

Chapter 2

Different Ways of Describing Plasma Dynamics

2.1 General Particle Description, Liouville and Klimontovich Equations

In order to realize which approximations that are made in the descriptions of plasmas that we generally use [1–25], it is instructive to start from the most general description which includes all individual particles and their correlations in the six dimensional phase space $(\mathbf{r}, \mathbf{v})$. In the absence of particle sources or sinks we must have a continuity equation for the delta function density N :

$$N(X, t) = \sum_{i=1}^N (X - X_i(t)) \quad X = (\mathbf{r}, \mathbf{v}) \quad (2.1)$$

$$\frac{\partial}{\partial t} N + \sum_i \frac{\partial}{\partial r_i} \left(N \frac{\partial r_i}{\partial t} \right) + \sum_i \frac{\partial}{\partial v_i} \left(N \frac{\partial v_i}{\partial t} \right) = 0, \quad (2.2)$$

Since we have included all particles this system conserves energy if we ignore radiation.

Thus there must be a Hamiltonian for the system and we use the Hamilton equations:

$$\frac{\partial r_i}{\partial t} = \frac{\partial H}{\partial v_i} \quad ; \quad \frac{\partial v_i}{\partial t} = -\frac{\partial H}{\partial r_i} \quad (2.3a)$$

Leading to the form

$$\frac{\partial}{\partial t} N + \sum_i \frac{\partial r_i}{\partial t} \frac{\partial N}{\partial r_i} + \sum_i \frac{\partial v_i}{\partial t} \frac{\partial N}{\partial v_i} = 0, \quad (2.3b)$$

Using acceleration due to the Lorenz force we then get

$$\left\{ \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{r}} + \frac{e}{m} \vec{E} \cdot \frac{\partial}{\partial \vec{v}} + \Omega(\vec{v} \times \hat{e}_{\parallel}) \cdot \frac{\partial}{\partial \vec{v}} \right\} N(X, t) = 0 \quad (2.4)$$

Where we wrote $\frac{e\mathbf{B}_i}{m} = \Omega_c \hat{\mathbf{e}}_{\parallel}$. This can easily be generalized to the electromagnetic case. Equation 2.4 is written as a conservation along orbits in phase space. i.e.

$$\frac{DN}{Dt} = 0$$

Where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{r}} + \frac{e}{m} \vec{E} \cdot \frac{\partial}{\partial \vec{v}} + \Omega(\vec{v} \times \hat{e}_{\parallel}) \cdot \frac{\partial}{\partial \vec{v}}$$

Is the total operator in (2.4). Equation 2.4 is the Liouville or Klimontovich equation. Since $N(X, t)$, as given by (2.1), contains the simultaneous location of all particles in phase space, it can be considered as a probability density in phase space. It gives the probability of finding a particle in the location $(\mathbf{r}, \mathbf{v})$ given the simultaneous locations $(\mathbf{r}_i, \mathbf{v}_i)$ of all the other particles. This is an enormous amount of information which is usually not needed. This information can be reduced by integrating over the positions of several other particles giving an hierarchy of distribution functions (the BBGKY hierarchy) where the evolution of each distribution function, giving the probability of the simultaneous distribution of n particles, depends on that of $n + 1$ particles. Thus we need to close this hierarchy in some way. This is usually done by expanding in the plasma parameter

$$g = \frac{1}{n\lambda_d^3} \quad ; \quad \lambda_d = \sqrt{\frac{T}{4\pi en}}$$

Which is the inverse number of particles in a Debyesphere. When the plasma-parameter tends to zero only collective interactions remain between the particles. The effect is as if the particles were smeared out in phase space. When we study the equation of the one particle distribution function and include effects of the two particle distribution function (describing pair collisions) as expanded in g we get the equation:

$$\left\{ \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{r}} + \frac{e}{m} \vec{E} \cdot \frac{\partial}{\partial \vec{v}} + \Omega(\vec{v} \times \hat{e}_{\parallel}) \cdot \frac{\partial}{\partial \vec{v}} \right\} f(\mathbf{r}, \mathbf{v}, t) = \left(\frac{\partial f}{\partial t} \right)_{coll} \quad (2.5)$$

where f is the one particle distribution function and the right hand side approximates close collisions (first order in g). Here various approximations like Boltzmanns or the Fokker-Planck collision terms are used. If we can ignore close collisions completely we have the Vlasov equation:

$$\left\{ \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{r}} + \frac{e}{m} \vec{E} \cdot \frac{\partial}{\partial \vec{v}} + \Omega(\vec{v} \times \hat{e}_{\parallel}) \cdot \frac{\partial}{\partial \vec{v}} \right\} f(\mathbf{r}, \mathbf{v}, t) = 0 \quad (2.6)$$

