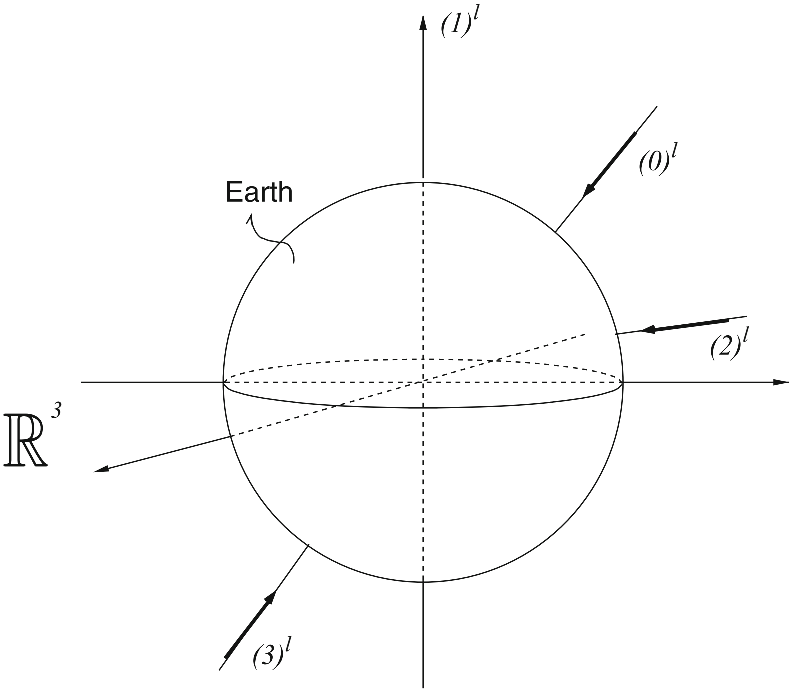


Fig. 2.10 Three massive particles falling freely in space under Earth's gravity

approximately constant (with no angular motion). The particles are initially located at three different locations. Each of the particles follows trajectories governed by (2.81). (See Fig. 2.10 where the 3-acceleration vectors are *highlighted*.)

It is obvious from Fig. 2.10 that the three 3-acceleration vectors have distinct directions. Therefore, the observers falling freely along the lines ${}_{(0)}l$, ${}_{(2)}l$ and ${}_{(3)}l$ possess nonzero relative 3-accelerations. Thus, gravitational effects *cannot be eliminated (or enhanced) nonlocally!* That is to say, there is no global accelerating reference frame which will yield apparent weightlessness for all observers. Let us quote Synge's comments on this topic [242]:

The principle of equivalence performed the essential office of midwife at the birth of general relativity, but, as Einstein remarked, the infant would never have got beyond its long-clothes had it not been for Minkowski's concept. I suggest that the midwife now be buried with appropriate honours and the facts of absolute space–time faced.

Let us go back to Fig. 2.10 and furnish a space and time version of the same phenomenon. (Recall that (2.81) yields a parabola in space and time.) Thus, we put forward Fig. 2.11, ignoring the third particle pursuing ${}_{(3)}l$. In Fig. 2.11, parabolic world lines ${}_{(0)}L$ and ${}_{(2)}L$ have constant 3-accelerations and *a nonzero relative 3-acceleration*. Moreover, we included a vertical world line ${}_{(e)}L := {}_{(1)}L$ representing a static idealized observer on the Earth's surface. However, each of the two

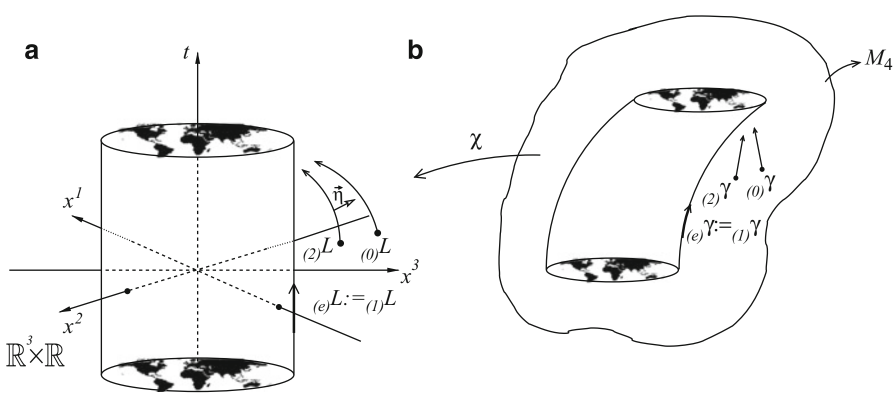


Fig. 2.11 (**a**) Space and time trajectories of two geodesic particles freely falling towards the Earth. (**b**) A similar figure but adapted to the geodesic motion of the two freely falling observers in curved space–time M_4

observers pursuing trajectories ${}_{(0)}L$ and ${}_{(2)}L$ experience no apparent gravitational force. Thus, they rightfully consider themselves as *inertial observers* following geodesic world lines. (Such world lines have zero 4-accelerations.) However, the Earth-bound observer, following the straight world line (in the sense depicted in the figure) ${}_{(e)}L$, experiences his own weight under gravity. Therefore, he concludes that he is *not an inertial observer* and is pursuing *a nongeodesic trajectory in space and time*. Now, in a Minkowskian coordinate chart of *a flat space–time*, neither of the parabolic curves ${}_{(0)}L$ and ${}_{(2)}L$ could be considered geodesics. As well, neither could the straight line ${}_{(e)}L$ be nongeodesic. Hence, we are confronted with a dilemma. The only logical way out is to recognize the Fig. 2.11a as the arena of *a coordinate chart for a curved pseudo-Riemannian space–time manifold*. (See the Fig. 2.11b where we *qualitatively represent* a similar diagram but with the two geodesic observers pursuing straight lines.) Furthermore, the fact that relative 4-acceleration components $\frac{D^2\eta^i(\tau)}{\mathrm{d}\tau^2}$ between two geodesics ${}_{(0)}L$ and ${}_{(2)}L$ *are not all zero*, implies, by the geodesic deviation equations (1.191), that $R^i{}_{ljk} \not\equiv 0$.

In Einstein's theory of gravitation, either there *is an intrinsic gravitational field or there is none*, according to whether the Riemann curvature tensor of space–time vanishes or not. This is an absolute property; *it has nothing to do with apparent gravitational effects* such as those due to nongeodesic motion of observers. In other words, both the astronauts in the international space station, experiencing apparent weightlessness, and Einstein, standing in the Prussian Academy of Sciences experiencing his weight, are immersed in a nonzero gravitational field since $R^i{}_{ljk} \neq 0$ in their vicinities. However, very far away from gravitating bodies, straight lines in a Minkowskian coordinate chart are geodesics, $R^i{}_{ljk} \to 0$, and gravitational effects are vanishingly small. So we conclude, symbolically speaking, *intrinsic gravitation* $\equiv$ *nonzero Riemann curvature tensor*.

It is well known that the Newtonian theory of gravitation explains usual phenomena involving terrestrial and planetary motions under gravitation. We naturally

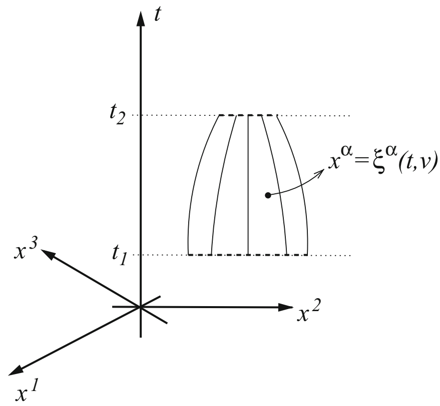


Fig. 2.12 Qualitative representation of a swarm of particles moving under the influence of a gravitational field

ask: "What is the significance of the nonzero curvature tensor of Einstein's theory of gravitation in terms of the Newtonian gravitational potential?" We answer this legitimate question in the following example.

Example 2.2.1. Consider a swarm of massive particles in a gravitational field following Newtonian equations of motion (2.79ii) and subject only to gravity. Let these particles span a very smooth world surface in space and time. Moreover, let this surface be parametrically represented by the equations

$$x^\alpha = \xi^\alpha(t, v),$$
$$t \in [t_1, t_2], \quad v \in [v_1, v_2],$$

(see Fig. 2.12).

We assume that the functions ξ^α are of class C^3 and that the potential W is of class C^2. By Newton's equations of motion (2.79ii), we have

$$\frac{\partial^2 \xi^\alpha(t,v)}{\partial t^2} + \delta^{\alpha\beta} \left.\frac{\partial W(\mathbf{x})}{\partial x^\beta}\right|_{x^\mu=\xi^\mu(\cdot\cdot)} = 0. \tag{2.83}$$

Differentiating the above equation with respect to v and commuting the order of differentiations, we obtain

$$\frac{\partial^3 \xi^\alpha(t,v)}{\partial t^2 \partial v} + \delta^{\alpha\beta} \left[\frac{\partial^2 W(\cdot\cdot)}{\partial x^\beta \partial x^\gamma}\right]_{|\cdot\cdot} \cdot \frac{\partial \xi^\gamma(t,v)}{\partial v} = 0. \tag{2.84}$$

Denoting *the relative separation components* by $\eta^\alpha(t,v) := \frac{\partial \xi^\alpha(t,v)}{\partial v}$, (2.84), goes over into

$$\frac{\partial^2 \eta^\alpha(t,v)}{\partial t^2} + \delta^{\alpha\beta} \left[\frac{\partial^2 W(\cdot\cdot)}{\partial x^\beta \partial x^\gamma}\right]_{|\cdot\cdot} \cdot \eta^\gamma(t,v) = 0. \tag{2.85}$$

The second term in the equation above gives rise to the usual *tidal forces* caused by gravitation. Now, going back to the pseudo-Riemannian space–time M_4, the spatial components of the geodesic deviation equation (1.192) yield

$$\frac{D^2\eta^\alpha(\tau,\nu)}{\partial\tau^2} + \left[R^\alpha{}_{ljk}(\cdot\cdot)\right]_{|\cdot\cdot} \cdot \frac{\partial\xi^l(\cdot\cdot)}{\partial\tau} \cdot \eta^j(\cdot\cdot) \cdot \frac{\partial\xi^k(\cdot\cdot)}{\partial\tau} = 0. \tag{2.86}$$

Comparing (2.85) and (2.86), we conclude that nonzero curvature components represent *relativistic tidal forces of the intrinsic gravitation*.

Now, let us explore another line of reasoning. In case tidal forces vanish, (2.84) implies that $\frac{\partial^2 W(\mathbf{x})}{\partial x^\beta \partial x^\gamma} = 0$. Solving the partial differential equations, we obtain the general solution as $W(\mathbf{x}) = c_{(0)} + \sum_{\mu=1}^{3} c_{(\mu)} x^\mu$. (Here, $c_{(0)}, c_{(\mu)}$'s are arbitrary constants of integration.) Thus, equations of motion (2.79ii) yields $\frac{\mathrm{d}^2\mathcal{X}^\alpha(t)}{\mathrm{d}t^2} = -\delta^{\alpha\beta} c_{(\beta)} = \text{const}$. Therefore, we are *back to the scenario of constant 3-acceleration!* By free fall, the apparent gravity can be eliminated. Thus, as expected, the intrinsic gravitation, equivalently the curvature tensor, must vanish. □

We shall now elaborate on the curved pseudo-Riemannian space–time M_4. (It is suggested to glance through the definition of a differentiable manifold in p. 1–3.) *The first assumption* on the pseudo-Riemannian manifold M_4 is that it is a connected, four-dimensional, Hausdorff, C^4-manifold with a Lorentz metric (of signature $+2$). The assumptions of Hausdorff topology and Lorentz metric imply that M_4 is paracompact. Thus, M_4 has a countable basis of open sets [150] providing a C^4-atlas.

In mathematical physics, the mathematical treatments should be rigorous and the mathematical symbols should be ultimately related to experimental observations. Consider an idealized event $p \in U \subset M_4$. The corresponding coordinates in the chart (χ, U) are provided by $x^i = \chi^i(p) := \left[\pi^i \circ \chi\right](p) = \pi^i(x) \in \mathbb{R}$ for $x \in D \subset \mathbb{R}^4$. We have to construct devices and methods to measure these four numbers x^i. For that purpose, we shall first investigate idealized histories of particles in $U \subset M_4$, equivalently in $D \subset \mathbb{R}^4$, by parametrized curves called world lines. In the sequel, we shall use arc separation parameters in (1.186) and (1.187) for timelike curves, which are furnished by $x^i = \mathcal{X}^i(s)$. (We *drop the symbol* #.) For a null curve, we express $x^i = \mathcal{X}^i(\alpha)$, where we use an affine parameter α. We assume the parametrized curve $\mathcal{X}$ to be of class C^3. Therefore, we can express from (1.186) that

$$\begin{aligned} g_{ij}\left(\mathcal{X}(s)\right) \frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s} \frac{\mathrm{d}\mathcal{X}^j(s)}{\mathrm{d}s} &\equiv -1, \\ g_{ij}\left(\mathcal{X}(\alpha)\right) \frac{\mathrm{d}\mathcal{X}^i(\alpha)}{\mathrm{d}\alpha} \frac{\mathrm{d}\mathcal{X}^j(\alpha)}{\mathrm{d}\alpha} &\equiv 0. \end{aligned} \tag{2.87}$$

As a consequence of the above equation, we derive that

$$g_{ij}\ (\mathcal{X}(s))\,\frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s}\left[\frac{\mathrm{d}^2\mathcal{X}^j(s)}{\mathrm{d}s^2}+\left\{\begin{matrix} j \\ k\ l\end{matrix}\right\}_{|..}\frac{\mathrm{d}\mathcal{X}^k(s)}{\mathrm{d}s}\,\frac{\mathrm{d}\mathcal{X}^l(s)}{\mathrm{d}s}\right]\equiv 0. \tag{2.88}$$

Therefore, *4-velocity is always orthogonal to 4-acceleration, whether or not the curve is a geodesic*.

Now, we investigate *the proper time and proper length* along a nonnull differentiable curve. By (1.186) and (2.17), the proper time along a timelike curve is furnished by

$$s=\mathcal{S}(t):=\int_{t_0}^{t}\sqrt{-g_{ij}\ (\mathcal{X}(u))\,\frac{\mathrm{d}\mathcal{X}^i(u)}{\mathrm{d}u}\,\frac{\mathrm{d}\mathcal{X}^j(u)}{\mathrm{d}u}}\,\mathrm{d}u. \tag{2.89}$$

(The proper time is also denoted by τ by some authors.)

The proper length along a spacelike curve is given by

$$s=\mathcal{S}(l):=\int_{l_0}^{l}\sqrt{g_{ij}\ (\mathcal{X}(w))\,\frac{\mathrm{d}\mathcal{X}^i(w)}{\mathrm{d}w}\,\frac{\mathrm{d}\mathcal{X}^j(w)}{\mathrm{d}w}}\,\mathrm{d}w. \tag{2.90}$$

(The arc separation along a null curve is exactly zero.)

Example 2.2.2. Consider the pseudo-Riemannian space–time manifold M_4 and an orthogonal coordinate chart (χ, U). Let the corresponding domain $D \subset \mathbb{R}^4$ contain the origin $(0,0,0,0)$. Moreover, let the metric tensor components satisfy $g_{11}(x)> 0,\ g_{22}(x) > 0,\ g_{33}(x) > 0$, and $g_{44}(x) < 0$. Then, the proper time along the x^4-axis is given by

$$\tau \equiv s=\int_{0}^{x^4}\sqrt{-g_{44}(0,0,0,t)}\,\mathrm{d}t. \tag{2.91}$$

The proper length along the x^1-axis is furnished by

$$l^1 \equiv s=\int_{0}^{x^1}\sqrt{g_{11}(w,0,0,0)}\,\mathrm{d}w. \tag{2.92}$$

Similar equations hold for the x^2-axis and x^3-axis.

The proper two-dimensional area of the coordinate rectangle $(0, x^1) \times (0, x^2)$ is given by

$$A := \int_0^{x^1}\int_0^{x^2} \sqrt{\gamma(u^1, u^2)}\, \mathrm{d}u^1\, \mathrm{d}u^2,$$

$$\gamma(u^1, u^2) := \det\begin{bmatrix} g_{11}(u^1, u^2, 0, 0) & g_{12}(u^1, u^2, 0, 0) \\ g_{12}(u^1, u^2, 0, 0) & g_{22}(u^1, u^2, 0, 0) \end{bmatrix}. \tag{2.93}$$

(We assume that $\gamma(u^1, u^2) > 0$.)

The proper three-dimensional volume of the "coordinate rectangular solid" $(0, x^1) \times (0, x^2) \times (0, x^3)$ on the spacelike three-dimensional hypersurface $x^4 = 0$ is furnished by

$$V := \int_0^{x^1}\int_0^{x^2}\int_0^{x^3} \sqrt{\overline{g}(u^1, u^2, u^3)}\, \mathrm{d}u^1\, \mathrm{d}u^2\, \mathrm{d}u^3,$$

$$\overline{g}(u^1, u^2, u^3) := \det\left[g_{\alpha\beta}(u^1, u^2, u^3, 0)\right]. \tag{2.94}$$

(We assume that $\overline{g}(u^1, u^2, u^3) > 0$.)

In a flat space–time, relative to a Minkowskian coordinate chart, the right-hand sides of (2.91)–(2.94) will reduce to x^4, x^1, $x^1 x^2$, and $x^1 x^2 x^3$, respectively. (We have tacitly assumed in this example that the matrix $[g_{ij}(x)]$ has four real eigenvalues $\lambda_{(1)}(x) > 0$, $\lambda_{(2)}(x) > 0$, $\lambda_{(3)}(x) > 0$, and $\lambda_{(4)}(x) < 0$.) □

Now, we discuss an idealized observer following a future-pointing timelike world line. He or she carries a regular (point) clock to measure the proper time along the world line. An orthonormal basis set or a tetrad is transported along with the observer. He or she may be an inertial or noninertial observer. The inertial observer obviously follows a *timelike geodesic* furnished by

$$\mathcal{U}^i(s) := \frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s}, \tag{2.95i}$$

$$\frac{D\mathcal{U}^i(s)}{\mathrm{d}s} = 0, \tag{2.95ii}$$

$$g_{ij}\,(\mathcal{X}(s))\,\mathcal{U}^i(s)\,\mathcal{U}^j(s) \equiv -1, \tag{2.95iii}$$

$$g_{ij}\,(\mathcal{X}(s))\,\mathcal{U}^i(s)\frac{D\mathcal{U}^j(s)}{\mathrm{d}s} \equiv 0. \tag{2.95iv}$$

Equation (2.95i) defines *the 4-velocity components* $\mathcal{U}^i(s)$. The geodesic equation (2.95ii) indicates that 4-velocity components $\mathcal{U}^i(s)$ undergo *parallel transportations*. (See (1.175) and (1.178i,ii).)

It is convenient to choose the timelike unit vector $\vec{\mathbf{e}}_{(4)}(\mathcal{X}(s))$ of the orthonormal tetrad as $e^i_{(4)}(\mathcal{X}(s)) = \mathcal{U}^i(s)$. The choice of the other three spacelike unit vectors of the tetrad is arbitrary up to a rotation or a reflection induced by an element of the orthogonal group $O(3, \mathbb{R})$. Let the choice of orthonormality be made at the initial event $\mathcal{X}(0)$. Therefore,

$$g_{ij}(\mathcal{X}(0))\, e^i_{(a)}(\mathcal{X}(0))\, e^j_{(b)}(\mathcal{X}(0)) = d_{(a)(b)}. \tag{2.96}$$

Assume now that each of the four 4-vectors is *transported parallelly* along the geodesic, that is,

$$\frac{De^i_{(a)}(\mathcal{X}(s))}{\mathrm{d}s} = 0. \tag{2.97}$$

By the Leibniz rule for the derivative $\frac{D}{\mathrm{d}s}$ in (1.175), we deduce that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{ij}(\mathcal{X}(s))\, e^i_{(a)}(\mathcal{X}(s))\, e^j_{(b)}(\mathcal{X}(s))\right] \\
&= \frac{D}{\mathrm{d}s}\left[g_{ij}(\cdot\cdot)\, e^i_{(a)}(\cdot\cdot)\, e^j_{(b)}(\cdot\cdot)\right] \\
&= g_{ij}(\cdot\cdot)\left\{\left[\frac{D}{\mathrm{d}s}\, e^i_{(a)}(\cdot\cdot)\right]\cdot e^j_{(b)}(\cdot\cdot) + e^i_{(a)}(\cdot\cdot)\cdot\left[\frac{D}{\mathrm{d}s}\, e^j_{(b)}(\cdot\cdot)\right]\right\} \\
&= 0,
\end{aligned}$$

or,

$$g_{ij}(\mathcal{X}(s))\, e^i_{(a)}(\mathcal{X}(s))\, e^j_{(b)}(\mathcal{X}(s)) = \text{const.}$$

By the initial conditions (2.96), the constants must be $d_{(a)(b)}$. Thus, the orthonormality of the tetrad is preserved under parallel transport along a geodesic. Such a tetrad is essential for the inertial observer to measure physical components like $T_{(a)(b)}(\mathcal{X}(s))$ and physical (or orthonormal) components of other tensor fields.

Now, let us investigate *a noninertial observer pursuing a nongeodesic timelike world line*. An obvious example is an idealized (point) scientist working in a fixed location of a laboratory on Earth's surface. What kind of orthonormal tetrad can this observer carry along their world line? We can explore the Frenet-Serret orthonormal tetrad introduced in (1.196). Recapitulating the formulas, we provide the following equations:

$$\begin{aligned}
\lambda^i_{(1)}(s) &:= \frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s} \equiv \mathcal{U}^i(s), \\
\frac{D\lambda^i_{(1)}(s)}{\mathrm{d}s} &= \kappa_{(1)}(s)\lambda^i_{(2)}(s),
\end{aligned}$$

$$\frac{D\lambda^{i}_{(2)}(s)}{\mathrm{d}s} = \kappa_{(2)}(s)\lambda^{i}_{(3)}(s) + \kappa_{(1)}(s)\lambda^{i}_{(1)}(s),$$
$$\frac{D\lambda^{i}_{(3)}(s)}{\mathrm{d}s} = \kappa_{(3)}(s)\lambda^{i}_{(4)}(s) - \kappa_{(2)}(s)\lambda^{i}_{(2)}(s),$$
$$\frac{D\lambda^{i}_{(4)}(s)}{\mathrm{d}s} = -\kappa_{(3)}(s)\lambda^{i}_{(3)}(s). \tag{2.98}$$

Here, $\kappa_{(1)}(s)$ is the *principal* or *first curvature*. Moreover, $\kappa_{(2)}(s)$ and $\kappa_{(3)}(s)$ are *the second and third curvature*, respectively. The spacelike unit vectors $\vec{\boldsymbol{\lambda}}_{(2)}(s), \vec{\boldsymbol{\lambda}}_{(3)}(s)$, and $\vec{\boldsymbol{\lambda}}_{(4)}(s)$ are *the first, second, and third normal* to the curve. For a nongeodesic curve, $\kappa_{(1)}(s) \neq 0$. Therefore, by (2.98), the orthonormal Frenet-Serret tetrad $\{\vec{\boldsymbol{\lambda}}_{(a)}(s)\}_1^4$ is *not parallelly transported*. Thus, we look for another mode of transportation of an orthonormal tetrad or frame along a *nongeodesic* curve (or an accelerating observer). We define the *Fermi derivative* of a vector field along a curve by the following string of equations:

$$\mathcal{U}^i(s) = \frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s},$$
$$\kappa_{(1)}(s)N^i(s) := \frac{D\mathcal{U}^i(s)}{\mathrm{d}s},$$
$$\frac{D_F\vec{\mathbf{V}}(\mathcal{X}(s))}{\mathrm{d}s} := \frac{D\vec{\mathbf{V}}(\mathcal{X}(s))}{\mathrm{d}s} - \kappa_{(1)}(s)\left[\mathbf{g}_{..}(\mathcal{X}(s))\left(\vec{\mathbf{N}},\vec{\mathbf{V}}\right)\right]\vec{\mathbf{U}}(s)$$
$$+\kappa_{(1)}(s)\left[\mathbf{g}_{..}(\mathcal{X}(s))\left(\vec{\mathbf{U}},\vec{\mathbf{V}}\right)\right]\vec{\mathbf{N}}(s). \tag{2.99}$$

(Consult [126, 243].)

The *Fermi-Walker transport* (*F–W transport* in short) is defined by

$$\frac{D_F\vec{\mathbf{V}}(\mathcal{X}(s))}{\mathrm{d}s} = \vec{\mathbf{O}}(\mathcal{X}(s))$$

or
$$\frac{DV^i(\mathcal{X}(s))}{\mathrm{d}s} = \kappa_{(1)}(s)\left[\mathcal{U}^i(s)N^j(s) - \mathcal{U}^j(s)N^i(s)\right]V_j(\mathcal{X}(s))$$
$$= \left[\mathcal{U}^i(s)\cdot\frac{D\mathcal{U}^j(s)}{\mathrm{d}s} - \mathcal{U}^j(s)\frac{D\mathcal{U}^i(s)}{\mathrm{d}s}\right]V_j(\mathcal{X}(s)). \tag{2.100}$$

An important property of F–W transport is discussed next.

Theorem 2.2.3. *Let $\mathcal{X}$ be a parametrized timelike curve of class C^2 with the image $\Gamma \subset D \subset \mathbb{R}^4$. Let a differentiable tetrad $\left\{\vec{\mathbf{e}}_{(a)}(\mathcal{X}(s))\right\}_1^4$ be orthonormal at the initial point $\mathcal{X}(0)$. Furthermore, let each of the vectors $\vec{\mathbf{e}}_{(a)}(\mathcal{X}(s))$ be subjected to F–W transport along the curve. Then, the orthonormality of the tetrad $\left\{\vec{\mathbf{e}}_{(a)}(\mathcal{X}(s))\right\}_1^4$ is preserved for all $s \in [0, s_1]$.*

Proof. By the assumptions made and (2.100), it follows that

$$g_{ij}\left(\mathcal{X}(0)\right) e^i_{(a)}\left(\mathcal{X}(0)\right) e^j_{(b)}\left(\mathcal{X}(0)\right) = d_{(a)(b)},$$
$$\frac{De^i_{(a)}(\mathcal{X}(s))}{\mathrm{d}s} = \kappa_{(1)}(s)\, g_{jk}(\cdot\cdot)\left[\mathcal{U}^i(s)N^j(s) - \mathcal{U}^j(s)N^i(s)\right] e^k_{(a)}\left(\mathcal{X}(s)\right). \tag{2.101}$$

Therefore, we obtain by (1.175) and the Leibniz property, that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left[g_{ij}\left(\mathcal{X}(s)\right) e^i_{(a)}\left(\mathcal{X}(s)\right) e^j_{(b)}\left(\mathcal{X}(s)\right)\right] \\
&= \frac{D}{ds}\left[g_{ij}(\cdot\cdot)\, e^i_{(a)}(\cdot\cdot)\, e^j_{(b)}(\cdot\cdot)\right] \\
&= g_{ik}(\cdot\cdot)\left\{\left[\frac{De^i_{(a)}(\cdot\cdot)}{\mathrm{d}s}\right]\cdot e^k_{(b)}(\cdot\cdot) + e^i_{(a)}(\cdot\cdot)\cdot\left[\frac{De^k_{(b)}(\cdot\cdot)}{\mathrm{d}s}\right]\right\} \\
&= \kappa_{(1)}(s)\, g_{ik}(\cdot\cdot)\, g_{jl}(\cdot\cdot)\left[e^k_{(b)}(\cdot\cdot)\, e^l_{(a)}(\cdot\cdot) + e^k_{(a)}(\cdot\cdot)\, e^l_{(b)}(\cdot\cdot)\right] \\
&\quad\times\left[\mathcal{U}^i(\cdot\cdot)N^j(\cdot\cdot) - \mathcal{U}^j(\cdot\cdot)N^i(\cdot\cdot)\right] \equiv 0.
\end{aligned}$$

The zero on the right-hand side resulted from the double contraction of a symmetric tensor with an antisymmetric one. Thus, we conclude that

$$g_{ij}\left(\mathcal{X}(s)\right) e^i_{(a)}\left(\mathcal{X}(s)\right) e^j_{(b)}\left(\mathcal{X}(s)\right) = \text{ const. } = d_{(a)(b)}. \qquad \blacksquare$$

It can be noted by (2.100), (2.95iv), and (2.98) that the tangent vector field $\mathcal{U}^i(s)$ undergoes F–W transport *automatically*. Moreover, in the limit $\kappa_{(1)}(s) \to 0$, the F–W transport $\to$ parallel transport. In Fig. 2.13, we compare and contrast parallel transport versus F–W transport.

In Fig. 2.13b, the tangent vector $\mathcal{U}^i(s) = e^i_{(4)}\left(\mathcal{X}(s)\right)$ *remains a tangent vector to the nongeodesic curve* under F–W transport. Thus, F–W transport provides a noninertial observer with an orthonormal tetrad. Furthermore, it also provides an orthonormal (spatial) triad orthogonal to the 4-velocity vector $\mathcal{U}^i(s)$. This triad forms a spatial frame of reference for the observer. It is this frame of reference which provides us a correct generalization of the Newtonian concept of *a nonrotating moving frame*.

Example 2.2.4. Consider a *conformally flat space–time* M_4. (Consult Theorem 1.3.33 and Appendix 4.) The metric is furnished by

$$\mathrm{d}s^2 = [\phi(x)]^2\, d_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j,$$
$$\phi \in C^3(D \subset \mathbb{R}^4; \mathbb{R}), \quad \phi(x) > 0, \quad \partial_4\phi \neq 0. \tag{2.102i}$$

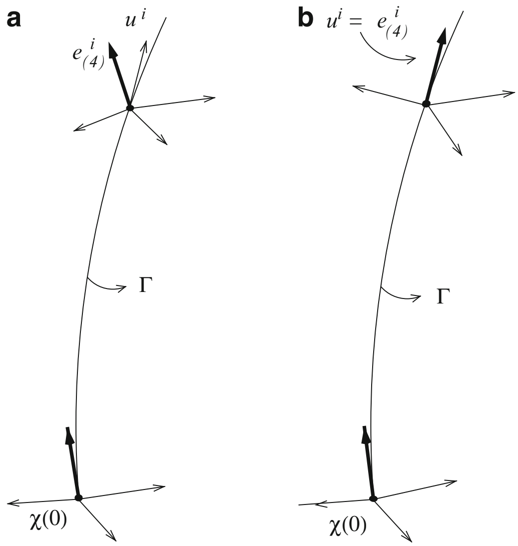


Fig. 2.13 (**a**) shows the parallel transport along a nongeodesic curve. (**b**) depicts the F–W transport along the same curve

$$\mathbf{g}_{..}(x) = [\phi(x)]^2\, d_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j, \tag{2.102ii}$$

$$\vec{\mathbf{e}}_{(a)}(x) = [\phi(x)]^{-1}\, \delta^i_{(a)} \frac{\partial}{\partial x^i}, \tag{2.102iii}$$

$$g_{ij}(x)\, e^i_{(a)}(x)\, e^j_{(b)}(x) = d_{(a)(b)}. \tag{2.102iv}$$

Consider the image Γ of a differentiable curve given by

$$\begin{aligned} x^\alpha &= \mathcal{X}^\alpha(s) \equiv 0, \\ x^4 &= \mathcal{X}^4(s) = \mathcal{S}^{-1}(s), \\ s &= \mathcal{S}(x^4) := \int_0^{x^4} \phi(0,0,0,t)\, \mathrm{d}t. \end{aligned} \tag{2.103}$$

(The existence of $\mathcal{S}^{-1}$ is assured by the fact that $\frac{\mathrm{d}\mathcal{S}(x^4)}{\mathrm{d}x^4} = \phi(0,0,0,x^4) > 0$.) Therefore, Γ is a portion of the x^4-axis. The unit tangential vector along Γ is given by (2.103) and (2.102iii) as

$$\begin{aligned} \mathcal{U}^i(s) &= \frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s} = [\phi(\mathcal{X}(s))]^{-1}\, \delta^i_{(4)} = e^i_{(4)}(\mathcal{X}(s)), \\ \mathcal{U}^\alpha(s) &\equiv 0, \quad \mathcal{U}^4(s) > 0. \end{aligned} \tag{2.104}$$

Now, from the metric (2.102ii), the Christoffel symbols can be computed as

$$\left\{ {i \atop j\ k} \right\} = [\phi(x)]^{-1} \cdot d^{\,il} \cdot \left[d_{jl}\partial_k\phi + d_{lk}\partial_j\phi - d_{kj}\partial_l\phi \right]. \tag{2.105}$$

Thus, by (2.104) and (2.105), we obtain

$$\frac{D\mathcal{U}^i(s)}{\mathrm{d}s} = \left\{[\phi(x)]^{-3}\left(\delta^i_{(4)}\partial_4\phi + d^{\,ij}\partial_j\phi\right)\right\}_{|x=\mathcal{X}(s)}. \tag{2.106}$$

The above equation demonstrates that the principle curvature $\kappa_{(1)}(s)$ (and therefore the 4-acceleration) is nonzero, unless $\partial_\alpha\phi \equiv 0$.

Equation (2.100) implies that the tangential vector $\mathcal{U}^i(s)\left[\frac{\partial}{\partial x^i}\right]_{|..}$ *automatically undergoes F–W transport along Γ.*

For the other three spacelike vector fields $e^i_{(\alpha)}(\mathcal{X}(s))$, the left-hand side of (2.100), by consequences of (2.104), (2.102iii), and (2.105), yields

$$\begin{aligned}\frac{De^i_{(\alpha)}(\cdot\cdot)}{\mathrm{d}s} &= \frac{de^i_{(\alpha)}(\cdot\cdot)}{\mathrm{d}s} + \left\{\begin{matrix} i \\ j\ k \end{matrix}\right\}_{|..}\mathcal{U}^j(s)\,e^k_{(\alpha)}(\cdot\cdot) \\ &= \left\{[\phi(x)]^{-3}\cdot\delta^i_{(4)}\cdot\partial_\alpha\phi\right\}_{|..}.\end{aligned} \tag{2.107}$$

On the other hand, the right-hand side of (2.100), by use of (2.2.4,ii), (2.104), and (2.106), implies that

$$\begin{aligned}&g_{jk}(\cdot\cdot)\left\{\mathcal{U}^i(s)\,e^j_{(\alpha)}(\cdot\cdot)\frac{D\mathcal{U}^k(s)}{\mathrm{d}s} - \mathcal{U}^j(s)\,e^k_{(\alpha)}(\cdot\cdot)\frac{D\mathcal{U}^i(s)}{\mathrm{d}s}\right\} \\ &= \left\{[\phi(x)]^{-3}\cdot\delta^i_{(4)}\cdot\partial_\alpha\phi\right\}_{|..}.\end{aligned} \tag{2.108}$$

Comparing (2.107) and (2.108), we conclude that the orthonormal tetrad $\left\{\vec{\mathbf{e}}_{(a)}(\mathcal{X}(s))\right\}_1^4$ indeed undergoes the F–W transport along Γ. □

(Conformally flat space–times are discussed in detail in Appendix 4.)

We shall now consider an operational method for measurement of the proper length of a spacelike curve. Recall that *the arc separation function* $\sum$ of a differentiable curve $\mathcal{X}$ was discussed in # 8 of Exercise 1.3. It is defined by

$$\sum(\mathcal{X}) := \int_{t_{(1)}}^{t_{(2)}}\sqrt{\left|g_{ij}(\mathcal{X}(t))\frac{\mathrm{d}\mathcal{X}^i(t)}{\mathrm{d}t}\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\right|}\,\mathrm{d}t. \tag{2.109}$$

(The integral above is *invariant under reparametrization as well as under a general coordinate transformation.*)

Consider an observer with an orthonormal tetrad, clock, and extremely small devices to emit and receive photons. (See the Fig. 2.14.)

