

Basics of O-minimality and Hardy Fields

Chris Miller

Abstract This paper consists of lecture notes on some fundamental results about the asymptotic analysis of unary functions definable in o-minimal expansions of the field of real numbers.

Mathematics Subject Classification (2010): Primary 03C64, Secondary 26A12

Preface

This paper consists of lecture notes for the first of three modules of a graduate course, “Topics in O-minimality”, held at the Fields Institute (Toronto) as part of the Thematic Program on O-minimal Structures and Real Analytic Geometry (January through June of 2009). The “and” from the title indicates the intersection of the two subjects, not their union; only the most fundamental theorems and a few illustrative applications thereof are presented. Most, but not all, of the results are from my Ph.D. thesis (*Polynomially bounded o-minimal structures*, University of Illinois at Urbana-Champaign, 1994), and have also been published previously in various journals.

The reader is assumed to be familiar with basic graduate-level model theory and o-minimality. (A brief description of structures in the syntactic sense is given in the introduction for readers who might only be familiar with definability theory.) For basic model theory, see the first few chapters of Marker [7]; for model-theoretic o-minimality in particular, the survey paper by van den Dries [29] is a good start, or one can go directly to seminal papers by Pillay and Steinhorn [16]

C. Miller (✉)
Department of Mathematics, The Ohio State University,
231 West 18th Avenue, Columbus, OH 43210, USA
e-mail: miller@math.osu.edu

and Knight et al. [5]. The paper [24] by Speissegger is also useful, as it is based on lecture notes from an earlier Fields Institute program (Algebraic Model Theory, August 1996–July 1997). An extensive account of definability-theoretic o-minimality requiring little or no background in logic is found in [31]; see also van den Dries and Miller [34] for an exposition aimed at geometers.

Some global notation and conventions. The set of nonnegative integers is denoted by $\mathbb{N}$. We regard $\mathbb{R}^0$ as the one-point space $\{0\}$, and identify maps $f: \mathbb{R}^0 \rightarrow \mathbb{R}^n$ with the corresponding points $f(0) \in \mathbb{R}^n$.

For any additively-written abelian group G and $A \subseteq G$, we denote the nonzero elements of A by A^* . If G is ordered, then $A^{>0}$ denotes the positive elements of A .

Generally, boldface fonts used within text indicate definitions. To illustrate, **ultimately** abbreviates “for all sufficiently large positive arguments”, or grammatical variants thereof as appropriate. Example of usage: Ultimately, $x^2 < e^x$.

1 Introduction

We begin with a brief description of first-order expansions in the syntactic sense of the ordered field of real numbers $\mathbb{R} := (\mathbb{R}, <, +, -, \cdot, 0, 1)$. Readers already familiar with model theory should note that the approach here is specialized to expansions of the real field in extensions of the language of ordered rings $\{<, +, -, \cdot, 0, 1\}$ by function (including constant) symbols only.

For each $n \in \mathbb{N}$, let $\mathcal{F}_n$ be a (possibly empty) set of functions $\mathbb{R}^n \rightarrow \mathbb{R}$. We regard the real numbers as an ordered field equipped with these extra functions, that is, as the **structure** (in the syntactic sense) $\mathfrak{R} := (\mathbb{R}, (f)_{f \in \bigcup \mathcal{F}_n})$. We also call $\mathfrak{R}$ an **expansion** (in the syntactic sense) of $\mathbb{R}$. The **primitive functions** of $\mathfrak{R}$ are the 0-ary functions 0 and 1, the unary (that is, $\mathbb{R} \rightarrow \mathbb{R}$) function $x \mapsto -x$, the binary functions addition and multiplication, and the members of the $\mathcal{F}_n$. The **language** L of $\mathfrak{R}$ (or $L(\mathfrak{R})$ if needed for clarity) consists of the symbols $\{<, +, -, \cdot, 0, 1\}$ together with symbols representing the functions from the $\mathcal{F}_n$. All symbols are assumed to be pairwise distinct. Generally, we do not distinguish by notation the symbols in the language of $\mathfrak{R}$ and the objects that they denote.

For each $n \in \mathbb{N}$, let $\mathcal{T}_n$ be the smallest subring of functions $\mathbb{R}^n \rightarrow \mathbb{R}$ that contains the coordinate projections $x \mapsto x_i: \mathbb{R}^n \rightarrow \mathbb{R}$ for $i = 1, \dots, n$ and is “closed under composition”, that is, for all $m \in \mathbb{N}$, $g \in \mathcal{F}_m$ and $f_1, \dots, f_m \in \mathcal{T}_n$, we have $g \circ (f_1, \dots, f_m) \in \mathcal{T}_n$.

We construct collections $\mathfrak{R}_{n,k}$ of subsets of $\mathbb{R}^n$ for each $n \in \mathbb{N}$ by induction on k :

- For $k = 0$ and $n \in \mathbb{N}$ put $\mathfrak{R}_{n,0} = \{f^{-1}(0) : f \in \mathcal{T}_n\}$.
- For all $n \in \mathbb{N}$, let $\mathfrak{R}_{n,k+1}$ be the boolean algebra of subsets of $\mathbb{R}^n$ generated by

$$\mathfrak{R}_{n,k} \cup \{\text{pr}(X) : X \in \mathfrak{R}_{n+1,k}\},$$

where $\text{pr}: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ denotes projection on the first n coordinates.

A set $X \subseteq \mathbb{R}^n$ is **$\emptyset$ -definable**¹ in $\mathfrak{R}$ if there exists $k \in \mathbb{N}$ such that $X \in \mathfrak{R}_{n,k}$. Given $A \subseteq \mathbb{R}$, we say that X is **A -definable in $\mathfrak{R}$** if X is $\emptyset$ -definable in the expanded structure $(\mathfrak{R}, (a)_{a \in A})$, where the notation indicates that we have enlarged (“expanded”) $\mathcal{F}_0$ by the points in A regarded as functions $\mathbb{R}^0 \rightarrow \mathbb{R}$. For the most part, only the cases $A = \mathbb{R}$ and $A = \emptyset$ will matter to us in this paper. We usually drop the “in $\mathfrak{R}$ ” whenever the structure under consideration is understood, and we abbreviate “ $\mathbb{R}$ -definable” by just “definable”.² If X is a singleton $\{x\}$, we usually drop the set braces and talk about the point $x \in \mathbb{R}^n$ as being A -definable (and so on). A map $f: X \rightarrow \mathbb{R}^p$ is A -definable if its graph $\{(x, f(x)) : x \in X\} \subseteq \mathbb{R}^{n+p}$ is A -definable.

The distinction between “ A -definable” and “definable” is often extremely important in model-theoretic arguments. We shall be paying far more attention to this distinction than is customary in o-minimal analytic geometry.

Let $A \subseteq \mathbb{R}$. If X is A -definable, then so is its closure and interior. Given an A -definable function $f: X \rightarrow \mathbb{R}$, its domain X is A -definable, the set of points in its interior where f is differentiable is A -definable, and if X is open and f is differentiable on X , then the graph of each partial derivative of f is A -definable. Throughout this work we use many such basic facts. The proofs consist of showing that the various sets and functions are defined by formulas built up from variables, quantifiers, boolean connectives and symbols from the language of $(\mathfrak{R}, (a)_{a \in A})$. See Appendices A and B of [34] for details and more information.