2.2 Kinetic Theory as Generally Used by Plasma Physicists

The kinetic equations 2.5 and 2.6 are the equations usually used by plasma physicists. Equation 2.6 is reversible like (2.4). This means that processes can go back and forth. Equation 2.6 describes only collective motions. An example of this is wave propagation. It is also able to describe temporary damping (in the linearized case) of waves, so called Landau damping, due to resonances between particles and waves. Since the plasma parameter g in typical laboratory plasmas is of the order 10^{-8} collective phenomena usually dominate over phenomena related to close collisions. We mentioned above the Fokker-Planck collision term for close collisions. However, in a random phase situation also turbulent collisions can be described by a Fokker-Planck equation. It can be written:

$$\left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right) f(x, v, t) = \frac{\partial}{\partial v} \left[\beta v + D^v \frac{\partial}{\partial v} \right] f(x, v, t) \quad (2.7)$$

The Fokker-Planck equation is fundamental and of interest in many contexts. One aspect is that it, in its original form is Markovian (particles have forgotten previous events) but recently has been generalized to the non-Markovian case [23]. For constant coefficients it has an exact analytical solution [1]. It is interesting to note that already two waves leads to stochasticity of particles, giving quasilinear transport [17]. However, the solution of the Fokker-Planck equation, including friction, leads to a saturation of the mean square deviation of velocity after a time of order $1/\beta$. A solution is shown in Fig. 2.1. For the turbulent case (β and D^v depending on intensities of the turbulent waves), the initial linear growth of $\langle (\Delta v)^2 \rangle$ is according to quasilinear theory and would be predicted by the Chirikov results. The saturation follows from Dupree-Weinstock theory [12, 13] which is a strongly nonlinear renormalized theory. Thus in the flat region nonlinearities have introduced correlations. This is analogous to correlations between three wave packets introduced by nonlinearities in the Random Phase approximation [11].

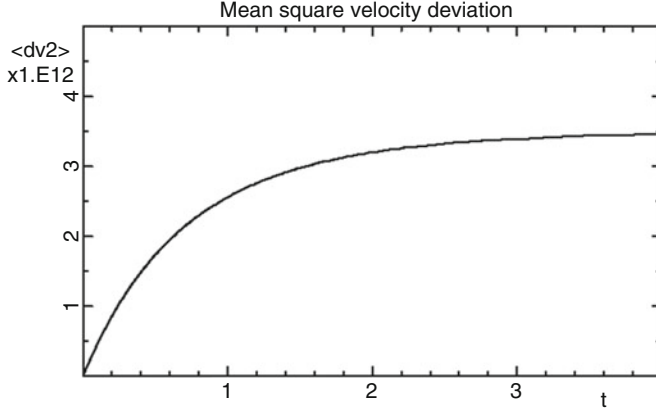


Fig. 2.1 Mean square velocity deviation $\langle (\Delta v)^2 \rangle$ as a function of time showing initial quasilinear linear growth and later saturation at $t \sim 1/\beta$

2.3 Gyrokinetic Theory

For low frequency ($\omega \ll \Omega_c$) we can average the kinetic equation over the gyromotion. This leads to the gyrokinetic equation:

$$\begin{aligned}
 & (\omega - \omega_D(v_{\parallel}^2, v_{\perp}^2) - k_{\parallel} v_{\parallel}) \left(f_{k,\omega}^{(1)} + \frac{q\phi_{k,\omega}}{T} f_0 \right) e^{-iL_k} \\
 &= \left[(\omega - \omega_{\bullet}) \frac{q}{T} (\phi_{k,\omega} - v_{\parallel} \mathbf{A}_{\parallel}) J_0(\xi_k) - i \frac{v_{\perp}}{k_{\perp}} (\hat{\mathbf{e}}_{\parallel} \times \mathbf{k}) \cdot \mathbf{A}_{\mathbf{k}} J_0' \right] f_0 \quad (2.8a)
 \end{aligned}$$

We will return to the derivation of this equation, and its nonlinear extension in Chap. 5.

However, we will mention that the magnetic drift is the sum of gradient B and curvature drifts as:

$$\mathbf{v}_D = \mathbf{v}_{\nabla B} + \mathbf{v}_{\kappa} \quad (2.8b)$$

where

$$\mathbf{v}_{\nabla B} = \frac{v_{\perp}^2}{2\Omega_c} (\hat{\mathbf{e}}_{\parallel} \times \nabla \ln B) \quad (2.8c)$$

$$\mathbf{v}_{\kappa} = \frac{v_{\parallel}^2}{\Omega_c} (\hat{\mathbf{e}}_{\parallel} \times \kappa) \quad (2.8d)$$

and

$$\kappa = (\hat{e}_{\parallel} \cdot \nabla) \hat{e}_{\parallel} \quad (2.8e)$$

The drifts defined by (2.8c) and (2.8d) are the gradB and curvature drifts respectively and (2.8e) defines the curvature vector. We here use $\hat{e}_{\parallel} = \mathbf{B}/B$ as a space dependent unity vector along the magnetic field. In a slab geometry with fixed magnetic field we instead use the unity vector $\hat{z}$.