In Fig. 2.14, $\Gamma_{(1)}$ and $\Gamma_{(2)}$ are images of curves $\mathcal{X}_{(1)}$ and $\mathcal{X}_{(2)}$ representing the same observer. Null curves (or photon trajectories) are represented by $\mathcal{X}_{(3)}$ and $\mathcal{X}_{(4)}$ with images $\Gamma_{(3)}$ and $\Gamma_{(4)}$ respectively. Moreover, Γ is the image of a spacelike

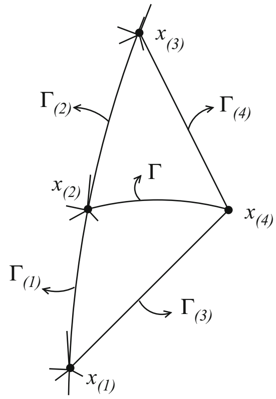


Fig. 2.14 Measurement of a spacelike separation along the image Γ

curve $\mathcal{X}$ under scrutiny. In special relativity, the measurement of a spacelike distance is much simpler to analyze than in the case of curved space–time. The procedure was briefly mentioned after (2.18). Now, we shall deal with the problem in *flat space–time* in greater detail. (This will allow us to have better insight into the corresponding analysis when the space–time is curved.) Let us choose Minkowskian coordinates and each of the five curves in Fig. 2.14 as geodesic or *straight lines*. We choose proper time parameter $s \equiv \tau$ for the inertial observer, proper length parameter l for the spacelike straight line $\mathcal{X}$, and an affine parameter α for the null straight lines $\mathcal{X}_{(3)}$ and $\mathcal{X}_{(4)}$. Therefore, we can express

$$\begin{aligned}
\mathcal{X}^i_{(1)}(s) &= t^i_{(1)}s + c^i_{(1)}, \quad s \in [s_1, s_2), d_{ij}\, t^i_{(1)}t^j_{(1)} = -1\,; \\
\mathcal{X}^i_{(2)}(s) &= t^i_{(2)}s + c^i_{(2)}, \quad s \in [s_2, s_3], d_{ij}\, t^i_{(2)}t^j_{(2)} = -1, \quad t^i_{(2)} = t^i_{(1)}\,; \\
\mathcal{X}^i(l) &= t^i l + c^i, \quad l \in [l_1, l_2], d_{ij}\, t^i t^j = 1\,; \\
\mathcal{X}^i_{(3)}(\alpha) &= t^i_{(3)}\alpha + c^i_{(3)}, \quad \alpha \in [\alpha_1, \alpha_2), d_{ij}\, t^i_{(3)}t^j_{(3)} = 0\,; \\
\mathcal{X}_{(4)}(\alpha) &= t^i_{(4)}\alpha + c^i_{(4)}, \quad \alpha \in (\alpha_2, \alpha_3], d_{ij}\, t^i_{(4)}t^j_{(4)} = 0.
\end{aligned} \tag{2.110}$$

Here, $t^i_{(\cdot\cdot)}$'s and $c^i_{(\cdot\cdot)}$'s are const. By (2.109), (2.110) yields

$$\sum(\mathcal{X}_{(1)}) = \sqrt{-d_{ij}\, t^i_{(1)}t^j_{(1)}} \cdot (s_2 - s_1) = \sqrt{-d_{ij}\left(x^i_{(2)} - x^i_{(1)}\right)\left(x^j_{(2)} - x^j_{(1)}\right)},$$

$$\sum(\mathcal{X}_{(2)}) = \sqrt{-d_{ij}\left(x^i_{(3)} - x^i_{(2)}\right)\left(x^j_{(3)} - x^j_{(2)}\right)},$$

$$\begin{aligned}
\sum(\mathcal{X}) \quad &= \sqrt{d_{ij}\, t^i t^j} \cdot (l_2 - l_1) = \sqrt{d_{ij}\left(x^i_{(4)} - x^i_{(2)}\right)\left(x^j_{(4)} - x^j_{(2)}\right)},\\
\sum(\mathcal{X}_{(3)}) &= \sqrt{d_{ij}\left(x^i_{(4)} - x^i_{(1)}\right)\left(x^j_{(4)} - x^j_{(1)}\right)} = 0,\\
\sum(\mathcal{X}_{(4)}) &= \sqrt{d_{ij}\left(x^i_{(3)} - x^i_{(4)}\right)\left(x^j_{(3)} - x^j_{(4)}\right)} = 0.
\end{aligned} \tag{2.111}$$

Now, we can express

$$\begin{aligned}
x^i_{(4)} - x^i_{(3)} &= \left(x^i_{(4)} - x^i_{(2)}\right) - \left(x^i_{(3)} - x^i_{(2)}\right),\\
x^i_{(4)} - x^i_{(1)} &= \left(x^i_{(4)} - x^i_{(2)}\right) + \left(x^i_{(2)} - x^i_{(1)}\right),\\
x^i_{(2)} - x^i_{(1)} &= \lambda\left(x^i_{(3)} - x^i_{(2)}\right),\\
&\lambda := (s_2 - s_1)\big/(s_3 - s_2) > 0.
\end{aligned} \tag{2.112}$$

Putting the above equations into the null separations of (2.111), we obtain

$$\begin{aligned}
&\lambda\Big\{d_{ij}\Big[\left(x^i_{(4)} - x^i_{(2)}\right)\left(x^j_{(4)} - x^j_{(2)}\right) + \left(x^i_{(3)} - x^i_{(2)}\right)\left(x^j_{(3)} - x^j_{(2)}\right)\\
&\quad - 2\left(x^i_{4)} - x^i_{(2)}\right)\left(x^j_{(3)} - x^j_{(2)}\right)\Big]\Big\} = 0,\\
&d_{ij}\Big[\left(x^i_{(4)} - x^i_{(2)}\right)\left(x^j_{(4)} - x^j_{(2)}\right) + \lambda^2\left(x^i_{(3)} - x^i_{(2)}\right)\left(x^j_{(3)} - x^j_{(2)}\right)\\
&\quad + 2\lambda\left(x^i_{(4)} - x^i_{(2)}\right)\left(x^j_{(3)} - x^j_{(2)}\right)\Big] = 0.
\end{aligned} \tag{2.113}$$

Adding the above equations and cancelling the factor $(1 + \lambda) > 0$, we deduce that

$$d_{ij}\left(x^i_{(4)} - x^i_{(2)}\right)\left(x^j_{(4)} - x^j_{(2)}\right) = -\lambda d_{ij}\left(x^i_{(3)} - x^i_{(2)}\right)\left(x^j_{(3)} - x^j_{(2)}\right), \tag{2.114}$$

or,

$$\left[\sum(\mathcal{X})\right]^2 = \lambda\left[\sum\left(\mathcal{X}_{(2)}\right)\right]^2 = \sum\left(\mathcal{X}_{(1)}\right)\cdot\sum\left(\mathcal{X}_{(2)}\right).$$

Therefore, *the (inertial) observer can measure spatial distances by the readings of his clock!* [55, 242].

Now, let us try to compute the spacelike separation along the image Γ (Fig. 2.14), in the case when the curvature tensor is nonzero. We employ Riemann normal coordinates for the point $x_{(2)}$ in Fig. 2.14. (See p. 67 for the definition.) Let the domain of validity $D \subset \mathbb{R}^4$ encompass the space–time diagram in Fig. 2.14. So, we assume that

$$\begin{aligned}
&x_{(2)} = (0,0,0,0),\\
&g_{ij}(0,0,0,0) = d_{ij},\\
&\partial_k g_{ij}(x)_{|(0,0,0,0)} = 0.
\end{aligned} \tag{2.115}$$

From the point of view of physics, we refer to this coordinate chart as *local Minkowskian coordinates*.

Let us assume that the observer is inertial. In such a case, geodesics originating at $x_{(2)}$ are furnished by straight lines [55, 242]

$$\begin{aligned}
&\mathcal{X}^i_{(1)}(s) = t^i_{(1)}\cdot s, \quad s \in [-s_1, 0),\\
&d_{ij}\, t^i_{(1)}\, t^j_{(1)} = -1, \quad \mathcal{X}^i_{(1)}(0) = (0,0,0,0) = x^i_{(2)}\,;\\
&\mathcal{X}^i_{(2)}(s) = t^i_{(2)}\cdot s = t^i_{(1)}\cdot s, \quad s \in [0, s_3]\,;\\
&\mathcal{X}^i(s) = t^i\cdot l, \;\; l \in [0, l_2].
\end{aligned} \tag{2.116}$$

In the first of the above equations, we have admitted negative values of the parameters, indicating that it is *a signed arc separation parameter*. The null geodesics with images $\Gamma_{(3)}$ and $\Gamma_{(4)}$ are *not* necessarily straight lines. They are governed by equations

$$\begin{aligned}
&\frac{\mathrm{d}^2\mathcal{X}^i_{(3)}(\alpha)}{\mathrm{d}\alpha^2} + \left\{\begin{matrix} i \\ j\;k \end{matrix}\right\}_{|\mathcal{X}_{(3)}(\alpha)} \cdot \frac{\mathrm{d}\mathcal{X}^j_{(3)}(\alpha)}{\mathrm{d}\alpha}\,\frac{\mathrm{d}\mathcal{X}^k_{(3)}(\alpha)}{\mathrm{d}\alpha} = 0, \quad \alpha \in [\alpha_1, \alpha_2)\,;\\
&\frac{\mathrm{d}^2\mathcal{X}^i_{(4)}(\alpha)}{\mathrm{d}\alpha^2} + \left\{\begin{matrix} i \\ j\;k \end{matrix}\right\}_{|\mathcal{X}_{(4)}(\alpha)} \cdot \frac{\mathrm{d}\mathcal{X}^j_{(4)}(\alpha)}{\mathrm{d}\alpha}\,\frac{\mathrm{d}\mathcal{X}^k_{(4)}(\alpha)}{\mathrm{d}\alpha} = 0, \quad \alpha \in (\alpha_2, \alpha_3].
\end{aligned} \tag{2.117}$$

(Recall that α is an affine parameter.) Now, computing arc separations along various curves, we obtain

$$\begin{aligned}
\sum(\mathcal{X}_{(1)}) &= \int_{-s_1}^{0} \sqrt{-g_{ij}(\mathcal{X}(t))\frac{\mathrm{d}\mathcal{X}^i_{(1)}(t)}{\mathrm{d}t}\,\frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}}\,\mathrm{d}t \\
&= \sqrt{-g_{ij}\left(\mathcal{X}_{(1)}(0)\right)t^i_{(1)}\,t^j_{(1)}}\cdot(s_1) \\
&= \sqrt{-d_{ij}\left(x^i_{(2)}-x^i_{(1)}\right)\left(x^j_{(2)}-x^j_{(1)}\right)}, \\
\sum(\mathcal{X}_{(2)}) &= \sqrt{-d_{ij}\left(x^i_{(3)}-x^i_{(2)}\right)\left(x^j_{(3}-x^j_{(2)}\right)}, \\
\sum(\mathcal{X}) \quad &= \sqrt{d_{ij}\left(x^i_{(4)}-x^i_{(2)}\right)\left(x^j_{(4)}-x^j_{(2)}\right)}.
\end{aligned} \tag{2.118}$$

Comparing the above equations with the corresponding equations in (2.111), we conclude that the agreements are *exact*. However, null curves, which are assumed to be of class C^3, pose some complications. Expressing the 4-velocity components $\mathcal{U}^i_{(3)}(\alpha)=\frac{\mathrm{d}\mathcal{X}^i_{(3)}(\alpha)}{\mathrm{d}\alpha}$ along the null curve $\mathcal{X}_{(3)}$, we derive from (2.117) that

$$\frac{\mathrm{d}\mathcal{U}^i_{(3)}(\alpha)}{\mathrm{d}\alpha} = -\left\{ {i \atop j\ k} \right\}_{\big|\mathcal{X}_{(3)}(\alpha)}\cdot\mathcal{U}^j_{(3)}(\alpha)\,\mathcal{U}^k_{(3)}(\alpha), \quad \alpha\in[\alpha_1,\alpha_2). \tag{2.119}$$

By the mean value theorem [32], we can deduce from (2.119) that

$$\begin{aligned}
\mathcal{X}^i_{(3)}(\alpha) = \ & x^i_{(1)} + (\alpha-\alpha_1)\,\mathcal{U}^i_{(3)}(\alpha_1) + \frac{1}{2}(\alpha-\alpha_1)^2\left[\frac{\mathrm{d}\mathcal{U}^i_{(3)}(\alpha)}{\mathrm{d}\alpha}\right]_{\big|\beta}, \\
& \beta := \alpha_1+\theta(\alpha-\alpha_1), \quad 0<\theta<1.
\end{aligned} \tag{2.120}$$

Using (2.119) in (2.120), we conclude that

$$\begin{aligned}
(\alpha_2-\alpha_1)\,\mathcal{U}^i_{(3)}(\alpha_1) &= x^i_{(3)} - x^i_{(1)} - r^i\left(|\alpha_2-\alpha_1|\,,\,\partial_k\,g_{ij\,|\beta_2}\right), \\
r^i(\cdot\cdot) &:= \frac{(\alpha_2-\alpha_1)^2}{2}\left[\left\{ {i \atop j\ k} \right\}\mathcal{U}^j_{(3)}\,\mathcal{U}^k_{(3)}\right]_{\big|\beta}.
\end{aligned} \tag{2.121}$$

For a "small" domain and "weak" intrinsic gravitation, the remainder term $\left|r^i(\cdot\cdot)\right|$ is very small compared to $\left|x^i_{(3)}-x^i_{(1)}\right|$. It should be remarked, however, that (2.121) is *exact*. Since we are dealing with geodesic triangles in a curved manifold [266], we do expect some deviations from the conclusions in the flat manifold. (We have tacitly assumed here that there are *no conjugate points for geodesics* in the domain of consideration.)

Example 2.2.5. Consider a Riemannian normal or local Minkowskian coordinate chart. By the conditions

$$g_{ij}(0,0,0,0) = d_{ij},$$
$$\partial_k g_{ij}(x)_{|(0,0,0,0)} = 0,$$

we can derive from (1.141ii)

$$R_{ijkl}(0,0,0,0) = (1/2)\left[\partial_j \partial_k g_{il} + \partial_i \partial_l g_{jk} - \partial_j \partial_l g_{ik} - \partial_i \partial_k g_{jl}\right]_{|(0,0,0,0)}. \quad (2.122)$$

With the above equations, we can easily prove the algebraic identities of the Riemann–Christoffel tensor stated in Theorem 1.3.19.

We can furthermore prove that

$$\begin{aligned}\left\{\partial_l \partial_k g_{ij} - \partial_i [kl, j] - \partial_j [kl, i]\right\}_{|(0,0,0,0)} &= -\left[R_{ikjl} + R_{iljk}\right]_{|(0,0,0,0)} \\ &=: 3S_{ijkl}(0,0,0,0). \end{aligned} \quad (2.123)$$

(See Problem # 4 of Exercise 2.2.) □

Synge, in his book [243], dealt extensively with the problem of physical measurements with help of the world function, $\Omega(x_2, x_1)$ of p. 86, the *symmetrized curvature tensor* (in (2.123)), the principal curvature of observer's world line, etc. The book [184] by Misner et al. also deals with such topics from slightly different perspectives.

We shall now turn our attention to the equations of motion of a particle in curved space–time. We consider the general case of a possibly variable mass $M(s) > 0$ for the particle. We generalize the special relativistic equations given in # 6 of Exercise 2.1. From the definition of force as the rate of change of momentum, *we postulate the equations of motion* along a future-pointing timelike curve as

$$\begin{aligned} M(s)\left[\frac{\mathrm{d}^2\mathcal{X}^i(s)}{\mathrm{d}s^2} + \left\{ {i \atop j\,k} \right\}_{|..} \frac{\mathrm{d}\mathcal{X}^j(s)}{\mathrm{d}s}\frac{\mathrm{d}\mathcal{X}^k(s)}{\mathrm{d}s}\right] + \frac{\mathrm{d}M(s)}{\mathrm{d}s}\frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s} \\ = F^i(x,u)\Big|_{\substack{x^k=\mathcal{X}^k(s) \\ u^k=\frac{\mathrm{d}\mathcal{X}^k(s)}{\mathrm{d}s}}}, \end{aligned} \quad (2.124)$$

$$\text{or} \quad \frac{D}{\mathrm{d}s}\left[M(s)\mathcal{U}^i(s)\right] = F^i(x,u)_{|..}.$$

On the right-hand sides of (2.124), the 4-force vector components $F^i(\cdot\cdot)_{|..}$ are solely due to *nongravitational effects*.

In terms of *physical* (*or orthonormal*) *components*, (2.124) yields

$$M(s)\,\mathcal{U}^{(b)}(s)\left[\partial_{(b)}u^{(a)}(x)-\gamma^{(a)}{}_{(d)(b)}(x)\cdot u^{(d)}(x)\right]\Big|_{\substack{x^k=\mathcal{X}^k(s)\\ u^{(e)}=\mathcal{U}^{(e)}(s)}}$$

$$+\frac{\mathrm{d}M(s)}{\mathrm{d}s}\cdot\mathcal{U}^{(a)}(s)=F^{(a)}(x,u)_{|..},\tag{2.125i}$$

$$\frac{\mathrm{d}M(s)}{\mathrm{d}s}=-d_{(a)(b)}\big[F^{(a)}(\cdot\cdot)\big]_{|..}\cdot\mathcal{U}^{(b)}(s).\tag{2.125ii}$$

Here, $\gamma^{(a)}{}_{(d)(b)}(x)$ are components of *the Ricci rotation coefficients* in (1.124ii). Although (2.124) and (2.125i) are mathematically equivalent, (2.125i) are preferable from the point of view of observation, as they correspond to the physical frame. Equation (2.125ii) is the relativistic generalization of the Newtonian power law (2.19ii). With help from (2.125ii), we can prove that *the mass function $M(s)$ is constant-valued if and only if vector fields $\mathcal{U}^i(s)$ and $F^i(\cdot\cdot)_{|..}$ are orthogonal.* (Compare with Problem # 6(i) of Exercise 2.1.) In such a scenario, (2.124), (2.98), and (2.99) yield

$$M(s)=m=\text{positive const.},\tag{2.126}$$

$$\frac{D\mathcal{U}^i(s)}{\mathrm{d}s}=(m)^{-1}\,F^i(\cdot\cdot)_{|..}=\kappa_{(1)}(s)\,N^i(s).\tag{2.127}$$

Therefore, *nongeodesic or 4-accelerated motions* of particles and observers, with $\kappa_{(1)}(s)\neq 0$, must be caused by *nongravitational forces*.

There is a special class of nongravitational forces called *monogenic forces* [159]. A monogenic force is derivable from *a single work function* $\mathcal{W}(x,u)$ of class C^2 such that

$$F^i(\cdot\cdot)_{|..}:=g^{ij}(\mathcal{X}(s))\left\{\nabla_j\mathcal{W}(\cdot\cdot)_{|..}-\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\partial\mathcal{W}(\cdot\cdot)}{\partial u^j}\right]_{|..}\right\}.\tag{2.128}$$

We elaborate on the consequences of a monogenic force in the following example.

Example 2.2.6. We can derive equations of motion (2.127) with (2.128) *from a variational principle*. (See (1.187) and Appendix 1.) Consider the Lagrangian function L of eight variables in the following:

$$L(x,u):=-m\sqrt{-g_{ij}(x)\,u^i u^j}+\mathcal{W}(x,u).\tag{2.129}$$

Therefore, taking partial derivatives, we obtain

$$
\begin{aligned}
\frac{\partial L(\cdot\cdot)}{\partial x^i} &= \frac{m}{2}\,\frac{(\partial_i g_{jk})u^j u^k}{\sqrt{-g_{kl}u^k u^l}} + \frac{\partial \mathcal{W}(\cdot\cdot)}{\partial x^i},\\
\frac{\partial L(\cdot\cdot)}{\partial u^i} &= \frac{m g_{ij}(\cdot\cdot)u^j}{\sqrt{-g_{kl}u^k u^l}} + \frac{\partial \mathcal{W}(\cdot\cdot)}{\partial u^i}.
\end{aligned}
\tag{2.130}
$$

The Euler–Lagrange equations, derived from (2.130), yield

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\partial L(\cdot\cdot)}{\partial u^i}\right]_{|\cdot\cdot} - \frac{\partial L(\cdot\cdot)}{\partial x^i}_{|\cdot\cdot}\\
&= m\left[g_{ij}(\mathcal{X}(s))\frac{\mathrm{d}\mathcal{U}^j(s)}{\mathrm{d}s} + \partial_k g_{ij}{}_{|\cdot\cdot}\cdot\mathcal{U}^j(s)\,\mathcal{U}^k(s)\right] + \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{W}(\cdot\cdot)_{|\cdot\cdot}\\
&\quad - \frac{m}{2}\cdot\partial_i g_{jk}{}_{|\cdot\cdot}\cdot\mathcal{U}^j(s)\,\mathcal{U}^k(s) - \partial_i\mathcal{W}(\cdot\cdot)_{|\cdot\cdot},
\end{aligned}
$$

or, $$m\left[g_{ij}(\cdot\cdot)\frac{\mathrm{d}\mathcal{U}^j(s)}{\mathrm{d}s} + [jk,i]_{|\cdot\cdot}\,\mathcal{U}^j(s)\,\mathcal{U}^k(s)\right] = \partial_i\mathcal{W}_{|\cdot\cdot} - \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\partial\mathcal{W}}{\partial u^i}\right]_{|\cdot\cdot},$$

or, $$m\frac{D\mathcal{U}^i(s)}{\mathrm{d}s} = g^{ij}(\cdot\cdot)_{|\cdot\cdot}\left\{\nabla_j\mathcal{W}_{|\cdot\cdot} - \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\partial\mathcal{W}}{\partial u^j}\right]_{|\cdot\cdot}\right\}. \tag{2.131}$$

□

We shall now consider a useful class of monogenic forces by requiring that the work function $\mathcal{W}(x,u)$ is a *positive, homogeneous function of degree one*. That is, for an arbitrary positive number $\lambda > 0$, we must have from (2.129),

$$
\begin{aligned}
\mathcal{W}(x,\lambda u) &= \lambda\mathcal{W}(x,u)\,;\\
L(x,\lambda u) &= \lambda L(x,u).
\end{aligned}
\tag{2.132}
$$

(Consult [55, 171, 242].)

By Euler's theorem on homogeneous functions [176], we derive from (2.132) that

$$
\begin{aligned}
u^i\frac{\partial\mathcal{W}(x,u)}{\partial u^i} &= \mathcal{W}(x,u)\,;\\
u^i\frac{\partial L(x,u)}{\partial u^i} &= L(x,u).
\end{aligned}
\tag{2.133}
$$

We may assume another condition on the above Lagrangian, namely,

$$\det\left\{\frac{\partial^2[L(x,u)]^2}{\partial u^i\,\partial u^j}\right\} \neq 0. \tag{2.134}$$

Equation (2.133), along with the inequality (2.134) and the differentiability $L \in C^2(D \subset \mathbb{R}^8; \mathbb{R})$, leads to a *Finsler metric* [2],

$$\begin{aligned} f_{ij}(x,u) &:= -(1/2)\frac{\partial^2[L(x,u)]^2}{\partial u^i\,\partial u^j} \equiv f_{ji}(x,u), \\ \det\left[f_{ij}(\cdot\cdot)\right] &\neq 0. \end{aligned} \tag{2.135}$$

Now, we shall define conjugate 4-momentum vector components from (2.130) and (2.132) as

$$p_i = P_i(x,u) := \frac{\partial L(x,u)}{\partial u^i} = \frac{m g_{ij} u^j}{\sqrt{-g_{kl}u^k u^l}} + \frac{\partial \mathcal{W}(x,u)}{\partial u^i}. \tag{2.136}$$

We can deduce from the equations above that

$$g^{ij}(x)\left[p_i - \frac{\partial \mathcal{W}(x,u)}{\partial u^i}\right]\left[p_j - \frac{\partial \mathcal{W}(x,u)}{\partial u^j}\right] = -m^2. \tag{2.137}$$

This equation is the generalization of the special relativistic quadratic constraint (2.33). It is *the generalized mass-shell condition*. For a massless particle like the photon, we can put $m = 0$ on the right-hand side of (2.137).

We shall now investigate the arena of *relativistic Hamiltonian mechanics*. First, we will follow nonrelativistic Hamiltonian mechanics in order to introduce the relativistic version of the *Legendre transformation*. Thus, we define *the relativistic Hamiltonian* as

$$\widehat{\mathcal{H}}(x,p) := p_i u^i - L(x,u) = u^i\,\frac{\partial L(\cdot\cdot)}{\partial u^i} - L(\cdot\cdot). \tag{2.138}$$

Let us *assume* now (2.128) and (2.132). Therefore, by the conditions (2.136) and (2.133), we conclude that $\widehat{\mathcal{H}}(x,p) \equiv 0$. Thus, the obvious approach fails! So, we look for *another definition* of a relativistic Hamiltonian. We notice that by (2.136) and (2.133), the 4-momentum function $P_i(x,u)$ is a homogeneous polynomial of *degree zero* in the variables u^i. Therefore, for the positive number $\lambda := (u^4)^{-1}$, and the variables $v^\alpha := u^\alpha (u^4)^{-1}$, we obtain

$$p_i = P_i(x,u) = P_i(x,(u^4)^{-1}u) = P_i(x; v^1, v^2, v^3, 1)). \tag{2.139}$$

The above equations locally yield a parametrized seven-dimensional hyperspace in the eight-dimensional "covariant phase space" [55] coordinatized by the (x, u)-chart. Alternatively, the elimination of three variables v^α in (2.139) will result locally in a *holonomic constraint*

$$\mathcal{H}(x, p) = 0. \tag{2.140}$$

Since we have already derived such a constraint in (2.137), we identify *the relativistic Hamiltonian, or super-Hamiltonian*, as

$$\mathcal{H}(x, p) := \frac{1}{2m}\left\{g^{ij}(x)\left[p_i - \frac{\partial \mathcal{W}(x,u)}{\partial u^i}\right]\left[p_j - \frac{\partial \mathcal{W}(x,u)}{\partial u^j}\right] + m^2\right\}, \quad \text{(2.141i)}$$

$$\frac{\partial}{\partial u^k}\left\{g^{ij}(x)\left[p_i - \frac{\partial \mathcal{W}(x,u)}{\partial u^i}\right]\left[p_j - \frac{\partial \mathcal{W}(x,u)}{\partial u^j}\right]\right\} \equiv 0. \quad \text{(2.141ii)}$$

Remark: The super-Hamiltonian above is, of course, *different from that in (2.138)*.

The variationally derived equations of motion in (2.131) are obtainable by an alternative Lagrangian that involves the super-Hamiltonian $\mathcal{H}(x, p)$.

Let a parametrized curve of class C^2 in a domain of *a 13-dimensional space* be specified by:

$$x^i = \overline{\mathcal{X}}^i(t), \quad u^i = \frac{d\overline{\mathcal{X}}^i(t)}{dt}, \quad p_i = \overline{\mathcal{P}}_i(t), \quad \lambda = \overline{\Lambda}(t), \quad t \in [t_1, t_2]. \tag{2.142}$$

(Here, t is just a parameter.)

Let a Lagrangian of class C^2 be furnished by

$$\mathcal{L}_{(\mathrm{I})}(x, u; p; \lambda) := p_i u^i - \lambda \mathcal{H}(x, p). \tag{2.143}$$

Here, λ is the *Lagrange multiplier*. The corresponding nine Euler–Lagrange equations are given by

$$\begin{aligned} 0 &= \frac{d}{dt}\left[\frac{\partial \mathcal{L}_{(\mathrm{I})}(\cdot\cdot)}{\partial u^i}\right]_{|..} - \left[\frac{\partial \mathcal{L}_{(\mathrm{I})}(\cdot\cdot)}{\partial x^i}\right]_{|..} \\ &= \frac{d\mathcal{P}_i(t)}{dt} - \left[\lambda \frac{\partial \mathcal{H}(\cdot\cdot)}{\partial x^i}\right]_{|..}, \end{aligned} \quad \text{(2.144i)}$$

$$0 = 0 - \left[\frac{\partial \mathcal{L}_{(\mathrm{I})}(\cdot\cdot)}{\partial p_i}\right]_{|..} = -\frac{d\overline{\mathcal{X}}^i(t)}{dt} + \left[\lambda \frac{\partial \mathcal{H}(\cdot\cdot)}{\partial p_i}\right]_{|..}, \quad \text{(2.144ii)}$$

$$0 = 0 - \left[\frac{\partial \mathcal{L}_{(\mathrm{I})}(\cdot\cdot)}{\partial \lambda}\right]_{|..} = \mathcal{H}(x, p)_{|..}. \quad \text{(2.144iii)}$$

We reparametrize the curve by setting

$$s = \mathcal{S}(t) := \int_{t_1}^{t} \overline{\Lambda}(w)\,\mathrm{d}w,$$
$$\mathcal{X}^i(s) = \overline{\mathcal{X}}^i(t), \quad \mathcal{P}_i(s) = \overline{\mathcal{P}}_i(t). \tag{2.145}$$

Equations (2.144i–iii) go over into

$$\frac{\mathrm{d}\mathcal{P}_i(s)}{\mathrm{d}s} = -\frac{\partial\mathcal{H}(\cdot\cdot)}{\partial x^i}_{|..}, \tag{2.146i}$$

$$\frac{\partial\mathcal{X}^i(s)}{\mathrm{d}s} = \frac{\partial\mathcal{H}(\cdot\cdot)}{\partial p_i}_{|..}, \tag{2.146ii}$$

$$\text{and} \quad \mathcal{H}(x,p)_{|..} = 0\,. \tag{2.146iii}$$

The equations above are called the *relativistic Hamiltonian equations of motion* or *relativistic canonical equations of motion*. (See [243].)

We assert that the equations of motion in (2.146i–iii) are *equivalent to those in* (2.131).

Example 2.2.7. We explore the motion of a massive, charged particle in an external electromagnetic field. (See Example 1.2.22 and (2.67).) We choose for the work function as

$$\mathcal{W}(x,u) := e\,A_k(x)\,u^k, \quad \mathcal{W}(x,\lambda u) = \lambda\mathcal{W}(x,u),$$
$$\frac{\partial\mathcal{W}(x,u)}{\partial u^k} = eA_k(x). \tag{2.147}$$

Equations (2.129), (2.130), and (2.136) yield

$$L(x,u) = -m\sqrt{-g_{ij}(x)u^i u^j} + e\,A_k(x)\,u^k,$$
$$p_i = \frac{\partial L(x,\,u)}{\partial u^i} = \frac{m\,g_{ij}(x)\,u^j}{\sqrt{-g_{kl}\,u^k u^l}} + e\,A_i(x),$$
$$\left[p_i - e\,A_i(x)\right]_{|..} = m\,g_{ij}\,(\mathcal{X}(s))\,\mathcal{U}^j(s). \tag{2.148}$$

The relativistic super-Hamiltonian function from (2.141i) is furnished by

$$\mathcal{H}(x,p) = (2m)^{-1}\left\{g^{kl}(x)\left[p_k - e\,A_k(x)\right]\left[p_l - e\,A_l(x)\right] + m^2\right\}. \tag{2.149}$$

Hamilton's canonical equations of motion (2.146i,ii) yield

$$
\begin{aligned}
\frac{d\mathcal{X}^i(s)}{ds} &= \frac{\partial \mathcal{H}(\cdot\cdot)}{\partial p_i}_{|..} = m^{-1}\{g^{ij}(x)[p_j - e\,A_j(x)]\}_{|..} = \mathcal{U}^i(s)\,;\\
\frac{d\mathcal{P}_i(s)}{ds} &= -\frac{\partial \mathcal{H}(\cdot\cdot)}{\partial x^i}_{|..} = -(2m)^{-1}\{\partial_i g^{kl}[p_k - eA_k][p_l - eA_l]\\
&\qquad - 2e\, g^{kl}\cdot \partial_i A_k[p_l - eA_l]\}_{|..}\\
&= (m)^{-1}\left\{\left(g^{lm} g^{kn}\partial_i g_{mn}\right)\left(\frac{m^2}{2}u_k u_l\right) + emg^{kl}\cdot\partial_i A_k\cdot u_l\right\}_{|..},\\
\text{or,}\quad & \frac{d}{ds}\left[m\, g_{ik}u^i + eA_i\right]_{|..} = \left[\frac{m}{2}\partial_i g_{kl}\cdot u^k u^l + e\partial_i A_k\cdot u^k\right]_{|..},\\
\text{or,}\quad & m\left[\frac{d^2\mathcal{X}^i(s)}{ds^2} + \left\{\begin{matrix} i \\ j\;k\end{matrix}\right\}_{|..}\frac{d\mathcal{X}^j(s)}{ds}\,\frac{d\mathcal{X}^k(s)}{ds}\right]\\
&= e\left\{g^{ij}(x)\left[\partial_j A_k - \partial_k A_j\right]\right\}_{|..}\frac{d\mathcal{X}^k(s)}{ds}\\
&= e\left[g^{ij}(x)\,F_{jk}(s)\right]_{|..}\frac{d\mathcal{X}^k(s)}{ds}.
\end{aligned}
\tag{2.150}
$$

The above equations are the correct generalization of the relativistic Lorentz equations of motion in (2.67) to curved space–time. □

There exist alternative Lagrangians for the variational derivation of Hamilton's canonical equations (2.146i,ii). We cite two such Lagrangians in the following:

$$L_{(\mathrm{II})}(x; p, p'; \lambda) := x^i\, p_i' + \lambda\mathcal{H}(x, p), \tag{2.151i}$$

$$L_{(\mathrm{III})}(x, u; p, p'; \lambda) := (1/2)(p_i u^i - x^i\, p_i') - \lambda\mathcal{H}(x, p). \tag{2.151ii}$$

(Here, the prime denotes the derivative $p'_{i|...} := \frac{d\mathcal{P}(t)}{dt}$.) The Lagrangian function $\mathcal{L}_{(\mathrm{III})}$ in the equation above explicitly reveals the symplectic structure of the canonical equations (2.146i–iii). (See Problem # 8(ii) of Exercise 2.2.)

So far, in this chapter, we have assumed that the space–time is a pseudo-Riemannian manifold M_4 and has, in general terms, attributed its curvature to the intrinsic gravitation. However, we have set up no field equations by which the curvature of space–time is made to depend on material sources. That is, we do not yet have the relativistic analog of (1.156). Now, we shall pursue this project.

Recall the Newtonian equations of motion in an external gravitational field furnished by (2.79ii). In a curvilinear coordinate chart of the Euclidean space $\mathbb{E}_3$, these equations go over into

$$\overline{\mathbf{g}}_{..}(\mathbf{x}) = \overline{g}_{\alpha\beta}(\mathbf{x})\, \mathrm{d}x^{\alpha} \otimes \mathrm{d}x^{\beta}, \tag{2.152i}$$

$$\overline{\mathbf{R}}^{\cdot}_{\ldots}(\mathbf{x}) \equiv \overline{\mathbf{O}}^{\cdot}_{\ldots}(\mathbf{x}), \tag{2.152ii}$$

$$\begin{aligned} &\frac{\mathrm{d}^2\mathcal{X}^{\alpha}(t)}{\mathrm{d}t^2} + \overline{\begin{Bmatrix} \alpha \\ \beta\,\gamma \end{Bmatrix}}_{|..} \frac{\mathrm{d}\mathcal{X}^{\beta}(t)}{\mathrm{d}t}\, \frac{\mathrm{d}\mathcal{X}^{\gamma}(t)}{\mathrm{d}t} = -\overline{g}^{\alpha\beta}(\mathbf{x})\, \partial_{\beta} W(\mathbf{x})_{|..} \\ &\quad = -\overline{\nabla}^{\alpha} W(\mathbf{x})_{|..}. \end{aligned} \tag{2.152iii}$$

The equations of motion (2.152iii) is variationally derivable from the Lagrangian

$$L_{(N)}(\mathbf{x}, \mathbf{v}) := (m/2)\left[\overline{g}_{\alpha\beta}(\mathbf{x})\, v^{\alpha} v^{\beta} - 2W(\mathbf{x})\right]. \tag{2.153}$$

On the other hand, in the absence of *nongravitational forces*, a particle follows a geodesic world line in *the curved space–time*. Such a geodesic is variationally derivable from the Lagrangian in (2.129) as

$$L(x, u) = -m\sqrt{-g_{ij}(x) u^i u^j}. \tag{2.154}$$

If we use *any affine parameter*, including s, *an alternative Lagrangian* for the geodesic curve is provided by

$$\begin{aligned} L_{(2)}(x, u) &:= (m/2)\, g_{ij}(x)\, u^i u^j \\ &= (m/2)\left[g_{\alpha\beta}(x)\, u^{\alpha} u^{\beta} + 2g_{\alpha 4}(x)\, u^{\alpha} u^4 + g_{44}(x)(u^4)^2\right]. \end{aligned} \tag{2.155}$$

Defining *coordinate 3-velocity components* by $v^{\alpha} := u^{\alpha}/u^4$, the Lagrangian above can be expressed as

$$L_{(2)}(\cdot\cdot) = (m/2)\left[g_{\alpha\beta}(x) v^{\alpha} v^{\beta} + 2g_{\alpha 4}(x) v^{\alpha} + g_{44}(x)\right] (u^4)^2. \tag{2.156}$$

Comparing Lagrangians $L_{(N)}(\cdot\cdot)$ in (2.153) and $L_{(2)}(\cdot\cdot)$ in (2.156), we conclude that in the low velocity regime, the metric tensor component $-g_{44}(x)$ corresponds to twice the Newtonian potential $W(\mathbf{x})$ plus an undetermined constant. In a weak gravitational field, this relationship is (approximately) given by $g_{44}(x) = -[1 + 2W(\mathbf{x})] < 0$.

Now, we shall explore the field equation for the Newtonian potential inside material sources. We choose physical units such that *the Newtonian constant of gravitation* $G = 1$. (Recall that we have already chosen units so that the speed of light $c = 1$.) In such units, physical quantities are expressible as powers of the spatial unit of “length” or else “time.”)