A subset X of $\mathbb{R}^n$ is **described by** a collection $\mathcal{H}$ of functions $\mathbb{R}^n \rightarrow \mathbb{R}$ if X is a finite union of sets of the form $\{x \in \mathbb{R}^n : f(x) = 0, g_1(x) < 0, \dots, g_l(x) < 0\}$ with $f, g_1, \dots, g_l \in \mathcal{H}$.

The structure $\mathfrak{R}$:

- has (or admits) **quantifier elimination**—QE, for short—if, for all $n \in \mathbb{N}$, every $\emptyset$ -definable set in $\mathbb{R}^n$ is described by $\mathcal{T}_n$;
- is **model complete** if for all $n \in \mathbb{N}$ and $\emptyset$ -definable $X \subseteq \mathbb{R}^n$ there exist $p \in \mathbb{N}$ and $Y \subseteq \mathbb{R}^{n+p}$ described by $\mathcal{T}_{n+p}$ such that X is the projection of Y on the first n coordinates.
- is **polynomially bounded** if for each definable unary function f there exists $N \in \mathbb{N}$ such that ultimately $|f(x)| \leq x^N$.
- is **exponential** if it defines the real exponential function $e^x: \mathbb{R} \rightarrow \mathbb{R}$.
- is **exponentially bounded** if for each definable unary function f there exists $N \in \mathbb{N}$ such that ultimately $|f(x)| \leq \exp_N(x)$, where $\exp_N$ denotes the N -th compositional iterate of e^x .

¹Often called “0-definable” in the model-theoretic literature.

²In many older papers in model theory, the default is that “definable” means “ $\emptyset$ -definable”.

- is **o-minimal** if every definable subset of $\mathbb{R}$ is a finite union of points and open intervals.³ In fact, it is enough to require that every $\emptyset$ -definable subset of $\mathbb{R}$ be a finite union of points and open intervals; see 2.1 below.

Observe that QE implies model completeness (but not vice versa, as we shall soon see). Evidently, $\mathfrak{R}$ is exponentially bounded if it is polynomially bounded, and not polynomially bounded if it is exponential.

Note. Syntactic (language-dependent) notions such as quantifier elimination and model completeness are formulated in model theory for *theories* (in a given language), not *structures*. For example, rather than saying that $\mathfrak{R}$ is model complete, we should say that its complete theory $\text{Th}(\mathfrak{R})$ is model complete in the language in which $\mathfrak{R}$ is presented. While this level of precision is sometimes needed in abstract model theory, we adopt the convention in this paper that whenever we apply a syntactic notion to a structure, we mean to apply it to the complete theory of the structure under consideration in some given fixed language that should be clear from context.

For $r \in \mathbb{R}$, x^r denotes the **power function**

$$x \mapsto \begin{cases} x^r, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

Although the notation x^r is ambiguous for $r \in \mathbb{N}$, this will not cause any complications. The extension of the domain to $(-\infty, 0]$ is only to satisfy the technicality that primitive functions of structures must, by definition, be totally defined. Much of the time, this syntactic requirement has no impact on the underlying mathematics, where we care as usual only about the restriction to $\mathbb{R}^{>0}$. Power functions play a crucial role in this paper. The set of all $r \in \mathfrak{R}$ such that the function x^r is definable is a subfield of $\mathbb{R}$ (exercise); we call it the **field of exponents** of $\mathfrak{R}$. If $\mathfrak{R}$ is exponential, then $\mathbb{R}$ is its field of exponents (exercise).

Before proceeding any further, let us consider a few examples in order to help justify all these definitions.

The structure $\overline{\mathbb{R}}$ admits QE by the Tarski-Seidenberg theorem—see [31, Chap. 2] for an interesting proof due to Łojasiewicz—hence so does the expansion $(\overline{\mathbb{R}}, (r)_{r \in \mathbb{R}})$ (exercise); the notation indicates that $\mathcal{F}_0 = \mathbb{R}$ and all other $\mathcal{F}_n$ are empty. For $(\overline{\mathbb{R}}, (r)_{r \in \mathbb{R}})$, the sets $\mathcal{T}_n$ consist precisely of the real polynomials on $\mathbb{R}^n$, and then sets described by $\mathcal{T}_n$ are called **semialgebraic** (or sometimes $\mathbb{R}$ -semialgebraic). Thus, the sets definable in $\overline{\mathbb{R}}$ are exactly the real semialgebraic sets. It follows from QE that $\overline{\mathbb{R}}$ has field of exponents $\mathbb{Q}$ and is o-minimal, as then every unary definable function is ultimately algebraic, and every set of the form

$$\{x \in \mathbb{R} : f(x) = 0, g_1(x) < 0, \dots, g_l(x) < 0\}$$

³In other words, the only definable subsets of $\mathbb{R}$ are those that must be there by virtue of the usual ordering of the real line. Hence, the structure is “order-minimal”, thus accounting for the abbreviation “o-minimal” and the use of a plain text font for the “o”.

where $f, g_1, \dots, g_l$ are unary polynomials is a finite union of points and open intervals.

The **real exponential field** is the structure $(\overline{\mathbb{R}}, (r)_{r \in \mathbb{R}}, e^x)$. Here, $\mathcal{F}_0 = \mathbb{R}$, $\mathcal{F}_1 = \{e^x\}$ and all other $\mathcal{F}_n$ are empty. By Wilkie [39], the structure $(\overline{\mathbb{R}}, e^x)$ is model complete, hence so is $(\overline{\mathbb{R}}, (r)_{r \in \mathbb{R}}, e^x)$ (exercise). Indeed, for every set $X \subseteq \mathbb{R}^n$ $\emptyset$ -definable in $(\overline{\mathbb{R}}, e^x)$, there exist $p \in \mathbb{N}$ and a polynomial $P: \mathbb{R}^{2(n+p)} \rightarrow \mathbb{R}$ with integer coefficients such that X is the projection on the first n coordinates of the set $\{(x, y) \in \mathbb{R}^{n+p} : P(x, y, e^x, e^y) = 0\}$, where $e^x = (e^{x_1}, \dots, e^{x_n})$ and $e^y = (e^{y_1}, \dots, e^{y_p})$. By Hovanskiĭ [4], such sets have only finitely many connected components; o-minimality follows. Quantifier elimination fails for $(\overline{\mathbb{R}}, e^x)$, indeed, no proper⁴ expansion of $\overline{\mathbb{R}}$ by real-analytic primitive functions has QE (van den Dries [26]; see also 4.6 below). The structure $(\overline{\mathbb{R}}, e^x)$ is exponentially bounded, but this is far from obvious; it was first established by van den Dries and Miller in [33], but a stronger and more general result was later established by Lion et al. [6].