Since $\hat{e}_{\parallel}$ it is generally used in inhomogeneous systems we will need to solve eigenvalue equations. Then $k_{\parallel}$ and sometimes even the magnetic drift frequency will become operators. Since then the eigenvalue problem depends on the particular velocity we are considering, the total eigenvalue solution will have to be averaged over velocity space. Thus we have an integral eigenvalue problem. The fact that magnetic curvature is destabilizing on the outside and stabilizing on the inside of a torus will show in a dependence of ω_D on the poloidal angle. The density perturbation from (2.8a) will be obtained by dividing by the first factor and integrating over velocity.

$$\frac{\delta n_i}{n_i} = -\frac{e\phi}{T_i} \left[1 - \frac{1}{n_0} \int_0^{\infty} \frac{\omega - \omega_{*i} [1 + \eta_i (m_i v^2 / 2T_i - 3/2)]}{\omega - k_{\parallel} v_{\parallel} - \omega_{Di} (v_{\parallel}^2 + v_{\perp}^2 / 2) / v_{th}^2} J_0(\xi)^2 f_0 d^3 v \right] \quad (2.9)$$

We here took the electrostatic approximation just for the purpose of illustration. The integral in (2.9) will have resonances corresponding to wave particle resonances. However, as will be discussed later, in the nonlinear regime, nonlinear frequency shifts may detune these resonances.

2.4 Fluid Theory as Obtained by Taking Moments of the Vlasov Equation

An alternative to making the full kinetic calculation is to first derive fluid equations by taking moments of (2.5) or (2.8a) (of course collisions can be added also to [2.8]). Clearly, in general (2.9) contains less information than (2.5). However, if we expand the fluid equations obtained from (2.5) in the low frequency limit the results obtained from (2.5) and (2.8a) the results will be identical. The equations obtained by taking moments of (2.5) are called *fluid equations* and the equations obtained by taking moments of (2.8a) are called *gyrofluid equations*.

Fluid equations really describe a continuum where the local velocities have been averaged over the particle distribution at every point. This leads to the presence of **fluid drifts** that are **not guiding centre drifts** in an inhomogeneous plasma. However, the macroscopic properties like the time derivative of the density are, of course, the same whether we use fluid or gyrofluid equations. Another aspect

which is not either a really dividing property is the fact that several authors have added the linear kinetic resonances to gyrofluid equations. These are then called Gyro-Landau Fluid resonances. However, this is just a question of habits of different authors and, of course, there is nothing that prevents us from adding linear kinetic resonances to fluid equations.

2.4.1 The Maxwell Equations

Since the ordinary fluid equations are what we will mainly use in this book we will here start by including the Maxwells equations.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (2.10a)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \mu_0 \quad (2.10b)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.10c)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad (2.10d)$$

Here (2.10a) is the induction law and (2.10b) is the ampere law. Here the last term is the displacement current which will generally be neglected here since we consider low frequencies where quasineutrality holds. Equation 2.10c is general and tells us that there are no magnetic charges while (2.10d) will mostly be replaced by the quasineutrality condition.

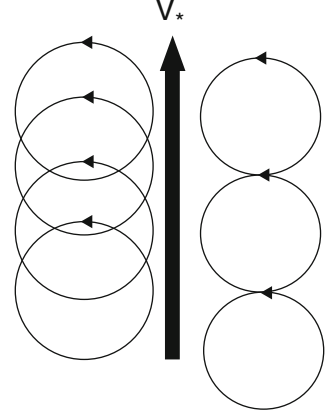
2.4.2 The Low Frequency Expansion

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \frac{1}{mn} (\nabla P + \nabla \cdot \pi) + \vec{g} = 0 \quad (2.11a)$$

$$\omega \ll \Omega_c \Rightarrow$$

$$\mathbf{v}_\perp = \mathbf{v}_E + \mathbf{v}_p + \mathbf{v}_* + \vec{v}_a + \vec{v}_g \quad (2.11b)$$

$$\mathbf{v}_E = \frac{1}{B} (\mathbf{E} \times \hat{\mathbf{z}}) \quad (2.11c)$$

Fig. 2.2 Diamagnetic drift

$$\mathbf{v}_* = \frac{1}{qnB} (\hat{\mathbf{z}} \times \nabla P) \quad (2.11d)$$

$$\mathbf{v}_p = \frac{1}{\Omega_c} \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) (\hat{\mathbf{z}} \times \mathbf{v}) \quad (2.11e)$$

A usual approximation is to substitute the $\mathbf{E} \times \mathbf{B}$ drift into (2.11e)

This gives:

$$\mathbf{v}_p = \frac{1}{B\Omega_c} \left(\frac{\partial}{\partial t} + \mathbf{v}_E \cdot \nabla \right) \mathbf{E} \quad (2.11f)$$

$$\vec{v}_g = \frac{\vec{g} \times \hat{\mathbf{z}}}{\Omega_c} \quad (2.11g)$$

However, this needs to be generalized when we include Finite Larmor Radius (FLR) effects.

