Chapter 2
Mathematical and Physical Pendulum

2.1 The Mathematical Pendulum

A particle of mass m connected by a rigid, weightless rod (or a thread) of length l to a base by means of a pin joint that can oscillate and rotate in a plane we call a *mathematical pendulum* (Fig. 2.1).

Let us resolve the gravity force into the component along the axis the rod and the component perpendicular to this axis, where both components pass through the particle of mass m. The normal component does not produce the particle motion. The component tangent to the path of the particle, being the arc of a circle of radius l, is responsible for the motion. Writing the equation of moments about the pendulum's pivot point we obtain

$$ml^2\ddot{\varphi} + mgl\sin\varphi = 0, \tag{2.1}$$

where ml^2 is a mass moment of inertia with respect to the pivot point.

From (2.1) we obtain

$$\ddot{\varphi} + \alpha^2\sin\varphi = 0, \tag{2.2}$$

where

$$\alpha = \sqrt{\frac{g}{l}}\left[\frac{\text{rad}}{\text{s}}\right].$$

If it is assumed that we are dealing only with small oscillations of the pendulum, the relationship $\sin\alpha \approx \alpha$ holds true and (2.2) takes the form

$$\ddot{\varphi} + \alpha^2\varphi = 0. \tag{2.3}$$

It is the second-order linear differential equation describing the circular motion of a particle (Sect. 4.2 of [1]). Let us recall that its general solution has the form

$$\varphi = \phi\sin(\alpha t + \Theta_0), \tag{2.4}$$

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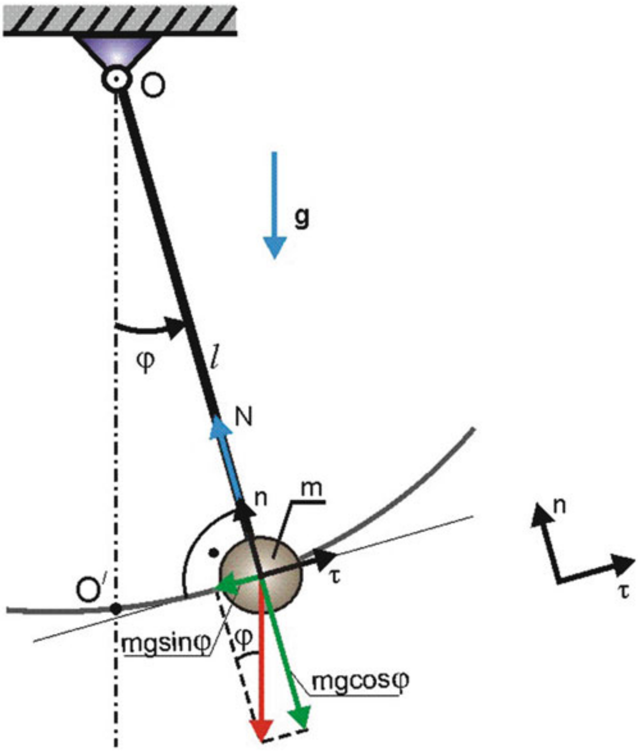


Fig. 2.1 Mathematical pendulum

which means that we are dealing with small harmonic oscillations of the period $T = \frac{2\pi}{\alpha} = 2\pi\sqrt{\frac{l}{g}}$. Let us note that in the case of small oscillations of a pendulum, their period does not depend on the initial angle of deflection of the pendulum but exclusively on the length of pendulum l. We say that such a motion is isochronous. That observation, however, is not valid for the big initial angle of deflection. Such a conclusion can be drawn on the basis of the following calculations (see also [2, 3]).

Let us transform (2.2) into the form

$$\begin{aligned} \dot{\varphi} &= \gamma, \\ \dot{\gamma} &= -\alpha^2 \sin\varphi. \end{aligned} \tag{2.5}$$

Let us note that

$$\dot{\gamma} = \frac{d\gamma}{dt} = \frac{d\gamma}{d\varphi}\frac{d\varphi}{dt} = \frac{d\gamma}{d\varphi}\gamma = \frac{1}{2}\frac{d}{d\varphi}\left(\gamma^2\right), \tag{2.6}$$

and substituting (2.6) into the second equation of system (2.5) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\gamma^2\right) = -2\alpha^2 \sin\varphi. \tag{2.7}$$

Separating the variables and integrating we obtain

$$\dot{\varphi}^2 = 2\alpha^2 \cos\varphi + 2C, \tag{2.8}$$

where C is the integration constant.

Let us emphasize that we were able to conduct the integration thanks to the fact that the investigated system is conservative (it was assumed that the medium in which the vibrations take place introduced no damping). Equation (2.8) is the *first integral* of the non-linear differential equation (2.2) since it relates the functions $\varphi(t)$ and $\dot{\varphi}(t)$. In other words, it is the non-linear equation of reduced order with respect to the original equation (2.2).

Let us introduce the following initial conditions: $\varphi(0) = \varphi_0$, $\dot{\varphi}(0) = \dot{\varphi}_0$, and following their substitution into (2.8), in order to determine the integration constant, we obtain

$$2C = \dot{\varphi}_0^2 - 2\alpha^2 \cos\varphi_0. \tag{2.9}$$

From (2.8) we obtain

$$\dot{\varphi} = \pm\sqrt{2\left(C + \alpha^2 \cos\varphi\right)}, \tag{2.10}$$

where C is given by (2.9).

The initial condition $\dot{\varphi}_0$ determines the selection of the sign in formula (2.10). If $\dot{\varphi}_0 > 0$, then we select a plus sign, and if $\dot{\varphi}_0 < 0$, then we select a minus sign. If $\dot{\varphi}_0 = 0$, then the choice of the sign in front of the square root should agree with the sign of acceleration $\ddot{\varphi}(0)$.

Following separation of the variables in (2.10) and integration we have

$$t = \pm\int_{\varphi_0}^{\varphi} \frac{\mathrm{d}\varphi}{\sqrt{2\left(C + \alpha^2 \cos\varphi\right)}}. \tag{2.11}$$

Unfortunately, it is not possible to perform the integration of the preceding equation using elementary functions. The preceding integral is called the *elliptic integral*. Equation (2.11) describes the time plot $\varphi(t)$, and the form of the function (the solution) depends on initial conditions as shown subsequently.

Below we will consider two cases of selection of the initial conditions [4].

Case 1. Let us first consider a particular form of the initial condition, namely, let $0 < \varphi_0 < \pi$ and $\dot{\varphi}_0 = 0$. For such an initial condition from (2.9) we obtain

$$C = -\alpha^2 \cos\varphi_0. \tag{2.12}$$

Note that (2.7) indicates that $|C| \le \frac{g}{l}$.

Substituting the constant thus obtained into (2.11) we obtain

$$t = \pm\frac{1}{\alpha}\int_{\varphi_0}^{\varphi}\frac{\mathrm{d}\varphi}{\sqrt{2\left(\cos\varphi - \cos\varphi_0\right)}}. \tag{2.13}$$

The outcome of the actual process determines the sign of the preceding expression, i.e., we have $t \geq 0$. From observations it follows that following introduction of the aforementioned initial condition (or similarly for $-\pi < \varphi_0 < 0$) the angle $\varphi(t)$ decreases, that is, $\cos\varphi_0 > \cos\varphi$; therefore, one should select a minus sign in (2.13). The angle $\varphi(t)$ will be a decreasing function until the second extreme position $\varphi = -\varphi_0$ is attained. From that instant we will be dealing with similar calculations since the initial conditions are determined by the initial angle $-\varphi_0$ and the speed $\dot{\varphi}\left(\frac{T}{2}\right) = 0$, where T is a period of oscillations.

Starting from the aforementioned instant, the angle $\varphi(t)$ will increase from the value $-\varphi_0$ to the value $+\varphi_0$; therefore, in that time interval one should select a plus sign in (2.13).

Note that

$$\begin{aligned}\cos\varphi - \cos\varphi_0 &= 1 - 2\sin^2\frac{\varphi}{2} - \left(1 - 2\sin^2\frac{\varphi_0}{2}\right)\\ &= 2\left(\sin^2\frac{\varphi_0}{2} - \sin^2\frac{\varphi}{2}\right),\end{aligned} \tag{2.14}$$

and hence from (2.13) (minus sign) we obtain

$$t = -\frac{1}{2\alpha}\int_{\varphi_0}^{\varphi}\frac{\mathrm{d}\varphi}{\sqrt{\sin^2\frac{\varphi_0}{2} - \sin^2\frac{\varphi}{2}}} = -\frac{1}{2\alpha}\int_{\varphi_0}^{\varphi}\frac{\mathrm{d}\varphi}{\sin\frac{\varphi_0}{2}\sqrt{1 - \frac{\sin^2\frac{\varphi}{2}}{\sin^2\frac{\varphi_0}{2}}}}. \tag{2.15}$$

For the purpose of further transformations let us introduce a new variable ξ of the form

$$\sin\xi = \frac{\sin\frac{\varphi}{2}}{\sin\frac{\varphi_0}{2}}. \tag{2.16}$$

Differentiating both sides of the preceding equation we obtain

$$\cos\xi\mathrm{d}\xi = \frac{\cos\frac{\varphi}{2}}{2\sin\frac{\varphi_0}{2}}\mathrm{d}\varphi. \tag{2.17}$$

Taking into account that the introduction of the new variable ξ leads also to a change in the limits of integration, i.e., $\xi_0 = \frac{\pi}{2}$ corresponds to φ_0 [see (2.16)], and taking into account relationships (2.16) and (2.17) in (2.15) we obtain

$$t = \frac{1}{2\alpha}\int_{\xi}^{\frac{\pi}{2}}\frac{2\sin\frac{\varphi_0}{2}\cos\xi\mathrm{d}\xi}{\cos\frac{\varphi}{2}\sin\frac{\varphi_0}{2}\sqrt{1 - \sin^2\xi}}$$

$$
\begin{aligned}
&= \frac{1}{\alpha}\int_{\xi}^{\frac{\pi}{2}} \frac{\mathrm{d}\xi}{\sqrt{\frac{(1-\sin^2\xi)\cos^2\frac{\varphi}{2}}{\cos^2\xi}}} \\
&= \frac{1}{\alpha}\int_{\xi}^{\frac{\pi}{2}} \frac{\mathrm{d}\xi}{\sqrt{\frac{\cos^2\frac{\varphi}{2}-(1-\cos^2\xi)\cos^2\frac{\varphi}{2}}{\cos^2\xi}}} = \frac{1}{\alpha}\int_{\xi}^{\frac{\pi}{2}} \frac{\mathrm{d}\xi}{\sqrt{\cos^2\frac{\varphi}{2}}} \\
&= \frac{1}{\alpha}\int_{\xi}^{\frac{\pi}{2}} \frac{\mathrm{d}\xi}{\sqrt{1-\sin^2\frac{\varphi_0}{2}\sin^2\xi}},
\end{aligned} \tag{2.18}
$$

because according to (2.16) we have

$$
\sin^2\frac{\varphi}{2} = \sin^2\xi\sin^2\frac{\varphi_0}{2}. \tag{2.19}
$$

The change in the angle of oscillations from φ_0 to zero corresponds to the time interval $T/4$, which after using (2.18) leads to the determination of the period of oscillations

$$
T = \frac{4}{\alpha}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\xi}{\sqrt{1-\sin^2\frac{\varphi_0}{2}\sin^2\xi}}. \tag{2.20}
$$

Let $x = \sin^2\frac{\varphi_0}{2}\sin^2\xi$; then $|x| < 1$, and the integrand can be expanded in a Maclaurin series about $x = 0$ in the following form:

$$
f(x) = \frac{1}{\sqrt{1-x}} = 1 - \frac{1}{2}x + \cdots = 1 - \frac{1}{2}\sin^2\frac{\varphi_0}{2}\sin^2\xi + \cdots . \tag{2.21}
$$

Taking into account (2.20) and (2.21) we obtain

$$
\begin{aligned}
T &= 4\sqrt{\frac{l}{g}}\left(\int_0^{\frac{\pi}{2}}\mathrm{d}\xi - \frac{1}{2}\sin^2\frac{\varphi_0}{2}\int_0^{\frac{\pi}{2}}\sin^2\xi\mathrm{d}\xi\right) \\
&= 4\sqrt{\frac{l}{g}}\left(\frac{\pi}{2} - \frac{1}{4}\cdot\frac{\pi}{2}\sin^2\frac{\varphi_0}{2}\right) \\
&= 2\pi\sqrt{\frac{l}{g}}\left(1-\frac{1}{4}\sin^2\frac{\varphi_0}{2}\right) \approx 2\pi\sqrt{\frac{l}{g}}\left(1-\frac{1}{16}\varphi_0^2\right),
\end{aligned} \tag{2.22}
$$

where in the last transformation the relationship $\sin\frac{\varphi_0}{2} \approx \frac{\varphi_0}{2}$ was used.

Case 2. Now let us consider the case where apart from the initial amplitude φ_0 the particle (bob of pendulum) was given the speed $\dot{\varphi}_0$ big enough that, according to (2.9), the following inequality is satisfied:

$$C = \frac{1}{2}\dot{\varphi}_0^2 - \frac{g}{l}\cos\varphi_0 > \frac{g}{l}. \tag{2.23}$$

Then the radicand in (2.10) is always positive. This means that the function $\varphi(t)$ is always increasing (plus sign) or decreasing (minus sign). The physical interpretation is such that the pendulum rotates clockwise (plus sign) or counterclockwise (minus sign).