This ends our brief description of the syntactical point of view. *From now on:* We revert to using standard terminology and conventions from mathematical logic and model theory, in particular, we allow structures to have primitive relations other than just $<$, though we tend to replace relations by characteristic functions whenever convenient. We usually employ “fraktur” font to indicate structures $\mathfrak{A}$, $\mathfrak{B}$ and so on) with corresponding math roman caps (A , B and so on) for the underlying sets (or even underlying ordered rings, depending on context), except for $\mathfrak{R}$, which will always denote an expansion of $\overline{\mathbb{R}}$.

I next outline the main results presented in these notes. *For the rest of this section*, let $\mathfrak{R}$ be an o-minimal expansion of $\overline{\mathbb{R}}$ with field of exponents K .

There is a striking dichotomy in the asymptotic behavior of the definable unary functions:

Theorem 1.1 (Growth Dichotomy [8]). *$\mathfrak{R}$ is either polynomially bounded or exponential.*

Our focus in these notes will be on the case that $\mathfrak{R}$ is polynomially bounded. The next result is fundamental.

Theorem 1.2 (Piecewise Uniform Asymptotics [9]). *Let $\mathfrak{R}$ be polynomially bounded and $f: \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$ be definable. Then there is a finite $S \subseteq K$ such that for each $a \in \mathbb{R}^m$ either the function $x \mapsto f(a, x): \mathbb{R} \rightarrow \mathbb{R}$ is ultimately equal to 0 or there exists $r \in S$ such that $\lim_{x \rightarrow +\infty} f(a, x)/x^r \in \mathbb{R}^*$.*

An easy consequence (see 4.5 below for details):

1.3 ([10]). *If $\mathfrak{R}$ is polynomially bounded and $U \subseteq \mathbb{R}^n$ is definable, open and connected, then the ring of all definable C^∞ functions $U \rightarrow \mathbb{R}$ is a quasianalytic class, that is, if $f: U \rightarrow \mathbb{R}$ is definable and C^∞ , then $f = 0$ iff there exists $x_0 \in U$ such that all partials of f vanish at x_0 .*

⁴A **proper** expansion of $\overline{\mathbb{R}}$ is one that defines a non-semialgebraic set.

If $K \neq \mathbb{R}$, then it is immediate from Growth Dichotomy that $\mathfrak{R}$ is polynomially bounded. We are then interested in what happens when we expand $\mathfrak{R}$ by more power functions. Let us make this precise. Given an expansion $\mathfrak{A}$ of $\mathbb{R}$ and $S \subseteq \mathbb{R}$, put $\mathfrak{A}^S = (\mathfrak{A}, (r)_{r \in S}, (x^r)_{r \in S})$, the expansion of $\mathfrak{A}$ by constants from S and power functions with exponents from S . Since every power function is definable in $(\mathbb{R}, e^x)$, every set definable in $\mathfrak{A}^S$ is definable in $(\mathfrak{A}, e^x)$. By o-minimality of the Pfaffian closure ([23] or [25]), $(\mathfrak{R}, e^x)$ is o-minimal, hence so is $\mathfrak{R}^S$ for any $S \subseteq \mathbb{R}$. The question: If $\mathfrak{R}$ is polynomially bounded, is the same true of $\mathfrak{R}^S$, and is the field of exponents what we would hope for, namely, the field $K(S)$? In this generality, we do not know, but we have a reasonable partial answer:

Theorem 1.4 ([9] and [12, 4.1]). *Let $\mathfrak{R}$ be polynomially bounded and $S \subseteq \mathbb{R}$ be such that the restriction $x^r \upharpoonright [1, 2]$ is $\emptyset$ -definable in $\mathfrak{R}$ for each $r \in S$.*

1. *If $\mathfrak{R}$ has QE and is $\forall$ -axiomatizable⁵, then the same is true of $\mathfrak{R}^{K(S)}$.*
2. *If $\mathfrak{R}$ is model complete, then the same is true of $\mathfrak{R}^S$.*
3. *Every function definable in $\mathfrak{R}^{K(S)}$ is given piecewise by iterated compositions of functions definable in $\mathfrak{R}$ and powers from $K(S)$.*
4. *$\mathfrak{R}^S$ is polynomially bounded with field of exponents $K(S)$.*

The interval $[1, 2]$ is chosen only for convenience and concreteness: If $r \in \mathbb{R}$ and I is any infinite interval of positive real numbers such that $x^r \upharpoonright I$ is $\emptyset$ -definable, then $x^r \upharpoonright [1, 2]$ is $\emptyset$ -definable (by density of $\mathbb{Q}$ in $\mathbb{R}$ and the multiplicative properties of power functions).

Remark. With a bit more work, one can show that $\mathfrak{R}^S$ is o-minimal without appeal to Pfaffian closure; see [38] for the main idea (indeed, for the overarching strategy of the proof of Theorem 1.4).

Let us consider some concrete applications.

1.5 ([9]). *Put $\phi(x) = 1/(1 + x^2)$ for $x \in \mathbb{R}$. For each $S \subseteq \mathbb{R}$, all of the following are model complete and polynomially bounded with field of exponents $\mathbb{Q}(S)$:*

- $\overline{\mathbb{R}}^S$
- $(\overline{\mathbb{R}}^S, e^\phi)$
- $(\overline{\mathbb{R}}^S, \arctan)$
- $(\overline{\mathbb{R}}^S, e^\phi, \arctan)$.

Proof. By results from [27] and [39], all of the following are o-minimal, have field of exponents $\mathbb{Q}$ and are model complete:

- $(\overline{\mathbb{R}}, ((1 + \phi)^r)_{r \in S}, (r)_{r \in S})$
- $(\overline{\mathbb{R}}, ((1 + \phi)^r)_{r \in S}, (r)_{r \in S}, e^\phi)$
- $(\overline{\mathbb{R}}, ((1 + \phi)^r)_{r \in S}, (r)_{r \in S}, \arctan)$
- $(\overline{\mathbb{R}}, ((1 + \phi)^r)_{r \in S}, (r)_{r \in S}, e^\phi, \arctan)$

⁵That is, its theory is axiomatizable by universal sentences.

For each $r \in \mathbb{R}$, the function $x^r \upharpoonright [1, 2]$ is $\emptyset$ -definable in $(\overline{\mathbb{R}}, (1+\phi)^r)$, and $(1+\phi)^r$ is both existentially and universally definable in $(\overline{\mathbb{R}}, x^r)$. The result is now immediate from Theorem 1.4.2. $\square$

Remark. (i) Every compact trajectory of any real linear vector field is definable in the structure $(\overline{\mathbb{R}}, e^\phi, \arctan)$, which partly explains our interest in it; see [12, 28] for more information. (ii) By Bianconi [1], $\arctan$ is not definable in $(\overline{\mathbb{R}}, e^x)$, nor is e^ϕ definable in $(\overline{\mathbb{R}}, \arctan)$.