Due to the bending of field lines we also have an electromagnetic drift

$$\mathbf{v}_{\delta B} = v_{||} \frac{\delta \mathbf{B}_{\perp}}{\mathbf{B}_{||}} \quad (2.11h)$$

Here the diamagnetic drift is a pure fluid drift, i.e. it does not move particles (Fig. 2.2).

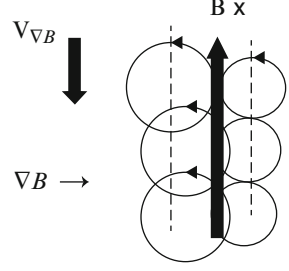
Since the diamagnetic drift does not move particles it does not cause a density perturbation i.e. (Fig. 2.2)

$$\nabla \cdot (n\mathbf{v}_*) = 0 \quad (2.12)$$

Equation 2.12 is the lowest order consequence of the fact that the diamagnetic drift does not move particles. In the momentum equation the stress tensor cancels

Fig. 2.3 Magnetic drift.

The particle drift is compensated by the fact that more particles contribute from the side with weaker magnetic field in such a way that there is no fluid drift



convective diamagnetic effects. Such effects are cancelled also in the energy equation as we will soon see.

The magnetic drift is not a fluid drift because the guiding centre drift is compensated by the fluid effect of having more particles from one side (Fig 2.3)

2.4.3 The Energy Equation

The highest order moment equation that we are going to make use of is the energy equation. It is most commonly written as an equation for the pressure variation as:

$$\frac{3}{2} \left(\frac{\partial}{\partial t} + \mathbf{v}_j \cdot \nabla \right) P_j + \frac{5}{2} P_j \nabla \cdot \mathbf{v}_j = -\nabla \cdot \mathbf{q}_j + \sum_{j=i} Q_{ji} \quad (2.13)$$

where $\mathbf{q}_j$ is the heat flux and Q_{ij} is the heat transferred from species i to species j by means of collisions. This energy exchange typically contains effects like Ohmic heating and temperature equilibrium terms. It will be neglected in the following. The heat flux $\mathbf{q}$ is for the collision dominated case ($\lambda \gg l_f$) according to Braginskii:

$$\mathbf{q}_j = 0.71 n_j T_j \mathbf{U}_{||} - \kappa_{||} \nabla_{||} T - \kappa_{\perp} \nabla_{\perp} T + \mathbf{q}_{*i} + \frac{3}{2} v_j \frac{n_j T_j}{\Omega_{cj}} (\hat{\mathbf{e}}_{||} \times \mathbf{U}) \quad (2.14)$$

where $\mathbf{U}$ is the relative velocity between species j and i . The thermal conductivities for electrons are given by

$$\kappa_{||e} = 3.16 \frac{n_e T_e}{m_e v_e} \quad \kappa_{\perp e} = 4.66 \frac{n_e T_e v_e}{m_e \Omega_{ce}^2}$$

and for ions by

$$\kappa_{||i} = 3.9 \frac{n_i T_i}{m_i v_i} \quad \kappa_{\perp i} = 2 \frac{n_i T_i v_i}{m_i \Omega_{ci}^2}$$

and

$$\mathbf{q}_{*j} = \frac{5}{2} \frac{P_j}{m_j \Omega_{cj}} (\mathbf{e}_{||} \times \nabla T_j) \quad (2.15)$$

If we neglect the full right hand side of (2.14) we obtain the adiabatic equation of state for three dimensional motion, i.e.

$$\frac{d}{dt} \left(P n^{5/3} \right) = 0 \quad (2.16)$$

which holds for processes that are so rapid that the heat flux does not have time to develop. When $\text{div } \mathbf{v} = 0$ which is a rather common situation, the pressure perturbation can be taken as due to convection in a background gradient. This will be further discussed later.

Another usual form of the energy equation is that obtained after subtracting the continuity equation. It may be written as:

$$\frac{3}{2} n_j \left(\frac{\partial}{\partial t} + \mathbf{v}_j \cdot \nabla \right) T_j + P_j \nabla \cdot \mathbf{v}_j = -\nabla \cdot \mathbf{q}_j \quad (2.17)$$

Equations 2.13 and 2.17 are fluid equations and the velocities thus contain the diamagnetic drifts. As it turns out these drifts cancel in a way similar to that in the momentum equation but now due to the heat flow terms, i.e.

$$\frac{3}{2} n \mathbf{v}_* \cdot \nabla T - T \mathbf{v}_* \cdot \nabla n = \frac{5}{2} n \mathbf{v}_* \cdot \nabla T \quad (2.18a)$$

$$\frac{3}{2} n \mathbf{v}_* \cdot \nabla T - T \mathbf{v}_* \cdot \nabla n = -\nabla \cdot \mathbf{q}_* \quad (2.18b)$$

