Chapter 2
Statics

2.1 A Concept of Equilibrium

If the velocity and acceleration of every particle in a material system are equal to zero, such a system is at rest. On the other hand, as we recall, from Newton's first law it follows that a particle having mass to which no force is applied or the applied forces are balanced is either at rest or in uniform motion along a straight line.

If a material system acted on by a force system does not change its position during an arbitrarily (infinitely) long time, we say that it is in equilibrium under the action of that force system.

Statics is the branch of mechanics that focuses on the equilibrium of material bodies (particles) under the action of forces (moments of force). It deals with the analysis of forces acting on material systems at rest or moving in uniform motion along a straight line and, as will become clear later, may be treated as a special case of *dynamics*. This chapter, devoted to statics, might be extended by material presented in books devoted to classical mechanics like, for instance, [1–25].

It turns out that for a material system to remain in equilibrium under the action of a certain force system, that force system must satisfy the so-called *equilibrium conditions*.

In order to emphasize the lack of time influence on equilibrium conditions, in statics the term *state of static equilibrium* is often used. In general, statics can be divided into *elementary statics* and *analytical statics*. In the case of elementary statics, during analysis of static equilibrium states vector algebra and graphical methods are applied. On the other hand, in the case of *analytical statics* the concepts of *virtual displacement* and *principle of virtual work* are used, which in part was discussed in Chap. 1.

Let us assume that the nth particle is subjected to K forces $\mathbf{F}_k$, L reactions $\mathbf{F}_l^R$, and M internal forces $\mathbf{F}_m^i$ (Fig. 2.1). The equilibrium condition has the following form:

$$\mathbf{F}_n + \mathbf{F}_n^R + \mathbf{F}_n^i = \mathbf{0}, \tag{2.1}$$

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where

$$\mathbf{F}_n = \sum_{k=1}^{K} \mathbf{F}_k, \quad \mathbf{F}_n^R = \sum_{l=1}^{L} \mathbf{F}_l^R, \quad \mathbf{F}_n^i = \sum_{m=1}^{M} \mathbf{F}_m^i. \tag{2.2}$$

After multiplying both sides of (2.1) by the unit vectors $\mathbf{E}_j$ of the coordinate system $OX_1X_2X_3$ (scalar product) and taking into account (2.2), we obtain the so-called *analytical conditions of equilibrium* of the form

$$\sum_{k=1}^{K} F_{kx_j} + \sum_{l=1}^{L} F_{lx_j}^R + \sum_{m=1}^{M} F_{mx_j}^i = 0, \quad j = 1, 2, 3. \tag{2.3}$$

Because in general the forces may be projected onto the axes of any curvilinear coordinate system, the following theorem is valid.

Theorem 2.1. *A particle is in equilibrium if the sum of projections of external, internal, and reaction forces (acting on this particle) onto axes of the adopted coordinate system is equal to zero.*

The three conditions (2.3) are necessary, but not sufficient, for the particle to remain at rest, as follows from Newton's first law.

In order to formulate the equilibrium conditions for a whole material system or a body (infinite number of particles), one should formulate such equations for every particle $n \in [1, N]$ ($N = \infty$ in the case of the body) and add them together, obtaining

$$\mathbf{S} = \sum_{n=1}^{N} (\mathbf{F}_n + \mathbf{F}_n^R + \mathbf{F}_n^i) = \mathbf{0}, \tag{2.4}$$

where N is the number of particles of the material system.

A system of particles will be in equilibrium if the sum of projections of external, internal, and reaction forces, acting on every particle of the system, onto three axes of the adopted coordinate system is equal to zero. After projection we obtain $3N$ analytical equilibrium conditions for the system of N particles.

According to Newton's third law, the internal forces are the effect of action and reaction and they are *pairs of opposite forces*, which means that they cancel out one another, that is, $\sum_{n=1}^{N} \mathbf{F}_n^i = \mathbf{0}$.

Observe that the result of action of a given force on a rigid body remains unchanged when that force is applied at any point of its line of action (the so-called *principle of transmissibility*), and hence the forces acting on a rigid body can be represented by *sliding vectors*. In other words, if instead of a force $\mathbf{F}$ at a given point of the rigid body we apply a force $\mathbf{F}'$ of the same magnitude and direction at a point A' ($A \neq A'$), then the equilibrium state, or motion of a rigid body, is not affected provided that the two forces have the same line of action. The principle of transmissibility is based on experimental evidence, and the mentioned forces $\mathbf{F}$ and $\mathbf{F}'$ are called *equivalent*.

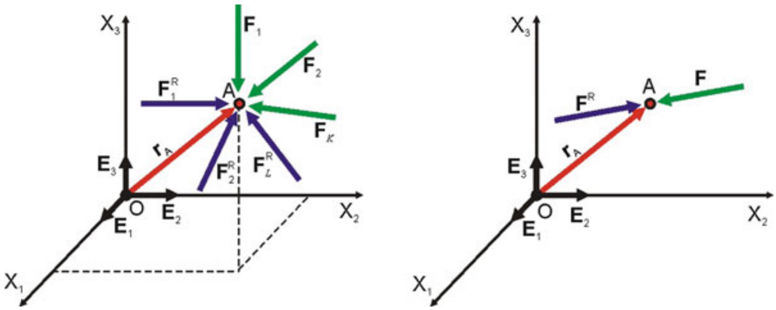


Fig. 2.1 A particle n (point A) under the action of forces $\mathbf{F}_k$ and reactions $\mathbf{F}_l^R$, and the same particle under the action of the resultant forces $\mathbf{F}$ and reactions $\mathbf{F}^R$ (internal forces not shown)

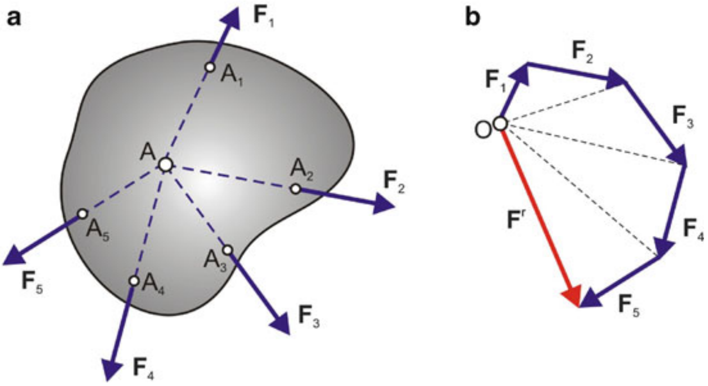


Fig. 2.2 Concurrent forces acting on a rigid body (**a**) and a force polygon (**b**)

In Fig. 2.1 it was shown that all forces are applied at a single point A. Now we will consider a more general case of a rigid body loaded with a discrete system of forces applied at points A_n of magnitudes F_n, $n = 1, \ldots, N$ and of lines of action passing through a certain point A (Fig. 2.2).

The force system that we call *concurrent* and those forces need not lie in a common plane. Because, by assumption, the lines of action of the forces intersect at point A, the system in Fig. 2.2a is equivalent to the force system depicted in Fig. 2.2b. Adding successively the force vectors and using the "triangle rule," that is, replacing every two forces by their resultant force (marked by a dotted line) we obtain a so-called *force polygon*. The action of the resultant vector $\mathbf{F}^r$ is equivalent to the simultaneous action of all forces, that is,

$$\mathbf{F}^r = \sum_{i=n}^{N} \mathbf{F}_n. \tag{2.5}$$

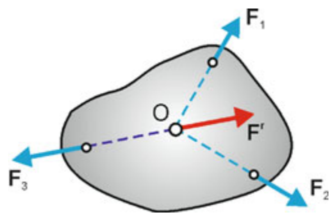


Fig. 2.3 Geometrical interpretation of the three-forces theorem

The method of construction of a force polygon indicates that in order to obtain vector $\mathbf{F}^r$ we can add the vectors $\mathbf{F}_n$ together directly (i.e., the vectors denoted by dotted lines can be omitted during addition).

The vectors $\mathbf{F}_n$ are called *sides* of a force polygon and the vector $\mathbf{F}^r$ is called a *closing vector* of a force polygon. Moreover, the sign of the sum (sigma) denotes addition of vectors from left to right, that is, from $\mathbf{F}_1$, $\mathbf{F}_2, \ldots$ to $\mathbf{F}_N$.

The presented construction is easy in the case of a planar force system. In space, the force polygon is a broken line whose sides are the force vectors and the resultant $\mathbf{F}^r$ connects the tail of the first force vector to the tip of the last force vector of the given force system. In this section we will take up the analysis of the planar force polygon.

From the foregoing considerations it follows directly that because the action of many forces can be replaced by the action of one resultant force, a rigid body remains in equilibrium under the action of a concurrent force system if $\mathbf{F}^r = \mathbf{0}$, that is, according to (2.5) if

$$\sum_{n=1}^{N} \mathbf{F}_n = \mathbf{0}. \tag{2.6}$$

If the above equation is satisfied, the polygon of forces $\mathbf{F}_n$ is closed, that is, the tail of the first force vector coincides with the tip of the last force vector.

Theorem 2.2. *(On three forces) If a body remains in equilibrium under the action of only three non-parallel coplanar forces, then their lines of action must intersect at a single point, that is, the system of forces must be concurrent.*

To prove the above theorem it is enough to observe that if we have three forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, then the action of any two of them, e.g., $\mathbf{F}_1$ and $\mathbf{F}_2$, can be replaced with their resultant $\mathbf{F}^r$. According to Newton's first law, the body is in equilibrium if $\mathbf{F}^r = -\mathbf{F}_3$ and these vectors are collinear. Therefore, the lines of action of the three forces must intersect at a single point (Fig. 2.3).

Let us note that if we are dealing with a system of three concurrent forces and a body under the action of these forces is in equilibrium, all of these forces must be coplanar (i.e., lie in one plane).

Later we will show (by making use of Theorem 2.2) how to reduce an arbitrary three-dimensional system of non-concurrent forces to a so-called *equivalent system of two skew forces*.

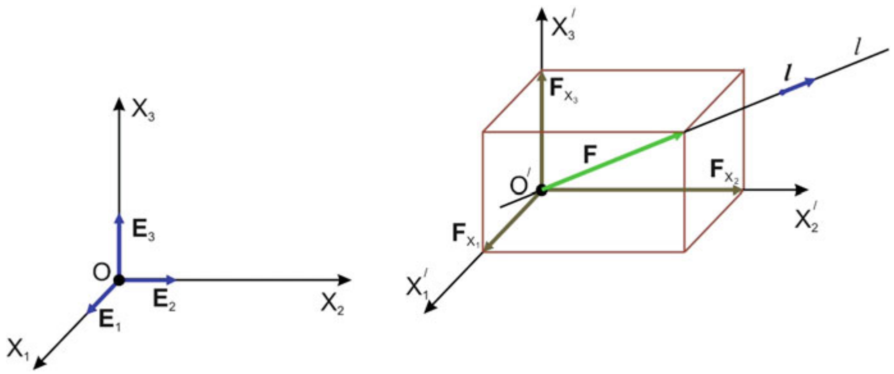


Fig. 2.4 Coordinate systems $OX_1X_2X_3$ and $O'X_1'X_2'X_3'$ and the force vector **F**

In practice many problems of statics boil down to the construction of the force polygon. The approach is like we call a *geometrical* approach and is complemented by the use of trigonometric functions to find unknown quantities. In order to use this method the magnitudes, the lines of action, and the senses of forces must be known.

Apart from the geometrical, the *analytical* approach is commonly applied. Let us introduce the fixed Cartesian coordinate system $OX_1X_2X_3$. In order to analytically describe a force **F** one should know its point of application $O'(x_1, x_2, x_3)$ and magnitudes of its projections onto axes OX_1, OX_2, and OX_3 of the form F_{x_1}, F_{x_2}, and F_{x_3}.

In Fig. 2.4 the location of point O' in the coordinate system $O'X_1'X_2'X_3'$ and the vectors $\mathbf{F}$, $\mathbf{F}_{X_i}$ $(i = 1, 2, 3)$, are shown.

Let the vector of force **F** lie on the l axis of unit vector $\boldsymbol{l}$, and let $\mathbf{E}_i$ and $\mathbf{E'}_i$ be unit vectors respectively of the axes OX_i and $O'X_i'$, and because the axes of the systems $OX_1X_2X_3$ and $O'X_1'X_2'X_3'$ are parallel, we have $\mathbf{E}_i \parallel \mathbf{E}_i'$. According to the introduced notation we have

$$\mathbf{F} \equiv \boldsymbol{l}F = F_{x_1}\mathbf{E}_1 + F_{x_2}\mathbf{E}_2 + F_{x_3}\mathbf{E}_3 = \mathbf{F}_{x_1} + \mathbf{F}_{x_2} + \mathbf{F}_{x_3}. \tag{2.7}$$

In other words, a force **F** is said to have been resolved into three rectangular components $\mathbf{F}_{x_i}$, $i = 1, 2, 3$ if they are perpendicular to each other and directed along the coordinate axes.

After multiplication (scalar product) of the above equation in turn by $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ we obtain

$$\begin{aligned}
F_{x_1} &= F\boldsymbol{l} \circ \mathbf{E}_1 = F\cos(\boldsymbol{l}, \mathbf{E}_1),\\
F_{x_2} &= F\boldsymbol{l} \circ \mathbf{E}_2 = F\cos(\boldsymbol{l}, \mathbf{E}_2),\\
F_{x_3} &= F\boldsymbol{l} \circ \mathbf{E}_3 = F\cos(\boldsymbol{l}, \mathbf{E}_3),
\end{aligned} \tag{2.8}$$

because

$$|\boldsymbol{l}| = |\mathbf{E}_1| = |\mathbf{E}_2| = |\mathbf{E}_3| = 1.$$

If we know the vector $\mathbf{F}$, that is, its magnitude F and direction defined by the unit vector $\boldsymbol{l}$, the coordinates of the components of the vector are described by (2.8).

After squaring (2.7) by sides we obtain

$$F = \sqrt{F_{x_1}^2 + F_{x_2}^2 + F_{x_3}^2}, \tag{2.9}$$

and after taking into account (2.9) in (2.8) we calculate the cosines of the angles formed by the force vector with the axes of the coordinate system (called direction cosines)

$$\begin{aligned}
\cos(\boldsymbol{l}, \mathbf{E}_1) &= \frac{F_{x_1}}{F} = \frac{F_{x_1}}{\sqrt{F_{x_1}^2 + F_{x_2}^2 + F_{x_3}^2}}, \\
\cos(\boldsymbol{l}, \mathbf{E}_2) &= \frac{F_{x_2}}{F} = \frac{F_{x_2}}{\sqrt{F_{x_1}^2 + F_{x_2}^2 + F_{x_3}^2}}, \\
\cos(\boldsymbol{l}, \mathbf{E}_3) &= \frac{F_{x_3}}{F} = \frac{F_{x_3}}{\sqrt{F_{x_1}^2 + F_{x_2}^2 + F_{x_3}^2}}.
\end{aligned} \tag{2.10}$$

If F_{x_1}, F_{x_2} and F_{x_3} are known, then on the basis of (2.9) and (2.10) we can determine vector $\mathbf{F}$, that is, its magnitude $|\mathbf{F}| = F$ and its direction defined by the direction cosines. It follows directly from (2.10) that $\cos^2(\boldsymbol{l}, \mathbf{E}_1) + \cos^2(\boldsymbol{l}, \mathbf{E}_2) + \cos^2(\boldsymbol{l}, \mathbf{E}_3) = 1$, and hence angles describing a position of the force $\mathbf{F}$ in relation to the Cartesian axes depend on each other.

Let us now return to Fig. 2.2, where the system of concurrent forces acts solely on the rigid body. If such a body is in equilibrium, according to (2.5), we have $\mathbf{F}^r = \mathbf{0}$, and after multiplying (2.5) by $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ we obtain

$$\begin{aligned}
F_{1x_1} + F_{2x_1} + \cdots + F_{Nx_1} &= 0, \\
F_{1x_2} + F_{2x_2} + \cdots + F_{Nx_2} &= 0, \\
F_{1x_3} + F_{2x_3} + \cdots + F_{Nx_3} &= 0.
\end{aligned} \tag{2.11}$$

The equilibrium condition of the rigid body subjected to the action of a three-dimensional system of concurrent forces is described by three algebraic (2.11), in view of the fact that the number of unknowns should not exceed three (or, in the case of the planar system of forces, two) for the considered force system to be *statically determinate*. In Fig. 2.5 a symmetrical system of concurrent forces formed after suspending the body of weight $\mathbf{G}$ from three ropes is shown.

After introducing the Cartesian coordinate system and including the system geometry we can (through some trigonometric relations) determine three unknowns

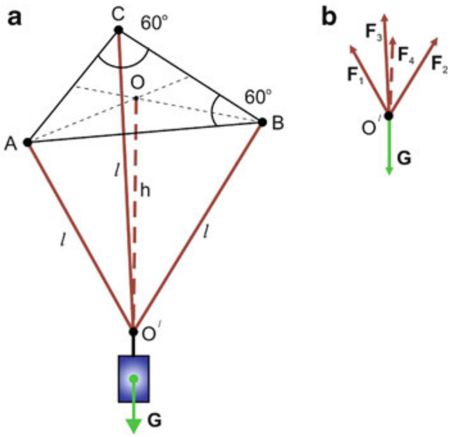


Fig. 2.5 A weight **G** suspended from three (four) ropes in space (**a**) and a system of concurrent forces (**b**)

from three equations and, consequently, describe the forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$. Let us assume now that at the center O of the triangle ABC an extra fourth rope was attached. If the length of that rope differs even slightly from the distance h, either that rope exclusively carries the total weight **G** (if slightly shorter) or it carries no load at all (if slightly longer).

If we assume that the weight **G** is carried by four ropes, it is impossible to determine the four forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ in the ropes; such a problem is *statically indeterminate*. This appears also in the planar system of concurrent forces if point C becomes coincident with point B and then $AO = OB$. The problem is *statically determinate* if the weight **G** is suspended from two ropes AO' and BO' and statically indeterminate if, additionally, we introduce the third rope OO' and all three ropes are loaded.

One deals with a statically indeterminate problem when the number of unknown values of forces (torques) denoted by N is larger than the number of equilibrium equations N_r. The difference $N - N_r$ is called a *degree of static indeterminacy*. As will be shown later, additional equilibrium equations are obtained after taking into account deformations of the investigated mechanical system. In general, while solving a statically indeterminate problem, the *method of forces* and the *method of displacements* are often applied.

The first approach consists of three stages [26]:

1. Determination of degree of static indeterminacy.
2. Transformation of a statically indeterminate system to a statically determinate one with unknown values of loads but known character and point of application.
3. Determination of a set of desired force values from condition of displacement continuity at the force application points.

The second method, i.e., the *displacements method*, is to use the relationship between external forces, displacement nodes of the construction, displacements of the ends of particular ropes (rods) and their geometric and material properties.

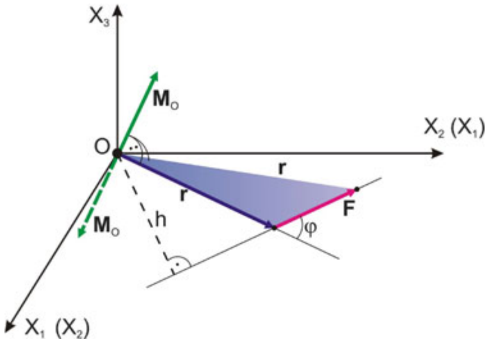


Fig. 2.6 Graphical representation of a moment of force **F** about a point O

It takes advantage of the emergence of movements of rod ends in a strict dependence resulting from the continuity of the structure.

The displacement method will be described in Example 2.5.

The static indeterminacy of a mechanical system is often due to the introduction of the so-called technologically justified assembling stress (e.g., stresses resulting from the initial stretch of the ropes supporting the structure). In addition, the state of stress may appear as a result of non-uniform heating of the system. The loading state of a mechanical system following from the introduction of assembly and thermal stresses is an independent and additional loading of the considered mechanical system.

In a general case a non-concurrent force system may act on a rigid body. In order to describe the equilibrium conditions in this case it is necessary to introduce the notions of moment of force about a point and moment of force about an axis (which was discussed in Sect. 1.2).

The notion of moment of force about a point was first used by Archimedes, but it concerned planar systems.

The moment of force **F** about point O is defined by a vector product of the form

$$\mathbf{M}_O \equiv \mathbf{M}_O(\mathbf{F}) = \mathbf{r} \times \mathbf{F}, \tag{2.12}$$

which after introduction of the Cartesian coordinate system $OX_1X_2X_3$ is illustrated in Fig. 2.6.

According to the definition of vector product [see (1.13), (1.14)] and Fig. 2.6, we have

$$\mid \mathbf{M}_O \mid \equiv M_O = rF \sin\varphi = Fh, \tag{2.13}$$

that is, the magnitude of the moment of force is equal to the doubled area of a triangle (in blue) of sides r and F. Its direction is given by the right-hand rule, that is, the arrow of the vector $\mathbf{M}_O$ points toward an eye of a person looking if the vector $\mathbf{r}$ is rotated toward the vector $\mathbf{F}$ counterclockwise (positive sense) provided that the coordinate system is right-handed. However, if the system of coordinates is left-handed (its axes in parentheses in Fig. 2.6), the sense of the vector $\mathbf{M}_O$ changes (vector marked by a dashed line in Fig. 2.6).

Let us note that vectors describing a real physical quantity (e.g., force, velocity, acceleration) do not change when the adopted coordinate system is changed. In the considered case the vector of the moment of force $\mathbf{M}_O$ changes when the coordinate system is changed from right-handed to left-handed. Vectors having such a property are called *pseudovectors*.

Theorem 2.3. *(Varignon*[1]*) The moment of a resultant force of a system of concurrent forces about an arbitrary point O (a pole) is equal to the sum of individual moments of each force of the system about that pole.*

Proof of the above theorem is obvious if one uses the property of distributivity of a vector product with respect to addition.

Let us note that Varignon's theorem includes also the special case of concurrent forces when the point of intersection of their lines of action is situated in infinity (in that case we are dealing with a system of parallel forces). Moreover, let us note that the introduction of any force $\mathbf{F}'$ of the direction along that of $\mathbf{r}$ does not change the moment since we have

$$\mathbf{M}_O = \mathbf{r} \times (\mathbf{F} + \mathbf{F}') = \mathbf{r} \times \mathbf{F} + \mathbf{r} \times \mathbf{F}' = \mathbf{r} \times \mathbf{F}, \tag{2.14}$$

because $\mathbf{r} \times \mathbf{F}' = \mathbf{0}$.

This trivial observation will be exploited later during the reduction of an arbitrary three-dimensional force system to two skew forces in space (forces that do not lie in one plane).

If by a resultant force we understand the force replacing the action of the system of concurrent forces, such a notion loses its meaning in the case of an arbitrary force system in space. Then the closing vector of a three-dimensional polygon of forces is called a *main force vector*.

The components of the moment of force vector about a pole O is obtained directly from the definition using the determinant, that is,

$$\begin{aligned}
\mathbf{M}_O = \mathbf{r} \times \mathbf{F} &= \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ r_{x_1} & r_{x_2} & r_{x_3} \\ F_{x_1} & F_{x_2} & F_{x_3} \end{vmatrix} \\
&= \mathbf{E}_1(r_{x_2}F_{x_3} - r_{x_3}F_{x_2}) + \mathbf{E}_2(-r_{x_1}F_{x_3} + r_{x_3}F_{x_1}) + \mathbf{E}_3(r_{x_1}F_{x_2} - r_{x_2}F_{x_1}) \\
&\equiv M_{Ox_1}\mathbf{E}_1 + M_{Ox_2}\mathbf{E}_2 + M_{Ox_3}\mathbf{E}_3.
\end{aligned} \tag{2.15}$$

Let us introduce now the concept of the moment of force $\mathbf{F}$ with respect to an axis l of unit vector $\boldsymbol{l}$ (Fig. 2.7).

[1]Pierre Varignon (1654–1722), French mathematician (friend of Newton, Leibniz, and the Bernoulli family).

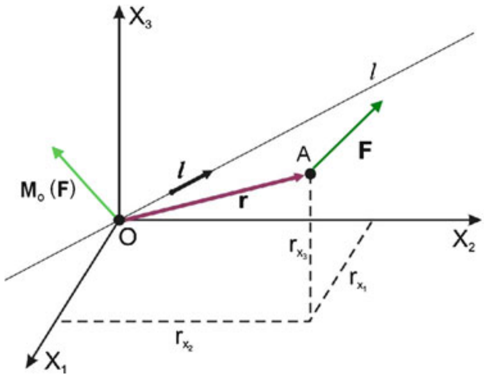


Fig. 2.7 Moment of force **F** about an axis l

Definition 2.1. *The magnitude of the moment of force about an axis l is equal to the scalar product of the moment of force about an arbitrary point O on that axis and a unit vector $\boldsymbol{l}$ of the axis (thus it is a scalar that respects the sign determined by the sense in agreement or opposite to the unit vector $\boldsymbol{l}$).*

According to the above definition, and after taking into account (2.15), we have

$$\begin{aligned}
M_l(\mathbf{F}) &= \boldsymbol{l} \circ \mathbf{M}_0(\mathbf{F}) \\
&= M_{Ox_1}\cos(\boldsymbol{l},\mathbf{E}_1) + M_{Ox_2}\cos(\boldsymbol{l},\mathbf{E}_2) + M_{Ox_3}\cos(\boldsymbol{l},\mathbf{E}_3) \\
&= (r_{x_2}F_{x_3} - r_{x_3}F_{x_2})\cos(\boldsymbol{l},\mathbf{E}_1) + (-r_{x_1}F_{x_3} + r_{x_3}F_{x_1})\cos(\boldsymbol{l},\mathbf{E}_2) \\
&\quad + (r_{x_1}F_{x_2} - r_{x_2}F_{x_1})\cos(\boldsymbol{l},\mathbf{E}_3) \\
&= \begin{vmatrix} \cos(\boldsymbol{l},\mathbf{E}_1) & \cos(\boldsymbol{l},\mathbf{E}_2) & \cos(\boldsymbol{l},\mathbf{E}_3) \\ r_{x_1} & r_{x_2} & r_{x_3} \\ F_{x_1} & F_{x_2} & F_{x_3} \end{vmatrix}.
\end{aligned} \tag{2.16}$$

Observe that $M_l(\mathbf{F})$ is the magnitude of the moment of force about axis l, that is, it depends on the versor $\boldsymbol{l}$ sense.

Let us resolve $\mathbf{F}(\mathbf{r})$ into two rectangular components $_{\|}\mathbf{F}(_{\|}\mathbf{r})$ and $_{\perp}\mathbf{F}(_{\perp}\mathbf{r})$ (lying in a plane perpendicular to $\boldsymbol{l}$). Then from (2.16) we obtain $M_l(\mathbf{F}) = \boldsymbol{l} \circ (_{\|}\mathbf{r} + {}_{\perp}\mathbf{r}) \times (_{\|}\mathbf{F} + {}_{\perp}\mathbf{F}) = \boldsymbol{l} \circ (_{\perp}\mathbf{r} \times {}_{\perp}\mathbf{F})$. In other words the moment $M_l(\mathbf{F})$ describes the tendency of the force $\mathbf{M}$ to give to the rigid body a rotation about the fixed axis l.

At the end of this section we will present definitions of a couple and a moment of a couple and introduce the basic properties of a couple.

Definition 2.2. *A system of two parallel forces with opposite senses and equal values we call a **couple of forces**. A plane in which these forces lie is called a **plane of a couple of forces**.*

In Fig. 2.8 a couple of forces and a moment of a couple of forces are presented graphically.

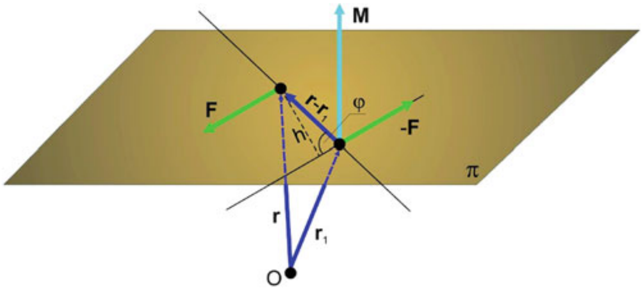


Fig. 2.8 A couple of forces and its moment about a point O

The moment of a couple of forces about a point O is the geometric sum of moments of each of the forces about point O, that is,

$$\begin{aligned}\mathbf{M} &= \mathbf{M}_O(\mathbf{F}) + \mathbf{M}_O(-\mathbf{F}) = \mathbf{r} \times \mathbf{F} + \mathbf{r}_1 \times (-\mathbf{F}) \\ &= \mathbf{r} \times \mathbf{F} - \mathbf{r}_1 \times \mathbf{F} = (\mathbf{r} - \mathbf{r}_1) \times \mathbf{F}. \end{aligned} \tag{2.17}$$

From (2.17) it follows that the moment $\mathbf{M}$ of a couple of forces about point O does not depend on the location of this point, but only on the vector $\mathbf{r}-\mathbf{r}_1$ describing the relative position of points of application of forces forming the couple and lying in a plane of the couple of forces π.

The magnitude of moment of a couple of forces is equal to

$$M = F|\mathbf{r} - \mathbf{r}_1| \sin\varphi = Fh, \tag{2.18}$$

where h is the distance between lines of action of the couple and is called the *arm of a couple of forces*. Thus, the magnitude of a moment of a couple of forces corresponds to the area of a rectangle with sides F and h. It is clear that a couple applied to a body tends to rotate it. Since $\mathbf{r} - \mathbf{r}_1$ is independent of the choice of the origin O, the moment $\mathbf{M}$ of a couple is a *free vector*.

Let us note that $\mathbf{M}$ is perpendicular to the plane π and its sense is determined by senses of vectors of the couple of forces. The static action of a couple is equivalent to the moment of a couple. What follows are some theorems regarding a couple of forces (proofs are left to the reader).

Theorem 2.4. *The action of an arbitrary couple of forces on a rigid body is invariant with respect to the rotation of the plane of the couple through an arbitrary angle.*

Theorem 2.5. *The action of an arbitrary couple of forces on a rigid body is invariant with respect to the choice of an arbitrary plane parallel to the plane of the couple.*

Theorem 2.6. *The action of an arbitrary couple of forces on a rigid body is invariant if the product Fh [see (2.18)] remains unchanged, that is, it is possible to vary the magnitude of force and its arm as long as Fh remains constant.*

Theorem 2.7. *An arbitrary system of couples of forces in R^3 space is statically equivalent to a single couple of forces whose moment is the geometrical (vector) sum of moments coming from each couple of forces in the system.*

The reader is encouraged to prove that two couples possessing the same moment $\mathbf{M}$ are equivalent. Since the couples can be presented by vectors, they can be summed up in a geometrical manner. In addition, any given force $\mathbf{F}$ acting on a rigid body can be resolved into a force at an arbitrary given pole O and a couple, that is, at point O we have an equivalent force–couple $(\mathbf{F} - \mathbf{M}_O)$ system provided that the couple's moment is equal to the moment of $\mathbf{F}$ about O. In what follows we apply the statements and comments introduced thus far.

Let us emphasize that the net result of a couple relies on the production of a moment $\mathbf{M}$ (couple vector). Since $\mathbf{M}$ is independent of the point about which it takes place (free vector), then in practice it should be computed about a most convenient point for analysis. One may add two or more couples in a geometric way. One may also replace a force with (a) an equivalent force couple at a specified point; (b) a single equivalent force provided that $\mathbf{F} \perp \mathbf{M}$, which is satisfied in all two-dimensional problems.

In Fig. 2.1 the position of a particle is described by the vector $\mathbf{r}_n$ in the adopted Cartesian coordinate system. In the case of a system of particles, each particle n of the system is described by a radius vector $\mathbf{r}_n$, $n = 1, \ldots, N$. After multiplying equilibrium condition (2.1) by $\mathbf{r}_n$ (cross product) and adding together the obtained equations (after assuming $\mathbf{F}_n^i = \mathbf{0}$) we obtain

$$\mathbf{M}_O = \sum_{n=1}^{N} (\mathbf{r}_n \times \mathbf{F}_n) + \sum_{n=1}^{N} (\mathbf{r}_n \times \mathbf{F}_n^R) = \mathbf{0}. \tag{2.19}$$

Let us note that (2.19) is also valid for $\mathbf{F}_n^i \neq \mathbf{0}$ because $\sum_{n=1}^{N} (\mathbf{r}_n \times \mathbf{F}_n^i) = \mathbf{0}$ (internal forces of the system exist in pairs that are equal, act along the same line of action, but have opposite senses; therefore, taken together they all yield zero moment about any point).

According to (2.4) and (2.19) a material system is in equilibrium if the system of forces and reactions, and the main moment produced by these forces and reactions, is equal to zero, that is, we have

$$\mathbf{S} = \mathbf{0}, \quad \mathbf{M}_O = \mathbf{0}. \tag{2.20}$$

Let us note that the above condition is a necessary condition, but in general it is not a sufficient one. According to Newton's first law, particles can move along

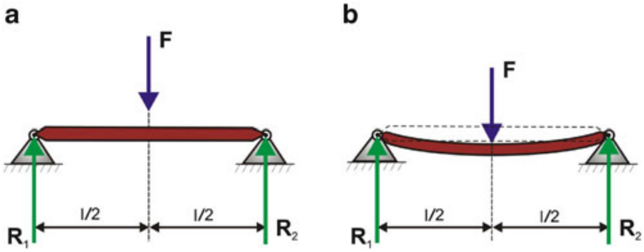


Fig. 2.9 A rigid (**a**) and flexible (**b**) beam with a concentrated force applied in the middle of their lengths

straight lines in uniform motion, that is, they can change their relative position in spite of being subjected to the action of balanced forces. If we are dealing with a rigid material system or a rigid body, then the relative motion of the particles of the body is impossible and conditions (2.20) are both necessary and sufficient conditions.

If, however, we consider a continuous material system (CMS), such as deformable solid bodies, gases, or liquids, then conditions (2.20) are necessary conditions, but not sufficient ones.

In a general case for an arbitrary material system the necessary and sufficient equilibrium conditions are described by the following equations for every particle n:

$$\mathbf{F}_n^e + \mathbf{F}_n^i + \mathbf{F}_n^R = \mathbf{0}, \quad n = 1, 2 \ldots, N. \tag{2.21}$$

Although (2.21) seems to be simpler, in fact it is more complicated, since it involves determination of internal forces $\mathbf{F}_n^i$. In order to do so one should release from constraints the internal points of the material system and consider the equilibrium of external, internal, and reaction forces.

As an example we will consider a beam of length l loaded at the midpoint with the force $\mathbf{F}$ on the assumption that it is a rigid beam (Fig. 2.9a) and a flexible beam (Fig. 2.9b) [9].

The absence of deflection in the first case (Fig. 2.9a) follows from the fact that the beam is a rigid body. The presence of deflection in the second case (Fig. 2.9b) is caused by the beam's flexibility. The conditions of equilibrium are the same in both cases, although the equilibrium positions are different. In both cases internal forces are developed, but only the second case is accompanied by the beam's deformation. Observe that in the case of a rigid body (beam), each of the forces may be shifted in an arbitrary way along its line of action, but this approach is prohibited for a flexible body (beam), that is, the transmissibility principle is violated.

The above example is associated with the so-called *freezing principle*. Equilibrium of forces acting on a flexible body is not violated when the flexible beam becomes "frozen."