It turns out that for certain special values of initial conditions, namely,

$$C = \frac{1}{2}\dot{\varphi}_0^2 - \frac{g}{l}\cos\varphi_0 = \frac{g}{l}, \tag{2.24}$$

we can perform the integration given by (2.11).

From that equation, and taking into account (2.24) and assuming $\dot{\varphi}_0 > 0$, we obtain

$$\begin{aligned} t &= \sqrt{\frac{l}{g}}\int_{\varphi_0}^{\varphi}\frac{\mathrm{d}\varphi}{\sqrt{2(1+\cos\varphi)}} = \sqrt{\frac{l}{g}}\int_{\varphi_0}^{\varphi}\frac{\mathrm{d}\varphi}{\sqrt{2\cos\frac{\varphi}{2}}} = \\ &= \sqrt{\frac{l}{g}}\ln\left[\frac{\tan\left(\frac{\pi}{4}-\frac{\varphi_0}{4}\right)}{\tan\left(\frac{\pi}{4}-\frac{\varphi}{4}\right)}\right], \end{aligned} \tag{2.25}$$

because

$$\sqrt{2(1+\cos\varphi)} = \sqrt{2\left(1+\cos^2\frac{\varphi}{2}-\sin^2\frac{\varphi}{2}\right)} = 2\cos\frac{\varphi}{2}.$$

In (2.25) we have a singularity since when $\varphi \to \pi$, the time $t \to +\infty$. This means that for the initial condition (2.24) the pendulum attains the vertical position for $t \to +\infty$. The foregoing analysis leads to the diagram presented in Fig. 2.2.

As was already mentioned, (2.1) cannot be solved in an analytical way using elementary functions. However, it will be shown that using the natural coordinates $\boldsymbol{\tau}, \mathbf{n}$ one may determine an exact value of reaction $\mathbf{N}$ (Fig. 2.1).

The Euler equations of motion are as follows:

$$\begin{aligned} m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} &= F_\tau = -mg\sin\varphi, \\ \frac{mv^2}{\rho} &= F_n + N = -mg\cos\varphi + N. \end{aligned} \tag{2.26}$$

Taking into account $s = l\varphi$, the first equation of (2.26) takes the form of (2.2). The second equation of (2.26) yields

$$N = \frac{mv^2}{l} + mg\cos\varphi. \tag{2.27}$$

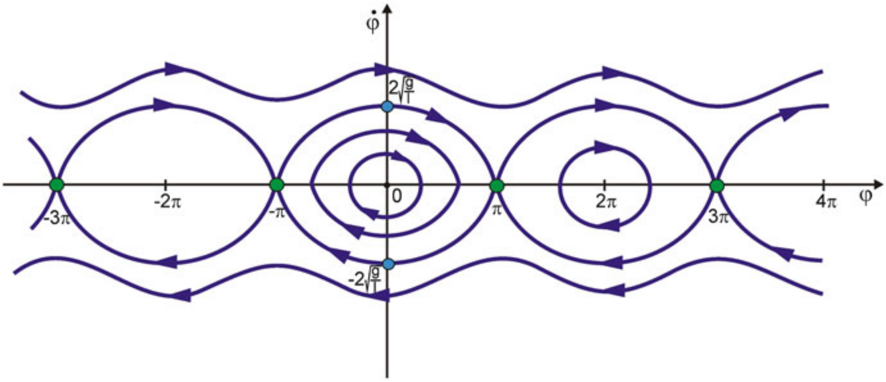


Fig. 2.2 Small oscillations about an equilibrium position, a critical case described by the initial condition (2.24), and pendulum rotations

Observe that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}\varphi} \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{v}{l}\frac{\mathrm{d}v}{\mathrm{d}\varphi}, \tag{2.28}$$

and from (2.2) one obtains

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \alpha^2 \sin\varphi = \frac{\mathrm{d}v}{l\,\mathrm{d}t} + \alpha^2 \sin\varphi = \frac{v}{l^2}\frac{\mathrm{d}v}{\mathrm{d}\varphi} + \alpha^2 \sin\varphi = 0 \tag{2.29}$$

or, equivalently,

$$v\frac{\mathrm{d}v}{\mathrm{d}\varphi} = -gl\sin\varphi. \tag{2.30}$$

Separation of variables and integration give

$$\frac{(v^2 - v_0^2)}{2} = gl(\cos\varphi - \cos\varphi_0) \tag{2.31}$$

or, equivalently,

$$\frac{v^2}{l} = \frac{v_0^2}{l} + 2g(\cos\varphi - \cos\varphi_0). \tag{2.32}$$

Substituting (2.32) into (2.27) yields

$$N = \frac{mv_0^2}{l} + 2mg(\cos\varphi - \cos\varphi_0).$$

A minimum force value can be determined from the equation

$$\frac{\mathrm{d}N}{\mathrm{d}\varphi} = -2mg\sin\varphi = 0, \tag{2.33}$$

which is satisfied for $\varphi = 0, \pm\pi, \pm 2\pi, \ldots$.

Because

$$\frac{\mathrm{d}^2 N}{\mathrm{d}\varphi^2} = -2mg\cos\varphi, \tag{2.34}$$

then

$$\left.\frac{\mathrm{d}^2 N}{\mathrm{d}\varphi^2}\right|_0 = -2mg, \qquad \left.\frac{\mathrm{d}^2 N}{\mathrm{d}\varphi^2}\right|_\pi = 2mg. \tag{2.35}$$

This means that in the lower (upper) pendulum position the thread tension achives its maximum (minimum).

The force minimum value is computed from (2.27):

$$N_{min} = \frac{mv_0^2}{l} + 2mg(-1 - \cos\varphi_0). \tag{2.36}$$

The thread will be stretched when $N_{min} \geq 0$, i.e., for

$$v_0 \geq \sqrt{2gl(1 + \cos\varphi_0)}. \tag{2.37}$$

At the end of this subsection we will study a *pendulum resultant motion*.

Let us assume that an oscillating mathematical pendulum undertakes a flat motion in plane Π, which rotates about a vertical axis crossing the pendulum clamping point (Fig. 2.3).

The equation of a relative pendulum motion expressed through the natural coordinates $\boldsymbol{\tau}, \mathbf{n}$ has the following form (projections of forces onto the tangent direction):

$$m(p_\tau^w + p_\tau^u + p_\tau^C) = -mg\sin\varphi, \tag{2.38}$$

and since the Coriolis acceleration $\mathbf{p}_\tau^C \perp \Pi$, then $p_\tau^C = 0$.

Projection of the translation acceleration onto a tangent to the particle trajectory is $p_\tau^u = (\omega^2 l \sin\varphi)\cos\varphi$, and finally (2.38) takes the form

$$ml\ddot{\varphi} = -mg\sin\varphi + m\omega^2 l \sin\varphi\cos\varphi \tag{2.39}$$

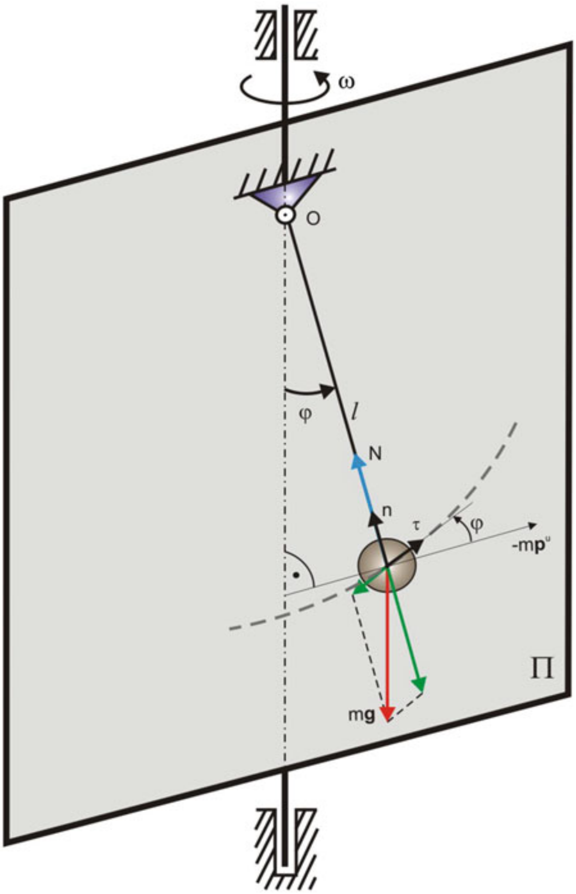
or

$$\ddot{\varphi} = (\omega^2\cos\varphi - \alpha^2)\sin\varphi. \tag{2.40}$$

Observe that now we have two sets of equilibrium positions yielded by the equation

$$(\omega^2\cos\varphi - \alpha^2)\sin\varphi = 0. \tag{2.41}$$

Fig. 2.3 Resultant mathematical pendulum motion oscillating in plane Π rotating with angular velocity ω



Besides the previously discussed set governed by the equation $\sin\varphi = 0$, i.e., $\varphi_n = n\pi, \quad n \in C$, the additional set of equilibrium positions is given by the following formula:

$$\varphi_n = \pm \arccos \frac{\alpha^2}{\omega^2} + 2\pi n, \quad n \in C \tag{2.42}$$

if $\omega \geq \alpha$.

Strongly non-linear equations of motion of the pendulum (2.2) can also be solved in an exact way. Following the introduction of non-dimesional time $\tau = \alpha t$ we obtain

$$\varphi'' + \sin\varphi = 0, \tag{2.43}$$

and multiplying by sides through φ' we have

$$\varphi'\varphi'' \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\varphi'^2}{2}\right) = -\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\sin\varphi \tag{2.44}$$

or

$$\mathrm{d}\left(\frac{\varphi'^2}{2}\right) = -\sin\varphi\,\mathrm{d}\varphi. \tag{2.45}$$

Integration of (2.45) yields

$$\frac{\varphi'^2}{2} = -[(-\cos\varphi) + \cos\varphi_0], \tag{2.46}$$

that is,

$$\varphi' = \pm\sqrt{2}\sqrt{|\cos\varphi - \cos\varphi_0|}, \tag{2.47}$$

where $\cos\varphi_0$ is a constant of integration.

Because

$$\cos\varphi = 1 - 2\sin^2\left(\frac{\varphi}{2}\right), \tag{2.48}$$

(2.47) takes the form

$$\varphi' = -\sqrt{2}\sqrt{2\left|\sin^2\left(\frac{\varphi_0}{2}\right) - \sin^2\left(\frac{\varphi}{2}\right)\right|}, \tag{2.49}$$

and following separation of the variables and integration we have

$$-\int_{\varphi_0}^{\varphi} \frac{\mathrm{d}\left(\frac{\varphi}{2}\right)}{\sqrt{\left|\sin^2\left(\frac{\varphi_0}{2}\right) - \sin^2\left(\frac{\varphi}{2}\right)\right|}} = \int_0^t \mathrm{d}\tau. \tag{2.50}$$

The obtained integral cannot be expressed in terms of elementary functions and is called an elliptic integral because it also appears during calculation of the length of an elliptical curve.

For the purpose of its calculation we introduce two parameters – k, called the *elliptic modulus*, and u, called the *amplitude* – according to the following equations:

$$\begin{aligned}
\sin\left(\frac{\varphi}{2}\right) &= \sin\left(\frac{\varphi_0}{2}\right)\sin\theta = k\sin\theta, \qquad k = \sin\left(\frac{\varphi_0}{2}\right),\\
\cos\left(\frac{\varphi}{2}\right) &= \sqrt{1 - \sin^2\left(\frac{\varphi}{2}\right)} = \sqrt{1 - k^2\sin^2\theta},\\
\mathrm{d}\left(\sin\left(\frac{\varphi}{2}\right)\right) &= \cos\left(\frac{\varphi}{2}\right)\mathrm{d}\left(\frac{\varphi}{2}\right) = k\mathrm{d}(\sin\theta) = k\cos\theta\,\mathrm{d}\theta,\\
\mathrm{d}\left(\frac{\varphi}{2}\right) &= \frac{k\cos\theta\,\mathrm{d}\theta}{\cos\left(\frac{\varphi}{2}\right)} = \frac{k\cos\theta\,\mathrm{d}\theta}{\sqrt{1 - k^2\sin^2\theta}}.
\end{aligned} \tag{2.51}$$

From the first equation of (2.51) we have

$$\sin^2\left(\frac{\varphi_0}{2}\right) = k^2, \tag{2.52}$$

and the integral from (2.50) takes the form

$$\int_{\varphi(t)}^{\varphi_0} \frac{k\cos\theta \mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}\sqrt{k^2-k^2\sin^2\theta}} = \int_{\varphi(t)}^{\varphi_0} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}. \tag{2.53}$$

In the foregoing integral one should additionally alter the limits of integration. We have here a conservative system; let oscillations of the system be characterized by period T (we consider the case of a lack of rotation of the pendulum). After time $T/2$ starting from the initial condition φ_0 according to the first equation of (2.51) we have $\sin\theta = -1$, which implies $\theta(T/2) = 3\pi/2$. In turn, at the instant the motion began according to that equation we have $\sin\theta = 1$, that is, $\theta(0) = \pi/2$. Finally, (2.50) takes the form

$$\int_{\pi/2}^{3\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} = 2\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{T}{2}. \tag{2.54}$$

The desired period of the pendulum oscillations is equal to

$$T = 4\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}. \tag{2.55}$$

The integral

$$F(\theta^*, k) = \int_0^{\theta^*} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{2.56}$$

is called an *elliptic incomplete integral of the first kind*. Introducing variable

$$z = \sin\theta \tag{2.57}$$

we can represent integral (2.56) in the following equivalent form:

$$\int_0^{\theta^*} = \frac{\mathrm{d}z}{\cos\theta\sqrt{1-k^2z^2}} \int_0^{\sin\theta^*} \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{2.58}$$

A graph of dependency $T(k)$ on the basis of (2.55) is presented in Fig. 2.4.