Where $\omega_* = \mathbf{k} \bullet \mathbf{v}_*$. We can then write the energy equation in the form:

$$\frac{3}{2} n_j \left(\frac{\partial}{\partial t} + \mathbf{v}_{gcj} \cdot \nabla \right) T_j - T_j \left(\frac{\partial n_j}{\partial t} + \mathbf{v}_{gcj} \cdot \nabla n_j \right) = -\nabla \cdot \mathbf{q}_{gcj} \quad (2.19)$$

Where $\mathbf{v}_{gc}$ here is defined as the guiding centre part of the fluid velocity, i.e. without the magnetic drift and $\mathbf{q}_{gcj}$ is $\mathbf{q}_j$ as defined in (2.14) but without the diamagnetic heatflow, i.e. (2.15). As we will see later, in a curved magnetic field also (2.15) will contain a guiding centre part. Equation 2.19 shows that the relevant convective velocity in the energy equation is the guiding centre part of the fluid velocity. The term coming from $\text{div } \mathbf{v}$ is

$$\frac{\partial n_j}{\partial t} + \mathbf{v}_{gcj} \cdot \nabla n_j = -n \nabla \cdot \mathbf{v}_{gcj} - \nabla \cdot (n \mathbf{v}_{*j})$$

Where the last term is a pure magnetic drift effect. From this follows also that the convective velocity in (2.16) does not contain the diamagnetic drift.

Another useful equation of state may be obtained at low frequencies and small collision rates for electrons. In this case the energy equation is dominated by the $\text{div } \mathbf{q}$ term so that the lowest order equation of state is $\mathbf{q} = 0$ or $\kappa_{\parallel} \nabla_{\parallel} T = 0$. Now $\nabla_{\parallel} = (1/B)(\mathbf{B}_0 + \delta\mathbf{B}) \cdot \nabla$ so that, after linearization

$$\mathbf{B}_0 \cdot \nabla \delta T + \delta\mathbf{B} \cdot \nabla T_0 = 0 \quad (2.20)$$

If the perpendicular perturbation in $\mathbf{B}$ is represented by a parallel vector potential we obtain the equation of state:

$$\delta T_j = -\eta_j \frac{\omega_{*j}}{k_{\parallel}} q_j A_{\parallel} \quad (2.21)$$

where $\eta_j = d \ln T_j / d \ln n_j$

Although the above expression for $\mathbf{q}$ has been derived by assuming domination of collisions along $B(\lambda \gg l_f)$ the equation of state (2.21) can also be used to reproduce the electron density response in the limit $\omega \ll k_{\parallel} v_{\parallel}$ obtained from the Vlasov equation. The reason for this is that it arises as a limiting case that does not depend on the explicit form of $\kappa_{\parallel}$.

With regard to the cancellation of the diamagnetic drifts this effect is very important for vortex modes since typically the perturbed part of $\mathbf{v}_*$ is of the same order as $\mathbf{v}_E$. The application of (2.16) for such modes thus depends strongly on this cancellation and the relevant convective velocity in d/dt is the guiding centre part of the fluid velocity.

2.5 Gyrofluid Theory as Obtained by Taking Moments of the Gyrokinetic Equation

We will now consider equations obtained by taking moments of (2.8a). These are in principle equivalent to fluid equations. Finite Larmor Radius (FLR) effects are included to all orders in gyrofluid equations already at taking the moments while FLR effects in fluid equations have to be obtained by extensive work with convective diamagnetic and stress tensor effects. We refer the reader to Ref [25] in order to see how FLR effects are included in gyrofluid theory. An important difference between gyrofluid and fluid equations is that gyrofluid equations do not contain the pressure term perpendicular to the magnetic field. This simplifies a lot although as mentioned above, taking the moments of the gyrokinetic equation, involving magnetic drifts and Bessel functions is more complicated in itself.

Averaging the magnetic drift (2.8b) over a Maxwellian velocity distribution we get:

$$\mathbf{v}_D = \mathbf{v}_{\nabla B} + \mathbf{v}_K \quad (2.22a)$$

where

$$\mathbf{v}_{\nabla \mathbf{B}} = \frac{T}{m\Omega_c} (\hat{\mathbf{e}}_{\parallel} \times \nabla \ln B) \quad (2.22b)$$

$$\mathbf{v}_{\kappa} = \frac{T}{m\Omega_c} (\hat{\mathbf{e}}_{\parallel} \times \kappa) \quad (2.22c)$$

The main particle drifts in gyrofluid theory are the ExB drift, $\mathbf{v}_{\mathbf{E}}$, the polarization drift, $\mathbf{v}_{\mathbf{p}}$ and the magnetic drift $\mathbf{v}_{\mathbf{D}}$. Gyrofluid theory is a theory for the motion of guiding centres so diamagnetic or stress tensor drifts are not present. While the perpendicular motion is pretty much given by the drifts just mentioned (the Coriolis drift is added in combination with a toroidal flow) the parallel motion (without flow) is given by (2.23) [25]. This equation is interesting since the parallel motion should be the same for guiding centres and ordinary fluid while fluid equations do not have a convective magnetic drift.