2.2 Geometrical Equilibrium Conditions of a Planar Force System

As has already been mentioned, in a general case formulation of geometrical equilibrium conditions is not an easy problem, and usually this approach is applied to systems of forces lying in a plane. As was shown earlier, according to the first equation of (2.20) a polygon constructed from force vectors should be closed.

In the case where a material system is acted upon by a system of three non-parallel forces equivalent to zero, these forces are coplanar and their lines of action intersect at one point, forming a triangle (Theorem 2.2).

In a general case for an arbitrary plane force system geometrical equilibrium conditions boil down to the *force polygon* becoming closed and to satisfying the condition of the so-called *funicular polygon*. As an introduction let us first consider three arbitrary forces $\mathbf{F}_n$, $n = 1, 2, 3$ lying in one plane (Fig. 2.10).

From the construction depicted in Fig. 2.10 it follows that, at first, any two of the three forces (in this case $\mathbf{F}_1$ and $\mathbf{F}_2$) are moved to the point of intersection of their lines of action O' and then added geometrically, yielding the force $\mathbf{F}_1 + \mathbf{F}_2$. One proceeds similarly with the vector $\mathbf{F}_1 + \mathbf{F}_2$ and the vector $\mathbf{F}_3$. The lines of action of these vectors, again, determine the point O to which both vectors are moved. Next, using the parallelogram law we obtain a resultant vector $\mathbf{F}^r$ replacing the action of those forces. We also obtain the line of action of $\mathbf{F}^r$ ($\mathbf{F}^r$ is a *sliding vector*).

A similar procedure may be applied for any number of force vectors acting in a plane (coplanar and non-parallel vectors). However, in practice sometimes it turns out that the task is not easy. One reason for this is that the forces may intersect at the points outside the drawing space. Moreover, their small angles of intersection may lead to large inaccuracies.

Below a different graphical method called *funicular polygon method* will be presented.

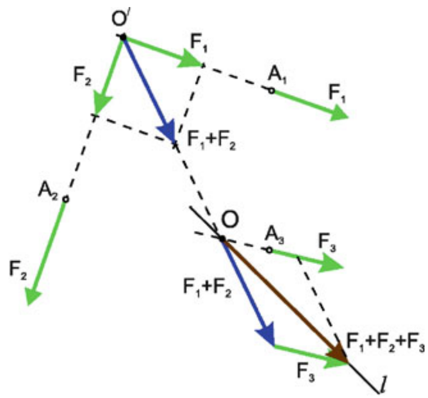


Fig. 2.10 Determination of magnitude, sense, and direction of a resultant $\mathbf{F}^r = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$ for the case of a planar system of three non-parallel forces

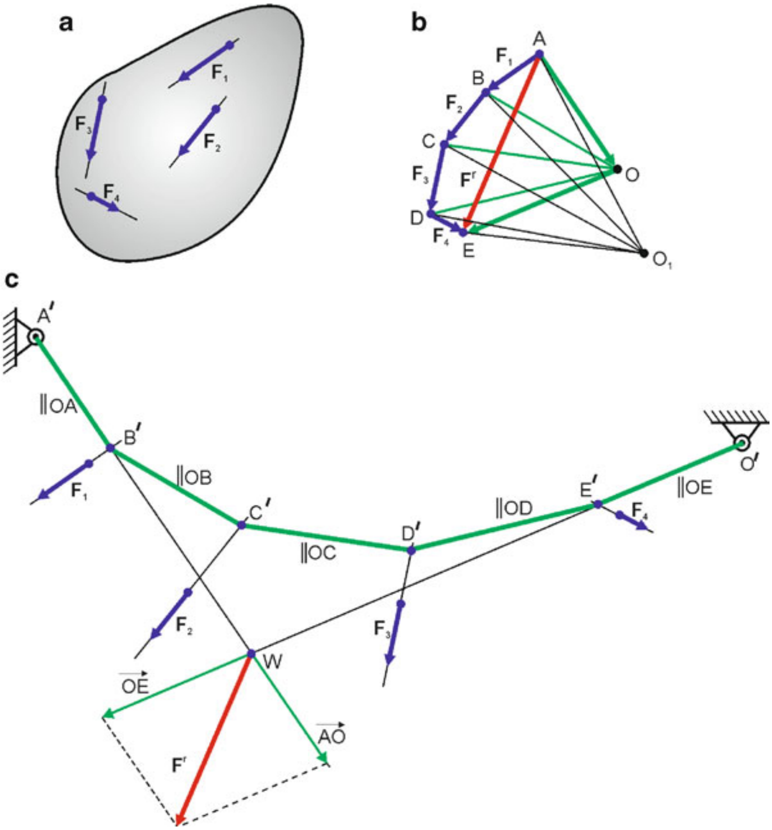


Fig. 2.11 An arbitrary system of four forces (**a**), a force polygon (**b**), and a funicular polygon (**c**)

Let us consider now an arbitrary system of forces acting in a plane on a certain rigid body (Fig. 2.11). In order to preserve clarity of the drawing we limit ourselves to consideration of four forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$.

The procedure of construction of a force polygon is as follows. In a plane we take an arbitrary point A, being the tail of force vector $\mathbf{F}_1 = \overrightarrow{AB}$. Next, at the tip of that vector we attach the vector $\mathbf{F}_2 = \overrightarrow{BC}$, and so on. Next, we take an arbitrary point O called a *pole* and connect it with points A, B, C, D, and E. Let us note that

$$
\begin{aligned}
\mathbf{F}^r &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 \\
&= \left(\overrightarrow{OB} - \overrightarrow{OA}\right) + \left(\overrightarrow{OC} - \overrightarrow{OB}\right) + \left(\overrightarrow{OD} - \overrightarrow{OC}\right) + \left(\overrightarrow{OE} - \overrightarrow{OD}\right) \\
&= \overrightarrow{OE} - \overrightarrow{OA} = \overrightarrow{AO} + \overrightarrow{OE}.
\end{aligned} \tag{2.22}
$$

The segments OA, OB, OC, OD, and OE are called *rays*, but in (2.10) the vectors of their lengths appeared as forces, which means $|\overrightarrow{OA}| = OA$, and so on.

The force polygon allows for determination of the resultant $\mathbf{F}^r$, that is, the force replacing the action of forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$. However, additionally we have to determine the line of action of the resultant force (it must be parallel to the direction of the force $\mathbf{F}^r$ obtained using the force polygon method, see Fig. 2.11b). In practice this means that one should determine a point W through which the force $\mathbf{F}^r$ would pass.

To this end we make use of the so-called *funicular polygon*. Let us take an arbitrary point A' lying on the line of action of the force $\mathbf{F}_1$ (construction is temporarily conducted in a separate drawing). Next, we draw a line passing through that point and parallel to OA (cf. the force polygon). Then, we step off on that line a segment $A'B'$ of length $|\overrightarrow{AO}|$ (one may limit oneself to the determination of the direction of that force). Next, through point B' we draw a line parallel to OB and step off the segment $B'C' = |\overrightarrow{OB}|$. Through the obtained point C' we draw a line parallel to OC and step off the segment $C'D' = |\overrightarrow{OC}|$, obtaining in this way point D'. Through that point we draw a line parallel to OD and step off the segment $D'E' = |\overrightarrow{OD}|$. Through the obtained point E' we draw a line parallel to OE and step off the segment $E'O' = |\overrightarrow{EO}|$. According to (2.22) the resultant can be determined using the parallelogram law, since the force vectors $\overrightarrow{AO}$ and $\overrightarrow{OE}$ are known [the forces represented by the other vectors (rays) cancel each other]. One may also move the vectors $\overrightarrow{AO}$ and $\overrightarrow{OB}$, $\overrightarrow{BO}$ and $\overrightarrow{OC}$, $\overrightarrow{CO}$ and $\overrightarrow{OD}$ respectively to points B', C', D', and E'. It is easy to notice that after their geometrical addition only the vectors $\overrightarrow{AO}$ and $\overrightarrow{OE}$ remain. It follows that extending the lines passing through points A' and B', and through O' and E', leads to determination of point W, through which passes the line of action of the resultant $\mathbf{F}^r$. It follows also that it is possible to determine the magnitude of that force (after geometrical addition of $|\overrightarrow{A'B'}|$ and $|\overrightarrow{O'E'}|$).

If we took a perfectly flexible cable of negligible weight and length $AO + OB + OC + OD + OE$ and fixed it at points A' and O', then after application of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ at points B', C', D', and E' (Fig. 2.11c), the cable would remain in equilibrium. That is where the name of funicular polygon comes from. At each of these points act three forces that are in equilibrium. It is easy to check that the choice of some other pole (point O_1 in Fig. 2.11b) leads to the determination of a different point of intersection, but it must lie on the line of action of the force $\mathbf{F}^r$.

Let us now consider the particular case of the force polygon depicted in Fig. 2.11b, namely, when $\mathbf{F}^r \to \mathbf{0}$, that is, when $E \to A$, which means that $|\overrightarrow{EO}| = |\overrightarrow{OA}|$ and the vectors $\overrightarrow{OE}$ and $\overrightarrow{AO}$ become collinear at the limit while retaining the opposite senses. Since the polygon $ABCDA$ is closed, points A and E are coincident, that is, $A \equiv E$.

Such a situation is depicted in Fig. 2.12a, and the corresponding force polygon is shown in Fig. 2.12b.

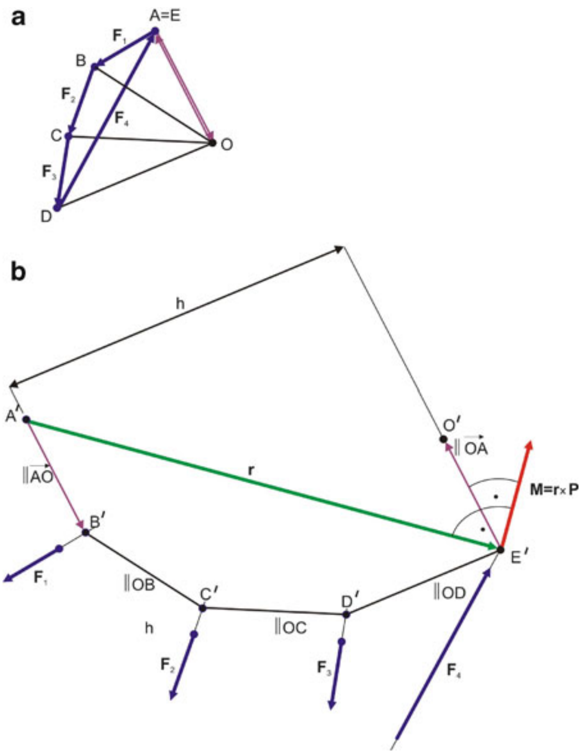


Fig. 2.12 A closed polygon of four forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, $\mathbf{F}_4$ (**a**) and the corresponding funicular polygon (**b**) (vector **r** is not a part of the funicular polygon but was drawn only for interpretation of a moment of force vector $\mathbf{M} = \mathbf{r} \times \mathbf{P}$ perpendicular to the drawing's plane)

From the figure it follows that the vectors of forces $\overrightarrow{A'B'} = \overrightarrow{AO}$ and $\overrightarrow{E'O'} = \overrightarrow{OE}$ possess the same magnitudes and opposite senses, so they produce the moment of a force of magnitude $|\mathbf{M}| = |\mathbf{r} \times \mathbf{P}| = Pr\sin\varphi = Ph$, where $\mathbf{P} = \overrightarrow{A'B'}$ and φ denotes the angle formed between vectors **r** and **P**. It is impossible to determine the point of application of the force $\mathbf{F}^r$ because it is a singular case where $\mathbf{F}^r = \mathbf{0}$ and $A'B' \parallel O'E'$. In this case the force system $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ is equivalent to the couple of forces.

Let us now consider another special case of the force polygon and funicular polygon relying on the considerations regarding Fig. 2.12. Let point $E' \to A'$, which means that $\mathbf{r} \to \mathbf{0}$, that is, $\mathbf{r} \times \mathbf{P} = \mathbf{M} \to \mathbf{0}$. When the endpoints of the funicular polygon are coincident, that is, $E' \equiv A'$, the moment $\mathbf{M} = \mathbf{0}$ (this case is presented in Fig. 2.13). Because the force polygon remains unchanged (Fig. 2.12a), in Fig. 2.13 only the funicular polygon is shown. In this case the force system $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ remains in equilibrium.

From the considerations and drawings in Figs. 2.11–2.13 the following conclusions result:

1. If neither the force polygon nor the funicular polygon is closed, then the investigated planar force system is equivalent to a resultant force.

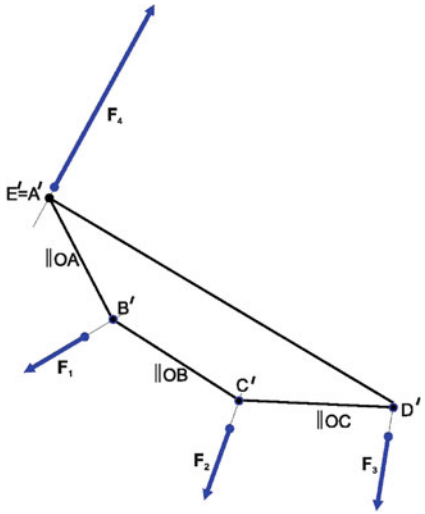


Fig. 2.13 Closed funicular polygon ($\mathbf{M} = \mathbf{0}$)

2. If the funicular polygon is not closed but the force polygon is closed, then planar force system is equivalent to a moment of force (couple).
3. If both force and funicular polygons are closed, then the force system is equivalent to zero.

The final geometrical equilibrium condition for a two-dimensional force system reads:

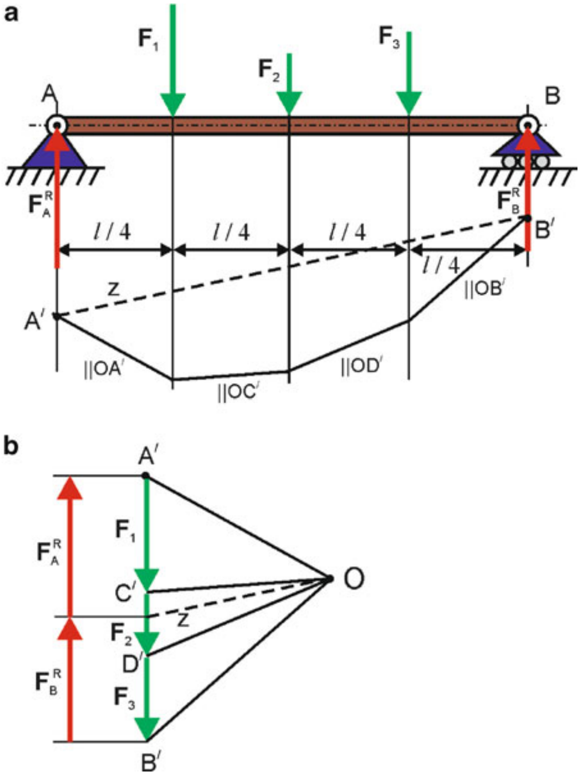
A planar force system is equivalent to zero if both the force polygon and funicular polygon are closed.

Example 2.1. Three vertical forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ act on a horizontal beam of length l depicted in Fig. 2.14. The beam is pin supported at point A and has a roller support at point B. Determine the reactions in beam supports.

In this case the forces $\mathbf{F}_n$ are parallel, so the unknown reactions $\mathbf{F}_A^R$ and $\mathbf{F}_B^R$ are parallel as well and lie on the vertical lines through points A and B. According to the previous considerations let us first construct the force polygon in a certain assumed scale of the drawing. Next, after taking pole O we draw the rays from the pole to the tail and the tip of every force vector. We take an arbitrary point A' on the vertical line passing through A. After connecting points A' and B' we obtain the closing line of the force polygon (dashed line). Next, we translate it in parallel, so that it passes through the pole. Then its point of intersection with segment $A'B'$ in the force polygon defines the unknown reactions.

Since the tip of reaction $\mathbf{F}_A^R$ must coincide with the tail of the force $\mathbf{F}_1$, we complement the force polygon with reactions $\mathbf{F}_A^R$ and $\mathbf{F}_B^R$ and, as can be seen, $\mathbf{F}_A^R + \mathbf{F}_B^R + \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 = \mathbf{0}$. □

Fig. 2.14 A horizontal beam loaded with forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ (**a**) and a force polygon (**b**)



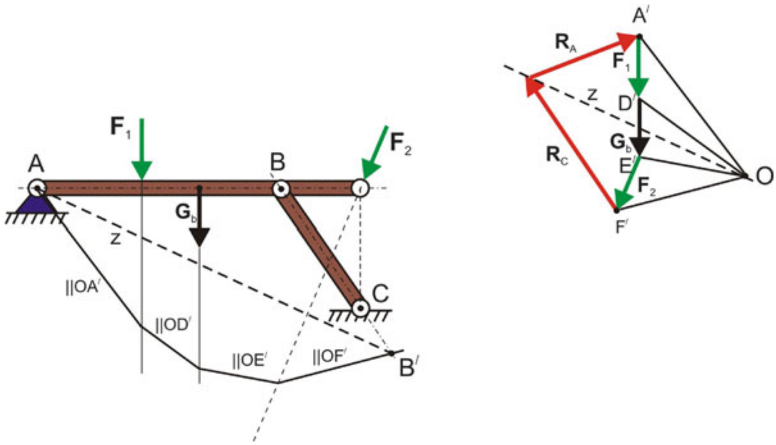


Fig. 2.15 Schematic for calculations of a supported beam, a force polygon, and a funicular polygon

Example 2.2. A horizontal beam is loaded with forces $\mathbf{F}_1$ and $\mathbf{F}_2$ (Fig. 2.15). The beam is pin supported at point A and at point B supported by the rod with a pin joint. Determine the reactions in pin joints A and C assuming that the beam has a homogeneous mass distribution and weight $\mathbf{G}_b$.

Because in this case only the vectors $\mathbf{F}_1$ and $\mathbf{G}_b$ are parallel to each other but are not parallel to $\mathbf{F}_2$, the reactions $\mathbf{F}_A^R$ and $\mathbf{F}_C^R$ are not parallel (the direction of reaction $\mathbf{F}_C^R$ is defined by the axis of rod BC). We construct the force polygon for $\mathbf{F}_1, \mathbf{G}_b, \mathbf{F}_2$.

The construction of the funicular polygon we start from point A (reaction $\mathbf{F}_A^R$ must pass through point A, but its direction is unknown). Doing the construction in the way described earlier, the line parallel to OF' intersects segment BC at point B' and segment AB' is the closing line of the funicular polygon. After drawing a line parallel to AB' that passes through pole O (line z) and drawing from the tip of the vector $\mathbf{F}_2$ the line parallel to BC, the point of their intersection determines the unknown reactions $\mathbf{F}_A^R$ and $\mathbf{F}_C^R$. □

2.3 Geometrical Equilibrium Conditions of a Space Force System

As distinct from the previously analyzed case of the concurrent force system shown in Fig. 2.2, we will consider now a non-concurrent force system.

Our aim is to reduce the force system $\mathbf{F}_1, \ldots, \mathbf{F}_N$ to an arbitrary chosen point O of the body (or rigidly connected to the body) called a *pole*.

The method given below was presented already by Poinsot[2] and henceforth is called *Poinsot's method*. We will show that, according to Poinsot's method, the action of the force $\mathbf{F}_1$ on a rigid body with respect to pole O is equivalent to the action of the force $\mathbf{F}_1$ applied at point O and a couple $\mathbf{F}_1$ and $\mathbf{F}_1^{(2)}$ applied at points A_1 and O, respectively, and $\mathbf{F}_1^{(2)} = -\mathbf{F}_1$.

In other words at point O we apply the forces $\mathbf{F}_1^{(1)}$ and $\mathbf{F}_1^{(2)}$, where $\mathbf{F}_1^{(1)}$ denotes the vector $\mathbf{F}_1$ moved in a parallel translation to point O, and $\mathbf{F}_1^{(2)} = -\mathbf{F}_1^{(1)}$. Point O is in equilibrium under the action of two vectors of the directions along the direction of $\mathbf{F}_1$ and having the same magnitudes but opposite senses.

The action of force at point A_1 manifests at point O as the action of the force $\mathbf{F}_1$ at that point and the couple $\mathbf{F}_1$ (applied at point A_1) and $-\mathbf{F}_1$ (applied at point O). In turn, the action of the couple $\mathbf{F}_1$ and $-\mathbf{F}_1$ is equivalent to the action of a moment of the couple, which according to (2.17) is equal to $\mathbf{M}_1 = \overrightarrow{OA_1} \times \mathbf{F}_1$.

In Fig. 2.16a at point O only the two forces $\mathbf{F}_1^{(1)}$ and $\mathbf{F}_1^{(2)}$ canceling one another are marked, and in Fig. 2.16b reduction of the whole force system $\mathbf{F}_1, \ldots, \mathbf{F}_N$ is

[2] Louis Poinsot (1777–1859), French mathematician and physicist, precursor of geometrical mechanics.

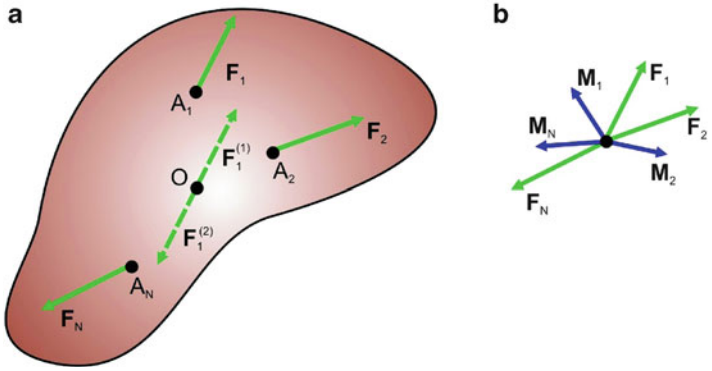


Fig. 2.16 Space force system $\mathbf{F}_1, \ldots, \mathbf{F}_N$ acting on a rigid body (**a**) and its reduction to a pole O (**b**)

shown. The latter boils down to applying at point O all the vectors of forces $\mathbf{F}_1, \ldots, \mathbf{F}_N$ and moments $\mathbf{M}_1, \ldots, \mathbf{M}_N$ determined in a way that is analogous to that shown for the case of force $\mathbf{F}_1$.

The application of Poinsot's method led to obtaining the concurrent force and moment system, which is schematically depicted in Fig. 2.16b. With the aid of the conducted construction we can develop a force polygon and a moment of the force polygon, which leads to the determination of the *main force vector* and *main moment of the force vector* of the forms

$$\mathbf{S} = \sum_{i=n}^{N} \mathbf{F}_n, \tag{2.23}$$

$$\mathbf{M}_O = \sum_{n=1}^{N} \mathbf{M}_n. \tag{2.24}$$

The technique described above is equivalent to the *theorem on parallel translation of force*.

Let us note that if the force system $\mathbf{F}_1, \ldots, \mathbf{F}_N$ was concurrent and acted, e.g., at point A_1, the vector $\mathbf{S}$ according to (2.23) would be the resultant of forces applied at point A_1. The process of reduction regards pole O, and the notion of the resultant of forces at that point concerns the sum of forces $\mathbf{F}_1^{(1)}, \mathbf{F}_2^{(1)}, \ldots, \mathbf{F}_N^{(1)}$, and therefore the notion of the main moment of force vector was introduced instead of using the notion of resultant of forces.

Let us now proceed to the analysis of (2.24). Let us consider the moment of the couple $\mathbf{F}_1$ and $\mathbf{F}_1^{(2)}$, which is equal to

$$\mathbf{M}_1 = \mathbf{M}_O(\mathbf{F}_1) + \mathbf{M}_O(\mathbf{F}_1^{(2)}) = \mathbf{M}_O(\mathbf{F}_1), \tag{2.25}$$

because the force $\mathbf{F}_1^{(2)}$ passes through pole O. Similar considerations concern the remaining forces, and finally (2.24) takes the following form:

$$\mathbf{M}_O = \sum_{n=1}^{N} \mathbf{M}_O(\mathbf{F}_n). \tag{2.26}$$

The result of the considerations carried out above leads to the formulation of the general theorem of statics of a rigid body.

Theorem 2.8. *An arbitrary system of non-concurrent forces in space acting on a rigid body is statically equivalent to the action of the main force vector (2.23) applied at an arbitrary point (pole) and the main moment of force vector (2.24).*

2.4 Analytical Equilibrium Conditions

An analytical form of equilibrium conditions relies on the vectorial (2.20). Because in the Euclidean space each of the vectors possesses three projections, the equilibrium conditions boil down to the following three conditions concerning the forces:

$$S_{x_j} = \sum_{n=1}^{N} F_{x_j n} = 0, \quad j = 1, 2, 3, \tag{2.27}$$

and because

$$\begin{aligned}
\mathbf{M}_O &= \sum_{n=1}^{N} (r_n \times \mathbf{F}_n) = \sum_{n=1}^{N} \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ x_{1n} & x_{2n} & x_{3n} \\ F_{x_{1n}} & F_{x_{2n}} & F_{x_{3n}} \end{vmatrix} \\
&= \sum_{n=1}^{N} \Big[\mathbf{E}_1 \big(x_{2n} F_{x_3 n} - x_{3n} F_{x_2 n} \big) + \mathbf{E}_2 \big(x_{3n} F_{x_1 n} - x_{1n} F_{x_3 n} \big) \\
&\quad + \mathbf{E}_3 \big(x_{1n} F_{x_2 n} - x_{2n} F_{x_1 n} \big) \Big],
\end{aligned} \tag{2.28}$$

we have an additional three equilibrium equations of the form

$$\begin{aligned}
M_{Ox_1} &\equiv \sum_{n=1}^{N} \big(x_{2n} F_{x_3 n} - x_{3n} F_{x_2 n} \big) = 0, \\
M_{Ox_2} &\equiv \sum_{n=1}^{N} \big(x_{3n} F_{x_1 n} - x_{1n} F_{x_3 n} \big) = 0, \\
M_{Ox_3} &\equiv \sum_{n=1}^{N} \big(x_{1n} F_{x_2 n} - x_{2n} F_{x_1 n} \big) = 0,
\end{aligned} \tag{2.29}$$

where $\mathbf{M}_O = \sum_{j=1}^{3} \mathbf{E}_j M_{Ox_j}$. In general, the main moment changes with a change of the reduction point (the pole).

From the equations above it follows that the force system of the lines of action laid out arbitrarily in space is in equilibrium if the algebraic equations (2.27) and (2.29) are satisfied. From the mentioned equations it is easy to obtain the relations regarding the forces lying in the selected planes $(O - X_1 - X_2)$ or $(O - X_1 - X_3)$. In the equations above the forces $\mathbf{F}_n$ denote external forces and reactions.

External forces can be divided into *concentrated forces* (applied to points of a body), *surface forces* (applied over certain areas), and *volume forces* (applied to all particles of a body). An example of volume forces are the forces caused by the body weight, and of surface forces—the forces generated by the surface of contact between the bodies being loaded. A set of external forces consists of known forces (*active forces*) and the forces that are subject to determination (*passive forces*).

The set of external (active and passive) forces is called the *loading of a mechanical system*.

Let us consider a wheeled vehicle (a car with four wheels) with an extra load placed on its roof. Here the active forces are the car weight $\mathbf{G}_c$ and load weight $\mathbf{G}_l$. At the points of contact between each wheel and road surface appear four reactions $\mathbf{F}_i^R$. Here the external forces are the forces $\mathbf{G}_c$ and $\mathbf{G}_l$ (active forces) and reactions $\mathbf{F}_i^R$ (passive forces). The remaining forces acting within the system isolated from its environment (i.e., the car and load) are called *internal forces*. According to Newton's third law these forces mutually cancel each other, and for their determination it is necessary to employ the so-called *imaginary cut technique*. For instance, in order to examine the action of the load on the car one should "cut" the system at the points of contact of the load with the car and replace their interaction with reactions, which are the internal forces. Thus we carry out the imaginary division of the analyzed mechanical system into two subsystems in static equilibrium. Then, from the equilibrium equations of either subsystem we determine the desired internal forces.

In order to determine the analytical equilibrium conditions one may also make use of the so-called *three-moments theorem*.

Theorem 2.9. *If we choose three different non-collinear points A_j, then the equilibrium conditions of a material system are*

$$\mathbf{M}_{A_j} = \mathbf{M}_{A_j}\left(\sum_{n=1}^{N} \mathbf{F}_n\right) = \mathbf{0}, \qquad j = 1, 2, 3. \tag{2.30}$$

This means that the main moment of force vectors of the force system about three arbitrary but non-collinear points are equal to zero. The proof follows. Let the equation be satisfied for A_1 (the moment about that point equals zero) and equations for two other points not be satisfied. This means that the system is not equivalent to the couple but to the resultant force that must pass through point A_1.

Now let two equations of moments about points A_1 and A_2 be satisfied. Because it is possible to draw a line through points A_1 and A_2, a non-zero resultant force must lie on that line. However, if additionally the third equation related to point A_3 is satisfied and points A_1, A_2, and A_3 are not collinear (by assumption), then the resultant force must be equal to zero. That completes the proof.

Projecting the moments of forces (2.30) on the three axes of Cartesian coordinate system we obtain nine algebraic equations of the form

$$M_{x_n A_j} = 0, \quad n, j = 1, 2, 3. \tag{2.31}$$

Here, we are dealing with the apparent contradiction because there are nine (2.31) and six (2.27) and (2.29). However, for (2.31), for each force nine coordinates defining its distance from the chosen points A_1, A_2, and A_3 are needed. In the rigid body the distances A_1A_2, A_2A_3, and A_1A_3 are constant, which reduces the number of independent coordinates to six. So, one may proceed in a different way. The six mutually independent axes should be taken in such a way as to formulate six independent equations only.

The equilibrium conditions (2.27) and (2.29) consisting of six equations resulted from projecting the main force vector $\mathbf{S}$, and the main moment of force vector $\mathbf{M}_O$ reduced to an arbitrary point O (Sect. 2.3), where the point of reduction O is the origin of the Cartesian coordinate system.

The state of equilibrium of the considered rigid body means that it does not make any *displacement*, that is, neither translation nor rotation about point O and consequently with respect to the adopted coordinate system $OX_1X_2X_3$. We assume that in the absence of forces $\mathbf{F}_1, \ldots, \mathbf{F}_N$ the rigid body does not move with respect to the adopted coordinate system $OX_1X_2X_3$, and also remains unmoved under the action of the force system $\mathbf{F}_1, \ldots, \mathbf{F}_N$ if (2.27) and (2.29) are satisfied. If under the action of an arbitrary force system $\mathbf{F}_1, \ldots, \mathbf{F}_N$ the rigid body remains in equilibrium with respect to $OX_1X_2X_3$, these forces must satisfy (2.27) and (2.29). Since, if $\mathbf{S} \neq \mathbf{0}$, $\mathbf{M}_O = \mathbf{0}$, point O would be subjected to an action of the force $\mathbf{S}$ leading to the loss of static equilibrium state.

If we had $\mathbf{S} = \mathbf{0}$, $\mathbf{M}_O \neq \mathbf{0}$, then the main moment $\mathbf{M}_O$ would cause the rotation of the rigid body, leading to the loss of its static equilibrium. Equilibrium equations of the form

$$\begin{gathered} \mathbf{S} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_N = \mathbf{0}, \\ \mathbf{M}_O = \mathbf{M}_O(\mathbf{F}_1) + \mathbf{M}_O(\mathbf{F}_2) + \ldots + \mathbf{M}_O(\mathbf{F}_N) = \mathbf{0} \end{gathered} \tag{2.32}$$

represent the necessary and sufficient equilibrium conditions for a free (unconstrained) rigid body subjected to the action of an arbitrary three-dimensional force system. The above conditions transform into conditions (2.27) and (2.29) after the introduction of the Cartesian coordinate system of origin at O.

In other words, if we reduce an arbitrary three-dimensional system of forces $\mathbf{F}_n$ applied at the points $A_n(x_{1n}, x_{2n}, x_{3n})$, $n = 1, \ldots, N$ to an arbitrary point (the reduction pole) O, we obtain the main force vector $\mathbf{S}$ and main moment of force vector $\mathbf{M}_O$, and after adopting the Cartesian coordinate system at point O, we have

$$\begin{aligned}\mathbf{S} &= \mathbf{E}_1 \sum_{n=1}^{N} F_{1n} + \mathbf{E}_2 \sum_{n=1}^{N} F_{2n} + \mathbf{E}_3 \sum_{n=1}^{N} F_{3n} \\ &= \mathbf{E}_1 S_1 + \mathbf{E}_2 S_2 + \mathbf{E}_3 S_3,\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}\mathbf{M}_O &= \mathbf{E}_1 \sum_{n=1}^{N} [F_{3n}(x_{2n} - x_{20}) - F_{2n}(x_{3n} - x_{30})] \\ &\quad + \mathbf{E}_2 \sum_{n=1}^{N} [F_{1n}(x_{3n} - x_{30}) - F_{3n}(x_{1n} - x_{10})] \\ &\quad + \mathbf{E}_3 \sum_{n=1}^{N} [F_{2n}(x_{1n} - x_{10}) - F_{1n}(x_{2n} - x_{20})] \\ &\equiv \mathbf{E}_1 M_{01} + \mathbf{E}_2 M_{02} + \mathbf{E}_3 M_{03}.\end{aligned} \tag{2.34}$$

In the case of reduction of a three-dimensional force system there exist two *reduction invariants*. The first is the main vector $\mathbf{S}$ and the second the projection of vector $\mathbf{M}_O$ onto the direction of the vector $\mathbf{S}$. The first invariant means that the reduction of spatial forces being in fact a geometrical addition of vectors gives the same result for an arbitrarily chosen point of reduction. The second invariant means that for the arbitrarily chosen point of reduction the projection of the main moment vector onto the direction of the main force vector is constant.

In the latter case it is possible to find such a direction (a straight line) where if the points of reduction lie on that line, the magnitude of $\mathbf{M}_O$ is minimum. On that line we can place the vector $\mathbf{S}$, which as the invariant may be freely moved in space. Such a line, after assigning to it the sense defined by the sense of $\mathbf{S}$, we call the central axis (axis of a wrench). Every set of force vectors and moment of force vectors can have only one central axis. The system of two vectors $\mathbf{S}$ and $\mathbf{M}_O$ lying on the central axis we call a *wrench*.

The general equilibrium conditions (2.27) and (2.29) may be simplified and then boil down to the special cases of the field of forces considered earlier, which we will briefly describe below.

A concurrent force system in space. Taking pole O at the point of intersection of the lines of action of these forces, (2.29) are identically equal to zero, and the equilibrium conditions are described only by three equations (2.27).

An arbitrary force system in a plane. An arbitrary planar force system may be reduced to the main force vector $\mathbf{S}$ and main moment of force vector $\mathbf{M}_O$. After choosing the axis OX_3 perpendicular to the plane of action of the forces the problem of determination of static equilibrium reduces to the analysis of algebraic equations of the form

$$S_1 = \sum_{n=1}^{N} F_{1n} = 0, \quad S_2 = \sum_{n=1}^{N} F_{2n} = 0,$$
$$M_{O_3} = \sum_{n=1}^{N} M_{O_{3n}}(\mathbf{F}_n) = 0. \tag{2.35}$$

A system of parallel forces in space. Let the axis OX_3 be perpendicular to the vector field of parallel forces. From (2.27) and (2.29) for the considered case we obtain

$$\sum_{n=1}^{N} F_{3n} = 0, \quad \sum_{n=1}^{N} M_{O_{1n}}(\mathbf{F}_n) = 0, \quad \sum_{n=1}^{N} M_{O_{2n}}(\mathbf{F}_n) = 0. \tag{2.36}$$

The last case will be analyzed in more detail because we are going to refer to it by the introduction of the notion of a mass center of a system of particles and a mass center of a rigid body.