The Newtonian potential $W(\mathbf{x})$, in curvilinear coordinates (2.152i,ii), is governed by *Poisson's equation*

$$\overline{\nabla}^2 W(\mathbf{x}) := \overline{g}^{\alpha\beta}(\mathbf{x})\overline{\nabla}_\alpha \overline{\nabla}_\beta W = 4\pi\rho(\mathbf{x}) \tag{2.157i}$$

$$\text{or,} \quad \overline{\nabla}^2 [-(1 + 2W(\mathbf{x}))] = -8\pi\rho(\mathbf{x}). \tag{2.157ii}$$

Here, $\rho(\mathbf{x})$ is the nonrelativistic *mass density*. In the last row of the table in p. 127, the nonrelativistic mass density, momentum density, and stress density are *all unified in relativity theory by the symmetric energy–momentum–stress tensor* $T_{ij}(x)$. Therefore, in the right-hand side of the relativistic gravitational field equations, we would like to have $-8\pi\, T_{ij}(x)$, instead of *just* $-8\pi\rho(\mathbf{x})$ as in (2.157ii). This choice forces us to consider *a symmetric tensor* $\theta_{ij}(\cdot\cdot)$ for the left-hand side of the field equations. Since the left-hand side of (2.157ii) is (approximately) given by $\overline{\nabla}^2 g_{44}(\cdot\cdot)$, our tensor $\theta_{ij}(\cdot\cdot)$ should contain up to and including second partial derivatives of $g_{ij}(x)$. Moreover, the curvature tensor plays an important role in the intrinsic gravitation. Therefore, we would like $\theta_{ij}(\cdot\cdot)$ to be expressible by *various contractions of the curvature tensor*. Moreover, the energy–momentum–stress tensor satisfies the useful differential conservation equations $\nabla_j T^{ij} = 0$. Therefore, we shall also require $\theta_{ij}(\cdot\cdot)$ to satisfy $\nabla_j \theta^{ij} = 0$. Now, we have accumulated enough criteria to derive $\theta_{ij}(\cdot\cdot)$ explicitly in the following theorem [90].

Theorem 2.2.8. *Let the metric field components $g_{ij}(x)$ be of class C^3 in a domain $D_{(e)} \subset D \subset \mathbb{R}^4$. Let a symmetric, differentiable tensor field $\theta^{ij}(x)$ in $D_{(e)}$ be defined by*

$$\theta^{ij}(x) := k\left\{R^{ij}(x) + g^{ij}(x)\cdot[\phi(x)\,R(x) + \psi(x)]\right\}.$$

Here, k is a nonzero constant, $R_{ij}(\cdot\cdot)$ are components of the Ricci tensor, $R(\cdot\cdot)$ is the curvature invariant, $\phi(x)$ and $\psi(x)$ are two differentiable scalar fields. Then, $\nabla_j\theta^{ij} = 0$, if and only if $\theta^{ij}(x) = k\left[G^{ij}(x) - \Lambda\, g^{ij}(x)\right]$, where $G^{ij}(x)$ are components of the Einstein tensor and Λ is a constant.

Proof. (i) Assume that $\theta^{ij}(x) = k\left[G^{ij}(x) - \Lambda\, g^{ij}(x)\right]$. Then, $\nabla_j\theta^{ij} = k \times \left[\nabla_j G^{ij} - \Lambda\nabla_j\, g^{ij}\right] \equiv 0$ by (1.150ii) and (1.131ii).

(ii) Now, assume that $\nabla_j\theta^{ij} = 0$. Then, from the definition of $\theta^{ij}(x)$, we conclude that

$$0 = k\left\{\nabla_j R^{ij} + g^{ij}(x)\cdot\partial_j\,[\phi(x)\,R(x) + \psi(x)]\right\}.$$

Using $\nabla_j R^{ij} = (1/2)\, g^{ij}(x)\,\partial_j R$, we obtain that

$$0 = g^{ij}(x)\cdot\partial_j\,[(1/2)\,R(x) + \phi(x)\,R(x) + \psi(x)]\,.$$

Therefore,

$$(1/2)\,R(x) + \phi(x)\,R(x) + \psi(x) = -\Lambda = \text{const.}$$

Thus,

$$\theta^{ij}(x) = k\left[G^{ij}(x) - \Lambda\, g^{ij}(x)\right]. \qquad \blacksquare$$

Choosing the constant $k = 1$ for physical reasons (e.g., an acceptable Newtonian law for weak fields), we *assume* that the *gravitational field equations* are furnished by

$$\theta_{ij}(x) = G_{ij}(x) - \Lambda\, g_{ij}(x) = -8\pi\, T_{ij}(x),$$
$$x \in D_{(e)} \subset D \subset \mathbb{R}^4. \tag{2.158}$$

These field equations constitute the second assumption of the gravitational theory of this book. The constant Λ is called the *cosmological constant*, due to the fact that it was useful to Einstein in yielding a static universe when applying his field equations to the universe as a whole.

The right-hand side of (2.158) will read, in the common c.g.s. units, as $-\kappa\, T_{ij}(x)$, where $\kappa := 8\pi G/c^4 = 2.07 \times 10^{-48}\,\mathrm{gm}^{-1}\,\mathrm{cm}^{-1}\,\mathrm{s}^2$. In the sequel, we shall *continue to use the constant* κ, although we shall *still employ geometrized units* where $c = G = 1$. (Thus, in these units, $\kappa = 8\pi$.)

Einstein did not consider the cosmological constant Λ in his original papers. He published as field equations the following (or equivalent):

$$G_{ij}(x) = \begin{cases} -\kappa\, T_{ij}(x) & \text{inside material sources,} \\ 0 & \text{outside material sources} \end{cases} \tag{2.159i}$$

or

$$G_{(a)(b)}(x) = \begin{cases} -\kappa\, T_{(a)(b)}(x) & \text{inside material sources,} \\ 0 & \text{outside material sources.} \end{cases} \tag{2.159ii}$$

The *Einstein field equations* above are *great intellectual achievements* of the twentieth century! Outside material sources, the *vacuum field equations* reduce to

$$R_{ij}(x) = 0, \tag{2.160i}$$

$$\text{or,} \qquad R_{(a)(b)}(x) = 0. \tag{2.160ii}$$

(Recall that the above are the conditions for *Ricci flatness* as discussed in p. 70.)

Now, alternative versions of field equations (2.159i,ii) will be provided. Following Lichnerowicz [166], we shall assign "names" to the field equations. Denoting by $T(x) := g^{ij}(x)\, T_{ij}(x)$, we furnish the alternative versions as

$$\mathcal{E}_{ij}(x) := G_{ij}(x) + \kappa\, T_{ij}(x) = 0, \tag{2.161i}$$

$$\mathcal{E}_{(a)(b)}(x) := G_{(a)(b)}(x) + \kappa\, T_{(a)(b)}(x) = 0\,; \tag{2.161ii}$$

$$\begin{aligned} \tilde{\mathcal{E}}_{ij}(x) &:= R_{ij}(x) + \kappa\left[T_{ij}(x) - (1/2)\, g_{ij}(x)\, T(x)\right] \\ &= R^{k}{}_{ijk}(x) + \kappa\left[T_{ij}(x) - (1/2)\, g_{ij}(x)\, T(x)\right] = 0, \end{aligned} \tag{2.162i}$$

$$\tilde{\mathcal{E}}_{(a)(b)}(x) := R_{(a)(b)}(x) + \kappa\left[T_{(a)(b)}(x) - (1/2)\, d_{(a)(b)}\, T(x)\right]$$
$$= R^{(c)}{}_{(a)(b)(c)}(x) + \kappa\left[T_{(a)(b)}(x) - (1/2)\, d_{(a)(b)}\, T(x)\right] = 0\,; \quad (2.162ii)$$

$$\mathcal{E}^{l}{}_{ijk}(x) := R^{l}{}_{ijk}(x) - C^{l}{}_{ijk}(x)$$
$$+ (\kappa/2)\Big\{\delta^{l}{}_{k}\, T_{ij}(x) - \delta^{l}{}_{j}\, T_{ik}(x) + g_{ij}(x)\, T^{l}{}_{k}(x) - g_{ik}(x)\, T^{l}{}_{j}(x)$$
$$+ (2/3)\cdot\left[\delta^{l}{}_{j}\, g_{ik}(x) - \delta^{l}{}_{k}\, g_{ij}(x)\right]\cdot T(x)\Big\} = 0, \quad (2.163i)$$

$$\mathcal{E}^{(d)}{}_{(a)(b)(c)}(x) := R^{(d)}{}_{(a)(b)(c)}(x) - C^{(d)}{}_{(a)(b)(c)}(x)$$
$$+ (\kappa/2)\Big\{\delta^{(d)}{}_{(c)}\, T_{(a)(b)}(x) - \delta^{(d)}{}_{(b)}\, T_{(a)(c)}(x)$$
$$+ d_{(a)(b)} T^{(d)}{}_{(c)}(x) - d_{(a)(c)}\, T^{(d)}{}_{(b)}(x)$$
$$+ (2/3)\cdot\left[\delta^{(d)}{}_{(b)}\, d_{(a)(c)} - \delta^{(d)}{}_{(c)}\, d_{(a)(b)}\right]\cdot T(x)\Big\} = 0. \quad (2.163ii)$$

Here, $C^{l}{}_{ijk}(x)$ (and $C^{(d)}{}_{(a)(b)(c)}(x)$) are components of Weyl's conformal tensor, as defined in (1.169i, ii).

In the material "vacuum", $T_{ij}(x) \equiv 0$ (or, $T_{(a)(b)}(x) \equiv 0$). Therefore, the field equations in (2.161i)–(2.163ii) reduce to their vacuum (sourceless) versions

$$\tilde{\mathcal{E}}^{(0)}{}_{ij}(x) := R_{ij}(x) = 0, \quad (2.164i)$$
$$\tilde{\mathcal{E}}^{(0)}{}_{(a)(b)}(x) := R_{(a)(b)}(x) = 0\,; \quad (2.164ii)$$

$$\tilde{\mathcal{E}}^{(0)l}{}_{ijk}(x) := R^{l}{}_{ijk}(x) - C^{l}{}_{ijk}(x) = 0, \quad (2.165i)$$
$$\tilde{\mathcal{E}}^{(0)(d)}{}_{(a)(b)(c)}(x) := R^{(d)}{}_{(a)(b)(c)}(x) - C^{(d)}{}_{(a)(b)(c)}(x) = 0. \quad (2.165ii)$$

The system of field equations (2.161i)–(2.163ii) must be further augmented by the following equations:

$$\mathcal{T}^{i}(x) := \nabla_{j}\, T^{ij} = 0, \quad (2.166i)$$
$$\mathcal{T}^{(a)}(x) := \nabla_{(b)}\, T^{(a)(b)} = 0\,; \quad (2.166ii)$$

$$\nabla_{j}\,\mathcal{E}^{ij} \equiv \kappa\, \mathcal{T}^{i}(x), \quad (2.167i)$$
$$\nabla_{(b)}\mathcal{E}^{(a)(b)} \equiv \kappa\, \mathcal{T}^{(a)}(x)\,; \quad (2.167ii)$$

$$\mathcal{C}^i\left(g_{jk}, \partial_l g_{jk}\right) = 0, \tag{2.168i}$$

$$\mathcal{C}^{(a)}\left(e^i{}_{(b)}, \partial_j e^i{}_{(b)}\right) = 0. \tag{2.168ii}$$

The equations $\mathcal{C}^i(\cdot\cdot) = 0$ (or $\mathcal{C}^{(a)}(\cdot\cdot) = 0$) stand for four possible *coordinate conditions*. (e.g., four equations $g_{\alpha 4}(x) \equiv 0$ and $g_{44}(x) \equiv -1$ can be *locally imposed* to obtain a geodesic normal (or Gaussian normal) coordinate chart, (as discussed in (1.160)).)

The field equations (2.161i), (2.166i), (2.167i), and (2.168i) (or their orthonormal counterparts) constitute a system of *semilinear*, *second-order*, *coupled*, *partial differential equations* in the domain $D_{(e)} \subset D \subset \mathbb{R}^4$. (See Appendix 2.) It is useful to take a count of the number of unknown functions versus the number of (algebraically and differentially) *independent equations*. The counting process reveals whether the system is *underdetermined*, *overdetermined*, or *determinate*. (In an underdetermined system, we can *prescribe suitably some unknown functions*.) In the most general case, we assume that the ten functions $T_{ij}(x)$ are *unknown*, although often physical considerations may place constraints on them. In the scenario where the $T_{ij}(x)$ are unknown, the counting will be symbolically expressed as the following:

No. of unknown functions: $10(g_{ij}) + 10(T_{ij}) = 20.$

No. of equations: $10(\mathcal{E}_{ij} = 0) + 4(\mathcal{T}^i = 0) + 4(\mathcal{C}^i = 0) = 18.$

No. of differential identities: $4(\nabla_j \mathcal{E}^{ij} \equiv \kappa \mathcal{T}^i) = 4.$

No. of independent equations: $18 - 4 = 14.$

Therefore, *the most general system is underdetermined* and *six out of the twenty unknown functions can at most be prescribed*.

Now, we shall explore an isolated material body surrounded by the vacuum. Let it be represented by the world tube $D_{(b)} \subset D_{(e)} \subset D \subset \mathbb{R}^4$. (See Fig. 2.15.)

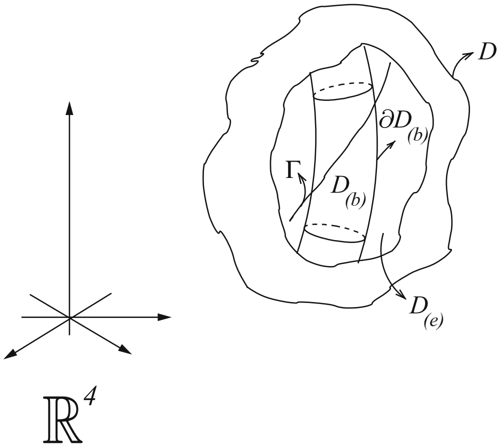
The relevant field equations from (2.159i,ii) are

$$G_{ij}(x) = \begin{cases} -\kappa\, T_{ij}(x) & \text{for } x \in D_{(b)} \subset D_{(e)} \subset \mathbb{R}^4, \\ 0 & \text{for } x \in D_{(e)} - D_{(b)}. \end{cases} \tag{2.169}$$

We assume certain jump discontinuities $\left[T_{ij}(x)\right] \not\equiv 0$ are allowed across the timelike three-dimensional boundary hypersurface $\partial D_{(b)}$. (*We do not deal with infinite discontinuities.*) There are two common jump conditions available in the literature. *Synge's junction conditions* [243] are the following:

$$\begin{aligned} \left[g_{ij}(x)\right]_{|\partial D_{(b)}} &= \left[\partial_k\, g_{ij}\right]_{|\partial D_{(b)}} \equiv 0, \\ \left[T_{ij}(x)\, n^j\right]_{|\partial D_{(b)}} &= 0. \end{aligned} \tag{2.170}$$

Fig. 2.15 A material world tube in the domain $D_{(b)}$



Here, $n^i \frac{\partial}{\partial x^i}{}_{|\partial D_{(b)}}$ denotes the unit, spacelike normal vector to the hypersurface $\partial D_{(b)}$. Moreover, the notation $[\ldots]$ indicates the measure of the jump discontinuity. Physically, this condition may be interpreted as implying that there should be no flux of matter or energy off of the junction surface (which, however, may not be static).

The other popular junction conditions are due to Israel, Sen, Lanczos, and Darmois [45, 140, 156, 234]. We shall call these conditions the *I–S–L–D jump conditions* to match the nomenclature often used in the literature. Suppose that the intrinsic metric components of $\partial D_{(b)}$ are given by $\overline{g}_{\mu\nu}(u)$ of (1.223). Moreover, *the extrinsic curvature* is furnished by the components $K_{\mu\nu}(u) = -(1/2)\left[\nabla_i\, n_j + \nabla_j\, n_i\right]\left(\partial_\mu\, \xi^i\right)\, \left(\partial_\nu\, \xi^j\right)$ in (1.234). Then, the I–S–L–D junction conditions are characterized by

$$\begin{aligned} \left[g_{ij}(x)\right]_{|\partial D_{(b)}} &\equiv 0, \\ \left[\nabla_i\, n_j + \nabla_j\, n_i\right]_{|\partial D_{(b)}} &\equiv 0. \end{aligned} \tag{2.171}$$

This junction condition implies the absence of surface layers (thin "δ-function" shells) of matter and will be further motivated when we consider the variational principle applied to gravitation. (See Appendix 1.)

Remark: The conditions (2.170) and (2.171) also apply to junctions separating different types of material (*not just matter-vacuum boundaries*).

Now, let us consider an (not necessarily inertial) observer with world line Γ as in Fig. 2.15. This observer is penetrating the material world tube $D_{(b)}$ with a parallelly transported (or F–W transported) orthonormal tetrad. The observer has devices to measure the physical components $G_{(a)(b)}(\mathcal{X}(s))$ (via measurements of $R_{(a)(b)(c)(d)}$)

and $T_{(a)(b)}(\mathcal{X}(s))$ along the world line $x^i = \mathcal{X}^i(s)$. Thus, he or she can validate field equations (2.169) along the world line by actual experiments. Therefore, from the point of view of physics, the field equations (2.161ii), (2.164ii), (2.166ii), etc., in terms of *physical components* are more relevant than the corresponding coordinate components!

Next, consider a special coordinate chart, namely, Riemann normal coordinates or local Minkowskian coordinates in Example 2.2.5. The vacuum equations (2.164i), in this chart, yields

$$\begin{aligned} R_{jk}(0,0,0,0) &= \left[g^{il}(x)\, R_{ijkl}(x)\right]_{|(0,0,0,0)} \\ &= (1/2)\left\{\Box\, g_{jk}(x) + \partial_j \partial_k \left(d^{il}\, g_{il}\right) - \partial^i \left(\partial_j\, g_{ik}\right) - \partial^l \left(\partial_k\, g_{jl}\right)\right\}_{|(0,0,0,0)} = 0. \end{aligned} \tag{2.172}$$

Here, *the wave operator (or D'Alembertian)* is defined as $\Box := d^{il}\partial_i \partial_l$. This linearized reduction is at *the single point* $(0,0,0,0)$. However, (2.172) is *exact*. Moreover, there is a glimpse of *gravitational waves* in (2.172). (Gravitational waves shall be briefly discussed in Appendix 5.)

The vacuum equations $G_{ij}(x) = 0$ (or, $R_{ij}(x) = 0$) are also derivable from a variational principle involving the *Lagrangian density* $\mathcal{L}\left(g_{ij}, \partial_k g_{ij}, \partial_l \partial_k g_{ij}\right) := \sqrt{-g(x)} \cdot R(x)$, known as the *Einstein-Hilbert Lagrangian density* (see the Appendix 1). (Hilbert discovered the vacuum equations *independently* [131], after attending Einstein's earlier seminars on the problem of gravitation.)

Tensorial field equations (2.161i), (2.162i), (2.163i), (2.166i), etc., *retain their forms intact* under a general coordinate transformation in (1.37). Similarly, field equations (2.161ii), (2.162ii), (2.163ii), (2.166ii), etc., preserve their forms under a variable Lorentz transformation of the tetrad fields. Both of these "covariances" are called "general covariances" of the field equations. We have already mentioned general covariances of (2.74)–(2.78iii) in *the theory of special relativity*. However, historically, the notion of general covariance or *general relativity* has been used *only in the context of the curved space–time*. We shall also continue to use "general relativity" only in curved space–time (by popular demand)!

Example 2.2.9. Consider a (static) metric of class C^{ω} in the following [243]:

$$\begin{aligned} \mathbf{g}_{..}(x) &:= [1 - 2W(\mathbf{x})]\, \delta_{\alpha\beta}\, \mathrm{d}x^{\alpha} \otimes \mathrm{d}x^{\beta} - [1 + 2W(\mathbf{x})]\, \mathrm{d}x^4 \otimes \mathrm{d}x^4, \\ \mathrm{d}s^2 &= [1 - 2W(\mathbf{x})]\, \delta_{\alpha\beta}\, \mathrm{d}x^{\alpha}\mathrm{d}x^{\beta} - [1 + 2W(\mathbf{x})]\, (\mathrm{d}x^4)^2, \\ (x) &\equiv (\mathbf{x}, x^4) \in \mathbf{D} \times \mathbb{R} =: D \subset \mathbb{R}^4, \\ &|W(\mathbf{x})| < (1/2). \end{aligned} \tag{2.173}$$

Computation of the Einstein tensor components from (2.173), by (1.141i), (1.147ii), (1.148i), and (1.149i), leads to

$$\begin{aligned}
G_{\alpha\beta}(\cdot\cdot) = {} & \frac{4W(\cdot\cdot)}{[1-4W^2]}\left[\delta_{\alpha\beta}\nabla^2 W - \partial_\alpha\partial_\beta W\right] \\
& - 2\left[\frac{1+4W+12W^2}{(1-4W^2)^2}\right]\partial_\alpha W\cdot\partial_\beta W \\
& + \left[\frac{3+4W+12W^2}{(1-4W^2)^2}\right]\delta_{\alpha\beta}\delta^{\mu\nu}\partial_\mu W\cdot\partial_\nu W \\
=: {} & -\kappa\, T_{\alpha\beta}(\mathbf{x}), && \text{(2.174i)} \\
G_{\alpha 4}(\cdot\cdot) \equiv {} & 0 =: -\kappa\, T_{\alpha 4}(\mathbf{x}), && \text{(2.174ii)} \\
G_{44}(\cdot\cdot) = {} & -2\left[\frac{1+2W}{1-2W)^2}\right]\cdot\nabla^2 W - 3\left[\frac{1+2W}{(1-2W)^3}\right]\cdot\delta^{\mu\nu}\partial_\mu W\cdot\partial_\nu W \\
=: {} & -\kappa\, T_{44}(\mathbf{x}), && \text{(2.174iii)} \\
\nabla^2 W := {} & \delta^{\mu\nu}\partial_\mu\partial_\nu W. && \text{(2.174iv)}
\end{aligned}$$

It turns out that the above energy–momentum–stress components satisfy

$$\begin{aligned}
& T_{ij}(\cdot\cdot) = \rho(\cdot\cdot)\, u_i(\cdot\cdot)\, u_j(\cdot\cdot) + p_{ij}(\cdot\cdot), \\
& u^i(\cdot\cdot) := [1+2W]^{-1/2}\cdot\delta^i{}_{(4)}, \\
& g_{ij}(\cdot\cdot)\, u^i(\cdot\cdot)\, u^j(\cdot\cdot) \equiv -1, \\
& p_{ij}(\cdot\cdot)\, u^j(\cdot\cdot) \equiv 0, \\
& \rho(\mathbf{x}) := \frac{\kappa^{-1}}{(1-2W)^2}\left[2\nabla^2 W + 3\frac{\delta^{\mu\nu}\partial_\mu W\cdot\partial_\nu W}{(1-2W)}\right]. && \text{(2.175)}
\end{aligned}$$

The above energy–momentum–stress tensor represents a complicated visco-anisotropic fluid, which may or may not exist in nature. (We are just exploring a mathematical model.)

Further, let $W(\mathbf{x})$ in the metric (2.173) satisfy the Newtonian gravitational equations

$$\nabla^2 W = \begin{cases} (\kappa/2)\,\mu(\mathbf{x}) > 0 & \text{for } \mathbf{x}\in\mathbf{D}_{(1)}\cup\cdots\cup\mathbf{D}_{(n)}, \\ 0 & \text{for } \mathbf{x}\in\mathbf{D}-\left\{\mathbf{D}_{(1)}\cup\cdots\cup\mathbf{D}_{(n)}\right\}; \end{cases}$$

$$W(\mathbf{x}) = -\int\limits_{\mathbf{D}_{(1)}\cup\cdots\cup\mathbf{D}_{(n)}} \frac{\mu(\mathbf{y})\,\mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3}{\|\,\mathbf{x}-\mathbf{y}\,\|}. \qquad \text{(2.176)}$$

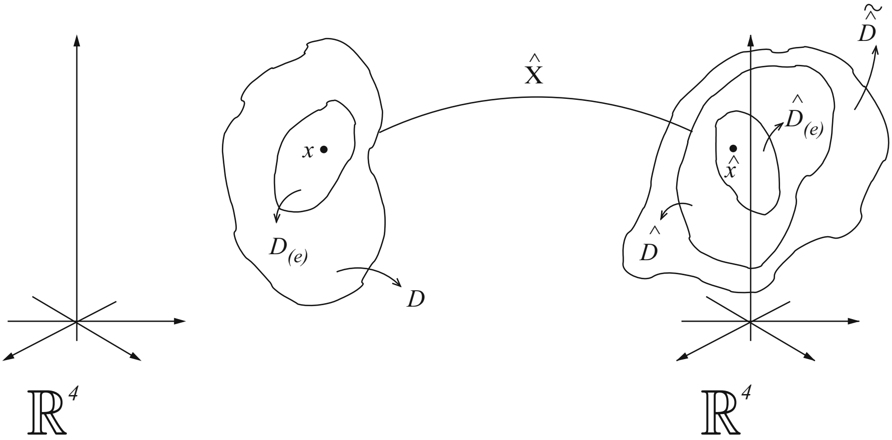


Fig. 2.16 Analytic extension of solutions from the original domain $D_{(e)}$ into $\hat{\tilde{\tilde{D}}}$

Therefore, the Newtonian potential $W(\mathbf{x})$ is due to superposition of gravitational potentials of n static, massive, extended bodies. These bodies are in equilibrium by virtue of mutual cancellations of gravitational attractions and visco-anisotropic repulsions. The metric (2.173), with help of (2.176), depicts *the exact general relativistic generalization of this Newtonian phenomenon*. □

The field equations (2.161i)–(2.163ii), (2.164i)–(2.165ii), and (2.166i,ii), which are partial differential equations involving metric tensor components or tetrad components, are posed in a suitable domain $D_{(e)} \subset D \subset \mathbb{R}^4$. If we can solve these equations in the domain $D_{(e)}$, then we can try to extend the solutions analytically to the whole coordinate domain $D \subset \mathbb{R}^4$. In cases where we succeed in this endeavor, we make a suitable coordinate transformation to the domain $\widehat{D} \subset \mathbb{R}^4$, where the solutions are automatically known by virtue of the transformation laws in (1.107i–iii). In the hatted coordinate chart, we try to analytically extend the solutions into a "larger" domain $\hat{\tilde{\tilde{D}}} \subset \mathbb{R}^4$. (See Fig. 2.16.)

We can try to extend solutions for the metric components into more and more coordinate charts and hope to finally construct an atlas for the differentiable pseudo-Riemannian space–time manifold M_4. *We may or may not succeed* in this endeavor. Global extensions and global analysis of the space–time continuum are quite involved as they require knowledge of the entire history and future of all gravitating bodies involved. (See the book by Hawking and Ellis [126] on this subject.) Here, we merely cite some toy models of two-dimensional curved surfaces, embedded in three-dimensional Euclidean space, to illustrate myriads of *strange possibilities* that global structures of space–time may acquire. (See Fig. 2.17.)

In Fig. 2.17a, the surface is that of a "bugle." It is a surface of negative Gaussian curvature, and it is *geodesically incomplete*. (For *a geodesically complete manifold*, geodesics can be extended for all (real) values of affine parameters.

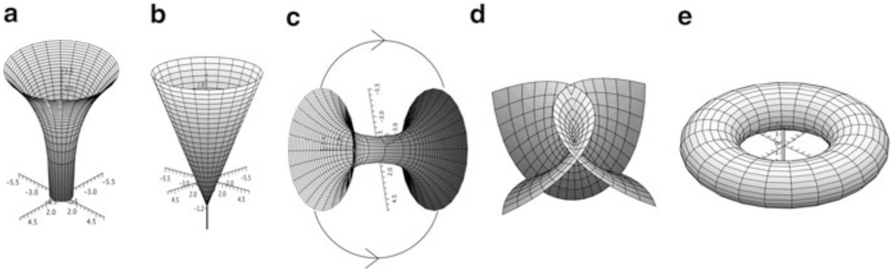


Fig. 2.17 Five two-dimensional surfaces with some peculiarities

(See [56, 126, 266].)) The surface in Fig. 2.17b represents a circular cone with a "hair" or *singularity* on the bottom which is *not a differentiable manifold*. Figure 2.17c represents a manifold with a "throat" and identifications. Figure 2.17d depicts a surface with *self-intersection*. It is an example of *a Riemannian variety* [90], which is *not a Riemannian manifold*. The closed surface in Fig. 2.17e has one handle, or equivalently, it is a surface of *genus* 1 [266]. In general relativity, 2.17a can qualitatively represent a submanifold of the space outside of a spherical star. Figure 2.17b can qualitatively represent the gravitational collapse of a spherical matter distribution to form a singularity, and Fig. 2.17c can be a spatial representation of an exotic object known as a *wormhole*. (See Appendix 6.)

Finally, let us be reminded again of two aspects of gravitational forces. Firstly, *apparent gravity* is always due to the 4-acceleration of the observer. Secondly, *intrinsic gravity* (the actual gravitational field) is always caused by tidal forces or the (nonzero) curvature of the space–time. To be honest, we have to admit that in the general theory of relativity, the notion of "gravitational forces" is just a myth! The curvature effects of the space–time are *erroneously misinterpreted* as "gravitational forces." (In the sequel, *whenever we use the word gravitation, we really imply curvature effects*.) We conclude this section with Wheeler's quotation: "Matter tells space how to curve, space tells matter how to move."

(Note that in the quotation above, the word "gravitation" is conspicuously *absent*!)

Exercises 2.2

1. Consider the Newtonian potential

$$W(\mathbf{x}) := (1/2)\left[(a+b)(x^1)^2 - a(x^2)^2 - b(x^3)^2\right],$$

where $a > 0,\ b > 0$ are constants.

Integrate differential equations (2.85) for the components $\eta^\alpha(t, v)$ of the relative separation vector field.

2. Consider the generalized Frenet-Serret formula (2.98). In the case of *a hyperbola of constant curvature*, we put $\kappa_{(1)}(s) = b = \text{const.} \neq 0,\ \kappa_{(2)}(s) \equiv 0,\ \kappa_{(3)}(s) \equiv 0$. In the case where the space–time is *flat*, obtain a class of general solutions to differential equations (2.98).
3. Let a differential tensor field ${}^{r}_{s}\mathbf{T}(x)$ be restricted to a differentiable timelike curve $x^i = \mathcal{X}^i(s)$. *The Fermi derivative* of the tensor field is defined by

$$\begin{aligned}\frac{D_F T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(\mathcal{X}(s))}{\mathrm{d}s} :=& \frac{D T^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}(\cdot\cdot)}{\mathrm{d}s}\\ &+ \sum_{\alpha=1}^{r}\left[\mathcal{U}_k \cdot \frac{D\mathcal{U}^{i_\alpha}}{\mathrm{d}s} - \mathcal{U}^{i_\alpha} \cdot \frac{\mathrm{d}\mathcal{U}_k}{\mathrm{d}s}\right] \cdot T^{i_1,\ldots,k,\ldots,i_r}{}_{j_1,\ldots,j_s}\\ &+ \sum_{\beta=1}^{s}\left[\mathcal{U}_{j_\beta} \cdot \frac{D\mathcal{U}^{l}}{\mathrm{d}s} - \mathcal{U}^{l} \cdot \frac{D\mathcal{U}_{j_\beta}}{\mathrm{d}s}\right] \cdot T^{i_1,\ldots,i_r}{}_{j_1,\ldots,l,\ldots,j_s}.\end{aligned}$$

   Prove the Leibniz property:

$$\begin{aligned}\frac{D_F\left[\left(A^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}\right)\left(B^{k_1,\ldots,k_p}{}_{l_1,\ldots,l_q}\right)\right]}{\mathrm{d}s} =& \left[\frac{D_F A^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s}}{\mathrm{d}s}\right] \cdot B^{k_1,\ldots,k_p}{}_{l_1,\ldots,l_q}\\ &+ A^{i_1,\ldots,i_r}{}_{j_1,\ldots,j_s} \cdot \left[\frac{D_F B^{k_1,\ldots,k_p}{}_{l_1,\ldots,l_q}}{\mathrm{d}s}\right].\end{aligned}$$

4. The *symmetrized curvature tensor* $\mathbf{S}_{\cdot\cdot\cdot\cdot}(x)$ is defined by the components

$$S_{ijkl}(x) := -(1/3)\left[R_{ikjl}(x) + R_{iljk}(x)\right].$$

   (i) Prove that

$$S_{ijlk}(x) \equiv S_{ijkl}(x) \equiv S_{jikl}(x) \equiv S_{klij}(x)\,;$$

$$S_{ijkl}(x) + S_{iklj}(x) + S_{iljk}(x) \equiv 0.$$

   (ii) Compute the number of linearly independent components of $\mathbf{S}_{\cdot\cdot\cdot\cdot}(x)$.

5. Show that the relativistic equation of motion (2.124) implies

$$M(s) = M(0) - \int_0^s g_{ij}(\mathcal{X}(t))\,F^i(\cdot\cdot)_{|\cdot\cdot} \cdot \frac{\mathrm{d}\mathcal{X}^j(t)}{\mathrm{d}t}\,\mathrm{d}t.$$

(*Remarks*: The equation above proves that external gravitational forces, or combined external gravitational and electromagnetic forces alone, *cannot* alter the proper mass or rest energy of a (classical) particle.)

6. Consider the following Lagrangian function [55]:

$$L(x,u) := \left[1 + c_{(0)} V(x)\right] \Big\{ eA_i(x)u^i - m\sqrt{|g_{ij}(x)u^i u^j|} + c_{(2)} \sqrt[3]{|\phi_{ijk}(x)u^i u^j u^k|} + c_{(3)} \sqrt[4]{|\psi_{ijkl}(x)u^i u^j u^k u^l|} \Big\}.$$

Here, $c_{(0)}, e, m > 0, c_{(2)}, c_{(3)}$, are constants. The functions $V(x)$, $A_i(x)$, $g_{ij}(x)$, $\phi_{ijk}(x)$, $\psi_{ijkl}(x)$ are of class C^3 in $D \subset \mathbb{R}^4$. Moreover, tensor fields $g_{ij}(x)$, $\phi_{ijk}(x)$, and $\psi_{ijkl}(x)$ are symmetric and totally symmetric fields. Using the fact that $L(x, \lambda u) = \lambda L(x,u)$ for $\lambda > 0$, construct the corresponding Finsler metric $f_{ij}(x,u)$, according to (2.135).

7. Prove the differential identities (2.141ii) involving the super-Hamiltonian function $\mathcal{H}(\cdot\cdot)$.
8. Consider the 4×4 identity matrix $[\,I\,] := [\delta^i{}_j]$. Define an antisymmetric, numerical 8×8 matrix by $[A] := \left[\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right]$.

   Consider a general, differentiable coordinate transformation in *the extended phase space* (*or cotangent bundle* [38,55,56]) as

$$\hat{x}^i = \widehat{\xi}^i(x;p),$$

$$\hat{p}_i = \widehat{\eta}_i(x;p),$$

$$\underset{8\times 8}{[J]} := \left[\begin{array}{c|c} \partial\widehat{x}^i/\partial x^j & \partial\widehat{x}^i/\partial p_j \\ \hline \partial\widehat{p}_i/\partial x^j & \partial\widehat{p}_i/\partial p_j \end{array}\right], \quad \det[J] \neq 0.$$

   Let the (extended) Jacobian matrix satisfy $[J][A][J]^{\mathrm{T}} = [A]$. Prove that under such a transformation, relativistic canonical equation (2.146i,ii) remains intact (or "covariant").

(*Remarks*: (i) Transformations satisfying $[J][A][J]^{\mathrm{T}} = [A]$ are called *relativistic canonical transformations*.
(ii) The subset of linear, homogeneous, canonical transformations constitutes a Lie group called the *symplectic group* $S_p(8;\mathbb{R})$. (See [123, 160].))

9. Consider gravitational field equations (2.162ii), (2.164ii), and (2.166ii). Show that in terms of directional derivatives, Ricci rotation coefficients, and *the physical* (*or tetrad*) *components* $T_{(a)(b)}(x)$, the field equations are equivalent to the following:

$$\partial_{(c)}\gamma^{(c)}{}_{(a)(b)} - \partial_{(b)}\gamma^{(c)}{}_{(a)(c)} + \gamma^{(c)}{}_{(a)(d)}\cdot\gamma^{(d)}{}_{(b)(c)} - \gamma^{(c)}{}_{(d)(c)}\cdot\gamma^{(d)}{}_{(a)(b)}$$
$$= \begin{cases} -\kappa\left[T_{(a)(b)}(x) - (1/2)\, d_{(a)(b)}\cdot d^{(c)(d)}T_{(c)(d)}(x)\right], & \text{inside matter} \\ 0, \quad \text{outside matter}; \end{cases}$$

$$\partial_{(b)}T^{(a)(b)} - \gamma^{(a)}{}_{(d)(b)}T^{(d)(b)}(x) - \gamma^{(b)}{}_{(d)(b)}T^{(a)(b)}(x) = 0, \quad \text{inside matter.}$$

10. *Harmonic coordinate conditions* (or the *harmonic gauge*) are characterized by four differential equations $\partial_j\left(\sqrt{|g|}\, g^{ij}\right) = 0$. (See [155].)
    (i) Consider a differentiable scalar field defined by $\phi(x) := x^i$. (Here, the index i takes *one of the values* in $\{1, 2, 3, 4\}$.) Prove that in the harmonic gauge,
    $$\Box\phi := g^{ij}(x)\nabla_i\nabla_j\phi = 0.$$
    (ii) Prove that in the harmonic gauge, gravitational field equations reduce to
    $$\frac{1}{2}\, g^{kl}(x)\,\partial_k\partial_l\, g^{ij} - \begin{Bmatrix} i \\ k\ p \end{Bmatrix}\begin{Bmatrix} j \\ l\ q \end{Bmatrix} g^{kl}(x)\, g^{pq}(x)$$
    $$= \begin{cases} -\kappa\left[T^{ij}(x) - \frac{1}{2}\, g^{ij}(x)\, T^k{}_k(x)\right] & \text{inside material sources,} \\ \quad 0 & \text{outside material sources.} \end{cases}$$

(*Remarks*: The harmonic gauge conditions are analogous to the *Lorentz gauge condition* $\nabla_i A^i = \frac{1}{\sqrt{|g|}}\,\partial_i[\sqrt{|g|}\,A^i] = 0$ in electromagnetic field theory.)