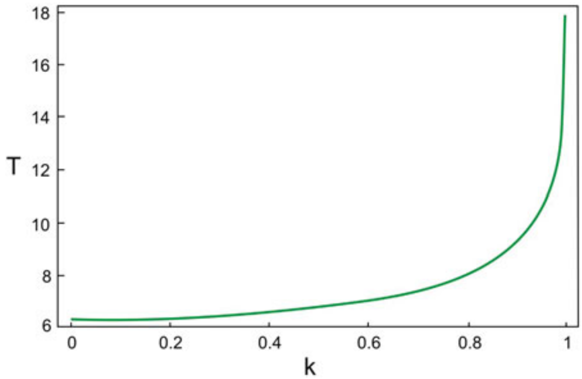


Fig. 2.4 Graph of dependency of period of pendulum oscillations on initial deflection $k = \sin(\varphi_0/2)$

2.2 The Physical Pendulum

In this section we will consider the plane motion of a material body suspended on the horizontal axis and allowed to rotate about it. It is the physical pendulum depicted in Fig. 2.5.

The pendulum is hung at point O, and the straight line passing through that point and the pendulum mass center C defines the axis OX_1. We assume that the angle of rotation of pendulum φ is positive and the sense of the axis perpendicular to the OX_1 axis is taken in such a way that the Cartesian coordinate system $OX_1X_2X_3$ is a right-handed one (the X_3 axis is perpendicular to the plane of the drawing).

Neglecting the resistance to motion in a radial bearing O and the resistance of the medium, the only force producing the motion is the component of gravity force tangent to a circle of radius s. The equation of moments of force about point O has the form

$$I_O\ddot{\varphi} = -msg\sin\varphi, \tag{2.59}$$

where I_O is the moment of inertia of the physical pendulum with respect to pivot point O, i.e., the axis X_3.

By analogy to the equation of motion of a mathematical pendulum (2.2), we will reduce (2.59) to the form

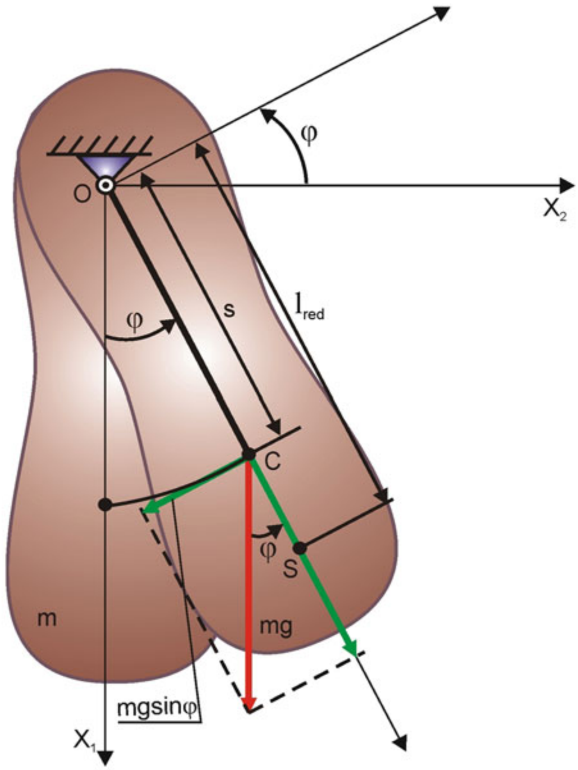
$$\ddot{\varphi} + \alpha_f^2\sin\varphi = 0, \tag{2.60}$$

where

$$\alpha_f^2 = \frac{mgs}{I_O} = \frac{g}{l_{\text{red}}}. \tag{2.61}$$

From (2.61) it follows that the introduced *reduced length of a physical pendulum* equals $l_{\text{red}} = \frac{I_C}{ms}$. All the calculations conducted in Sect. 2.1 hold also in this case

Fig. 2.5 Physical pendulum of mass m



owing to the same mathematical model. In particular, the period of small oscillations of the physical pendulum about the equilibrium position $\varphi = 0$ equals

$$T = \frac{2\pi}{\alpha_f} = 2\pi\sqrt{\frac{I_O}{mgs}} = 2\pi\sqrt{\frac{l_{\text{red}}}{g}}. \tag{2.62}$$

Let the moment of inertia with respect to the axis parallel to X_3 and passing through the mass center of the physical pendulum be I_C. According to the parallel axis theorem the moment of inertia I_O reads

$$I_O = I_C + ms^2 = m\left(\frac{I_O}{m} + s^2\right) = m\left(i_C^2 + s^2\right), \tag{2.63}$$

where i_C is a radius of gyration with respect to the axis passing through the mass center of the pendulum.

According to the previous calculations the reduced length of the physical pendulum is equal to

$$l_{\text{red}} = \frac{I_O}{ms} = s + \frac{i_C^2}{s}. \tag{2.64}$$

From the preceding equation it follows that the reduced length of a physical pendulum is a function of s. If $s \to 0$, then $l_{\mathrm{red}} \to \infty$, whereas if $s \to \infty$, then $l_{\mathrm{red}} \to \infty$. According to (2.62) the period of small oscillations of the pendulum $T \to \infty$ for $s \to 0$ and $T \to 0$ for $s \to \infty$. Our aim will be to determine such a value of s, i.e., the distance of the point of rotation of the pendulum from its mass center, for which the period is minimal. According to (2.64)

$$\frac{\mathrm{d}l_{\mathrm{red}}}{\mathrm{d}s} = 1 - \frac{i_C^2}{s^2} = 0, \tag{2.65}$$

$$\frac{\mathrm{d}^2 l_{\mathrm{red}}}{\mathrm{d}s^2} = \frac{2}{s^3} i_C^2 > 0. \tag{2.66}$$

From the two foregoing equations it follows that the function $l_{\mathrm{red}}(s)$ attains the minimum for the value $s = i_C$ because its second derivative for such s is positive.

In Fig. 2.5 point S is marked at the distance l_{red} from the axis of rotation. We will call that point the *center of swing corresponding to the pivot point O*. In other words, if we concentrate the total mass of the physical pendulum at point S, then we will obtain a mathematical pendulum of length l_{red}.

Let us suspend the pendulum at point S obtained in that way and determine the corresponding center of swing S^*. The length reduced to point S, according to (2.64), is equal to

$$l_{\mathrm{red}}^* = SS^* = CS + \frac{i_C^2}{CS}. \tag{2.67}$$

Because

$$CS = OS - OC = l_{\mathrm{red}} - s = \frac{i_C^2}{s}, \tag{2.68}$$

from (2.67) we have

$$l_{\mathrm{red}}^* = l_{\mathrm{red}} - s + s = l_{\mathrm{red}}. \tag{2.69}$$

From the foregoing calculations it follows that the pivot point of the pendulum and the corresponding center of swing play an identical role with respect to one another.

In Sect. 2.1 we mentioned the *first integral of motion*. Now, as distinct from that approach, we will determine the relationships between the velocities and the displacement of the pendulum based on the theorem of the conservation of mechanical energy. Let the mechanical energy of a physical pendulum be given by

$$T(t) + V(t) = C + mgs, \tag{2.70}$$

where C is a certain constant, i.e., the stored energy of the pendulum introduced by the initial conditions, and mgs denotes the potential energy of the system in the equilibrium position $\varphi = 0$.

Because we are dealing with a conservative system, the sum of kinetic energy T and potential energy V does not change in time and is constant for every time instant t in the considered case (Fig. 2.5):

$$T = \frac{1}{2} I_O \dot{\varphi}^2, \tag{2.71}$$

$$V = mgs(1 - \cos\varphi), \tag{2.72}$$

where V is the potential energy of the pendulum deflected through the angle φ. Substituting (2.71) and (2.72) into (2.70) we have

$$\frac{I_O \dot{\varphi}^2}{2} - mgs\cos\varphi = C, \tag{2.73}$$

and hence

$$\dot{\varphi}^2 = \frac{2}{I_O}(C + mgs\cos\varphi). \tag{2.74}$$

Let us note that (2.74) is analogous to the previously obtained equation for a mathematical pendulum (2.8) since we have

$$\dot{\varphi}^2 = 2C_f + 2\alpha_f^2 \cos\varphi, \tag{2.75}$$

where $C_f = C/I_O$ and α_f^2 is defined by (2.61).

If the initial conditions of the pendulum have the form $\varphi(0) = \varphi_0$, $\dot{\varphi}(0) = 0$, then from (2.73) we obtain

$$C = -mgs\cos\varphi_0. \tag{2.76}$$

This means that the initial energy is associated only with the potential energy, and taking into account (2.76) in (2.74) we have

$$\dot{\varphi}^2 = \frac{2mgs}{I_O}(\cos\varphi - \cos\varphi_0). \tag{2.77}$$

The last equation corresponds to (2.8).

2.3 Planar Dynamics of a Triple Physical Pendulum

2.3.1 Equations of Motion

Our goal is to introduce a mathematical model of a 2D triple physical pendulum. The *mathematical model* [3] is a description of the system dynamics with the aid of equations, in this case, ordinary differential equations. The mathematical model is a mathematical expression of physical laws valid in the considered system. In order to proceed with writing the equations, we must first have a physical model understood as a certain conception of physical phenomena present in the system. One should remember that the physical model to be presented below does not exist in reality but is an idea of a triple physical pendulum. If there existed a real object, which we would also call a *triple physical pendulum*, it would be able to correspond to our physical model approximately at best. By saying here that the real system and the model correspond to each other approximately, we mean that all physical phenomena taken into account in the model occur in the real object. Additionally, the influence of the physical phenomena occurring in the real system and not taken into account in the model (they can be treated as disturbances) on the observed (measured) quantities that are of interest to us is negligible. Those observed quantities in the triple pendulum can be, for instance, three angles describing the position of the pendulum at every time instant. Clearly, there exists a possibility of further development of this model so that it incorporates increasingly more physical phenomena occurring in a certain existing real system and, consequently, becomes closer to that real system. An absolute agreement, however, will not be attainable. On the other hand, one may also think about the opposite situation, in which to the theoretically created idea of a pendulum we try to match a real object, that is, the test stand. Then we build it in such a way that in the stand only the laws and physical phenomena assumed in the model are, to the best possible approximation, valid. The influence of other real phenomena on the quantities of interest should be negligible.

Our physical model of a triple pendulum (Fig. 2.6) consists of three ($i = 1, 2, 3$) absolutely rigid bodies moving in a vacuum in a uniform gravitational field of lines that are parallel and directed against the axis X_2 of the global coordinate system $O_1X_1X_2X_3$, connected to each other by means of revolute joints O_i and connected to an absolutely rigid base [5]. Those joints have axes perpendicular to the plane $O_1X_1X_2$ so that the whole system moves in planar motion. We assume that in the joints there exists viscous damping, that is, that the resistive moment counteracting the relative motion of two pendulums connected to each other is proportional (with a certain proportionality factor c_i) to their relative angular velocity. We also assume that mass centers of particular pendulums (C_i) lie in planes determined by the axes of joints by which the given pendulum is connected to the rest of the system (this does not apply to the third pendulum). This last assumption allows for a decrease in the number of model parameters – to be precise, the number of parameters establishing the positions of mass centers of particular pendulums. Each of the pendulums has its own local coordinate system $C_iX_1^{(i)}X_2^{(i)}X_3^{(i)}$ ($i = 1, 2, 3$) of the origin at the mass center of the given pendulum and the axis $X_3^{(i)}$ perpendicular to the plane of motion. Geometrical parameters that determine the positions of mass centers (e_i) and the distances between the joints (l_1, l_2) are indicated in the figure. Moreover, each of the pendulums possesses mass m_i and mass moment of inertia I_i with respect to the axis C_iX_3 passing through the mass center (centroidal axis) and perpendicular to the plane of motion. The first pendulum is acted upon by an external moment $M_e(t)$. The configuration of the system is uniquely described by three angles ψ_i, as shown in Fig. 2.6.