$$\frac{\partial \delta u_{\parallel}}{\partial t} + 2\mathbf{v}_{\mathbf{D}} \cdot \nabla \delta u_{\parallel} = -\hat{\mathbf{e}}_{\parallel} \cdot \nabla (\delta p + en\phi) \quad (2.23)$$

2.6 One Fluid Equations

A characteristic property of the low frequency expansion of the two fluid equations (2.11a–h) is that the dominant guiding centre drift, the $\mathbf{E} \times \mathbf{B}$ drift, is the same for electrons and ions. Thus in some sense we expect the plasma to move as one fluid. Now we know that this can only be an approximation since the drift velocities due to pressure gradients are different for electrons and ions. However, for the strong, global, Magnetohydrodynamic instabilities, the instability is much faster than the drift frequencies introduced by the density and temperature gradients. In this limit it can be useful to introduce one fluid equations. These are derived by adding or subtracting the equations for electrons and ions after multiplication by the respective masses. Of course, this is a formal procedure that can be used to introduce also the individual drift motions of ions and electrons. Then, however, the equations are no longer one fluid equations. The basic one fluid equations are:

$$\rho \frac{d\mathbf{v}}{dt} = \mathbf{J} \times \mathbf{B} - \nabla P \quad (2.24a)$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J} \quad (2.24b)$$

$$\frac{d}{dt}(Pn^{\gamma}) = 0 \quad (2.24c)$$

Here we used the convective derivative $d/dt = \partial/\partial t + \mathbf{v} \cdot \text{grad}$, ρ is the mass density, η is the conductivity and γ is the adiabaticity index usually taken as 5/3. Equation 2.24a is the equation of motion, (2.24b) is usually called Ohms law and (2.24c) is the equation of state.

Here (2.24a) retains both ion and electron inertia although electron inertia can almost always be ignored. Ion inertia corresponds to including the ion polarization drift in the two fluid equations. The one fluid equations have been used extensively in order to determine MHD stability of various magnetic configurations. In particular an energy principle method was introduced which was used for pioneering work in the beginning of plasma fusion research.

In the present book we will consider both the global MHD instabilities and microinstabilities important for transport. Since two fluid, or kinetic descriptions, will be needed for microinstabilities, it will thus be more convenient to use a two fluid approach in order to obtain a unified description.

2.7 Finite Larmor Radius Effects in a Fluid Description

Up to now we have neglected diamagnetic contributions to the polarization drift and the stress tensor drift. As it turns out these are related to finite Larmor radius (FLR) effects. We shall show here how the lowest order FLR effects can be obtained by a systematic inclusion of these terms.

We will initially for simplicity neglect temperature gradients and temperature perturbations. This leads to the relation

$$\nabla \cdot \mathbf{v}_* = \frac{T}{qB} \nabla \cdot (\hat{\mathbf{z}} \times \nabla n/n) = 0 \quad (2.25)$$

Since also $\nabla \cdot \mathbf{v}_0 = 0$ we can to leading order use the incompressibility condition $\nabla \cdot \mathbf{v}_0 = 0$ when substituting drifts into $\mathbf{v}_p$ and $\mathbf{v}_\pi$. We will also assume large mode numbers, i.e. $k \gg \kappa = d \ln n_0 / dx$ and $d\kappa/dx = 0$.

From the stress tensor as given by Braginskii we can obtain effects of viscosity related to friction between particles and collisionless gyroviscosity, which is a pure FLR effect.

The relevant gyroviscous components are:

$$\pi_{xy} = \pi_{yx} = \frac{nT}{2\Omega_c} \left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} \right) + \frac{1}{4\Omega_c} \left(\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} \right) \quad (2.26a)$$

$$\pi_{yy} = -\pi_{xx} = \frac{nT}{2\Omega_c} \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) + \frac{1}{4\Omega_c} \left(\frac{\partial q_x}{\partial y} + \frac{\partial q_y}{\partial x} \right) \quad (2.26b)$$

Here $\mathbf{q}$ is determined by the fluid truncation and will include higher order FLR effects. We note, however, that the part of $\mathbf{q}_*$ corresponding to a flux of perpendicular energy is (compare Eq. (6.30))

$$\mathbf{q}_*^\perp = 2 \frac{P_j}{m_j \Omega_{cj}} (\mathbf{e}_{||} x \nabla T_\perp)$$