Answers and Hints to Selected Exercises

1.

$$\eta^1(t, v) = A^1(v)\cos\left(\sqrt{a+b}\cdot t\right) + B^1(v)\sin\left(\sqrt{a+b}\cdot t\right),$$
$$\eta^2(t, v) = A^2(v)\,\mathrm{e}^{\sqrt{a}\,t} + B^2(v)\,\mathrm{e}^{-\sqrt{a}\,t},$$
$$\eta^3(t, v) = A^3(v)\,\mathrm{e}^{\sqrt{b}\,t} + B^3(v)\,\mathrm{e}^{-\sqrt{b}\,t}.$$

Six arbitrary functions $A^i(v)$, $B^i(v)$ resulted from integration.

2.

$$\mathcal{X}^1(s) = (b)^{-1}\ \cosh(bs) + c^1,$$
$$\mathcal{X}^2(s) = c^2, \quad \mathcal{X}^3(s) = c^3,$$
$$\mathcal{X}^4(s) = b^{-1}\ \sinh(bs) + c^4.$$

There are four arbitrary constants c^i's of integration.

4. (ii) The number is 20, which is the same number of independent components for $R_{ijkl}(x)$.
7. Using (2.133), we get

$$u^i\,\frac{\partial\mathcal{W}(\cdot\cdot)}{\partial u^i} = \mathcal{W}(\cdot\cdot),$$

$$\text{or} \qquad u^i\,\frac{\partial^2\mathcal{W}(\cdot\cdot)}{\partial u^k\,\partial u^i} \equiv 0.$$

Now, employing (2.136), we obtain

$$\frac{\partial}{\partial u^k}\left\{g^{ij}(x)\left[p_i - \frac{\partial\mathcal{W}(\cdot\cdot)}{\partial u^i}\right]\left[p_j - \frac{\partial\mathcal{W}(\cdot\cdot)}{\partial u^j}\right]\right\}$$
$$= -2g^{ij}(x)\cdot\frac{\partial^2\mathcal{W}(\cdot\cdot)}{\partial u^k\,\partial u^i}\cdot\left[p_j - \frac{\partial\mathcal{W}(\cdot\cdot)}{\partial u^j}\right]$$
$$= -\frac{2m}{\sqrt{-g_{mn}u^m u^n}}\cdot\left[u^i\,\frac{\partial^2\mathcal{W}(\cdot\cdot)}{\partial u^k\,\partial u^i}\right] \equiv 0.$$

8. Write canonical equations (2.146i,ii) as block-matrix equations [243]:

$$\underset{8\times 1}{\left[\begin{array}{c}\dfrac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s}\\ \hline \dfrac{\mathrm{d}\mathcal{P}_i(s)}{\mathrm{d}s}\end{array}\right]} = \underset{8\times 8}{\left[\begin{array}{c|c}0 & I\\ \hline -I & 0\end{array}\right]}\underset{8\times 1}{\left[\begin{array}{c}\dfrac{\partial\mathcal{H}(\cdot\cdot)}{\partial x^i}\Big|_{\cdot\cdot}\\ \hline \dfrac{\partial\mathcal{H}(\cdot\cdot)}{\partial p_i}\Big|_{\cdot\cdot}\end{array}\right]} = \underset{8\times 8}{[A]}\underset{8\times 1}{\left[\begin{array}{c}\dfrac{\partial\mathcal{H}(\cdot\cdot)}{\partial x^i}\\ \hline \dfrac{\partial\mathcal{H}(\cdot\cdot)}{\partial p_i}\end{array}\right]}.$$

Transforming into hatted coordinates by the chain rule, the equations above imply that

$$\left[\begin{array}{c|c} \dfrac{\partial x^i}{\partial \widehat{x}^j} & \dfrac{\partial x^i}{\partial \widehat{p}_j} \\ \hline \dfrac{\partial p_i}{\partial \widehat{x}^j} & \dfrac{\partial p_i}{\partial \widehat{p}_j} \end{array}\right] \left[\begin{array}{c} \dfrac{\mathrm{d}\widehat{\mathcal{X}}^j(s)}{\mathrm{d}s} \\ \hline \dfrac{\mathrm{d}\widehat{\mathcal{P}}_j(s)}{\mathrm{d}s} \end{array}\right] = [A] \left[\begin{array}{c|c} \dfrac{\partial \widehat{x}^j}{\partial x^i} & \dfrac{\partial \widehat{p}_j}{\partial x^i} \\ \hline \dfrac{\partial \widehat{x}^j}{\partial p_i} & \dfrac{\partial \widehat{p}_j}{\partial p_i} \end{array}\right] \left[\begin{array}{c} \dfrac{\partial \widehat{\mathcal{H}}(\cdot\cdot)}{\partial \widehat{x}^j}_{|..} \\ \hline \dfrac{\partial \widehat{\mathcal{H}}(\cdot\cdot)}{\partial \widehat{p}_j}_{|..} \end{array}\right],$$

$$\text{or,}\qquad [J]^{-1} \left[\begin{array}{c} \dfrac{\mathrm{d}\widehat{\mathcal{X}}^j(s)}{\mathrm{d}s} \\ \hline \dfrac{\mathrm{d}\widehat{\mathcal{P}}_j(s)}{\mathrm{d}s} \end{array}\right] = [A][J]^{\mathrm{T}} \left[\begin{array}{c} \dfrac{\partial \widehat{\mathcal{H}}(\cdot\cdot)}{\partial \widehat{x}^j}_{|..} \\ \hline \dfrac{\partial \widehat{\mathcal{H}}(\cdot\cdot)}{\partial \widehat{p}_j}_{|..} \end{array}\right],$$

$$\text{or,}\qquad \left[\begin{array}{c} \dfrac{\mathrm{d}\widehat{\mathcal{X}}^j(s)}{\mathrm{d}s} \\ \hline \dfrac{\mathrm{d}\widehat{\mathcal{P}}_j(s)}{\mathrm{d}s} \end{array}\right] = [A] \left[\begin{array}{c} \dfrac{\partial \widehat{\mathcal{H}}(\cdot\cdot)}{\partial \widehat{x}^j}_{|..} \\ \hline \dfrac{\partial \widehat{\mathcal{H}}(\cdot\cdot)}{\partial \widehat{p}_j}_{|..} \end{array}\right].$$

2.3 General Properties of T_{ij}

We shall now explore *the algebraic classification* of orthonormal components of the energy–momentum–stress tensor field $T_{(a)(b)}(x)$ at a particular event, corresponding to $x_0 \in D \subset \mathbb{R}^4$. The 4×4 real, symmetric matrix $\left[T_{(a)(b)}(x_0)\right]$ has the (usual) *characteristic polynomial equation*

$$p(\lambda) := \det\left[T_{(a)(b)}(x_0) - \lambda \cdot \delta_{(a)(b)}\right] = 0. \tag{2.177}$$

The corresponding roots, which are *the usual eigenvalues*, are *all real but non-relativistic*! However, physically relevant, relativistic *invariant eigenvalues* are furnished by another polynomial equation

$$p^{\#}(\lambda) := \det\left[T_{(a)(b)}(x_0) - \lambda \cdot d_{(a)(b)}\right] = 0.$$
$$\text{or}\quad \det\left[T^{(a)}{}_{(b)}(x_0) - \lambda \cdot \delta^{(a)}{}_{(b)}\right] = 0. \tag{2.178}$$

As discussed in Appendix 3, *the relativistic or Lorentz-invariant eigenvalues need neither be real nor the* 4×4 *matrix* $\left[T_{(a)(b)}(x_0)\right]$ *be diagonalizable*. In the special cases where $\left[T_{(a)(b)}(x_0)\right]$ admits only *real invariant eigenvalues*, the classification of the matrix is provided by the following types. (See Example A3.8 for more details.)

$$\text{Type-I:} \qquad \left[T_{(a)(b)}(x_0)\right]_{(J)} = \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 0 & 0 \\ 0 & 0 & \lambda_{(3)} & 0 \\ 0 & 0 & 0 & \lambda_{(4)} \end{bmatrix}. \tag{2.179}$$

This matrix is already diagonalized. The 4×1 column vectors representing the relativistic eigenvectors are (obviously) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. These vectors are isomorphic to the (natural) orthonormal basis vectors in $\left\{\vec{\mathbf{e}}_{(a)}(x_0)\right\}_1^4$. The energy–momentum–stress tensor is expressible as

$$\begin{aligned} \mathbf{T}^{\cdot\cdot}(x_0) &= \sum_{a=1}^{4} \lambda_{(a)} \left[\vec{\mathbf{e}}_{(a)}(x_0) \otimes \vec{\mathbf{e}}_{(a)}(x_0)\right], \\ T^{ij}(x_0) &= \lambda_{(1)} e^i_{(1)}(x_0) e^j_{(1)}(x_0) + \lambda_{(2)} e^i_{(2)}(x_0) e^j_{(2)}(x_0) \\ &\quad + \lambda_{(3)} e^i_{(3)}(x_0) e^j_{(3)}(x_0) + \lambda_{(4)} e^i_{(4)}(x_0) e^j_{(4)}(x_0). \end{aligned} \tag{2.180}$$

In case invariant eigenvalues $\lambda_{(1)}, \lambda_{(2)}, \lambda_{(3)}, -\lambda_{(4)}$ are all distinct, the Segre characteristic is[6] $[1, 1, 1, 1]$. (See (A3.4).) This type of the energy–momentum–stress tensor is known as Type-I$_{(a)}$.

In the case of Type-I$_{(b)}$, we assume that $\lambda_{(1)} = \lambda_{(2)}$ and $\lambda_{(1)}, \lambda_{(3)}, -\lambda_{(4)}$ are distinct. The Segre characteristic is $[(1, 1), 1, 1]$. The components $T^{ij}(x_0)$ are given by

$$\begin{aligned} T^{ij}(x_0) &= \lambda_{(1)} \left[e^i_{(1)}(x_0) e^j_{(1)}(x_0) + e^i_{(2)}(x_0) e^j_{(2)}(x_0)\right] \\ &\quad + \lambda_{(3)} e^i_{(3)}(x_0) e^j_{(3)}(x_0) + \lambda_{(4)} e^i_{(4)}(x_0) e^j_{(4)}(x_0) \\ &= \lambda_{(1)} g^{ij}(x_0) + \left[\lambda_{(3)} - \lambda_{(1)}\right] e^i_{(3)}(x_0) e^j_{(3)}(x_0) \\ &\quad + \left[\lambda_{(1)} + \lambda_{(4)}\right] e^i_{(4)}(x_0) e^j_{(4)}(x_0). \end{aligned} \tag{2.181}$$

Here, we have used equations (1.105) and *the notation* $e^i_{(a)}(x_0) \equiv \lambda^i_{(a)}(x_0)$.

The Type-I$_{(c)}$ is characterized by $\lambda_{(1)} = \lambda_{(2)}$, $\lambda_{(3)} = -\lambda_{(4)}$, and $\lambda_{(1)} \neq -\lambda_{(4)}$. The Segre characteristic is $[(1, 1), (1, 1)]$.

[6]Consult Appendix 3 for the definition of a Segre characteristic.

For the Type-I$_{(d)}$, $\lambda_{(1)} = \lambda_{(2)} = \lambda_{(3)}$ and $\lambda_{(1)} \neq -\lambda_{(4)}$. The Segre characteristic is furnished by $[(1,1,1),1]$. The energy–momentum–stress components are given by

$$\begin{aligned} T^{ij}(x_0) &= \lambda_{(1)}\left[e^i_{(1)}(x_0)e^j_{(1)}(x_0) + e^i_{(2)}(x_0)e^j_{(2)}(x_0) + e^i_{(3)}(x_0)e^j_{(3)}(x_0)\right] \\ &\quad + \lambda_{(4)}e^i_{(4)}(x_0)e^j_{(4)}(x_0) \\ &= \lambda_{(1)}g^{ij}(x_0) + \left[\lambda_{(1)} + \lambda_{(4)}\right] e^i_{(4)}(x_0)e^j_{(4)}(x_0). \end{aligned} \tag{2.182}$$

Here, we have made use of (1.105).

The Type-I$_{(e)}$ is characterized by $\lambda_{(1)} = \lambda_{(2)} = \lambda_{(3)} = -\lambda_{(4)}$. (See (A3.4).) The Segre characteristic is $[(1,1,1,1)]$. The energy–momentum–stress tensor field is provided by

$$\begin{aligned} T^{ij}(x) &= \lambda_{(1)}\Big[e^i_{(1)}(x)e^j_{(1)}(x) + e^i_{(2)}(x)e^j_{(2)}(x) \\ &\qquad + e^i_{(3)}(x)e^j_{(3)}(x) - e^i_{(4)}(x)e^j_{(4)}(x)\Big] \\ &= \lambda_{(1)}g^{ij}(x). \end{aligned} \tag{2.183}$$

The Type-II is characterized by

$$\begin{aligned} \left[T_{(a)(b)}(x_0)\right] &= \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 0 & 0 \\ 0 & 0 & \lambda_{(3)}+1 & 1 \\ 0 & 0 & 1 & 1-\lambda_{(3)} \end{bmatrix}, \\ \text{or,} \quad \left[T^{(a)}{}_{(b)}(x_0)\right] &= \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 0 & 0 \\ 0 & 0 & \lambda_{(3)}+1 & 1 \\ 0 & 0 & -1 & \lambda_{(3)}-1 \end{bmatrix} \neq \left[T^{(a)}{}_{(b)}(x_0)\right]^{\mathrm{T}}. \end{aligned} \tag{2.184}$$

The relativistic eigencolumn vectors for the above matrix are *isomorphic to spacelike vectors* $\vec{\mathbf{e}}_{(1)}(x_0), \vec{\mathbf{e}}_{(2)}(x_0)$, and *the null vector* $\vec{\mathbf{e}}_{(3)}(x_0) - \vec{\mathbf{e}}_{(4)}(x_0)$. In the case when $\lambda_{(1)}, \lambda_{(2)}$, and $\lambda_{(3)}$ are *distinct*, the Segre characteristic is furnished by $[1,1,2]$. (See (A3.5).)

For Type-II$_{(b)}$, assume that $\lambda_{(1)} = \lambda_{(2)}$, $\lambda_{(1)} \neq \lambda_{(3)}$. The Segre characteristic is provided by $[(1,1),2]$.

For Type-II$_{(c)}$, assume that $\lambda_{(1)} \neq \lambda_{(3)}$ and $\lambda_{(2)} = \lambda_{(3)}$. The Segre characteristic is furnished by $[1,(1,2)]$.

For Type-II$_{(d)}$, assume that $\lambda_{(1)} = \lambda_{(2)} = \lambda_{(3)}$. The Segre characteristic is $[(1,1,2)]$.

The Type-III$_{(a)}$ is characterized by

$$\left[T^{(a)}_{\ (b)}(x_0)\right] = \begin{bmatrix} \lambda_{(1)} & 1 & 0 & 0 \\ 0 & \lambda_{(1)} & 0 & 0 \\ 0 & 0 & \lambda_{(2)} & 1 \\ 0 & 0 & 0 & \lambda_{(2)} \end{bmatrix}. \tag{2.185}$$

Here, we assume that $\lambda_{(1)} \neq \lambda_{(2)}$ and the Segre characteristic is $[2, 2]$.

For the Type-III$_{(b)}$, we assume that $\lambda_{(1)} = \lambda_{(2)}$ and the Segre characteristic is $[(2, 2)]$.

The Type-IV$_{(a)}$ is characterized by

$$\left[T^{(a)}_{\ (b)}(x_0)\right] = \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 1 & 0 \\ 0 & 0 & \lambda_{(2)} & 1 \\ 0 & 0 & 0 & \lambda_{(2)} \end{bmatrix}. \tag{2.186}$$

In this case, $\lambda_{(1)} \neq \lambda_{(2)}$ and the corresponding Segre characteristic is $[1, 3]$.

For Type-IV$_{(b)}$, we assume that $\lambda_{(1)} = \lambda_{(2)}$. The Segre characteristic is $[(1, 3)]$.

Type-V is characterized by

$$\left[T^{(a)}_{\ (b)}(x_0)\right] = \begin{bmatrix} \lambda_{(1)} & 1 & 0 & 0 \\ 0 & \lambda_{(1)} & 1 & 0 \\ 0 & 0 & \lambda_{(1)} & 1 \\ 0 & 0 & 0 & \lambda_{(1)} \end{bmatrix}. \tag{2.187}$$

The Segre characteristic is [4]. (Consult (A3.4)–(A3.10) of Appendix 3.)

Example 2.3.1. Consider the case of a *perfect fluid* defined by the energy–momentum–stress tensor field

$$\begin{aligned} T^{ij}(x) &:= p(x)g^{ij}(x) + [\rho(x) + p(x)]\, U^i(x)U^j(x) \\ &\equiv [\rho(x) + p(x)]\, U^i(x)U^j(x) + p(x)g^{ij}(x), \\ &g_{ij}(x)U^i(x)U^j(x) \equiv -1. \end{aligned} \tag{2.188}$$

Here, $\rho(x)$ is *the proper mass density* and $p(x)$ is the *pressure*. (See (2.45).) By (2.182), it is clear that the Segre characteristic of $T_{ij}(x)$ in (2.188) throughout the space–time domain of consideration is $[(1, 1, 1), 1]$.

We can deduce from (2.188) that

$$T_{ij}(x)U^j(x) = -\rho(x)g_{ij}(x)U^j(x),$$
$$T_{ij}(x)V^j(x) = p(x)g_{ij}(x)V^j(x). \tag{2.189}$$

Here, $V^j(x)$ represents an arbitrary, nonzero spacelike vector field satisfying the orthogonality $U_i(x)V^i(x) \equiv 0$.

Therefore, we conclude that $p(x)$ and $-\rho(x)$ are *the invariant eigenvalues* of the energy–momentum–stress tensor field $T_{ij}(x)$. (Note that in this example, $T_{ij}(x)U^i(x)U^j(x) = \rho(x)$.) □

In a macroscopic domain of the space–time universe, we usually experience that the proper mass density is nonnegative, or $\rho(x) \geq 0$. To generalize this concept, invariant criteria on $T_{ij}(x)$, called *energy conditions*, are introduced. (See [126].)

I. The *Weak/Null Energy Condition*: This condition is furnished by the weak inequality

$$T_{ij}(x)W^i(x)W^j(x) \geq 0 \tag{2.190}$$

for *every timelike/null vector field* $W^i(x)\frac{\partial}{\partial x^i}$. Physically speaking, this inequality, for timelike $W^i(x)\frac{\partial}{\partial x^i}$, implies that the energy density of a material source as measured by any observer pursuing a timelike curve, must be nonnegative.

II. The *Dominant Energy Condition*: This condition is characterized by

(i) $$T_{ij}(x)W^i(x)W^j(x) \geq 0 \tag{2.191}$$

and (ii) $T^{ij}(x)W_j(x)\frac{\partial}{\partial x^i}$ is nonspacelike.

($W^i(x)\frac{\partial}{\partial x^i}$ is *an arbitrary timelike or null vector field*.)

The above conditions may be physically interpreted as the local energy density is always nonnegative and the local energy flux is always nonspacelike.

III. The *Strong Energy Condition*: This condition is governed by the weak inequality

$$T_{ij}(x)W^i(x)W^j(x) \geq (1/2)T^i{}_i(x)W^j(x)W_j(x) \tag{2.192}$$

for every timelike (or null) vector field $W^i(x)\frac{\partial}{\partial x^i}$.

The energy conditions are reasonable in a space–time domain containing regular, macroscopic, matter distribution. However, these conditions likely do not hold in the neighborhood of an extreme pressure and density, such as close to a singularity. Moreover, in the microcosm, the arena of quantum effects, these conditions are not meaningful. An investigation on the energy conditions may be found in [14].

In the case of a diagonalizable energy–momentum–stress tensor field given in (2.180), the energy conditions can be considerably sharpened.

Theorem 2.3.2. *Let the energy–momentum–stress tensor field $T^{ij}(x)$ be diagonalizable as*

$$T^{ij}(x) = \rho(x)U^i(x)U^j(x) + \sum_{\mu=1}^{3} p_{(\mu)}(x)V^i_{(\mu)}(x)V^j_{(\mu)}(x),$$
$$U^i(x)U_i(x) \equiv -1, \quad U_i(x)V^i_{(\mu)}(x) \equiv 0,$$
$$g_{ij}(x)V^i_{(\mu)}(x)V^j_{(\nu)}(x) \equiv \delta_{(\mu)(\nu)}. \tag{2.193}$$

Then, the weak energy condition (2.190) *is equivalent to*

$$\rho(x) \geq 0\,; \quad \rho(x) + p_{(\mu)}(x) \geq 0, \quad \mu \in \{1, 2, 3\}, \tag{2.194i}$$

and the null energy condition is equivalent to only the second condition in (2.194i).
The dominant energy condition (2.191) *is equivalent to*

$$\left|p_{(\mu)}(x)\right| \leq \rho(x), \quad \mu \in \{1, 2, 3\}. \tag{2.194ii}$$

Furthermore, the strong energy condition is equivalent to

$$\rho(x) + p_{(\mu)}(x) \geq 0$$

and

$$\rho(x) + \sum_{\mu=1}^{3} p_{(\mu)}(x) \geq 0. \tag{2.194iii}$$

We shall skip the proof. (See [126, 257].)

Remarks: (i) The invariant eigenvalues $p_{(\mu)}(x)$ are called *principal pressures*.
(ii) The negative of principal pressures, namely, $\sigma_{(\mu)}(x) := -p_{(\mu)}(x)$ are called *principal tensions*.
(iii) A cosmological constant source can violate energy conditions.

Example 2.3.3. One can enforce the energy inequalities in (2.194i–iii) with help of *four*, *arbitrary*, *slack functions*.

Let five functions $q, f, h_{(1)}, h_{(2)}, h_{(3)}$ be arbitrary continuous functions in $x \in D \subset \mathbb{R}^4$. The weak/null energy inequality (2.194i) is solved by putting

$$\rho(x) := q(x), \quad p_{(\mu)}(x) := \left[h_{(\mu)}(x)\right]^2 - q(x).$$

In the case of the weak energy condition, the function $q(x)$ is subject to the extra restriction $q(x) = [f(x)]^2$ to ensure a nonnegative energy density.

The dominant energy condition (2.194ii) is solved by substituting

$$\rho(x) := [f(x)]^2, \quad p_{(\mu)}(x) := [f(x)]^2 \cdot \cos\left[\theta_{(\mu)}(x)\right].$$

Here, $f, \theta_{(1)}, \theta_{(2)}, \theta_{(3)}$ are four, arbitrary, continuous functions.

The strong energy condition (2.194iii) is solved by expressing

$$\begin{aligned}
\rho(x) &:= [f(x)\cdot\sinh\chi(x)]^2\,,\\
p_{(1)}(x) &:= 2\,[f(x)\cdot\cosh\chi(x)\cdot\sin\theta(x)\cdot\cos\phi(x)]^2\\
&\quad -[f(x)\cdot\sinh\chi(x)]^2\,,\\
p_{(2)}(x) &:= 2\,[f(x)\cdot\cosh\chi(x)\cdot\sin\theta(x)\cdot\sin\phi(x)]^2\\
&\quad -[f(x)\cdot\sinh\chi(x)]^2\,,\\
p_{(3)}(x) &:= 2\,[f(x)\cdot\cosh\chi(x)\cdot\cos\phi(x)]^2\\
&\quad -[f(x)\cdot\sinh\chi(x)]^2\,.
\end{aligned}$$

Here, four continuous functions f, χ, θ, ϕ are otherwise arbitrary. □

Now we shall investigate *macroscopic materials* in general. Constituent particles of such materials follow *a timelike* 4-*velocity field* $U^i(x)\frac{\partial}{\partial x^i}$ satisfying *the invariant eigenvalue equations*:

$$\begin{aligned}
T_{ij}(x)U^j(x) &= -\rho(x)U_i(x),\\
U_i(x)U^i(x) &\equiv -1.
\end{aligned} \tag{2.195}$$

The vector field $\vec{\mathbf{U}}(x) = U^i(x)\frac{\partial}{\partial x^i}$ is tangential to motion curves representing *stream lines*. (See (2.42) and Fig. 2.6.) It can be proved that the energy–momentum–stress tensor field $T_{ij}(x)$ in (2.195) must be of Type-I in (2.179). (See Problem # 2 of Exercise 2.3.)

We define a symmetric tensor field by

$$S_{ij}(x) := \rho(x)U_i(x)U_j(x) - T_{ij}(x). \tag{2.196}$$

It follows that

$$S_{ij}(x)U^j(x) = -\rho(x)U_i(x) - T_{ij}(x)U^j(x) \equiv 0. \tag{2.197}$$

We call $S_{ij}(x)$ *the general relativistic stress-tensor field*. (The special relativistic version was discussed in (2.45).)

We now define the *projection tensor* field

$$\begin{aligned}
\mathcal{P}^i{}_j(x) &:= \delta^i{}_j + U^i(x)U_j(x),\\
\mathcal{P}^i{}_j(x)U^j(x) &\equiv 0.
\end{aligned} \tag{2.198}$$

For *an arbitrary, nonzero vector field* $V^i(x)\frac{\partial}{\partial x^i}$, the projected vector $\mathcal{P}^i{}_j(x)$ $V^j(x)\frac{\partial}{\partial x^i}$ is *spacelike and orthogonal to* $U^i(x)\frac{\partial}{\partial x^i}$. (See Theorems 2.1.2 and 2.1.4.)

We shall now analyze *the relativistic kinematics* of material streamlines. We need to define several vector and tensor fields derived from the 4-velocity field $U^i(x)\frac{\partial}{\partial x^i}$. Assuming that streamlines are curves of class C^3, we define the *4-acceleration field*, *vorticity tensor*, *expansion tensor*, *expansion scalar*, and *shear tensor*, respectively, by

$$\dot{U}^i(x) := U^j \nabla_j U^i, \tag{2.199i}$$

$$2\omega_{ij}(x) := (\nabla_l U_k - \nabla_k U_l) \cdot \mathcal{P}^k{}_i(x) \cdot \mathcal{P}^l{}_j(x), \tag{2.199ii}$$

$$2\Theta_{ij}(x) := (\nabla_l U_k + \nabla_k U_l) \cdot \mathcal{P}^k{}_i(x) \cdot \mathcal{P}^l{}_j(x), \tag{2.199iii}$$

$$\Theta(x) := \Theta^k{}_k(x) = \nabla_k U^k, \tag{2.199iv}$$

$$\sigma_{ij}(x) := \Theta_{ij}(x) - (1/3)\Theta(x) \cdot \mathcal{P}_{ij}(x). \tag{2.199v}$$

Remarks: (i) The expansion scalar $\Theta(x)$ provides the expansion (or contraction) of a material domain (or body) containing streamlines.

(ii) *The symmetric shear tensor* $\sigma_{ij}(x)$ represents change of the shape of a material body, without any change of 3-volume content. (*Caution*: σ_{ij} is not to be confused with the nonrelativistic stress $\sigma_{\alpha\beta}$ of (2.38).)

(iii) The vanishing of *the antisymmetric vorticity tensor* $\omega_{ij}(x) \equiv 0$ represents *irrotational motions* of stream lines. (In such a case, we can prove that a family of three-dimensional hypersurfaces exists such that $U^i(x)\frac{\partial}{\partial x^i}$ are unit normals [86, 257].)

(iv) In the case where we have $\Theta(x) \equiv 0$, $\sigma_{ij}(x) \equiv 0$, and $\omega_{ij}(x) \neq 0$, the streamlines experience (relativistic) *rigid motions*. (See [86].)

Now, we shall state and prove a theorem about $\dot{U}^i(x)$, $\omega_{ij}(x)$, $\Theta_{ij}(x)$, $\Theta(x)$, and $\sigma_{ij}(x)$ fields.

Theorem 2.3.4. *Let the various fields $\dot{U}^i(x)$, $\omega_{ij}(x)$, $\Theta_{ij}(x)$, $\Theta(x)$, and $\sigma_{ij}(x)$ be defined according to* (2.199i–v) *for $x \in D \subset \mathbb{R}^4$. Then, the following identities hold:*

$$\dot{U}_j(x)U^j(x) = \omega_{ij}(x)U^j(x) = \Theta_{ij}(x)U^j(x) = \sigma_{ij}(x)U^j(x) \equiv 0; \tag{2.200}$$

$$\nabla_j U_i \equiv \omega_{ij}(x) + \sigma_{ij}(x) + (1/3)\Theta(x) \cdot \mathcal{P}_{ij}(x) - \dot{U}_i(x)U_j(x). \tag{2.201}$$

Proof. Equation (2.200) follows from definitions (2.199i–v) and the identity $\mathcal{P}^i{}_j(x)$ $U^j(x) \equiv 0$ in (2.198).

The right-hand side of (2.201) yields

$$\begin{aligned}
\text{R.H.S.} = {} & (1/2)\left[\nabla_j U_i - \nabla_i U_j + \dot{U}_i U_j - \dot{U}_j U_i\right] \\
& + (1/2)\left[\nabla_j U_i + \nabla_i U_j + \dot{U}_i U_j + \dot{U}_j U_i\right] \\
& - \dot{U}_i U_j \equiv \nabla_j U_i.
\end{aligned}$$

■

Now, we shall derive the *Raychaudhuri-Landau equation*. (See [214, 215, 257].)

Theorem 2.3.5. *Let the 4-velocity field $U^i(x)\frac{\partial}{\partial x^i}$ be of class C^2 in the domain $D \subset \mathbb{R}^4$. Then, the following differential equations hold:*

$$U^j(x)\nabla_j \Theta = \nabla_j \dot{U}^j + \omega^{jk}(x)\cdot\omega_{jk}(x) - \sigma^{jk}(x)\cdot\sigma_{jk}(x) \\ -(1/3)\cdot[\Theta(x)]^2 + R_{jk}(x)\cdot U^j(x)\cdot U^k(x), \tag{2.202i}$$

$$\frac{\mathrm{d}\Theta(\mathcal{X}(s))}{\mathrm{d}s} = \Big\{\nabla_j \dot{U}^j + \omega^{jk}(x)\cdot\omega_{jk}(x) - \sigma^{jk}(x)\cdot\sigma_{jk}(x) \\ -(1/3)\cdot[\Theta(x)]^2 + R_{jk}(x)\cdot U^j(x)\cdot U^k(x)\Big\}\Big|_{x^i=\mathcal{X}^i(s)}. \tag{2.202ii}$$

Proof. We start from the Ricci identity (1.145i) to express

$$(\nabla_k\nabla_l - \nabla_l\nabla_k)\,U_j = R^h{}_{jlk}(x)\cdot U_h(x).$$

Contracting with $g^{jl}(x)\cdot U^k(x)$, we deduce that

$$g^{jl}U^k\nabla_k\nabla_l U_j = g^{jl}\left\{\nabla_l\left[U^k\cdot\nabla_k U_j\right] - \left(\nabla_l U^k\right)\cdot\left(\nabla_k U_j\right)\right\} + R_{hk}U^hU^k \\ = \nabla_j\dot{U}^j - \left(\nabla^j U^k\right)\cdot\left(\nabla_k U_j\right) + R_{hk}U^hU^k.$$

Now, we use (2.201) for $\left(\nabla_k U_j\right)$. Substituting this expression in the middle term of the last equation and simplifying, we derive (2.202i). Restricting (2.202i) into the streamline given by $\frac{\mathrm{d}\mathcal{X}^i(s)}{\mathrm{d}s} = U^i\left(\mathcal{X}(s)\right)$, we obtain the other equation (2.202ii). ■

Remark: Equation (2.202ii) is called the *Raychaudhuri-Landau equation*, and it is very relevant in proving the singularity theorems which will be discussed briefly later in the book.

Example 2.3.6. In this example, we deal with *incoherent dust* (pressureless fluid) characterized by

$$T^{ij}(x) := \rho(x)U^i(x)U^j(x), \\ U_i(x)U^i(x) \equiv -1. \tag{2.203}$$

(See Example 2.1.11.) The proper mass density $\rho(x)$ is assumed to be strictly positive.

By the conservation equations (2.166i), we obtain that

$$0 = \nabla_j T^{ij} = \rho(x)U^j(x)\nabla_j U^i + U^i(x)\nabla_j\left[\rho U^j\right]. \tag{2.204}$$

Multiplying the above by $U_i(x)$ and contracting, we get

$$(1/2)\rho U^j\nabla_j\left(U_iU^i\right) + \left[U_iU^i\right]\cdot\nabla_j\left[\rho U^j\right] \\ = 0 - \nabla_j\left[\rho U^j\right] = 0, \\ \text{or,}\quad \nabla_j\left[\rho U^j\right] = 0. \tag{2.205}$$

The above is *the general relativistic continuity equation*. Substituting (2.205) into (2.204) and dividing by $\rho(x) > 0$, we derive

$$U^j(x)\nabla_j U^i = \dot{U}^i(x) = 0, \tag{2.206i}$$

$$U^j(x)\nabla_j U^i{}_{|x^i = \mathcal{X}^i(s)} = \frac{D\mathcal{U}^i(s)}{\mathrm{d}s} \tag{2.206ii}$$

$$= \frac{\mathrm{d}^2\mathcal{X}^i(s)}{\mathrm{d}s^2} + \left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\}_{|..} \cdot \frac{\mathrm{d}\mathcal{X}^j(s)}{\mathrm{d}s} \cdot \frac{\mathrm{d}\mathcal{X}^k(s)}{\mathrm{d}s} = 0. \tag{2.206iii}$$

Therefore, by (2.95ii), the streamlines follow *timelike geodesics* (as expected for a free pressureless fluid). Thus, the geodesic equations of motion of dust particles emerged from *conservation equations* (*which are consequences of the gravitational field equations* (2.159i).)

Now, supplementing the above condition of $\dot{U}_i(x) = 0$, *we assume further that the dust particles are undergoing irrotational motion*, that is,

$$\omega_{ij}(x) \equiv 0. \tag{2.207}$$

The gravitational field equation (2.162i) yields, from (2.203),

$$R_{ij}(x) = -\kappa\left[\rho(x)U_i(x)U_j(x) + (1/2)g_{ij}(x)\cdot\rho(x)\right],$$
$$R_{ij}(x)U^i(x)U^j(x) = -(\kappa/2)\cdot\rho(x) < 0. \tag{2.208}$$

Substituting (2.206ii), (2.207), and (2.208) into (2.202ii), we derive, assuming $\sigma^{jk}(x)\sigma_{jk}(x) \geq 0$, that

$$\frac{\mathrm{d}\Theta(\cdot\cdot)}{\mathrm{d}s} = -\left\{\sigma^{jk}(x)\sigma_{jk}(x) + (1/3)[\Theta(\cdot\cdot)]^2 + (\kappa/2)\rho(x)\right\}_{|x^i=\mathcal{X}^i(s)} < 0. \tag{2.209}$$

The above inequality demonstrates that the rate of expansion of the dust body (with particles following timelike geodesics) slows down with (proper) time. That fact proves the attractive aspects of gravitational forces. □

(*Remarks*: Consult # 5(ii) of Exercise 2.3 for the proof of $\sigma^{jk}(x)\sigma_{jk}(x) \geq 0$.)

Now, we shall investigate the streamlines of *a general material continuum*. The energy–momentum–stress tensor is furnished by (2.196) and (2.197). The differential conservation equation (2.166i) yields

$$0 = \nabla_j T^{ij} = \nabla_j\left[\rho U^i U^j - S^{ij}\right]$$
$$= \rho U^j \nabla_j U^i + U^i\left[\nabla_j\left(\rho U^j\right)\right] - \nabla_j S^{ij}. \tag{2.210}$$

Multiplying the above by $U_i(x)$ and using $U_i(x)U^i(x) \equiv -1$, we derive that

$$\nabla_j\left[\rho U^j\right] + U_i\nabla_j S^{ij} = 0. \tag{2.211}$$

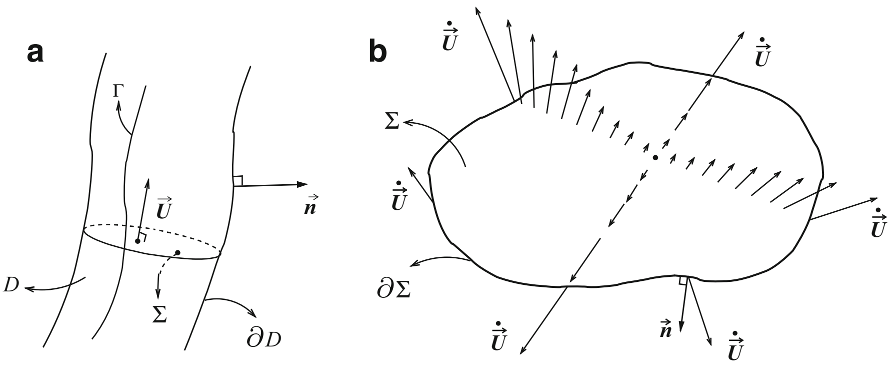


Fig. 2.18 (**a**) shows a material world tube. (**b**) shows the continuous $\dot{\vec{\mathbf{U}}}$ field over Σ

Substituting the equation above into (2.210), we deduce that

$$\rho U^j \nabla_j U^i = \left[\delta^i{}_k + U^i U_k\right] \nabla_j S^{kj}, \tag{2.212i}$$

$$\rho\left[\mathcal{X}(s)\right] \cdot \frac{D\mathcal{U}^i(s)}{\mathrm{d}s} = \rho \dot{U}^i{}_{|..} = \left[\mathcal{P}^i{}_k(x) \cdot \nabla_j S^{kj}\right]_{|..} . \tag{2.212ii}$$

This expression provides the governing equations for the time evolution of streamlines.