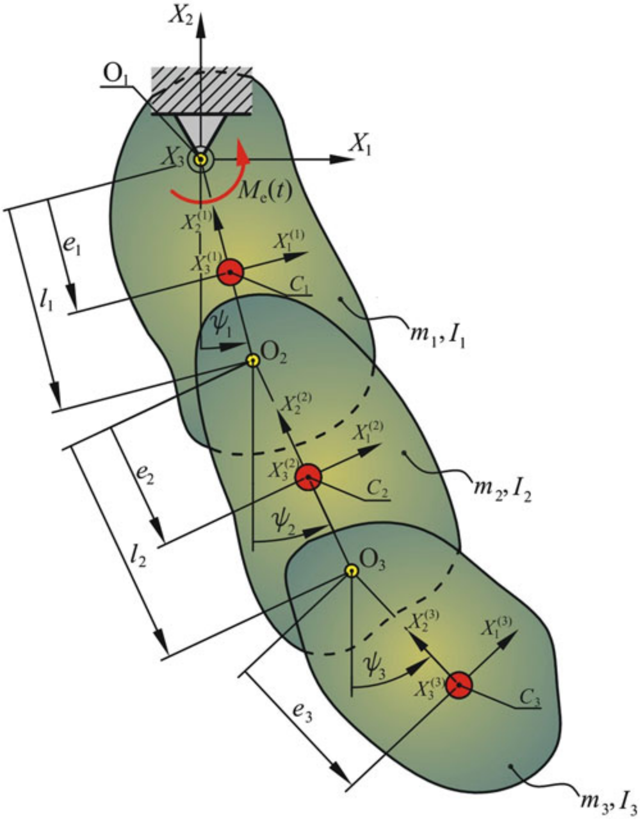


Fig. 2.6 A triple physical pendulum

For the derivation of equations of motion we will make use of Lagrange's equations of the second kind having the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_n}\right) - \frac{\partial T}{\partial q_n} + \frac{\partial V}{\partial q_n} = Q_n, \quad n = 1, \ldots, N, \tag{2.78}$$

where N is the number of generalized coordinates, q_n the nth generalized coordinate, T the kinetic energy of the system, V the potential energy, and Q_n the nth generalized force. In the case of the considered triple pendulum one may choose three angles ψ_1, ψ_2, and ψ_3 as generalized coordinates (describing uniquely the system configuration). Then, (2.78) take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\psi}_n}\right) - \frac{\partial T}{\partial \psi_n} + \frac{\partial V}{\partial \psi_n} = Q_n. \tag{2.79}$$

Gravity forces can be included in two ways. Firstly, one may put them into equations as appropriate components of the generalized forces Q_n. Secondly, after taking into account that the gravity forces are conservative, one can put them into equations as the appropriate potential energy. The latter method is more convenient and will be applied here. The potential energy (of gravity forces) is as follows:

$$V = V_0 + \sum_{n=1}^{3} m_n g x_{2C_n}, \tag{2.80}$$

where V_0 is an arbitrary constant, g the acceleration of gravity, and x_{2Cn} the coordinate determining the position along the X_2 axis of the mass center of the nth pendulum.

The kinetic energy of the system is the sum of the kinetic energies of each of the bodies. In turn, the kinetic energy of a body is the sum of its kinetic energy for the translational motion (with velocity of the mass center) and for the motion about the mass center. That relative motion, generally, is the motion about a point (that is, the instantaneous rotational motion), whereas in our special case of planar motion it is the rotational motion of the nth pendulum (of the axis of rotation $C_i X_3^{(n)}$). Thus, the kinetic energy of the system of three connected pendulums is equal to

$$T = \frac{1}{2}\sum_{n=1}^{3} m_n \left(\dot{x}_{1C_n}^2 + \dot{x}_{2C_n}^2\right) + \frac{1}{2}\sum_{n=1}^{3} I_n \dot{\psi}_n^2. \tag{2.81}$$

The coordinates of mass centers occurring in expressions (2.80) and (2.81) are equal to

$$\begin{aligned}
x_{1C_1} &= e_1 \sin\psi_1,\\
x_{1C_2} &= l_1 \sin\psi_1 + e_2 \sin\psi_2,\\
x_{1C_3} &= l_1 \sin\psi_1 + l_2 \sin\psi_2 + e_3 \sin\psi_3,\\
x_{2C_1} &= -e_1 \cos\psi_1,\\
x_{2C_2} &= -l_1 \cos\psi_1 - e_2 \cos\psi_2,\\
x_{2C_3} &= -l_1 \cos\psi_1 - l_2 \cos\psi_2 - e_3 \cos\psi_3,
\end{aligned} \tag{2.82}$$

whereas their time derivatives

$$\begin{aligned}
\dot{x}_{1C_1} &= e_1 \dot{\psi}_1 \cos\psi_1,\\
\dot{x}_{1C_2} &= l_1\, \dot{\psi}_1 \cos\psi_1 + e_2\, \dot{\psi}_2 \cos\psi_2,\\
\dot{x}_{1C_3} &= l_1\, \dot{\psi}_1 \cos\psi_1 + l_2\, \dot{\psi}_2 \cos\psi_2 + e_3\, \dot{\psi}_3 \cos\psi_3,
\end{aligned}$$

$$\dot{x}_{2C_1} = e_1\,\dot{\psi}_1 \sin\psi_1,$$
$$\dot{x}_{2C_2} = l_1\,\dot{\psi}_1 \sin\psi_1 + e_2\,\dot{\psi}_2 \sin\psi_2,$$
$$\dot{x}_{2C_3} = l_1\,\dot{\psi}_1 \sin\psi_1 + l_2\,\dot{\psi}_2 \sin\psi_2 + e_3\,\dot{\psi}_3 \sin\psi_3. \tag{2.83}$$

Inserting relationships (2.82) and (2.83) into expressions (2.80) and (2.81), applying suitable operations, using certain trigonometric identities, and grouping the terms, we obtain

$$V = -\sum_{n=1}^{3} M_n \cos\psi_n \tag{2.84}$$

and

$$T = \frac{1}{2}\sum_{n=1}^{3} B_n \dot{\psi}_n^2 + \sum_{n=1}^{2}\sum_{j=n+1}^{3} N_{nj}\,\dot{\psi}_n \dot{\psi}_j \cos\left(\psi_n - \psi_j\right), \tag{2.85}$$

where the following symbols were used:

$$M_1 = m_1 g e_1 + (m_2 + m_3)\, g l_1,$$
$$M_2 = m_2 g e_2 + m_3 g l_2$$
$$M_3 = m_3 g e_3,$$
$$B_1 = I_1 + {e_1}^2 m_1 + {l_1}^2 (m_2 + m_3)\,,$$
$$B_2 = I_2 + {e_2}^2 m_2 + {l_2}^2 m_3,$$
$$B_3 = I_3 + {e_3}^2 m_3,$$
$$N_{12} = m_2 e_2 l_1 + m_3 l_1 l_2,$$
$$N_{13} = m_3 e_3 l_1,$$
$$N_{23} = m_3 e_3 l_2. \tag{2.86}$$

Inserting relations (2.84) and (2.85) into the left-hand sides of (2.79) and differentiating, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\psi}_1}\right) - \frac{\partial T}{\partial \psi_1} + \frac{\partial V}{\partial \psi_1} &= B_1 \ddot{\psi}_1 + N_{12}\cos(\psi_1 - \psi_2)\,\ddot{\psi}_2 \\ &+ N_{13}\cos(\psi_1 - \psi_3)\,\ddot{\psi}_3 \\ &+ N_{12}\sin(\psi_1 - \psi_2)\,\dot{\psi}_2^2 + N_{13}\sin(\psi_1 - \psi_3)\,\dot{\psi}_3^2 \\ &+ M_1 \sin\psi_1, \end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\psi}_2}\right) - \frac{\partial T}{\partial \psi_2} + \frac{\partial V}{\partial \psi_2} &= B_2\ddot{\psi}_2 + N_{12}\cos\left(\psi_1 - \psi_2\right)\ddot{\psi}_1 \\
&+ N_{23}\cos\left(\psi_2 - \psi_3\right)\ddot{\psi}_3 \\
&- N_{12}\sin\left(\psi_1 - \psi_2\right)\dot{\psi}_1^2 + N_{23}\sin\left(\psi_2 - \psi_3\right)\dot{\psi}_3^2 \\
&+ M_2\sin\psi_2, \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\psi}_3}\right) - \frac{\partial T}{\partial \psi_3} + \frac{\partial V}{\partial \psi_3} &= B_3\ddot{\psi}_3 + N_{13}\cos\left(\psi_1 - \psi_3\right)\ddot{\psi}_1 \\
&+ N_{23}\cos\left(\psi_2 - \psi_3\right)\ddot{\psi}_2 \\
&- N_{13}\sin\left(\psi_1 - \psi_3\right)\dot{\psi}_1^2 - N_{23}\sin\left(\psi_2 - \psi_3\right)\dot{\psi}_2^2 \\
&+ M_3\sin\psi_3. \qquad (2.87)
\end{aligned}
$$

Now we should determine the right-hand sides, that is, what kinds of generalized forces act along particular generalized coordinates. When using generalized forces Q_n one should take into account in the equations all non-conservative forces acting on the system. This will be the moment of force $M_e(t)$ acting on the first pendulum but also moments of resistive forces at the connections of the pendulums. Because the generalized coordinates are angles, the generalized forces must be moments of force. Moreover, generalized coordinates describe the absolute angular positions of individual pendulums; therefore, the generalized forces will be the moments of force acting on particular pendulums. Therefore, it is already known that the moment $M_e(t)$ will be the component of the first generalized force. Also, on particular pendulums additionally act the moments associated with viscous damping at particular revolute joints. We may write it in the following way:

$$
\begin{aligned}
Q_1 &= M_e(t) + M_{01} + M_{21}, \\
Q_2 &= M_{12} + M_{32}, \\
Q_3 &= M_{23}, \qquad (2.88)
\end{aligned}
$$

where M_{ij} is the moment of force with which the ith pendulum or the base ($i = 0$) acts on the jth pendulum by means of viscous damping at a joint connecting two pendulums. Positive directions of particular moments are consistent with the adopted positive direction common for all generalized coordinates. Clearly, according to Newton's third law (action and reaction principle), $M_{ij} = -M_{ji}$ must hold. For viscous damping (proportional to the relative velocity) we have

$$
\begin{aligned}
M_{01} &= -c_1\dot{\psi}_1, \\
M_{12} &= -c_2\left(\dot{\psi}_2 - \dot{\psi}_1\right) = -M_{21}, \\
M_{23} &= -c_3\left(\dot{\psi}_3 - \dot{\psi}_2\right) = -M_{32}. \qquad (2.89)
\end{aligned}
$$

While taking the appropriate signs in the preceding formulas we keep in mind that the moment of damping has to counteract the relative motion of the connected pendulums. Finally, the generalized forces read

$$
\begin{aligned}
Q_1 &= M_e(t) - c_1\dot{\psi}_1 + c_2\left(\dot{\psi}_2 - \dot{\psi}_1\right),\\
Q_2 &= -c_2\left(\dot{\psi}_2 - \dot{\psi}_1\right) + c_3\left(\dot{\psi}_3 - \dot{\psi}_2\right),\\
Q_3 &= -c_3\left(\dot{\psi}_3 - \dot{\psi}_2\right),
\end{aligned}
\tag{2.90}
$$

where c_i is the viscous damping coefficient at joint O_i.

Equating formulas (2.90) to (2.87) and moving also the damping force to the left-hand side of every equation we obtain a system of three second-order ordinary differential equations (the mathematical model) describing the dynamics of the triple pendulum:

$$
\begin{aligned}
&B_1\ddot{\psi}_1 + N_{12}\cos(\psi_1-\psi_2)\,\ddot{\psi}_2 + N_{13}\cos(\psi_1-\psi_3)\,\ddot{\psi}_3\\
&\quad + N_{12}\sin(\psi_1-\psi_2)\,\dot{\psi}_2^2 + N_{13}\sin(\psi_1-\psi_3)\,\dot{\psi}_3^2\\
&\quad + c_1\dot{\psi}_1 - c_2\left(\dot{\psi}_2-\dot{\psi}_1\right) + M_1\sin\psi_1 = M_e(t),\\
&B_2\ddot{\psi}_2 + N_{12}\cos(\psi_1-\psi_2)\,\ddot{\psi}_1 + N_{23}\cos(\psi_2-\psi_3)\,\ddot{\psi}_3\\
&\quad - N_{12}\sin(\psi_1-\psi_2)\,\dot{\psi}_1^2 + N_{23}\sin(\psi_2-\psi_3)\,\dot{\psi}_3^2\\
&\quad + c_2\left(\dot{\psi}_2-\dot{\psi}_1\right) - c_3\left(\dot{\psi}_3-\dot{\psi}_2\right) + M_2\sin\psi_2 = 0,\\
&B_3\ddot{\psi}_3 + N_{13}\cos(\psi_1-\psi_3)\,\ddot{\psi}_1 + N_{23}\cos(\psi_2-\psi_3)\,\ddot{\psi}_2\\
&\quad - N_{13}\sin(\psi_1-\psi_3)\,\dot{\psi}_1^2 - N_{23}\sin(\psi_2-\psi_3)\,\dot{\psi}_2^2\\
&\quad + c_3\left(\dot{\psi}_3-\dot{\psi}_2\right) + M_3\sin\psi_3 = 0.
\end{aligned}
\tag{2.91}
$$

Equation (2.91) can also be represented in a more concise and clear form using matrix notation

$$
\mathbf{M}(\boldsymbol{\psi})\,\ddot{\boldsymbol{\psi}} + \mathbf{N}(\boldsymbol{\psi})\,\dot{\boldsymbol{\psi}}^2 + \mathbf{C}\dot{\boldsymbol{\psi}} + \mathbf{p}(\boldsymbol{\psi}) = \mathbf{f}_e(t),
\tag{2.92}
$$

where

$$
\mathbf{M}(\boldsymbol{\psi}) = \begin{bmatrix} B_1 & N_{12}\cos(\psi_1-\psi_2) & N_{13}\cos(\psi_1-\psi_3)\\ N_{12}\cos(\psi_1-\psi_2) & B_2 & N_{23}\cos(\psi_2-\psi_3)\\ N_{13}\cos(\psi_1-\psi_3) & N_{23}\cos(\psi_2-\psi_3) & B_3 \end{bmatrix},
$$

$$
\mathbf{N}(\boldsymbol{\psi}) = \begin{bmatrix} 0 & N_{12}\sin(\psi_1-\psi_2) & N_{13}\sin(\psi_1-\psi_3)\\ -N_{12}\sin(\psi_1-\psi_2) & 0 & N_{23}\sin(\psi_2-\psi_3)\\ -N_{13}\sin(\psi_1-\psi_3) & -N_{23}\sin(\psi_2-\psi_3) & 0 \end{bmatrix},
$$

$$\mathbf{C} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}, \quad \mathbf{p}(\boldsymbol{\psi}) = \begin{bmatrix} M_1 \sin \psi_1 \\ M_2 \sin \psi_2 \\ M_3 \sin \psi_3 \end{bmatrix}, \quad \mathbf{f}_e(t) = \begin{bmatrix} M_e(t) \\ 0 \\ 0 \end{bmatrix},$$

$$\boldsymbol{\psi} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix}, \quad \dot{\boldsymbol{\psi}} = \begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \end{bmatrix}, \quad \ddot{\boldsymbol{\psi}} = \begin{bmatrix} \ddot{\psi}_1 \\ \ddot{\psi}_2 \\ \ddot{\psi}_3 \end{bmatrix}, \quad \dot{\boldsymbol{\psi}}^2 = \begin{bmatrix} \dot{\psi}_1^2 \\ \dot{\psi}_2^2 \\ \dot{\psi}_3^2 \end{bmatrix}. \tag{2.93}$$

The unknowns of a system of differential equations describing the dynamics of a triple pendulum [in the form of (2.91) or (2.92)] are the functions $\psi_1(t)$, $\psi_2(t)$, and $\psi_3(t)$, which means that the solution of those equations describes the motion of the pendulum.