Result 1.5 was published in the year 1994. There is now (2011) a fairly large body of literature on model completeness and o-minimality results for expansions of $\mathbb{R}$ by differential rings of “restricted” functions. It is fair to say that the current state of the art appears in Rolin et al. [18], see also [17], to which we refer the reader for details and history. By combining their technology with ours, we obtain:

1.6. *Let $\mathfrak{R}_0$ be the expansion of $\overline{\mathbb{R}}$ by all functions of the form*

$$x \mapsto \begin{cases} f(x), & x \in [-1, 1]^n \\ 0, & x \in \mathbb{R}^n \setminus [-1, 1]^n, \end{cases}$$

where $n \in \mathbb{N}$ and $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is definable in $\mathfrak{R}$ and C^∞ on some open neighborhood of $[-1, 1]^n$. Suppose that $\mathfrak{R}_0$ is polynomially bounded. Let $S \subseteq \mathbb{R}$ be such that, for each $r \in S$, the restriction $x^r \upharpoonright [1, 2]$ is $\emptyset$ -definable in $\mathfrak{R}$. Then $\mathfrak{R}_0^S$ is model complete and polynomially bounded with field of exponents $\mathbb{Q}(S)$.

Proof. By 1.3 and [18], $\mathfrak{R}_0$ is model complete and has field of exponents $\mathbb{Q}$. Apply Theorem 1.4.2. $\square$

Concrete examples of applications of Theorem 1.4.1 take a bit more work to state; for more detailed information on this material, see [9, 28]. A **convergent Weierstrass system** (over $\mathbb{R}$) is a family of rings $W := (\mathbb{R}_L X_1, \dots, X_{n+1})_{n \in \mathbb{N}}$ such that the following hold for all n , where $X = (X_1, \dots, X_n)$.

- $\mathbb{R}[X] \subseteq \mathbb{R}_L X_\perp \subseteq \mathbb{R}[[X]]$.
- If σ is a permutation of $\{1, \dots, n\}$ and $f(X) \in \mathbb{R}_L X_\perp$, then

$$f(X_{\sigma(1)}, \dots, X_{\sigma(n)}) \in \mathbb{R}_L X_\perp.$$

- If $F \in \mathbb{R}_L X_\perp$ is a unit in $\mathbb{R}[[X]]$, then it is also a unit in $\mathbb{R}_L X_\perp$.
- (Weierstrass division) If $F \in \mathbb{R}_L X, X_{n+1}_\perp$ and $F(0, X_{n+1}) \in \mathbb{R}[[X_{n+1}]]$ is nonzero of order d , then for every $G \in \mathbb{R}_L X, X_{n+1}_\perp$ there exist $Q \in \mathbb{R}_L X, X_{n+1}_\perp$ and $R_0, \dots, R_{d-1} \in \mathbb{R}_L X_\perp$ such that

$$G = QF + \sum_{i=0}^{d-1} R_i X_{n+1}^i.$$

- Every $F \in \mathbb{R}_L X_{\perp}$ converges to an analytic function f on some box neighborhood U of the origin in $\mathbb{R}^n$ such that for all $a \in U$ the series

$$F(a + X) = \sum_{\mu \in \mathbb{N}^n} \frac{\partial^{|\mu|} f}{\partial X^{\mu}}(a) \frac{X^{\mu}}{\mu!} \in \mathbb{R}_L X_{\perp}$$

belongs to $\mathbb{R}_L X_{\perp}$.

Some examples:

- The family of rings of algebraic real power series is the smallest convergent Weierstrass system.
- The family of rings of convergent real power series is the largest convergent Weierstrass system.
- A power series $f \in \mathbb{R}[[X_1, \dots, X_n]]$ is **differentially algebraic** if the integral domain $\mathbb{R}[\partial^{|\mu|} f / \partial X^{\mu} : \mu \in \mathbb{N}^n] \subseteq \mathbb{R}[[X_1, \dots, X_n]]$ generated by the partial derivatives of f over the field of constants has finite transcendence degree over $\mathbb{R}$. The family of rings of convergent differentially algebraic real power series is a convergent Weierstrass system.

Given a convergent Weierstrass system W , we obtain a structure $\overline{\mathbb{R}}_W$ by expanding $\overline{\mathbb{R}}$ by all functions of the form

$$x \mapsto \begin{cases} F(x), & x \in [-1, 1]^n \\ 0, & x \in \mathbb{R}^n \setminus [-1, 1]^n, \end{cases}$$

where $n \in \mathbb{N}$ and all series $F(a + X)$ belong to W for all $a \in [-1, 1]^n$. (For $n = 0$, we take this to mean that we expand by all real constants.) By [27], $\overline{\mathbb{R}}_W$ is o-minimal and has field of exponents $\mathbb{Q}$. By [28], $\overline{\mathbb{R}}_W$ is model complete. By arguing as in Denef and van den Dries [2], $(\overline{\mathbb{R}}_W, x^{-1})$ has QE. Finally, by arguing as in van den Dries et al. [38, Sect. 2], $\overline{\mathbb{R}}_W^{\mathbb{Q}}$ is $\forall$ -axiomatizable. Consequently, given a subfield K of $\mathbb{R}$ such that the series $\sum_{k \geq 0} \binom{r}{k} X_1^k$ belongs to $\mathbb{R}_L X_{\perp}$ for all $r \in K$, we have by Theorem 1.4.1 that $\mathbb{R}_W^K$ admits quantifier elimination and is universally axiomatizable. This holds in particular for the case that W extends the family of rings of differentially algebraic convergent real power series. (For each $r \in \mathbb{R}$, the series $\sum_{k \geq 0} \binom{r}{k} X_1^k$ is convergent and differentially algebraic.)

We arrive at our last main result. As previously mentioned, we do not know if polynomial bounds are always preserved in passing from $\mathfrak{A}$ to $\mathfrak{A}^S$. But exponential bounds are if $\mathfrak{A}$ is polynomially bounded, indeed, the Pfaffian closure of $\mathfrak{A}$ is then exponentially bounded. We provide a part of the proof (see [25, Theorem B and Proposition 1.14] for the rest):

Theorem 1.7 ([6]). *Let $\mathfrak{A}$ be polynomially bounded, $G: \mathbb{R}^{N+2} \rightarrow \mathbb{R}$ be definable in $\mathfrak{A}$, and g be a solution on some ray (a, ∞) to the differential equation*

$$y^{(N+1)} = G(x, y, y', \dots, y^{(N)}).$$

If $(\mathfrak{R}, g)$ is o -minimal, then ultimately $|g(x)| \leq C \exp_{N+1}(x^r)$ for some $C > 0$ and $r \in K$.

Consideration of the functions $g := C \exp_{N+1}(x^r)$ for $C > 0$ and $r \in K$ shows that the upper bound is optimal,⁶ but the result can be sharpened in other ways; see [6].

Here is an outline of the rest of this paper. Section 2 consists of a collection of exercises for the reader, many of which are crucial for later developments. In Sect. 3, Hardy fields are introduced and their connection to o -minimal expansions of $\mathbb{R}$ is explained, culminating in the proof of Theorem 1.1. Section 4 contains a few special results about the polynomially bounded case of Growth Dichotomy, beginning with Theorem 1.2. Sections 5 and 6 are devoted to the proofs of Theorems 1.4 and 1.7. We conclude with some suggestions for further study.

2 Some Exercises

The reader is strongly recommended to at least attempt these exercises before moving on. Many of them will be used without proof, or even further mention, in the sequel. Let $\mathfrak{R}$ be an expansion of $\overline{\mathbb{R}}$.