In Example 2.3.6 involving *incoherent dust*, we concluded that streamlines pursue *timelike geodesics*. The proof for this fact emerged just from the gravitational field equations. Research has been pursued on the question of motion of an extended isolated material body [79].

We shall now provide some insight into such a problem under simplified assumptions. We assume the usual field equations and junction conditions:

$$G_{ij}(x) = \begin{cases} -\kappa\left[\rho(x) U_i(x) U_j(x) - S_{ij}(x)\right] & \text{inside } D, \\ 0 & \text{outside } D\,; \end{cases} \tag{2.213i}$$

$$G_{ij}(x)\, n^j(x)_{|\partial D} = 0. \tag{2.213ii}$$

(See Fig. 2.18a.)

We define another scalar field on $\partial\Sigma$ by

$$\left[\Phi(x)\right]_{|\partial\Sigma} := \left[\rho(x) n_i(x) \dot{U}^i(x)\right]_{|\partial\Sigma}. \tag{2.214}$$

Theorem 2.3.7. *Let the world tube of a material continuum be contained in the domain* $D \subset \mathbb{R}^4$. *Let the field equations and the junction conditions* (2.213i) *and* (2.213ii) *hold. Moreover, let the timelike* 4*-velocity tangent vector field* $U^i(x)\frac{\partial}{\partial x^i}$

be of class C^1 and $\dot{U}^i(x)\frac{\partial}{\partial x^i}|_{\partial\Sigma} \neq \vec{\mathbf{O}}(x)_{|\partial\Sigma}$. Furthermore, let the streamlines be irrotational so that $\omega_{ij}(x) \equiv 0$ in D. Then there exists at least one timelike continuous geodesic curve inside D.

Proof. Irrotational motion implies the existence of a one-parameter family of orthogonal, three-dimensional hypersurfaces [86, 126, 257]. One of these hypersurfaces, Σ, is shown in both of Figs. 2.18a, b. Since the 4-acceleration $\dot{U}^i(x)\frac{\partial}{\partial x^i}$ is orthogonal to timelike 4-velocity vector $U^i(x)\frac{\partial}{\partial x^i}$, it must be either the zero 4-vector or else a spacelike 4-vector inside Σ. Assuming $\rho(x) > 0$ in (2.213i), we conclude from (2.214) that $\text{sgn}\,[\Phi(x)]_{|\partial\Sigma} = \text{sgn}\big[n_i(x)\dot{U}^i(x)\big]_{|\partial\Sigma}$. By the assumption $\dot{U}^i(x)\frac{\partial}{\partial x^i}|_{\partial\Sigma} \neq \vec{\mathbf{O}}(x)_{|\partial\Sigma}$, we conclude that $\Phi(x)_{|\partial\Sigma} \neq 0$. Therefore, either $\Phi(x)_{|\partial\Sigma} > 0$ or else $\Phi(x)_{|\partial\Sigma} < 0$. Assume that $\Phi(x)_{|\partial\Sigma} > 0$. (The case of $\Phi(x)_{|\partial\Sigma} < 0$ can be treated in a similar fashion.) Now, the condition $\Phi(x)_{|\partial\Sigma} > 0$ implies that the 4-vector $\dot{U}^i(x)\frac{\partial}{\partial x^i}|_{\partial\Sigma}$ points outward everywhere on the continuous, piecewise-differentiable closed boundary curve *on* $\partial\Sigma$. (See Fig. 2.18b.) It is clear that *the winding number, or index*, of the continuous 4-vector field $\dot{U}^i(x)\frac{\partial}{\partial x^i}|_{\partial\Sigma}$ around the closed contour $\partial\Sigma$ is exactly one. Therefore, the fixed-point theorem[7] tells us that the 4-vector field $\dot{U}(x)\frac{\partial}{\partial x^i}|_{\Sigma}$ must be zero at *an interior point* of Σ. Since the world tube D contains a one-parameter family of spacelike hypersurfaces Σ, we have inside the world tube D, the image Γ of a continuous curve of zero 4-acceleration, or *a geodesic*. ■

Remarks: (i) The image of a curve Γ inside the material tube *need not be a stream line*.

(ii) A spinning body might not possess an interior geodesic. (See [46].)

Example 2.3.8. Consider the 1-form $\tilde{\mathbf{U}}(x) := U_i(x)\,\mathrm{d}x^i$. Assume that it is a *closed form*, or, $d\,\tilde{\mathbf{U}}(x) = \mathbf{O}_{..}(x)$. By (1.61), $\partial_i U_j - \partial_j U_i = 0$. Thus, the vorticity tensor components $\omega_{ij}(x) \equiv 0$. By Theorem 1.2.21, there exists a differentiable function ψ such that $U_i(x) = \partial_i\psi$. The function ψ must satisfy *the Hamilton–Jacobi equation* $g^{ij}(x)\cdot\partial_i\psi\cdot\partial_j\psi \equiv -1$. (See Example A2.3.) The one-parameter spacelike hypersurfaces Σ are provided by $\psi(x) = k =$ const. A typical two-dimensional boundary is $\partial\Sigma = \Sigma \cap \partial D$. Now, consider the equation $N^i(x)\partial_i\psi_{|\partial\Sigma} = 0$. Solving this underdetermined system, we arrive at a set of solutions:

$$-\vec{\mathbf{N}}_{(1)}(x) := (\partial_4\psi)\cdot\frac{\partial}{\partial x^1} - (\partial_1\psi)\cdot\frac{\partial}{\partial x^4},$$

$$-\vec{\mathbf{N}}_{(2)}(x) := (\partial_4\psi)\cdot\frac{\partial}{\partial x^2} - (\partial_2\psi)\cdot\frac{\partial}{\partial x^4},$$

[7] Let $\vec{\mathbf{V}}(\cdot\cdot)$ be a continuous vector field defined on Σ such that $\vec{\mathbf{V}}(\cdot\cdot) \neq \vec{\mathbf{0}}(\cdot\cdot)$ for any point x on $\partial\Sigma$. If the index or winding number of $\vec{\mathbf{V}}(\cdot\cdot)$ around $\partial\Sigma$ is not zero, then there exists at least one $x_0 \in \Sigma$ such that $\vec{\mathbf{V}}(x_0) = \vec{\mathbf{0}}(x_0)$. (See [37].)

$$
\begin{aligned}
-\vec{\mathbf{N}}_{(3)}(x) &:= (\partial_4\psi)\cdot\frac{\partial}{\partial x^3} - (\partial_3\psi)\cdot\frac{\partial}{\partial x^4},\\
\vec{\mathbf{n}}(x)_{|\partial\Sigma} &= \sum_{\mu=1}^{3}\left[c^{(\mu)}(x)\vec{N}_{(\mu)}(x)\right]_{|\partial\Sigma},\\
\sum_{\mu}\sum_{\nu}&\left[c^{(\mu)}(x)\cdot c^{(\nu)}(x)\cdot g_{ij}(x)N^i_{(\mu)}(x)N^j_{(\nu)}(x)\right]_{|\partial\Sigma} \equiv 1\,.
\end{aligned}
$$

The last algebraic equation is underdetermined and solution functions $c^{(\mu)}(x)$ exist. By (2.214) and (2.199i),

$$
\Phi(x)_{|\partial\Sigma} = \left[\rho(x)\cdot\left(\Sigma\, c^{(\mu)}(x)N^i_{(\mu)}(x)\right)\cdot g^{jk}(x)\nabla_k\psi\cdot\nabla_j\nabla_i\psi\right]_{|\partial\Sigma}\,.
$$

In case $\Phi(x)_{|\partial\Sigma} \neq 0$, Theorem 2.3.7 asserts the existence of an interior geodesic. The complicated expression of $\Phi(x)$ can be considerably reduced for *a special case*. We *assume the metric* may be cast in the form

$$
\begin{aligned}
\mathbf{g}_{..}(x) &= g_{\alpha\beta}(x)\,\mathrm{d}x^\alpha\otimes\mathrm{d}x^\beta + g_{44}(x)\,\mathrm{d}x^4\otimes\mathrm{d}x^4,\\
\mathrm{d}s^2 &= g_{\alpha\beta}(x)\,\mathrm{d}x^\alpha\mathrm{d}x^\beta - |g_{44}(x)|\,(\mathrm{d}x^4)^2.
\end{aligned}
$$

Moreover, we make a simple choice of Σ as

$$
\Sigma:\quad \psi(x) := -x^4 = k = \text{ const.}
$$

Therefore,

$$
\begin{aligned}
&U_\alpha(x)\equiv 0,\quad U_4(x)\equiv -1,\\
&g^{ij}(x)U_i(x)U_j(x) = g^{44}(x)\equiv -1,\\
\text{or,}\quad &g_{44}(x)\equiv -1.
\end{aligned}
$$

The vectors $\vec{\mathbf{N}}_{(\mu)}(x)$ and $\vec{\mathbf{n}}(x)_{|\partial\Sigma}$ satisfy

$$
\begin{aligned}
&\vec{\mathbf{N}}_{(\mu)}(x) = \frac{\partial}{\partial x^\mu},\quad \vec{\mathbf{n}}(x)_{|\partial\Sigma} = \sum_{\mu} c^{(\mu)}(x)\frac{\partial}{\partial x^\mu},\\
&\sum_{\mu}\sum_{\nu}\left[g_{\mu\nu}(x)c^{(\mu)}(x)c^{(\nu)}(x)\right]_{|\partial\Sigma} \equiv 1.
\end{aligned}
$$

The function $\Phi(x)_{|\partial\Sigma}$ reduces to

$$\begin{aligned}
&\left[\rho(x)\sum_{\mu} c^{(\mu)}(x)\delta^{i}{}_{\mu}g^{jk}(x)\nabla_k\psi\cdot\nabla_j\nabla_i\psi\right]_{|\partial\Sigma}\\
&=\rho(x)\sum_{\mu} c^{(\mu)}(x)g^{j4}(x)\left[0-\left\{\begin{matrix} i \\ j\ \mu\end{matrix}\right\}\partial_i\psi\right]_{|..}\\
&=\left[\rho(x)\sum_{\mu} c^{(\mu)}(x)\left\{\begin{matrix} 4 \\ 4\ \mu\end{matrix}\right\}\right]_{|..}\equiv 0.
\end{aligned}$$

Therefore, streamlines on $\partial\Sigma$ are *all timelike geodesics*. If we analyze the metric under consideration

$$\mathrm{d}s^2 = g_{\alpha\beta}(x)\,\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta} - (\mathrm{d}x^4)^2,$$

it turns out to be *a geodesic normal coordinate chart*. (Compare with (1.160).) Therefore, the streamlines *coincide* (inside and on the boundary of the whole domain D) with x^4-coordinate curves, which are all *timelike geodesics*. □

We have defined and discussed in Theorem 2.1.10, the total 4-momentum of an extended body in flat space–time. We would like to generalize those definitions for *curved space–time*. We can still use the Fig. 2.7 for the present purpose. The main difficulty in this endeavor is the problem of tensor transformations for a spatial integral under a general coordinate transformation in (1.2) and (1.37). The only logical choice is to express these integrals as *tensorially invariant entities*.

We start from four conservation equations (2.166i) explicitly stated as $\nabla_j T^{ij} = 0$. We introduce an additional differentiable vector field $\vec{\mathbf{V}}(x)$ satisfying

$$2\nabla_j\left[T^{ij}V_i\right] = T^{ij}(x)\left[\nabla_j V_i + \nabla_i V_j\right] = 0 \quad (2.215\text{i})$$

or

$$G^{ij}(x)\left[\nabla_j V_i + \nabla_i V_j\right] = 0. \quad (2.215\text{ii})$$

The above equation (2.215ii) *generalizes the Killing vector equations* (1.171iii). The existence of each of the vector fields $\vec{\mathbf{V}}(x)$, satisfying (2.215i), gives rise to a conserved integral. Before we prove such a statement, consider the material world tube in Fig. 2.19. (Compare with Fig. 2.7.)

Now we shall state and prove the following theorem on integral conservations.

Theorem 2.3.9. *Let the components $T^{ij}(x)$ of the differentiable energy-momentum-stress tensor field be nonzero inside the domain D of the world tube and vanish outside. Let the nonnull boundary $\partial D := \partial D_{(3)} \cup \Sigma_{(0)} \cup \Sigma$ be continuous, piecewise-differentiable, orientable, and closed. Moreover, let the junction*

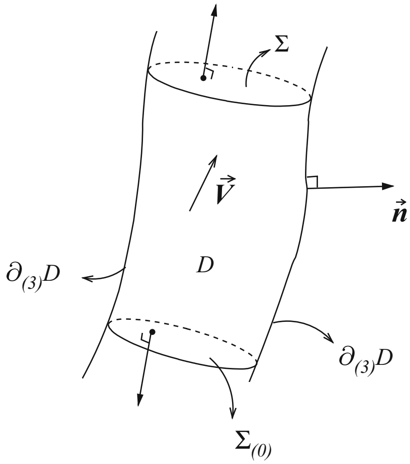


Fig. 2.19 A doubly sliced world tube of an isolated, extended material body

conditions $T^{ij}(x)n_j(x)_{|\partial D_{(3)}} = 0$ *hold. Furthermore, let a differentiable vector field* $\vec{\mathbf{V}}(x)$ *exist inside* D *satisfying* (2.215ii). *Then, there exists an invariant, conserved integral:*

$$P := -\int_{\Sigma} \left[T^{ij}(x)V_j(x)n_i(x)\right]_{|\Sigma} \cdot \mathrm{d}^3 v = \text{ const.} \tag{2.216}$$

Proof. Applying Gauss' Theorem 1.3.27, and the differential equation (2.215i), we obtain

$$\begin{aligned} 0 &= \int_D \nabla_i \left[T^{ij}V_j\right] \cdot \mathrm{d}^4 v = \int_{\partial D} \left[T^{ij}V_j n_i\right]_{|\partial D} \cdot \mathrm{d}^3 v \\ &= \int_{\Sigma_{(0)}} \left[T^{ij}V_j n_i\right]_{|\Sigma_{(0)}} \cdot \mathrm{d}^3 v + \int_{\Sigma} \left[T^{ij}V_j n_i\right]_{|\Sigma} \cdot \mathrm{d}^3 v + \int_{\partial D_{(3)}} \left[T^{ij}V_j n_i\right]_{|\partial D_{(3)}} \cdot \mathrm{d}^3 v. \end{aligned}$$

or,

$$\begin{aligned} \int_{\Sigma} \left[T^{ij}V_j n_i\right]_{|\Sigma} \cdot \mathrm{d}^3 v + 0 &= -\int_{\Sigma_{(0)}} \left[T^{ij}V_j n_i\right]_{|\Sigma_{(0)}} \cdot \mathrm{d}^3 v \\ &= \text{const.} \end{aligned}$$

■

Consider the scenario where the space–time domain D has some *symmetries*, or, admits groups of motion. Then, there will exist some Killing vector $\vec{\mathbf{K}}(x)$ satisfying $\nabla_j K_i + \nabla_i K_j = 0$. (See (1.171iii).) Therefore, in a domain with symmetry, *one possible solution* of (2.215i) is $\vec{\mathbf{V}}(x) = \vec{\mathbf{K}}(x)$.

Example 2.3.10. Consider *the flat space–time* and a global Minkowskian chart with $g_{ij}(x) = d_{ij}$. It is mentioned in p. 135 that there exist ten Killing vector fields furnished by

$$\vec{\mathbf{K}}_{(A)}(x) := \delta^i_{(A)} \frac{\partial}{\partial x^i}, \tag{2.217i}$$

$$\vec{\mathbf{K}}_{(A)(B)}(x) := d_{(A)i}\delta^j_{(B)} x^i \frac{\partial}{\partial x^j} - d_{(B)j}\delta^i_{(A)} x^j \frac{\partial}{\partial x^i}, \tag{2.217ii}$$

Here, $A, B \in \{1, 2, 3, 4\}$ are just *labels*. (We still use the summation convention for capital Roman indices.)

According to (2.216), there exist the following ten conserved invariant integrals:

$$P_{(A)} = -\int_{\Sigma} \left[d_{(A)i} T^{ij}(x) n_j(x)\right]_{|\Sigma} \cdot \mathrm{d}^3 v, \tag{2.218i}$$

$$J_{(A)(B)} := \int_{\Sigma} d_{(A)k} \cdot d_{(B)j} \left\{\left[x^j T^{ki}(x) - x^k T^{ji}(x)\right] n_i(x)\right\}_{|\Sigma} \cdot \mathrm{d}^3 v. \tag{2.218ii}$$

(Compare the equations above with (2.49) and (2.50).)

Consider *a constant-valued Lorentz transformation given by* [55]

$$\begin{aligned} &\widehat{\vec{\mathbf{K}}}_{(A)} = l^{(B)}{}_{(A)} \vec{\mathbf{K}}_{(B)}(x), \\ &l^{(A)}{}_{(B)}\, d_{(A)(C)}\, l^{(C)}{}_{(E)} = d_{(B)(E)}, \\ &l^{(4)}{}_{(4)} \geq 1. \end{aligned} \tag{2.219}$$

(This class of transformation is known as *orthochronus*.) Note that for *constant-valued* $l^{(A)}{}_{(B)}$, the transformed vector $\widehat{\vec{\mathbf{K}}}_{(A)}(x)$ *is also a Killing vector field*. Therefore, by (2.217i), (2.218i), and (2.219), we derive transformation rules:

$$\widehat{P}_{(A)} = l^{(B)}{}_{(A)} P_{(B)}.$$

Thus, we identify *invariant constants* $P_{(A)}$'s and $J_{(A)(B)}$'s as components of the total 4-momentum and the total relativistic angular momentum of an extended material body. □

Example 2.3.11. Assume that $T^{ij}(x)$ has an invariant eigenvalue $\lambda(x) \neq 0$ satisfying $T^{ij}(x) e_j(x) = \lambda(x) e^i(x)$. Moreover, assume that

$$V_j(x) = \sigma(x) e_j(x), \quad \sigma(x) \neq 0.$$

Equation (2.215i) yields

$$\begin{aligned}0 &= T^{ij}\left[\sigma \cdot \nabla_i e_j + (\nabla_i \sigma) \cdot e_j\right]\\ &= \sigma\left[\nabla_i\left(T^{ij} e_j\right)\right] + (\nabla_i \sigma) \cdot (\lambda e^i)\\ &= \sigma\left[\lambda\left(\nabla_i e^i\right) + e^i \cdot \nabla_i \lambda\right] + (\lambda e^i) \cdot \nabla_i \sigma.\end{aligned}$$

or,

$$e^i \nabla_i (\ln|\sigma|) + \left[(\nabla_j e^j) + e^j \nabla_j (\ln|\lambda|)\right] = 0.$$

The above equation is a linear, first-order p.d.e. with the unknown function $\ln|\sigma(x)|$.

By the discussions in Appendix 2, especially by the equations in (A2.13), solutions of this equation exist in principle. Therefore, the corresponding conserved integral is

$$P := -\int_{\Sigma} \left[\sigma(x)\lambda(x)e^i(x)n_i(x)\right]_{|\Sigma} \cdot \mathrm{d}^3 v. \qquad \square$$

Now, we shall investigate the generalization (2.215i,ii) of Killing vector equation (1.171iii). A very general class of solutions of (2.215i,ii) is furnished by[8]

$$\begin{aligned}G^{ij}(x)V_j(x) = -\kappa T^{ij}(x)V_j(x) = \nabla^i h + \nabla_j A^{ij},\\ \square h \equiv \nabla_i \nabla^i h = 0, \quad A^{ji}(x) \equiv -A^{ij}(x).\end{aligned} \tag{2.220}$$

Here, the scalar wave field $h(x)$ and the antisymmetric field $A^{ij}(x)$ are of class C^2 and otherwise *arbitrary*. There are *infinitely many solutions* in (2.220).

The physical implications of the solutions (2.220) are *not obvious*.

[8] 1. The Helmotz theorem [159] on differentiable vector field $\vec{\mathbf{W}}(\mathbf{x})$ in a three-dimensional domain allows the decomposition

$$W^\alpha(\mathbf{x}) = \nabla^\alpha h + \eta^{\alpha\beta\gamma}(\mathbf{x})\left[\nabla_\gamma A_\beta - \nabla_\beta A_\alpha\right].$$

2. For *a closed, differential p-form* of (1.58), the Hodge decomposition theorem [104] asserts that

$$\begin{aligned}W_{i_1,\ldots,i_p}(x)\,\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p} &= h_{i_1,\ldots,i_p}(x)\,\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_p}\\ &\quad + \mathrm{d}\left[\alpha_{i_1,\ldots,i_{p-1}}(x)\,\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_{p-1}}\right],\\ \nabla_j \nabla^j h_{i_1,\ldots,i_p} &= 0.\end{aligned}$$

Exercises 2.3

1. Consider *the flat metric in double-null coordinates* of Example 2.1.17. In a similar fashion, the metric can be expressed as

$$\begin{aligned} ds^2 = d_{ij}\,dx^i dx^j &= (dx^1)^2 + (dx^2)^2 + (dx^3 - dx^4)(dx^3 + dx^4) \\ &=: \left(d\widehat{x}^1\right)^2 + \left(d\widehat{x}^2\right)^2 + 2\, d\widehat{x}^3 d\widehat{x}^4 =: \eta_{ij}\, d\widehat{x}^i d\widehat{x}^j . \end{aligned}$$

   Consider an energy–momentum–stress tensor matrix $\left[T_{ij}(x_0)\right]$ of Segre class $[1,\ 3]$. Show that the transformed energy–momentum–stress tensor matrix

$$\left[\widehat{T}_{ij}\,(\widehat{x}_0)\right] := \begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 1 & 0 \\ 0 & 1 & 0 & \lambda_{(2)} \\ 0 & 0 & \lambda_{(2)} & 0 \end{bmatrix}$$

   remains of Segre class $[1, 3]$ (for $\lambda_{(1)} \neq \lambda_{(2)}$). Moreover, prove that the invariant triple eigenvector with respect to η^{ij} is along *the null direction* $\begin{bmatrix} 0 \\ 0 \\ 0 \\ t \end{bmatrix}$, $t \neq 0$.

2. Let the matrix $\left[T_{(a)(b)}(x_0)\right]$ admit a timelike eigenvector $U^{(b)}(x_0)$ satisfying $T_{(a)(b)}(x_0)U^{(b)}(x_0) = -\rho(x_0)U_{(a)}(x_0),\ d_{(a)(b)}U^{(a)}(x_0)U^{(b)}(x_0) = -1$. Consider the case that every invariant eigenvalue of $\left[T_{(a)(b)}(x_0)\right]$ is *real*. Prove that the matrix belongs to the Type-I of (2.179).
3. Prove Theorem 2.3.2.
4. (i) Let a projection tensor be defined as:

$$\mathcal{P}^i{}_j(x) := \delta^i{}_j - \varepsilon(v)V^i(x)V_j(x), \quad V_iV^j = \varepsilon(v) = \pm 1.$$

   Show that $\mathcal{P}^i{}_k(x)\mathcal{P}^k{}_j(x) = \mathcal{P}^i{}_j(x)$.

   (ii) Deduce that the invariant eigenvalues of $\mathcal{P}_{ij}(x)$ are exactly zero and one.
5. (i) Derive that $g^{ij}(x)\sigma_{ij}(x) = d^{(a)(b)}\sigma_{(a)(b)}(x) \equiv 0$.

   (ii) Prove that $\omega_{ij}(x) \cdot \omega^{ij}(x) \geq 0$ and $\sigma_{ij}(x) \cdot \sigma^{ij}(x) \geq 0$.
6. Show that in the case of vanishing expansion, $\Theta(x) \equiv 0$, and shear tensor $\sigma_{ij}(x)\, dx^i \otimes dx^j \equiv \mathbf{O}_{..}(x)$, *the Lie derivative of the projection tensor* $\mathcal{P}_{..}(x) := \left[g_{ij}(x) + U_i(x)U_j(x)\right]dx^i \otimes dx^j$ reduces to $L_{\vec{\mathbf{U}}}\left[\mathcal{P}_{..}(x)\right] \equiv \mathbf{O}_{..}(x)$.
7. Deduce that

$$\begin{aligned} U^k\nabla_k\left[\nabla_l U_j + \nabla_j U_l\right] &= \left[\nabla_l \dot{U}_j + \nabla_j \dot{U}_l\right] \\ &\quad - \left[\omega_j{}^k + \Theta_j{}^k - \dot{U}^k U_j\right] \cdot \left[\omega_{kl} + \Theta_{kl} - \dot{U}_k U_l\right] \end{aligned}$$

$$- \left[\omega_l{}^k + \Theta_l{}^k - \dot{U}^k U_l\right] \cdot \left[\omega_{kj} + \Theta_{kj} - \dot{U}_k U_j\right]$$
$$+ \left[R_{hjlk} + R_{hljk}\right] \cdot U^h U^k.$$

8. Consider a domain of material continuum with $T^{ij}(x) = \rho(x)U^i(x)U^j(x) - S^{ij}(x)$, $U_i(x)U^i(x) \equiv -1$. Prove that along a streamline, the rate of change of the proper mass density is given by

$$\frac{\mathrm{d}\rho[\mathcal{X}(s)]}{\mathrm{d}s} = -\left[\rho(x) \cdot (\nabla_i U^i) + U_i(x) \cdot \nabla_j S^{ij}\right]_{|..}.$$

9. Consider the material world tube depicted in Fig. 2.18a. Let the 4-acceleration spacelike vector on the boundary be given by

$$\dot{U}^i(x)_{|\partial\Sigma} = C^i = \text{ const.}, \quad |C^1| + |C^2| + |C^3| > 0.$$

In the case where the 4-acceleration vector field $\dot{\vec{\mathbf{U}}}$ is continuous in $\Sigma \cup \partial\Sigma$, determine whether or not there exists a geodesic inside the world tube.

10. An *anti-de Sitter space–time* domain (of constant negative curvature) is characterized by the metric (with cosmological constant set equal to -3 for notational convenience, since this corresponds to a radius of curvature of -1):

$$\mathbf{g}_{..}(x) := \left[1 + (x^1)^2\right]^{-1} \cdot \mathrm{d}x^1 \otimes \mathrm{d}x^1 + (x^1)^2 \cdot \mathrm{d}x^2 \otimes \mathrm{d}x^2$$
$$+ (x^1)^2 \cdot (\sin x^2)^2 \cdot \mathrm{d}x^3 \otimes \mathrm{d}x^3 - \left[1 + (x^1)^2\right] \cdot \mathrm{d}x^4 \otimes \mathrm{d}x^4,$$

$$\mathrm{d}s^2 = (1 + r^2)^{-1}\mathrm{d}r^2 + r^2\left[\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right] - (1 + r^2)\,\mathrm{d}t^2.$$

Prove that the following conserved integrals, representing the total energy and total "angular momentum components," respectively, exist:

$$P_{(t)} := -\int_{D_{(t)}} T^{44}(\cdot\cdot)(1 + r^2)\,r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi = \text{ const.},$$

$$J_{(1)} := \int_{D_{(t)}} \left[\sin\varphi \cdot T^{24}(\cdot\cdot) + \sin\theta \cdot \cos\theta \cdot \cos\varphi \cdot T^{34}(\cdot\cdot)\right] r^4 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

$$J_{(2)} := \int_{D_{(t)}} \left[\cos\varphi \cdot T^{24}(\cdot\cdot) - \sin\theta \cdot \cos\theta \cdot \sin\varphi \cdot T^{34}(\cdot\cdot)\right] r^4 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

$$J_{(3)} := \int_{D_{(t)}} T^{34}(\cdot\cdot) \cdot r^4 \sin^3\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

Answers and Hints to Selected Exercises

1. The 4×4 matrix to be investigated is given by:

$$\left[\widehat{T}_{ij}\left(\widehat{x}_0\right) - \lambda\eta_{ij}\right] := \begin{bmatrix} \lambda_{(1)} - \lambda & 0 & 0 & 0 \\ 0 & \lambda_{(2)} - \lambda & 1 & 0 \\ 0 & 1 & 0 & \lambda_{(2)} - \lambda \\ 0 & 0 & \lambda_{(2)} - \lambda & 0 \end{bmatrix}.$$

2. From (2.196) and (2.197), $S_{(a)(b)} := \rho(x_0) \cdot U_{(a)}(x_0) \cdot U_{(b)}(x_0) - T_{(a)(b)}(x_0)$, $S_{(a)(b)}(x_0) \cdot U^{(b)}(x_0) = 0$. Let $\lambda(x_0)$ be a real, nonzero invariant eigenvalue so that $S_{(a)(b)}(x_0) \cdot \mathrm{e}^{(b)}(x_0) = \lambda(x_0) \cdot e_{(a)}(x_0)$. Therefore, $\lambda(x_0) \cdot [e_{(a)}(x_0) \cdot U^{(a)}(x_0)] = [S_{(a)(b)}(x_0) \cdot U^{(a)}(x_0)] \cdot \mathrm{e}^{(b)}(x_0) = 0$. Thus, for $\lambda(x_0) \neq 0$, the corresponding eigenvector $\mathrm{e}^{(a)}(x_0)$ is *spacelike* and orthogonal to $U^{(a)}(x_0)$. In case $S_{(a)(b)}(x_0)$ has three nonzero invariant eigenvalues, there exist three spacelike eigenvectors orthogonal to $U^{(a)}(x_0)$. Thus,

$$T^{(a)(b)}(x_0) = \rho(x_0) \cdot U^{(a)}(x_0) \cdot U^{(b)}(x_0) + \sum_{\mu=1}^{3} \lambda_{(\mu)}(x_0) \cdot \mathrm{e}_{(\mu)}^{(a)} \cdot \mathrm{e}_{(\mu)}^{(b)}.$$

Therefore, $[T_{ab}(x_0)]$ is of Type-I (even if some or all of $\lambda_{(\mu)}(x_0) = 0$.)

4. (i)

$$\begin{aligned}&\left[\delta^i{}_k - \varepsilon(v) U^i U_k\right] \cdot \left[\delta^k{}_j - \varepsilon(v) U^k U_j\right] \\ &= \delta^i{}_j + (-1 - 1 + 1) \cdot \varepsilon(v) \cdot U^i U_j.\end{aligned}$$

5. (ii) Consider the vector field transformation equations

$$\widehat{U}^i\left(\widehat{x}\right) = \frac{\partial \widehat{X}^i(x)}{\partial x^j} \cdot U^j(x).$$

Choosing three spatial equations from above, investigate

$$0 = \widehat{U}^\alpha\left(\widehat{x}\right) = U^j(x) \frac{\partial \widehat{X}^\alpha(x)}{\partial x^j}.$$

Considering the equations above as linear, first-order, partial differential equations for *unknown functions*, $\widehat{X}^\alpha(x)$ conclude that solutions exist locally. (See Appendix 2.) In new coordinates, the condition $\widehat{U}_i\left(\widehat{x}\right) \cdot \widehat{U}^i\left(\widehat{x}\right) \equiv -1$ yields $\widehat{g}_{44}\left(\widehat{x}\right)\left[\widehat{U}^4\left(\widehat{x}\right)\right]^2 \equiv -1$. Therefore, $\widehat{g}_{44}\left(\widehat{x}\right) < 0$ and $\widehat{U}^4\left(\widehat{x}\right) \neq 0$. (The choice $\widehat{U}^4\left(\widehat{x}\right) > 0$ is the usual one.) Such coordinates constitute a *comoving coordinate chart*. Choosing the orthonormal tetrad

$\left\{\vec{e}_{(1)}, \vec{e}_{(2)}, \vec{e}_{(3)}, [|g_{44}|]^{-\frac{1}{2}} \cdot \delta^i{}_{(4)} \partial_i\right\}$, the components $\widehat{\omega}_{(\alpha)(4)}(\widehat{x}) \equiv 0$. Thus,

$$\omega_{ij}(x) \cdot \omega^{ij}(x) = \widehat{\omega}_{(a)(b)}(\widehat{x}) \cdot \widehat{\omega}^{(a)(b)}(\widehat{x}) = \widehat{\omega}_{(\alpha)(\beta)}(\widehat{x}) \cdot \widehat{\omega}^{(\alpha)(\beta)}(\widehat{x}) + 0$$
$$= 2\left\{\left[\widehat{\omega}_{(1)(2)}(\widehat{x})\right]^2 + \left[\widehat{\omega}_{(2)(3)}(\widehat{x})\right]^2 + \left[\widehat{\omega}_{(3)(1)}(\widehat{x})\right]^2\right\} \geq 0.$$

7. By the Ricci identity (1.145i), obtain

$$(\nabla_k \nabla_l - \nabla_l \nabla_k) U_j = R_{hjlk} U^h,$$
$$U^k \left[\nabla_k \nabla_l U_j\right] = \nabla_l \dot{U}_j - \left(\nabla_l U^k\right) \cdot \left(\nabla_k U_j\right) + R_{hjlk} \cdot U^h U^k.$$

8. Use (2.211).
9. The winding number or the index of $\dot{\vec{\mathbf{U}}}(x)$ around $\partial \Sigma$ is exactly zero. There exist no $x_0 \in \Sigma$ such that $\dot{\vec{\mathbf{U}}}(x_0) = \vec{\mathbf{0}}(x_0)$.
10. There exist *ten generators* for the Killing vector fields [129]. Out of these, for the present problem, the following four are relevant (with cosmological constant set equal to -3):

$$\vec{\mathbf{K}}_{(t)}(\cdot\cdot) := \frac{\partial}{\partial t},$$
$$\vec{\mathbf{K}}_{(1)}(\cdot\cdot) := \sin\varphi \cdot \frac{\partial}{\partial \theta} + \cot\theta \cdot \cos\varphi \cdot \frac{\partial}{\partial \varphi},$$
$$\vec{\mathbf{K}}_{(2)}(\cdot\cdot) := \cos\varphi \cdot \frac{\partial}{\partial \theta} - \cot\theta \cdot \sin\varphi \cdot \frac{\partial}{\partial \varphi},$$
$$\vec{\mathbf{K}}_{(3)}(\cdot\cdot) := \frac{\partial}{\partial \varphi}.$$

Substitute the above vectors for $\vec{\mathbf{V}}(x) = \vec{\mathbf{K}}_{(..)}(\cdot\cdot)$ in (2.216).