Let us also draw attention to the parameters of those equations. While there are 15 of the physical parameters of the pendulum (l_1, l_2, e_1, e_2, e_3, m_1, m_2, m_3, I_1, I_2, I_3, g, c_1, c_2, and c_3 – for now we omit parameters of excitation), in the equations there are actually 11 independent parameters (M_1, M_2, M_3, B_1, B_2, B_3, N_{13}, N_{23}, c_1, c_2, and c_3), because out of quantities (2.86) one is dependent on the remaining ones:

$$\frac{N_{12}}{N_{13}} = \frac{M_2}{M_3} = \frac{m_2 e_2}{m_3 e_3} + \frac{l_2}{e_3}, \tag{2.94}$$

and hence

$$N_{12} = N_{13} \frac{M_2}{M_3}. \tag{2.95}$$

From the fact that fewer parameters (11) occur in equations of motion than the number of physical parameters of a pendulum (15) it follows that the same pendulum in the sense of dynamics (i.e., behaving in the same way) one may build in an infinite number of ways.

2.3.2 Numerical Simulations

Differential equations (2.92) are strongly non-linear equations and do not have an exact analytical solution. For the investigation of their solutions, numerical and analytical approximate methods remain at our disposal. Here one of the most popular and effective numerical methods for the solution of ordinary differential equations – the fourth-order Runge–Kutta method [6] – was applied.

However, the mathematical model of a triple pendulum (2.91) or (2.92) has the form of a system of second-order equations. In order to be able to apply classical algorithms for integration of differential equations, we have to reduce these equations to the form of a system of first-order equations. Let us take a system state vector (the vector of state variables) as

$$\mathbf{x} = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\psi} \\ \dot{\boldsymbol{\psi}} \end{bmatrix}, \tag{2.96}$$

then the system of six first-order differential equations has the following general form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) = \begin{bmatrix} \dot{\boldsymbol{\psi}} \\ \ddot{\boldsymbol{\psi}} \end{bmatrix}. \tag{2.97}$$

Let us note that we need angular accelerations of the pendulums given explicitly as functions of system state $\mathbf{x}$ and time t. Then we have to solve (2.92) with respect to $\ddot{\psi}$, treating them as algebraic equations. Because it is a system of linear equations, we have

$$\ddot{\boldsymbol{\psi}} = [\mathbf{M}(\boldsymbol{\psi})]^{-1} \left[\mathbf{f}_e(t) - \mathbf{N}(\boldsymbol{\psi})\, \dot{\boldsymbol{\psi}}^2 - \mathbf{C}\dot{\boldsymbol{\psi}} - \mathbf{p}(\boldsymbol{\psi}) \right]. \tag{2.98}$$

Inserting relation (2.98) into (2.97), we eventually obtain

$$\dot{\mathbf{x}} = \left. \begin{bmatrix} \dot{\boldsymbol{\psi}} \\ [\mathbf{M}(\boldsymbol{\psi})]^{-1} \left[\mathbf{f}_e(t) - \mathbf{N}(\boldsymbol{\psi})\, \dot{\boldsymbol{\psi}}^2 - \mathbf{C}\dot{\boldsymbol{\psi}} - \mathbf{p}(\boldsymbol{\psi}) \right] \end{bmatrix} \right|_{\boldsymbol{\psi} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\, \dot{\boldsymbol{\psi}} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix},\, \dot{\boldsymbol{\psi}}^2 = \begin{bmatrix} x_4^2 \\ x_5^2 \\ x_6^2 \end{bmatrix}}. \tag{2.99}$$

In every step of the integration we must perform an inversion of the matrix $\mathbf{M}(\boldsymbol{\psi})$. Because of its size, it is possible to use for this purpose an exact analytical expression. In the case of a slightly larger system, in practice there would remain only the possibility of using one of the existing numerical methods for inverting a matrix. If we investigate (integrate) differential equations (2.92) by means of a numerical method, in fact not only will the mathematical model consist of these equations, but also the method itself should be considered as an integral part of the model. Then a system with continuous time is approximated by a system with discrete time, and the differential equations themselves are approximated by difference equations.

In order to conduct an illustrative numerical simulation of a pendulum (i.e., to find the numerical solution of the model), we have to adopt some concrete values for model parameters and initial conditions. Let us assume that this special case of

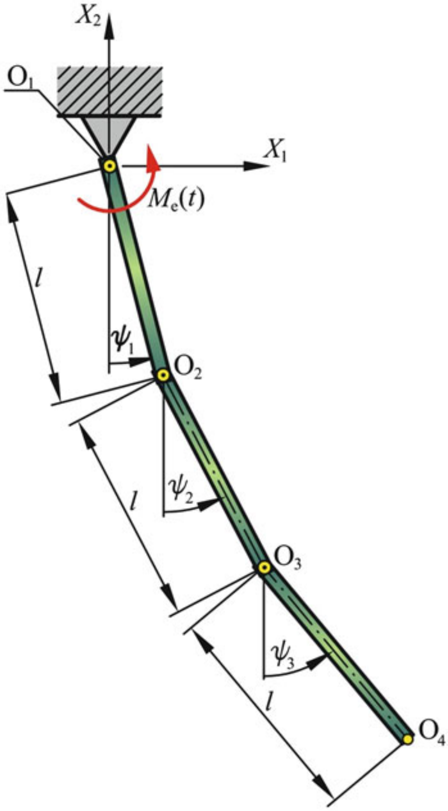


Fig. 2.7 Special case of a triple physical pendulum – a system of three identical pin-joined rods

a triple physical pendulum are three identical rods connected by means of joints located at their ends, as shown in Fig. 2.7.

Then we will have

$$
\begin{aligned}
l_1 &= l_2 = l,\\
m_1 &= m_2 = m_3 = m,\\
e_1 &= e_2 = e_3 = \frac{l}{2},\\
I_1 &= I_2 = I_3 = \frac{ml^2}{12},
\end{aligned}
\tag{2.100}
$$

where l is the length of a single pendulum and m its mass, and expressions (2.86) will take the form

$$
M_1 = \frac{5}{2}mgl, \qquad M_2 = \frac{3}{2}mgl, \qquad M_3 = \frac{1}{2}mgl,
$$

$$
B_1 = \frac{7}{3}ml^2, \qquad B_2 = \frac{4}{3}ml^2, \qquad B_3 = \frac{1}{3}ml^2,
$$

$$N_{12} = \frac{3}{2}ml^2, \qquad N_{13} = \frac{1}{2}ml^2, \qquad N_{23} = \frac{1}{2}ml^2. \tag{2.101}$$

In turn, setting $g = 10\,\mathrm{m/s^2}$, $m = 1\,\mathrm{kg}$ and $l = 1\,\mathrm{m}$ we obtain

$$\begin{aligned}
M_1 &= 25\,\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-2}, & M_2 &= 15\,\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-2}, & M_3 &= 5\,\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-2},\\
B_1 &= \frac{7}{3}\,\mathrm{kg}\cdot\mathrm{m}^2, & B_2 &= \frac{4}{3}\,\mathrm{kg}\cdot\mathrm{m}^2, & B_3 &= \frac{1}{3}\,\mathrm{kg}\cdot\mathrm{m}^2,\\
N_{12} &= \frac{3}{2}\,\mathrm{kg}\cdot\mathrm{m}^2, & N_{13} &= \frac{1}{2}\,\mathrm{kg}\cdot\mathrm{m}^2, & N_{23} &= \frac{1}{2}\,\mathrm{kg}\cdot\mathrm{m}^2.
\end{aligned} \tag{2.102}$$

Viscous damping in the joints are taken as

$$c_1 = c_2 = c_3 = 1\,\mathrm{N}\cdot\mathrm{m}\cdot\mathrm{s}. \tag{2.103}$$

We take the moment acting on the first pendulum as harmonically varying in time

$$M_e(t) = q\sin(\omega t), \tag{2.104}$$

where q is the amplitude and ω the angular frequency of pendulum excitation. These two parameters will vary for different simulation examples shown later, whereas the remaining parameters from (2.102) to (2.103) will be constant.

The free motion of a pendulum (the pendulum is not subjected to external excitation, i.e., $q = 0$) for initial conditions $\psi_1(0) = \psi_2(0) = \psi_3(0) = \dot{\psi}_1(0) = \dot{\psi}_2(0) = 0$ and $\dot{\psi}_3(0) = 1\,\mathrm{rad/s}$ is shown in Fig. 2.8. Vibrations decay because the energy of the pendulum is dissipated through damping in the joints, and no new energy is simultaneously supplied (no excitation). Therefore, the solution tends to a stable equilibrium position, and the only stable equilibrium position in this system is $\psi_1 = \psi_2 = \psi_3 = 0$.

In Fig. 2.9, in turn, we present the excited transient motion of a pendulum ($q = 25\,\mathrm{N}\cdot\mathrm{m}$ and $\omega = 3\,\mathrm{rad/s}$), which starts at the time instant $t = 0$ from zero initial conditions (bright) and tends to the stable periodical solution (by analogy to the stable equilibrium position) marked dark. The solution is marked bright for the time $t \in (0,\ 150\,\mathrm{s})$, whereas for $t \in (150\,\mathrm{s},\ 200\,\mathrm{s})$ it is marked dark. Clearly, for the time $t = 150\,\mathrm{s}$ only a pendulum with a good approximation moves on a periodic solution, whereas in reality it constantly approaches it and reaches it for $t \to \infty$.

The motion of a pendulum is presented in Figs. 2.9–2.11 as the motion of the tip of the third rod of the pendulum (point O_4 in Fig. 2.7) in the plane of motion of the pendulum (coordinates x_{1O_4} and x_{2O_4} describe the position of point O_4 in the coordinate system $O_1X_1X_2$). One should remember, however, that the space (plane) $x_{1O_4} - x_{2O_4}$ is a 2D subspace of the system phase space, which is actually 7D (apart from three angles ψ_1, ψ_2, and ψ_3, and three angular velocities $\dot{\psi}_1$, $\dot{\psi}_2$, and $\dot{\psi}_3$ we add here a phase of the periodic excitation $M_e(t)$). The graphs presented in the coordinates $x_{1O_4} - x_{2O_4}$ are projections of phase trajectories onto this subspace and

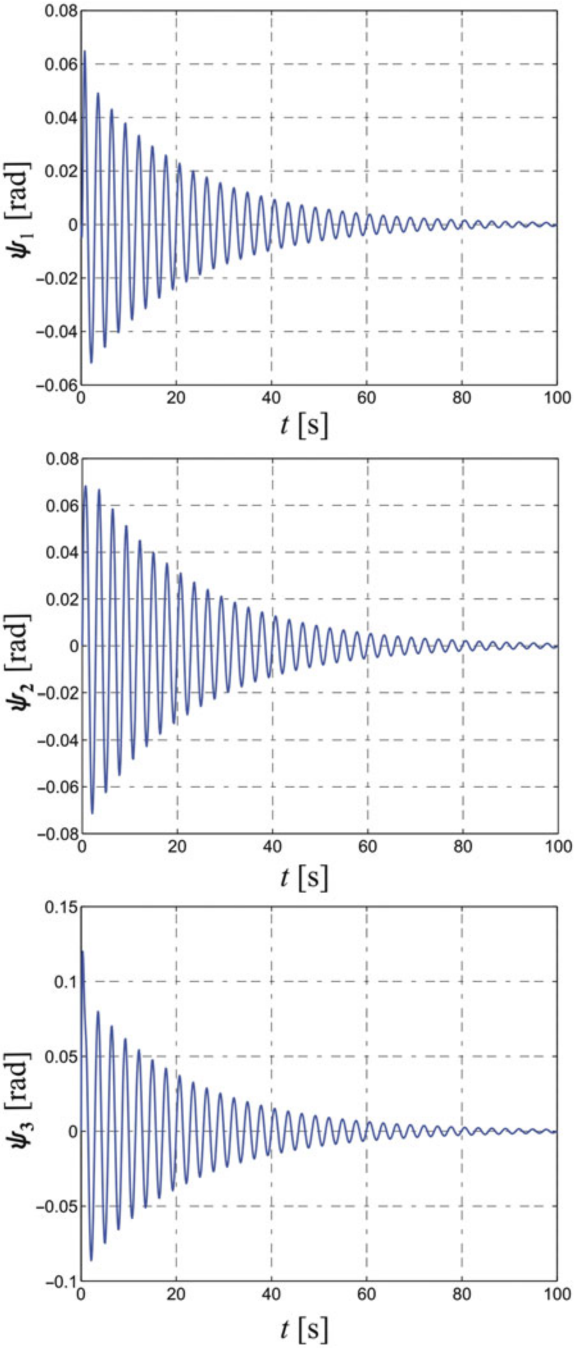


Fig. 2.8 Decaying motion of pendulum not subjected to external excitation ($q = 0$)