We will start by including only density background gradient

$$\begin{aligned} \nabla \cdot (\pi)_x &= \frac{\partial \pi_{xx}}{\partial x} + \frac{\partial \pi_{xy}}{\partial y} = -\frac{nT}{2\Omega_c} \Delta v_y - \frac{T}{2\Omega_c} \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \frac{dn}{dx} \\ \nabla \cdot (\pi)_y &= \frac{\partial \pi_{yx}}{\partial x} + \frac{\partial \pi_{yy}}{\partial y} = \frac{nT}{2\Omega_c} \Delta v_x + \frac{T}{2\Omega_c} \left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} \right) \frac{dn}{dx} \end{aligned}$$

These equations can be written in a more compact form as:

$$\nabla \cdot (\pi) = \frac{nT}{2\Omega_c} \left[\hat{\mathbf{z}} \times \Delta_\perp \mathbf{v} + \kappa (\nabla v_y - \hat{\mathbf{z}} \times v_x) \right]$$

We now obtain:

$$\mathbf{v}_\pi = \frac{1}{enB} \frac{nT}{2\Omega_c} \hat{\mathbf{z}} \times \nabla \cdot \pi = -\frac{1}{4} \rho^2 \Delta_\perp \mathbf{v} + \frac{1}{4} \rho^2 \kappa (\hat{\mathbf{z}} \times v_y + \nabla v_x)$$

Here ρ is the gyroradius of a general species. Since we are usually interested in substituting our drifts into the equation $\mathbf{div} \mathbf{j} = 0$ we need to calculate expressions of the form $\mathbf{div}(n\mathbf{v})$. We then find, including only linear terms in κ ,

$$\begin{aligned} \nabla \cdot (n\mathbf{v}_\pi) &= \mathbf{v}_\pi \cdot \nabla n_0 + n_0 \nabla \cdot \mathbf{v}_\pi \\ &= -\frac{1}{4} \rho^2 \nabla n_0 \cdot \Delta_\perp \mathbf{v} - \frac{1}{4} \rho^2 n_0 \Delta_\perp \nabla \cdot \mathbf{v} + \frac{1}{4} \rho^2 \kappa n_0 \Delta_\perp v_x \end{aligned}$$

Now assuming $\mathbf{div} \mathbf{v} = 0$ We obtain:

$$\nabla \cdot (n\mathbf{v}_\pi) = -\frac{1}{2} \rho^2 \nabla n_0 \cdot \Delta \mathbf{v} \quad (2.27)$$

The polarization drift can be written in the form:

$$\mathbf{v}_p = \frac{1}{\Omega_c} \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) (\hat{\mathbf{z}} \times \mathbf{v})$$

We start by observing that due to our large mode number approximation only perturbed drifts will enter in the last $\mathbf{v}$. Then in the linear approximation the $\mathbf{v}$ term in the convective derivative can only be a background $\mathbf{v}$. The only background $\mathbf{v}$

that we are interested in here is the diamagnetic drift. We will then start by considering the contribution from this term to $\mathbf{div}(\mathbf{n}\mathbf{v}_p)$. It is:

$$\begin{aligned} \frac{n}{\Omega_c}(\mathbf{v}_* \cdot \nabla) \nabla \cdot (\hat{\mathbf{z}} \times \mathbf{v}) &= \frac{n}{\Omega_c}(\mathbf{v}_* \cdot \nabla) \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) \\ &= -\frac{1}{2} \kappa n \rho^2 \frac{\partial}{\partial y} \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) \end{aligned} \quad (2.28a)$$

Now adding (2.27) and (2.28a) we find (with $\nabla n_0 = -\kappa n_0 \hat{\mathbf{x}}$):

$$\begin{aligned} \nabla \cdot (\mathbf{n}\mathbf{v}_\pi) + \nabla \cdot \left(\frac{n}{\Omega_c}(\mathbf{v}_* \cdot \nabla) \nabla \cdot (\hat{\mathbf{z}} \times \mathbf{v}) \right) \\ = \frac{1}{2} \rho^2 \kappa n \Delta v_x - \frac{1}{2} \kappa n \rho^2 \frac{\partial}{\partial y} \left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \right) = \frac{1}{2} \rho^2 \kappa n \frac{\partial}{\partial x} \nabla \cdot \mathbf{v} = 0 \end{aligned} \quad (2.28b)$$

We thus find that *convective diamagnetic contributions to $\mathbf{div}(\mathbf{n}\mathbf{v}_p)$ are exactly cancelled by the stress tensor contribution $\mathbf{div}(\mathbf{n}\mathbf{v}_\pi)$.*

This result can easily be understood from a physical point of view since the diamagnetic drift is not a particle drift and cannot transfer information by convection. *We now have the general result:*

$$\nabla \cdot [n(\mathbf{v}_p + \mathbf{v}_\pi)] = \nabla \cdot \left[\frac{n}{\Omega_c} \frac{\partial}{\partial t} (\hat{\mathbf{z}} \times \mathbf{v}) \right] \quad (2.29)$$

It is also interesting to compare (2.29) with (2.12). In order to obtain a result corresponding to (2.12) for the gyroviscous part of the stress tensor drift $\mathbf{v}_\pi$ it is necessary to add the convective diamagnetic parts of the polarization drift which are of the same order in $k^2 \rho^2$. We may thus consider (2.29) to express the same kind of physics as (2.12) but for drifts that are first order in the FLR parameter $k^2 \rho^2$. Since (2.12) is no longer true in the presence of curvature (compare 6.23) the same is expected for (2.29).