In the general case an arbitrary force system acting on a rigid body is equivalent to either an action of main force vector or main moment of force vector.

Up to this point the equivalence of two force systems (sets of forces) was formulated in a descriptive way. The criterion of equivalence can, however, be stated as a theorem.

Theorem 2.10. *The necessary and sufficient condition of equivalence of two force systems, acting on a rigid body, with respect to a certain point (pole) is that these systems have identical main force vectors and main moment of force vectors with respect to that point.*

The above theorem can be proved based on the principle of virtual work [27].

The system of parallel forces in three dimensions includes four special cases:

1. The forces may lie on parallel lines and possess opposite senses and different magnitudes.
2. The forces may lie on parallel lines and possess opposite senses but the same magnitudes.
3. The forces may lie on parallel lines and possess the same senses but different magnitudes.
4. The forces may lie on parallel lines and possess the same senses and magnitudes.

Let us limit our considerations to two forces representing the cases listed above. These forces always lie in one plane as they are parallel.

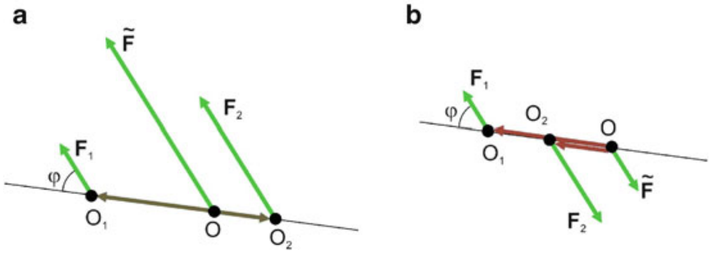


Fig. 2.17 Sketches illustrating Theorem 2.12 (**a**) and Theorem 2.13 (**b**)

We have proved already that if we have two forces of the same magnitudes and opposite senses, we are dealing with a couple. The action of a couple is equivalent to the action of a *moment of a couple*. If these forces were collinear, they would have no effect on the body because they would cancel one another (case 2). If the forces are parallel, had opposite senses and different or identical magnitudes, and arenot collinear, these cases will be considered below. For that purpose we first introduce the notion of an *equivalent equilibrant force*.

If the force system $(\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_N)$ applied to a rigid body is equivalent to only one force $\mathbf{F}$, we will call such a force an *equivalent equilibrant force*.

Theorem 2.11. *If a system of forces possesses an equivalent equilibrant force, then the vector of this force* $\tilde{\mathbf{F}}$ *is equal to the main force vector* $\mathbf{F}$ *of the force system, and its moment about an arbitrary pole is equal to the main moment of force of the force system about that pole.*

We will define now an equivalent equilibrant force for the case of two forces of different magnitudes and arbitrary senses [27].

Theorem 2.12. *A system of two parallel forces* $(\mathbf{F}_1, \mathbf{F}_2)$ *having the same senses applied to a rigid body possesses an equivalent equilibrant force* $\tilde{\mathbf{F}} = \mathbf{F}_1 + \mathbf{F}_2$*. The force* $\tilde{\mathbf{F}}$ *lies in a plane defined by the forces* $(\mathbf{F}_1, \mathbf{F}_2)$*, is parallel to them, and its line of action divides in the inside a segment* O_1O_2*, which connects the tails of vectors of those forces, into two parts inversely proportional to their magnitudes* F_1 *and* F_2 *(Fig. 2.17a).*

Theorem 2.13. *A system of two parallel forces* $(\mathbf{F}_1, \mathbf{F}_2)$ *having opposite senses and different magnitudes applied to a rigid body possesses an equivalent equilibrant force* $\tilde{\mathbf{F}} = \mathbf{F}_1 + \mathbf{F}_2$*. The force* $\tilde{\mathbf{F}}$ *lies in a plane defined by the forces* $(\mathbf{F}_1, \mathbf{F}_2)$*, is parallel to them, and its line of action divides on the outside a segment* O_1O_2*, which connects the tails of vectors of those forces, into two parts inversely proportional to their magnitudes* F_1 *and* F_2 *(Fig. 2.17b).*

We will prove Theorem 2.13. According to Varignon's theorem for point O we have

$$\overrightarrow{OO_1} \times \mathbf{F}_1 - \overrightarrow{OO_2} \times \mathbf{F}_2 = \mathbf{0} \times \tilde{\mathbf{F}} \equiv \mathbf{0}$$

and

$$\tilde{\mathbf{F}} = \mathbf{F}_1 + \mathbf{F}_2.$$

From the first equation it follows for both cases that after passing to scalars we have

$$|\overrightarrow{OO_1}| \cdot |\mathbf{F}_1| \sin\varphi - |\overrightarrow{OO_2}| \cdot |\mathbf{F}_2| \sin(180^\circ - \varphi) = 0,$$

that is,

$$|OO_1| \cdot |\mathbf{F}_1| = |OO_2| \cdot |\mathbf{F}_2|.$$

Let us consider now the special case that follows from Fig. 2.17b, that is, when $\mathbf{F}_1$ is applied at point O_1 and at point O_2 the force $\mathbf{F}_2 = -\mathbf{F}_1$, that is, we are dealing with a couple of forces. It is easy to observe that a couple does not possess an equivalent equilibrant force because $\tilde{\mathbf{F}} = \mathbf{F}_1 - \mathbf{F}_1 = \mathbf{0}$. The couple $(\mathbf{F}_1, \mathbf{F}_2)$ does have, however, a different interesting property, which we have already mentioned. The moment produced by the couple depends not on the choice of the pole in space, but only on the distance of the points of application of the forces $\mathbf{F}_1$ and $\mathbf{F}_2 = -\mathbf{F}_1$. It is possible to prove that couples having the same moment of force are *equivalent couples*.

Let us now consider the mixed (hybrid) case of a system of parallel forces in space acting on a rigid body, where some forces have opposite senses and some the same senses.

As was considered earlier on examples of two parallel forces having either the same or opposite senses, we can reduce the whole system of parallel forces to two parallel forces $\tilde{\mathbf{F}}_1$ and $\tilde{\mathbf{F}}_2$, called an *equivalent equilibrant force*.

The force vectors $\mathbf{F}_n$, $n = 1, \ldots, N$ we will treat as bound vectors (as distinct from the free vectors used so far), and the force $\tilde{\mathbf{F}}_1$ represents an equivalent equilibrant force for the forces having senses opposite to the force $\tilde{\mathbf{F}}_2$, replacing the action of the second group of forces. Finally, the problem of reduction in this case is reduced to the problem presented in Fig. 2.17b, which means that we are able to determine the location of the point $O = C$, where an equivalent equilibrant force $(\tilde{\mathbf{F}}_1, \tilde{\mathbf{F}}_2)$ is applied.

The case depicted in Fig. 2.17b enables us to draw certain further conclusions.

Let us observe that according to the proof of Theorem 2.13 at the point $O = C$, the vector of force $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}_1 + \tilde{\mathbf{F}}_2$ (Fig. 2.18) is applied and the location of that point depends exclusively on the magnitudes of force vectors, that is, from F_1/F_2. It follows that the locations of points C, O_1, and O_2 do not change at the rotation of force vectors through the same angle α. Let, after the rotation through the same angle α, the lines of action of the mentioned forces be parallel to the axis OX_2 of the adopted Cartesian coordinate system, that is, all the previously mentioned forces after reduction lie in a plane parallel to the plane OX_1X_2.

Let us write an equation of moments about an axis OX_3 (perpendicular to the plane of the drawing in which the forces lie).

If the number of parallel forces is N, then we will denote them as $\mathbf{F}_1, \mathbf{F}_2, \ldots,$ $\mathbf{F}_N$, where those having senses opposite to the positive direction of the axis OX_2

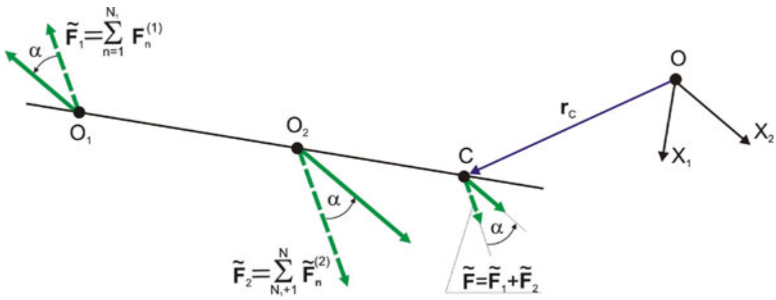


Fig. 2.18 Rotation of forces $\tilde{\mathbf{F}}_1, \tilde{\mathbf{F}}_2$, and $\tilde{\mathbf{F}}$ through an angle α

are defined as $\mathbf{F}_n \circ \mathbf{E}_2 = -F_n$. The sum of moments about a point O is equal to

$$\mathbf{M}_0 = \mathbf{r}_C \times \tilde{\mathbf{F}} = \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ x_{1C} & x_{2C} & x_{3C} \\ 0 & \tilde{F} & 0 \end{vmatrix} = -\mathbf{E}_1 \tilde{F} x_{3C} + \mathbf{E}_3 \tilde{F} x_{1C},$$

$$\begin{aligned}
\mathbf{M}_1 &= \sum_{n=1}^{N_1} \left(\mathbf{r}_n \times \tilde{\mathbf{F}}_n^{(1)}\right) = \sum_{n=1}^{N_1} \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ x_{1n} & x_{2n} & x_{3n} \\ 0 & -F_n & 0 \end{vmatrix} \\
&= \mathbf{E}_1 \sum_{n=1}^{N_1} F_n x_{3n} - \mathbf{E}_3 \sum_{n=1}^{N_1} F_n x_{1n}, \\
\mathbf{M}_2 &= \sum_{n=N_1+1}^{N} \left(\mathbf{r}_n \times \tilde{\mathbf{F}}_n^{(2)}\right) = \sum_{n=N_1+1}^{N} \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ x_{1n} & x_{2n} & x_{3n} \\ 0 & F_n & 0 \end{vmatrix} \\
&= -\mathbf{E}_1 \sum_{n=N_1+1}^{N} F_n x_{3n} + \mathbf{E}_3 \sum_{n=N_1+1}^{N} F_n x_{1n},
\end{aligned} \tag{2.37}$$

and hence we obtain

$$\begin{aligned}
\tilde{F} x_{1C} &= \sum_{n=N_1+1}^{N} F_n x_{1n} - \sum_{n=1}^{N_1} F_n x_{1n}, \\
\tilde{F} x_{3C} &= \sum_{n=N_1+1}^{N} F_n x_{3n} - \sum_{n=1}^{N_1} F_n x_{3n},
\end{aligned} \tag{2.38}$$

where

$$\tilde{F} = \sum_{n=N_1+1}^{N} F_n - \sum_{n=1}^{N_1} F_n. \tag{2.39}$$

If we deal only with N parallel forces (we assume $N_1 = 0$) of the same senses, from (2.38) we obtain

$$\left(\sum_{n=1}^{N} F_n\right) x_{1C} = \sum_{n=1}^{N} F_n x_{1n},$$
$$\left(\sum_{n=1}^{N} F_n\right) x_{3C} = \sum_{n=1}^{N} F_n x_{3n}, \tag{2.40}$$

which defines the position of the center C of the parallel forces having the same senses consistent with the sense of $\mathbf{E}_2$ in the adopted coordinate system.

The third missing equation is

$$\left(\sum_{n=1}^{N} F_n\right) x_{2C} = \sum_{n=1}^{N} F_n x_{2n},$$

which allows us, through the suitable choice of two out of the three presented equations, to determine the position of the center of parallel forces in each of the planes OX_1X_2, OX_2X_3, and OX_1X_3.

If now in the selected points $n = 1, \ldots, N$ we apply the vectors of parallel forces $m_n\mathbf{g}$ (weights), where $\mathbf{g}$ is the acceleration of gravity, we can determine the center of gravity of those forces after introducing the Cartesian coordinate system such that $\mathbf{F}_n = m_n g\mathbf{E}_3$.

The gravity center is coincident with the mass center of the given discrete mechanical system in the gravitational field.

Equation (2.40) listed above can be obtained from the following equation:

$$\mathbf{r}_C \times \sum_{n=1}^{N} \mathbf{F}_n = \sum_{n=1}^{N} \mathbf{r}_n \times \mathbf{F}_n. \tag{2.41}$$

2.5 Mechanical Interactions, Constraints, and Supports

In Chap. 1 the concept of material system as a collection of particles was introduced. In many cases such simplification is not sufficient. Particles are a special case of rigid bodies whose geometrical dimensions were reduced to zero and only their masses were left. A natural consequence of expansion of the concept of system of particles is a *system of rigid bodies*. Such bodies can act on each other depending on how they are connected through *forces* and *moments of forces*. As was discussed earlier, forces in the problems of statics are treated as *sliding vectors* and moments of forces as *free vectors*.

The introduced system of rigid bodies is a system isolated from its surroundings in the so-called *modeling process*. In view of that, the surroundings act mechanically on the isolated system of rigid bodies. Such modeling leads naturally to the introduction of the notions of *active* and *passive* mechanical interactions.

Active mechanical interactions are forces and moments coming from the surroundings and acting on the considered system of bodies (they include the interactions produced by gravitational fields, by various pneumatic or hydraulic actuators, by engines, etc.).

Active mechanical interactions produce, according to Newton's third law, passive mechanical interactions, that is, interactions between the rigid bodies in the considered system. The passive interactions (forces and moments of reactions) we determine by performing a mental release from constraints of the bodies of the system interacting mechanically.

According to the other classification criterion of mechanical interactions we divide them into *external* (the counterpart of active) and *internal* (the counterpart of passive). The classification of the system of bodies depending on their number in the system is also introduced. If we are dealing with a single body (many bodies), the system is called *simple* (*complex*, *multibody*).

The simple system is one rigid body that can be in static equilibrium under the action of either external interactions exclusively or *hybrid* interactions, i.e., *external* and *internal*. Let us consider two bodies and assume that one of them was fixed. Next, the internal interaction of these bodies is replaced by the reaction forces and reaction moments of forces coming from the interactions (supports) of the fixed body (now called the base) on the free body, and now the mentioned supports are treated as external.

In the case of simple problems (one rigid body), the problem of statics boils down to releasing the body from supports, and the introduction of the mentioned reaction forces and reaction moments of forces, and then to application of (2.27) and (2.29) and their solution. During the solution of statics equations the following three cases may occur:

1. The number of equations is equal to the number of unknowns (then the problem is statically determinate and the solution of linear algebraic equations for the forces and moments of forces can be done easily).
2. The number of equations is smaller than the number of unknowns (then the problem is statically indeterminate; in order to solve the problem it is necessary to know the relation between deformation and force (stress) fields).
3. The number of equations is greater than the number of unknowns (then the body becomes a *mechanism* and the excessive forces and moments of forces can be treated as driving ones).

If we are dealing with complex system of bodies and we have a base isolated in this system, we first detach the system from the base and after that we proceed similarly as in the case of the simple system. Equilibrium conditions of such an inextricable force system are *necessary equilibrium conditions*. In the next step we mentally divide the system into subsystems detaching one by one the bodies interacting mechanically until we isolate the body and have the forces and moments of forces coming from the interactions with other bodies clearly defined. The equilibrium conditions of each of the isolated bodies now constitute the necessary and sufficient conditions for the equilibrium of the isolated body.

Let us note that the solution of the statically determinate problem is reduced to the determination of both the equilibrium position (equilibrium configuration) of the system and the unknown forces and moments of forces keeping the system in an equilibrium position.

The topic of constraints and degrees of freedom of a system will be covered in detail later; however, here we will introduce some basic notions essential to solving problems of statics.

A particle in space has three possibilities of motion (three degrees of freedom), whereas a rigid body in space has six possibilities of motion (six degrees of freedom). Since, if we introduce the geometry of the "point" (the dimension), the body gains the possibility of three independent rotations. Such a body (a particle) we call *free*. Its contact with another body occurs by means of constraints that in the analyzed problems of *statics* are called *supports*.

Now, if a rigid body (particle) during its motion is in contact with a plane at all times, the rigid body has three degrees of freedom (two translations and one rotation), whereas the particle has two degrees of freedom (two translations in the adopted coordinate system).

The mentioned plane plays the role of support, which in the case of a rigid body eliminates three degrees of freedom, and in the case of a particle, one degree of freedom. Because the particle and the rigid body by definition possess mass and mass moments of inertia, an arbitrary support (in our case the plane) produces a mechanical (supporting) interaction, that is, support forces and moments of support forces.

Depending on the geometry (shape) of the supports, they can eliminate different types of body (bodies) motion and thus create different reactions and support moments.

Below we will briefly characterize some of the supports often used in mechanical systems. Because we have already distinguished plane and space force systems, we will also classify the supports in accordance with the generation of planar and spatial force systems by them.

In Fig. 2.19 a few examples of supports with a two-dimensional *force* system are presented.

In Fig. 2.19a a sliding support is shown. The unknowns are the magnitude of reaction $\mathbf{F}^R$, because its direction is perpendicular to the radius of curvature of a base, and the reaction torque $\mathbf{M}_R$ (rotation is not possible). A similar role is played by a sleeve and telescope. In the case of Fig. 2.19b, the pin support is also present (see also Fig. 2.19c), where we have two unknowns (magnitude and direction of reaction $\mathbf{F}^R$). In order to diminish the resistance to motion (friction) between the systems in contact often rollers treated as massless elements and moving with no

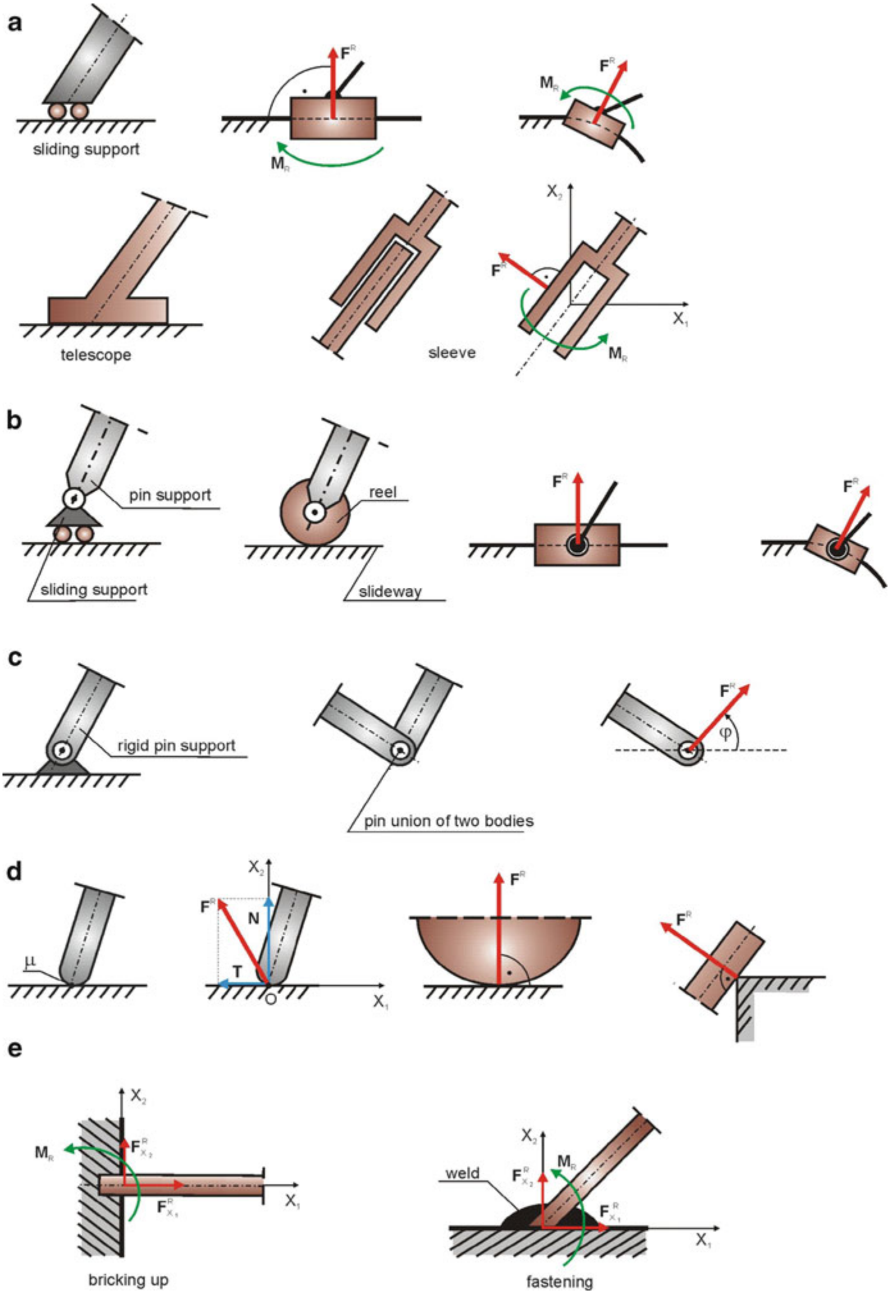


Fig. 2.19 Examples of supports with two-dimensional systems of forces, their schematic designations, and unknown reactions (moments)

resistance to motion are used (Fig. 2.19c). Rollers are often introduced in order to decrease the motion resistance (friction) between contacting bodies, and they are treated as massless bodies that are movable without motion resistance. In Fig. 2.19d the contact supports with and without friction are shown. In the latter case we are dealing with one unknown, that is, the magnitude of reaction $\mathbf{F}^R$, since its direction is defined.

In the case of a rigidly clamped edge and restraint (Fig. 2.19e), we have three unknowns, that is, two components of reaction and the reaction torque.

The supports discussed above that enable the point (the element) of a rigid body in contact with a base to move in the specific (assumed in advance) direction (technical and technological manufacture of the elements of bodies in contact) we call *directional support*. In this case the motion of the body can take place along a straight line or a curve.

Until now, our considerations have not taken into account the phenomenon of friction between the elements of a body in contact. The phenomenon of friction is the subject of the next section; here it is only necessary to emphasize that in the case of so-called fully developed friction ($T = \mu N$) we are dealing with one unknown, and in case where $T < \mu N$ (not developed friction), two unrelated forces T and N have to be determined.

In Fig. 2.20a, a ball-and-socket joint and a point contact are shown, and if a rigid body makes an arbitrary motion while in contact with a plane, we are to determine three unknown reactions $F_{x_1}^R$, $F_{x_2}^R$, and $F_{x_3}^R$. If the connection between the bodies is realized by ball-and-socket support, it prevents the translation of a body in every direction but allows for rotation about any axis passing through the center of the support (ball), and after introduction of the Cartesian coordinate system, three magnitudes of support reaction undergo determination. If the body moves on a rough surface with undeveloped friction, where $\sqrt{(T_1^2 + T_2^2)} < \mu N$, then three unknowns, T_1, T_2, and N, must also be determined. Finally, let us note that if the contact surface between the contacting parts of the bodies is from one side a sphere and from the other a cylindrical groove (the guide), the support is a ball-cylinder joint (two unknown reactions). If, instead, the contact of bodies takes place over a cylindrical surface, such a support we call a cylindrical joint (two unknown reaction forces and two reaction moments).

If a rigid body is fixed in three dimensions (Fig. 2.20b), such a support prevents any translation and any rotation of the body, and the determination of six unknown quantities, that is, three support reactions and three reaction moments, is necessary.

If the contact of a body with a plane takes place by means of a roller moving on a rough surface with undeveloped friction, two forces T and N are subject to determination. If the roller moves along the guide, this kind of support generates two unknown reactions as well (Fig. 2.20c).

In Fig. 2.20d–g, short and long radial and angular bearings are depicted along with the corresponding reaction forces and reaction moments; their diagrams are also shown.

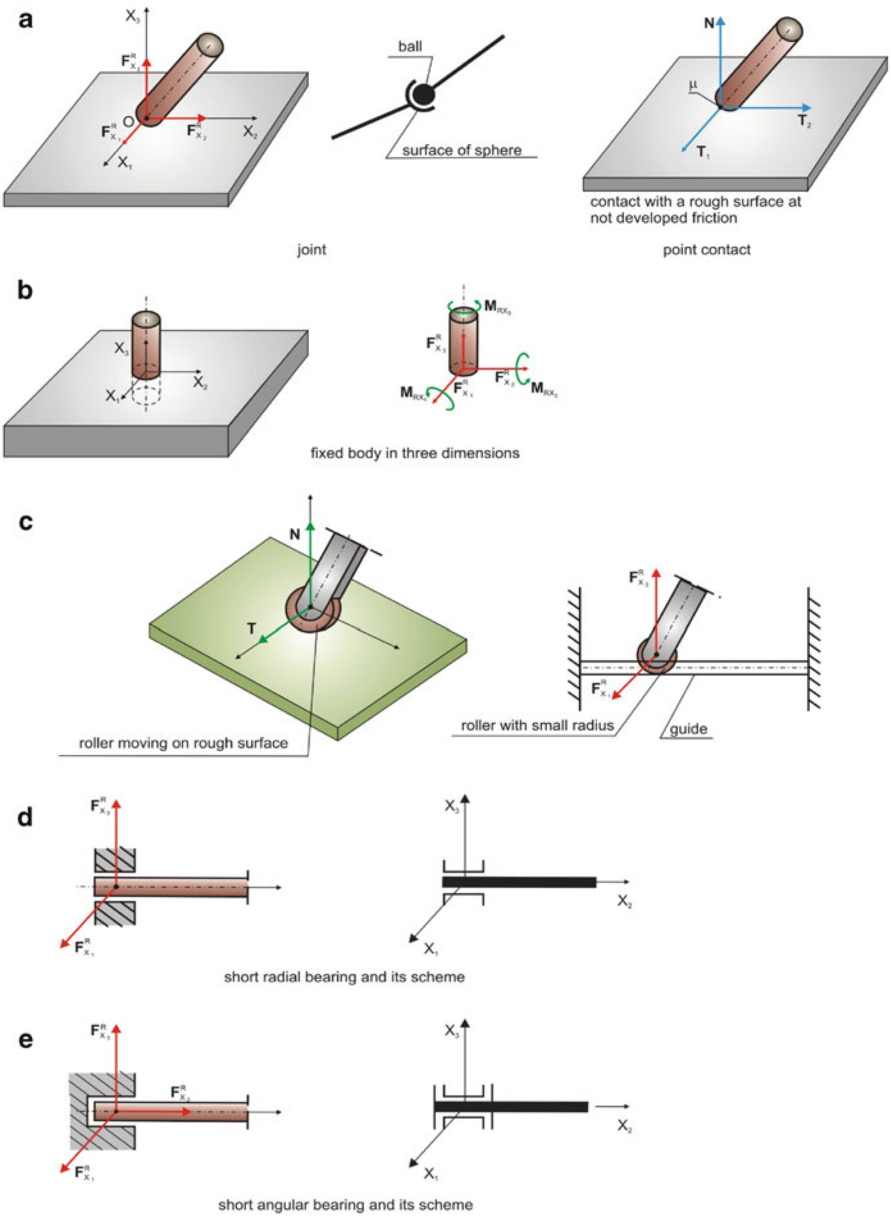


Fig. 2.20 Examples of supports carrying three-dimensional systems of forces

In Fig. 2.20h, the Cardan universal joint is shown. The quantities $F^R_{x_1}$, $F^R_{x_3}$, and M_{Rx_2}, which act on the right-hand side of the joint and undergo determination, are also marked.

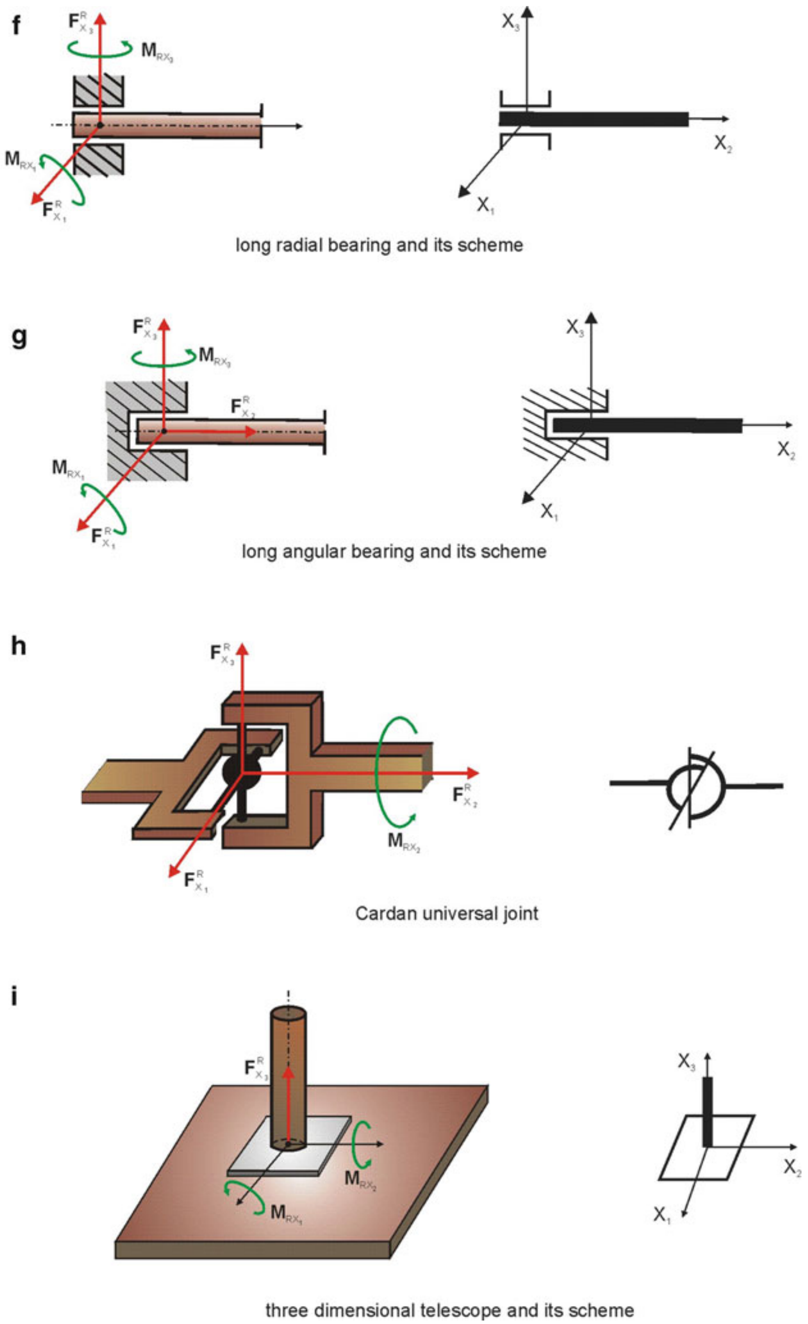


Fig. 2.20 (continued) Examples of supports carrying three dimensional systems of forces

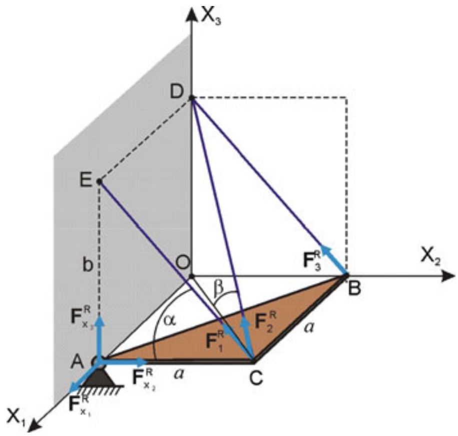


Fig. 2.21 Triangular plate suspended by three massless rods CE, CD, and BD

In Fig. 2.20i, a three-dimensional telescope is depicted along with three unknown quantities that are subject to determination. Figures 2.19 and 2.20 were prepared based on the textbooks [28, 29].

In practice, in order to determine the reactions of the base on a rigid body we choose systems of axes suitable for the particular problem and perform the projections of reaction forces and reaction moments on these axes. Let us note that the axes do not have to be mutually perpendicular, but not all of them can lie in one plane or be parallel.

In this way we solve a problem of statics in several stages.

Example 2.3. A homogeneous steel plate of the shape of a right, isosceles triangle and of the weight $\mathbf{G}$ is fixed by means of three steel weightless cables and a ball-and-socket joint at a point A (Fig. 2.21), and $AO = OB = BC = CA = a$, $AE = OD = b$. Determine reactions at point A and forces in the cables.

From equations of equilibrium with respect to the axes of the coordinate system we have

$$\begin{aligned}
OX_1: &\quad F^R_{x_1} - F^R_{2x_1} = 0,\\
OX_2: &\quad F^R_{x_2} - F^R_{1x_2} - F^R_{2x_1} - F^R_{3x_2} = 0,\\
OX_3: &\quad F^R_{x_3} + F^R_{1x_3} + F^R_{2x_3} + F^R_{3x_3} - G = 0,
\end{aligned}$$

and from equilibrium conditions of the moments of forces we obtain

$$\begin{aligned}
M_{x_1}: &\quad a\left(F^R_{1x_3} + F^R_{2x_3} + F^R_{3x_3}\right) - \frac{2}{3}aG = 0,\\
M_{x_2}: &\quad -a\left(F^R_{1x_3} + F^R_{2x_3} - F^R_{x_3}\right) + \frac{2}{3}aG = 0,\\
M_{x_3}: &\quad a\left(-F^R_{1x_2} + F^R_{x_2}\right) = 0,
\end{aligned}$$

because the gravity center of the plate is situated at the distance marked in Fig. 2.22a.