Remark: The remaining Killing vectors for the anti-de Sitter space–time (again with cosmological constant set equal to -3) are the following:

$$\vec{\mathbf{K}}_{(4)}(\cdot\cdot) := -r \cdot \sin(t) \cdot \sin\theta \cdot \cos\varphi \cdot (1 + r^2)^{-\frac{1}{2}} \cdot \frac{\partial}{\partial t}$$
$$+ (1 + r^2)^{\frac{1}{2}} \cdot \cos(t) \cdot \sin\theta \cdot \cos\varphi \cdot \frac{\partial}{\partial r}$$
$$+ r^{-1} \cdot (1 + r^2)^{\frac{1}{2}} \cdot \cos(t) \cdot \left[\cos\theta \cdot \cos\varphi \cdot \frac{\partial}{\partial \theta} - \frac{\sin\varphi}{\sin\theta} \cdot \frac{\partial}{\partial \varphi}\right],$$

$$
\begin{aligned}
\vec{\mathbf{K}}_{(5)}(\cdot\cdot) := & -r\cdot\sin(t)\cdot\sin\theta\cdot\sin\varphi\cdot(1+r^2)^{-\frac{1}{2}}\cdot\frac{\partial}{\partial t}\\
& + (1+r^2)^{\frac{1}{2}}\cdot\cos(t)\cdot\sin\theta\cdot\sin\varphi\cdot\frac{\partial}{\partial r}\\
& + r^{-1}\cdot(1+r^2)^{\frac{1}{2}}\cdot\cos(t)\cdot\left[\cos\theta\cdot\sin\varphi\cdot\frac{\partial}{\partial\theta}+\frac{\cos\varphi}{\sin\theta}\cdot\frac{\partial}{\partial\varphi}\right],\\
\vec{\mathbf{K}}_{(6)}(\cdot\cdot) := & -r\cdot\sin(t)\cdot\cos\theta\cdot(1+r^2)^{-\frac{1}{2}}\cdot\frac{\partial}{\partial t}\\
& + (1+r^2)^{\frac{1}{2}}\cdot\cos(t)\cdot\cos\theta\cdot\frac{\partial}{\partial r} - r^{-1}\cdot(1+r^2)^{\frac{1}{2}}\cdot\cos(t)\cdot\sin\theta\cdot\frac{\partial}{\partial\theta}\,,\\
\vec{\mathbf{K}}_{(7)}(\cdot\cdot) := & \ r\cdot\cos(t)\cdot\sin\theta\cdot\cos\varphi\cdot(1+r^2)^{-\frac{1}{2}}\cdot\frac{\partial}{\partial t}\\
& + (1+r^2)^{\frac{1}{2}}\cdot\sin(t)\cdot\sin\theta\cdot\cos\varphi\cdot\frac{\partial}{\partial r}\\
& + r^{-1}\cdot(1+r^2)^{\frac{1}{2}}\cdot\sin(t)\cdot\left[\cos\theta\cdot\cos\varphi\cdot\frac{\partial}{\partial\theta}-\frac{\sin\varphi}{\sin\theta}\cdot\frac{\partial}{\partial\varphi}\right],\\
\vec{\mathbf{K}}_{(8)}(\cdot\cdot) := & \ r\cdot\cos(t)\cdot\sin\theta\cdot\sin\varphi\cdot(1+r^2)^{-\frac{1}{2}}\cdot\frac{\partial}{\partial t}\\
& + (1+r^2)^{\frac{1}{2}}\cdot\sin(t)\cdot\sin\theta\cdot\sin\varphi\cdot\frac{\partial}{\partial r}\\
& + r^{-1}\cdot(1+r^2)^{\frac{1}{2}}\cdot\sin(t)\cdot\left[\cos\theta\cdot\sin\varphi\cdot\frac{\partial}{\partial\theta}-\frac{\cos\varphi}{\sin\theta}\cdot\frac{\partial}{\partial\varphi}\right],\\
\vec{\mathbf{K}}_{(9)}(\cdot\cdot) := & \ r\cdot\cos(t)\cdot\cos\theta\cdot(1+r^2)^{-\frac{1}{2}}\cdot\frac{\partial}{\partial t}\\
& + (1+r^2)^{\frac{1}{2}}\cdot\sin(t)\cdot\cos\theta\cdot\frac{\partial}{\partial r} - r^{-1}\cdot(1+r^2)^{\frac{1}{2}}\cdot\sin(t)\cdot\sin\theta\cdot\frac{\partial}{\partial\theta}\,.
\end{aligned}
$$

2.4 Solution Strategies, Classification, and Initial-Value Problems

Let us consider a differentiable coordinate transformation (1.2) in a regular domain $D \subset \mathbb{R}^4$:

$$
\begin{aligned}
\widehat{x}^i &= \widehat{X}^i(x),\\
\widehat{g}_{ij}\,(\widehat{x}) &= \frac{\partial X^k\,(\widehat{x})}{\partial\widehat{x}^i}\cdot\frac{\partial X^l\,(\widehat{x})}{\partial\widehat{x}^j}\cdot g_{kl}(x)\,.
\end{aligned}
\tag{2.221}
$$

We pose four reasonable coordinate conditions

$$\widehat{\mathcal{C}}^m\left(\widehat{g}_{ij}\left(\widehat{x}\right)\right) = 0, \tag{2.222i}$$

$$\text{or,} \qquad \widehat{\mathcal{C}}^m\left[\frac{\partial X^k\left(\widehat{x}\right)}{\partial \widehat{x}^i} \cdot \frac{\partial X^l\left(\widehat{x}\right)}{\partial \widehat{x}^j} \cdot g_{kl}(x)\right] = 0. \tag{2.222ii}$$

The four nonlinear partial differential equations in (2.222ii), for *the four unknown functions* $X^k\left(\widehat{x}\right)$, *will locally admit solutions*. (See Appendix 2.) That is why, in a four-dimensional pseudo-Riemannian (or Riemannian) manifold, coordinate charts exist which admit *at most four (reasonable) coordinate conditions*.[9]

We recapitulate Einstein's interior field equations (2.161i), (2.166i), and differential identities (2.167i):

$$\mathcal{E}_{ij}(x) := G_{ij}(x) + \kappa\, T_{ij}(x) = 0, \tag{2.223i}$$

$$\mathcal{T}^i(x) := \nabla_j\, T^{ij} = 0, \tag{2.223ii}$$

$$\nabla_j\, \mathcal{E}^{ij} \equiv \kappa\, \mathcal{T}^i(x). \tag{2.223iii}$$

As in p. 164, we count the number of unknown functions versus the number of independent equations (without including coordinate conditions).

No. of unknown functions: $(10 g_{ij}) + 10(T_{ij}) = 20.$

No. of differential equations: $10(\mathcal{E}_{ij} = 0) + 4(\mathcal{T}^i = 0) = 14.$

No. of differential identities: $4(\nabla_j \mathcal{E}^{ij} - \kappa \mathcal{T}^i \equiv 0) = 4.$

No. of independent equations: $14 - 4 = 10.$

The system of semilinear, second- and first-order partial differential equations (2.223i,ii) is definitely *underdetermined*. We are allowed to prescribe *ten* out of *twenty* unknown functions to make the system *determinate*. We can make such choices in *11 distinct ways*.

Remarks: (i) In strategy-I, ten functions of $T_{ij}(x)$ are prescribed and ten functions of $g_{ij}(x)$ are treated as unknown. It is the most difficult strategy mathematically. However, from the perspective of physics, it is the most useful. This method is sometimes called the T-method [243].

(ii) In strategy-II, ten functions of $g_{ij}(x)$ are prescribed and ten functions of $T_{ij}(x) := -(\kappa)^{-1} G_{ij}(x)$ are treated as unknown. Thus, mathematically, it is

[9] In an N-dimensional domain, at most N (reasonable), coordinate conditions hold. Therefore, a two-dimensional metric can be locally reduced to *a conformally flat form*. A three-dimensional metric can be locally brought to *an orthogonal form*. However, orthogonal coordinates *may not exist* in dimensions $N > 3$. The coordinate conditions $\widehat{\mathcal{C}}^m\left(\widehat{g}_{ij}\left(\widehat{x}\right)\right) = 0$ are *not* tensor field equations.

the simplest method. Although energy conditions (2.190) or (2.191) or (2.192) may be difficult to satisfy, there may arise occasion when this strategy is useful. This method is sometimes called the g-method [243].

(iii) In strategy-III there exist *nine mixed methods* [18] where $10 - s$ of functions among the $T_{ij}(x)$ and s of the metric functions are prescribed for $1 \leq s \leq 9$. Moreover, $10-s$ functions among the $g_{ij}(x)$ and s functions among the $T_{ij}(x)$ are treated as unknown.

(iv) In the case of *the vacuum field equations* (2.159i), the addition of the four coordinate conditions $\mathcal{C}^i(g_{kl}, \partial_l g_{jk}) = 0$, makes *the system determinate*.

(v) In the case when the space–time domain has some *symmetries* (or admits groups of motions), the counting has to be completely revised, as the symmetries impose extra conditions. The general scheme, however, remains the same. We will address some specific cases in later sections.

It should be noted that there is a relation among the coordinate conditions $\mathcal{C}^i(g_{kl}, \partial_l g_{jk}) = 0$ and the prescription of metric functions $g_{ij}(x)$. *In case four coordinate conditions are imposed, the above strategies undergo revisions.*

Example 2.4.1. We shall furnish an example of the g-method. Let the metric tensor components be *prescribed as*

$$g_{ij}(x) := d_{kl} \cdot \frac{\partial f^k(x)}{\partial x^i} \cdot \frac{\partial f^l(x)}{\partial x^j}. \tag{2.224}$$

Here, four prescribed functions $f^k(x)$ are of class C^4. By (1.161), the space–time domain is *flat*. Therefore,

$$T_{ij}(x) := -(\kappa)^{-1} G_{ij}(x) \equiv 0.$$

Thus, the choice (2.224) of $g_{ij}(x)$ has *annihilated the possibility of material sources completely*. □

Another example of the g-method has been given in Example 2.2.9.

Example 2.4.2. In the domain of consideration, we choose *four coordinate conditions* as $g_{\alpha 4}(x) \equiv 0$ and $g_{44}(x) \equiv -1$. Thus, the metric is expressible as

$$\mathbf{g}_{..}(x) = g_{\alpha\beta}(x)\, \mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta - \mathrm{d}x^4 \otimes \mathrm{d}x^4,$$

$$\mathrm{d}s^2 = g_{\alpha\beta}(x)\, \mathrm{d}x^\alpha \mathrm{d}x^\beta - (\mathrm{d}x^4)^2. \tag{2.225}$$

This is *the geodesic normal or Gaussian normal coordinate chart* of (1.160).

On the three-dimensional spacelike hypersurface characterized by $x^4 = T$, the intrinsic metric is furnished by (2.225) as

$$\begin{aligned} \overline{\mathbf{g}}_{..}(\mathbf{x}, T) &= g_{\alpha\beta}(\mathbf{x}, T)\, \mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta \\ &=: \overline{g}_{\alpha\beta}(\mathbf{x}, T)\, \mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta. \end{aligned} \tag{2.226}$$

Here, we have a slight difference of notation with (1.223). (Compare (A1.35i) and (A1.35ii).)

The Gauss' equations (1.242i) *in this example* (with notations $\overline{A}^{\alpha}(\cdot\cdot) := \overline{g}^{\alpha\beta}(\cdot\cdot)A_{\beta}(\cdot\cdot)$ and $\overline{\nabla}_{\beta}A_{\alpha} := \partial_{\beta}A_{\alpha} - \left\{\begin{matrix}\gamma \\ \alpha\ \beta\end{matrix}\right\} \cdot A_{\gamma}(\cdot\cdot)$) yield the following equations:

$$\begin{aligned} R_{\rho\mu\nu\sigma} &= \overline{R}_{\rho\mu\nu\sigma} + K_{\mu\sigma}\cdot K_{\rho\nu} - K_{\mu\nu}\cdot K_{\rho\sigma} \\ &= \overline{R}_{\rho\mu\nu\sigma} + \frac{1}{4}\left[\partial_4\overline{g}_{\mu\sigma}\cdot\partial_4\overline{g}_{\rho\nu} - \partial_4\overline{g}_{\mu\nu}\cdot\partial_4\overline{g}_{\rho\sigma}\right]; \end{aligned} \tag{2.227}$$

$$\begin{aligned} G_{\mu\nu} &= \overline{G}_{\mu\nu} - \frac{1}{2}\,\partial_4\partial_4\overline{g}_{\mu\nu} - \overline{K}^{\sigma}_{\ \sigma}\cdot K_{\mu\nu} + 2\overline{K}^{\alpha}_{\ \mu}\cdot K_{\alpha\nu} \\ &\quad + \frac{1}{2}\,\overline{g}_{\mu\nu}\left[\left(\overline{K}^{\sigma}_{\ \sigma}\right)^2 - 3\overline{K}^{\rho}_{\ \sigma}\cdot\overline{K}^{\sigma}_{\ \rho} + \overline{g}^{\alpha\beta}\cdot\partial_4\partial_4\overline{g}_{\alpha\beta}\right] \\ &= \overline{G}_{\mu\nu} - \frac{1}{2}\,\partial_4\partial_4\overline{g}_{\mu\nu} - \frac{1}{4}\,\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\rho\sigma}\cdot\partial_4\overline{g}_{\mu\nu} + \frac{1}{2}\,\overline{g}^{\alpha\beta}\cdot\partial_4\overline{g}_{\mu\beta}\cdot\overline{g}_{\nu\alpha} \\ &\quad + \overline{g}_{\mu\nu}\left[\frac{1}{8}\left(\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\rho\sigma}\right)^2 - \frac{3}{8}\,\overline{g}^{\alpha\beta}\cdot\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\alpha\rho}\cdot\partial_4\overline{g}_{\beta\sigma}\right. \\ &\qquad\left. + \frac{1}{2}\,\overline{g}^{\rho\sigma}\cdot\partial_4\partial_4\overline{g}_{\rho\sigma}\right] = -\kappa\, T_{\mu\nu}\,; \end{aligned} \tag{2.228}$$

$$\begin{aligned} G_{\mu4} &= \partial_{\mu}\left[\overline{K}^{\sigma}_{\ \sigma}\right] - \overline{\nabla}^{\sigma}\left(K_{\mu\sigma}\right) \\ &= \frac{1}{2}\,\partial_{\mu}\left[\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\rho\sigma}\right] - \frac{1}{2}\,\overline{\nabla}^{\sigma}\left[\partial_4\overline{g}_{\mu\sigma}\right] = -\kappa\, T_{\mu4}\,; \end{aligned} \tag{2.229}$$

$$\begin{aligned} G_{44} &= \frac{1}{2}\,\overline{R} - \frac{1}{2}\left(\overline{K}^{\sigma}_{\ \sigma}\right)^2 + \frac{1}{2}\,\overline{K}^{\rho}_{\ \sigma}\cdot\overline{K}^{\sigma}_{\ \rho} \\ &= \frac{1}{2}\,\overline{R} - \frac{1}{8}\left(\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\rho\sigma}\right)^2 + \frac{1}{8}\,\overline{g}^{\mu\nu}\cdot\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\mu\rho}\cdot\partial_4\overline{g}_{\nu\sigma} \\ &= -\,\kappa\, T_{44}\,. \end{aligned} \tag{2.230}$$

In this strategy, we *prescribe six* $T_{\alpha\beta}(\cdot\cdot)$ and *define* $T_{44} := -(\kappa)^{-1}G_{44}$ and $T_{\mu4} := -(\kappa)^{-1}G_{\mu4}$. Moreover, we solve *ten differential equations* (2.228) and $\mathcal{T}^i = 0$ for *six unknown functions* $\overline{g}_{\alpha\beta}(\cdot\cdot)$. (In these ten differential equations, there exist *four differential identities*.) Thus, this strategy is mathematically simpler than Strategy-I (the T-method).

In this method, the main energy condition $T_{44}(\mathbf{x}, T) \geq 0$ depends on the sign of the expression $\left[\frac{1}{2}\,\overline{R} - \frac{1}{2}\left(\overline{K}^{\sigma}_{\ \sigma}\right)^2 + \frac{1}{2}\overline{K}^{\rho}_{\ \sigma}\cdot\overline{K}^{\sigma}_{\ \rho}\right]$.

The field equation (2.230) is supposed to be a refinement on Poisson's equation (2.157i) of the Newtonian potential $W(\mathbf{x})$. Therefore, the Newtonian potential must be well *hidden in the six metric components* $\overline{g}_{\alpha\beta}(\mathbf{x}, T)$!

The *six field equations* (2.228) happen to be *subtensor field equations* [55, 244]. *These are covariant under the (restricted) transformations of spatial coordinates alone*. □

Let us go back to the system of the first- and second-order semilinear partial differential equations (2.223i, ii) representing field equations. *The most general solutions* of the system will involve *24 arbitrary functions of integration*! Out of these 24 functions, *four arbitrary functions* can be absorbed by coordinate functions. The most general solutions of *the vacuum field equations* contain *20 arbitrary functions of integration*. These arbitrary functions can be adjusted to match the prescribed initial-boundary value problems.

Example 2.4.3. Consider the space–time domain corresponding to

$$D := \left\{x : a < x^1 < b,\ x^2 \in \mathbb{R},\ x^3 > 2,\ x^4 \in \mathbb{R}\right\}.$$

We investigate the vacuum field equations

$$R_{ij}(x) = 0$$

in this domain.

A *class of* (*nonflat*) *general solutions*, [3, 205], is furnished by

$$\begin{aligned}
\mathbf{g}_{..}(x) :=& \left(\ln|x^3|\right) \cdot \exp\left[F(x^1)\right] \cdot \mathrm{d}x^1 \otimes \mathrm{d}x^1 \\
&+ \left(x^3\right)^2 \cdot \exp\left[2\alpha(x^2)\right] \cdot \mathrm{d}x^2 \otimes \mathrm{d}x^2 + \mathrm{d}x^3 \otimes \mathrm{d}x^3 \\
&+ \exp\left[2\beta(x^4)\right] \cdot \left(\mathrm{d}x^1 \otimes \mathrm{d}x^4 + \mathrm{d}x^4 \otimes \mathrm{d}x^1\right),
\end{aligned}$$

$$\text{or,} \qquad \begin{aligned}
\mathrm{d}s^2 =& \left(\ln|x^3|\right) \cdot \exp\left[F(x^1)\right] \cdot \left(\mathrm{d}x^1\right)^2 + \left(x^3\right)^2 \cdot \exp\left[2\alpha(x^2)\right]\left(\mathrm{d}x^2\right)^2 \\
&+ \left(\mathrm{d}x^3\right)^2 + 2\exp\left[2\beta(x^4)\right] \cdot \mathrm{d}x^1 \mathrm{d}x^4,
\end{aligned}$$

$$D := \left\{(x^1, x^2, x^3, x^4) : -\infty < x^1 < \infty, -\infty < x^2 < \infty, 0 < x^3, -\infty < x^4 < \infty\right\}.$$

Here, $F(x)$, $\alpha(x^2)$, and $\beta(x^4)$ are assumed to be *arbitrary functions* of class C^2. The arbitrary functions $\alpha(x^2)$ and $\beta(x^4)$ can be absorbed by the coordinate transformation:

$$\widehat{x}^1 = x^1, \quad \widehat{x}^2 = \int \mathrm{e}^{\alpha(x^2)} \cdot \mathrm{d}x^2, \quad \widehat{x^3} = x^3,$$

$$\widehat{x}^4 = \int \mathrm{e}^{2\beta(x^4)} \cdot \mathrm{d}x^4.$$

Thus, the vacuum solution involving one arbitrary function is given by

$$ds^2 = \left(\ln|\widehat{x}^3|\right) \cdot \exp\left[F(\widehat{x}^1)\right] \cdot \left(d\widehat{x}^1\right)^2 + \left(\widehat{x}^3\right)^2 \cdot \left(d\widehat{x}^2\right)^2 + 2\, d\widehat{x}^1 d\widehat{x}^4.$$

The "initial-value problem"

$$\widehat{g}_{11}\left(0, \widehat{x}^2, \widehat{x}^3, \widehat{x}^4\right) = \frac{\partial \widehat{g}_{11}(\widehat{x})}{\partial \widehat{x}^1}\Big|_{(0,\widehat{x}^2,\widehat{x}^3,\widehat{x}^4)} = \ln|\widehat{x}^3|$$

yields one possible metric (not involving arbitrary functions) as

$$ds^2 = \left(\ln|\widehat{x}^3|\right) \cdot e^{\widehat{x}^1} \cdot \left(d\widehat{x}^1\right)^2 + \left(\widehat{x}^3\right)^2 \cdot \left(d\widehat{x}^2\right)^2 + \left(d\widehat{x}^3\right)^2 + 2\, d\widehat{x}^1 d\widehat{x}^4. \qquad \square$$

The classification of a semilinear (or a quasi-linear) second-order partial differential equation has been discussed in Appendix 2 after (A2.44). The classification of a system of first-order, semilinear (or quasi-linear) partial differential equations has been touched upon in (A2.51). (For a detailed treatment, we refer to the book by Courant and Hilbert [43].)

To illustrate briefly, we deal with *a couple of toy models* in the following:

Example 2.4.4. Consider a domain of *the flat space–time and the wave equation*

$$\square V = d^{ij}\partial_i\partial_j V = 0. \tag{2.231}$$

(See Problem #4 of Exercise 2.1.) By the discussions after (A2.44), it is clear that (2.231) is *hyperbolic*. The second-order, linear p.d.e. (2.231) is equivalent to the following system of four first-order p.d.e.s:

$$\partial_4\omega_\alpha - \partial_\alpha\omega_4 = 0, \tag{2.232i}$$

$$d^{ij}\partial_i\omega_j = 0, \tag{2.232ii}$$

$$\omega_j = \partial_j V. \tag{2.232iii}$$

(Compare with (A2.48).) We can combine (2.232ii) and (2.232i) into the following form:

$$\Gamma_{ij}{}^{k} \cdot \partial_k\omega_j = 0,$$

$$\Gamma_{1j}{}^{k} := d^{jk}, \quad \Gamma_{21}{}^{4} = \Gamma_{32}{}^{4} = \Gamma_{43}{}^{4} := 1,$$

$$\Gamma_{24}{}^{1} = \Gamma_{34}{}^{2} = \Gamma_{44}{}^{3} := -1\,;$$

$$\text{otherwise, } \Gamma_{ij}{}^{k} \equiv 0. \tag{2.233}$$

(Here, the index j is *also* summed.) Compare (2.233) with (A2.49). *Caution*: The components $\Gamma_{ij}{}^{k}$ are not tensorial.

The characteristic matrix of (2.233) is provided by

$$\underset{4\times 4}{\left[\Gamma_{ij}\right]} := \left[{\Gamma_{ij}}^{k} \cdot \partial_k \phi\right] = \begin{bmatrix} \partial_1\phi & \partial_2\phi & \partial_3\phi & -\partial_4\phi \\ \partial_4\phi & 0 & 0 & -\partial_1\phi \\ 0 & \partial_4\phi & 0 & -\partial_2\phi \\ 0 & 0 & \partial_4\phi & -\partial_3\phi \end{bmatrix}$$

$$\begin{aligned} \det[\Gamma_{ij}] &= -(\partial_4\phi)^2\left[(\partial_1\phi)^2 + (\partial_2\phi)^2 + (\partial_3\phi)^2 - (\partial_4\phi)^2\right] \\ &= -(\partial_4\phi)^2 \cdot \left[d^{ij}\,\partial_i\phi \cdot \partial_j\phi\right]. \end{aligned} \tag{2.234}$$

(See (2.51).)

Therefore, for the case $\partial_4\phi \neq 0$, the equation

$$\det[\Gamma_{ij}] = 0$$

yields *the characteristic* (*three-dimensional*) *hypersurfaces* governed by

$$d^{ij} \cdot \partial_i\phi \cdot \partial_j\phi = 0. \tag{2.235}$$

By the criteria in [43], *the system (2.233) is hyperbolic*. The p.d.e. (2.235) stands for *a null hypersurface* (like a null cone). (*Shock waves* of the wave equation (2.231) travel along such a hypersurface [43].) □

Example 2.4.5. Consider the analogous problem in a domain of the four-dimensional *Euclidean manifold*. *A harmonic function* $V(x)$ satisfies *the potential equation*:

$$\delta^{ij}\,\partial_i\partial_j V = 0. \tag{2.236}$$

The equivalent first-order system is furnished by

$$\Gamma_{ij}^{k}\,\partial_k\omega_j = 0. \tag{2.237}$$

(Here, we have set $\omega_j = \partial_j V$.) The characteristic matrix is provided by

$$\underset{4\times 4}{\left[\Gamma_{ij}\right]} = \begin{bmatrix} \partial_1\phi & \partial_2\phi & \partial_3\phi & \partial_4\phi \\ \partial_4\phi & 0 & 0 & -\partial_1\phi \\ 0 & \partial_4\phi & 0 & -\partial_2\phi \\ 0 & 0 & \partial_4\phi & -\partial_3\phi \end{bmatrix}. \tag{2.238}$$

The equation for the characteristic criterion is:

$$\det[\Gamma_{ij}] = -(\partial_4\phi)^2\left[(\partial_1\phi)^2 + (\partial_2\phi)^2 + (\partial_3\phi)^2 + (\partial_4\phi)^2\right] = 0. \tag{2.239}$$

Therefore, for the case $\partial_4\phi \neq 0$, the solutions are given by $\phi(x) = k = \text{const}$.

There exists *no nondegenerate characteristic hypersurface*. The system (2.237) is called *elliptic* [43]. (The harmonic function $V(x)$ must be *real-analytic*.) □

Example 2.4.6. Now, we shall classify gravitational field equations (2.223i). We use *harmonic coordinates* (or *the harmonic gauge*) to simplify calculations. (Recall the harmonic coordinate chart as introduced in Problem #10 of Exercise 2.2.) The corresponding field equations are given by

$$\begin{aligned} & g^{kl}(x)\,\partial_k\partial_l\, g^{ij} - 2\begin{Bmatrix} i \\ k\ p \end{Bmatrix}\begin{Bmatrix} j \\ l\ q \end{Bmatrix} g^{kl}(x)\cdot g^{pq}(x) \\ &= \begin{cases} -2\kappa\left[T^{ij}(x) - \frac{1}{2}g^{ij}(x)T^k_{\ k}(x)\right] & \text{inside material sources,} \\ 0 & \text{outside material sources.} \end{cases} \end{aligned} \tag{2.240}$$

Now, let us obtain a system of first-order quasi-linear p.d.e.s which is *equivalent to the second-order system in (2.240)*. Following Example 2.4.4, and putting $\omega^{ij}_{\ \ k} = \partial_k g^{ij}$, we arrive at the following system of *40 equations*:

$$\begin{aligned} \omega^{ij}_{\ \ k}(x) &\equiv \omega^{ji}_{\ \ k}(x), \\ \partial_\alpha\omega^{ij}_{\ \ 4} - \partial_4\omega^{ij}_{\ \ \alpha} &= 0, \\ g^{kl}\partial_k\omega^{ij}_{\ \ l} - h^{ij}(g^{kl},\omega^{kl}_{\ \ p}) &= \begin{cases} -2\kappa\left[T^{ij} - \frac{1}{2}g^{ij}T^k_{\ k}\right] & \text{inside,} \\ 0 & \text{outside.} \end{cases} \end{aligned} \tag{2.241}$$

We define the 4×4 submatrix as

$$\underset{4\times4}{[\gamma]} = [\gamma_{ij}^{\ \ k}\cdot\partial_k\phi] := \begin{bmatrix} g^{1k}\partial_k\phi & g^{2k}\partial_k\phi & g^{3k}\partial_k\phi & g^{4k}\partial_k\phi \\ \partial_4\phi & 0 & 0 & -\partial_1\phi \\ 0 & \partial_4\phi & 0 & -\partial_2\phi \\ 0 & 0 & \partial_4\phi & -\partial_3\phi \end{bmatrix}. \tag{2.242}$$

The characteristic matrix and its determinant for the system (2.241) are furnished by

$$\underset{40\times40}{[\Gamma]} := \begin{bmatrix} \boxed{\gamma} & & & & & & & & & \\ & \boxed{\gamma} & & & & & & & & \\ & & \boxed{\gamma} & & & & & & & \\ & & & \boxed{\gamma} & & & 0 & & & \\ & & & & \boxed{\gamma} & & & & & \\ & & & & & \boxed{\gamma} & & & & \\ & 0 & & & & & \boxed{\gamma} & & & \\ & & & & & & & \boxed{\gamma} & & \\ & & & & & & & & \boxed{\gamma} & \\ & & & & & & & & & \boxed{\gamma} \end{bmatrix}, \tag{2.243i}$$

$$\det[\Gamma] = \left[(\partial_4\phi)^2\cdot(g^{ij}\cdot\partial_i\phi\cdot\partial_j\phi)\right]^{10}. \tag{2.243ii}$$

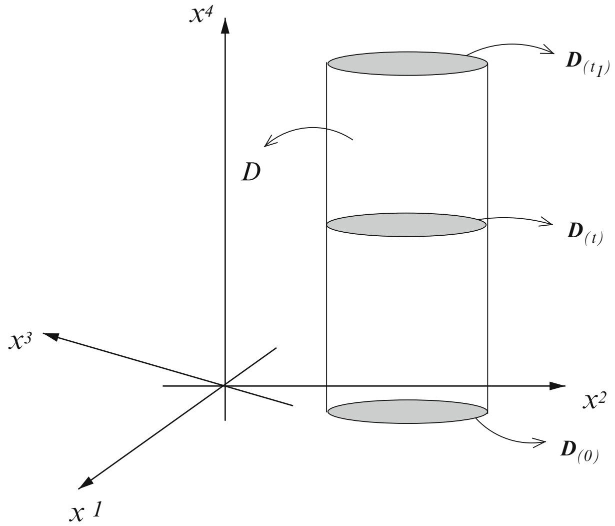


Fig. 2.20 Domain $D := \mathbf{D}_{(0)} \times (0, t_1) \subset \mathbb{R}^4$ for the initial-value problem

In the case $\partial_4 \phi \neq 0$, the equation $\det[\Gamma] = 0$ yields

$$g^{ij}(x) \cdot \partial_i \phi \cdot \partial_j \phi = 0. \tag{2.244}$$

Above is the governing equation for *a three-dimensional null hypersurface* in the space–time metric of signature +2. Obviously, the system (2.241) is *hyperbolic*. □

We shall now explore the *initial-value problem* (or the *Cauchy problem*) of gravitational field equations (2.159i). However, we firstly examine a toy model of one hyperbolic p.d.e., namely, the usual wave equation:

$$\nabla_i \nabla^i V = 0. \tag{2.245}$$

Let the domain of validity $D := \mathbf{D}_{(0)} \times (0, t_1)$ have one boundary $\mathbf{D}_{(0)}$ as the initial hypersurface $x^4 = 0$. (See Fig. 2.20.)

The corresponding *Cauchy-Kowalewski theorem* is stated below:

Theorem 2.4.7. *Let metric components $g^{ij}(x)$ be real analytic with $g^{44}(x) < 0$ in a space–time domain $D := \mathbf{D}_{(0)} \times (0, t_1) \subset \mathbb{R}^4$ with one boundary at $x^4 = 0$. Moreover, let this boundary hypersurface contain the origin* $(0, 0, 0, 0)$. *Furthermore, let $\mathbf{D}_{(0)}$ be the projection of D onto the hypersurface $x^4 = 0$. Given real-analytic functions $f(\mathbf{x})$ and $h(\mathbf{x})$ in $\mathbf{x} \in \mathbf{D}_{(0)}$, there exists a half-neighborhood $\mathcal{N}_\delta^+(0, 0, 0, 0)$ with a unique solution $V(x)$ of the partial differential equation* (2.245) *such that $\lim_{x^4 \to 0_+} V(\mathbf{x}, x^4) = f(\mathbf{x})$ and $\lim_{x^4 \to 0_+} \left[\frac{\partial V(\mathbf{x}, x^4)}{\partial x^4} \right] = h(\mathbf{x})$.*

For the proof, consult [43].

As a preliminary to the investigation of initial-value problems for general relativity, we state the following *Lichnerowicz-Synge lemma* [166, 243]:

Lemma 2.4.8. *Consider gravitational field equations* (2.223i,ii) *in a domain* $D := \mathbf{D}_{(0)} \times (0, t_1)$. *Then, the following two statements are mathematically equivalent:*

$$\text{(A)} \qquad \mathcal{E}_{ij}(x) = 0 \quad \textit{in} \quad x \in D. \tag{2.246i}$$

$$\text{(B)} \qquad \mathcal{E}_{\alpha\beta}(x) - \frac{1}{2}\, g_{\alpha\beta}(x)\, \mathcal{E}^{i}_{\ i}(x) = 0 \tag{2.246ii}$$

$$\textit{and} \qquad \nabla_j \mathcal{E}^{ij} = 0 \quad \textit{in} \quad x \in D \tag{2.246iii}$$

$$\textit{with} \qquad \mathcal{E}^{4}_{\ i}(\mathbf{x}, 0) = 0 \quad \textit{for} \quad \mathbf{x} \in \mathbf{D}_{(0)}. \tag{2.246iv}$$

For the proof, see [243].

Now, we shall state and prove a theorem on the solution of the initial-value problem in general relativity.

Theorem 2.4.9. *Let the energy–momentum–stress tensor components* $T_{ij}(x)$ *and metric tensor components* $g_{ij}(x)$ *be real analytic with* $g_{44}(x) < 0$ *in a space–time domain* $D := \mathbf{D}_{(0)} \times (0, t_1) \subset \mathbb{R}^4$ *with one boundary hypersurface at* $x^4 = 0$. *Given 30 real-analytic functions,* $g^{\#}_{ij}(\mathbf{x})$, $\psi_{ij}(\mathbf{x})$, $T^{\#}_{\alpha\beta}(\mathbf{x})$, *and* $\theta_i(\mathbf{x}) = T^{\#}_{4i}(\mathbf{x})$ *for* $\mathbf{x} \in \mathbf{D}_{(0)}$ *be prescribed. Moreover, let the initial constraints* $\left[G^{4}_{\ i}(x) + \kappa T^{4}_{\ i}(x)\right]_{|x^4=0} = 0$ *hold. Then, there exist 20 unique solutions* $g_{ij}(x)$ *and* $T_{ij}(x)$ *of the field equations* $\mathcal{E}_{ij}(x) = 0$ *in* $\mathbf{D}_{(0)} \times (0, t_1)$ *such that* $\lim_{x^4 \to 0_+} g_{ij}(x) = g^{\#}_{ij}(\mathbf{x})$, $\lim_{x^4 \to 0_+} \left[\partial_4 g_{ij}\right] = \psi_{ij}(\mathbf{x})$, $\lim_{x^4 \to 0_+} T_{\alpha\beta}(x) = T^{\#}_{\alpha\beta}(\mathbf{x})$, *and* $\lim_{x^4 \to 0_+} T_{i4}(x) = \theta_i(\mathbf{x})$.