If $X \subseteq \mathbb{R}^n$ is definable, then there exist $m \in \mathbb{N}$, $a \in \mathbb{R}^m$ and $\emptyset$ -definable $Y \subseteq \mathbb{R}^{m+n}$ such that $X = \{x \in \mathbb{R}^n : (a, x) \in Y\}$. Note that X is A -definable, where A is the union of the coordinates of a . To put this another way: Any set definable in $\mathfrak{R}$ is $\emptyset$ -definable in some expansion of $\mathfrak{R}$ by finitely many constants. In practice, this observation is often used to reduce worrying about “ A -definable versus definable” to the case that $A = \emptyset$ when dealing with some fixed definable set X .

Each point of a finite $\emptyset$ -definable set is $\emptyset$ -definable. (There is a lexicographically least element of the set if it is nonempty. And so on.)

Every polynomial map with coefficients from the real-closed subfield of $\mathbb{R}$ generated by $\mathcal{F}_0$ is $\emptyset$ -definable (where $\mathcal{F}_0$ is as in the definition of “structure”, given in the second paragraph of the Introduction).

Every set described by $\mathcal{T}_n$ is $\emptyset$ -definable.

Let $X \subseteq \mathbb{R}^n$ be described by $\mathcal{T}_n$. Then there exist $p \in \mathbb{N}$ and $f \in \mathcal{T}_{n+p}$ such that X is the projection on the first n coordinates of $f^{-1}(0)$. (*Hint.* For functions $f, g: \mathbb{R}^n \rightarrow \mathbb{R}$ and $x \in \mathbb{R}^n$ we have $f(x) = 0$ and $g(x) = 0$ iff $f(x)^2 + g(x)^2 = 0$, and $f(x) = 0$ or $g(x) = 0$ iff $f(x)g(x) = 0$.)

If $\mathfrak{R}$ has QE and $A \subseteq \mathbb{R}$, then $(\mathfrak{R}, (a)_{a \in A})$ has QE, and similarly with “model complete” in place of “QE”.

If an injective map $f: A \rightarrow B$ is definable, then the compositional inverse $f^{-1}: B \rightarrow A$ is definable.

The field of exponents is indeed a field.

⁶Exercise. Prove it.

The field of exponents of $(\overline{\mathbb{R}}, e^x)$ is $\mathbb{R}$.

Given $A \subseteq \mathbb{R}^{n+k}$ and $x \in \mathbb{R}^n$, let A_x denote the **fiber** $\{y \in \mathbb{R}^k : (x, y) \in A\}$ of A over x . Show that if A is $\emptyset$ -definable, then so are the following sets:

- A_x , if x is $\emptyset$ -definable (in particular, if $x \in \mathbb{Q}^n$).
- $\{x \in \mathbb{R}^n : A_x \text{ is closed}\}$.
- $\{x \in \mathbb{R}^n : A_x \text{ is bounded}\}$.
- $\{x \in \mathbb{R}^n : A_x \text{ is discrete}\}$.
- $\{x \in \mathbb{R}^n : A_x \text{ is finite}\}$.
- $\{(x, y) \in \mathbb{R}^{n+k} : y \text{ is in the boundary of } A_x\}$.
- $\{x \in \mathbb{R}^n : \text{the boundary of } A_x \text{ is finite}\}$.
- $\{x \in \mathbb{R}^n : A_x \text{ is a finite union of points and open intervals}\}$, if $k = 1$.

2.1. If every $\emptyset$ -definable subset of $\mathbb{R}$ is a finite union of points and open intervals, then $\mathfrak{A}$ is o-minimal. (By previous exercises, for every $n \in \mathbb{N}$ and $\emptyset$ -definable $A \subseteq \mathbb{R}^{n+1}$, the set of all $x \in \mathbb{R}^n$ such that A_x is a finite union of points and open intervals is dense and $\emptyset$ -definable. Finish by an appropriate induction on n . See [13, Proposition 1] for details. The proof works for all expansions of the real line $(\mathbb{R}, <)$ whose $\emptyset$ -definable points are dense in $\mathbb{R}$. See [3, 1.14] for a rather different proof, due to van den Dries, that works for expansions of arbitrary densely ordered groups.)

2.2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$. Show that:

- $Y := \{y \in \mathbb{R}^{>0} : \lim_{x \rightarrow +\infty} (f(xy) - f(x)) \in \mathbb{R}\}$ is a multiplicative group.
- $Z := \{z \in \mathbb{R} : \exists y \in \mathbb{R}^{>0}, \lim_{x \rightarrow +\infty} (f(xy) - f(x)) = z\}$ is an additive group.
- The function $L(f)(y) = \lim_{x \rightarrow +\infty} (f(xy) - f(x)): Y \rightarrow Z$ is a surjective homomorphism. The notation is to suggest that $L(f)$ is somehow the “logarithmic part” of f , but this should not be taken too seriously, as we could easily have $Y = \{1\}$ and $Z = \{0\}$.
- The sets Y, Z and the function $L(f)$ are $\emptyset$ -definable in $(\overline{\mathbb{R}}, f)$.
- If $(\overline{\mathbb{R}}, f)$ is o-minimal and $\lim_{x \rightarrow +\infty} (f(2x) - f(x)) \in \mathbb{R}^*$, then $Y = \mathbb{R}^{>0}$ and $L(f) = \log_a$ for some $a \in \mathbb{R}^{>0}$. (Recall that every subgroup of $(\mathbb{R}, +)$ is either cyclic or dense, and every endomorphism of $(\mathbb{R}, +)$ is either nowhere continuous or linear.) Conclude that $\log$ is definable, hence so is e^x .

2.3. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be ultimately nonzero. Show that:

- The sets

$$Y := \{y \in \mathbb{R}^{>0} : \lim_{x \rightarrow +\infty} f(xy)/f(x) \in \mathbb{R}\}$$

$$Z := \{z \in \mathbb{R}^{>0} : \exists y \in \mathbb{R}^{>0}, \lim_{x \rightarrow +\infty} f(xy)/f(x) = z\}$$

are multiplicative groups.

- The function $P(f)(y) = \lim_{x \rightarrow +\infty} (f(xy)/f(x)): Y \rightarrow Z$ is a surjective homomorphism. The notation is to suggest that $P(f)$ is somehow the “power part” of f , but again, this should not be taken too seriously. We tend to write just Pf as convenient.

- The sets Y, Z and the function Pf are $\emptyset$ -definable in $(\overline{\mathbb{R}}, f)$.
- If $(\overline{\mathbb{R}}, f)$ is o-minimal and $2 \in Y$, then $Y = \mathbb{R}^{>0}$ and Pf is a power function.
- If there exists $r \in \mathbb{R}$ such that $\lim_{x \rightarrow +\infty} f(x)/x^r \in \mathbb{R}^*$, then $Y = \mathbb{R}^{>0}$ and $Pf = x^r$.
- Calculate Y, Z and Pf directly for the functions $\log x, x^r \log x, (\log x)^{\log x}$.