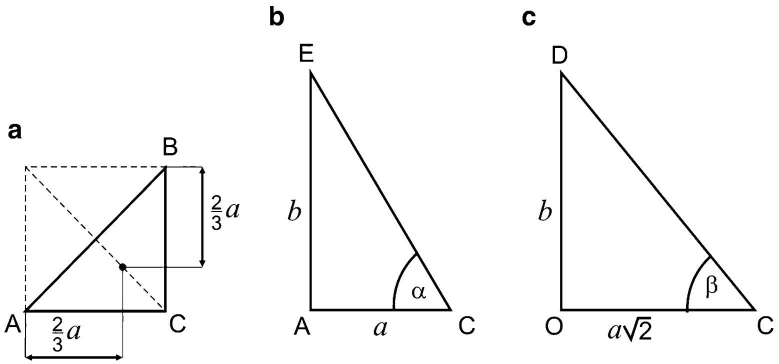


Fig. 2.22 Auxiliary sketches: (**a**) location of gravity center of plate; geometry of triangles related to rod CE (**b**) and CD (**c**)

The angles α and β are determined from the equations $\tan\beta = b/(a\sqrt{(2)})$ and $\tan\alpha = b/a$ and the equilibrium equations assume the following form:

$$\begin{cases} F_{x_1}^R - \dfrac{\sqrt{2}}{2}F_2^R\cos\beta = 0, \\ F_{x_2}^R - F_1^R\cos\alpha - \dfrac{\sqrt{2}}{2}F_2^R\cos\beta - F_3^R\cos\alpha = 0, \\ F_{x_3}^R + F_1^R\sin\alpha + F_2^R\sin\beta + F_3^R\sin\alpha - G = 0, \\ a(F_1^R\sin\alpha + F_2^R\sin\beta + F_3^R\sin\alpha) - \dfrac{2}{3}aG = 0, \\ -a(F_1^R\sin\alpha + F_2^R\sin\beta) - aF_{x_3}^R + \dfrac{2}{3}aG = 0, \\ a(-F_1^R\cos\alpha + F_{x_2}^R) = 0. \end{cases} \qquad (*)$$

From the third equation of the system $(*)$ we obtain

$$-F_{x_3}^R + G = F_1^R\sin\alpha + F_2^R\sin\beta + F_3^R\sin\alpha,$$

and substituting this into the fourth equation $(*)$ we obtain

$$-F_{x_3}^R + G - \frac{2}{3}G = 0,$$

that is,

$$F_{x_3}^R = \frac{1}{3}G.$$

Substituting $F_{x_3}^R$ into the fifth equation of $(*)$ we obtain

$$F_1^R \sin\alpha + F_2^R \sin\beta = \frac{1}{3}G, \qquad (**)$$

and substituting into the third equation of $(*)$ we have

$$F_3^R = \frac{G}{3\sin\alpha}.$$

According to the sixth equation of the system $(*)$, the second equation of $(*)$ takes the form

$$\frac{\sqrt{2}}{2}F_2^R \cos\beta + F_3^R \cos\alpha = 0,$$

and hence after substituting the already known F_3^R we obtain

$$F_2^R = -\frac{2G\cos\alpha}{3\sqrt{2}\sin\alpha\cos\beta} = -\frac{2}{3\sqrt{2}}G\frac{\cot\alpha}{\cos\beta},$$

and substituting F_2^R into $(**)$ we obtain

$$F_1^R = \frac{G}{3\sin\alpha}\left(1 + \sqrt{2}\cot\alpha\tan\beta\right).$$

Using the sixth equation of the system $(*)$ we have

$$F_{x_2}^R = \frac{G\cot\alpha}{3}\left(1 + \sqrt{2}\cot\alpha\tan\beta\right),$$

and on the basis of the first equation of $(*)$ we finally arrive at

$$F_{x_1}^R = -\frac{G}{3}\cot\alpha. \qquad \square$$

Example 2.4. Determine the reactions in support bearings of a rigid body supported along the vertical axis at two points A and B by means of ball bearings (Fig. 2.23) and loaded with an arbitrary force system $\mathbf{F}_1, \ldots, \mathbf{F}_N$.

As a result of the projection of forces on the axes of the adopted coordinate system we obtain

$$F_{Ax_1}^R + F_{Bx_1}^R + \sum_{n=1}^{N} F_{nx_1} = 0,$$

$$F_{Ax_2}^R + F_{Bx_2}^R + \sum_{n=1}^{N} F_{nx_2} = 0,$$

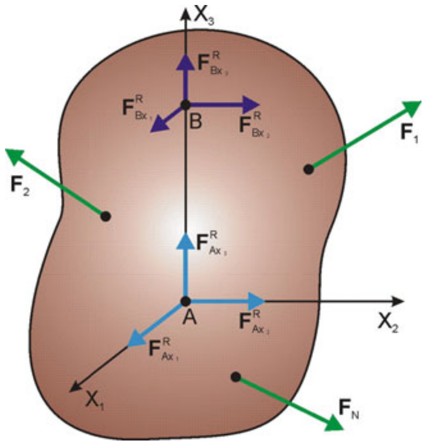


Fig. 2.23 Rigid body supported by ball bearings

$$F^R_{Ax_3} + F^R_{Bx_3} + \sum_{n=1}^{N} F_{nx_3} = 0.$$

The equations of moments about axes AX_1 and AX_2 have the following form:

$$-F^R_{Bx_2} AB + \sum_{n=1}^{N} M_{x_1}(\mathbf{F}_n) = 0,$$

$$F^R_{Bx_1} AB + \sum_{n=1}^{N} M_{x_2}(\mathbf{F}_n) = 0.$$

From the two equations above we determine $F^R_{Bx_1}$ and $F^R_{Bx_2}$, and from the first and second equations of the previous system of equations we determine $F^R_{Ax_1}$ and $F^R_{Ax_2}$.

The equation of moments about the axis AX_3 reads

$$\sum_{n=1}^{N} M_{x_3}(\mathbf{F}_n) = 0,$$

and it defines the relations between the force vectors so that the body remains in static equilibrium.

From the considerations above it follows that the third equation of the first system of equations allows for the determination of the sum of reactions $F^R_{Ax_3}$ and $F^R_{Bx_3}$, but it does not enable us to determine the individual magnitudes of $F^R_{Ax_3}$ and $F^R_{Bx_3}$. The problem becomes statically determinate if at point B the short radial bearing is applied (Fig. 2.20d). Then $F^R_{Bx_3} = 0$, and it is possible to determine the reaction $F^R_{Ax_3}$.

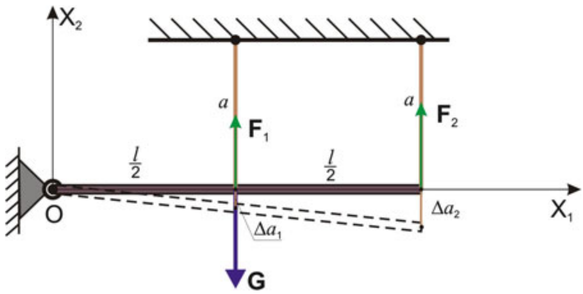


Fig. 2.24 A beam of weight **G** suspended by two steel ropes

We will now show using an example in what way statically indeterminate problems require a knowledge of kinematics. Without knowing the basic relationships of kinematics it is impossible to solve this category of problems. In Chap. 6 we generalize our observations, i.e., we solve the problem of kinematics of an elastic body (a body deformable within the limits of the linear theory). □

Example 2.5. A homogeneous beam of weight **G** and length l was suspended from two steel ropes of lengths a and cross sections S, shown in Fig. 2.24.

The sum of moments about point O leads to the following equation:

$$F_1\frac{l}{2} + F_2 l = G\frac{l}{2},$$

that is,

$$F_1 + 2F_2 = G. \tag{$*$}$$

From the equation of moments relative to the center of beam, the vertical reaction at point O is equal to F_2. Let the rigid beam be in equilibrium after deformation of both ropes. According to Hooke's law and displacement compatibility (the same angle of rotation of the beam $\Delta\varphi$) we have

$$\sigma_1 = E\varepsilon_1, \qquad \sigma_2 = E\varepsilon_2,$$

where E is Young's modulus. In turn,

$$\frac{\sigma_1}{E} \equiv \varepsilon_1 = \frac{\Delta a_1}{a} = \frac{l\Delta\varphi}{2a}, \qquad \frac{\sigma_2}{E} \equiv \varepsilon_2 = \frac{\Delta a_2}{a} = \frac{l\Delta\varphi}{a}.$$

Thus

$$\frac{F_1}{E} = \frac{l\Delta\varphi S}{2a}, \qquad \frac{F_2}{E} = \frac{l\Delta\varphi S}{a},$$

and hence

$$\frac{2aF_1}{ElS} = \frac{aF_2}{ElS},$$

that is, $2F_1 = F_2$. After substituting this relation into ($*$) we obtain successively $F_1 = \frac{1}{5}G$, $F_2 = \frac{2}{5}G$. □

2.6 Reduction of a Space Force System to a System of Two Skew Forces

We have already mentioned the possibility of reduction of a space force system into a so-called *equivalent system of three forces*, and then to a system of two skew forces, i.e., the forces of non-intersecting lines of action.

Let a rigid body be loaded at points B_n with forces $\mathbf{F}_n$, $n = 1, \ldots, N$. Let us take one of the points on the line of action of the force $\mathbf{F}_n$ and denote it O_n. Let us take three arbitrary, non-collinear points B_1, B_2, and B_3 and link them with point O_n (Fig. 2.25).

We attach the free vector $\mathbf{F}_n$ at point O_n and resolve it into components along the axes $\overrightarrow{O_n B_1}$, $\overrightarrow{O_n B_2}$, and $\overrightarrow{O_n B_3}$, that is,

$$\mathbf{F}_n = \mathbf{F}_{nB_1} + \mathbf{F}_{nB_2} + \mathbf{F}_{nB_3}. \tag{2.42}$$

We will proceed similarly regarding other forces, i.e., on each of their lines of action we will take a point, where we will attach the mentioned force vector, and again we will resolve it into components that link the selected point with the fixed points B_1, B_2, and B_3. In that way we will reduce all forces to three forces attached at the three chosen points B_1, B_2, and B_3, i.e., we have

$$\mathbf{F}_{B_1} = \sum_{n=1}^{N} \mathbf{F}_{nB_1}, \quad \mathbf{F}_{B_2} = \sum_{n=1}^{N} \mathbf{F}_{nB_2}, \quad \mathbf{F}_{B_3} = \sum_{n=1}^{N} \mathbf{F}_{nB_3}. \tag{2.43}$$

In a general case these three forces are situated in space and do not intersect each other. Now we will show how to reduce those forces to two skew forces (Fig. 2.26).

At the mentioned points B_1, B_2, and B_3 in Fig. 2.26 we attach the force vectors $\mathbf{F}_{B_1}$, $\mathbf{F}_{B_2}$, and $\mathbf{F}_{B_3}$ determined earlier.

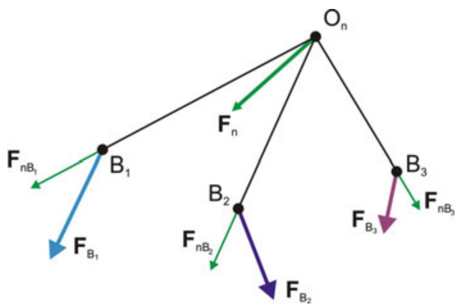


Fig. 2.25 Reduction of a space force system to forces $\mathbf{F}_{B_1}$, $\mathbf{F}_{B_2}$, and $\mathbf{F}_{B_3}$

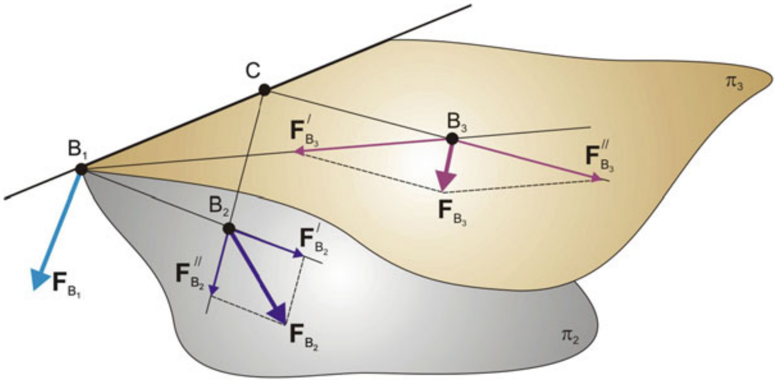


Fig. 2.26 Reduction of a force system to two skew forces

Point B_1 and vector $\mathbf{F}_{B_2}$ determine the plane π_2, and point B_1 and vector $\mathbf{F}_{B_3}$ determine the plane π_3. These planes intersect along an edge on which lies point B_1. Let us take now point C on that edge and connect both points B_1 and C lying on the edge with points B_2 and B_3. We will resolve the vector of force $\mathbf{F}_{B_2}$ in the plane π_2 into components along the rays CB_2 and B_1B_2, and the vector $\mathbf{F}_{B_3}$ in the plane π_3 into components along the rays CB_3 and B_1B_3 using the parallelogram law.

Because we are dealing with a rigid body, we will attach vectors $\mathbf{F}'_{B_2}$ and $\mathbf{F}'_{B_3}$ at point B_1 and vectors $\mathbf{F}''_{B_2}$ and $\mathbf{F}''_{B_3}$ at point C. Let us note that at points B_1 and C we obtain three-dimensional force systems that are concurrent. According to the considerations made earlier, the action of concurrent forces at those points can be replaced with resultant forces

$$\begin{aligned} \mathbf{F}^r_{B_1} &= \mathbf{F}_{B_1} + \mathbf{F}'_{B_2} + \mathbf{F}'_{B_3}, \\ \mathbf{F}^r_{C} &= \mathbf{F}''_{B_2} + \mathbf{F}''_{B_3}. \end{aligned} \tag{2.44}$$

These forces do not intersect and their lines of action are not parallel. The system of parallel forces $\mathbf{F}_n$, $n = 1, \ldots, N$ acting on a rigid body will be in equilibrium if these forces are collinear and $\mathbf{F}^r_{B_1} + \mathbf{F}^r_C = \mathbf{0}$. It is difficult to satisfy such a condition proceeding with a geometrical construction.

On the other hand, according to the previous construction, taking an arbitrary pole O we can reduce these forces to the main force vector $\mathbf{F} = \mathbf{F}^r_{B_1} + \mathbf{F}^r_C$ and main moment of force vector $\mathbf{M}_O = \mathbf{M}_O(\mathbf{F}^r_{B_1}) + \mathbf{M}_O(\mathbf{F}^r_C)$.

2.7 Reduction of a Space Force System to a Wrench

Expanding the earlier considerations concerning a plane force system, one can perform the reduction of an arbitrary space force system $\mathbf{F}_n$ attached at the points $A_n(x_{1n}, x_{2n}, x_{3n})$, $n = 1, \ldots, N$ to a certain arbitrarily chosen reduction pole $O(x_{10}, x_{20}, x_{30})$.

Let us consider the rigid body depicted in Fig. 2.11a; now the force system $\mathbf{F}_n$ is three-dimensional. In that figure we did not introduce the moments of force as an external load (they can be generated by the force vectors of the same lengths and opposite senses acting in parallel directions). In this case (2.27) will remain unchanged (now $\mathbf{S} \neq \mathbf{0}$), but (2.29) will assume the form

$$M_{x_1} = \sum_{n=1}^{N} \left[(x_{2n} - x_{20}) F_{x_3 n} - (x_{3n} - x_{30}) F_{x_2 n} \right],$$

$$M_{x_2} = \sum_{n=1}^{N} \left[(x_{3n} - x_{30}) F_{x_1 n} - (x_{1n} - x_{10}) F_{x_3 n} \right],$$

$$M_{x_3} = \sum_{n=1}^{N} \left[(x_{1n} - x_{10}) F_{x_2 n} - (x_{2n} - x_{20}) F_{x_1 n} \right]. \tag{2.45}$$

In general, the magnitudes of the components of the main moment of force vector depend on the choice of pole O, but the main force vector $\mathbf{S}$ does not change with the position of pole O (it is the *first invariant* of reduction in statics). Often the notion of that invariant would be introduced in the form $I_1 = S_{x_1}^2 + S_{x_2}^2 + S_{x_3}^2$. It turns out that the *second invariant* of reduction is the projection of the main moment of force vector $\mathbf{M}$ on the direction of the main force vector $\mathbf{S}$, which then has the form $I_2 = \mathbf{M} \circ \mathbf{S}$.

Let us consider now the projection of vector $\mathbf{M}$ on the direction of the vector $\mathbf{S}$ of the following form:

$$\mathbf{M} \circ \mathbf{S} = |\mathbf{M}||\mathbf{S}| \cos\varphi = M_S |\mathbf{S}|. \tag{2.46}$$

Bearing in mind that $\dfrac{\mathbf{S}}{|\mathbf{S}|}$ is the unit vector, we have

$$\mathbf{M}_S = M_S \frac{\mathbf{S}}{|\mathbf{S}|} = \frac{\mathbf{M} \circ \mathbf{S}}{|\mathbf{S}|^2} \mathbf{S}, \tag{2.47}$$

where $\frac{\mathbf{M} \circ \mathbf{S}}{|\mathbf{S}|^2} = p$ is called the pitch of the wrench. Now, let us change the pole from point O to some other point O' (Fig. 2.27).

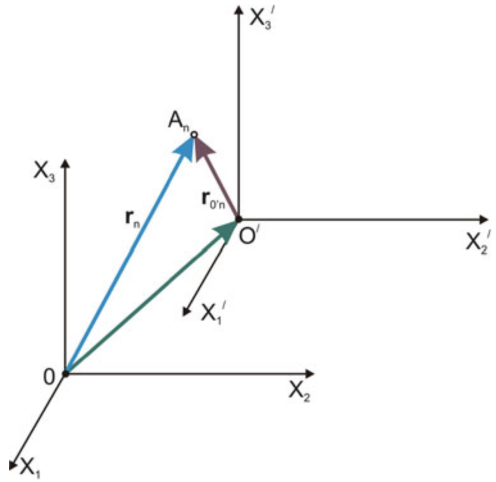


Fig. 2.27 Two poles O and O' with corresponding Cartesian coordinate systems

According to Fig. 2.27, the main moment of force vector $\mathbf{M}_O$ about pole O' is equal to

$$\begin{aligned}\mathbf{M}_{O'} &= \sum_{n=1}^{N}\left(\mathbf{r}_{O'n}\times\mathbf{F}_n\right) = \sum_{n=1}^{N}\left(\mathbf{r}_n+\overrightarrow{O'O}\right)\times\mathbf{F}_n\\ &= \sum_{n=1}^{N}\left(\mathbf{r}_n\times\mathbf{F}_n\right)+\sum_{n=1}^{N}\left(\overrightarrow{O'O}\times\mathbf{F}_n\right)\\ &= \mathbf{M}+\overrightarrow{O'O}\times\sum_{n=1}^{N}\mathbf{F}_n = \mathbf{M}+\overrightarrow{O'O}\times\mathbf{S},\end{aligned} \tag{2.48}$$

where $\mathbf{M}$ is the main moment about point O.

After projection of the vector $\mathbf{M}_{O'}$ on the direction of $\mathbf{S}$ we obtain

$$\begin{aligned}\mathbf{M}_{O'}\circ\mathbf{S} &= \left[\mathbf{M}+\left(\overrightarrow{O'O}\times\mathbf{S}\right)\right]\circ\mathbf{S}\\ &= \mathbf{M}\circ\mathbf{S}+\left(\overrightarrow{O'O}\times\mathbf{S}\right)\circ\mathbf{S} = \mathbf{M}\circ\mathbf{S}\end{aligned} \tag{2.49}$$

because the vector $\overrightarrow{O'O}\times\mathbf{S}$ is perpendicular to vector $\mathbf{S}$ (the scalar product $(\overrightarrow{O'O}\times\mathbf{S})\circ\mathbf{S} = 0$). Thus, we showed that the change of the pole does not change the projection of a main moment of force vector on the direction of a main force vector. Let us take once more some other arbitrary pole. Let this be point O to which we reduced, in the already described fashion, the force system. Through point O we will draw a line l parallel to vector $\mathbf{S}$ and attach at that point vector $\mathbf{M}$. Vectors $\mathbf{M}$ and $\mathbf{S}$ will determine a certain plane π (Fig. 2.28).

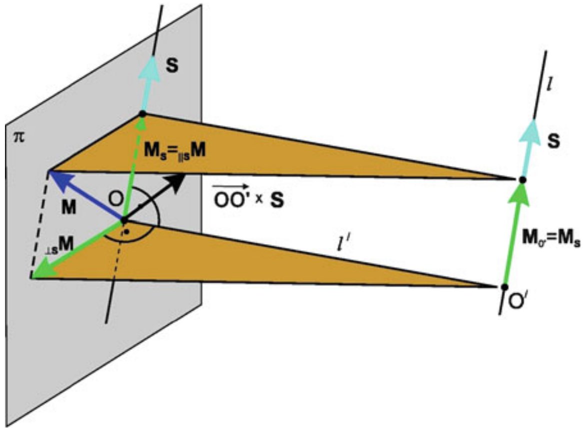


Fig. 2.28 A construction leading to the determination of an axis of a wrench l

In that plane we will resolve vector $\mathbf{M}$ into two components: $_{\|\mathbf{S}}\mathbf{M}$ and $_{\perp\mathbf{S}}\mathbf{M}$. Now, through point O we will draw a line l' perpendicular to plane π and we choose as the new pole O' (belonging to both l and l') with respect to which we will perform the reduction of $\mathbf{M}$ and $\mathbf{S}$. Let us note that

$$\mathbf{M}_{O'} = \mathbf{M} + \overrightarrow{O'O} \times \mathbf{S}, \tag{2.50}$$

and since

$$\mathbf{M} = {}_{\|\mathbf{S}}\mathbf{M} + {}_{\perp\mathbf{S}}\mathbf{M}, \tag{2.51}$$

we have

$$\begin{aligned}\mathbf{M}_{O'} &= {}_{\|\mathbf{S}}\mathbf{M} + {}_{\perp\mathbf{S}}\mathbf{M} + \overrightarrow{O'O} \times \mathbf{S} \\ &= \mathbf{M}_\mathbf{S} + {}_{\perp\mathbf{S}}\mathbf{M} + \overrightarrow{O'O} \times \mathbf{S}.\end{aligned} \tag{2.52}$$

The component $_{\perp\mathbf{S}}\mathbf{M} + \overrightarrow{O'O} \times \mathbf{S}$ undergoes a change with the change in position of point O'. Let us take point O' so that $\mathbf{M}'_O = \mathbf{M}_S$. From (2.52) it follows that

$$\begin{aligned}\mathbf{M}_\mathbf{S} &= {}_{\perp\mathbf{S}}\mathbf{M} + \mathbf{M}_\mathbf{S} + \overrightarrow{O'O} \times \mathbf{S}, \\ {}_{\perp\mathbf{S}}\mathbf{M} &= \overrightarrow{OO'} \times \mathbf{S}.\end{aligned} \tag{2.53}$$

From (2.53) it follows that

$$|\overrightarrow{OO'}| = \frac{|{}_{\perp\mathbf{S}}\mathbf{M}|}{|\mathbf{S}|}, \tag{2.54}$$

which defines the distance of point O from point O'.

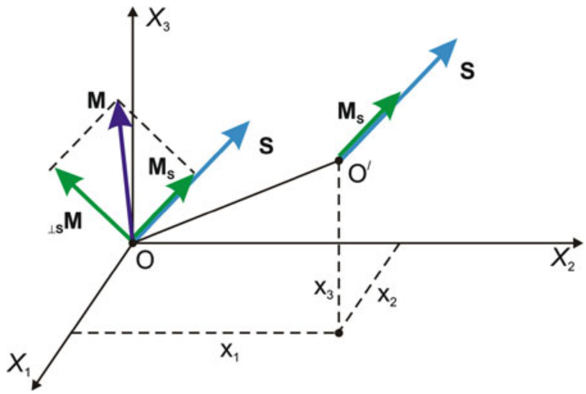


Fig. 2.29 Force S, main moment of force, and location of central axis

The axis passing through point O' and parallel to vector **S** we call a *central axis*. By the choice of an arbitrary pole O it was shown how to determine the location of the wrench axis by means of construction through the proper choice of point O'.

From the foregoing considerations we can draw an important conclusion.

The action of an arbitrary space force (and moments) system can be replaced by the action of a single force (being the geometric sum of all forces) and one main moment of force, and both vectors lie on a common line called the *central axis*.

Equations (2.53) allow for the determination of the central axis in space. We obtain from them the following equation:

$$\mathbf{M}-\mathbf{M_S} = \sum_{i=1}^{3}\left(M_{x_i}-M_{S_{x_i}}\right)\mathbf{E}_i = \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ x_1 & x_2 & x_3 \\ S_{x_1} & S_{x_2} & S_{x_3} \end{vmatrix}$$

$$= \mathbf{E}_1\left(x_2 S_{x_3} - x_3 S_{x_2}\right) + \mathbf{E}_2\left(x_3 S_{x_1} - x_1 S_{x_3}\right) + \mathbf{E}_3\left(x_1 S_{x_2} - x_2 S_{x_1}\right), \quad (2.55)$$

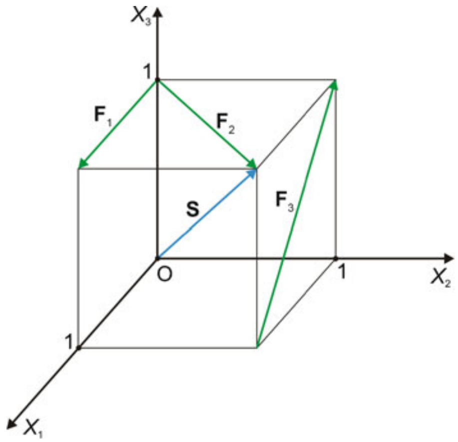
and hence

$$\begin{aligned} M_{x_1} - M_{S_{x_1}} &= x_2 S_{x_3} - x_3 S_{x_2}, \\ M_{x_2} - M_{S_{x_2}} &= x_3 S_{x_1} - x_1 S_{x_3}, \\ M_{x_3} - M_{S_{x_3}} &= x_1 S_{x_2} - x_2 S_{x_1}. \end{aligned} \quad (2.56)$$

The method of determination of the central axis is illustrated in Fig. 2.29.

If we treated the coordinates of the point $O'(x_1, x_2, x_3)$ in the coordinate system $OX_1X_2X_3$ as that determined from (2.56), then knowing the coordinates of vector **S**, we may find the equation of the central axis.

Fig. 2.30 Three forces and location of a pole O



If the coordinate system is introduced at some other point of the body and the coordinates of the pole in that system are denoted as (x_{10}, x_{20}, x_{30}), then the point belonging to the central axis can be found from the following equations:

$$\begin{aligned} M_{x_1} - M_{S_{x_1}} &= (x_2 - x_{20})S_{x_3} - (x_3 - x_{30})S_{x_2}, \\ M_{x_2} - M_{S_{x_2}} &= (x_3 - x_{30})S_{x_1} - (x_1 - x_{10})S_{x_3}, \\ M_{x_3} - M_{S_{x_3}} &= (x_1 - x_{10})S_{x_2} - (x_2 - x_{20})S_{x_1}. \end{aligned} \tag{2.57}$$

Let us assume that we reduce the space force system about pole O. We are dealing with the following special cases:

1. $\mathbf{S} \neq \mathbf{0}$, $\mathbf{M}_O \neq \mathbf{0}$ and $\mathbf{M}_O \perp \mathbf{S}$.
   Then $\mathbf{M}_S = \mathbf{M}_O \circ \mathbf{S} = \mathbf{0}$. The resultant force $\mathbf{F}^r$ is parallel to $\mathbf{S}$ but cannot pass through point O. It is located on the central axis.
2. $\mathbf{S} \neq \mathbf{0}$, $\mathbf{M}_O = \mathbf{0}$.
   The resultant $\mathbf{F}^r = \mathbf{S}$ and passes through pole O (because it does not produce a moment).
3. $\mathbf{S} = \mathbf{0}$, $\mathbf{M}_O \neq \mathbf{0}$.
   The moment of force vector is a free vector, so it can be attached at an arbitrary point in space.
4. $\mathbf{S} = \mathbf{0}$, $\mathbf{M}_O = \mathbf{0}$.
   The analyzed rigid body is in equilibrium.

Example 2.6. Reduce the three forces of magnitudes $F_1 = 1$, $F_2 = \sqrt{2}$, and $F_3 = \sqrt{2}$ that act as shown in Fig. 2.30.

The axes of the coordinate system are taken along the edges of the cube. We aim to determine the components of vector $\mathbf{S}$ and moment $\mathbf{M}$, which are equal to:

$$
\begin{aligned}
S_{x_1} &\equiv F_1 + \frac{\sqrt{2}}{2}F_2 - F_3\frac{\sqrt{2}}{2} = 1,\\
S_{x_2} &= F_2\frac{\sqrt{2}}{2} = 1,\\
S_{x_3} &= F_3\frac{\sqrt{2}}{2} = 1,\\
M_{x_1} &= -F_2\frac{\sqrt{2}}{2}\cdot 1 + F_3\frac{\sqrt{2}}{2}\cdot 1 = 0,\\
M_{x_2} &= F_1\cdot 1 - F_3\frac{\sqrt{2}}{2}\cdot 1 + F_2\frac{\sqrt{2}}{2} = 1,\\
M_{x_3} &= F_3\frac{\sqrt{2}}{2}\cdot 1 = 1.
\end{aligned}
$$

In order to make use of formula (2.56) we need to know additionally the values of M_{Sx_1}, M_{Sx_2}, and M_{Sx_3}. For the determination of the coordinates of vector $\mathbf{M}_S$ we will multiply (2.47) in turn by $\mathbf{E}_i$, $i = 1, 2, 3$.

Since

$$
\begin{aligned}
\mathbf{M} \circ \mathbf{S} &= M_{x_1}S_{x_1} + M_{x_2}S_{x_2} + M_{x_3}S_{x_3} = 2,\\
|\mathbf{S}|^2 &= S_{x_1}^2 + S_{x_2}^2 + S_{x_3}^2 = 3,
\end{aligned}
$$

after using (2.47) we obtain

$$
M_{S_{x_1}} = \frac{2}{3}, \quad M_{S_{x_2}} = \frac{2}{3}, \quad M_{S_{x_3}} = \frac{2}{3}.
$$

According to (2.56) we have

$$
x_2 - x_3 = -\frac{2}{3}, \quad -x_1 + x_3 = \frac{1}{3}, \quad x_1 - x_2 = \frac{1}{3}.
$$

In order to determine the point of intersection of three planes we will calculate the determinant of the system

$$
\Delta = \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 0 + 1\cdot(-1)^3(-1) - 1\cdot(-1)^4(1) = 1 - 1 = 0,
$$

and the minors

$$\Delta_{x_1} = \begin{vmatrix} -\frac{2}{3} & 1 & -1 \\ \frac{1}{3} & 0 & 1 \\ \frac{1}{3} & -1 & 0 \end{vmatrix} = \frac{1}{3}\cdot(-1)^3(-1) + 1\cdot(-1)^5\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{1}{3} - \frac{1}{3} = 0,$$

$$\Delta_{x_2} = \begin{vmatrix} 0 & -\frac{2}{3} & -1 \\ -1 & \frac{1}{3} & 1 \\ 1 & \frac{1}{3} & 0 \end{vmatrix} = -\frac{2}{3}(-1)^3(-1) - 1\cdot(-1)^4\left(-\frac{2}{3}\right) = -\frac{2}{3} + \frac{2}{3} = 0,$$

$$\Delta_{x_3} = \begin{vmatrix} 0 & 1 & -\frac{2}{3} \\ -1 & 0 & \frac{1}{3} \\ 1 & -1 & \frac{1}{3} \end{vmatrix} = 1\cdot(-1)^3\left(-\frac{2}{3}\right) - \frac{2}{3}\cdot(-1)^4(1) = \frac{2}{3} - \frac{2}{3} = 0.$$

Recall that if $\Delta \neq 0$, then the three planes intersect at one point. If $\Delta = \Delta x_1 = \Delta x_2 = \Delta x_3 = 0$, then the three planes have a common line.

Let us take $x_2 = 0$. We calculate $x_1 = \frac{1}{3}$ and $x_3 = \frac{2}{3}$. Point $A(\frac{1}{3}, 0, \frac{2}{3})$ belongs to the line constituting the central axis. We determine the central axis by drawing a line parallel to $\mathbf{S}$ through point A.

If the planes intersect at one point, we determine that point solving the system of three algebraic non-homogeneous equations. Let the point of intersection have the coordinates $x_{1A},\ x_{2A},\ x_{3A}$. The central axis passes through that point and is parallel to the vector $\mathbf{S} = \mathbf{E}_1 S_{x1} + \mathbf{E}_2 S_{x2} + \mathbf{E}_3 S_{x3}$. In view of that, the equation of central axis has the following form:

$$\frac{x_1 - x_{1A}}{S_{x_1}} = \frac{x_2 - x_{2A}}{S_{x_2}} = \frac{x_3 - x_{3A}}{S_{x_3}}. \qquad \square$$

The steps illustrated in Sects. 2.1–2.7 that aimed to reduce and simplify force systems (providing a lack of friction) can be briefly summarized in the following manner:

(a) Reducing a force system to a resultant force $\mathbf{F} = \sum_{n=1}^{N} \mathbf{F}_n$ and a couple at given point O (resultant moment):

$$\mathbf{M}_O = \sum_{n=1}^{N} (\mathbf{r} \times \mathbf{F}_n).$$

(b) Moving a force–couple system from point O_1 to point O_2, expressed by the formula

$$\mathbf{M}_{O_2} = \mathbf{M}_{O_1} + \overrightarrow{O_2O_1} \times \mathbf{F}.$$

(c) By equivalency checking, reducing to the one point O two systems of forces, which are equivalent if the two force–couple systems are identical, i.e.,

$$\sum_{n=1}^{N} \mathbf{F}_n = \sum_{k=1}^{K} \mathbf{F}_k,$$

$$\sum_{n=1}^{N} (\mathbf{r_n} \times \mathbf{F_n}) = \sum_{k=1}^{K} (\tilde{\mathbf{r}}_k \times \tilde{\mathbf{F}}_k).$$

(d) Reducing a given force system to a single force; this takes place only if $\mathbf{S} \perp \mathbf{M}_O$, i.e., when force $\mathbf{S}$ and couple vector $\mathbf{M}_O$ are mutually perpendicular (which happens for concurrent coplanar and parallel forces). If position vector $\mathbf{r}$ depicted from point O to any point on the line of action of the single force $\mathbf{F}$ satisfies the equation $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$.
(e) Applying force and funicular polygons (Sect. 2.2).
(f) Applying the three-moments theorem (Sect. 2.4).
(g) Reducing a system of parallel forces (Sect. 2.4).
(h) Reducing a three-dimensional force system to two skew forces (Sect. 2.6).
(i) Reducing a given force system to a wrench.

In general for a three-dimensional force system (not concurrent, coplanar, or parallel) one may introduce the reduction on a basis of items (f), (h), and (i). Observe that the equivalent force system consisting of the resultant force $\mathbf{S}$ and the couple vector $\mathbf{M}_O$ are not mutually perpendicular and hence, although they cannot be reduced to a single force, they can be reduced to a wrench since they constitute the combination of the force $\mathbf{S}$ and a couple vector $\mathbf{M}_S$ $(\mathbf{S} \parallel \mathbf{M}_S)$ that lie on the wrench (case i). In other words, after reduction of a given force–couple system $(\mathbf{S}, \mathbf{M}_O)$ and after determination of the pitch $p = \frac{\mathbf{S}_O \circ \mathbf{M}_O}{S^2}$, $\mathbf{M}_s$ satisfies the equation $\mathbf{M}_S + \mathbf{r} \times \mathbf{S} = \mathbf{M}_O$. It allows one to find that point where the line of action of the wrench intersects a special plane ($\mathbf{r}$ is directed from O to that point).

2.8 Friction

The phenomenon of friction is common both in mechanics and in everyday life. Many scientific works, including monographs, cover this topic (citation is omitted here). Also, the present author and his coworkers have published monographs on the problems of classical friction [30–35].

In general, friction is a force that opposes a body's motion and possesses a sense opposite to that of the velocity of the body's relative motion. In order to ensure sliding of one surface with respect to another if both remain continuously in contact, it is necessary to act all the time on the moving body with a certain force. This requirement is connected with the resistance to motion called *sliding friction*. This is because it turns out that even apparently very smooth surfaces possess irregularities that cause resistance to motion. Friction is divided into *static* and *dynamic* friction. We deal with the former when we want to move one body with respect to another, e.g., remaining at rest. In the majority of cases, static friction is greater than dynamic friction. Sliding friction depends on the condition of the contacting surfaces (i.e., whether they are dry or lubricated), their smoothness and wear resistance abilities, and humidity and temperature.

In cases where two bodies roll with respect to one another, the friction is inversely proportional to the radius of the object that rolls. Such friction is called *rolling friction*.