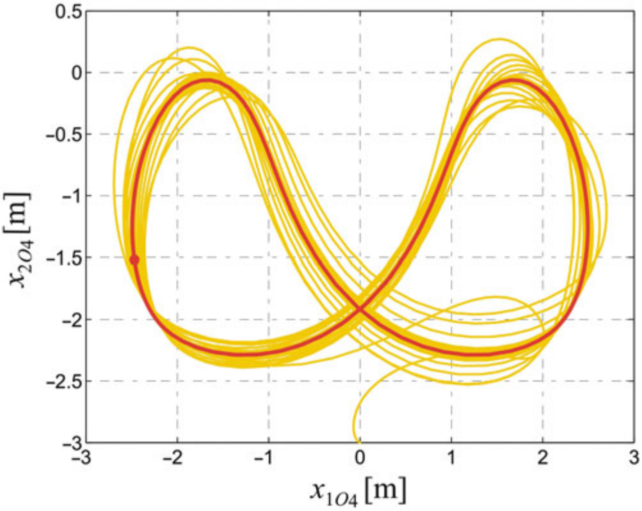


Fig. 2.9 Excited motion of pendulum tending to stable periodic solution (dark) for $q = 25\,\mathrm{N} \cdot \mathrm{m}$ and $\omega = 3\,\mathrm{rad/s}$

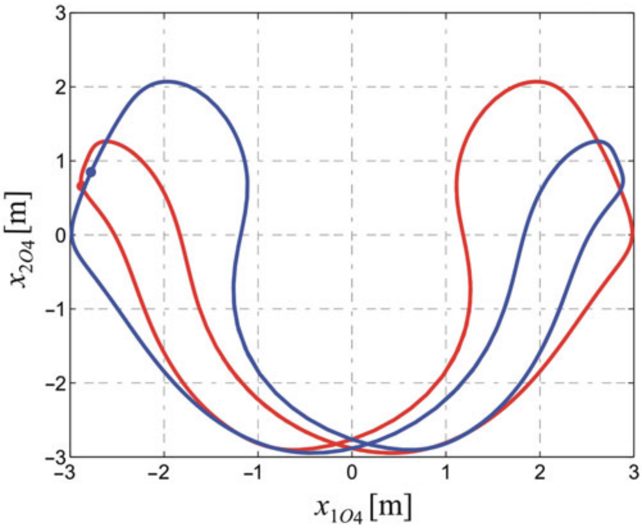


Fig. 2.10 Two coexisting periodic solutions for $q = 25\,\mathrm{N} \cdot \mathrm{m}$ and $\omega = 2.022\,\mathrm{rad/s}$ attained from different initial conditions

do not contain complete information. If we observe in Fig. 2.9 a periodic solution (dark) in the form of a closed line (i.e., the motion is repetitive), then the tip of the third rod moving on this line returns to its previous position (e.g., to the point marked with a dark circle), and the values of all state variables repeat themselves (positions and angular velocities and the phase of the periodic excitation).

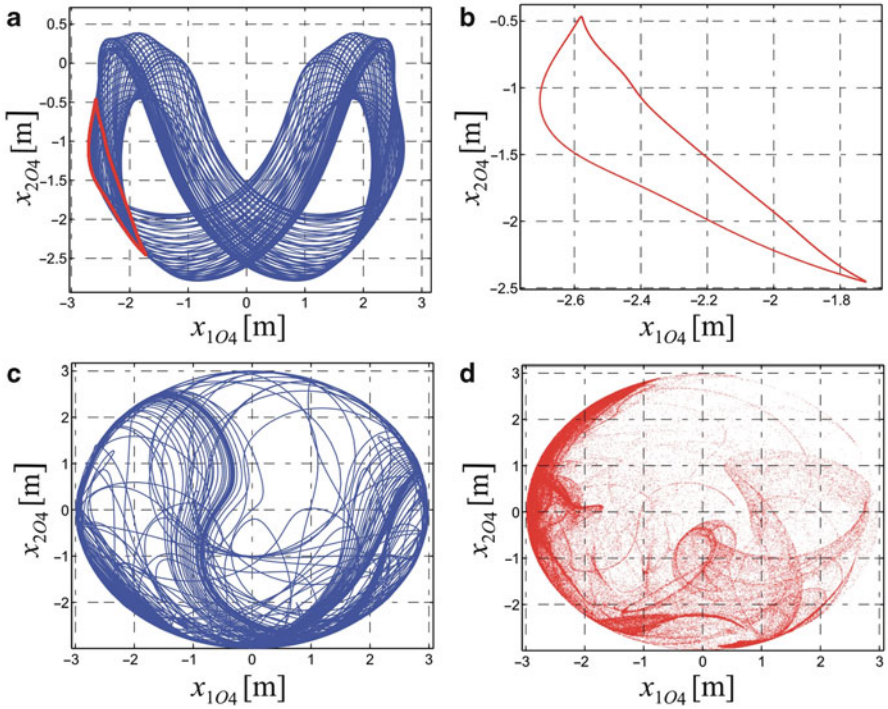


Fig. 2.11 Quasiperiodic solution for $\omega = 2.8\,\text{rad/s}$ [trajectory (**a**), Poincaré section (**b**] and chaotic solution for $\omega = 2\,\text{rad/s}$ [trajectory (**c**), Poincaré section (d)]; excitation amplitude $q = 25\,\text{N} \cdot \text{m}$

In non-linear systems the coexistence of many solutions (for the same parameters) is possible. An example is two periodic solutions for the excitation parameters $q = 25\,\text{N} \cdot \text{m}$ and $\omega = 2.022\,\text{rad/s}$ shown in Fig. 2.10. Each of the solutions is attained from different initial conditions: the solution marked in bright from zero initial conditions at the instant $t = 0$, and the solution marked in dark from initial conditions $\psi_1(0) = \psi_2(0) = \psi_3(0) = \dot{\psi}_3(0) = 0$ and $\dot{\psi}_1(0) = \dot{\psi}_2(0) = -1\,\text{rad/s}$. On the graph the initial transient motion for $t \in (0,\, 150\,\text{s})$ is neglected, and only the motion in the time interval $t \in (150,\, 200\,\text{s})$ is shown, when it has already taken place in a sufficient approximation on the appropriate periodic solution. Each solution is asymmetrical, whereas the system is symmetrical. In turn, the two solutions together form an object symmetrical with respect to the axis of symmetry of the pendulum. The symmetry of the system implies that asymmetrical solutions may appear only in such twin pairs.

From the previous examples we see that the typical behavior of a damped and periodically excited pendulum is that after waiting some time and neglecting certain initial transient motion, the pendulum starts to move periodically. This period is always a multiple of the period of excitation. However, it happens sometimes that the pendulum never starts its periodic motion regardless of the length of the transient period we would like to skip. An example is the solution shown in Fig. 2.11a for the excitation parameters $q = 25\,\mathrm{N} \cdot \mathrm{m}$ and $\omega = 2.8\,\mathrm{rad/s}$. The pendulum starts at the time instant $t = 0$ from zero initial conditions. The transient motion for $t \in (0, 200\,\mathrm{s})$ was skipped and only the motion for $t \in (200, 350\,\mathrm{s})$ is shown on the graph. One may check that after skipping a time interval of transient motion of an arbitrary length, the pendulum still is going to behave qualitatively in the same way as shown in Fig. 2.11a.

For a more detailed analysis of aperiodic motions a tool called a *Poincaré section* (also a *Poincaré map*) is very useful. In the case of a system with periodic excitation, the simplest way to create such a section is by sampling the state of the system in the intervals equal to the period of excitation. A Poincaré section of the solution from Fig. 2.11a, obtained by sampling the position of the tip of the third rod of the pendulum at time instances $t_i = iT$ $(i = 1, 2, 3, \ldots)$, where $T = 2\pi/\omega$ is the period of excitation, is shown in Fig. 2.11b.

On this occasion we obviously skip an appropriate number of initial points in order to remove the transient motion. The Poincaré section shown in Fig. 2.11b contains 3,500 points. As can be seen, these points form a continuous line. This is a characteristic of *quasiperiodic motion*.

In Fig. 2.11c, d another case of aperiodic motion of the pendulum is shown for the excitation parameters $q = 25\,\mathrm{N}{\cdot}\mathrm{m}$ and $\omega = 2\,\mathrm{rad/s}$. The pendulum starts its motion at the time instant $t = 0$ from zero initial conditions. The transient motion for $t \in (0, 200\,\mathrm{s})$ was skipped. The motion of the tip of the third rod for $t \in (200, 400\,\mathrm{s})$ is shown in Fig. 2.11c, whereas Fig. 2.11d shows the corresponding Poincaré section obtained in the same way as for the quasiperiodic solution, this time composed of 10^6 points. It is a typical section for *chaotic motion*, that is, the set of points approximating this motion is an infinite set.

In the end, we should add that a Poincaré section for a *periodic solution* of period nT (only those kinds of solutions are possible in a system with periodic excitation), where T is a period of excitation and n is an integer number, will form a set consisting of n separate points. An example would be the individual points plotted in Figs. 2.9 and 2.10.

2.3.3 Dynamic Reactions in Bearings

Lagrange's equations enable a relatively easy derivation of equations of motion of complex dynamical systems, since, for instance, they allow for avoiding the direct determination of dynamic reactions in a system. However, when these reactions have to be determined, it turns out that in equations of motion alone there is not enough information, and additional analysis of the physical system is required. That is precisely the case for a triple physical pendulum. If we want to determine the dynamic reactions in its three joints, we have to consider separately the motion

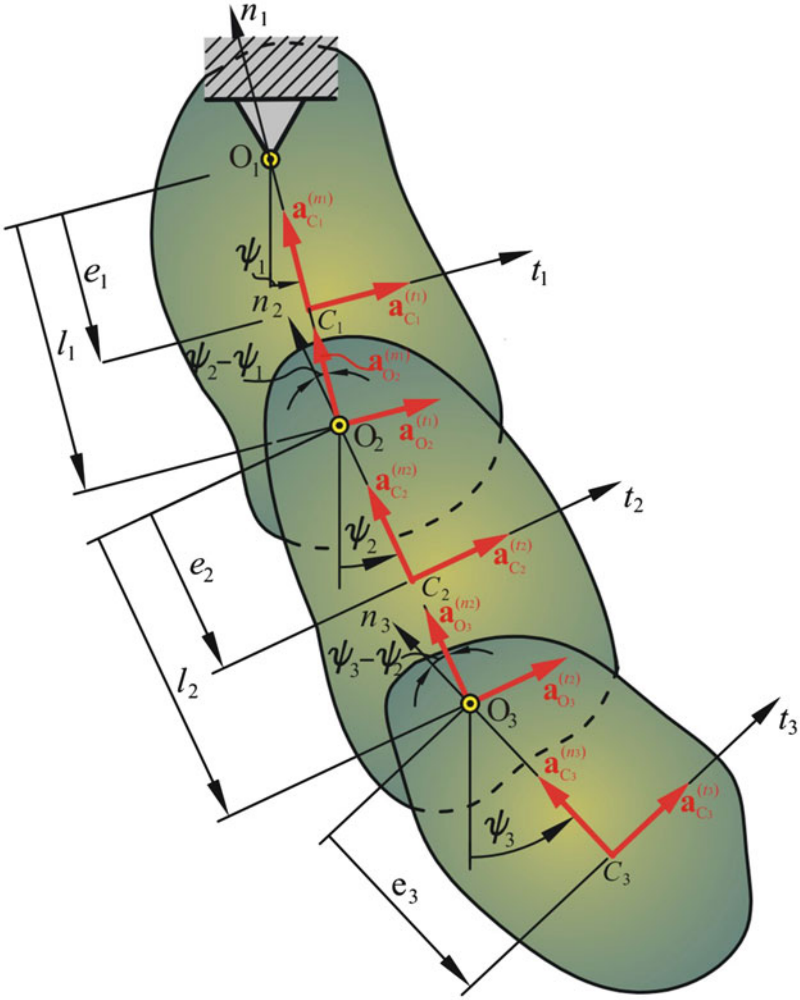


Fig. 2.12 Accelerations of characteristic points of a pendulum and their decomposition in local coordinate systems

of each of the bodies under the action of external forces. In order to obtain the relationships allowing us to determine the reactions, we will have to find the accelerations of the mass centers of particular bodies.