Proof. Instead of using (2.246i), we use the equivalent equations (2.246ii), (2.246iii), and (2.246iv). Moreover, instead of using the T-method of p. 197, we use a mixed method of the p. 197 to solve the field equations. Therefore, by Example 2.4.2, we make use of geodesic normal coordinates (2.225) yielding

$$\mathbf{g}_{..}(x) = \overline{g}_{\alpha\beta}(x)\, \mathrm{d}x^{\alpha} \otimes \mathrm{d}x^{\beta} - \mathrm{d}x^4 \otimes \mathrm{d}x^4.$$

Moreover, we prescribe $T_{\alpha\beta}(x)$ in D and solve for $\overline{g}_{\alpha\beta}(x)$ and $T_{i4}(x)$ from (2.246ii–iv). The field equations (2.246ii) and (2.246iii) yield, respectively,

$$\begin{aligned} &\overline{R}_{\mu\nu}(x) - \frac{1}{2}\left(\partial_4 \partial_4 \overline{g}_{\mu\nu}\right) - \frac{1}{4}\, \overline{g}^{\rho\sigma} \cdot \partial_4 \overline{g}_{\rho\sigma} \cdot \partial_4 \overline{g}_{\mu\nu} + \frac{1}{2}\, \overline{g}^{\alpha\beta} \cdot \partial_4 \overline{g}_{\mu\alpha} \cdot \partial_4 \overline{g}_{\nu\beta} \\ &\quad + \kappa \left[T_{\mu\nu} - \frac{1}{2}\, \overline{g}_{\mu\nu} \left(\overline{g}^{\alpha\beta} T_{\alpha\beta} - T_{44}\right)\right] = 0, \end{aligned} \tag{2.247i}$$

$$\partial_4 T_{j4}(x) = \partial_\alpha T^{\alpha}_{\ j}(x) + \begin{Bmatrix} i \\ i\ k \end{Bmatrix} T^{k}_{\ j}(x) - \begin{Bmatrix} k \\ i\ j \end{Bmatrix} T^{i}_{\ k}(x). \tag{2.247ii}$$

From (2.247i,ii), we derive the following string of partial differential equations by *successive differentiations*:

$$\begin{aligned}
\partial_4\partial_4\overline{g}_{\mu\nu}(x) &= 2\overline{R}_{\mu\nu}(x) - \frac{1}{2}\,\overline{g}^{\rho\sigma}\cdot\partial_4\overline{g}_{\rho\sigma}\cdot\partial_4\overline{g}_{\mu\nu} + \overline{g}^{\alpha\beta}\cdot\partial_4\overline{g}_{\mu\alpha}\cdot\partial_4\overline{g}_{\nu\beta} \\
&\quad + 2\kappa\left[T_{\mu\nu}(x) - \frac{1}{2}\,\overline{g}_{\mu\nu}\cdot\left(\overline{g}^{\alpha\beta}T_{\alpha\beta} - T_{44}\right)\right], \\
\partial_4\partial_4\partial_4\overline{g}_{\mu\nu} &= \partial_4\{\ldots\ldots\ldots\ldots\ldots\}, \qquad (2.248\text{i}) \\
&\vdots \qquad\qquad \vdots \\
\partial_4 T_{j4}(x) &= \partial_\alpha T^{\alpha}{}_{j}(x) + \left\{\begin{matrix} i \\ i\ k \end{matrix}\right\} T^{k}{}_{j}(x) - \left\{\begin{matrix} k \\ i\ j \end{matrix}\right\} T^{i}{}_{k}(x), \\
\partial_4\partial_4 T_{j4}(x) &= \partial_4\{\ldots\ldots\ldots\ldots\ldots\}, \qquad (2.248\text{ii}) \\
&\vdots \qquad\qquad \vdots
\end{aligned}$$

Now, *we prescribe 16 real-analytic initial values*

$$\begin{aligned}
\overline{g}_{\alpha\beta}(\mathbf{x},0) &= g^{\#}_{\alpha\beta}(\mathbf{x}), \\
\partial_4\overline{g}_{\alpha\beta}(x)_{|x^4=0} &= \psi_{\alpha\beta}(\mathbf{x}) \equiv 2K^{\#}_{\alpha\beta}(\mathbf{x}), \\
T_{j4}(\mathbf{x},0) &= \theta_j(\mathbf{x}). \qquad (2.249)
\end{aligned}$$

The field equations (2.248i,ii) yield on the initial hypersurface $x^4 = 0$,

$$\begin{aligned}
\partial_4\partial_4\overline{g}_{\mu\nu|x^4=0} &= 2R^{\#}_{\mu\nu}(\mathbf{x}) - \frac{1}{2}\,g^{\#\rho\sigma}(\mathbf{x})\cdot\psi_{\rho\sigma}(\mathbf{x})\cdot\psi_{\mu\nu}(\mathbf{x}) \\
&\quad + g^{\#\alpha\beta}(\mathbf{x})\cdot\psi_{\mu\alpha}(\mathbf{x})\cdot\psi_{\nu\beta}(\mathbf{x}) + 2\kappa\left[T^{\#}_{\mu\nu} - \frac{1}{2}\,g^{\#}_{\mu\nu}\left(g^{\#\alpha\beta}T^{\#}_{\alpha\beta} - T^{\#}_{44}\right)\right]_{|x^4=0} \\
&= (\text{initial values}), \\
\partial_4\partial_4\partial_4\overline{g}_{\mu\nu|x^4=0} &= (\text{initial values}), \\
&\vdots \qquad\qquad \vdots \qquad (2.250\text{i}) \\
\partial_4 T_{j4|x^4=0} &= (\text{initial values}) \\
\partial_4\partial_4 T_{j4|x^4=0} &= (\text{initial values}) \\
&\vdots \qquad\qquad \vdots \qquad (2.250\text{ii})
\end{aligned}$$

The prescribed functions in (2.249) *have to satisfy partial differential equations* (2.246iv) on the initial hypersurface. These are explicitly provided by:

$$\begin{aligned}
&\partial_\alpha \left[g^{\#\mu\nu}(\mathbf{x}) \cdot \psi_{\mu\nu}(\mathbf{x})\right] - \nabla^{\#\sigma}\left[\psi_{\alpha\sigma}(\mathbf{x})\right] + 2\kappa\, \theta_\alpha(\mathbf{x}) = 0,\\
\text{or,}\quad &\partial_\alpha \left[K^{\#\mu}{}_{\mu}(\mathbf{x})\right] - \nabla^{\#\sigma}\left[K_{\alpha\sigma}(\mathbf{x})\right] + \kappa\, \theta_\alpha(\mathbf{x}) = 0,\\
\text{and}\quad &R^{\#}(\mathbf{x}) - \frac{1}{4}\left[g^{\#\mu\nu} \cdot \psi_{\mu\nu}\right]^2 + \frac{1}{4}\left(g^{\#\mu\nu} \cdot g^{\#\rho\sigma} \cdot \psi_{\mu\rho} \cdot \psi_{\nu\sigma}\right) + 2\kappa\, \theta_4(\mathbf{x}) \quad = 0,\\
\text{or,}\quad &R^{\#}(\mathbf{x}) - \left[K^{\#\mu}{}_{\mu}(\mathbf{x})\right]^2 + K^{\#\nu}{}_{\rho}(\mathbf{x}) \cdot K^{\#\rho}{}_{\nu}(\mathbf{x}) + 2\kappa\, \theta_4(\mathbf{x}) = 0.
\end{aligned} \tag{2.251}$$

(Here, $K^{\#}_{\mu\nu}(\mathbf{x})$ are components of *the extrinsic curvature* as given in (1.251).) Assuming that (2.251) is satisfied, the corresponding Taylor series are expressed as:

$$\begin{aligned}
\overline{g}_{\alpha\beta}(x) &= g^{\#}_{\alpha\beta}(\mathbf{x}) + x^4 \cdot \psi_{\alpha\beta}(\mathbf{x}) + \frac{1}{2}\left(x^4\right)^2 \cdot \left(\partial_4\partial_4\overline{g}_{\alpha\beta}\right)_{|x^4=0} + \cdots,\\
T_{4j}(x) &= \theta_j(\mathbf{x}) + x^4 \cdot \left(\partial_4 T_{4j}\right)_{|x^4=0} + \frac{1}{2}\left(x^4\right)^2 \cdot \left(\partial_4\partial_4 T_{4j}\right)_{|x^4=0} + \cdots.
\end{aligned} \tag{2.252}$$

Using (2.250i,ii), one can explicitly construct the series in (2.252). The condition of real analyticity guarantees the absolute convergence of series in (2.252) for $0 \leq x^4 < t_1 := \min\left(\tau_{\alpha\beta}(\mathbf{x}), \tau_i(\mathbf{x})\right)$. Moreover, it is clear from (2.252) that $\lim_{x^4 \to 0_+} \overline{g}_{\alpha\beta}(x) = g^{\#}_{\alpha\beta}(\mathbf{x})$, $\lim_{x^4 \to 0_+} \partial_4 g_{\alpha\beta} = \psi_{\alpha\beta}(\mathbf{x})$, and $\lim_{x^4 \to 0_+} T_{4j}(x) = \theta_j(\mathbf{x})$. Thus, the initial-value problem of general relativity is solved, and Theorem 2.4.9 is proved. ■

Example 2.4.10. Consider the initial-value problem for vacuum equations. These are summarized from (2.248i) as

$$\partial_4\partial_4\overline{g}_{\mu\nu}(x) = 2\overline{R}_{\mu\nu}(x) - \frac{1}{2}\,\overline{g}^{\rho\sigma} \cdot \partial_4\overline{g}_{\rho\sigma} \cdot \partial_4\overline{g}_{\mu\nu} + \overline{g}^{\alpha\beta} \cdot \partial_4\overline{g}_{\mu\alpha} \cdot \partial_4\overline{g}_{\nu\beta}. \tag{2.253}$$

The initial data (*or Cauchy data*) $\overline{g}_{\mu\nu}(\mathbf{x}, 0) = g^{\#}_{\mu\nu}(\mathbf{x})$ and $\left[\partial_4\overline{g}_{\mu\nu}\right]_{|x^4=0} = \psi_{\mu\nu}(\mathbf{x})$ must satisfy, by (2.251), the following equations:

$$\nabla^{\#}_{\alpha}\left[\psi^{\#\alpha}{}_{\beta} - \delta^{\alpha}{}_{\beta}\psi^{\#\mu}{}_{\mu}\right] = 0, \tag{2.254i}$$

$$4R^{\#}(\mathbf{x}) - \left[\psi^{\#\mu}{}_{\mu}\right]^2 + \left[\psi^{\#\nu}{}_{\rho} \cdot \psi^{\#\rho}{}_{\nu}\right] = 0. \tag{2.254ii}$$

Equations (2.254i,ii) are physically important for *gravitational waves*. (See Appendix 5.) This system of four equations for 12 unknown functions is *undetermined*. It seems as if it would be easy to solve these, but in fact, it is not! □

Example 2.4.11. Consider a specific scenario for the initial-value problem for vacuum equations. We choose the initial data as

$$g^{\#}_{\alpha\beta}(\mathbf{x}) = \delta_{\alpha\beta}, \quad R^{\#}_{\alpha\beta}(\mathbf{x}) \equiv 0, \quad R^{\#}(\mathbf{x}) \equiv 0\,;$$

$$\left[\psi_{\alpha\beta}(\mathbf{x})\right] := \begin{bmatrix} 2c_{(1)} & 0 & 0 \\ 0 & 2c_{(2)} & 0 \\ 0 & 0 & 2c_{(3)} \end{bmatrix} = \left[2K^{\#}_{\mu\nu}(\mathbf{x})\right],$$

$$\nabla^{\#}_{\alpha}\left[\psi^{\#\alpha}{}_{\beta}\right] \equiv 0, \quad \nabla^{\#}_{\beta}\left[\psi^{\#\mu}{}_{\mu}\right] \equiv 0.$$

Here, *the constants* $c_{(1)}, c_{(2)}, c_{(3)}$ are assumed to satisfy the constraints:

$$c_{(1)} + c_{(2)} + c_{(3)} = 1 = \left[c_{(1)}\right]^2 + \left[c_{(2)}\right]^2 + \left[c_{(3)}\right]^2.$$

In this case, (2.254i,ii) are *identically satisfied*. Moreover, solving (2.253), we arrive at the special *Kasner metric* [147]

$$\begin{aligned} \mathbf{g}_{..}(x) = {} & \left(1 + x^4\right)^{2c_{(1)}} \cdot \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \left(1 + x^4\right)^{2c_{(2)}} \cdot \mathrm{d}x^2 \otimes \mathrm{d}x^2 \\ & + \left(1 + x^4\right)^{2c_{(3)}} \cdot \mathrm{d}x^3 \otimes \mathrm{d}x^3 - \mathrm{d}x^4 \otimes \mathrm{d}x^4, \\ \mathrm{d}s^2 = {} & \left(1 + x^4\right)^{2c_{(1)}} \cdot \left(\mathrm{d}x^1\right)^2 + \left(1 + x^4\right)^{2c_{(2)}} \cdot \left(\mathrm{d}x^2\right)^2 \\ & + \left(1 + x^4\right)^{2c_{(3)}} \cdot \left(\mathrm{d}x^3\right)^2 - \left(\mathrm{d}x^4\right)^2. \end{aligned} \tag{2.255}$$

□

We shall provide another example of the initial-value scheme in Example 6.2.3.

Exercises 2.4

1. Consider four harmonic coordinate conditions $\partial_j\left[\sqrt{|g|}\, g^{ij}\right] = 0$. Obtain a class of general solutions of these differential equations in terms of arbitrary functions.
2. Let six metric components be $g_{12}(x) = g_{13}(x) = g_{14}(x) = g_{23}(x) = g_{24}(x) = g_{34}(x) \equiv 0$ yielding an orthogonal coordinate chart:

$$\begin{aligned} \mathbf{g}_{..}(x) = {} & \left[h_{(1)}(x)\right]^2 \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \left[h_{(2)}(x)\right]^2 \mathrm{d}x^2 \otimes \mathrm{d}x^2 \\ & + \left[h_{(3)}(x)\right]^2 \mathrm{d}x^3 \otimes \mathrm{d}x^3 - \left[h_{(4)}(x)\right]^2 \mathrm{d}x^4 \otimes \mathrm{d}x^4, \\ \mathrm{d}s^2 = {} & \sum_i \sum_j h_{(i)} \cdot h_{(j)} \cdot d_{ij}\, \mathrm{d}x^i\, \mathrm{d}x^j. \end{aligned}$$

(Summation convention is temporarily suspended.) Express a special class of vacuum field equations $R_{ij}(x) = 0$ in terms of four functions $h_{(i)}(x) > 0$ which are of class C^3.

3. Consider the conformal tensor components of definition (1.169i)

$$C^l_{\ ijk}(x) := R^l_{\ ijk}(x) + \frac{1}{2}\left[\delta^l_{\ j} R_{ik} - \delta^l_{\ k} R_{ij} + g_{ik} R^l_{\ j} - g_{ij} R^l_{\ k}\right] + \frac{R(x)}{6}\left[\delta^l_{\ k}\, g_{ij} - \delta^l_{\ j}\, g_{ik}\right]$$

in a space–time domain. Solve for ten functions $g_{ij}(x)$ in the system of ten independent quasi-linear second-order partial differential equations $C^l_{\ ijk}(x) = 0$.
4. Recall the vacuum field equations $R_{ij}(x) = 0$. Using 50 functions $g_{ij}(x)$ and $\left\{ {k \atop ij} \right\}$, express an equivalent system of first-order partial differential equations.
5. Let the metric be expressed as:

$$\mathbf{g}_{..}(x) := \left[a(x^4)\right]^2 \cdot \delta_{\alpha\beta}\, \mathrm{d}x^\alpha \otimes \mathrm{d}x^\beta - \mathrm{d}x^4 \otimes \mathrm{d}x^4,$$
$$\mathrm{d}s^2 = \left[a(x^4)\right]^2 \cdot \delta_{\alpha\beta}\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta - \left(\mathrm{d}x^4\right)^2,$$
$$x \in D := \mathbf{D} \times (t_0, t_1) \subset \mathbb{R}^4, \quad t_0 > 1.$$

Moreover, the function $a(x^4) > 0$ is of class C^3. For the initial-value problem, the prescribed functions are assumed to be as follows: In $D \subset \mathbb{R}^4$, $T_{\alpha\beta}(x) \equiv 0$.

On the initial hypersurface at $x^4 = t_0$,

(i) $g^{\#}_{\alpha\beta}(\mathbf{x}) = (\alpha)^2 \cdot \delta_{\alpha\beta} = \text{const.}$,
(ii) $\psi_{\alpha\beta}(\mathbf{x}) = \beta \cdot \delta_{\alpha\beta} = \text{const.}$,
(iii) $\theta_\alpha(\mathbf{x}) \equiv 0$,
(iv) $\theta_4(\mathbf{x}) = \rho_0 = \text{const.}$

Prove that, with the notation $\dot{a}(x^4) := \frac{\mathrm{d}a(x^4)}{\mathrm{d}x^4}$, (2.247i,ii) and (2.251) reduce to

$$-\left[a(x^4)\cdot \ddot{a}(x^4) + 2\left(\dot{a}(x^4)\right)^2\right] + \frac{\kappa}{2}\left[a(x^4)\right]^2 \cdot T_{44}(x^4) = 0,$$
$$\partial_4 T_{\alpha 4} \equiv 0, \quad \partial_4 T_{44} = -\frac{3[\dot{a}(x^4)]}{a(x^4)} \cdot T_{44}(x^4),$$
$$\text{and, } -\frac{3}{2}\left(\frac{\beta}{\alpha^2}\right)^2 + 2\kappa\, \rho_0 = 0.$$

Answers and Hints to Selected Exercises

1. Consider fields with the following symmetries:

$$\Gamma^{jikl}(x) \equiv \Gamma^{ijkl}(x) \equiv \Gamma^{ijlk}(x) \equiv -\Gamma^{iljk}(x).$$

The functions $\Gamma^{ijkl}(x)$ are of class C^4. A class of general solutions is furnished by

$$\sqrt{|g|}\, g^{ij}(x) = \partial_k \partial_l \Gamma^{ijkl}.$$

(*Remarks:* (i) Harmonic coordinate conditions and the class of general solutions are *not tensor field equations*.
(ii) In fact, *no coordinate condition is expressible as a tensor field equation*.)

2. *Suspend summation convention* in this answer [56, 90].
   For $l \neq k$, and $i \neq k$,

$$R_{lk}(x) = \sum_{i \neq k,l}{}' \left[h_{(i)}(x)\right]^{-1} \cdot \Big\{\partial_l \partial_k h_{(i)} - \left[\partial_l h_{(i)}\right] \cdot \left[\partial_k \ln h_{(l)}\right]$$
$$- \left[\partial_k h_{(i)}\right] \cdot \left[\partial_l \ln h_{(k)}\right]\Big\} = 0\,;$$

$$R_{kk}(x) = d_{kk} \cdot \left[h_{(k)}(x)\right] \cdot \sum_{i \neq k}{}' \left[h_{(i)}(x)\right]^{-1} \cdot \Big\{ d^{ii} \cdot \partial_i \left[h_{(i)}^{-1} \cdot \partial_i h_{(k)}\right]$$
$$+ d^{kk} \cdot \partial_k \left[h_{(k)}^{-1} \cdot \partial_k h_{(i)}\right] + \sum_{l \neq i,k}{}' d^{ll} \cdot \left[h_{(l)}^{-2} \cdot \partial_l h_{(i)} \cdot \partial_l h_{(k)}\right]\Big\} = 0.$$

(*Remarks*:

(i) There exist four differential identities among ten equations above. However, the system is still overdetermined and difficult to solve.
(ii) The above equations hold in an N-dimensional manifold.
(iii) Orthogonal coordinates may not exist for $N > 3$.)

3. The most general solution of the system of p.d.e.s is provided by:

$$g_{ij}(x) = \exp\left[-2\mu(x)\right] \cdot d_{kl} \cdot \partial_i f^k \cdot \partial_j f^l.$$

   Here, $\mu(x)$ is of class C^3. Moreover, the four functions $f^k(x)$ are of class C^4, and they are functionally independent. These five functions are otherwise *arbitrary*. (Consult Theorem 1.3.32 and (1.162).)

(*Remark*: By the coordinate transformation $\widehat{x}^i = f^i(x)$, the metric components can be reduced to $\widehat{g}_{ij}\,(\widehat{x}) = \exp\left[-2\widehat{\mu}\,(\widehat{x})\right] \cdot d_{kl}$.)

4.

$$\partial_k g_{ij} = g_{jh}(x) \cdot \begin{Bmatrix} h \\ k\ i \end{Bmatrix} + g_{ih}(x) \cdot \begin{Bmatrix} h \\ k\ j \end{Bmatrix},$$

$$\partial_l \left[g_{jh} \cdot \begin{Bmatrix} h \\ k\ i \end{Bmatrix} + g_{ih} \cdot \begin{Bmatrix} h \\ k\ j \end{Bmatrix} \right] - [l \leftrightarrow k] = 0,$$

$$\partial_k \begin{Bmatrix} i \\ i\ j \end{Bmatrix} - \partial_i \begin{Bmatrix} i \\ k\ j \end{Bmatrix} + \begin{Bmatrix} i \\ k\ h \end{Bmatrix} \begin{Bmatrix} h \\ i\ j \end{Bmatrix} - \begin{Bmatrix} i \\ i\ h \end{Bmatrix} \begin{Bmatrix} h \\ k\ j \end{Bmatrix} = 0.$$

5. Use (2.247i,ii) and (2.252). Also, recall that the metric $\mathbf{g}^{\#}_{..} = (\alpha)^2 \cdot \mathbf{I}_{..}$ *is flat*.

(*Remarks*: Integration of the above equations leads to *the flat Friedmann model of cosmology* in Chap. 6.)

2.5 Fluids, Deformable Solids, and Electromagnetic Fields

We start with *a pressureless incoherent dust or a dust cloud*. In a flat space–time, we introduced the topic in Example 2.1.11. In a curved space–time, Example 2.3.6 dealt with such a material. The pertinent equations are the following:

$$\begin{aligned}
T_{ij}(x) &:= \rho(x)\cdot U_i(x)\cdot U_j(x), \quad \rho(x) > 0,\\
U_i(x)\cdot U^i(x) &\equiv -1,\\
\nabla_i\left(\rho U^i\right) &= 0,\\
\dot{U}^{(a)}\left[\mathcal{X}(s)\right] &= \frac{\mathrm{d}\mathcal{U}^{(a)}(s)}{\mathrm{d}s} - \gamma^{(a)}{}_{(d)(b)}\cdot\mathcal{U}^{(d)}(s)\cdot\mathcal{U}^{(b)}(s) = 0. \qquad (2.256)
\end{aligned}$$

The last equation shows that streamlines follow timelike geodesics.

Now, we shall investigate a *perfect fluid* (or, *an ideal fluid*). This fluid was already introduced in Example 2.3.1. The field equations, from Equations (2.161i), (2.166i), and (2.168i), are given by the following:

$$T^{ij}(x) := \left[\rho(x) + p(x)\right]\cdot U^i(x)\cdot U^j(x) + p(x)\cdot g^{ij}(x), \qquad (2.257\text{i})$$

$$\begin{aligned}
\mathcal{E}^{ij}(x) := G^{ij}(x) + \kappa\big\{&\left[\rho(x)+p(x)\right]\cdot U^i(x)\cdot U^j(x)\\
&+ p(x)\cdot g^{ij}(x)\big\} = 0, \qquad (2.257\text{ii})
\end{aligned}$$

$$\mathcal{T}^i(x) := \nabla_j\left\{\left[\rho(x)+p(x)\right]\cdot U^i(x)\cdot U^j(x) + p(x)\cdot g^{ij}(x)\right\} = 0, \qquad (2.257\text{iii})$$

$$\mathcal{U}(x) := g_{ij}(x)\cdot U^i(x)\cdot U^j(x) + 1 = 0, \qquad (2.257\text{iv})$$

$$\mathcal{C}^i\left(g_{kj},\ \partial_l g_{kj}\right) = 0. \qquad (2.257\text{v})$$

The counting of the number of unknown functions versus the number of independent equations is provided by the following:

No. of unknown functions: $10(g_{ij}) + 4(U^i) + 1(\rho) + 1(p) = 16$.

No. of equations: $10(\mathcal{E}^{ij} = 0) + 4(\mathcal{T}^i = 0) + 1(\mathcal{U} = 0) + 4(\mathcal{C}^i = 0) = 19$.

No. of identities: $4(\nabla_j\mathcal{E}^{ij} \equiv \kappa\mathcal{T}^i) = 4$.

No. of independent equations: $19 - 4 = 15$.

Therefore, the system is *underdetermined*. To make the system *determinate*, it is permissible to impose an *equation of state*

$$F(\rho, p) = 0,$$
$$\left[\frac{\partial F(\cdot\cdot)}{\partial \rho}\right]^2 + \left[\frac{\partial F(\cdot\cdot)}{\partial p}\right]^2 > 0. \tag{2.258}$$

Remark: Choosing four coordinate conditions $\mathcal{C}^i(\cdot\cdot) = 0$ is a usual practice. However, in p. 197, it is mentioned that there exist *three other strategies of solutions*. In each strategy, the counting has to be done separately.

Let the world tube of the fluid body be exhibited in Fig. 2.19. Synge's junction conditions (2.170) across the hypersurface $\partial_{(3)}D$ of jump discontinuities, with $U^i(x)n_i(x)_{|\partial_{(3)}D} = 0$, reduce to

$$\left[g_{ij}(x)\right]_{|\partial_{(3)}D} = 0 = \left[\partial_k g_{ij}\right]_{|\partial_{(3)}D};$$
$$[p(x)]_{|\partial_{(3)}D} = 0. \tag{2.259}$$

Energy conditions from Theorem 2.3.2 reduce in this case to the following:

(i) Weak energy conditions:

$$\rho(x) \geq 0, \quad \rho(x) + p(x) \geq 0. \tag{2.260i}$$

(ii) Dominant energy conditions:

$$|p(x)| \leq \rho(x). \tag{2.260ii}$$

(iii) Strong energy conditions:

$$\rho(x) \geq 0, \quad \rho(x) + p(x) \geq 0, \quad \rho(x) + 3p(x) \geq 0. \tag{2.260iii}$$

The four differential conservation equations (2.257iii), by (2.211) and (2.212i,ii), yield the following constitutive equations:

$$\nabla_j\left[\rho(x)\cdot U^j(x)\right] = -\,p(x)\cdot\nabla_j U^j, \tag{2.261i}$$

$$\dot{\rho}(x) := U^j(x)\cdot\nabla_j\rho = -\left[\rho(x) + p(x)\right]\cdot\nabla_j U^j, \tag{2.261ii}$$

$$\begin{aligned}\left[\rho(x) + p(x)\right]\dot{U}^i(x) &= \left[\rho(x) + p(x)\right]\cdot U^j(x)\cdot\nabla_j U^i \\ &= -\left[g^{ij}(x) + U^i(x)\cdot U^j(x)\right]\cdot\nabla_j p,\end{aligned} \tag{2.261iii}$$

$$\left[\rho(x) + p(x)\right]_{|\mathcal{X}(s)}\cdot\frac{D\,\mathcal{U}^i(s)}{ds} = -\left\{\left[g^{ij}(x) + U^i(x)\cdot U^j(x)\right]\cdot\nabla_j p\right\}_{|\cdot\cdot}. \tag{2.261iv}$$

The last equations (2.261iv) govern streamlines of a perfect fluid flow.

Now, we shall introduce *physical (or orthonormal) components* of the nonrelativistic 3-velocity vector field analogous to those of (2.24). These are provided by

$$\begin{aligned}
\mathcal{V}^{(\alpha)}(x) &:= \frac{U^{(\alpha)}(x)}{U^{(4)}(x)},\\
[\mathcal{V}(x)]^2 &:= \delta_{(\alpha)(\beta)}\mathcal{V}^{(\alpha)}(\cdot\cdot)\cdot\mathcal{V}^{(\beta)}(\cdot\cdot),\\
U^{(\alpha)}(x) &= \mathcal{V}^{(\alpha)}(x)\Big/\sqrt{1-\mathcal{V}^2(\cdot\cdot)},\\
U^{(4)}(x) &= 1\Big/\sqrt{1-\mathcal{V}^2(\cdot\cdot)}.
\end{aligned} \tag{2.262}$$

Remark: Components $\mathcal{V}^{(\alpha)}(x)$'s *are not actual components of any four-dimensional vector or tensor*.

Example 2.5.1. Consider the relativistic equations of motion (2.261iii). In terms of physical (or orthonormal) components and Ricci rotation coefficients of (1.138), the spatial components of (2.261iii) can be expressed as

$$\begin{aligned}
&\big[\rho(x)+p(x)\big]\cdot\Bigg\{\sqrt{1-\mathcal{V}^2(\cdot\cdot)}\cdot\left[\partial_{(4)}\left(\frac{\mathcal{V}^{(\alpha)}}{\sqrt{\cdots}}\right)+\mathcal{V}^{(\beta)}\cdot\partial_{(\beta)}\left(\frac{\mathcal{V}^{(\alpha)}}{\sqrt{\cdots}}\right)\right]\\
&\quad-\gamma^{(\alpha)}{}_{(4)(4)}(\cdot\cdot)-\left(\gamma^{(\alpha)}{}_{(4)(\beta)}+\gamma^{(\alpha)}{}_{(\beta)(4)}\right)\cdot\mathcal{V}^{(\beta)}-\gamma^{(\alpha)}{}_{(\delta)(\beta)}\cdot\mathcal{V}^{(\delta)}\cdot\mathcal{V}^{(\beta)}\Bigg\}\\
&\quad=-\big[1-\mathcal{V}^2(\cdot\cdot)\big]\cdot\delta^{(\alpha)(\beta)}\cdot\partial_{(\beta)}p-\big(\partial_{(4)}p+\mathcal{V}^{(\beta)}\cdot\partial_{(\beta)}p\big)\cdot\mathcal{V}^{(\alpha)}.
\end{aligned} \tag{2.263}$$

The equation above is *the exact general relativistic version of Euler's equation of (perfect) fluid flow* [243]. *A simplistic interpretation* of (2.263) is that

$$(\text{mass})\ \times\ (\text{acceleration})\ =\ -(\text{gradient of pressure}). \qquad \square$$

Now, we shall deal with a 4×4 matrix $\big[T_{ij}(x)\big]$ of Segre characteristic $[(1,1),1,1]$. Physically speaking, this class includes (1) an *anisotropic fluid,* (2) *a deformable solid with symmetry*, (3) *a perfect fluid plus a tachyonic dust* [62], and many other usual or exotic materials. By (2.181), we express

$$\begin{aligned}
T_{ij}(x) &= \big[\rho(x)+p_\perp(x)\big]\cdot U_i(x)\cdot U_j(x)+p_\perp(x)\cdot g_{ij}(x)\\
&\quad+\big[p_\parallel(x)-p_\perp(x)\big]\cdot S_i(x)\cdot S_j(x).
\end{aligned} \tag{2.264}$$

The corresponding gravitational field equations are furnished by

$$\begin{aligned}
\mathcal{E}_{ij}(x) &:= G_{ij}(x)+\kappa\,T_{ij}(x)=0,\\
\mathcal{T}^i(x) &:= \nabla_j T^{ij}=0,
\end{aligned}$$

$$\begin{aligned}
&\mathcal{U}(x) := U_i(x) \cdot U^i(x) + 1 = 0, \\
&\mathcal{S}(x) := S_i(x) \cdot S^i(x) - 1 = 0, \\
&\mathcal{P}(x) := U_i(x) \cdot S^i(x) = 0, \\
&\mathcal{C}^i\left(g_{jk}, \partial_l g_{jk}\right) = 0.
\end{aligned} \tag{2.265}$$

The above system is underdetermined and four subsidiary conditions can be imposed to make the system determinate.

The constitutive equations for the anisotropic fluid flow can be derived from $\mathcal{T}^i(x) = 0$, $U_i(x) \cdot \mathcal{T}^i(x) = 0$, and $S_i(x) \cdot \mathcal{T}^i(x) = 0$. These are explicitly given by

$$\begin{aligned}
&U^i \cdot \nabla_j\left[(\rho + p_\perp) \cdot U^j\right] + (\rho + p_\perp) \cdot U^j \cdot \nabla_j U^i + \nabla^i p_\perp \\
&\quad + S^i \cdot \nabla_j\left[(p_\parallel - p_\perp) S^j\right] + (p_\parallel - p_\perp)\, S^j \nabla_j S^i = 0,
\end{aligned} \tag{2.266i}$$

$$-\nabla_j\left[(\rho + p_\perp) U^j\right] + U_i \cdot \nabla^i p_\perp + (p_\parallel - p_\perp)\, U_i \cdot S^j \cdot \nabla_j S^i = 0, \tag{2.266ii}$$

$$(\rho + p_\perp) \cdot S_i U^j \cdot \nabla_j U^i + S_i \nabla^i p_\perp + \nabla_j\left[(p_\parallel - p_\perp) S^j\right] = 0. \tag{2.266iii}$$

Substituting (2.266ii,iii) into (2.266i), we finally deduce that

$$\begin{aligned}
&(\rho + p_\perp) \cdot (\delta^i{}_k - S^i \cdot S_k) \cdot U^j \cdot \nabla_j U^k \\
&\quad + (p_\parallel - p_\perp) \cdot (\delta^i{}_k + U^i \cdot U_k) \cdot S^j \cdot \nabla_j S^k \\
&\quad + (\delta^i{}_j + U^i \cdot U_j - S^i \cdot S_j) \cdot \nabla^j p_\perp = 0.
\end{aligned} \tag{2.267}$$

The equation above provides the streamlines of an anisotropic fluid flow.

The energy conditions of Theorem 2.3.2 reduce in this case to the following:

1. Weak energy conditions:
$$\rho \geq 0, \quad \rho + p_\perp \geq 0, \quad \rho + p_\parallel \geq 0. \tag{2.268i}$$
2. Dominant energy conditions:
$$\rho \geq |p_\perp|, \quad \rho \geq |p_\parallel|. \tag{2.268ii}$$
3. Strong energy conditions:
$$\rho \geq 0, \quad \rho + p_\perp \geq 0, \quad \rho + p_\parallel \geq 0, \quad \rho + 2p_\perp + p_\parallel \geq 0. \tag{2.268iii}$$

Synge's junction conditions (2.170) for jump discontinuities on the hypersurface $\partial D_{(3)}$, with $U^i(\cdot\cdot) n_i(\cdot\cdot)_{|..} = 0$, reduce to

$$\begin{aligned}
&\left[g_{ij}\right]_{|\partial_{(3)} D} = 0, \quad \left[\partial_k g_{ij}\right]_{|\partial_{(3)} D} = 0, \\
&\left[p_\perp \cdot \delta^i{}_j + (p_\parallel - p_\perp) \cdot S^i \cdot S_j\right] n^j{}_{|\partial_{(3)} D} = 0.
\end{aligned} \tag{2.269}$$

We shall provide some special examples of the anisotropic fluid in the next chapter. Also, see [63, 66] for a detailed treatment of fluids.

Now, we shall investigate the case of a *deformable solid* body. It is characterized by the energy–momentum–stress tensor components:

$$T^{ij}(x) := \rho(x) \cdot U^i(x) \cdot U^j(x) - S^{ij}(x),$$
$$S^{ij}(x) := \sum_{\alpha=1}^{3} \sigma_{(\alpha)}(x) \cdot e^i_{(\alpha)}(x) \cdot e^j_{(\alpha)}(x). \tag{2.270}$$

(We have already mentioned such equations in (2.183), (2.193), and (2.196).)

The gravitational field equations are furnished by

$$\mathcal{E}^{ij}(x) := G^{ij}(x) + \kappa \left[\rho \cdot U^i \cdot U^j - S^{ij}(x)\right] = 0, \tag{2.271i}$$

$$\mathcal{T}^i(x) := \nabla_j \left[\rho \cdot U^i \cdot U^j - S^{ij}\right] = 0, \tag{2.271ii}$$

$$\mathcal{U}(x) := U_i(x) U^i(x) + 1 = 0, \tag{2.271iii}$$

$$\mathcal{P}_{(\alpha)}(x) := U_i(x)\, e^i_{(\alpha)}(x) = 0, \tag{2.271iv}$$

$$\mathcal{N}_{(\alpha)(\beta)}(x) := g_{ij} \cdot e^i_{(\alpha)} \cdot e^j_{(\beta)} - \delta_{(\alpha)(\beta)} = 0, \tag{2.271v}$$

$$\mathcal{C}^i\left(g_{jk}, \partial_l g_{jk}\right) = 0. \tag{2.271vi}$$

The above system is underdetermined and five subsidiary conditions can be imposed.

The constitutive equations are

$$\nabla_j \left[\rho U^j\right] + U_i \cdot \nabla_j \left[\sum_{\alpha=1}^{3} \sigma_{(\alpha)} \cdot e^i_{(\alpha)} \cdot e^j_{(\alpha)}\right] = 0, \tag{2.272i}$$

$$\rho \cdot U^j \cdot \nabla_j U^i = \left(\delta^i{}_k + U^i \cdot U_k\right) \cdot \nabla_j \left[\sum_{\alpha=1}^{3} \sigma_{(\alpha)} \cdot \mathrm{e}^k_{(\alpha)} \cdot e^j_{(\alpha)}\right]. \tag{2.272ii}$$

(Compare the above equations with (2.211) and (2.212i).)

The energy conditions are provided by (2.194i–iii), and Synge's junction conditions (2.170), with $U^i(\cdot\cdot) n_i(\cdot\cdot)_{|..} = 0$, in this case reduce to

$$\left[g_{ij}\right]_{|\partial_{(3)} D} = 0, \quad \left[\partial_k g_{ij}\right]_{|\partial_{(3)} D} = 0,$$
$$\left[\sum_{\alpha=1}^{3} \sigma_{(\alpha)} \cdot e^i_{(\alpha)} \cdot e^j_{(\alpha)} \cdot n_j\right]_{|\partial_{(3)} D} = 0. \tag{2.273}$$

Example 2.5.2. Consider the physical or orthonormal components of equations of motion (2.272ii). The spatial components, with help of (2.262) and (2.263), yield

$$\rho(x)\cdot\Bigg\{\frac{1}{\sqrt{1-\mathcal{V}^2(\cdot\cdot)}}\cdot\Bigg[\partial_{(4)}\left(\frac{\mathcal{V}^{(\alpha)}}{\sqrt{\cdots}}\right)+\mathcal{V}^{(\beta)}\cdot\partial_{(\beta)}\left(\frac{\mathcal{V}^{(\alpha)}}{\sqrt{\cdots}}\right)$$
$$-\left[1-\mathcal{V}^2(\cdot\cdot)\right]^{-1}\cdot\Big[\gamma^{(\alpha)}{}_{(4)(4)}(\cdot\cdot)+\left(\gamma^{(\alpha)}{}_{(4)(\beta)}+\gamma^{(\alpha)}{}_{(\beta)(4)}\right)\cdot\mathcal{V}^{(\beta)}$$
$$+\gamma^{(\alpha)}{}_{(\beta)(\delta)}\cdot\mathcal{V}^{(\beta)}\cdot\mathcal{V}^{(\delta)}\Big]\Bigg]\Bigg\}$$
$$=\left[\delta^{(\alpha)}{}_{(b)}+\left(1-\mathcal{V}^2\right)^{-1/2}\cdot\mathcal{V}^{(\alpha)}U_{(b)}\right]\cdot\nabla_{(c)}S^{(b)(c)}. \tag{2.274}$$

(Compare with (2.41) and (2.125i).) For a class of *equilibrium of the deformable body*, we assume that $\mathcal{V}^{\alpha}(x)\equiv 0$. Thus, the conditions

$$0=U_{(a)}S^{(a)(b)}=0-\frac{1}{\sqrt{\cdots}}\cdot S^{(4)(b)},$$

or,
$$S^{(4)(b)}(x)\equiv 0$$

holds.