Moreover, we distinguish *aero* and *hydrodynamic* friction. The former concerns the motion of objects in gases, while the latter concerns that in liquids. For small speeds v of objects motion in a fluid (liquid or gas), the force of resistance is proportional to the speed, whereas for medium (high) speeds of motion, the force is proportional to v^2 (v^3). It should be emphasized that the character of the mentioned dependencies presents the problem rather qualitatively since the described phenomenon of friction depends on the temperature, viscosity, object shape, and fluid density. In general the friction phenomenon is not yet fully understood because it is a very complex process connected with motion resistance and accompanied by the heat generation and wear of contacting bodies. Friction can be treated, then, also as a certain energetic process.

In most cases the kinetic energy of objects subjected to the action of friction forces is turned into heat energy. For instance, it is widely known that objects moving in air such as airplanes, rockets, or spaceships are very hot, which demands application of new materials of high quality and high heat resistance. Also, the rims of wheeled vehicles heat and the striking of two firestones or rubbing of two pieces of wood leads to igniting a fire. In this case we are dealing with *energy dissipation*, which means that the mechanical energy (kinetic and potential) of the object is not conserved since part of it is converted into *internal energy* (in this case into heat).

The friction between two solid bodies, as can be seen from the preceding description, is a complicated physical phenomenon. Apart from the mentioned heating of bodies, often we are dealing with their electrification, changes in the surface of the rubbing bodies, or diffusion phenomena, i.e., migration of the molecules of one body into another body.

Generally speaking, it is possible to distinguish three basic types of frictional interactions during the contact of two bodies: (a) sliding friction connected with the translational motion (no rotation) of two bodies; (b) sliding friction connected with rotational motion; (c) rolling resistance, e.g., a train wheel on a rail; and (d) sliding torsion friction.

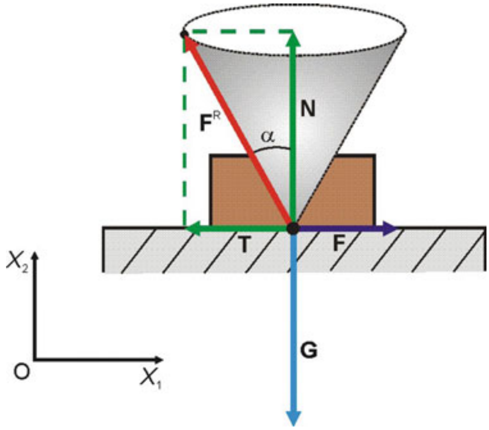


Fig. 2.31 A sliding friction and a cone of friction

Of the many researchers dealing with the friction phenomenon, the French physicist Amontons[3] should be counted as a pioneer in the field. He proved in the course of empirical investigations a lack of dependency of the friction force on the area of contact of rubbing bodies. The fundamental laws of friction were formulated by Coulomb.[4]

From the considerations above it follows that investigations aimed at finding the connection of these phenomena with the atomic scale are of great importance for a deeper understanding of the phenomenon. In this direction are moving such sciences as tribology, physics, and nanomechanics. Those investigations are important in the age of common miniaturization, nanotechnology, and mechatronics.

Friction depends on the pressure of one surface rubbing against another, but also on the relative speed of the bodies in contact. In this textbook we will consider only the classical laws of friction, but the reader particularly interested in this topic can refer to the works cited above.

Let us consider a classical example treated in many textbooks (Fig. 2.31).

A rigid body of rectangular cross section (its height is neglected) and weight $\mathbf{G}$ lies on a horizontal surface and is subjected to an action of a certain force $\mathbf{F}$. If the body remains at rest, after projection of the existing forces onto the axes OX_1 and OX_2 we obtain that $\mathbf{T} = -\mathbf{F}$ and $\mathbf{G} = -\mathbf{N}$, where $\mathbf{N}$ is the normal force and $\mathbf{T}$ the friction force (static reaction) that results from the contact between the bodies.

It is easy to imagine, and it follows also from everyday experiences, that the increase of the magnitude of force $\mathbf{F}$ will cause the increase of magnitude of tangential reaction (for $\mathbf{F} = \mathbf{0}$ there is no tangential reaction). Such a process will continue up to a certain threshold magnitude of force $\mathbf{F}_{th}$. After crossing this magnitude of force the body will start to slide over the other fixed body (the base),

[3] Guillaume Amontons (1663–1705), self-taught French scholar dealing with, among other things, thermodynamics and friction.

[4] Charles Augustin de Coulomb (1736–1806), French physicist.

and we will be dealing with kinetic friction. The friction corresponding to $\mathbf{F}_{\text{th}}$ will be called the *limiting friction force* or a *fully developed friction force*.

The reaction $\mathbf{F}^R$ caused by the forces $\mathbf{F}$ and $\mathbf{G}$ can be determined based on the normal force $\mathbf{N}$ and the friction force $\mathbf{T}$ after adding both vectors, i.e.,

$$\mathbf{F}^R = \mathbf{T} + \mathbf{N}. \tag{2.58}$$

It is easy to observe that if $\mathbf{T} \to \mathbf{T}_{\text{th}}$, the magnitude of the resultant force vector increases. The angle of inclination α of the reaction to the vertical line increases as well, and for $\mathbf{T} = \mathbf{T}_{\text{th}}$ it reaches a value called the *angle of repose*.

Applying the classical Coulomb and Morin[5] laws of friction one should remember that they are only a certain approximation of the complex phenomena characterizing the friction process. According to Fig. 2.31 we have $T_{\text{th}} = N \tan\alpha$ and after introducing the coefficient of sliding friction we finally obtain

$$T_{\text{th}} = \mu N. \tag{2.59}$$

This moment is well worth emphasizing because for the first time mechanics has to refer to the experiment in this case (it will happen a second time during the analysis of impact phenomena and will be connected with the determination of the so-called *coefficient of restitution*). It is not possible to determine the friction force without knowing the coefficient μ, which is obtained as a result of experimental research. For rough surfaces the coefficient is big and diminishes for smooth and lubricated surfaces. The coefficient of friction may depend on the manufacturing processes connected with a material's production, e.g., the value of coefficient of friction in the case of rolled steel depends on the direction of rolling. For certain materials (e.g., wood) friction has an anisotropic character, i.e., the coefficient μ depends on the direction of motion.

The Coulomb–Morin model distinguishes between *static* and *kinetic* (*sliding*) *friction*—the forces lying in a tangent plane at the point of contact of the bodies when the (tangential) relative velocity of the contacting bodies at that point (slip velocity) is respectively equal to zero (no slip) or greater than zero (slip). The magnitude of the static friction force can change from zero up to a certain maximum value proportional to the normal force, which is usually written as $T = \mu_s N$ (μ_s is the *coefficient of static friction*). This definition includes both the introduced "tangent reaction" and the "threshold friction force" (the tangent reaction is the friction force as well). The magnitude of the static friction force is usually not known a priori (the reaction of the friction constraints) and its sense is determined as being opposite to the tendency of sliding—sense of slip at the contact point for perfectly smooth surfaces. The force of kinetic friction is defined as $T = \mu_k N$ (μ_k is the *coefficient of kinetic friction*) and its sense is opposite to that of slip velocity, and $\mu_k < \mu_s$. A separate and little researched (not adequately mathematically described) problem is the transient phenomena for small slip velocities.

[5] Arthur Jules Morin (1795–1880), French general and professor of mechanics.

Although in Fig. 2.31 we have considered so far the plane system, we may consider friction in space as well. Then force $\mathbf{F}^R$ will change depending on the change in force $\mathbf{F}$. The vector $\mathbf{F}^R$ at the change of $\mathbf{F} \in [\mathbf{0}, \mathbf{F}_{\mathrm{th}}]$ will be located inside the cone whose generatrix and normal force form an angle of friction α. In the case of fully developed friction, the opening angle of the cone $2\alpha_{\mathrm{th}}$ is the greatest. If the reaction $\mathbf{F}^R$ is situated outside the cone, the state of rest of the investigated body will be interrupted, and the body will start sliding over the surface of the base body. A shift from a state of rest to motion, i.e., from statics to dynamics, will occur. The body will remain at relative rest (because in a general case both bodies can move with respect to the adopted fixed coordinate system) if the following inequality is satisfied:

$$T \leq \mu_{\mathrm{s}} N. \tag{2.60}$$

The cross section of a cone with a plane perpendicular to vector $\mathbf{N}$, depicted in Fig. 2.31, is circular. In the case of anisotropic friction, the cross section is not circular, but it might be, for example, an ellipse.

In further considerations we will make use of the three laws connected with friction, which had already been formulated by Coulomb.

1. The threshold friction force is proportional to the normal force, i.e., $T_{\mathrm{th}} = \mu_{\mathrm{s}} N$, where μ_{s} is the *coefficient of threshold static friction (fully developed)*.
2. The friction force does not depend on the size of area of contact between the bodies.
3. The coefficient of friction depends on the kind of material and the condition of surfaces of the bodies in contact.

If the body is in motion, usually the coefficient of friction μ_{k} (so-called kinetic friction) is smaller than the coefficient of developed friction μ_{s}, that is, $\mu_{\mathrm{k}} \leq \mu_{\mathrm{s}}$. This difference leads to certain theoretical difficulties connected with the transition from rest to motion (and vice versa) and with the mathematical modeling of the coefficient of friction because of the need to distinguish one state of the investigated system from another.

In Table 2.1 the average values of the coefficient of sliding friction are given for various materials of bodies being at rest or in motion for different surface conditions (oiled and dry). In textbooks on mechanics the so-called *rolling resistance* is considered as a separate problem. This topic will be described in accordance with the textbook [9].

We will consider two cases of a heavy cylinder lying on rigid (Fig. 2.32a) and flexible (Fig. 2.32b) planes.

In the first case (a), the cylinder is in contact with the plane along a line (the generatrix of the cylinder). Along this generatrix of the cylinder the normal and tangential forces will appear (in the cross section perpendicular to the cylinder's axis we will obtain vectors $\mathbf{N}$ and $\mathbf{T}$). The horizontal force $\mathbf{F}$ acts on the cylinder at distance h from the plane on which the cylinder is rolling. Writing the equation of moments about point A we have

$$M_A = Fh. \tag{2.61}$$

Table 2.1 Coefficients of sliding friction [9]

Material of contacting bodies	At rest (μ_s) Dry	At rest (μ_s) Oiled	At motion (μ_k) Dry	At motion (μ_k) Oiled
Steel vs. steel	0.15	0.1	0.1	0.01
Steel vs. cast iron or bronze	0.18	0.1	0.16	0.01
Cast iron vs. cast iron	0.45	0.25	0.2	0.05
Bronze vs. cast iron or bronze	0.21	–	0.18	–
Metal vs. wood	0.5–0.6	0.1	0.2–0.5	0.02–0.08
Wood vs. wood	0.65	0.2	0.2–0.4	0.04–0.16
Leather vs. metal	0.6	0.25	0.25	0.12
Steel vs. ice	0.03	–	0.02	–

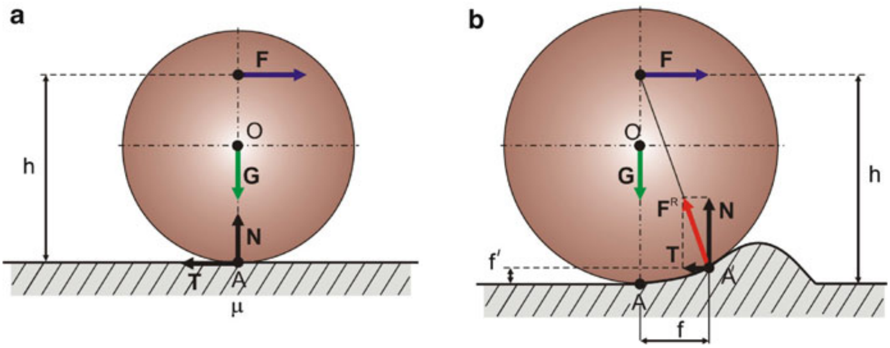


Fig. 2.32 Rolling of a heavy cylinder on a rigid (**a**) and a flexible (**b**) plane

The moment $\mathbf{M}_A = \mathbf{0}$ if $\mathbf{F} = \mathbf{0}$ or $h = 0$. Otherwise, an arbitrarily small, but non-zero, force $\mathbf{F}$ for $h > 0$ will initiate the cylinder motion. If the force $F > \mu N$, then the cylinder will slide on the plane. If $F < \mu N$, then the cylinder will roll on the plane.

The considered case is idealized since from the experiment it follows that a heavy cylinder causes deformation of the horizontal surface (Fig. 2.32b) and the action of force $\mathbf{F}$ creates a reaction connected with the force of rolling resistance $\mathbf{T}_{\text{res}}$ and the moment of force of rolling resistance $\mathbf{M}_{\text{res}}$. Force $\mathbf{F}$, which forces the cylinder to move, causes displacement of the point of application of the reaction of plane from point A to point A'. Writing the equation of moments about a point A we obtain now

$$M_A = Tf' + Nf - Fh. \tag{2.62}$$

Because $f' \ll f$ and $T < N$, the moment Tf' can be neglected, and from (2.62) we obtain

$$M_A = Nf - Fh. \tag{2.63}$$

Table 2.2 Rolling resistance coefficients [9]

Cylinder	Basis	$f(10^{-2}\text{ m})$
Cast iron	Cast iron	0.005
Steel	Steel rail	0.005
Wooden	Wood	0.8
Roller bearings and balls (hardened and ground)	Hardened and ground steel	0.0005–0.001

The motion will start when the moment Fh overcomes the moment $Nf = M_{\text{res}}$, called a *moment of rolling resistance*. The quantity f we call a *coefficient of rolling resistance* or an *arm of rolling resistance*. Let us note that, as distinct from the dimensionless coefficient μ, the quantity f possesses a dimension of length and characterizes the maximum distance at which the point of application of reaction can move preserving the state of rest of the body, i.e., when $Fh = Nf$. Several average values of the coefficient of rolling resistance f for various materials of the cylinder and ground are given in Table 2.2.

The third case of the classical division of friction, i.e., rotational friction, remains to be described. In the case of sliding friction the problem can be reduced to the point of contact. If, however, to one (or both) of the contacting bodies we apply a moment of force, the assumption of their contact only at the point is not valid. The moment of force is carried by the moment of friction forces that must occur over a certain (although very small) area. Thus, the moment of force is balanced not by a single force but by a couple of forces. After overcoming the moment of rotational friction, the moment of force will cause rotation of the body on which it acts (on the assumption that the other body remains fixed) about an axis perpendicular to the surface of contact of the bodies. The threshold magnitude of the moment of rotational friction is usually assumed to be proportional to the force $\mathbf{N}$ (which is the force compressing the bodies, normal to the surface of contact), and that threshold magnitude equals $M_{\text{F}} = \mu' N$, where now μ' denotes a *rotational friction coefficient*, and this time the coefficient μ' possesses a dimension of length. The problem is relatively complex since the moment of friction force depends on the normal stress in the area of contact (area size greater than zero) distribution, which must be uniform over the mentioned area. Additionally, the rotational friction coefficient depends on the coefficient of forward sliding friction because it is caused by the normal force $\mathbf{N}$. For example, from the theoretical considerations concerning the contact problem of the cylinder of radius r making contact with the fixed surface over the area πr^2, the coefficient of rotational friction $\mu' = 0.25\pi r\mu$, where μ denotes the sliding friction.

Let us consider a classical case of a rigid body having the shape of a rectangular prism (Fig. 2.33), but in the present case we assume that the dimensions of its cross section with a vertical plane, i.e., width $2b$ and height $2h$ of rectangle, are known. We will also assume that its geometrical center coincides with the center of mass (point C) and that at point C the gravity force $\mathbf{G}$ and horizontal force $\mathbf{F}$ are applied (Fig. 2.33).

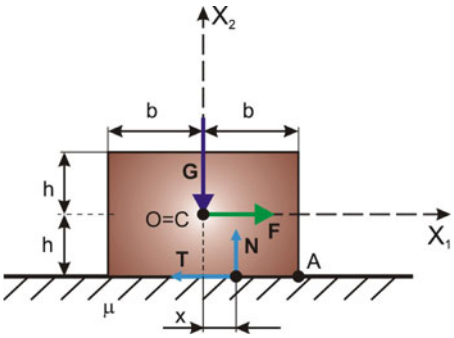


Fig. 2.33 Schematic of forces for static equilibrium of a block

The action of forces **G** and **F** will cause the reaction of the block in the form of the friction force **T** and normal force **N** as in Fig. 2.33. For the force system to be in equilibrium, on the assumption of finite area of contact, the point of application of force **N** must lie at a certain distance x from vector **G**.

From the projection of the forces onto the axes of the adopted coordinate system we obtain

$$G = N, \qquad T = F, \tag{2.64}$$

and writing the equation of moments about point C we have

$$Th = Nx, \tag{2.65}$$

which allows us to determine the unknown quantity

$$x = \frac{Th}{N} = \frac{Fh}{G}. \tag{2.66}$$

According to the law of friction $T = \mu N$, we obtain that $F < \mu G$, and next we have another inequality, that is, $x = \frac{Fh}{G} \leq b$, i.e., $F \leq \frac{b}{h}G$. Let us trace now the phenomena that accompany the process of increasing the magnitude of force **F**. If the coefficient of friction $\mu < b/h$, then the block will remain in equilibrium until the friction force reaches its threshold magnitude $T = \mu N = \frac{b}{h}N = \frac{b}{h}G$. After exceeding this value, the block will start sliding on the plane. If the coefficient of friction $\mu > b/h$, if only the magnitude of horizontal force $F > \frac{b}{h}G$, then the block will rotate about point (edge) A.

In four subsequent examples we will consider the case of a block lying on an inclined plane, this time neglecting its geometrical dimensions (Example 2.7), a homogeneous cylinder rolling on an inclined plane (Example 2.8), the more complex case of the block (wardrobe) taking into account its geometry and assuming the presence of friction along two edges (points) of contact (Example 2.9), and, finally, a shaft-bearing frictional problem (Example 2.10).

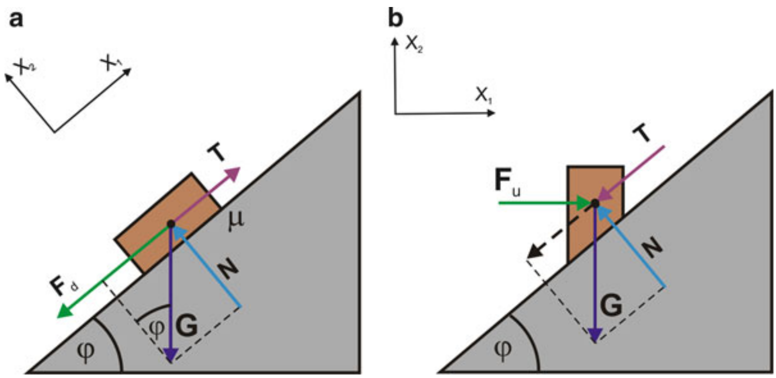


Fig. 2.34 Boundary conditions of equilibrium of bodies of masses m under the action of forces $\mathbf{F}_d$ (**a**) and $\mathbf{F}_u$ (**b**)

Example 2.7. Determine the minimum magnitude of the force $\mathbf{F}_d$ ($\mathbf{F}_u$) at the moment of block motion down (up) the inclined plane (Fig. 2.34).

In Fig. 2.34a, the equations of equilibrium read

$$-F_d + T - G\sin\varphi = 0,$$
$$N - G\cos\varphi = 0.$$

On the verge of equilibrium loss the friction is fully developed, so $T = \mu N$, and from the first equation we obtain

$$F_d = G(\mu\cos\varphi - \sin\varphi).$$

In this case, the condition $F_d = 0$ is possible, which is equivalent to $\tan\varphi = \mu$. For $\tan\varphi \leq \mu$, the block will remain motionless on the inclined plane, and upon increasing the angle of inclination, i.e., for $\tan\varphi > \mu$, the block will slide down.

In the second case (Fig. 2.34b), the equations of motion are as follows:

$$F_u - T\cos\varphi - N\sin\varphi = 0,$$
$$-G + N\cos\varphi - T\sin\varphi = 0.$$

Because $T = \mu N$, the equations take the form

$$F_u = N(\mu\cos\varphi + \sin\varphi),$$
$$G = N(\cos\varphi - \mu\sin\varphi),$$

so

$$F_u = G\frac{\mu\cos\varphi + \sin\varphi}{\cos\varphi - \mu\sin\varphi} = G\frac{\mu + \tan\varphi}{1 - \mu\tan\varphi}.$$

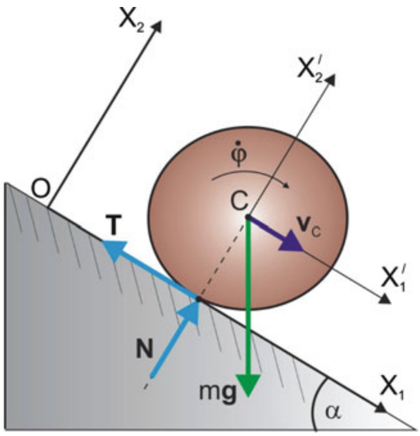


Fig. 2.35 Rolling of cylinder on inclined plane

If $\tan\varphi \to 1/\mu$, then $F_u \to \infty$. In such a case, no force is able to move the block up. We may say that a binding of the body occurs. □

Example 2.8. A homogeneous cylinder with mass m and radius r is rolling on an inclined plane without slipping. The plane forms an angle α with a horizontal line (Fig. 2.35). Determine the cylinder center acceleration and the angle value α, where a slip begins.

The equations of motion have the following form:

$$\begin{aligned} m\ddot{x}_{1C} &= mg\sin\alpha - T, \\ m\ddot{x}_{2C} &= -mg\cos\alpha + N, \\ I_C\ddot{\varphi} &= -Tr. \end{aligned} \tag{*}$$

Because $\ddot{x}_{2C} = 0$, or $x_{2C} = r$, $N = mg\cos\alpha$.

In turn, $\dot{x}_{1C} = -r\dot{\varphi}$ because the $\dot{\varphi}$ value is negative, and hence, after substituting into the third equation (*), we have

$$\frac{mr^2}{2}\left(-\frac{\ddot{x}_{1C}}{r}\right) = -Tr,$$

or $m\ddot{x}_{1C} = 2T$.

From the first equations $(*)$ we calculate

$$\ddot{x}_{1C} = \frac{2}{3}g\sin\alpha,$$

and then $T = \frac{1}{3}mg\sin\alpha$. For $\alpha = \alpha_{th}$, we obtain $\tan\alpha = 3\mu$. □

Example 2.9. Determine the range of equilibrium of a rigid body (a wardrobe) of weight **G** situated on an inclined plane (Fig. 2.36) with (without) friction present at points A and B.

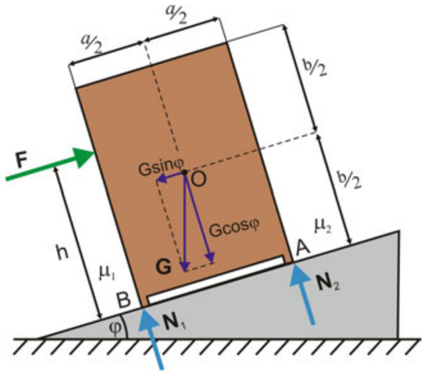


Fig. 2.36 A wardrobe standing on an inclined plane

Let us consider three cases.

(i) The case of no friction ($\mu_1 = \mu_2 = 0$):

$$F = G \sin\varphi,$$

$$N_1 + N_2 = G \cos\varphi,$$

$$M_A \equiv -Fh - N_1 a + G\frac{b}{2}\sin\varphi + G\frac{a}{2}\cos\varphi = 0.$$

In addition, the equation of moments about point A was written above. From the equations we determine

$$N_1 = \frac{1}{a}\left[-Fh + \frac{1}{2}G(b\sin\varphi + a\cos\varphi)\right],$$

$$N_2 = \frac{1}{a}\left[Fh + \frac{1}{2}G(a\cos\varphi - b\sin\varphi)\right].$$

If we want to move the wardrobe standing on a horizontal floor ($\varphi = 0$), from the preceding equations we obtain

$$F = 0, \quad N_1 = N_2 = \frac{G}{2},$$

that is, the wardrobe will move under the action of an arbitrarily small force $F \neq 0$.

(ii) Let us introduce now the friction forces $\mathbf{T}_1$ and $\mathbf{T}_2$ and let us find the force $\mathbf{F}_{\min}$ such that the wardrobe starts to move up the inclined plane.

The projections of force vectors on the OX_1 and OX_2 axes give

$$F = G\sin\varphi + T_1 + T_2,$$

$$N_1 + N_2 = G\cos\varphi.$$

The equation of moments about the center of mass of the wardrobe reads

$$M_O \equiv -F\left(h - \frac{b}{2}\right) - (T_1 + T_2)\frac{b}{2} - N_1\frac{a}{2} + N_2\frac{a}{2} = 0.$$

From the preceding equation, after assuming $F = T_1 = T_2$ and $\varphi = 0$, we obtain $N_1 = N_2 = \frac{G}{2}$, which agrees with the result obtained earlier (case i).

If we assume that $T_1 \in [0, \mu_1 N_1]$ and $T_2 \in [0, \mu_2 N_2]$, then the problem cannot be solved because we have four unknowns N_1, N_2, T_1, and T_2 but only three equations.

In this case, we have to assume in advance the possibility of motion of the wardrobe after equilibrium loss. While increasing the force $\mathbf{F}$ for certain parameters of the system the wardrobe can first either slide or rotate about point A. The problem, so far statically indeterminate, can be solved after assuming one of two possible options. Let us assume that there occurs a rotation about point A under the action of force $\mathbf{F}$. This means that $\mathbf{N}_1 = \mathbf{T}_1 = \mathbf{0}$. The equations of equilibrium of forces and moments for this case are as follows:

$$\begin{aligned}
F &= G\sin\varphi + T_2,\\
N_2 &= G\cos\varphi,\\
M_A &\equiv -Fh + G\frac{b}{2}\sin\varphi + G\frac{a}{2}\cos\varphi = 0.
\end{aligned}$$

Taking the angle $\varphi = 0$, we obtain

$$F = T_2, \quad N_2 = G, \quad Fh = G\frac{a}{2}.$$

If we assume now that the friction force $\mathbf{T}_2$ is not developed and is bounded to the range $0 < T_2 < \mu G$, we obtain the condition for rotation on horizontal ground, which can be realized for an arbitrary $\mathbf{T}_2$ from the aforementioned range if

$$T_2 h = G\frac{a}{2},$$

that is,

$$\mu h = \frac{a}{2}.$$

After transformation for $\varphi \neq 0$ we have

$$\begin{aligned}
F &= G(\sin\varphi + \mu_2\cos\varphi),\\
Fh &= \frac{G}{2}(b\sin\varphi + a\cos\varphi),
\end{aligned}$$

and, in turn,

$$(\sin\varphi + \mu_2 \cos\varphi) = \frac{1}{2h}(b \sin\varphi + a \cos\varphi),$$

that is,

$$\sin\varphi\left(1 - \frac{b}{2h}\right) + \cos\varphi\left(\mu_2 - \frac{a}{2h}\right) = 0.$$

Note that

$$\begin{aligned} a\cos(\varphi - \psi) &= a\cos\psi\cos\varphi + a\sin\psi\sin\varphi \\ &= A\cos\varphi + B\sin\varphi. \end{aligned}$$

For our case

$$A = a\cos\psi = \mu_2 - \frac{2}{2h}, \quad B = a\sin\psi = 1 - \frac{b}{2h}.$$

The equation that we are going to solve has the form

$$\cos(\varphi - \psi) = 0,$$

that is,

$$\varphi - \psi = (2k+1)\frac{\pi}{2}, \qquad k = 0, 1, 2.$$

For $k = 0$ we have $\varphi_0 = \psi + \frac{\pi}{2}$, which means that

$$\varphi_0 = \arctan\left(\frac{2h - b}{2h\mu_2 - a}\right) + \frac{\pi}{2}.$$

For $k = 1$, such a problem cannot be physically realized in the considered case.

Let us assume now that there will be a loss of equilibrium caused by moving the wardrobe up the inclined plane. In this case the equations will take the following form ($N_1 \neq 0$):

$$\begin{aligned} &F = G\sin\varphi + T_1 + T_2, \\ &N_1 + N_2 = G\cos\varphi, \\ &M_A \equiv -Fh + G\frac{b}{2}\sin\varphi + G\frac{a}{2}\cos\varphi - N_1 a = 0. \end{aligned}$$

For the angle $\varphi = 0$ we obtain

$$F = T_1 + T_2,$$
$$N_1 + N_2 = G,$$
$$-Fh + G\frac{a}{2} - N_1 a = 0.$$

From the last two equations we determine (it can be seen that the sum of friction forces is canceled out by force $\mathbf{F}$):

$$N_1 = \frac{G}{2} - F\frac{h}{a},$$

$$N_2 = \frac{G}{2} + F\frac{h}{a}.$$

The displacement will occur when $N_1 > 0$, that is, when

$$\frac{G}{2} > F\frac{h}{a}.$$

For the case $\mu_1 = \mu_2 = \mu$ we have

$$\frac{G}{2} > \mu G\frac{h}{a}, \quad \text{or} \quad \frac{a}{2\mu} > h.$$

Now, let us assume that $\varphi \neq 0$, and to simplify the analysis, let us take $\mu_1 = \mu_2 = \mu$. From the equations in question we obtain

$$F = G(\sin\varphi + \mu\cos\varphi),$$
$$N_1 = \frac{1}{2}G\left(\frac{b}{a}\sin\varphi + \cos\varphi\right) - F\frac{h}{a}.$$

In this case, force $\mathbf{F}$ cancels the action of friction forces and the component of weight vector parallel to the surface of the inclined plane.

The condition for displacement reads

$$\frac{1}{2}\left(\frac{b}{a}\sin\varphi + \cos\varphi\right) > (\sin\varphi + \mu\cos\varphi)\frac{h}{a}.$$

(iii) Let us introduce now the friction forces $\mathbf{T}_1$ and $\mathbf{T}_2$ and consider the case where we should determine the force $\mathbf{F}_{\text{min}}$ such that the wardrobe does not move down the inclined plane.

In the present case, the senses of friction forces will change and the equations will take the following form:

$$F = G\sin\varphi - T_1 - T_2,$$
$$N_1 + N_2 = G\cos\varphi.$$

Let us assume that as a result of action of force **F** the rotation of the wardrobe about point B will occur. The equation of moment is

$$M_B \equiv -Fh - \frac{a}{2}G\cos\varphi + \frac{b}{2}G\sin\varphi + N_2 a = 0.$$

On the assumption that $\mu_1 = \mu_2 = \mu$, from the equations above we determine

$$F = G(\sin\varphi - \mu\cos\varphi),$$
$$N_2 = F\frac{h}{a} + \frac{1}{2}G\left(\cos\varphi - \frac{b}{a}\sin\varphi\right).$$

Let us check if the rotation about point B is possible for $\varphi = 0$.

From the preceding equations ($N_2 = 0$) we obtain

$$F = -T_1,$$
$$N_1 = G,$$
$$Fh + G\frac{a}{2} = 0,$$

that is,

$$h = \frac{a}{2\mu}.$$

From the foregoing considerations it follows that there exists a possibility of equilibrium loss by the rotation of the wardrobe, and in the present case the boundary value of this condition is the same as the corresponding value for translation.

Let us assume now that $\varphi \neq 0$ and check if there possibly exists an angle for which the loss of equilibrium by rotation about point B takes place. In this case we have ($N_2 = 0$)

$$F = G\sin\varphi - T_1,$$
$$N_1 = G\cos\varphi,$$

$$-Fh - \frac{a}{2}G\cos\varphi + \frac{b}{2}G\sin\varphi = 0.$$

Then we have

$$h(\sin\varphi - \mu\cos\varphi) = \frac{1}{2}(-a\cos\varphi + b\sin\varphi),$$

and after transformations we obtain

$$\left(-h + \frac{b}{2}\right)\sin\varphi + \left(h\mu - \frac{a}{2}\right)\cos\varphi = 0.$$

Let us assume now that the equilibrium loss will occur by the motion of the wardrobe down the inclined plane, so the loss of equilibrium will result in sliding. The condition of the motion is $N_2 \geq 0$ and in the boundary case $N_2 = 0$. This condition is identical with the one for the rotation about point B.

Let us note that the current example allows for the analysis of two special cases. For $\varphi = 0$ we are dealing with the problem of equilibrium loss of the wardrobe situated on a horizontal surface (already presented). Let us assume now that $a = b = h = 0$, that is, we will be considering a particle of mass m situated on the inclined plane ($\mathbf{N}_1 + \mathbf{N}_2 = \mathbf{N}$).

From the considerations conducted for case (i) we obtain

$$F = mg\sin\varphi,$$
$$N = mg\cos\varphi.$$

This means that for the particle to remain at rest on the inclined plane one should apply to it force $\mathbf{F}$ of magnitude $mg\sin\varphi$.

In case (ii) we have

$$F^*_{\min} = G(\sin\varphi + \mu\cos\varphi),$$

whereas in case (iii) we have

$$F_{\min} = G(\sin\varphi - \mu\cos\varphi).$$

Let us note that for the forces within the range $F \in [F_{\min}, F^*_{\min}]$ the particle remains in equilibrium. Range $[F_{\min}, F^*_{\min}]$ we will call the *range of equilibrium loading*. □

Example 2.10. Determine the angle of friction φ and so-called *circle of friction* of radius r during contact of the shaft journal of radius R and the bearing shell on the assumption that the radius of the bearing shell is slightly larger than R and was omitted in Fig. 2.37.

Let us assume that the shaft journal is loaded with a couple of moment $\mathbf{M}$ and a force $\mathbf{F}$. The action of the moment causes the displacement of the point of contact A between the journal and the bearing shell upward, i.e., we are dealing with the phenomenon of shaft climbing.

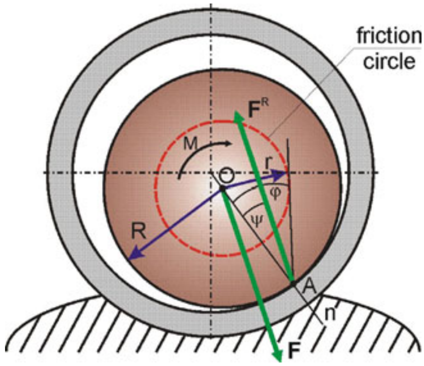


Fig. 2.37 Contact of a journal (rotor) with a bearing shell (plain bearing) with radii and forces marked

Let us consider the idealized case where there is no dependency of the friction force on the speed and no bearing lubrication; the rolling resistance can be neglected.

Because the shaft journal is in equilibrium under the action of the active force $\mathbf{F}$ and the couple moment $\mathbf{M}$ and we are dealing with a plane force system, it can be balanced with only one force (reaction) $\mathbf{F}^R$, which is equal to $-\mathbf{F}$ and is attached at point A. Then the value of the main vector of force is equal to $F^R - F = 0$. Moreover, the moment coming from the couple $(\mathbf{F}, \mathbf{F}^R)$ should be balanced with the moment coming from the pair of active forces $\mathbf{M}$, that is

$$F^R(R \sin \psi) = M,$$

which leads to the relation

$$M = FR \sin \psi.$$

This means that the magnitudes of M and F are related by the formula above through the angle ψ, i.e., the angle between the normal to the contact surface of the journal and the bearing shell and the reaction $\mathbf{F}_R$. If the magnitude of moment $\mathbf{M}$ is increased, for the boundary case defined by the angle of friction φ the equilibrium will be lost and the journal will start to rotate, sliding on the surface of the bearing shell.

In other words, the static equilibrium condition of the shaft journal is described by the following inequality:

$$M \le FR \sin \varphi,$$

where $\psi \le \varphi$.