The accelerations of the mass center of the first pendulum C_1 and the joint O_2 (Fig. 2.12) can be expressed in terms of tangential and normal components:

$$\mathbf{a}_{C_1} = \mathbf{a}_{C_1}^{(t_1)} + \mathbf{a}_{C_1}^{(n_1)}, \qquad \mathbf{a}_{O_2} = \mathbf{a}_{O_2}^{(t_1)} + \mathbf{a}_{O_2}^{(n_1)}, \tag{2.105}$$

where

$$a_{C_1}^{(t_1)} = \varepsilon_1 e_1 = \ddot{\psi}_1 e_1, \qquad a_{C_1}^{(n_1)} = \omega_1^2 e_1 = \dot{\psi}_1^2 e_1, \tag{2.106a}$$

$$a_{O_2}^{(t_1)} = \varepsilon_1 l_1 = \ddot{\psi}_1 l_1, \qquad a_{O_2}^{(n_1)} = \omega_1^2 l_1 = \dot{\psi}_1^2 l_1. \tag{2.106b}$$

In turn, the acceleration of the mass center of the second pendulum C_2 can be represented as

$$\mathbf{a}_{C_2} = \mathbf{a}_{O_2} + \mathbf{a}_{C_2/O_2}^{(t_2)} + \mathbf{a}_{C_2/O_2}^{(n_2)} \tag{2.107}$$

or, taking into account relationship (2.105), as

$$\mathbf{a}_{C_2} = \mathbf{a}_{O_2}^{(t_1)} + \mathbf{a}_{O_2}^{(n_1)} + \mathbf{a}_{C_2/O_2}^{(t_2)} + \mathbf{a}_{C_2/O_2}^{(n_2)}, \tag{2.108}$$

where

$$a_{C_2/O_2}^{(t_2)} = \varepsilon_2 e_2 = \ddot{\psi}_2 e_2, \qquad a_{C_2/O_2}^{(n_2)} = \omega_2^2 e_2 = \dot{\psi}_2^2 e_2. \tag{2.109}$$

The total acceleration of point C_2 can also be decomposed into the following two components (Fig. 2.12):

$$\mathbf{a}_{C_2} = \mathbf{a}_{C_2}^{(t_2)} + \mathbf{a}_{C_2}^{(n_2)}, \tag{2.110}$$

and the best way to determine them is to project the right-hand side of (2.108) onto the directions t_2 and n_2:

$$\begin{aligned} a_{C_2}^{(t_2)} &= a_{O_2}^{(t_1)} \cos(\psi_2 - \psi_1) + a_{O_2}^{(n_1)} \sin(\psi_2 - \psi_1) + a_{C_2/O_2}^{(t_2)}, \\ a_{C_2}^{(n_2)} &= -a_{O_2}^{(t_1)} \sin(\psi_2 - \psi_1) + a_{O_2}^{(n_1)} \cos(\psi_2 - \psi_1) + a_{C_2/O_2}^{(n_2)}, \end{aligned} \tag{2.111}$$

and when we take into account relationships (2.106b) and (2.109), the acceleration components take the form

$$\begin{aligned} a_{C_2}^{(t_2)} &= \ddot{\psi}_1 l_1 \cos(\psi_2 - \psi_1) + \dot{\psi}_1^2 l_1 \sin(\psi_2 - \psi_1) + \ddot{\psi}_2 e_2, \\ a_{C_2}^{(n_2)} &= -\ddot{\psi}_1 l_1 \sin(\psi_2 - \psi_1) + \dot{\psi}_1^2 l_1 \cos(\psi_2 - \psi_1) + \dot{\psi}_2^2 e_2. \end{aligned} \tag{2.112}$$

In an analogous way we can proceed with the acceleration of point O_3:

$$\mathbf{a}_{O_3} = \mathbf{a}_{O_2} + \mathbf{a}_{O_3/O_2}^{(t_2)} + \mathbf{a}_{O_3/O_2}^{(n_2)}, \tag{2.113a}$$

$$\mathbf{a}_{O_3} = \mathbf{a}_{O_2}^{(t_1)} + \mathbf{a}_{O_2}^{(n_1)} + \mathbf{a}_{O_3/O_2}^{(t_2)} + \mathbf{a}_{O_3/O_2}^{(n_2)}, \tag{2.113b}$$

$$\mathbf{a}_{O_3} = \mathbf{a}_{O_3}^{(t_2)} + \mathbf{a}_{O_3}^{(n_2)}, \tag{2.113c}$$

where

$$a^{(t_2)}_{O_3/O_2} = \varepsilon_2 l_2 = \ddot{\psi}_2 l_2, \qquad a^{(n_2)}_{O_3/O_2} = \omega_2^2 l_2 = \dot{\psi}_2^2 l_2. \tag{2.114}$$

Projecting (2.113b) onto directions t_2 and n_2 we obtain

$$\begin{aligned} a^{(t_2)}_{O_3} &= a^{(t_1)}_{O_2}\cos(\psi_2-\psi_1) + a^{(n_1)}_{O_2}\sin(\psi_2-\psi_1) + a^{(t_2)}_{O_3/O_2}, \\ a^{(n_2)}_{O_3} &= -a^{(t_1)}_{O_2}\sin(\psi_2-\psi_1) + a^{(n_1)}_{O_2}\cos(\psi_2-\psi_1) + a^{(n_2)}_{O_3/O_2}, \end{aligned} \tag{2.115}$$

and after taking into account relations (2.106b) and (2.114) we obtain

$$\begin{aligned} a^{(t_2)}_{O_3} &= \ddot{\psi}_1 l_1\cos(\psi_2-\psi_1) + \dot{\psi}_1^2 l_1\sin(\psi_2-\psi_1) + \ddot{\psi}_2 l_2, \\ a^{(n_2)}_{O_3} &= -\ddot{\psi}_1 l_1\sin(\psi_2-\psi_1) + \dot{\psi}_1^2 l_1\cos(\psi_2-\psi_1) + \dot{\psi}_2^2 l_2. \end{aligned} \tag{2.116}$$

The acceleration of point C_3 can be represented as

$$\mathbf{a}_{C_3} = \mathbf{a}_{O_3} + \mathbf{a}^{(t_3)}_{C_3/O_3} + \mathbf{a}^{(n_3)}_{C_3/O_3} \tag{2.117}$$

or, after taking into account (2.113b), as

$$\mathbf{a}_{C_3} = \mathbf{a}^{(t_1)}_{O_2} + \mathbf{a}^{(n_1)}_{O_2} + \mathbf{a}^{(t_2)}_{O_3/O_2} + \mathbf{a}^{(n_2)}_{O_3/O_2} + \mathbf{a}^{(t_3)}_{C_3/O_3} + \mathbf{a}^{(n_3)}_{C_3/O_3}, \tag{2.118}$$

where

$$a^{(t_3)}_{C_3/O_3} = \varepsilon_3 e_3 = \ddot{\psi}_3 e_3, \qquad a^{(n_3)}_{C_3/O_3} = \omega_3^2 e_3 = \dot{\psi}_3^2 e_3. \tag{2.119}$$

The total acceleration of point C_3 we also decompose into the following two components (Fig. 2.12):

$$\mathbf{a}_{C_3} = \mathbf{a}^{(t_3)}_{C_3} + \mathbf{a}^{(n_3)}_{C_3}, \tag{2.120}$$

and projecting (2.117) onto directions t_3 and n_3 we obtain

$$\begin{aligned} a^{(t_3)}_{C_3} &= a^{(t_1)}_{O_2}\cos(\psi_3-\psi_1) + a^{(n_1)}_{O_2}\sin(\psi_3-\psi_1) \\ &\quad + a^{(t_2)}_{O_3/O_2}\cos(\psi_3-\psi_2) + a^{(n_2)}_{O_3/O_2}\sin(\psi_3-\psi_2) + a^{(t_3)}_{C_3/O_3}, \\ a^{(n_3)}_{C_3} &= -a^{(t_1)}_{O_2}\sin(\psi_3-\psi_1) + a^{(n_1)}_{O_2}\cos(\psi_3-\psi_1) \\ &\quad - a^{(t_2)}_{O_3/O_2}\sin(\psi_3-\psi_2) + a^{(n_2)}_{O_3/O_2}\cos(\psi_3-\psi_2) + a^{(n_3)}_{C_3/O_3}, \end{aligned} \tag{2.121}$$

and taking into account relationships (2.106b), (2.114), and (2.119) we obtain

$$
\begin{aligned}
a_{C_3}^{(t_3)} &= \ddot{\psi}_1 l_1 \cos(\psi_3 - \psi_1) + \dot{\psi}_1^2 l_1 \sin(\psi_3 - \psi_1) \\
&\quad + \ddot{\psi}_2 l_2 \cos(\psi_3 - \psi_2) + \dot{\psi}_2^2 l_2 \sin(\psi_3 - \psi_2) + \ddot{\psi}_3 e_3, \\
a_{C_3}^{(n_3)} &= -\ddot{\psi}_1 l_1 \sin(\psi_3 - \psi_1) + \dot{\psi}_1^2 l_1 \cos(\psi_3 - \psi_1) \\
&\quad - \ddot{\psi}_2 l_2 \sin(\psi_3 - \psi_2) + \dot{\psi}_2^2 l_2 \cos(\psi_3 - \psi_2) + \dot{\psi}_3^2 e_3. \qquad (2.122)
\end{aligned}
$$

The dynamic reactions of the action of the links of a pendulum to one another and to the base can be represented as the sum of the following components (Fig. 2.13):

$$\mathbf{R}_{O_1} = \mathbf{R}_{O_1}^{(t_1)} + \mathbf{R}_{O_1}^{(n_1)}, \qquad (2.123a)$$

$$\mathbf{R}_{O_2} = \mathbf{R}_{O_2}^{(t_2)} + \mathbf{R}_{O_2}^{(n_2)}, \qquad (2.123b)$$

$$\mathbf{R}_{O_3} = \mathbf{R}_{O_3}^{(t_3)} + \mathbf{R}_{O_3}^{(n_3)}. \qquad (2.123c)$$

Due to space limitations, Fig. 2.13 does not contain the moments of forces of the actions of links on one another and of the base action through joints, since in the following calculationswe do not use moment equations but force equations only.

For each link the equations expressing the acceleration of its mass center under the action of external forces have the following form:

$$
\begin{aligned}
m_1 \mathbf{a}_{C_1} &= \mathbf{R}_{O_1} + m_1 \mathbf{g} - \mathbf{R}_{O_2}, \\
m_2 \mathbf{a}_{C_2} &= \mathbf{R}_{O_2} + m_2 \mathbf{g} - \mathbf{R}_{O_3}, \\
m_3 \mathbf{a}_{C_3} &= \mathbf{R}_{O_3} + m_3 \mathbf{g}, \qquad (2.124)
\end{aligned}
$$

and projecting these equations onto directions t_1, n_1, t_2, n_2, t_3, and n_3 we obtain

$$
\begin{aligned}
m_1 a_{C_1}^{(t_1)} &= R_{O_1}^{(t_1)} - m_1 g \sin(\psi_1) \\
&\quad - R_{O_2}^{(t_2)} \cos(\psi_2 - \psi_1) + R_{O_2}^{(n_2)} \sin(\psi_2 - \psi_1), \\
m_1 a_{C_1}^{(n_1)} &= R_{O_1}^{(n_1)} - m_1 g \cos(\psi_1) \\
&\quad - R_{O_2}^{(t_2)} \sin(\psi_2 - \psi_1) - R_{O_2}^{(n_2)} \cos(\psi_2 - \psi_1), \\
m_2 a_{C_2}^{(t_2)} &= R_{O_2}^{(t_2)} - m_2 g \sin(\psi_2) \\
&\quad - R_{O_3}^{(t_3)} \cos(\psi_3 - \psi_2) + R_{O_3}^{(n_3)} \sin(\psi_3 - \psi_2),
\end{aligned}
$$

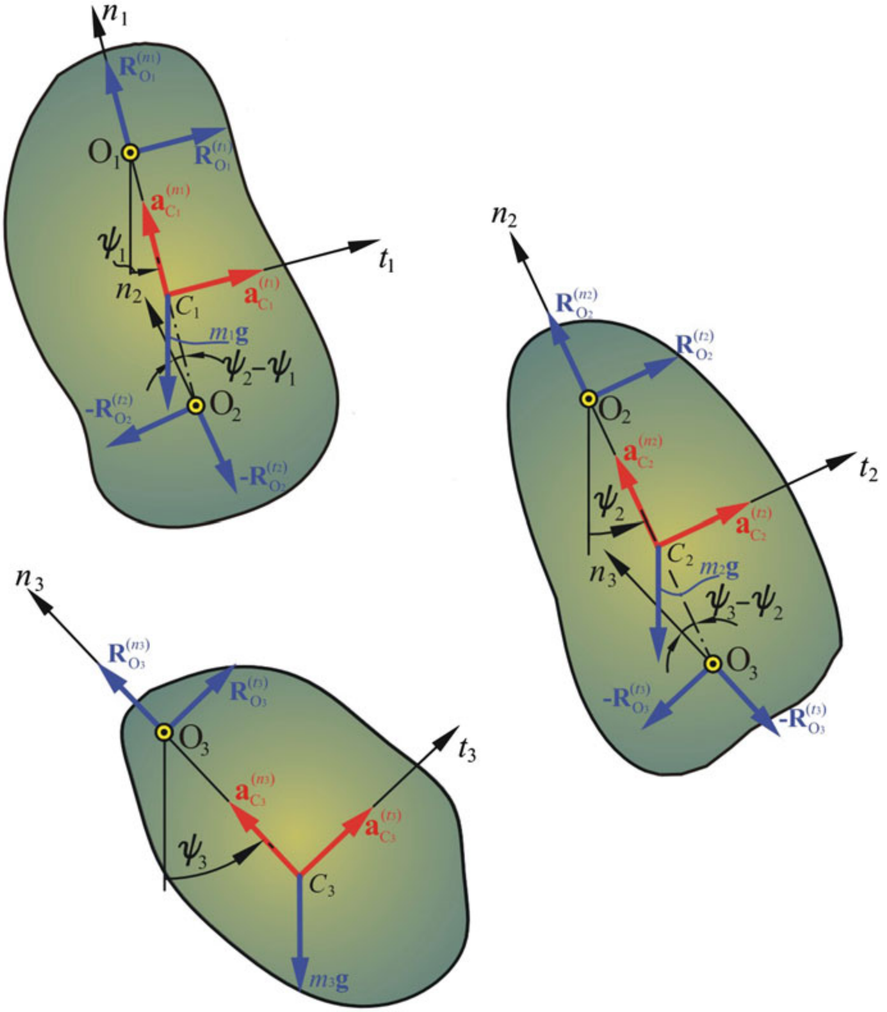


Fig. 2.13 External forces acting on particular links of a pendulum and the accelerations of the mass centers of the links (force couples acting at joints not shown)

$$
\begin{aligned}
m_2 a_{C_2}^{(n_2)} &= R_{O_2}^{(n_2)} - m_2 g \cos(\psi_2) \\
&\qquad - R_{O_3}^{(t_3)} \sin(\psi_3 - \psi_2) - R_{O_3}^{(n_3)} \cos(\psi_3 - \psi_2)\,, \\
m_3 a_{C_3}^{(t_3)} &= R_{O_3}^{(t_3)} - m_3 g \sin(\psi_3)\,, \\
m_3 a_{C_3}^{(n_3)} &= R_{O_3}^{(n_3)} - m_3 g \cos(\psi_3)\,. \qquad (2.125)
\end{aligned}
$$