The leading order linear contributions to (2.29) are now:

$$\begin{aligned} \frac{n}{\Omega_{ci}} \frac{\partial}{\partial t} \nabla \cdot (\hat{\mathbf{z}} \times \mathbf{v}_E) &= -\frac{1}{2} n \rho_i^2 \frac{\partial}{\partial t} \Delta \frac{e\phi}{T_i} \\ \frac{n}{\Omega_{ci}} \frac{\partial}{\partial t} \nabla \cdot (\hat{\mathbf{z}} \times \mathbf{v}_{*i}) &= -\frac{1}{2} n \rho_i^2 \frac{\partial}{\partial t} \Delta \delta n \end{aligned}$$

Here only the perturbation in density contributes to the last term. We now have to specialize further to a particular density response. For flute modes, which are of particular interest in this context, the simplest leading order density perturbation is the $\mathbf{E} \times \mathbf{B}$ convective, i.e.

$$\frac{\delta n}{n} = \frac{\omega_{*e}}{\omega} \frac{e\phi}{T_e} \quad (2.30)$$

We then obtain in $(\omega, \mathbf{k})$ space:

$$\nabla \cdot [n(\mathbf{v}_p + \mathbf{v}_\pi)] \approx -\frac{i}{2}nk^2\rho_i^2(\tau\omega + \omega_{*e})\frac{e\phi}{T_e} = -ink^2\rho^2(\omega - \omega_{*i})\frac{e\phi}{T_e} \quad (2.31)$$

The result (2.31) is in agreement with kinetic theory (compare Chap. 4). The FLR effect enters as a convective contribution to the polarization drift but is in fact due to the time variation of the perturbed diamagnetic drift.

2.7.1 Effects of Temperature Gradients

The main source of modification in the presence of temperature gradients is a compressibility of $\mathbf{v}_*$. Thus (2.24) is changed into:

$$\nabla \cdot \mathbf{v}_* = \frac{en}{qB} \nabla \cdot \left[\frac{1}{n} (\hat{\mathbf{z}} \times \nabla T) \right] \quad (2.32)$$

Since (2.25) contains n only in the combination $\mathbf{P} = n\mathbf{T}$, (2.26) remains unchanged if we change the definition of κ into $\kappa_p = -(1/P_0)dP_0/dx$. We then have:

$$\begin{aligned} \nabla \cdot (n\mathbf{v}_\pi) = & -\frac{1}{4}\rho^2\nabla n_0 \cdot \Delta_\perp \mathbf{v} - \frac{1}{4}\rho^2 n_0 \Delta_\perp \nabla \cdot \mathbf{v} + \frac{1}{4}\rho^2 \kappa n_0 \Delta_\perp v_x \\ & - \frac{1}{4}\rho^2 n_0 \frac{\nabla T}{T} \cdot \Delta_\perp \mathbf{v} - \frac{1}{4m\Omega_c^2} \Delta_\perp \nabla \cdot \mathbf{q}_*^\perp \end{aligned} \quad (2.33)$$

where the last term is due to the $\mathbf{q}$ parts of (2.25a,b). As it turns out it cancels the **div** $\mathbf{v}_*$ term. Thus (2.27) is changed into:

$$\nabla \cdot (n\mathbf{v}_\pi) = -\frac{1}{2}\rho^2 \frac{1}{T} \nabla P_0 \cdot \Delta \mathbf{v} \quad (2.34)$$

Since, in the presence of temperature gradients, $\mathbf{v}_*$ in the convective derivative of the polarization drift contains the full pressure gradient we now find that (2.28b) is unchanged (with our new definition of κ) and so is the conclusion in *italics* following it and (2.29). Since background pressure gradients in a natural way lead to convective pressure perturbations we now must write:

$$\frac{n}{\Omega_{ci}} \frac{\partial}{\partial t} \nabla \cdot (\hat{\mathbf{z}} \times \mathbf{v}_*) = -\frac{1}{2} \frac{\rho_i^2}{T} \frac{\partial}{\partial t} \Delta \delta P \quad (2.35)$$

For the convective pressure perturbation we have:

$$\frac{\delta P_i}{n} = -\frac{\omega_{*iT}}{\omega} \frac{e\phi}{T_i} \quad (2.36)$$

where ω_{*iT} is the diamagnetic drift frequency of ions due to the full background pressure gradient. Accordingly, (2.31) becomes:

$$\nabla \cdot [n(\mathbf{v}_p + \mathbf{v}_\pi)] = -ink^2 \rho^2 (\omega - \omega_{*iT}) \frac{e\phi}{T_e} \quad (2.37)$$

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