Denoting $R \sin \varphi = r$ we will introduce the notion of a friction circle of radius r and the center at point O (center of shaft journal). From Fig. 2.37 it follows that static equilibrium takes place when the reaction $\mathbf{F}^R$ crosses the friction circle and in the boundary case is tangent to it. □

2.9 Friction and Relative Motion

As was mentioned earlier in Sect. 2.8 (see description of cone of friction), in fact, the friction belongs to phenomena that should be modeled in three-dimensional space (3D). Moreover, in reality we are dealing with friction during the impact of two bodies treated as solids, and these two phenomena of mechanics are closely related to one another. The description of the rolling of two bodies is more connected with two-dimensional modeling. In the present section we will limit ourselves to the modeling of friction as a one-dimensional phenomenon, indicating certain mathematical difficulties already at this stage of elementary modeling. However, it should be emphasized that from a formal point of view, reduction of the friction phenomenon to a one-dimensional problem is justified only for "small" rotational motions of the body and in cases where it might be possible to separate the translational motion of the contacting bodies from their rotational motion.

The models of kinetic friction can be represented by the dependencies of the friction force on relative velocity (*static version*) or described by the first-order differential (*dynamic version*).

The dynamic version of friction modeling attempts to incorporate the results obtained during tribological research and allows for, e.g., an explanation of the so-called small displacements just before onset of sliding, which are observable for small speeds of motion.

We will now present some examples to show how the relative motion of bodies in contact with one another with friction may even lead to a paradoxical behavior.

1. *Supply of energy by means of friction.*

Let us consider the case of a rigid body lying on a stationary base, as was already discussed in Sect. 2.8, and subjected to the action of force **F** (Fig. 2.38).

Let the transfer of a force interaction to a block of weight m take place by means of a massless spring of stiffness k one of whose ends (point A) moves with speed v_A. The motion of point A causes an increase in force **F** acting on the block. After attaining the value $kx'_S = F_S$, the block will start to slide with respect to the stationary base in the fixed (environment) coordinate system OX. It turns out that

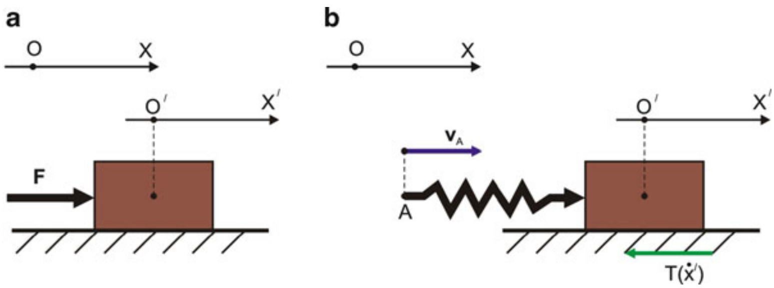


Fig. 2.38 Action of a force **F** on a block: directly (**a**) and through a spring (**b**)

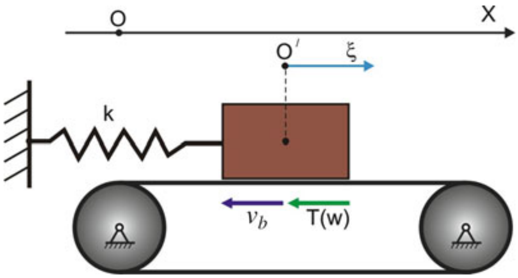


Fig. 2.39 A body of mass m lying on a conveyor moving at constant speed v_b

in many cases upon onset of motion after "breaking away" the static friction the sudden "forward jump" of the block follows. This phenomenon can be explained through the so-called decreasing part of the Coulomb friction versus the relative velocity of the moving bodies.

The analyzed system from Fig. 2.38b is analogous to the system depicted in Fig. 2.39, where the block lies on a belt moving at speed v_b and the spring in the static equilibrium position is compressed.

The equation of motion of a block (Fig. 2.39) in a fixed coordinate system reads

$$m\ddot{x} = -kx + T(w), \tag{2.67}$$

where

$$w = v_b - \dot{x} \tag{2.68}$$

denotes the relative velocity. In the case of static equilibrium $\dot{x} = \ddot{x} = 0$ and from (2.67) we obtain

$$kx_{\mathrm{st}} = T(v_b). \tag{2.69}$$

In the coordinate system connected to the point of the static equilibrium position of the block $\dot{x} = \ddot{x} = 0$ we have

$$m\ddot{\xi} = -k(x_{\mathrm{st}} + \xi) + T(v_b - \dot{\xi}), \tag{2.70}$$

where

$$\begin{aligned} x &= x_{\mathrm{st}} + \xi, \\ \dot{x} &= \dot{\xi}, \\ w &= v_b - \dot{x} = v_b - \dot{\xi}. \end{aligned} \tag{2.71}$$

Taking into account (2.69), (2.71) yields

$$m\ddot{\xi} + k\xi = T(v_b - \dot{\xi}) - T(v_b) \equiv Q(\dot{\xi}), \tag{2.72}$$

and it can be seen that the friction force $T(v_b)$ carries the static deflection of the spring.

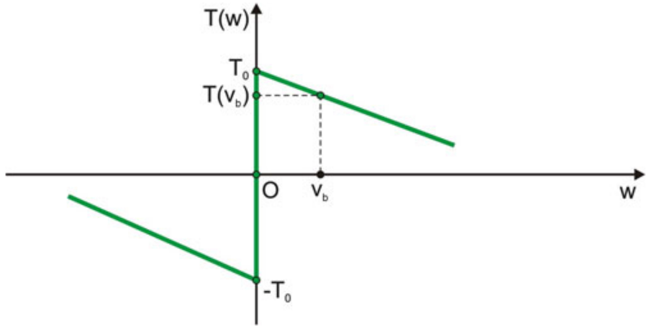


Fig. 2.40 Characteristic of friction $T(w)$

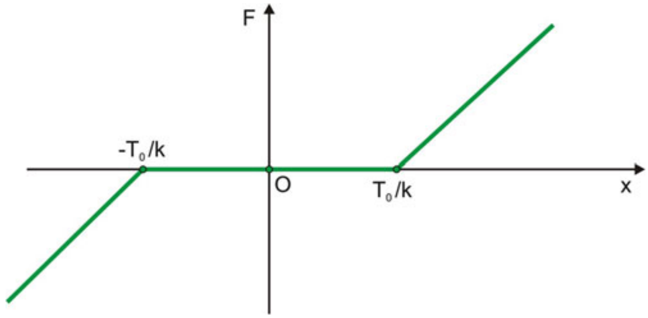


Fig. 2.41 A force in a spring $F(x)$ with so-called clearance interval marked: $\langle -T_0/k, T_0/k\rangle$ for $w = 0$

In both considered cases the vibrations of the block are observed and the only known way to explain this phenomenon thus far is by adoption of the so-called *decreasing characteristic* $T(w)$, assumed here to be a linear one (Fig. 2.40).

From the plot it can be seen that for $w = 0$, i.e., $\dot{x} = v_b$, there is no functional dependency since in the interval $\langle -T_0, T_0\rangle$ we have infinitely many points, i.e., there appears an indeterminability often understood as *functional inclusion*.

It turns out, however, that the magnitude of friction force $T(0) \in \langle -T_0, T_0\rangle$ is determined uniquely by the deflection of the spring on the assumption that $\ddot{\xi} = 0$, which is schematically shown in Fig. 2.41.

That characteristic does not exactly represent the real block behavior since when the spring tension reaches $kx = T_0$, we suddenly have $w = v_b$, that is, at that point (moment) the friction force is equal to $T(v_b)$. As can be seen in Fig. 2.40, the difference $T_0 - T(v_b)$, where $T_0 = T(0)$, is responsible for the appearance of the experimentally observed jump (acceleration) of the block.

We will now show that the characteristic of friction shown in Fig. 2.40 allows for the supply of energy to the system from its "reservoir" represented by the constant velocity v_b.

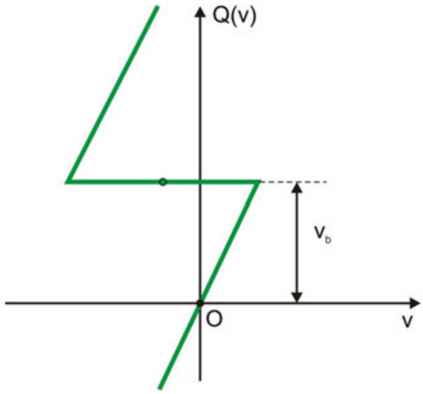


Fig. 2.42 Plot of a function $Q(v)$

From (2.72), after multiplying through by $\dot{\xi} = v$, we obtain

$$m\dot{v}v + k\xi v = Q(v)v. \tag{2.73}$$

The left-hand side of (2.73) can be cast in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(E_k + V) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mv^2}{2} + \frac{k\xi^2}{2}\right) = mv\dot{v} + k\xi v, \tag{2.74}$$

where E_k, V denotes kinetic and potential energy, respectively. After taking into account (2.74), (2.73) will assume the following form:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = Q(v)v, \tag{2.75}$$

where E is the total energy of the considered one-degree-of-freedom system.

The function $Q(v)$ is presented in Fig. 2.42.

From (2.75) and Fig. 2.42 it follows that in certain ranges we have $Q(v)v > 0$, and in the other $Q(v)v < 0$, that is, the energy is supplied/taken away to/from the system, which was the subject of a detailed analysis of the works [31, 36].

2. *Coulomb friction as a force exciting the motion of a rigid body.*

Let us consider a one-dimensional problem, where a flat steel slat lies on two cylinders rotating in opposite directions (Fig. 2.43a) and the dependency of coefficient of friction on speed $\mu(\dot{x})$ has the form shown in Fig. 2.43b.

The equation of motion of the slat reads

$$m\ddot{x} = T_A - T_B, \tag{2.76}$$

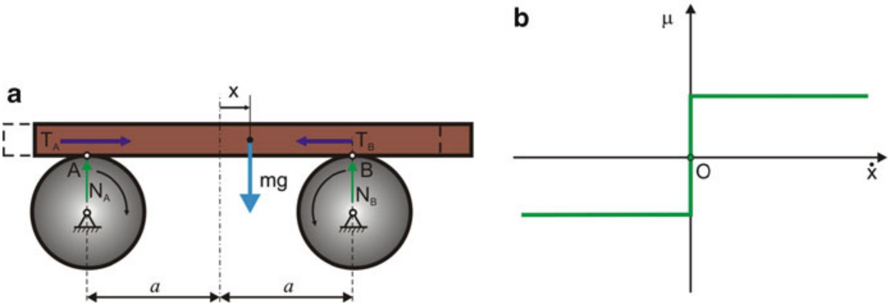


Fig. 2.43 A slat situated on two rotating cylinders (**a**) and a dependency $\mu(\dot{x})$ (**b**)

where

$$T_A = \mu N_A, \quad T_B = \mu N_B.$$

The equations of moments of forces about points A and B allow for the determination of the unknown normal reactions

$$N_A = \frac{(a-x)mg}{2a}, \quad N_B = \frac{(a+x)mg}{2a}.$$

From the equation of motion we obtain

$$m\ddot{x} = \frac{\mu mg}{2a}[a - x - (a + x)],$$

or

$$\ddot{x} + \frac{\mu g}{a}x = 0.$$

This means that the system vibrates harmonically with frequency $\alpha = \sqrt{\mu g a^{-1}}$. The equation obtained is often used to determine the coefficient of friction experimentally.

The careful reader may be somewhat confused by having the Coulomb friction presented in two different forms as in Figs. 2.40 and 2.43. In the first case it is easy to observe the self-excited vibration caused by friction, e.g., if one spreads a rosin film on the rubbing surfaces or plays on a string instrument. In the second case we are dealing with typical dry friction.

3. *Coulomb friction playing the role of viscous damping.*

Let a particle of mass m move in a plane at speed v (Fig. 2.44).

The velocity of particle v was resolved into two components:

$$\mathbf{v} = v_{x_1}\mathbf{E}_1 + v_{x_2}\mathbf{E}_2;$$

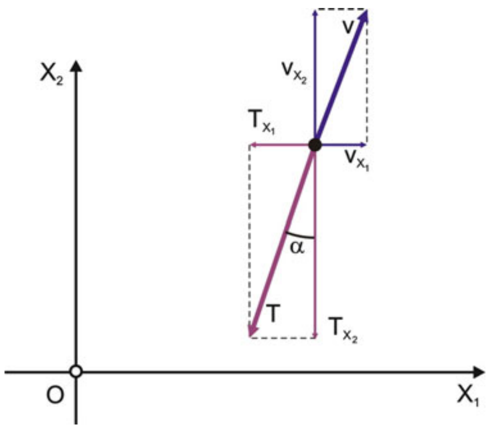


Fig. 2.44 Motion of a particle in plane OX_1X_2

the motion of the particle has two degrees of freedom and is described by two equations:

$$m\ddot{x}_1 = -T_{x_1},$$
$$m\ddot{x}_2 = -T_{x_2},$$

where

$$\frac{v_{x1}}{v} = \frac{T_{x1}}{T} = \sin\alpha, \quad \frac{v_{x2}}{v} = \frac{T_{x2}}{T} = \cos\alpha.$$

From the relationships above it follows that

$$T_{x_1} = T\alpha, \quad T_{x_2} = T,$$

on the assumption that α is a small angle; this means $\sin\alpha \approx \tan\alpha \approx \alpha$, $\cos\alpha \approx 1$.
This means that the particle moves at a high speed in the direction of the vertical axis and a low speed in the direction of the horizontal axis. In the considered case, the equations of motion assume the following form:

$$m\ddot{x}_1 = -T\alpha = -T\frac{v_{x1}}{v} = -T\frac{v_{x1}}{v_{x2}} = -cv_{x_1} = -c\dot{x}_1,$$
$$m\ddot{x}_2 = -T.$$

Assuming that only the component $\mathbf{v}_{x_1}$ of vector $\mathbf{v}$ is changed (keeping $\mathbf{v}_{x2}$ constant), one may introduce a resistant coefficient $c = T/v_{x_2}$. The resistant force in the direction OX_1 depends on the velocity v_{x_1}, and coefficient c is called a *viscous damping coefficient*.

2.10 Friction of Strings Wrapped Around a Cylinder

Let us consider a string (rope) wrapped around a cylinder of circular cross section (Fig. 2.45).

Let the cylinder and string have a rough surface. The weight of the string is negligible.

It follows from Fig. 2.45a that string ends are loaded by the forces $\mathbf{F}_a$ (active force) and $\mathbf{F}_p$ (passive force). Assume that force $\mathbf{F}_a$ achieved its critical value, i.e., infinitely small increase yields the string displacement coinciding with the $\mathbf{F}_a$ sense. Beginning from that time instant we are dealing with the so-called developed friction between the string and the cylinder.

Let us consider an element of the string defined by the angle $d\varphi$ (Fig. 2.45a). After performing the projection of forces onto the axes of the adopted coordinate system (Fig. 2.45b), we obtain

$$\begin{aligned} (F + dF)\cos\frac{d\varphi}{2} - F\cos\frac{d\varphi}{2} - dT &= 0, \\ dN - (F + dF)\sin\frac{d\varphi}{2} - F\sin\frac{d\varphi}{2} &= 0, \end{aligned} \tag{2.77}$$

where T denotes the friction between belt and cylinder. After assuming $\sin\frac{d\varphi}{2} \cong \frac{d\varphi}{2}$, $\cos\frac{d\varphi}{2} \cong 1$ we have

$$\begin{aligned} dF - dT &= 0, \\ dN - F\,d\varphi &= 0. \end{aligned} \tag{2.78}$$

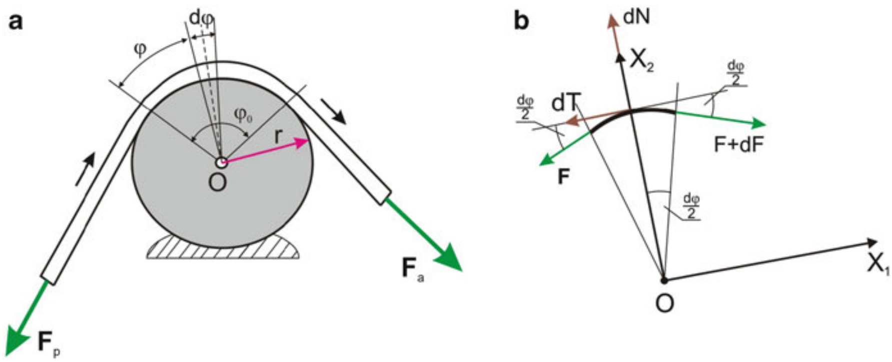


Fig. 2.45 Active ($\mathbf{F}_a$) and passive ($\mathbf{F}_p$) force acting on a string (**a**) and the computational scheme (**b**)

In what follows we take for $F = F_\mathrm{p}$ ($\varphi = 0$) and $F = F_\mathrm{a}$ ($\varphi = \varphi_0$) Because we are dealing with the fully developed friction $T = \mu N$, from the considerations above we will obtain

$$\int\limits_{F_\mathrm{p}}^{F_\mathrm{a}} \frac{\mathrm{d}F}{F} = \mu \int\limits_0^{\varphi_0} \mathrm{d}\varphi, \tag{2.79}$$

that is,

$$\ln F_\mathrm{a} - \ln F_\mathrm{p} = \mu\varphi_0, \tag{2.80}$$

where $F_\mathrm{a} > F_\mathrm{p}$ (meaning F_a overcomes friction resistance).

From the preceding equation we obtain the so-called Euler's belt formula of the form

$$F_\mathrm{a} = F_\mathrm{p} e^{\mu\varphi_0}. \tag{2.81}$$

From (2.81) it follows that force $\mathbf{F}_\mathrm{a}$ (active) balancing the action of force $\mathbf{F}_\mathrm{p}$ (passive), e.g., the weight hung at the end of the rope, does not depend on the radius of the cylinder on which it is pulled. For instance, assuming $\mu = 0.1$ (0.2) and the angle $\varphi_0 = \frac{\pi}{2}$ we obtain from the formula (2.81) the magnitude of active force $\mathbf{F}_\mathrm{a}$ equal to $2.19\,\mathbf{F}_\mathrm{p}$ ($4.76\,\mathbf{F}_\mathrm{p}$).

Observe that (2.81) can be interpreted in the following way. If the active force $\mathbf{F}_\mathrm{a}$ is going to keep only the weight $\mathbf{G}$ in equilibrium (i.e., its infinitely small decrease pushes the string displacement in the direction coinciding with the sense of vector $\mathbf{G}$), then the friction resistances help this force, and in this case this force value is found from the equation $F_\mathrm{a} = F_\mathrm{p} e^{-\mu\varphi_0}$.

We are dealing with a slightly different problem in the case of application of Euler's formula to conveyor systems with belts wrapped around the rotating drums, where transported material such as coal can be placed on the belt. It is desirable, then, to determine the maximum moment carried by the drum without slipping of the belt with respect to the drum. The friction of the strings is applied in belt transmissions because at the slight preliminary tension of the belt ($\mathbf{F}_\mathrm{p}$) the transmission even for a small angle of contact can carry significant torques.

We will show now a method to derive Euler's belt formula, which is different from that presented so far [37]. The considered contact of the drum with the rope can be approximated by means of the broken line shown in Fig. 2.46.

Let us cut out an element $OA_{m-1}A_mA_{m+1}$ from the drum and consider the equilibrium of the rope lying on $\Delta OA_{m-1}A_mA_{m+1}$, shown in Fig. 2.47. In the case depicted in Fig. 2.47a the problem cannot be solved. After neglecting the forces of friction between the rope and the drum in sections $A_{m-1}A_m$ and A_mA_{m+1}, the rope reaction will reduce to its reaction $\mathbf{F}^R$ caused by friction at the point A_m. However, at that point we do not know the direction of reaction because we do not know the direction of the normal to the surface at the contact point A_m (singular point) and $\mathbf{F}^R = \mathbf{N} + \mathbf{T}$. Therefore, we cannot apply the three-forces theorem to the forces $\mathbf{F}_\mathrm{p}$, $\mathbf{F}_\mathrm{a}$, and $\mathbf{F}^R$ in order to determine $\mathbf{F}_\mathrm{a}$. In this case we will treat the force $\mathbf{F}_\mathrm{a}$ as a force needed to keep the passive force $\mathbf{F}_\mathrm{p}$, that is, $|\mathbf{F}_\mathrm{a}| < |\mathbf{F}_\mathrm{p}|$. In order to avoid the

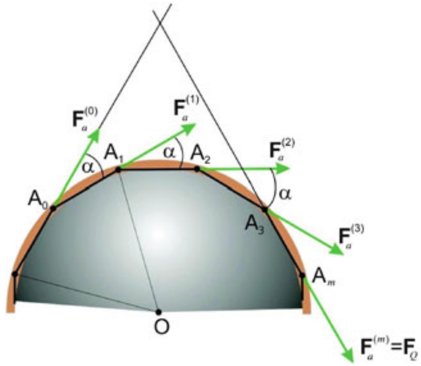


Fig. 2.46 Approximation of a rope by a broken line

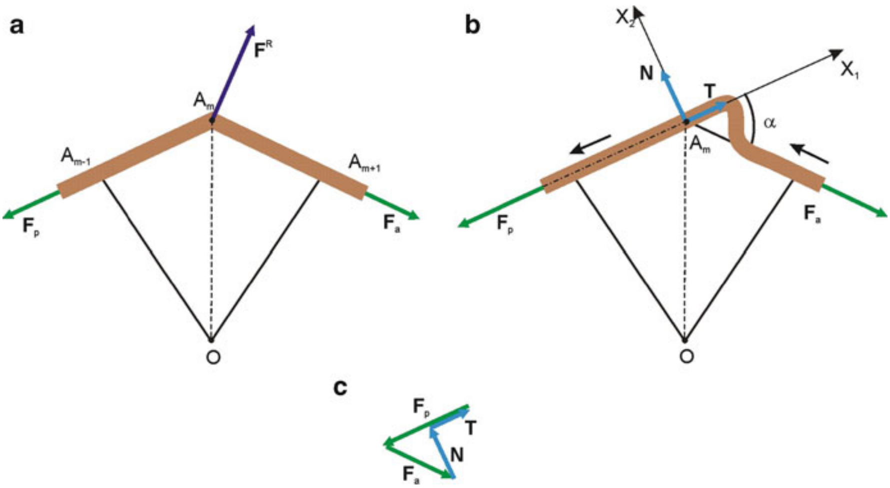


Fig. 2.47 A rope and reactions in a statically indeterminate case (**a**), a statically determinate case (**b**), and a force polygon (**c**)

singularity at the point A_m we can introduce the assumption of a high resistance to the bending of the rope and consequently assume that the rope will take the shape depicted in Fig. 2.47b around point A_m. Now we know the direction of the normal **N**, being perpendicular to $A_{m-1}A_m$, which enables us to construct the force polygon as shown in Fig. 2.47c.

Introducing the Cartesian coordinate system $A_m X_1 X_2$ (Fig. 2.47b), after projection of the force vectors we obtain

$$F_\mathrm{p} - T = F_\mathrm{a}\cos\alpha, \qquad N = F_\mathrm{a}\sin\alpha. \tag{2.82}$$

Considering the boundary case of fully developed friction we have $T = \mu N$. For a large number of sections M the angle α is small, and therefore we can assume that

$$\sin\alpha \cong \alpha, \qquad \cos\alpha \cong 1. \tag{2.83}$$

Consequently, from (2.82) we obtain

$$F_p = F_a(\cos\alpha + \mu\sin\alpha) = F_a(1+\mu\alpha). \tag{2.84}$$

We will apply the equation just obtained several times to the polygon constructed from M sections.

From Fig. 2.47 it follows that

$$\begin{aligned}
F_a^{(m-1)} &= F_a^{(m)}(1+\mu\alpha) = F_a(1+\mu\alpha),\\
F_a^{(m-2)} &= F_a^{(m-1)}(1+\mu\alpha) = F_a(1+\mu\alpha)^2,\\
F_a^{(m-k)} &= F_a^{(m-k-1)}(1+\mu\alpha) = F_a(1+\mu\alpha)^k,\\
&\dots\\
F_a^{(1)} &= F_a(1+\mu\alpha) = F_a(1+\mu\alpha)^{M-1},\\
F_a^{(0)} &= F_p = F_a^{(1)}(1+\mu\alpha) = F_a(1+\mu\alpha)^M.
\end{aligned} \tag{2.85}$$

Because $\varphi = M\alpha$, from the last equation of (2.85) we obtain

$$F_p = F_a(1+\mu\alpha)^M = F_a\left(1+\mu\frac{\varphi}{M}\right)^M. \tag{2.86}$$

Moving to the exact approximation of a circle with linear sections, i.e., for $M \to \infty$, from (2.86) we obtain Euler's formula:

$$F_p = F_a \lim_{M\to\infty}\left(1+\mu\frac{\varphi}{M}\right)^M = F_a e^{\mu\varphi}, \tag{2.87}$$

where e is the base of a natural logarithm.

From (2.87) we obtain $F_a = F_p e^{-\mu\varphi}$, which means that to support, for example, the weight $G = F_p$ hung at one end of the rope, one must use the force of $F_p e^{-\mu\varphi}$, which is smaller than the magnitude of the weight G. The situation will change if we want to pull the weight $G = F_p$ over the drum with friction. Then one should use the force of magnitude $Ge^{\mu\varphi}$. □

Example 2.11. Find the relationship between forces G and F in an equilibrium position for the braking system depicted in Fig. 2.48.

The massless rope is wrapped around a cylinder of radius r and pulled through massless pulleys 1 and 2 of negligible radii. The cylinder is connected to the braking drum of radius R, which is wrapped around by the braking belt attached to lever AB. The coefficient of friction on the braking drum is equal to μ.

After releasing from constraints we obtain two subsystems shown in Fig. 2.49.

Let us formulate the equilibrium condition for the lever writing the equation of moments about point O_3. Pulley 2 will be in equilibrium if $\mathbf{Q}_1 + \mathbf{Q}_2 = \mathbf{F}^i$ and for the equation of moments about O_3 the vertical component of $\mathbf{F}^i$, equal to $Q_1 + Q_2\cos\varphi$, is needed. The aforementioned equation of moments reads

$$Fl + F_a a_1 = (Q_1 + Q_2\cos\varphi)l_1 + F_p a_2.$$

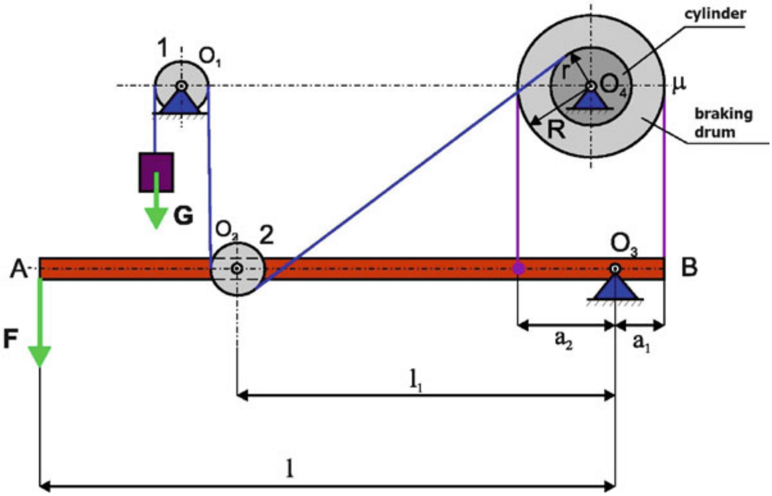


Fig. 2.48 A braking mechanism

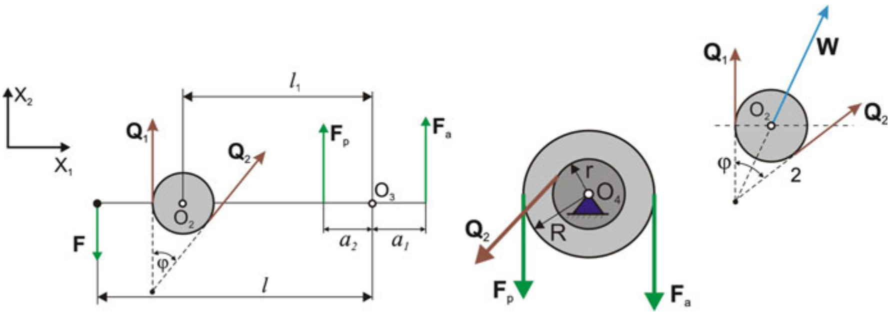


Fig. 2.49 Computational schemes

Because $Q_1 = Q_2 = G$, the equation of moments about point O_3 gives

$$Gr + F_p R = F_a R.$$

The relationship between forces F_a and F_p is defined by Euler's formula:

$$F_a = F_p e^{\mu\pi}.$$

From the preceding equation we have

$$Gr = F_p R(e^{\mu\pi} - 1),$$

that is,

$$F_{\text{p}} = \frac{Gr}{(e^{\mu\pi} - 1)R}.$$

Coming back to the equation with force F we obtain

$$F = \frac{G}{l}\left[\frac{r(a_2 - a_1 e^{\mu\pi})}{R(e^{\mu\pi} - 1)} + (1+\cos\varphi)l_1\right]. \qquad \square$$

2.11 Friction Models

2.11.1 Introduction

The aim of the present section is not an in-depth description of a friction phenomenon but only an emphasis on its complexity and a presentation of its engineering models. As has already been mentioned, the model of sliding friction $T = \mu N$ allows us to determine this force during macroscopic sliding of one of the analyzed surfaces on the other. In the general case the coefficient of friction depends on the contact pressure p (normal force N), the relative speed of sliding v_r, and temperature of the contact area θ, i.e., it is a function of the form

$$\mu = \mu(p, v_r, \theta). \tag{2.88}$$

In [38] many investigations of the dry friction phenomenon were conducted and computational methods serving to estimate this friction were proposed. In the cited works there are references to results of experimental investigations on the dependency of the friction coefficient on the speed of slippage. Various characteristics of that dependency were found, including the minimal and maximal values and with a monotonal decrease in value, as well as characteristics displaying a constant coefficient of friction for varying slip speed (Fig. 2.50).

Most frequently during the analyses the linear models of friction are used, which corresponds to a straight line 1 in Fig. 2.50.

The behavior of friction in the case of lubricated surfaces as a function of slip speed and in the case of stationary conditions is best described by friction characteristics called the Stribeck[6] curve. For low slip speeds, the friction force depends mainly on mechanical, structural, and physiochemical material properties of the rubbing surfaces (dry friction). For moderate slip speeds oil wedges form, and the resistance of a hybrid (fluid and dry) friction decreases. With further increases in slip speed complete separation of the rubbing surfaces takes place. Then only the fluid friction exists whose magnitude increases with increasing speed.

[6]Richard Stribeck (1861–1950), German engineer working mainly in Dresden, Köln, and Stuttgart.

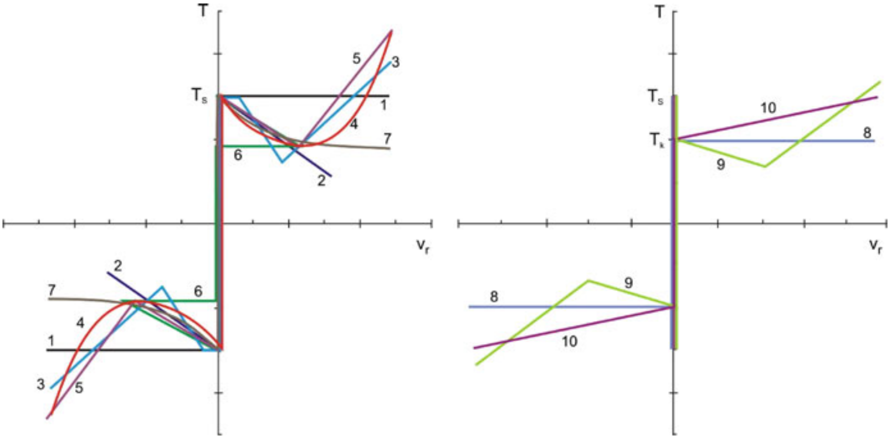


Fig. 2.50 Certain models of friction forces encountered in the literature

The Stribeck curve (curve 4 in Fig. 2.50) was used, for instance, in [32]. Simplified Stribeck curves (3 and 9 in Fig. 2.50) were the objects of analysis in [39]. The models of friction force represented by curves 2, 5, 6, 7, 8, and 10 were described among others in [36, 40]. In [33], in the numerical investigations of the dynamics of wheeled vehicles, the model of the *Magnum* type brake was used. This model relies on friction between road surface and tire. It pictures the mutual relations between the forces of adhesive friction (forces of adhesion), Coulomb friction, and viscous friction and the slip speed of the wheel, which are graphically represented by the Stribeck curve. For some road surfaces the modeling of friction with curve 7 (Fig. 2.50) is recommended.

The dependency describing curve 4 according to [41] possesses the following form:

$$\mu = \operatorname{sgn}(v_r)\Big(\big(a + b|v_r|\big)\exp\left(-c|v_r|\right) + d\Big), \tag{2.89}$$

and the dependency approximated by curve 7 [42] is expressed by

$$\mu = \operatorname{sgn}(v_r)\left(\frac{f_0 - f_{\min}}{1 + c|v_r|} + f_{\min}\right), \tag{2.90}$$

where $a, b, c, d, f_0, f_{\min}$ are constant parameters.

The Stribeck curve [43] (Fig. 2.51) attains its minimum at $v_r = v_{\min}$, and for $v_r < v_{\min}$ a characteristic drop in value of the coefficient of friction can be observed.

Starting from the classical works of Coulomb there has been much attention devoted to the problem of friction, which was treated as a classical process within the framework of Coulomb hypotheses and as a complex process involving wear and heat exchange between rubbing bodies [35]. Problems connected to one-dimensional Coulomb friction and the chaotic dynamics of simple mechanical

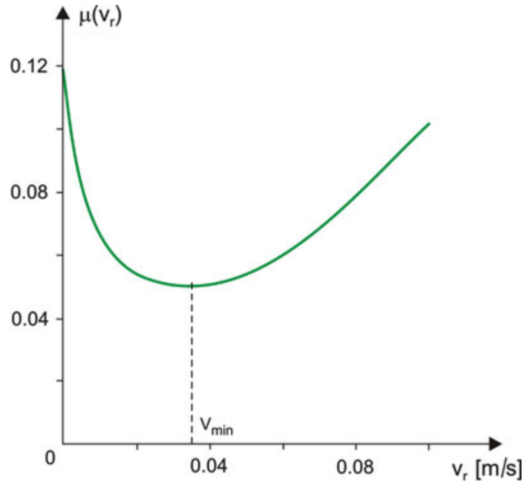


Fig. 2.51 A plot of the coefficient of kinetic friction against the relative speed of rubbing bodies

systems are also described in monographs [32] and [34], where an extensive literature on the topic was presented. However, the latest works of Contensou [44] and Zhuravlev [45, 46] point to the fact that the classical Coulomb model seen through the motion of contacting bodies in space can explain, to a large extent, many phenomena encountered in everyday life and in engineering associated with friction.