Those equations can be solved with respect to the components of the dynamic reactions, and taking into account relationships (2.106a), (2.112), and (2.122), and bearing in mind that $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$, we obtain

$$\begin{aligned}
R_{O_3}^{(t_3)} &= m_3 \left[g \sin\psi_3 + e_3\ddot{\psi}_3 + l_1 \left(\ddot{\psi}_1 \cos(\psi_1 - \psi_3) - \dot{\psi}_1^2 \sin(\psi_1 - \psi_3) \right) \right. \\
&\qquad \left. + l_2 \left(\ddot{\psi}_2 \cos(\psi_2 - \psi_3) - \dot{\psi}_2^2 \sin(\psi_2 - \psi_3) \right) \right], \\
R_{O_3}^{(n_3)} &= m_3 \left[g \cos\psi_3 + e_3\dot{\psi}_3^2 + l_1 \left(\ddot{\psi}_1 \sin(\psi_1 - \psi_3) + \dot{\psi}_1^2 \cos(\psi_1 - \psi_3) \right) \right. \\
&\qquad \left. + l_2 \left(\ddot{\psi}_2 \sin(\psi_2 - \psi_3) + \dot{\psi}_2^2 \cos(\psi_2 - \psi_3) \right) \right], \\
R_{O_2}^{(t_2)} &= m_2 \left[g \sin\psi_2 + e_2\ddot{\psi}_2 + l_1 \left(\ddot{\psi}_1 \cos(\psi_1 - \psi_2) - \dot{\psi}_1^2 \sin(\psi_1 - \psi_2) \right) \right] \\
&\qquad + R_{O_3}^{(n_3)} \sin(\psi_2 - \psi_3) + R_{O_3}^{(t_3)} \cos(\psi_2 - \psi_3), \\
R_{O_2}^{(n_2)} &= m_2 \left[g \cos\psi_2 + e_2\dot{\psi}_2^2 + l_1 \left(\ddot{\psi}_1 \sin(\psi_1 - \psi_2) + \dot{\psi}_1^2 \cos(\psi_1 - \psi_2) \right) \right] \\
&\qquad + R_{O_3}^{(n_3)} \cos(\psi_2 - \psi_3) - R_{O_3}^{(t_3)} \sin(\psi_2 - \psi_3), \\
R_{O_1}^{(t_1)} &= m_1 \left[g \sin\psi_1 + e_1\ddot{\psi}_1 \right] + R_{O_2}^{(n_2)} \sin(\psi_1 - \psi_2) + R_{O_2}^{(t_2)} \cos(\psi_1 - \psi_2), \\
R_{O_1}^{(n_1)} &= m_1 \left[g \cos\psi_1 + e_1\dot{\psi}_1^2 \right] + R_{O_2}^{(n_2)} \cos(\psi_1 - \psi_2) - R_{O_2}^{(t_2)} \sin(\psi_1 - \psi_2).
\end{aligned} \tag{2.126}$$

Now the absolute values of total reactions can be calculated as

$$\begin{aligned}
R_{O_1} &= \sqrt{\left(R_{O_1}^{(t_1)}\right)^2 + \left(R_{O_1}^{(n_1)}\right)^2}, \\
R_{O_2} &= \sqrt{\left(R_{O_2}^{(t_2)}\right)^2 + \left(R_{O_2}^{(n_2)}\right)^2}, \\
R_{O_3} &= \sqrt{\left(R_{O_3}^{(t_3)}\right)^2 + \left(R_{O_3}^{(n_3)}\right)^2}.
\end{aligned} \tag{2.127}$$

Some examples of time plots of dynamic reactions calculated from relations (2.126) and (2.127) are presented in Figs. 2.14–2.16.

The time plot shown in Fig. 2.14 corresponds to the solution shown in Fig. 2.8, that is, to the decaying motion of a pendulum without excitation. It can be seen that reactions decrease relatively quickly to a value close to a static reaction for a system at rest. In Fig. 2.15 we present the time plot of dynamic reactions in periodic motion of the pendulum shown in Fig. 2.9 ($q = 25\,\mathrm{N \cdot m}$ and $\omega = 3\,\mathrm{rad/s}$). Here greater values of reactions are visible. In turn, in Fig. 2.16 we present a certain select part of the time plot of dynamic reactions for the chaotic solution shown in

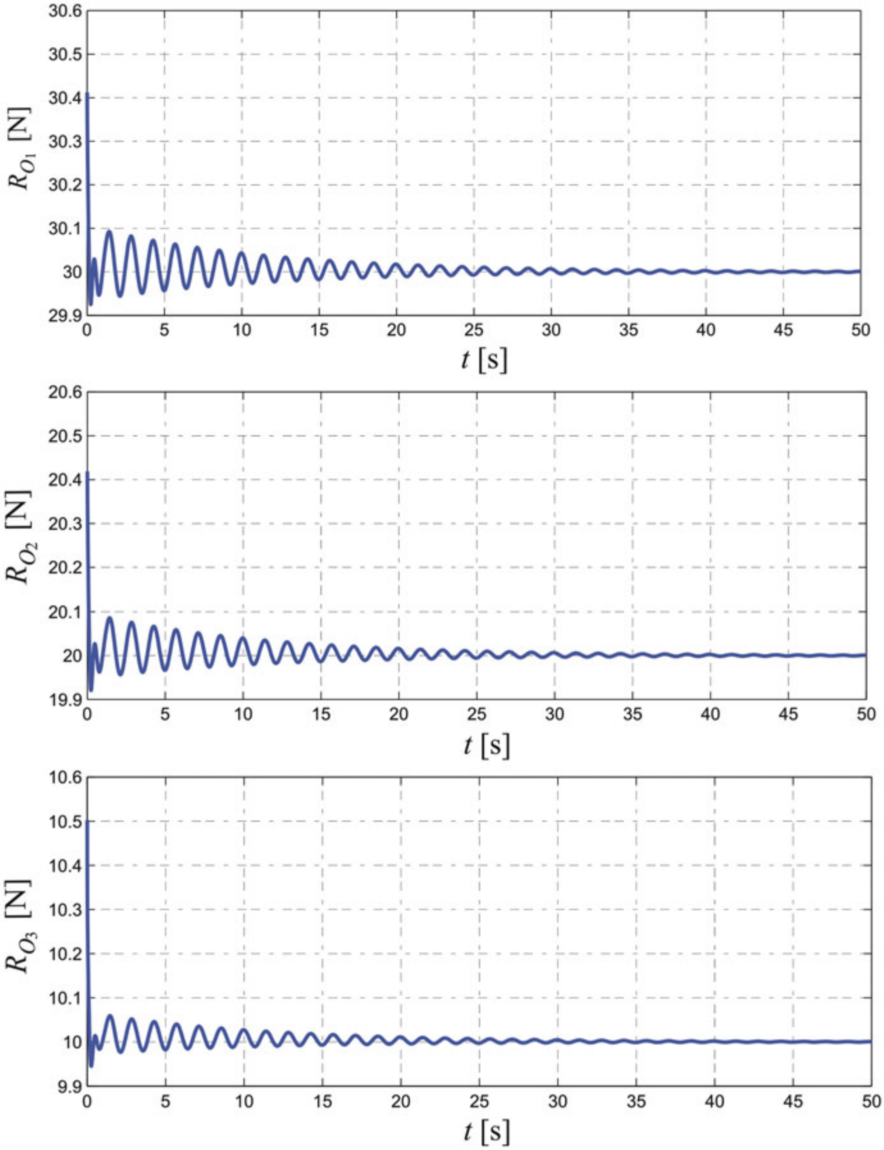


Fig. 2.14 Dynamic reactions in bearings for decaying motion of pendulum without external excitation ($q = 0$)

Fig. 2.11c, that is, for the parameters of excitation $q = 25\,\mathrm{N \cdot m}$ and $\omega = 2\,\mathrm{rad/s}$. The most rapid changes in the dynamic reactions of bearings are visible there. It should be emphasized, however, that it is only part of an irregular time plot, and the instantaneous values of reactions may be even greater.

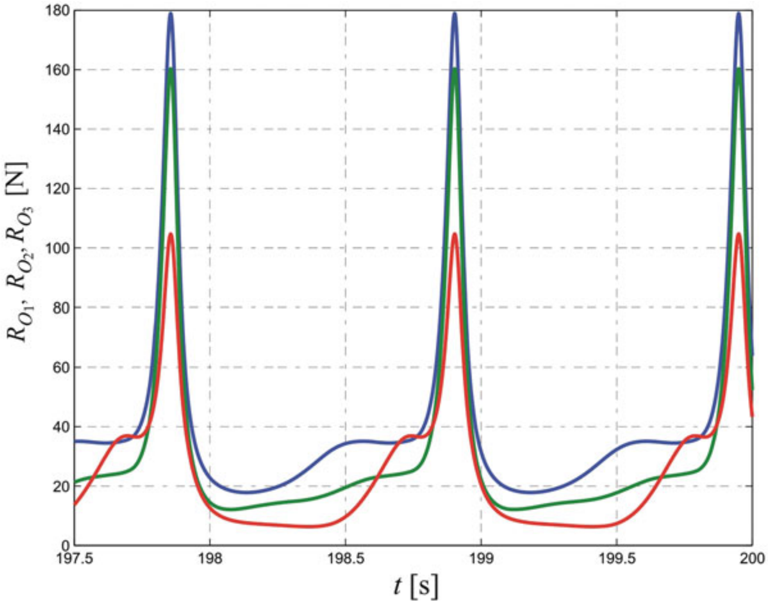


Fig. 2.15 Dynamic reactions in bearings for periodic motion of pendulum for $q = 25\,\mathrm{N \cdot m}$ and $\omega = 3\,\mathrm{rad/s}$; color codes: R_{O_1}—*1*, R_{O_2}—*2*, R_{O_3}—*3*

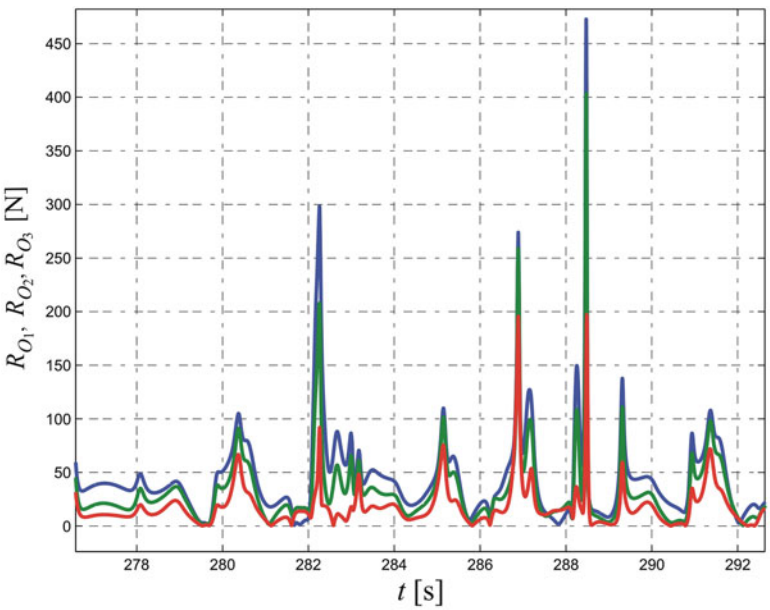


Fig. 2.16 Dynamic reactions in bearings for chaotic motion of pendulum for $q = 25\,\mathrm{N \cdot m}$ and $\omega = 2\,\mathrm{rad/s}$; color codes: R_{O_1}—*1*, R_{O_2}—*2*, R_{O_3}—*3*

Supplementary sources for the material in this chapter include [7–12]. In addition, numerous books are devoted to the periodic, quasiperiodic, and chaotic dynamics of lumped mechanical systems including [13–15].

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