2.4. In any expansion of $\overline{\mathbb{R}}$, the function e^x is $\emptyset$ -definable if it is definable. (*Hint:* e^x is the unique solution on $\mathbb{R}$ to the initial value problem $y' = y, y(0) = 1$.) Similarly, the function x^r is $\emptyset$ -definable if it is definable and r is $\emptyset$ -definable. (*Hint:* $r = (x^r)'(1)$.) More generally: Definable solutions to initial value problems are $\emptyset$ -definable if all of the data are $\emptyset$ -definable.

3 Hardy Fields, O-minimality and Growth Dichotomy

All of the main results listed in the introduction are tied to the asymptotic analysis of definable unary functions, as we now begin to describe. First, define an equivalence relation on the set of all functions $\mathbb{R} \rightarrow \mathbb{R}$ by relating functions f and g if they ultimately agree. The equivalence classes are called **germs** (at $+\infty$). Working with germs instead of functions allows us to ignore all but the asymptotics (at $+\infty$), that is, to focus on the “ultimate behavior” of the functions. We regard the set of all such germs as a ring, with $\text{germ}(f) + \text{germ}(g) = \text{germ}(f + g)$, and $\text{germ}(f) \cdot \text{germ}(g) = \text{germ}(f \cdot g)$. A **Hardy field** is a field of germs that is closed under differentiation, that is, if the germ of a function f belongs to the field, then there is a differentiable function g such that the germs of g and g' belong to the field and g is ultimately equal to f . Every Hardy field is naturally ordered by setting $\text{germ}(f) < \text{germ}(g)$ iff $g - f$ is ultimately positive. Observe that the set of germs of rational constant functions is a Hardy field, indeed, it is the smallest Hardy field. Of course, germs of constant functions are not very interesting, and one usually deals at least with Hardy fields that extend the (germs of) the field $\mathbb{Q}(x)$ of rational functions.

We are now ready for a result that begins to explain the title of this paper.

3.1. *If $\mathfrak{R}$ is an expansion of $\overline{\mathbb{R}}$, then the following are equivalent:*

- (1) $\mathfrak{R}$ is o-minimal.
- (2) The germs of definable unary functions form a Hardy field.
- (3) Every unary definable function is either ultimately zero or ultimately nonzero.

Proof. (1) $\Rightarrow$ (2) is immediate from the C^1 version of the Monotonicity Theorem (a fundamental result in o-minimality). (2) $\Rightarrow$ (3) is immediate from field structure and the definition of germ.

(3) $\Rightarrow$ (1).⁷ Let $A \subseteq \mathbb{R}$ be definable. We must show that A is a finite union of points and open intervals. For this, it suffices to show that boundary $\text{bd}(A)$ of A

⁷The argument is essentially from van den Dries et al. [38].

is finite, which in turn reduces via Bolzano-Weierstrass to showing that $\text{bd}(A)$ is bounded and discrete. (Recall that a subset of a topological space is discrete if all of its points are isolated.) Let f be the characteristic function of A . As A is definable, so is f (exercise). Then either f is ultimately equal to 1 or f is ultimately equal to 0, so there exists $b \in \mathbb{R}$ such that the ray (b, ∞) is either contained in A or disjoint from A . By the same reasoning applied to the set $\{-x : x \in A\}$, there exists $a \in \mathbb{R}$ such that $(-\infty, a)$ either is contained in A or disjoint from A . Hence, $\text{bd}(A)$ is bounded. Fix $x_0 \in \text{bd}(A)$. By arguing as above with the set $\{1/(a - x_0) : a \in A\}$, there exists $\epsilon > 0$ such that the interval $(x_0, x_0 + \epsilon)$ is either contained in A or disjoint from A , and similarly for $(x_0 - \epsilon, x_0)$. Hence, $\text{bd}(A) \cap (x_0 - \epsilon, x_0 + \epsilon) = \{x_0\}$, thus showing that $\text{bd}(A)$ is discrete. $\square$

Exercise. The above holds with “ $\emptyset$ -definable” in place of “definable”. (Recall 2.1.)

For the rest of this section, $\mathfrak{R}$ denotes an o-minimal expansion of $\overline{\mathbb{R}}$ with field of exponents K .

Let $\mathcal{H}$ denote the Hardy field of germs at $+\infty$ of the definable unary functions. We will not distinguish between functions and their germs by notation, relying instead upon context. We regard $\mathbb{R}$ as a subfield of $\mathcal{H}$ by identifying $r \in \mathbb{R}$ with the germ of the corresponding constant function. The germ of the identity function on $\mathbb{R}$ is denoted by x . We say that $f \in \mathcal{H}$ is **infinitely increasing** if $\lim_{x \rightarrow +\infty} f(x) = +\infty$.

Next are some crucial basic facts that the reader should verify before moving on.

3.2. • If $f \in \mathcal{H}$, then $\lim_{x \rightarrow +\infty} f(x) \in \mathbb{R} \cup \{\pm\}$.

- $\{(f, g) \in \mathcal{H}^* \times \mathcal{H}^* : \lim_{x \rightarrow +\infty} f(x)/g(x) \in \mathbb{R}^*\}$ is an equivalence relation. Denote the natural quotient map by ν . The image $\nu(\mathcal{H}^*)$ is an ordered abelian group by setting $\nu(f) + \nu(g) = \nu(fg)$ and $\nu(f) > 0$ iff $\lim_{x \rightarrow +\infty} f(x) = 0$. Denote the resulting absolute value on $\nu(\mathcal{H}^*)$ by $|\cdot|$. (Be careful: This does *not* mean that $|\nu(\cdot)| = \nu(|\cdot|)$.)
- If $f, g \in \mathcal{H}^*$ and $|f| \geq |g|$, then $\nu(f) \leq \nu(g)$. (This reversal of order might strike one as perverse, but the convention is firmly established in the literature.) The converse fails.
- If $f, g \in \mathcal{H}^*$ and $f \neq -g$, then $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$, with equality if $\nu(f) \neq \nu(g)$.
- If $f \in \mathcal{H}^*$ and $r \in \mathbb{R}^*$, then $\nu(rf) = \nu(f) = \nu(|f|)$.
- If $f \in \mathcal{H}^*$ and $\nu(f) \neq 0$, then exactly one of f , $1/f$, $-f$ or $-1/f$ is infinitely increasing, and $|\nu(f)| = |\nu(-f)| = |\nu(1/f)| = |\nu(-1/f)|$.
- If $f \in \mathcal{H} \setminus \mathbb{R}$, then

$$\nu(f'/f) = \nu((1/f)'/(1/f)) = \nu((-f)'/(f)) = \nu((-1/f)'/(1/f)).$$

Logarithmic differentiation—the map $a \mapsto a'/a : \mathcal{H}^* \rightarrow \mathcal{H}$ —plays an important role in Hardy field theory.

3.3. Let $a, b \in \mathcal{H}^*$ be such that $0 < |v(a)| \leq |v(b)|$. Then $v(a'/a) \geq v(b'/b)$.

Proof. Without altering $|v(a)|$, $|v(b)|$, $v(a'/a)$ or $v(b'/b)$, we replace a by $\pm a$ or $\pm 1/a$, and b by $\pm b$ or $\pm 1/b$, to reduce to the case that a and b are infinitely increasing.