Let an arbitrary rigid body be in contact with another fixed body (a base), and the contact between the bodies takes place at a point or over a circular patch of a very small radius. The rigid body moving on the base has five degrees of freedom, and its dynamics is described by the translational motion of the mass center (vector $\mathbf{v}$) and the rotational motion about the mass center (vector $\boldsymbol{\omega}$). As we will show below, the friction model from the perspective of simultaneous translational and rotational motion of a rigid body allows for the explanation of many phenomena, and especially allows for the elimination of the classical, often incorrect, results associated with the interpretation of motion of a rigid body as a combination of translational and rotational motion on the assumption that translational friction $\mathbf{T}$ (treated as a force of resistance to translational motion) and torsional friction $\mathbf{M}_T$ (treated as a moment of resistance to rotational motion) are seen as independent of each other. In the presented model further called CCZ (Coulomb–Contensou–Zhuravlev) $T = T(\mathbf{v}, \boldsymbol{\omega})$, $M_T = M_T(\mathbf{v}, \boldsymbol{\omega})$, and at the point $\mathbf{v} = \boldsymbol{\omega} = \mathbf{0}$ functions T and M_T do not possess a limit. The introduction of even a very small quantity $\boldsymbol{\omega}$ in the case $T(\mathbf{v}, \boldsymbol{\omega})$ and a very small quantity $\mathbf{v}$ in the case $M_T(\mathbf{v}, \boldsymbol{\omega})$ (which is close to real phenomena) leads, in many cases, to the elimination of non-holonomic constraints, which complicate many problems of classical mechanics.

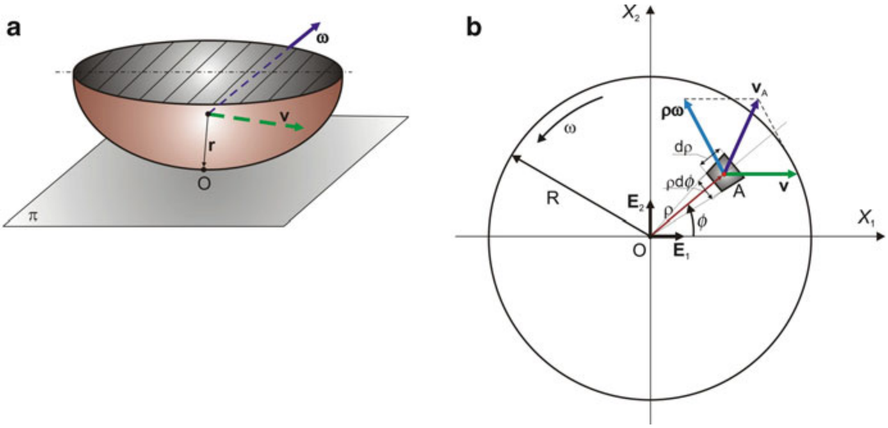


Fig. 2.52 Circular contact of rigid body and fixed plane (**a**) and velocity of an arbitrary point A of body contact patch (**b**)

2.11.2 A Modified Model of Coulomb Friction (CCZ Model)

In Fig. 2.52a the contact of a body with a fixed surface is shown, whereas Fig. 2.52b shows a body contact patch of circular shape in the neighborhood of point O.

The starting point for the construction of the CCZ model is the application of Coulomb's law. It is assumed that the differential of an elementary friction force $\mathrm{d}\mathbf{T}$ and elementary friction moment $\mathrm{d}\mathbf{M}_T$ is directed against the relative motion at point A according to the equations

$$\mathrm{d}\mathbf{M}_T = -\mu\sigma(\boldsymbol{\rho})\frac{\boldsymbol{\rho}\times\mathbf{v}_A}{|\mathbf{v}_A|}\mathrm{d}S, \qquad \mathrm{d}\mathbf{T} = -\mu\sigma(\boldsymbol{\rho})\frac{\mathbf{v}_A}{|\mathbf{v}_A|}\mathrm{d}S, \tag{2.91}$$

where $\mathbf{v}_A$ is the velocity of sliding at point A, that is, $\mathbf{v}_A = \mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}$ (Fig. 2.52a), (ρ, ϕ) are polar coordinates of point A (Fig. 2.52b), $\boldsymbol{\sigma}(\rho)\mathrm{d}S$ is a normal force dependent on the radius ρ describing the distance of point A from point O, and the elementary surface is equal to $\mathrm{d}S = \rho\mathrm{d}\rho\mathrm{d}\phi$.

In the adopted Cartesian coordinate system OX_1X_2 the velocity of point A is equal to

$$\mathbf{v}_A = (v - \omega\rho\sin\phi)\mathbf{E}_1 + \omega\rho\cos\phi\mathbf{E}_2. \tag{2.92}$$

In the general case a moment of force is calculated from the formula

$$\mathbf{M} = \boldsymbol{\rho}\times\mathbf{F} = \begin{vmatrix} \mathbf{E}_1 & \mathbf{E}_2 & \mathbf{E}_3 \\ \rho_1 & \rho_2 & \rho_3 \\ F_1 & F_2 & F_3 \end{vmatrix}$$
$$= \mathbf{E}_1(\rho_2F_3 - \rho_3F_2) - \mathbf{E}_2(\rho_1F_3 - \rho_3F_1) + \mathbf{E}_3(\rho_1F_2 - \rho_2F_1), \tag{2.93}$$

and hence, in the considered case, the differential of the friction moment is equal to

$$\mathrm{d}\mathbf{M}_T = \mathbf{E}_3(\rho_2 \mathrm{d}T_1 - \rho_1 \mathrm{d}T_2). \tag{2.94}$$

According to (2.91) and (2.92) we have

$$\begin{aligned}
\mathrm{d}T_1 &= -\frac{\mu\sigma(\rho)}{v_A}(v - \omega\rho\sin\phi)\rho\mathrm{d}\rho\mathrm{d}\phi, \\
\mathrm{d}T_2 &= -\frac{\mu\sigma(\rho)}{v_A}\omega\rho\cos\phi\rho\mathrm{d}\rho\mathrm{d}\phi, \\
v_A &= |\mathbf{v}_A| = \sqrt{v^2 - 2v\omega\rho\sin\phi + \omega^2\rho^2},
\end{aligned} \tag{2.95}$$

and in view of that

$$\mathrm{d}M_T = \frac{\mu\sigma(\rho)}{v_A}(\omega\rho^2 - v\rho\sin\phi)\rho\mathrm{d}\rho\mathrm{d}\phi. \tag{2.96}$$

We obtain the desired friction force $\mathbf{T}$ and friction moment $\mathbf{M}_T$ by means of integration of (2.91) and (2.96), and they are equal to

$$\begin{aligned}
\mathbf{T} &= -\mu\mathbf{E}_1 \int_0^R \int_0^{2\pi} \frac{(v - \omega\rho\sin\phi)\sigma(\rho)\rho\mathrm{d}\rho\mathrm{d}\phi}{\sqrt{v^2 - 2v\omega\rho\sin\phi + \omega^2\rho^2}} \\
&\quad -\mu\mathbf{E}_2 \int_0^R \int_0^{2\pi} \frac{\omega\rho\cos\phi\sigma(\rho)\rho\mathrm{d}\rho\mathrm{d}\phi}{\sqrt{v^2 - 2v\omega\rho\sin\phi + \omega^2\rho^2}}, \\
\mathbf{M}_T &= -\mu\mathbf{E}_3 \int_0^R \int_0^{2\pi} \frac{(\omega\rho^2 - v\rho\sin\phi)\sigma(\rho)\rho\mathrm{d}\rho\mathrm{d}\phi}{\sqrt{v^2 - 2v\omega\rho\sin\phi + \omega^2\rho^2}},
\end{aligned} \tag{2.97}$$

where in the considered case the second term of friction force in the first equation is equal to zero.

After the introduction of new variables $\rho^* = \frac{\rho}{R}$ and $u = \omega R$, where R is the radius of the circular contact patch between the bodies, (2.97) assume the following scalar form:

$$\begin{aligned}
T &= \mu R^2 \int_0^1 \rho^*\sigma(\rho^*) \int_0^{2\pi} \frac{(v - u\rho^*\sin\phi)\mathrm{d}\rho^*\mathrm{d}\phi}{\sqrt{u^2\rho^{*2} - 2uv\rho^*\sin\phi + v^2}}, \\
M &= \mu R^3 \int_0^1 \rho^*\sigma(\rho^*) \int_0^{2\pi} \frac{(u\rho^{*2} - v\rho^*\sin\phi)\mathrm{d}\rho^*\mathrm{d}\phi}{\sqrt{u^2\rho^{*2} - 2uv\rho^*\sin\phi + v^2}}.
\end{aligned} \tag{2.98}$$

It is well worth it to return to Fig. 2.52a and discuss the choice of the pole (point O) about which the reduction of the friction force and friction moment were carried out. In the general case, for the aforementioned choice of the reduction pole, integrals (2.98) cannot be expressed in terms of elementary functions but only in terms of elliptic integrals, which is inconvenient during the calculations. However, it turns out that physical observations associated with the kinematics and statics of the problem allow for (by the proper choice of the pole) the reduction of the problem to integrals expressed in terms of elementary functions. As will be presented, these integrals in the considered cases are expressed by simple analytical functions. An arbitrary choice of the reduction pole leads to the necessity of taking the force and moment of friction into account. However, it is possible to choose the pole at the point that is an *instantaneous center of rotation* (this case will be given a detailed treatment subsequently) and then, even though we are also dealing with the force and moment of friction, integrals (2.98) assume the form of elementary integrals. One can also choose the point of reduction such that the moment arm of friction force is equal to zero. Because in the general case $M = M(v, \omega)$, let us choose the reduction point such that $M(v, 0) = 0$. The physical interpretation of the pole like that coincides with the notion of mass center of the contact patch, where the role of mass is played by the normal stress $\sigma = \sigma(x_1, x_2)$ in the coordinate system rigidly connected to the contact surface. This point is defined by two algebraic equations:

$$\iint_D x_i \sigma(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 = 0, \qquad i = 1, 2, \tag{2.99}$$

where D is the area of contact of the bodies.

In the general case, (2.98) in the Cartesian coordinate system assume the form

$$\begin{aligned} T_1 &= -\mu R^2 \iint_D \frac{\sigma(x_1, x_2)(v - ux_2)\mathrm{d}x_1\mathrm{d}x_2}{\sqrt{(x_1^2 + x_2^2)u^2 + v^2 - 2uvx_2}}, \\ T_2 &= -\mu R^2 \iint_D \frac{\sigma(x_1, x_2)ux_1\mathrm{d}x_1\mathrm{d}x_2}{\sqrt{(x_1^2 + x_2^2)u^2 + v^2 - 2uvx_2}}, \\ M &= -\mu R^3 \iint_D \frac{\sigma(x_1, x_2)[(x_1^2 + x_2^2)u - vx_2]\mathrm{d}x_1\mathrm{d}x_2}{\sqrt{(x_1^2 + x_2^2)u^2 + v^2 - 2uvx_2}}, \end{aligned} \tag{2.100}$$

where the designation $u = \omega R$ was preserved.

As was emphasized earlier, in the case where region D is symmetrical with respect to the pole (e.g., a disc), the problem is two-dimensional since $T = T(u, v)$ and $M = M(u, v)$, and $T_2 = 0$. However, in the general case (for non-symmetrical regions), we have at our disposal three (2.100) and three kinematic quantities u, v_1, and v_2, that is, $T = T(u, v_1, v_2)$ and $M = M(u, v_1, v_2)$. Let us recall that earlier for the case of the circular contact patch we chose the axis of Cartesian coordinates as parallel to the velocity $\mathbf{v}$.

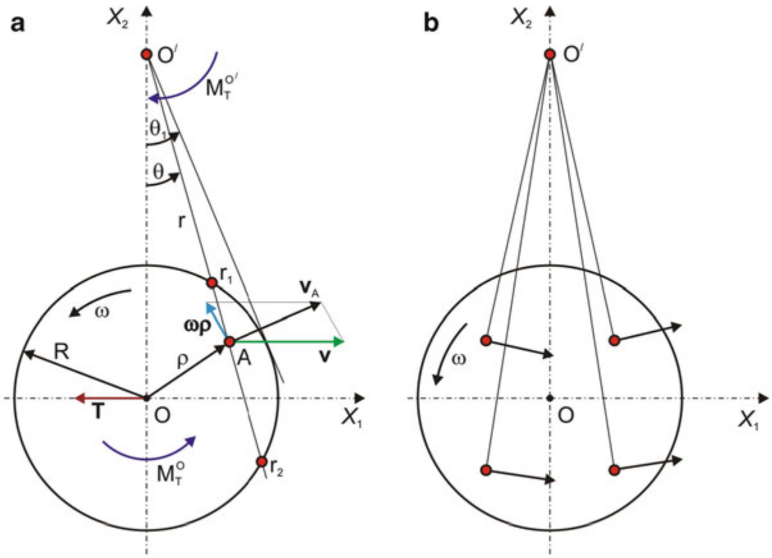


Fig. 2.53 A circular contact in the case of coordinates (r, θ) (**a**) and distributions of the velocity field of selected points of the contact patch (**b**)

Our next aim will be to determine the force and moment of friction in simple cases of symmetrical patches of contact between bodies. To this end we explore the following two cases, known from the literature, of contact stresses [47]:

(i) Circular contact patch of a disc with a plane, where the dependency of stress on the radius ρ is governed by the equation

$$\sigma(\rho) = \frac{3N}{2\pi R^2 \sqrt{1 - \left(\frac{\rho}{R}\right)^2}}; \tag{2.101}$$

(ii) Hertzian point contact, where

$$\sigma(\rho) = \frac{3N\sqrt{1 - \left(\frac{\rho}{R}\right)^2}}{2\pi R^2}. \tag{2.102}$$

In the preceding discussion, N is the normal force pressing the bodies against each other, R is the radius of the contact patch circle, and $\rho = OA$ (Figs. 2.52 and 2.53).

Case (i)

In order to avoid elliptic integrals we introduce the polar coordinate system (r, θ) shown in Fig. 2.53a.

Point O' is the instantaneous center of rotation, and $O'O = H$. In turn, from the velocity distribution in Fig. 2.53b it follows that in this case the friction component $T_2 = 0$.

Thus, by definition we have

$$\mathrm{d}T = -\mu\sigma(\rho)\frac{v_A \cos\theta}{v_A}\mathrm{d}S, \tag{2.103}$$

that is,

$$T = -\mu \iint_D \sigma(\rho)\cos\theta\, r\,\mathrm{d}\theta\mathrm{d}r. \tag{2.104}$$

In turn, on the basis of Fig. 2.53a one can formulate the following relationship:

$$\rho^2 = r^2\sin^2\theta + (H - r\cos\theta)^2. \tag{2.105}$$

From (2.104) and (2.101) we obtain

$$\begin{aligned}
T &= -\frac{3N\mu}{2\pi}\iint_D \frac{\cos\theta\, r\,\mathrm{d}\theta\mathrm{d}r}{R^2\sqrt{1-\left(\frac{r^2}{R^2}\sin^2\theta + \frac{H^2}{R^2} - 2\frac{H}{R}\frac{r}{R}\cos\theta + \frac{r^2}{R^2}\cos^2\theta\right)}} \\
&= -\frac{3N\mu}{2\pi}\iint_D \frac{\left(\frac{r}{R}\right)\cos\theta\mathrm{d}\theta\left(\frac{\mathrm{d}r}{R}\right)}{\sqrt{1-q^2\sin^2\theta - k^2 + 2qk\cos\theta - q^2\cos^2\theta}} \\
&= -\frac{3N\mu}{2\pi}\iint_D \frac{q\cos\theta\mathrm{d}\theta\mathrm{d}q}{\sqrt{1-q^2+2qk\cos\theta - k^2\sin^2\theta - k^2\cos^2\theta}} \\
&= -\frac{3N\mu}{2\pi}\iint_D \frac{q\cos\theta\mathrm{d}\theta\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2 + 1 - k^2\sin^2\theta}},
\end{aligned} \tag{2.106}$$

where we set $q = \frac{r}{R}$ and $k = \frac{H}{R}$.

The friction moment with respect to point O' is equal to

$$M_T^{O'} = HT + M_T^O. \tag{2.107}$$

From (2.107) it follows that by calculating $M_T^{O'}$ and with H known it is possible to determine M_T^O.

During the calculation of integral (2.106) one should consider two cases, that is, one where point O' lies outside the circle of radius R (then $k > 1$), and another where it lies inside or at the boundary of that circle (then $k \le 1$). Thus the problem boils down to the calculation of the following two integrals:

$$T=-\frac{3N\mu}{4\pi}\int\limits_{0}^{\pi}\cos\theta\mathrm{d}\theta\int\limits_{q_1}^{q_2}\frac{q\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}},\quad (k\le 1),$$

$$T=-\frac{3N\mu}{2\pi}\int\limits_{0}^{\theta_1}\cos\theta\mathrm{d}\theta\int\limits_{q_1}^{q_2}\frac{q\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}},\quad (k>1), \tag{2.108}$$

where

$$q_{1,2}=k\cos\theta\mp\sqrt{1-k^2\sin^2\theta},\qquad \theta_1=\arcsin\left(\frac{1}{k}\right).$$

Now, we will show how to calculate integrals (2.108). At first, let us calculate the indefinite integral

$$J^*=\int\frac{q\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}}$$
$$=\int\frac{q\mathrm{d}q}{\sqrt{-q^2+2qk\cos\theta+1-k^2}}=\int\frac{q\mathrm{d}q}{\sqrt{-q^2+2qa+b}}, \tag{2.109}$$

where $a=k\cos\theta$, $b=1-k^2$. Because the parameters a and b do not depend on q, in the course of the calculations they can be treated as constants. We successively have

$$J^*=-\frac{1}{2}\int\frac{-2q\mathrm{d}q}{\sqrt{-q^2+2qa+b}}=-\frac{1}{2}\int\frac{(-2q+2a-2a)\mathrm{d}q}{\sqrt{-q^2+2qa+b}}$$
$$=-\frac{1}{2}\int\frac{(-2q+2a)\mathrm{d}q}{\sqrt{-q^2+2qa+b}}+a\int\frac{\mathrm{d}q}{\sqrt{-q^2+2qa+b}}. \tag{2.110}$$

After setting $J^*=J_1+J_2$ we have

$$J_1=-\frac{1}{2}\int\frac{(-2q+2a)\mathrm{d}q}{\sqrt{-q^2+2qa+b}}=-\frac{1}{2}\int\frac{\mathrm{d}(-q^2+2qa+b)}{\sqrt{-q^2+2qa+b}}, \tag{2.111}$$

and the above integral is calculated with the aid of the substitution $t=-q^2+2qa+b$. Thus we have

$$J_1=-\frac{1}{2}\int t^{-\frac{1}{2}}\mathrm{d}t=-\frac{1}{2}\frac{t^{-\frac{1}{2}}}{\frac{1}{2}}=-\sqrt{t}=-\sqrt{-q^2+2qa+b}. \tag{2.112}$$

Now as we proceed to calculate the integral

$$J_2=a\int\frac{\mathrm{d}q}{\sqrt{-q^2+2qa+b}}. \tag{2.113}$$

We transform the denominator of the preceding expression into the form

$$\sqrt{-q^2+2qa+b} = \sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta} = \sqrt{-(q-a)^2+c}, \tag{2.114}$$

where $c = 1-k^2\sin^2\theta$. Thus we obtain

$$J_2 = a\int\frac{\mathrm{d}q}{\sqrt{c-(q-a)^2}} = a\int\frac{\mathrm{d}(q-a)}{\sqrt{\left(\sqrt{c}\right)^2-(q-a)^2}} = a\int\frac{\mathrm{d}z}{\sqrt{\left(\sqrt{c}\right)^2-z^2}} = a\arcsin\frac{z}{\sqrt{c}} = a\arcsin\frac{q-a}{\sqrt{c}}. \tag{2.115}$$

According to (2.110) we obtain

$$J^* = J_1 + J_2 = -\sqrt{-q^2+2qa+b} + a\arcsin\frac{q-a}{\sqrt{c}}. \tag{2.116}$$

Eventually, we obtain the desired definite integral

$$J_0^* = \int_{q_1}^{q_2}\frac{q\,\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}} = J^*\Big|_{q_1}^{q_2},$$

where $q_{1,2} = a \mp \sqrt{c}$. We have then

$$\begin{aligned} J_0^* &= -\sqrt{-q_2^2+2q_2a+b} + \arcsin\frac{q_2-a}{\sqrt{c}} + \sqrt{-q_1^2+2q_1a+b} \\ &\quad - \arcsin\frac{q_1-a}{\sqrt{c}} = -\sqrt{-a^2-2a\sqrt{c}-c+2a^2+2a\sqrt{c}+b} \\ &\quad + a\arcsin\frac{a+\sqrt{c}-a}{\sqrt{c}} + \sqrt{-a^2+2a\sqrt{c}-c+2a^2-2a\sqrt{c}+b} \\ &\quad - a\arcsin\frac{a-\sqrt{c}-a}{\sqrt{c}} = -\sqrt{a^2+b-c} + a\arcsin(1) \\ &\quad + \sqrt{a^2+b-c} - a\arcsin(-1) = a\frac{\pi}{2} - a\left(-\frac{\pi}{2}\right) = \pi a. \end{aligned} \tag{2.117}$$

The desired definite integral is equal to

$$\int_{q_1}^{q_2}\frac{q\,\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}} = \pi k\cos\theta. \tag{2.118}$$

Eventually, the unknown indefinite integral reads

$$\begin{aligned} J &= \int \pi k \cos\theta \cos\theta \mathrm{d}\theta = \pi k \int \cos^2\theta \mathrm{d}\theta \\ &= \frac{\pi k}{2} \int (1 + \cos 2\theta) \mathrm{d}\theta = \frac{\pi k}{2}\theta + \frac{\pi k}{4} \sin 2\theta, \end{aligned} \tag{2.119}$$

and its corresponding definite form is

$$\begin{aligned} J_0^{(1)} &= \int_0^{\pi} \cos\theta \mathrm{d}\theta \int_{q_1}^{q_2} \frac{q \mathrm{d}q}{\sqrt{-(q - k\cos\theta)^2 + 1 - k^2 \sin^2\theta}} \\ &= \left[\frac{\pi k}{2}\theta + \frac{\pi k}{4} \sin 2\theta \right]_0^{\pi} = \frac{\pi^2 k}{2}. \end{aligned} \tag{2.120}$$

The following integral remains to be calculated:

$$\begin{aligned} J_0^{(2)} &= \int_0^{\theta_1} \cos\theta \mathrm{d}\theta \int_{q_1}^{q_2} \frac{q \mathrm{d}q}{\sqrt{-(q - k\cos\theta)^2 + 1 - k^2 \sin^2\theta}} \\ &= \left[\frac{\pi k}{2}\theta + \frac{\pi k}{4} \sin 2\theta \right]_0^{\theta_1} = \frac{\pi k}{2}\theta_1 + \frac{\pi k}{4} \sin 2\theta_1. \end{aligned} \tag{2.121}$$

Since

$$\theta_1 = \arcsin\left(\frac{1}{k}\right), \tag{2.122}$$

we have

$$\begin{aligned} \sin 2\theta_1 &= 2\sin\theta_1 \cos\theta_1 = 2\sin\theta_1 \sqrt{1 - \sin^2\theta_1} \\ &= 2\sin\left(\arcsin\left(\frac{1}{k}\right)\right) \sqrt{1 - \sin^2\left(\arcsin\left(\frac{1}{k}\right)\right)} \\ &= \frac{2}{k}\sqrt{1 - \frac{1}{k^2}} = \frac{2\sqrt{k^2 - 1}}{k^2}, \end{aligned} \tag{2.123}$$

and finally

$$J_0^{(2)} = \frac{\pi k}{2} \arcsin\left(\frac{1}{k}\right) + \frac{\pi\sqrt{k^2 - 1}}{2k}. \tag{2.124}$$

After taking into account the results of the calculations above in (2.108) we obtain

$$T = -\frac{3N\mu}{2\pi}\begin{cases} \dfrac{\pi^2 k}{4}, & k \le 1, \\ \dfrac{\pi k}{2}\arcsin\left(\dfrac{1}{k}\right) + \dfrac{\pi}{2k}\sqrt{k^2-1}, & k > 1. \end{cases} \tag{2.125}$$

Case (ii)

In this case the problem boils down to the calculation of the friction force by means of the following definite integrals:

$$T = -\frac{3N\mu}{2\pi}\begin{cases} \dfrac{1}{2}\displaystyle\int_0^{\pi}\cos\theta \mathrm{d}\theta \int_{q_1}^{q_2}\sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}\, q\mathrm{d}q, \\ \displaystyle\int_0^{\theta_1}\cos\theta \mathrm{d}\theta \int_{q_1}^{q_2}\sqrt{-(q-k\cos\theta)^2+1-k^2\sin^2\theta}\, q\mathrm{d}q, \end{cases} \tag{2.126}$$

where further we adopt designations analogous to case (i), that is, $q_{1,2} = k\cos\theta \mp \sqrt{1-k^2\sin^2\theta}$, $\theta_1 = \arcsin\left(\frac{1}{k}\right)$.

At first, we calculate the indefinite integral

$$\begin{aligned} J &= \int \sqrt{-q^2+2qa+b}\, q\mathrm{d}q = -\frac{1}{2}\int \sqrt{-q^2+2qa+b}\,(-2q)\mathrm{d}q \\ &= -\frac{1}{2}\int \sqrt{-q^2+2qa+b}\,(-2q+2a-2a)\mathrm{d}q. \end{aligned} \tag{2.127}$$

Next, we have

$$J = J_1 + J_2, \tag{2.128}$$

where

$$\begin{aligned} J_1 &= -\frac{1}{2}\int \sqrt{-q^2+2qa+b}\,(-2q+2a)\mathrm{d}q, \\ J_2 &= a\int \sqrt{-q^2+2qa+b}\,\mathrm{d}q. \end{aligned} \tag{2.129}$$

Because

$$(-2q+2a)\mathrm{d}q = \mathrm{d}\left(-q^2+2aq\right) = \mathrm{d}\left(-q^2+2aq+b\right),$$

we have

$$J_1 = -\frac{1}{2}\int \left(-q^2+2aq+b\right)^{\frac{1}{2}}\mathrm{d}\left(-q^2+2aq+b\right). \tag{2.130}$$

In turn, after introducing the substitution

$$t = -q^2 + 2aq + b, \tag{2.131}$$

we obtain

$$J_1 = -\frac{1}{2}\int t^{\frac{1}{2}}\mathrm{d}t = -\frac{1}{2}\frac{t^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)} = -3t^{\frac{3}{2}}, \tag{2.132}$$

that is, eventually we have

$$J_1 = -3\left(\sqrt{-q^2 + 2aq + b}\right)^3. \tag{2.133}$$

We calculate the integral J_2 by parts:

$$\begin{aligned}
J_2 &= a\left[q\sqrt{-q^2 + 2aq + b} - \int q\mathrm{d}\left(\sqrt{-q^2 + 2aq + b}\right)\right] \\
&= a\left(q\sqrt{-q^2 + 2aq + b} - \int \frac{q(-2q + 2a)\mathrm{d}q}{2\sqrt{-q^2 + 2aq + b}}\right) \\
&= a\left(q\sqrt{-q^2 + 2aq + b} + \int \frac{q^2\mathrm{d}q}{\sqrt{-q^2 + 2aq + b}} - a\int \frac{q\mathrm{d}q}{\sqrt{-q^2 + 2aq + b}}\right) \\
&= a\left(q\sqrt{-q^2 + 2aq + b} + J_3 - aJ_4\right),
\end{aligned} \tag{2.134}$$

where

$$\begin{aligned}
J_3 &= \int \frac{q^2\mathrm{d}q}{\sqrt{-q^2 + 2aq + b}}, \\
J_4 &= \frac{q\mathrm{d}q}{\sqrt{-q^2 + 2aq + b}}.
\end{aligned} \tag{2.135}$$

According to (2.109) $J_4 = J^*$, that is,

$$J_4 = -\sqrt{-q^2 + 2aq + b} + \arcsin\frac{q-a}{\sqrt{c}}, \tag{2.136}$$

where $c = 1 - k^2\sin^2\theta$.

In the case of the integral J_3 we have

$$J_3 = -\int \frac{-q^2\mathrm{d}q}{\sqrt{-q^2 + 2aq + b}} = -\int \frac{(-q^2 + 2qa + b - 2qa - b)\mathrm{d}q}{\sqrt{-q^2 + 2aq + b}}$$

$$= -\int \frac{\left(\sqrt{-q^2+2aq+b}\right)^2 \mathrm{d}q}{\sqrt{-q^2+2aq+b}} + 2a\int \frac{q\mathrm{d}q}{\sqrt{-q^2+2aq+b}}$$
$$+b\int \frac{\mathrm{d}q}{\sqrt{-q^2+2aq+b}} = -\int \sqrt{-q^2+2aq+b}\,\mathrm{d}q + 2aJ_4 + bJ_5. \tag{2.137}$$

From (2.115) it follows that the integral J_5 is equal to

$$J_5 = \arcsin\frac{q-a}{\sqrt{c}}. \tag{2.138}$$

Let us note that the first integral in (2.137) is equal to $-\frac{1}{a}J_2$ [see (2.129)], that is, (2.137) assumes the form

$$J_3 = -\frac{1}{a}J_2 + 2aJ_4 + bJ_5, \tag{2.139}$$

where J_2 is given by (2.134).

From (2.134) and after taking into account (2.139) we obtain

$$J_2 = a\left(q\sqrt{-q^2+2aq+b} - \frac{1}{a}J_2 + 2aJ_4 + bJ_5 - aJ_4\right), \tag{2.140}$$

hence we obtain

$$J_2 = \frac{aq}{2}\sqrt{-q^2+2aq+b} + \frac{a^2}{2}J_4 + \frac{ab}{2}J_5. \tag{2.141}$$

Let us now proceed to the calculation of the definite integral $\tilde{J}$ corresponding to the indefinite integral (2.127). According to the Newton–Leibniz formula we have

$$\tilde{J} = \int\limits_{q_1}^{q_2} \sqrt{-q^2+2aq+b}\,q\mathrm{d}q = (J_1+J_2)\Big|_{q_1}^{q_2} = J_1\Big|_{q_1}^{q_2} + J_2\Big|_{q_1}^{q_2}. \tag{2.142}$$

We successively calculate

$$J_1\Big|_{q_1}^{q_2} = -3\left(\sqrt{-q^2+2aq+b}\right)^3\Big|_{q_1}^{q_2} = -3\left(\sqrt{-q^2+2aq+b}\right)^3\Big|_{a-\sqrt{c}}^{a+\sqrt{c}}$$
$$= -3\left[\left(\sqrt{-(a+\sqrt{c})^2+2a(a+\sqrt{c})+b}\right)^3\right.$$
$$\left. - \left(\sqrt{-(a-\sqrt{c})^2+2a(a-\sqrt{c})+b}\right)^3\right]$$
$$= -3\left[\left(\sqrt{a^2+b-c}\right)^3 - \left(\sqrt{a^2+b-c}\right)^3\right] \equiv 0, \tag{2.143}$$

$$
\begin{aligned}
J_2\Big|_{q_1}^{q_2} &= \frac{a}{2}\left(q\sqrt{-q^2+2aq+b}\right)\Big|_{q_1}^{q_2} + \frac{a^2}{2}J_4\Big|_{q_1}^{q_2} + \frac{ab}{2}J_5\Big|_{q_1}^{q_2} \\
&= \frac{a}{2}\left[(a+\sqrt{c})\sqrt{a^2+b-c} - (a-\sqrt{c})\sqrt{a^2+b-c}\right] \\
&\quad + \frac{a^2}{2}J_4\Big|_{q_1}^{q_2} + \frac{ab}{2}J_5\Big|_{q_1}^{q_2}. \qquad (2.144)
\end{aligned}
$$

The definite integral $J_4\Big|_{q_1}^{q_2}$ [see (2.136)] has already been obtained [see (2.117)] and is equal to

$$J_4\Big|_{q_1}^{q_2} = \pi a, \qquad (2.145)$$

and the integral $J_5\Big|_{q_1}^{q_2}$ [see (2.138)] has been also calculated earlier (see (2.117)) and reads

$$J_5\Big|_{q_1}^{q_2} = \pi. \qquad (2.146)$$

From (2.144) and after taking into account (2.145) and (2.146) we obtain

$$J_2\Big|_{q_1}^{q_2} = a\sqrt{c}\sqrt{a^2+b-c} + \frac{\pi}{2}a^3 + \frac{ab\pi}{2}. \qquad (2.147)$$

Returning to the original notation we have

$$
\begin{aligned}
\sqrt{a^2+b-c} &= \sqrt{k^2\cos^2\theta + 1 - k^2 - \left(1-k^2\sin^2\theta\right)}, \\
\frac{\pi}{2}a^3 &= \frac{\pi k^3}{2}\cos^3\theta, \\
\frac{ab\pi}{2} &= \frac{\pi}{2}\left(1-k^2\right)k\cos\theta, \qquad (2.148)
\end{aligned}
$$

and in view of that

$$J_2\Big|_{q_1}^{q_2} = \frac{\pi k^3}{2}\cos^3\theta + \frac{\pi}{2}(1-k^2)k\cos\theta.$$

The problem then boils down to the calculation of the integral

$$\int \tilde{J}\cos\theta \mathrm{d}\theta = \frac{\pi k^3}{2}\int \cos^4\theta \mathrm{d}\theta + \frac{\pi}{2}(1-k^2)k\int \cos^2\theta \mathrm{d}\theta. \qquad (2.149)$$

The elementary integrals are calculated in the following way:

$$\int \cos^2\theta \mathrm{d}\theta = \frac{1}{2}\int (1+\cos 2\theta)\mathrm{d}\theta = \frac{\theta}{2} + \frac{1}{4}\sin 2\theta,$$

$$\begin{aligned}\int \cos^4\theta \mathrm{d}\theta &= \frac{1}{4}\int (1+\cos 2\theta)^2\mathrm{d}\theta \\ &= \frac{1}{4}\int (1+2\cos 2\theta+\cos^2 2\theta)\mathrm{d}\theta \\ &= \frac{1}{4}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\int (1+\cos 4\theta)\mathrm{d}\theta \\ &= \frac{1}{4}\theta + \frac{1}{4}\sin 2\theta + \frac{\theta}{8} + \frac{1}{32}\sin 4\theta. \end{aligned} \tag{2.150}$$

In order to shorten the notation we calculate the following definite integrals:

$$\int\limits_0^{\pi} \cos^2\theta \mathrm{d}\theta = \left(\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right)\Bigg|_0^{\pi} = \frac{\pi}{2},$$

$$\begin{aligned}\int\limits_0^{\theta_1} \cos^2\theta \mathrm{d}\theta &= \left(\frac{\theta}{2} + \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta}\right)\Bigg|_0^{\arcsin\frac{1}{k}} \\ &= \frac{1}{2}\arcsin\frac{1}{k} + \frac{\sqrt{k^2-1}}{2k^2},\end{aligned}$$

$$\int\limits_0^{\pi} \cos^4\theta \mathrm{d}\theta = \left(\frac{3\theta}{8} + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right)\Bigg|_0^{\pi} = \frac{3\pi}{8},$$

$$\begin{aligned}\int\limits_0^{\theta_1} \cos^4\theta \mathrm{d}\theta &= \Bigg(\frac{3\theta}{8} + \frac{1}{2}\sin\theta\sqrt{1-\sin^2\theta} \\ &\quad + \frac{1}{8}\sin\theta\sqrt{1-\sin^2\theta}(1-2\sin^2\theta)\Bigg)\Bigg|_0^{\arcsin\frac{1}{k}} \\ &= \frac{3}{8}\arcsin\frac{1}{k} + \frac{\sqrt{k^2-1}}{2k^2} + \frac{(k^2-1)\sqrt{k^2-1}}{8k^4} \\ &= \frac{3}{8}\arcsin\frac{1}{k} + \frac{(5k^2-2)\sqrt{k^2-1}}{8k^4}.\end{aligned} \tag{2.151}$$

From (2.126) we eventually obtain

$$T = -\frac{3N\mu}{2\pi}\begin{cases} \int\limits_0^{\pi} \cos\theta \mathrm{d}\theta \int\limits_{q_1}^{q_2} \sqrt{-(q-k\cos\theta)^2 + 1 - k^2\sin^2\theta}\, q\mathrm{d}q \\ = \frac{\pi^2 k}{16}\left(4-k^2\right); \qquad (k \le 1) \\ \int\limits_0^{\theta_1} \cos\theta \mathrm{d}\theta \int\limits_{q_1}^{q_2} \sqrt{-(q-k\cos\theta)^2 + 1 - k^2\sin^2\theta}\, q\mathrm{d}q \\ = \frac{\pi k}{16}\left(4-k^2\right)\arcsin\frac{1}{k} + \frac{\pi\left(2+k^2\right)\sqrt{k^2-1}}{16k}; \qquad (k > 1). \end{cases} \tag{2.152}$$

Let us now return to the friction model presented in Figs. 2.52 and 2.53. The friction moment about point O' (the instantaneous center of rotation) we obtain by exploiting (2.106), where additionally the numerator of the integrand is multiplied by $\frac{Rr}{R} = Rq$ to obtain

$$M_T^{O'} = \mu N R \begin{cases} \frac{1}{2}\int\limits_0^{\pi} \mathrm{d}\theta \int\limits_{q_1}^{q_2} \frac{q^2\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2 + 1 - k^2\sin^2\theta}}, \qquad (k \le 1), \\ \int\limits_0^{\theta_1} \mathrm{d}\theta \int\limits_{q_1}^{q_2} \frac{q^2\mathrm{d}q}{\sqrt{-(q-k\cos\theta)^2 + 1 - k^2\sin^2\theta}}, \qquad (k > 1). \end{cases} \tag{2.153}$$

After using the previous calculations [see (2.135)] we obtain

$$M_T^{O'} = \mu N R \begin{cases} \frac{\pi^2(2+k^2)}{8}, \qquad k \le 1, \\ \frac{\pi^2(2+k^2)}{4}\arcsin\frac{1}{k} + \frac{3\pi}{4}\sqrt{k^2-1}, \qquad k > 1. \end{cases} \tag{2.154}$$

Our aim is to determine the moment of friction forces with respect to point O, which, according to the designations presented in Fig. 2.53a, is equal to

$$M_T^{O'} - M_T^{O} - HT = 0, \tag{2.155}$$

that is,

$$M_T^{O} = M_T^{O'} - HT, \tag{2.156}$$

where the friction force T is given by (2.106).