Substituting the above into (2.274), we derive that

$$0=\rho(x)\cdot\gamma^{(\alpha)}{}_{(4)(4)}(\cdot\cdot)+\delta^{(\alpha)}{}_{(\beta)}\cdot\nabla_{(\mu)}S^{(\beta)(\mu)}. \tag{2.275}$$

The equation above can be physically interpreted as

"the gravitational forces exactly balance the elastic forces." □

Now, we shall explore electromagnetic fields in a curved space–time manifold. We have already touched upon electromagnetic fields in Examples 1.2.19, 1.2.22, 1.3.6, 2.1.12, and 2.1.13 and (2.56i,ii), (2.60i–iii), (2.63), (2.67), (2.77i–iii), and (2.78i–iii). All these equations allow for *a straight forward generalization* in a domain of curved space–time. Recall that the electromagnetic field is represented by *an antisymmetric tensor field*

$$\mathbf{F}_{\cdot\cdot}(x)=F_{ij}(x)\,\mathrm{d}x^i\otimes\mathrm{d}x^j=(1/2)F_{ij}(x)\,\mathrm{d}x^i\wedge\mathrm{d}x^j. \tag{2.276}$$

Outside charged material sources, Maxwell's equations for an electromagnetic field in *a curved*, *background space–time*, are governed by

$$\nabla_j F^{ij}=0, \tag{2.277i}$$
$$\nabla_i F_{jk}+\nabla_j F_{ki}+\nabla_k F_{ij}\equiv\partial_i F_{jk}+\partial_j F_{ki}+\partial_k F_{ij}=0. \tag{2.277ii}$$

By (1.69) and (1.70), we deduce from (2.277ii) and (1.145i) that

$$F_{ij}(x) = \partial_i A_j - \partial_j A_i \equiv \nabla_i A_j - \nabla_j A_i, \tag{2.278i}$$

$$\nabla_k \nabla^j A_j - \nabla^j \nabla_k A_j = R^h{}_k(x) \cdot A_h(x). \tag{2.278ii}$$

Denoting the *generalized wave operator* (or the *generalized D'Alembertian*) by $\Box := \nabla_j \nabla^j$, Maxwell's equations (2.277i,ii) reduce to

$$\nabla^i \left[\nabla_j A^j\right] - R^{ij}(x) \cdot A_j(x) - \Box A^i = 0. \tag{2.279}$$

Now under *a gauge transformation* of (1.71ii), namely,

$$\widehat{A}_i(x) = A_i(x) - \partial_i \lambda = A_i(x) - \nabla_i \lambda, \tag{2.280}$$

the electromagnetic field remains unchanged, or $\widehat{F}_{ij}(x) \equiv F_{ij}(x)$. Let us choose *a special class* of $\lambda(x)$ such that

$$\Box \lambda = \nabla_i \nabla^i \lambda = \nabla_i A^i. \tag{2.281}$$

By (2.280) and (2.281),

$$\nabla_i \widehat{A}^i = \nabla_i A^i - \Box \lambda = 0. \tag{2.282}$$

The above is called the *Lorentz gauge condition* on the vector field $\vec{\widehat{\mathbf{A}}}(x)$. Maxwell's equations (2.279) simplify into

$$\Box \widehat{A}^i + R^{ij}(x) \cdot \widehat{A}_j(x) = 0. \tag{2.283}$$

Electromagnetic field equations (2.277i,ii) possess *an additional covariance under the conformal transformation* (1.166i). We shall state and prove the following theorem on this topic.

Theorem 2.5.3. *Let a conformal transformation be furnished by*

$$\begin{aligned} \overline{\mathbf{g}}_{..}(x) &= \exp[2\mu(x)] \cdot \mathbf{g}_{..}(x), \\ \overline{\mathbf{F}}_{..}(x) &:= \mathbf{F}_{..}(x), \\ \overline{F}^{ij}(x) &:= \overline{g}^{ik}(x)\, \overline{g}^{jl}(x) F_{kl}(x). \end{aligned} \tag{2.284}$$

Then, the electromagnetic field equations (2.277i,ii) *remain unchanged.*

Proof. By (2.284), it follows that

$$\det\left[\overline{g}_{ij}\right] = \exp[8\mu] \cdot \det\left[g_{ij}\right],$$

$$\overline{F}^{ij}(x) = \exp[-4\mu] \cdot F^{ij}(x),$$

$$\begin{aligned}\overline{\nabla}_j \overline{F}^{ij} &= \frac{1}{\sqrt{-\overline{g}}}\,\partial_j\left[\sqrt{-\overline{g}}\;\overline{F}^{ij}\right]\\ &= \mathrm{e}^{-4\mu}\cdot\left[\nabla_j F^{ij}\right].\end{aligned}$$

Therefore, (2.277i) remains intact. Since (2.277ii) can be expressed without the metric tensor, it remains valid automatically. ■

Example 2.5.4. Let us investigate electromagnetic field equations (2.277i,ii) in *a conformally flat*, background domain characterized by

$$\overline{g}_{ij}(x) = \exp\left[2\mu(x)\right]\cdot d_{ij}\,. \tag{2.285}$$

Equation (2.277i) still implies that $\overline{F}_{ij}(x) = \partial_i A_j - \partial_j A_i \equiv F_{ij}(x)$. The equations $\overline{\nabla}_j \overline{F}_i^{\ j} = 0$ yield

$$\nabla_j F_i^{\ j} = \partial_i\left[\partial_j A^j\right] - d^{kj}\partial_k\partial_j A_i = 0. \tag{2.286}$$

Assuming *the Lorentz gauge* condition:

$$\nabla_k A^k \equiv \partial_k A^k = 0, \tag{2.287}$$

the equations reduce to the wave equation

$$d^{kl}\partial_k\partial_l A^i = 0. \tag{2.288}$$

A general class of solutions of (2.287) and (2.288) is furnished (by the superposition of plane waves) as [55]:

$$\begin{aligned}&A^j(x) = \int\limits_{\mathbb{R}^3} \mathrm{Re}\left\{\alpha^j(\mathbf{k})\cdot\exp\left[\mathrm{i}\left(k_l x^l - \theta(\mathbf{k})\right)\right]\right\}\cdot \mathrm{d}^3\mathbf{k},\\ &k_4 = \nu(\mathbf{k}) := \sqrt{\delta^{\alpha\beta}k_\alpha k_\beta},\\ &\alpha^4(\mathbf{k}) := -k_\mu\alpha^\mu(\mathbf{k})\big/\nu(\mathbf{k}) \quad \text{for} \quad \nu(\mathbf{k}) > 0,\\ &\alpha^\mu(\mathbf{k}) \neq \lambda(\mathbf{k})\,k^\mu.\end{aligned} \tag{2.289}$$

Here, we have *assumed that the integrals in (2.289) converge absolutely and uniformly*. Moreover, *we assume that differentiations commute with integrals* [32]. The five functions $\alpha^j(\mathbf{k})$ and $\theta(\mathbf{k})$ are otherwise *arbitrary*. □

The classification of electromagnetic field equations (2.277i,ii) in flat space–time has been carried out in Appendix 2, Example A2.9. In case, $\partial_4 F_{ij} \neq 0$, the system is *hyperbolic*.

Now, we shall discuss the effects of electromagnetic energy–momentum–stress tensor on the curvature of the space–time manifold. The appropriate field equations for this investigation are the coupled *Einstein–Maxwell equations* as furnished in the following:

$$T^{ij}(x) := F^{ik}(x)F^{j}_{\ k}(x) - (1/4)g^{ij}(x)F_{kl}(x)F^{kl}(x), \tag{2.290i}$$

$$\mathcal{M}^{i}(x) := \nabla_j F^{ij} = 0, \tag{2.290ii}$$

$$^{*}\mathcal{M}^{i}(x) := \frac{1}{3!}\,\eta^{ijkl}(x)\left[\nabla_j F_{kl} + \nabla_k F_{lj} + \nabla_l F_{jk}\right] = 0, \tag{2.290iii}$$

$$\mathcal{E}^{ij}(x) := G^{ij}(x) + \kappa\, T^{ij}(x) = 0, \tag{2.290iv}$$

$$\mathcal{T}^{i}(x) := \nabla_j T^{ij} \equiv 0, \tag{2.290v}$$

$$\mathcal{C}^{i}\left(g_{jk}, \partial_l g_{jk}\right) = 0. \tag{2.290vi}$$

Remarks: (i) The special relativistic electromagnetic energy–momentum–stress was introduced in (2.63).

(ii) In (2.290iii), the definition of *the Hodge-star operation* in (1.113) is used.

(iii) A domain of space–time, in which (2.290i–vi) hold, is called an *electromagneto-vac domain*.

(iv) Equation (2.290i) implies that $T^{i}_{\ i}(x) \equiv 0$. Therefore, (2.290iv) yields that $R(x) \equiv 0$.

The number of unknown functions versus the number of independent equations in the system (2.290i–vi) is exhibited below:

(i) No. of unknown functions: $6(F_{ij}) + 10(g_{ij}) = 16$.

(ii) No. of equations: $4(\mathcal{M}_i = 0) + 4(^{*}\mathcal{M}_i = 0) + 10(\mathcal{E}^{ij} = 0) + 4(\mathcal{C}^{i} = 0) = 22$.

(iii) No. of identities: $1(\nabla_i \mathcal{M}^{i} \equiv 0) + 1(\nabla_i\, ^{*}\mathcal{M}^{i} \equiv 0) + 4(\nabla_j \mathcal{E}^{ij} \equiv 0) = 6$.

(iv) No. of independent equations: $22 - 6 = 16$.

Therefore, the system of (2.290i–vi) is *exactly determinate*!

Now, we shall discuss *the variational derivation* of the field equations (2.290i–vi). (Appendix 1 deals with variational derivation of differential equations.)

Example 2.5.5. Equations (2.290iii) yield (2.278i), which is

$$F_{ij}(x) = \partial_i A_j - \partial_j A_i \equiv \nabla_i A_j - \nabla_j A_i .$$

It turns out that the variational derivation demands the use of the *4-potential* $A^{i}(x)$, rather than $F_{ij}(x)$. Using (A1.25), we write the Lagrangian function for the coupled fields as

$$
\begin{aligned}
&L\left(a_i, y^{ij}, \gamma^k{}_{ij}; a_{ij}, y^{ij}{}_k, \gamma^k{}_{ijl}\right) \\
&:= y^{ij}\left[\gamma^k{}_{kij} - \gamma^k{}_{ijk} - \gamma^l{}_{lk}\gamma^k{}_{ij} + \gamma^l{}_{ik}\gamma^k{}_{lj}\right] \\
&\quad + (\kappa/2)\cdot y^{ij}y^{kl}\left(a_{lj} - a_{jl}\right)(a_{ki} - a_{ik}) \\
&=: \rho_{ij}(\cdot\cdot) + (\kappa/2)\cdot y^{ij}y^{kl}\left(a_{lj} - a_{jl}\right)(a_{ki} - a_{ik}), \\
&L(\cdots)\Big|_{\substack{y^{ij}=g^{ij}(x),\; y^{ij}{}_k=\partial_k g^{ij},\\ \gamma^k{}_{ij}=\left\{\begin{smallmatrix}k\\ ij\end{smallmatrix}\right\},\; \gamma^k{}_{ijl}=\partial_l\left\{\begin{smallmatrix}k\\ ij\end{smallmatrix}\right\},\\ a_{lj}=\partial_j A_l.}} = R(x) + (\kappa/2)\cdot F^{ij}(x)\cdot F_{ij}(x).
\end{aligned}
\tag{2.291}
$$

The corresponding action integral is provided by

$$
J(F) := \int_{D_4} L(\cdots)_{|\dots}\cdot\sqrt{-g(x)}\,\mathrm{d}^4x,
$$

$$
a_i = \pi_i \circ F(x) =: A_i(x). \tag{2.292}
$$

(See Fig. A1.2.). We vary independently $\widehat{y}^{ij} = g^{ij}(x) + \varepsilon h^{ij}(x)$, $\widehat{y}^k{}_{ij} = \left\{\begin{matrix}k\\ i\ j\end{matrix}\right\}$ $+\,\varepsilon h^k{}_{ij}(x)$, and $\widehat{a}_i = A_i(x) + \varepsilon h_i(x)$. Then, the vanishing of the variational derivative, that is,

$$
\lim_{\varepsilon\to 0}\frac{\Delta J(F)}{\varepsilon} = 0,
$$

yields, by (A1.26), etc., the field equations (2.290ii) and (2.290iv). Moreover, *the boundary terms* from (2.291), (A1.20ii), and (A1.29ii) imply that

$$
\int_{\partial D_4}\left\{g^{ij}(x)\cdot h^k{}_{ki}(x) - g^{ki}(x)\cdot h^j{}_{ki}(x) + 2\kappa\, F^{ij}(x)h_i(x)\right\} n_j(x)\cdot \mathrm{d}^3v = 0. \tag{2.293}
$$

Therefore, variations on the boundary ∂D_4 satisfying

$$
\left\{\left[g^{ij}(x)\cdot h^k{}_{ki}(x) - g^{ki}(x)\cdot h^j{}_{ki}(x) + 2\kappa\, F^{ij}(x)h_i(x)\right]n_j(x)\right\}_{|\partial D_4} = 0, \tag{2.294}
$$

imply (2.293). (But the converse is not true!). *A popular way* to write (2.293) is as follows:

$$\int_{\partial D_4} \left\{ \left[g^{ij}(x) \cdot \delta \begin{Bmatrix} k \\ k\, i \end{Bmatrix} - g^{ki} \cdot \delta \begin{Bmatrix} j \\ k\, i \end{Bmatrix} + 2\kappa\, F^{ij}(x) \cdot \delta A_i(x) \right] n_j(x) \right\} \mathrm{d}^3 v = 0. \tag{2.295}$$

Usually, boundary variations $\delta \begin{Bmatrix} k \\ j\, i \end{Bmatrix}_{|..} = 0$, $\delta A_i(x)_{|..} = 0$ are chosen (for the Dirichlet problem), but there exist *many more initial-boundary problems admitted by the variational principle*, as exhibited in (2.293) or (2.295). □

Now, we shall explore *coupled gravitational and electromagnetic fields*, generated by the distribution of a charged dust cloud (or a primitive plasma). In the flat space–time, we have already investigated *incoherent*, *charged dust* in (2.64i,ii). The pertinent field equations here are expressed in the following:

$$T^{ij}(x) := \rho(x) U^i(x) U^j(x) + F^{ik}(x) F_k^{\ j}(x) - (1/4) g^{ij}(x) F_{kl}(x) F^{kl}(x), \tag{2.296i}$$

$$\mathcal{M}^i(x) := \nabla_j F^{ij} - \sigma(x) U^i(x) = 0, \tag{2.296ii}$$

$$^*\mathcal{M}^i(x) := \frac{1}{3!}\, \eta^{ijkl}(x) \left[\nabla_j F_{kl} + \nabla_k F_{lj} + \nabla_l F_{jk} \right] = 0, \tag{2.296iii}$$

$$\mathcal{J}(x) := \nabla_i \left(\sigma U^i \right) = 0, \tag{2.296iv}$$

$$\mathcal{E}^{ij}(x) := G^{ij}(x) + \kappa\, T^{ij}(x) = 0, \tag{2.296v}$$

$$\mathcal{T}^i(x) := \nabla_j T^{ij} = 0, \tag{2.296vi}$$

$$\mathcal{U}(x) := U_i(x) U^i(x) + 1 = 0, \tag{2.296vii}$$

$$\mathcal{C}^i \left(g_{jk}, \partial_l g_{jk} \right) = 0. \tag{2.296viii}$$

We shall now count the number of unknown functions versus the number of independent equations.

No. of unknown functions: $6(F_{ij}) + 1(\sigma) + 1(\rho) + 4(U^i) + 10(g_{ij}) = 22.$

No. of equations: $4(\mathcal{M}^i = 0) + 4({}^*\mathcal{M}^i = 0) + 1(\mathcal{J} = 0) + 10(\mathcal{E}^{ij} = 0)$
$+ 4(\mathcal{T}^i = 0) + 1(\mathcal{U} = 0) + 4(\mathcal{C}^i = 0) = 28.$

No. of identities: $1(\nabla_i \mathcal{M}^i \equiv 0) + 1(\nabla_i\, {}^*\mathcal{M}^i \equiv 0) + 4(\nabla_j \mathcal{E}^{ij} \equiv \kappa \mathcal{T}^i) = 6.$

No. of independent equations: $28 - 6 = 22$.

Thus, *the system is determinate*.

Now, we shall derive the constitutive equations for the system in (2.296i–viii).

The differential conservation equations (2.296vi) yield

$$\begin{aligned} 0 &= \nabla_j \left[\rho U^i U^j + F^{ik} F^j_{\ k} - (1/4) g^{ij} F_{kl} F^{kl}\right] \\ &= U^i \nabla_j \left(\rho U^i\right) + \rho U^j \nabla_j U^i + F^{ik} \nabla_j F^j_{\ k} \\ &\quad + (1/2) g^{il} F^{jk} \left[\nabla_j F_{lk} + \nabla_l F_{kj} + \nabla_k F_{jl}\right] \\ \text{or} \qquad 0 &= U^i \nabla_j \left(\rho U^j\right) + \rho U^j \nabla_j U^i - \sigma F^{ik} U_k + 0. \end{aligned} \tag{2.297}$$

Multiplying the equation above by $U_i(x)$, we deduce that

$$\nabla_j \left(\rho U^j\right) = 0. \tag{2.298}$$

Thus, *the continuity of the material current still holds*. Substituting (2.298) into (2.297), we derive that

$$\begin{aligned} &\rho(x) \dot{U}^i(x) = \rho(x) U^j(x) \nabla_j U^i = \sigma(x) F^{ik}(x) U_k(x) \\ \text{or} \qquad &\rho\left[\mathcal{X}(s)\right] \cdot \frac{D\, \mathcal{U}^i(s)}{\mathrm{d}s} = \left[\sigma(x) F^i_{\ k}(x)\right]_{|..} \cdot \mathcal{U}^k(s). \end{aligned} \tag{2.299}$$

The above equation of a streamline is the appropriate (curved space–time) generalization of the Lorentz equation of motion (2.66) and (2.67) in the flat space–time scenario.

Here, we emphasize that the relativistic equations of motion for streamlines in (2.256)*,* (2.261iv)*, and* (2.267) *and equations of motion* (2.272ii) *for a solid particle are all consequences of Einstein's gravitational field equations. These equations are not added on as required in the corresponding non-general relativistic theories.*

Now, we shall touch upon briefly the algebraic properties of the antisymmetric electromagnetic tensor field

$$\mathbf{F}_{..}(x) = (1/2) F_{ij}(x)\, \mathrm{d}x^i \wedge \mathrm{d}x^j = F_{(a)(b)}(x)\, \tilde{\mathbf{e}}^{(a)}(x) \otimes \tilde{\mathbf{e}}^{(b)}(x).$$

The 4×4 antisymmetric matrix $\left[F_{(a)(b)}(x_0)\right]$ *does not possess any real, nonzero, usual eigenvalues*. However, the same matrix *has invariant, (real) eigenvalues*. The invariant eigenvalue probelm can be posed as:

$$F_{(a)(b)}(x_0) \cdot \nu^{(b)}(x_0) = \lambda\, d_{(a)(b)}\, \nu^{(b)}(x_0). \tag{2.300}$$

(Compare the above equation with equations in Example A3.5 of Appendix 3) It follows from (2.300) that

$$\lambda\, d_{(a)(b)}\, \nu^{(a)}(x_0)\, \nu^{(b)}(x_0) = 0. \tag{2.301}$$

Therefore, either the (invariant) eigenvalue $\lambda = 0$ or the "invariant eigenvector" $\nu^{(a)}(x_0)\, \vec{\mathbf{e}}_{(a)}(x_0)$ is *null or both*.

Now, consider two fields defined by

$$I_{(1)}(x) := F_{(a)(b)}(x) \cdot F^{(a)(b)}(x), \tag{2.302i}$$

$$I_{(2)}(x) := F_{(a)(b)}(x) \cdot {}^{*}F^{(a)(b)}(x). \tag{2.302ii}$$

(Here, the star stands for *the Hodge-star* operation in (1.113).) The field $I_{(1)}(x)$ is a scalar or invariant. However, the field $I_{(2)}(x)$ is *a pseudoscalar* and $\left[I_{(2)}(x)\right]^2$ is invariant.

We shall now introduce the physical or orthonormal components of the electric and magnetic field vectors. These are furnished by

$$E_{(\alpha)}(x) := F_{(\alpha)(4)}(x), \tag{2.303i}$$

$$H_{(\alpha)}(x) := \varepsilon_{(\alpha)(\beta)(\gamma)}\, F^{(\beta)(\gamma)}(x). \tag{2.303ii}$$

(Compare the equations above with (2.55).)

Remark: The components $E_{(\alpha)}(x)$ and $H_{(\alpha)}(x)$ are *not components of relativistic vector fields*.

The invariant in (2.302i) and the pseudoscalar in (2.302ii) can be expressed in terms of electric and magnetic vectors as

$$\begin{aligned} I_{(1)}(x) &= -2\delta^{(\alpha)(\beta)}\left[E_{(\alpha)}(x)E_{(\beta)}(x) - H_{(\alpha)}(x)H_{(\beta)}(x)\right] \\ &=: -2\left[\|\vec{\mathbf{E}}(x)\|^2 - \|\vec{\mathbf{H}}(x)\|^2\right], \end{aligned} \tag{2.304i}$$

$$I_{(2)}(x) = -4\delta^{(\alpha)(\beta)} E_{(\alpha)}(x) H_{(\beta)}(x) =: -4\left[\vec{\mathbf{E}}(x) \cdot \vec{\mathbf{H}}(x)\right]. \tag{2.304ii}$$

The energy–momentum–stress tensor of an electromagnetic field from (2.290i) is expressed as

$$T^{(a)(b)}(x) := F^{(a)(c)}(x)F^{(b)}{}_{(c)}(x) - (1/4)\, d^{(a)(b)} F_{(c)(d)}(x)F^{(c)(d)}(x). \tag{2.305}$$

We now choose the orthonormal tetrad $\left\{\vec{\mathbf{e}}_{(1)}(x), \vec{\mathbf{e}}_{(2)}(x), \vec{\mathbf{e}}_{(3)}(x), \vec{\mathbf{e}}_{(4)}(x)\right\}$ in a special way so that the vector $\vec{\mathbf{e}}_{(3)}(x)$ *is orthogonal to both* $\vec{\mathbf{E}}(x)$ *and* $\vec{\mathbf{H}}(x)$. (Rotation of spatial vectors $\{\vec{\mathbf{e}}_{(1)}(x), \vec{\mathbf{e}}_{(2)}(x), \vec{\mathbf{e}}_{(3)}(x)\}$ can always achieve this simplification.) Therefore, relative to this special class of frames,

$$E_{(3)}(x) = H_{(3)}(x) = 0. \tag{2.306}$$

Therefore, (2.304i,ii) simplify into

$$F_{(a)(b)}(x) F^{(a)(b)}(x) = -2\left[E_{(1)}^2 + E_{(2)}^2 - H_{(1)}^2 - H_{(2)}^2\right], \tag{2.307i}$$

$$F_{(a)(b)}(x)\, {}^*F^{(a)(b)}(x) = -4\left[E_{(1)} H_{(1)} + E_{(2)} H_{(2)}\right]. \tag{2.307ii}$$

Moreover, the simplified version of (2.305) yields *the nonzero components* as

$$\begin{aligned}
T_{(1)(1)}(x) &= -T_{(2)(2)}(x) = (1/2)\left[H_{(2)}^2 - H_{(1)}^2 + E_{(2)}^2 - E_{(1)}^2\right], \\
T_{(3)(3)}(x) &= T_{(4)(4)}(x) = (1/2)\left[H_{(1)}^2 + H_{(2)}^2 + E_{(1)}^2 + E_{(2)}^2\right], \\
T_{(1)(2)}(x) &= -\left[E_{(1)} E_{(2)} + H_{(1)} H_{(2)}\right], \\
T_{(3)(4)}(x) &= H_{(1)} E_{(2)} - E_{(1)} H_{(2)}.
\end{aligned} \tag{2.308}$$

The 4×4 symmetric matrix $\left[T_{(a)(b)}(x_0)\right]$ becomes *block diagonal*.

Now, we shall explore the invariant eigenvalue problem for the matrix $\left[T_{(a)(b)}(x_0)\right]$.

Theorem 2.5.6. *The invariant eigenvalues from the equation* $\det\left[T_{(a)(b)}(x_0) - \lambda\, d_{(a)(b)}\right] = 0$ *are furnished by four real numbers:*

$$\lambda = \lambda_0, -\lambda_0, \lambda_0, -\lambda_0\,;$$

$$\lambda_0 = (1/4)\left\{\left[F_{(a)(b)}(x_0) F^{(a)(b)}(x_0)\right]^2 + \left[F_{(a)(b)}(x_0)\, {}^*F^{(a)(b)}(x_0)\right]^2\right\}^{1/2}. \tag{2.309}$$

Proof. The determinantal equation for eigenvalues splits into two equations

$$\det\begin{bmatrix} T_{(1)(1)} - \lambda & T_{(1)(2)} \\ T_{(1)(2)} & T_{(2)(2)} - \lambda \end{bmatrix} = 0 \tag{2.310i}$$

$$\text{and} \qquad \det\begin{bmatrix} T_{(3)(3)} - \lambda & T_{(3)(4)} \\ T_{(3)(4)} & T_{(4)(4)} + \lambda \end{bmatrix} = 0. \tag{2.310ii}$$

Equations (2.310i,ii) yield, respectively,

$$\lambda^2 = \left[T_{(1)(1)}\right]^2 + \left[T_{(1)(2)}\right]^2, \tag{2.311i}$$

$$\lambda^2 = \left[T_{(4)(4)}\right]^2 - \left[T_{(3)(4)}\right]^2. \tag{2.311ii}$$

By (2.308) and (2.307i,ii), the right-hand sides of (2.311i) and (2.311ii) *coincide* so that

$$4\lambda^2 = \left[\left(E_{(1)}\right)^2 + \left(E_{(2)}\right)^2 - \left(H_{(1)}\right)^2 - \left(H_{(2)}\right)^2\right]^2 + 4\left[E_{(1)}H_{(1)} + E_{(2)}H_{(2)}\right]^2$$
$$= (1/4) \cdot \left\{\left[F_{(a)(b)}(x_0)F^{(a)(b)}(x_0)\right]^2 + \left[F_{(a)(b)}(x_0)\ {}^*F^{(a)(b)}(x_0)\right]^2\right\}.$$

Since the right-hand side of the equation is *an invariant*, the equation is valid in general for any other orthonormal tetrad. Thus, (2.309) is proved. ■

Remarks: (i) The proof for the similar theorem in *flat space–time* is exactly the same.
(ii) For $\lambda^2 > 0$, the Segre characteristic of $\left[T_{(a)(b)}(x_0)\right]$ is $[(1,1),(1,1)]$.

We define a *null electromagnetic field* by the condition $\lambda = 0$, which implies from (2.309) that

$$F_{(a)(b)}(x_0)F^{(a)(b)}(x_0) = 0,$$

and,

$$F_{(a)(b)}(x_0)\ {}^*F^{(a)(b)}(x_0) = 0. \tag{2.312}$$

(The Segre characteristic is $[(1,1,2)]$.)

In terms of electric and magnetic field vectors, (2.5) yields from (2.304i,ii)

$$\|\vec{\mathbf{E}}(x_0)\| = \|\vec{\mathbf{H}}(x_0)\|,$$
$$\vec{\mathbf{E}}(x_0) \cdot \vec{\mathbf{H}}(x_0) = 0. \tag{2.313}$$

Therefore, a null electromagnetic field physically represents the analog of *a plane electromagnetic wave* in flat space–time.

Exercises 2.5

1. Using (2.256), (2.298), and the junction condition $\rho U^i n_i|_{..} = 0$, prove the existence of the conserved, total (proper) mass

$$M := -\int_{\Sigma} \left[\rho(x)\, U^i(x)\, n_i(x)\right]_{|..} \cdot \mathrm{d}^3 v.$$

2. Consider a perfect fluid with the equation of state: $\rho = \mathcal{R}(p)$, $\nu(p) := \int dp \big/ \left[p + \mathcal{R}(p)\right]$, $\mu(x) := \nu\left[p(x)\right]$. Introduce a conformal transformation

$\overline{g}_{ij}(x) = \mathrm{e}^{2\mu(x)} \cdot g_{ij}(x)$, $W_i(x) := \mathrm{e}^{\mu} \cdot U_i(x)$, $\overline{W}^i(x) = \mathrm{e}^{-\mu} \cdot U^i(x)$. Prove that the equations of motion (2.261iii) reduce to

$$(\rho + p) \cdot U^j \cdot \overline{\nabla}_j U^i = U^i \cdot U^j \cdot \overline{\nabla}_j p.$$

3. Consider a perfect fluid whose energy–momentum–stress tensor is given by (2.257i) which obeys the conservation law (2.257iii).

   (i) Consider the projection of the conservation law in the direction of the fluid 4-velocity (i.e., $\mathcal{T}^i U_i = 0$). Show that this equation yields the relativistic continuity equation for a perfect fluid:

   $$\nabla_i \left(\rho U^i\right) + p \nabla_i U^i = 0.$$

   (ii) Consider now a projection which is orthogonal to the fluid 4-velocity. By projecting the conservation law in an orthogonal direction (i.e., $\mathcal{T}^i \mathcal{P}_{ij} = 0$, with $\mathcal{P}_{ij}$ being the perpendicular projection tensor of (2.198)), show that this implies the relativistic Euler equations for a perfect fluid:

   $$(\rho + p)\, U^i \nabla_i U_j + \nabla_j\, p + U^i U_j\, \nabla_i\, p = 0.$$

4. A relativistic fluid with the *heat flow vector* $\vec{\mathbf{q}}(x)$ is characterized by:

   $$T^{ij}(\cdot\cdot) := (\rho + p) U^i U^j + p g^{ij} + U^i q^j + U^j q^i + \pi^{ij},$$
   $$q_i U^i = 0, \quad \pi_{ij} U^j = 0, \quad \pi^i{}_i = 0, \quad \pi_{ij} = \pi_{ji}.$$

   Prove that the constitutive equations are furnished by

   (i) $\dot{\rho} + (\rho + p)\, \Theta + \nabla_i q^i - U_i \dot{q}^i + \pi^{ij} \sigma_{ij} = 0$,

   (ii) $(\rho + p)\dot{U}^i + \left(\delta^i{}_j + U^i U_j\right) \cdot \left(\nabla^j p + \dot{q}^j + \nabla_k \pi^{kj}\right)$
   $+ q_j \left(\sigma^{ij} + \omega^{ij}\right) + (4/3)\, \Theta\, q^i = 0.$

5. Consider a deformable solid body in general relativity characterized by

   $$T^{ij}(\cdot\cdot) := \rho U^i U^j - S^{ij} \equiv \rho U^i U^j - \sum_{\mu} \sigma_{(\mu)}\, e^i_{(\mu)}\, e^j_{(\mu)},$$
   $$S_{ij} U^j = 0.$$

   Prove that the constitutive equations are provided by

   (i) $\dot{\rho} = -\rho\, \Theta + S^{ij}\left[\sigma_{ij} + (1/3)\, \Theta\, g_{ij}\right]$,

   (ii) $\rho \dot{U}^i = \nabla_j S^{ij} - U^i S^{jk}\left[\sigma_{jk} + (1/3)\, \Theta\, g_{jk}\right]$.

6. Maxwell's equations and the Lorentz gauge condition are given in (2.283) and (2.282). Let *the background metric* satisfy $G_{ij}(x) = \Lambda g_{ij}(x)$. (Here, Λ is the nonzero cosmological constant.) Show that the electromagnetic 4-potential $\vec{\mathcal{A}}(x)$ satisfies a massive vector-boson equation

$$\Box A^i - \Lambda A^i(x) = 0.$$

7. Consider electromagneto-vac equations (2.290i–iv) with the Lorentz gauge condition (2.282). Derive that the 4-potential vector $\vec{\mathcal{A}}(x)$ satisfies gauge-field type of equations

$$\Box A^i - \kappa \left[F^{ik} F_{jk} - (1/4)\, \delta^i{}_j\, F_{kl} F^{kl} \right] A^j = 0.$$

8. The Hodge-dual operation in Example 1.3.6 yielded

$$^*F^{ij}(x) = (1/2)\, \eta^{ijkl}(x) \cdot F_{kl}(x).$$

Define a complex-valued, antisymmetric field by

$$\varphi^{kj}(x) := F^{kj}(x) - i\; {}^*F^{kj}(x).$$

 (i) Deduce that Maxwell's equations (2.277i,ii) reduce to $\nabla_j \varphi^{kj} = 0$.
 (ii) Prove that the energy–momentum–stress tensor (2.290i) yields $T^j_{\ k}(x) = (1/2) \cdot \mathrm{Re}[\overline{\varphi}^{jl}(x) \cdot \varphi_{kl}(x)]$. (Here, the bar denotes *complex-conjugation*.)

9. A global, *electromagnetic duality rotation* is furnished by

$$\widehat{\varphi}^{kj}(x) = \mathrm{e}^{i\alpha} \cdot \varphi^{kj}(x).$$

Here, α is *a real constant*. Show that electromagneto-vac equations (2.290ii,iii) are covariant and (2.290iv) are invariant under the duality rotation.

10. Prove that in terms of orthonormal or physical components, field equations (2.296v) for an incoherent charged dust are equivalent to

$$\begin{aligned} R^{(d)}{}_{(a)(b)(c)}(x) &= C^{(d)}{}_{(a)(b)(c)}(x) + (\kappa/2)\Big[\delta^{(d)}{}_{(b)} T_{(a)(c)} - \delta^{(d)}{}_{(c)} T_{(a)(b)} \\ &\quad + d_{(a)(c)} T^{(d)}{}_{(b)} - d_{(a)(b)} T^{(d)}{}_{(c)} + (2/3)\rho(x) \cdot \left(\delta^{(d)}{}_{(b)} d_{(a)(c)} - \delta^{(d)}{}_{(c)} d_{(a)(b)}\right)\Big]. \end{aligned}$$

11. Consider a null electromagnetic field given by (2.3.12).

 (i) Prove that the corresponding stress–energy–momentum tensor can be written as

$$T_{(a)(b)}(x_0) = \mu(x_0)\, \nu_{(a)}(x_0)\, \nu_{(b)}(x_0),$$

$$\text{with:} \qquad \mu(x_0) > 0, \quad \nu^{(a)}(x_0)\, \nu_{(a)}(x_0) = 0.$$

(ii) Show that the 4×4 matrix $\left[T_{(a)(b)}(x_0)\right]$ belongs to the Segre characteristic $[(1,1),2]$.

Answers and Hints to Selected Exercises

1. See Fig. 2.19 and use (2.216).
2.

$$\partial_i \mu = \partial_i p \big/ [p + \mathcal{R}(p)]\,;$$

$$\overline{\begin{Bmatrix} k \\ i\ j \end{Bmatrix}} = \begin{Bmatrix} k \\ i\ j \end{Bmatrix} + \delta^k{}_i \cdot \partial_j \mu + \delta^k{}_j \cdot \partial_i \mu - g_{ij}\, g^{kl} \cdot \partial_l \mu.$$

3. Note that $U^i U_i = -1$ and that $U^i \nabla_j U_i = 0$ (show this).
4. (i) Use (2.199i–v) and the consequent equation (2.201) as

$$\nabla_k U_j = \omega_{jk} + \sigma_{jk} + \frac{1}{3} \Theta \mathcal{P}_{jk} - \dot{U}_j U_k, \; \Theta = \nabla_k U^k.$$

Moreover, apply the following equations:

$$\nabla_j \left(U_i\, \pi^{ij}\right) = 0,$$

$$\begin{aligned} \text{or,} \qquad U_i \nabla_j \pi^{ij} &= 0 - \pi^{ij} \cdot \nabla_j U_i \\ &= -\pi^{ij}\left[\sigma_{ij} + \omega_{ij} + (1/3)\, \Theta\, \mathcal{P}_{ij} - \dot{U}_i U_j\right] \\ &= -\pi^{ij} \sigma_{ij} - 0 - 0 - 0. \end{aligned}$$

5. (i) Use (2.211). Moreoever,

$$0 = \nabla_k \left[S^{jk} U_j\right],$$

$$\begin{aligned} \text{or,} \quad U_j \nabla_k S^{jk} &= -S^{jk} \cdot \nabla_k U_j = -S^{jk}\left[\omega_{jk} + \sigma_{jk} + (1/3)\, \Theta\, \mathcal{P}_{jk} - \dot{U}_j U_k\right] \\ &= -S^{jk}\left[0 + \sigma_{jk} + (1/3)\, \Theta\, g_{jk} - 0\right]. \end{aligned}$$

7. Use (2.283) and $R^i{}_j(x) = -\kappa\, T^i{}_j(x)$.

8. (ii)

$$
\begin{aligned}
&- \eta^{abjl}(x) \cdot \eta_{cdkl}(x) \\
&= \delta^a_{\ c} \left(\delta^b_{\ d}\, \delta^j_{\ k} - \delta^b_{\ k}\, \delta^j_{\ d} \right) + \delta^a_{\ d} \left(\delta^b_{\ k}\, \delta^j_{\ c} - \delta^b_{\ c}\, \delta^j_{\ k} \right) \\
&\quad + \delta^a_{\ k} \left(\delta^b_{\ c}\, \delta^j_{\ d} - \delta^b_{\ d}\, \delta^j_{\ c} \right).
\end{aligned}
$$

10. Use field equations (2.163ii) and (2.296v).
11. (i) Choose a special orthonormal tetrad such that

$$
E_{(2)}(x_0) = E_{(3)}(x_0) = H_{(1)}(x_0) = H_{(3)}(x_0) = 0,
$$

$$
\mu(x) := (E_{(1)}(x_0))^2 \;=\; (H_{(2)}(x_0))^2 \;> 0. \text{Suppose} \quad E_{(1)}(x_0) = +H_{(2)}(x_0).
$$

Eigenvalue equations $T_{(a)(b)}(x_0)\, \nu^{(b)}(x_0) = 0$ reduce, from (2.308), to *one independent equation* $\nu^{(3)} - \nu^{(4)} = 0$.

The null eigenvector can be chosen as $\vec{\boldsymbol{\nu}} = [\delta^i_{\ (3)} + \delta^i_{\ (4)}] \cdot \frac{\partial}{\partial x^i}$. In terms of its covariant components, $T_{(a)(b)}(x_0) = \mu(x_0)\, \nu_{(a)}(x_0)\, \nu_{(b)}(x_0)$.

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