If $v(a) = v(b)$, then $v(a') = v(b')$ by l'Hôpital's Rule, so $v(a'/a) = v(a') - v(a) = v(b') - v(b) = v(b'/b)$.

Suppose that $v(b) < v(a)$. Then b/a is infinitely increasing, so all of a , a' , b' and $(b/a)'$ are positive, yielding $b'/b > a'/a > 0$ by the quotient rule. Hence, $v(a'/a) \geq v(b'/b)$. $\square$

The appeal to l'Hôpital's Rule above is worth explaining, as it might seem that we are making a classical “freshman's dream” mistake by “going the wrong way”. But because we are working in a Hardy field containing both a and b , we know that the function a'/b' has a limit $l \in \mathbb{R} \cup \{\pm\infty\}$. Hence, we must have $\lim_{x \rightarrow +\infty} a(x)/b(x) = l$ by l'Hôpital. This fact—that l'Hôpital's Rule “works both ways”—is crucial in Hardy field theory and is used often.

For $f, g: \mathbb{R} \rightarrow \mathbb{R}$, we write $f \sim g$ if g ultimately has no zeros and $\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = 1$. If $f, g \in \mathcal{H}^*$, then $v(f) = v(g)$ iff $f \sim cg$ for some $c \in \mathbb{R}^*$.

3.4. Let $a, b \in \mathcal{H}^*$ be such that $v(a) \geq 0$ and $v(b) \neq 0$. Then $v(a') > v(b'/b)$.

Proof. We may assume that $v(a) = 0$ by replacing a with $a + 1$ if otherwise. By l'Hôpital,

$$\frac{ab}{b} \sim \frac{ab' + a'b}{b'},$$

and so

$$a = \frac{ab}{b} \sim \frac{ab' + a'b}{b'} = a + a' \frac{b}{b'}.$$

Then $1 \sim 1 + (a'/a)/(b'/b)$, yielding $v(a'/a) > v(b'/b)$. Finish by observing that $v(a'/a) = v(a') - v(a)$, and $v(a) = 0$ by assumption. $\square$

Another important tool in Hardy field theory is **asymptotic integration**. Given $f \in \mathcal{H}^*$, there need not be $g \in \mathcal{H}^*$ such that $g' = f$. But for many purposes it is enough to have $g \in \mathcal{H}^*$ such that $g' \sim f$. Of course, there need not be such g either: Consider $\mathfrak{R} = \mathbb{R}$ and $f = 1/x$. We do have the following result, which will be enough for the purposes of this paper.

3.5. If $f \in \mathcal{H}^*$ and $v(f) < v(1/x)$, then there exists $g \in \mathcal{H}^*$ such that $g' \sim f$.

Proof. Note that $(xf)' \neq 0$.

Suppose that $v(f) \geq v((xf)')$, that is, $v(f/(xf)') \geq 0$. By 3.4, $v((f/(xf)'))' > v(1/x)$, that is, $v(x(f/(xf)'))' > 0$. Put $g_1 = xf^2/(xf)'$. Basic calculus shows that $g_1'/f = 1 + x(f/(xf)'))'$, so $g_1' \sim f$.

Suppose that $v(f) < v((xf)'),$ equivalently, $f'/f \sim -1/x$. Check that $g'_1 \neq 0$. Put $g_2 = fg_1/g'_1$ and check that $g'_2 \sim f$. (*Hint:*

$$\frac{g'_2}{f} - 1 = \frac{f'/f}{g'_1/g_1} - \frac{g''_1/g'_1}{g'_1/g_1},$$

and keep 3.3 in mind.) □

Exercise. With $f = (\log x)/x$ and g_1 as above, show directly that $g'_1 \not\sim f$.

3.6. So far, we have used only that $\mathcal{H}$ is a Hardy field, not that it arises from the germs of definable unary functions of an o-minimal expansion of $\mathbb{R}$. Hence, *results established so far hold in any Hardy field*. This now begins to change.

Our Hardy field $\mathcal{H}$ is “closed under composition”: If $f, g \in \mathcal{H}^*$ and f is infinitely increasing, then the composition $g \circ f$ lies in $\mathcal{H}^*$ as well. Evidently, the sign of $v(g \circ f)$ is the same as that of $v(g)$. Not all Hardy fields extending $\mathbb{R}(x)$ are closed under composition, e.g., the function field $\mathbb{R}(x, e^x)$, as a set of germs, is a Hardy field that does not contain $e^x \circ x^2$ (exercise). Also, $\mathcal{H}$ is “closed under compositional inverse”: If $f \in \mathcal{H}$ and $v(f) < 0$, then the germ f^{-1} of the ultimately-defined compositional inverse of f also belongs to $\mathcal{H}$. (We always use fraction-bar notation for reciprocals of elements of $\mathcal{H}^*$, so there is no ambiguity in the use of $^{-1}$.) Observe that $\sqrt{x} \notin \mathbb{R}(x)$, so closure under compositional inverse is also a special property.

Exercise. The map $(r, v(f)) \mapsto v(|f|^r): K \times v(\mathcal{H}^*) \rightarrow v(\mathcal{H}^*)$ is well defined. The ordered group $v(\mathcal{H}^*)$ together with this map is an ordered K -vector space. This is true for any Hardy field that is closed under taking powers from K of positive elements.

We are now ready to prove a stronger version of Growth Dichotomy (Theorem 1.1).

3.7. *Either $\mathfrak{R}$ is exponential or $v(\mathcal{H}^*) = K \cdot v(x)$.*

Proof. There are two cases to consider.

Case 1. There exists $f \in \mathcal{H}^*$ such that $v(f'/f) \neq v(1/x)$ and $v(f) \neq 0$.

We show that $\mathfrak{R}$ is exponential. By replacing f with $-f$, we take $f > 0$. By replacing f with $1/f$, we take f to be infinitely increasing. By replacing f with f^{-1} , we suppose that $v(f'/f) < v(1/x)$. By 3.5, there exists $g \in \mathcal{H}^*$ such that $g' \sim f'/f$. Put $h = g \circ f^{-1}$; then $h' \sim 1/x$. By the mean value theorem, we have

$$h \circ (2\xi) - h = \frac{x}{\xi} \cdot \xi h' \circ \xi$$

for some $\xi \in \mathcal{H}$ such that $x < \xi < 2x$. (*Exercise:* Why is the ultimately-defined function ξ definable?) Note that $v(x/\xi) = 0 = v(\xi h' \circ \xi)$, the latter by substituting ξ into xh' . Thus,

$$v(h \circ (2x) - h) = v(x/\xi) + v(\xi h' \circ \xi) = v(x/\xi) + v(xh') = 0.$$

By 2.2, $\mathfrak{R}$ is exponential.

Case 2. For all $f \in \mathcal{H}^*$, if $v(f'/f) \neq v(1/x)$, then $v(f) = 0$.

We show that $v(\mathcal{H}^*) = K.v(x)$. Let $f \in \mathcal{H}^*$.