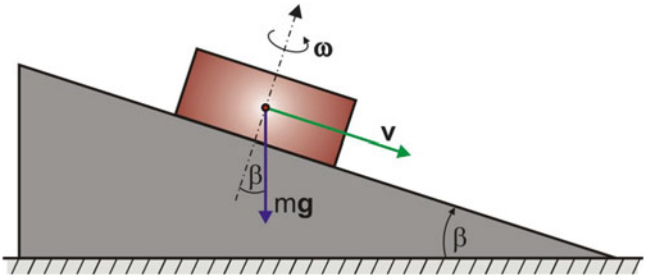


Fig. 2.54 Motion of a rigid body on an inclined plane with velocity of rotation (ω) and velocity of slip (v) included simultaneously

As an example of the application of the introduced theoretical considerations we will examine the motion of a disc on an inclined and a horizontal plane.

Let us consider the classical case of motion of a rigid body of mass m and mass moment of inertia I about the principal centroidal axis of inertia perpendicular to the surface of contact being inclined at an angle β to the horizontal surface (Fig. 2.54).

The equations of motion of the body have the form

$$\begin{aligned} m\dot{v} &= -T + mg\sin\beta, \\ I\dot{\omega} &= -M_T. \end{aligned} \tag{2.157}$$

The calculations will be conducted for the previously considered cases (i) and (ii) using the Padé approximation.

Let us now return to (2.98). As has already been mentioned, integrals (2.98) can be expressed in terms of elliptic integrals that can be checked easily, for instance, by employing the *Mathematica* software. However, it turns out that they can be approximated using the Padé approximation after introducing $\sigma(r)$ described by (2.101) and (2.102), which was demonstrated by Zhuravlev [46]. The calculations below were conducted using a *Mathematica* application for symbolic calculations. Let us apply the simplest form of the Padé approximation

$$\begin{aligned} M_T[u,v] &= M_0\frac{u}{u+av}, \\ T[u,v] &= T_0\frac{v}{v+bu}, \end{aligned} \tag{2.158}$$

where T_0, M_0, a, and b require determination.

The quantities T_0 and M_0 are determined in the following way:

$$T_0 = T[0,v], \qquad M_0 = M[u,0], \tag{2.159}$$

and from (2.98) we find

$$T_0 = \mu R^2 \int_0^1 \rho^* \sigma(\rho^*) \int_0^{2\pi} \mathrm{d}\rho^* \mathrm{d}\phi = 2\pi\mu R^2 \int_0^1 \rho^* \sigma(\rho^*) \mathrm{d}\rho^*,$$

$$M_0 = \mu R^3 \int_0^1 \rho^* \sigma(\rho^*) \int_0^{2\pi} \rho^* \mathrm{d}\rho^* \mathrm{d}\phi = 2\pi\mu R^3 \int_0^1 \rho^{*2} \sigma(\rho^*) \mathrm{d}\rho^*. \tag{2.160}$$

The quantities a and b are obtained after the differentiation of (2.158) and (2.98) with respect to variables u and v. After adequate differentiation of the equations we have

$$\frac{\partial T}{\partial v}[u, 0] = \frac{T_0}{bu}, \qquad \frac{\partial M}{\partial u}[0, v] = \frac{M_0}{av}, \tag{2.161}$$

and after differentiation of (2.98) we obtain

$$\frac{\partial T}{\partial v}[u, 0] = \frac{R\pi}{u} \int_0^1 \sigma(\rho^*) \mathrm{d}\rho^*,$$

$$\frac{\partial M}{\partial u}[0, v] = \frac{\pi}{Rv} \int_0^1 \rho^{*3} \sigma(\rho^*) \mathrm{d}\rho^*. \tag{2.162}$$

After equating (2.161) and (2.162) by sides we obtain the unknown coefficients

$$a = \frac{2\int_0^1 \rho^{*2} \sigma(\rho^*) \mathrm{d}\rho^*}{\int_0^1 \rho^{*3} \sigma(\rho^*) \mathrm{d}\rho^*}, \qquad b = \frac{2\int_0^1 \rho^{*} \sigma(\rho^*) \mathrm{d}\rho^*}{\int_0^1 \sigma(\rho^*) \mathrm{d}\rho^*}. \tag{2.163}$$

The desired coefficients were found using *Mathematica*.

Equations (2.157), after taking into account the Padé approximation, assume the form

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{T_0 v}{|v| + bu} + mg \sin\beta,$$

$$I\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{R M_0 u}{|u| + av}, \tag{2.164}$$

where $u = R\omega$.

In the case of a contact over the circular patch (case i) R denotes the radius, and in the case of the point contact $R \to 0$.

Following the transformation of (2.164) we obtain

$$\frac{m}{T_0}\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{v}{|v| + bu} + \frac{mg\sin\beta}{T_0},$$
$$\frac{I}{RM_0}\frac{\mathrm{d}u}{\mathrm{d}t} = -\frac{u}{|u| + av}. \tag{2.165}$$

The preceding equations assume the following form:

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = -\frac{v}{|v| + bu} + p,$$
$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = -\frac{eu}{|u| + av}, \tag{2.166}$$

where the following substitution for time and two dimensionless parameters were introduced:

$$\tau = \frac{T_0}{m}t, \qquad p = \frac{mg\sin\beta}{T_0}, \qquad e = \frac{RM_0 m}{T_0 I}. \tag{2.167}$$

For case (i) we assume $m = \rho_0 \pi R^2$ (ρ_0 is the surface density) and $I = \frac{1}{2}mR^2$, and from (2.160) and (2.163) we obtain

$$M_0 = \frac{1}{4}N\pi R\mu = \frac{1}{4}\pi R T_0, \qquad a = \frac{3\pi}{4},\ b = \frac{4}{\pi},$$

and for case (ii) we have

$$M_0 = \frac{3}{16}N\pi R\mu = \frac{3}{16}\pi R T_0, \qquad a = \frac{15\pi}{16},\ b = \frac{8}{3\pi}.$$

For case (i) $e = \frac{2R\pi R T_0 m}{4T_0 m R^2} = \frac{\pi}{2}$ [for case (ii) we have $M_0 = 0$ because $R = 0$].

Equations (2.166) for case (i) take the form

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = -\frac{v\pi}{|v|\pi + 4u} + p,$$
$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = -\frac{2\pi u}{4|u| + 3\pi v}, \tag{2.168}$$

and for case (ii) (2.166) are as follows:

$$\frac{\mathrm{d}v}{\mathrm{d}\tau} = -\frac{3v\pi}{3|v|\pi + 8u} + p,$$
$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = 0. \tag{2.169}$$

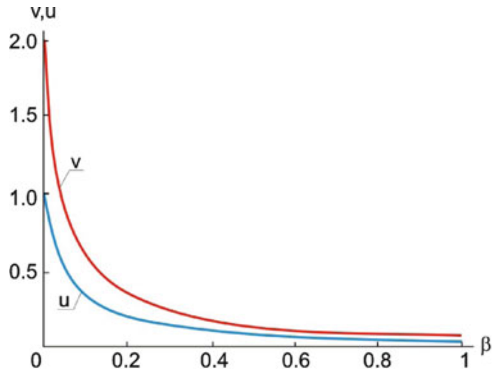


Fig. 2.55 Graphs of $v(\beta)$ and $u(\beta)$ dependencies

Let us consider (2.168) in more detail. After dividing these equations by sides we obtain

$$\frac{\mathrm{d}v}{\mathrm{d}u} = \frac{[(p-1)v + pbu](u+av)}{eu(v+bu)}, \tag{2.170}$$

where $a = \frac{3\pi}{4}$, $b = \frac{4}{\pi}$.

Mathematica cannot integrate (2.170) directly, although it is possible after the introduction of adequate substitution for the coordinates. For the case $p = 0$ (motion of a cylindrical disc on a horizontal surface), (2.170) is integrable by quadratures.

In the general case the numerical integration of (2.168) is made more difficult because of singularities of these equations (the denominators may tend to zero or to infinity). On the example of the case $p = 0$ we will show how to avoid the singularities of those equations by applying the substitution of the independent variable (time) according to the equation

$$\tau = \int_0^\beta (3\pi v + 4u)(\pi v + 4u)\mathrm{d}\beta. \tag{2.171}$$

The obtained equations

$$\frac{\mathrm{d}v}{\mathrm{d}\beta} = -v(3\pi v + 4u),$$
$$\frac{\mathrm{d}u}{\mathrm{d}\beta} = -2u(\pi v + 4u) \tag{2.172}$$

do not have singularities at the point $u = v = 0$ and can be easily integrated.

Below some illustrative results are shown.

From Fig. 2.55 it can be seen that $u(\beta) \to 0$, $v(\beta) \to 0$ for $\beta \to \infty$. In view of that, according to (2.171) τ attains a finite value. For certain finite

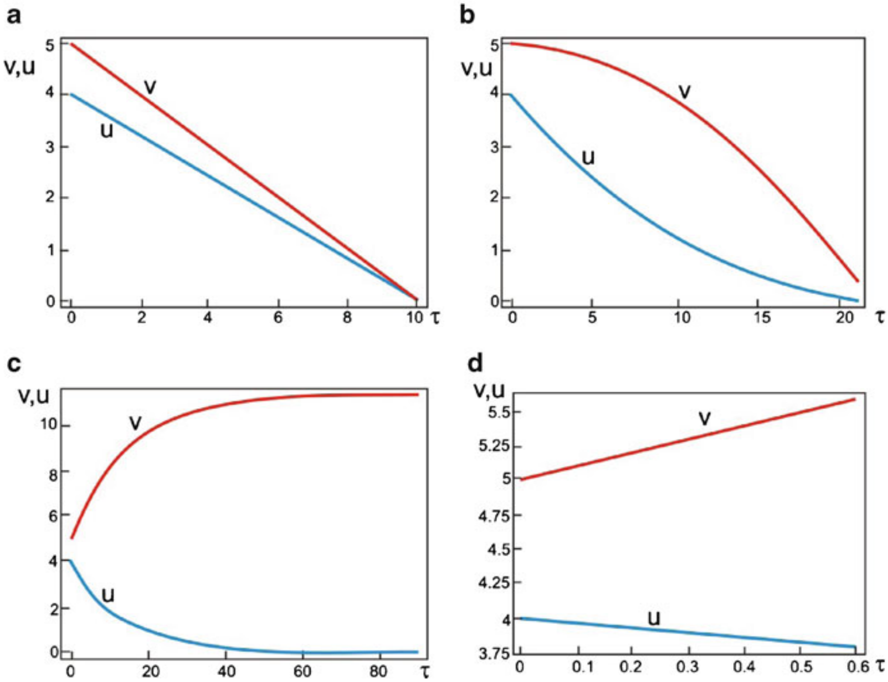


Fig. 2.56 Dependencies $u(\tau)$ and $v(\tau)$ for the following cases: (**a**) $p = 0$; (**b**) $p = 0.5$; (**c**) $p = 1$; (**d**) $p = 1.5$ for the circular contact

values of τ_* (which will be shown later with the aid of numerical calculations) $u(\tau_*) = v(\tau_*) = 0$. *This means that in the case of the applied Padé approximations both the sliding velocity v and the rotation velocity ω reach zero values at the same time instant.*

In Fig. 2.56 some illustrative graphs of $v(\tau)$ and $u(\tau)$ for $p = 0$, $p = 0.5$, $p = 1$, and $p = 1.5$ are presented. From the qualitative analysis conducted in [46] it follows that the behavior of the disc differs in a qualitative sense for $p < 1$ and for $p > 1$ (the critical case of $p = 1$).

In all the presented cases identical initial conditions are assumed: $v_0 = 5$, $u_0 = 4$. The time plots for $p = 0$ and $p = 1.5$ do not deviate much from straight lines, whereas for $p = 1.5$ we have $\omega(\tau) \to 0$ and $v(\tau) \to \infty$. In the case $p = 0.5$ it can be seen that $v(\tau)$ and $\omega(\tau)$ attain zero and their graphs are curves. In the critical case of $\tau \to \infty$ we have $\omega(\tau) \to 0$, whereas $v(\tau)$ reaches a steady value v_*, which can be easily estimated in an analytical way.

A natural and interesting question arises as to whether the introduction of the initial condition only for one variable causes the failure of the other variable to appear. This case is considered in a manner analogous to that of the previously analyzed example, and we assume respectively $v_0 = 0$, $u_0 = 0$, $v = 5$.

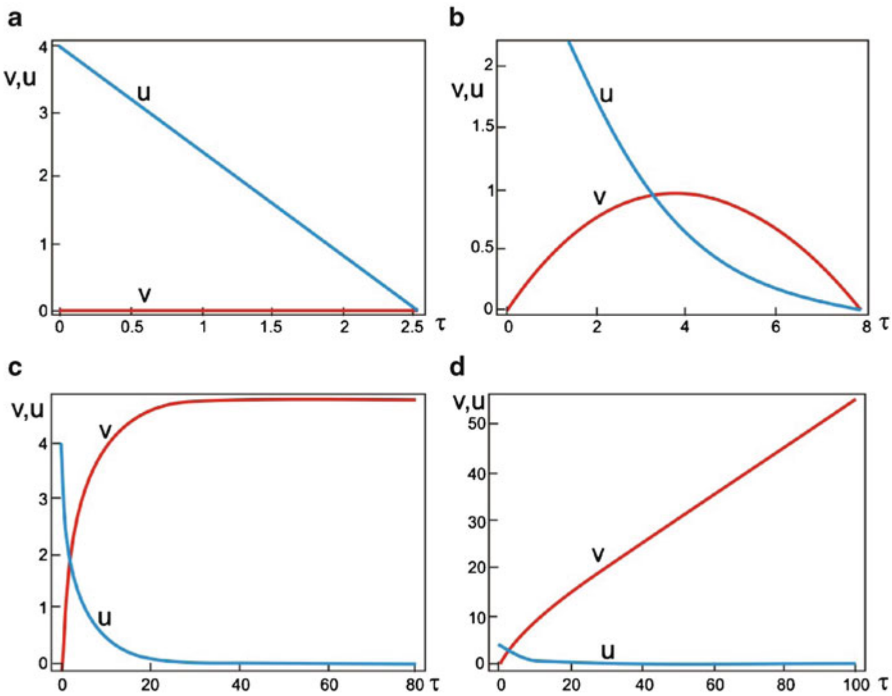


Fig. 2.57 Graphs of time characteristics $u(\tau)$ and $v(\tau)$ for the initial conditions $v_0 = 0$ and $u_0 = 4$ and cases (**a**) $p = 0$, (**b**) $p = 0.5$, (**c**) $p = 1$, (**d**) $p = 1.5$

From Fig. 2.57 it follows that during motion on the horizontal plane the disc only rotates with the rotation becoming gradually slower until it reaches a state of rest with no contribution from the slip velocity. In the $p = 0.5$ case the angular velocity attains zero along the curve, while the sliding velocity initially increases, reaches the maximum, and decreases to zero. In the critical case $p = 1$ the velocity $\omega(\tau)$ drops quickly to zero, and the sliding velocity reaches a steady value.

In the case $p = 0$ ($p = 0.5$) the velocity $\omega(\tau) \to 0$ for $\tau_* = 5$ ($\tau_* = 10$) at $v(\tau) = 0$. In the critical case $p = 1$ we have $v(\tau) = v_0$, $\omega(\tau) = \omega_0$, and in the case $p = 1.5$ we have $v(\tau) = 0$, while $v(\tau) \to \infty$ (Fig. 2.58).

Let us now return to the exact equations of disc motion for the case $p = 0$, which have the following form:

(i) The instantaneous center of rotation lies within the contact patch of the disc and inclined plane ($\omega R \geq v$):

$$\dot{v} = -\frac{\pi \mu p_0}{4 \rho_0 R^3} \frac{v}{\omega},$$

$$\dot{\omega} = -\frac{\pi \mu p_0}{4 \rho_0 R^3} \frac{\omega^2 R^2 - v^2}{\omega^2 R^2}; \tag{2.173}$$

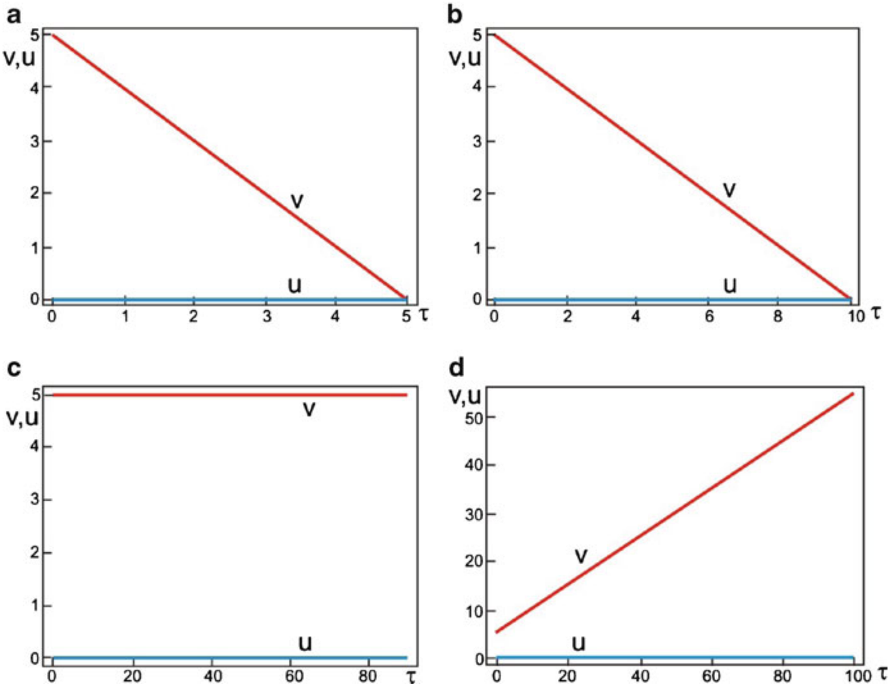


Fig. 2.58 Graphs of time characteristics $u(\tau)$ and $v(\tau)$ for initial conditions $v_0 = 5$ and $u_0 = 0$ and cases (**a**) $p = 0$, (**b**) $p = 0.5$, (**c**) $p = 1$, (**d**) $p = 1.5$

(ii) The instantaneous center of rotation lies outside the contact patch of the disc and inclined plane ($\omega R < v$):

$$\dot{v} = -\frac{\mu p_0}{2\rho_0 R^2}\left(\frac{v}{\omega R}\arcsin\frac{\omega R}{v} + \frac{\sqrt{v^2 - \omega^2 R^2}}{v}\right),$$

$$\dot{\omega} = -\frac{\mu p_0}{2\rho_0 R^2}\left(\frac{\omega^2 R^2 - v^2}{\omega^2 R^2}\arcsin\frac{\omega R}{v} + \frac{\sqrt{v^2 - \omega^2 R^2}}{\omega R}\right). \tag{2.174}$$

Above we have set $m = \rho_0 \pi R^2$, $I = \frac{1}{2}\rho_0 \pi R^4$.

In Fig. 2.59 three illustrative results of calculations are shown, where besides $v(t)$ and $\omega(t)$ the graph of $\omega(t)/v(t)$ is presented and where (we recall) the position of the instantaneous center of rotation is described by the equation $k = \frac{v(t)}{\omega(t)R} = \frac{r(t)}{R}$.

The position of the instantaneous center of rotation for $v(t) \to 0$ and $\omega(t) \to 0$ was analyzed in [48].

After dividing (2.173) and (2.174) respectively by sides we obtain

$$\frac{dv}{d\omega} = R\frac{k}{2 - k^2}, \qquad (k \le 1), \tag{2.175}$$

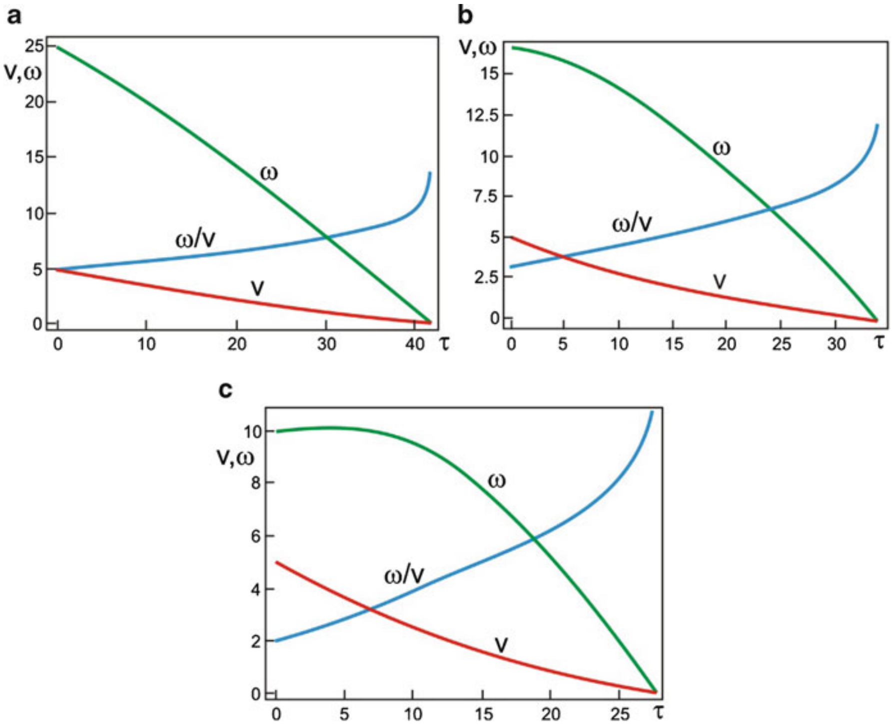


Fig. 2.59 Time histories of $v(t)$, $\omega(t)$, and $\omega(t)/v(t)$ for (**a**) $\omega_0 = 25$, (**b**) $\omega_0 = 16.666$, (**c**) $\omega_0 = 10$

$$\frac{\mathrm{d}v}{\mathrm{d}\omega} = R\frac{k \arcsin\left(\frac{1}{k}\right) + \frac{\sqrt{k^2-1}}{k}}{(2-k^2)\arcsin\left(\frac{1}{k}\right) + \sqrt{k^2-1}}, \qquad (k > 1). \tag{2.176}$$

The equation of a tangent to the trajectory of the system has the form $v/\omega = Rk$, and from (2.175) and (2.176) we obtain

$$k = \frac{k}{2-k^2}, \qquad (k \leq 1),$$

$$k = \frac{k \arcsin\left(\frac{1}{k}\right) + \frac{\sqrt{k^2-1}}{k}}{(2-k^2)\arcsin\left(\frac{1}{k}\right) + \sqrt{k^2-1}}, \qquad (k > 1). \tag{2.177}$$

The first equation possesses the solution $k = 1$, while the second one does not have any solutions.

In the end, let us return to case (ii) and (2.169). Some illustrative calculations for the point contact are given in Fig. 2.60 for different values of the parameter p. It can be seen that for $p \geq 1$ the motion changes in a qualitative sense.

As was mentioned, (2.158) at the points $u = 0$ and $v = 0$ are singular and the limits $\lim_{u,v\to 0} M_T$ and $\lim_{u,v\to 0} T$ do not exist. After the introduction of the designations $m = |M_T|/M_0$, $n = |T|/T_0$ (2.158) take the form

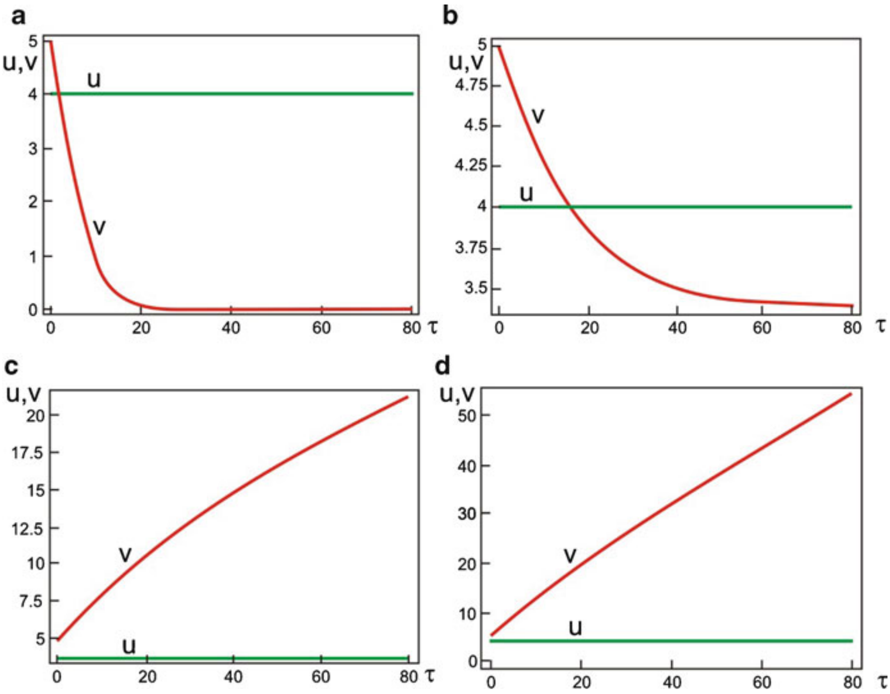


Fig. 2.60 Dependencies $u(\tau)$ and $v(\tau)$ for the point contact for the cases (**a**) $p = 0$, (**b**) $p = 0.5$, (**c**) $p = 1$, (**d**) $p = 1.5$

$$m = \frac{u}{u + av}, \qquad n = \frac{v}{v + bu}. \tag{2.178}$$

Because, as the numerical calculations showed, u and v tend to zero attaining this singular point simultaneously, let us introduce the relationship $v = \alpha u$. After taking into account this relation in (2.178) we obtain

$$m = \frac{u}{u + a\alpha v}, \qquad n = \frac{\alpha u}{\alpha u + bu}. \tag{2.179}$$

From (2.179) we obtain

$$m(1 + \alpha a) = 1, \qquad n\left(1 + \frac{b}{\alpha}\right) = 1. \tag{2.180}$$

From the first of (2.180) we calculate

$$\alpha = \frac{1 - m}{ma}, \tag{2.181}$$

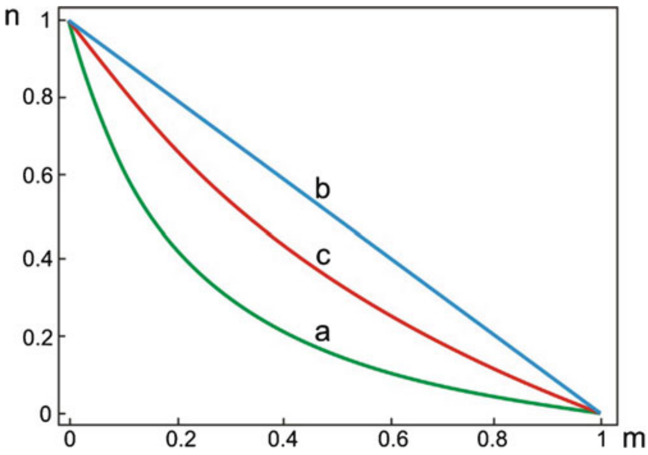


Fig. 2.61 Illustrative graphs of (2.182) for (**a**) $ab = 2$, (**b**) $ab = 17/3$, (**c**) $ab = 1$

and then, from the second one, after taking into account (2.181) we have

$$m + n[1 + (ab - 1)m] = 1, \tag{2.182}$$

and after transformations we obtain

$$n = \frac{1 - m}{(ab - 1)m + 1}.$$

In Fig. 2.61 the graphs of (2.182) are shown for the cases (a) for $ab = 2$, (b) for $ab = 17/3$, and for $ab = 1$ as a critical case (c).

From the graphs in Fig. 2.61 follows a very interesting conclusion, which is often observed in everyday life. In the case where a rigid body is at rest, the action of even a very small moment M_T leads to a rapid drop in magnitude of friction force T, which greatly facilitates the start of movement of a body. It is also associated with the frequently observed dangerous skid during cornering of a wheeled vehicle.

The CCZ model allows one also to pinpoint many characteristics of bodies moving with frictional contact. We will show that the friction does depend largely on the area represented by the radius R. To this end let us return to (2.173), which are solved for two values of R (Fig. 2.62).

When comparing the results presented in Figs. 2.59a and 2.62 it is clear that with increasing radius R the time needed for the disc to stop increases in a non-linear fashion. A similar situation can be observed also in the case of $\omega_0 = 10$ and two different values of R on the basis of (2.174) (Fig. 2.63).

Let us now analyze the effect of a two-dimensional friction on relative motion and self-excited vibrations.

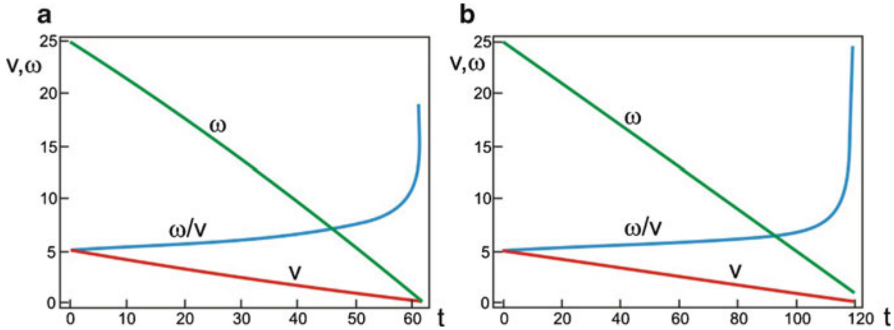


Fig. 2.62 Time histories $v(t)$, $\omega(t)$, and $\omega(t)/v(t)$ for $\omega_0 = 25$ and $R = 0.35$ (**a**) and $R = 0.45$ (**b**) (the remaining parameters as in Fig. 2.58)

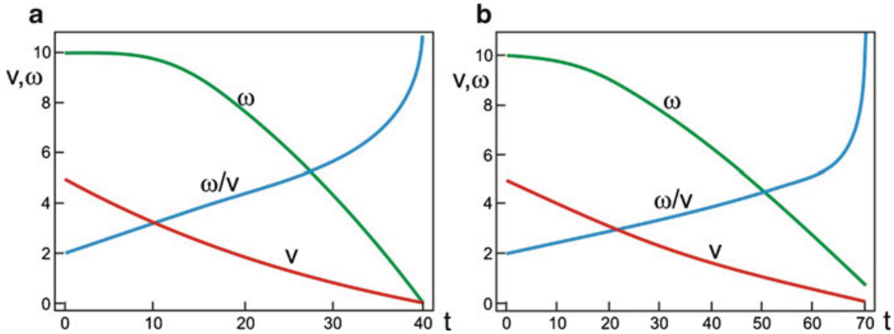


Fig. 2.63 Time histories $v(t)$, $\omega(t)$, and $\omega(t)/v(t)$ for $\omega_0 = 10$ and $R = 0.35$ (**a**) and $R = 0.45$ (**b**) (the remaining parameters as in Fig. 2.58)

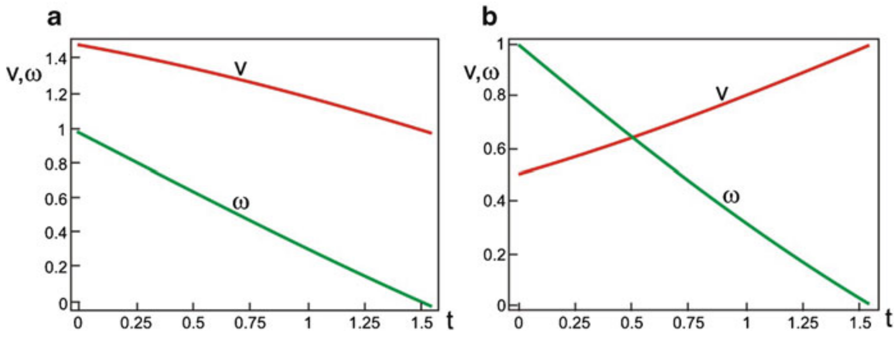


Fig. 2.64 Motion of disc determined by $\omega(t)$ and $v(t)$ for the cases $u_0 = 1$ and $v_0 = 1.5$ (**a**) and $v_0 = 0.5$ (**b**)

Let us consider the motion of a disc on an inclined plane, where now the motion takes place on an inclined plane remaining in contact with the moving disc and running with a constant velocity v_0. Equations of motion (2.168) assume the form

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}\tau} &= -\frac{\pi(v - v_0)}{\pi|v - v_0| + 4|u|} + p,\\ \frac{\mathrm{d}u}{\mathrm{d}\tau} &= -\frac{2\pi u}{3\pi|v - v_0| + 4|u|}.\end{aligned} \tag{2.183}$$

In Fig. 2.64 an example of calculations for the case $u_0 = 1$ and for two different initial velocities: (a) $v_0 = 1.5$ and (b) $v_0 = 0.5$.

From Fig. 2.64 it is seen that in both cases the velocities $\omega(t) \to 0$ and $v(t) \to 1$ attaining these values simultaneously in finite time.

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