We show first that $Pf = x^r$ for some $r \in K$, where Pf is as in 2.3. This is immediate if $v(f) = 0$ (for then $Pf = 1 = x^0$), so assume that $v(f) \neq 0$. Put $g = (f \circ (2x))/f \in \mathcal{H}^*$. Observe that

$$x \frac{g'}{g} = 2x \frac{f' \circ (2x)}{f \circ (2x)} - \frac{xf'}{f}.$$

Since $v(f) \neq 0$, we have $v(xf'/f) = 0$, and so $v(g'/g) > v(1/x)$. By the case assumption, $v(g) = 0$, that is, $f \circ (2x) \sim cf$ for some $c \in \mathbb{R}^*$. Now apply 2.3.

To finish the proof, we now let $f \in \mathcal{H}^*$ and show that $v(f) = v(Pf)$. Since $P((Pf)/f) = 1 \upharpoonright \mathbb{R}^{>0}$ (exercise), we are reduced to showing that if $Pf = 1 \upharpoonright \mathbb{R}^{>0}$, then $v(f) = 0$. By the case assumption, it suffices to show that $v(xf'/f) \neq 0$. By the mean value theorem,

$$\frac{f \circ (2x)}{f} - 1 = \frac{xf' \circ \xi}{f} = \frac{x}{\xi} \cdot \frac{\xi f' \circ \xi}{f \circ \xi} \cdot \frac{f \circ \xi}{f}$$

where $\xi \in \mathcal{H}^*$ and $x < \xi < 2x$. It suffices now to show that $v((\xi f' \circ \xi)/f \circ \xi) \neq 0$ (for then $v(xf'/f) \neq 0$ as well). Since $Pf(2) = 1$, we have

$$0 = v\left(\frac{f \circ (2x)}{f} - 1\right) = v\left(\frac{x}{\xi}\right) + v\left(\frac{\xi f' \circ \xi}{f \circ \xi}\right) + v\left(\frac{f \circ \xi}{f}\right).$$

Since $v(\xi) = v(x)$, it suffices now to show that $f \circ \xi \sim f$, which follows easily from monotonicity—either $f \leq f \circ \xi \leq f \circ (2x)$ or $f \geq f \circ \xi \geq f \circ (2x)$ —and that $Pf(2) = 1$ (that is, $f \circ (2x) \sim f$). $\square$

With Growth Dichotomy now established, the next natural step is to study the two resulting cases. But for the remainder of this paper, we deal for the most part only with the polynomially bounded case.

4 Basics of Polynomial Boundedness

Throughout this section, we assume that $\mathfrak{R}$ is o-minimal and polynomially bounded with field of exponents K .

We begin with Piecewise Uniform Asymptotics (Theorem 1.2), which we restate using the notation of the preceding section.

4.1. *Let $A \subseteq \mathbb{R}^m$ and $f: A \times \mathbb{R} \rightarrow \mathbb{R}$ be definable such that, for every $a \in A$, the function $x \mapsto f(a, x)$ is ultimately nonzero. Then there is a finite $S \subseteq K$ such that $\{v(f(a, x)) : a \in \mathbb{R}^m\} \subseteq S.v(x)$.*

Proof. By 3.7, for each $a \in A$ there exists unique $\rho(a) \in K$ such that $v(f(a, x)) = \rho(a).v(x) = v(x^{\rho(a)})$. We must show that $\rho(A)$ is finite; for this, it suffices to show that $\rho(A)$ is definable and has no interior. As the set $\{a \in A : v(f(a, x)) = 0\}$ is definable, we reduce to the case that $\rho(a) \neq 0$ for all $a \in A$. For each $a \in A$, the function $x \mapsto f(a, x)$ is ultimately differentiable. By l'Hôpital,

$$\rho(a) = \lim_{x \rightarrow +\infty} \frac{x(\partial f / \partial x)(a, x)}{f(a, x)}.$$

Thus, the function $\rho: A \rightarrow \mathbb{R}$ is definable, hence so is $\rho(A)$. We next show that $\rho(A)$ has no interior. Suppose otherwise; we derive a contradiction to finish the proof. Let I be an open interval contained in $\rho(A)$; then $K = \mathbb{R}$, since $\rho(A) \subseteq K$ and K is a subfield of $\mathbb{R}$. For all $a \in A$ and $x > 0$ we have

$$\lim_{t \rightarrow +\infty} \frac{f(a, tx)}{f(a, t)} = x^{\rho(a)},$$

so $(x, a) \mapsto x^{\rho(a)}: \mathbb{R}^{>0} \times A \rightarrow \mathbb{R}$ is definable. By Definable Choice [31, p. 94], there is a definable function $g: I \rightarrow A$ such that $\rho(g(y)) = y$ for all $y \in I$. Thus, the restriction of the function x^y to $\mathbb{R}^{>0} \times I$ is definable. By dividing x^y by some power function x^r with $r \in I$, we may assume that $0 \in I$. Then for some $\epsilon > 0$ the restriction of x^y to $\mathbb{R}^{>0} \times (0, \epsilon)$ is definable. Observe that $z = x^y$ if and only if $z^{1/y} = x$ for all $x > 0$ and $y \neq 0$, so the restriction of x^y to $\mathbb{R}^{>0} \times (1/\epsilon, \infty)$ is definable. Then $y \mapsto 2^y: (1/\epsilon, \infty) \rightarrow \mathbb{R}$ is also definable, so $\lim_{y \rightarrow +\infty} 2^y / y^r = c$ for some $r \in K$ and $c \in \mathbb{R}^*$. But then

$$c = \lim_{y \rightarrow +\infty} \frac{2^y}{y^r} = \lim_{y \rightarrow +\infty} \frac{2^{y+1}}{(y+1)^r} = \lim_{y \rightarrow +\infty} \frac{22^y}{(1 + (1/y))^r y^r} = 2c,$$

contradicting that $c \neq 0$. □

The following minor restatement is often useful.

4.2. *Let $A \subseteq \mathbb{R}^m$ and $f: A \times \mathbb{R}^{>0} \rightarrow \mathbb{R}$ be definable. Then there is a finite $S \subseteq K$ such that for all $a \in A$, either $f(a, t) = 0$ for all sufficiently small positive t (depending on a), or $\lim_{t \rightarrow 0^+} f(a, t)/t^r \in \mathbb{R}^*$ for some $r \in S$.*

(Apply 4.1 to the function $(a, t) \mapsto f(a, 1/t): A \times \mathbb{R}^{>0} \rightarrow \mathbb{R}$.)

We need some notation before stating the next result. For $x = (x_1, \dots, x_m) \in \mathbb{R}^m$, put $|x| = \sup\{|x_1|, \dots, |x_m|\}$. Given $A \subseteq \mathbb{R}^m$, y in the closure of A , and $f: A \rightarrow \mathbb{R}$, we write $f(x) = O(|x - y|^r)$ as $x \rightarrow y$ if there exist $C \in \mathbb{R}^{>0}$ and an open neighborhood $U \subseteq \mathbb{R}^m$ of y such that $|f(x)| \leq C|x - y|^r$ for all $x \in A \cap U$. If A is unbounded, we write $f(x) = O(|x|^r)$ as $x \rightarrow \infty$ if there exist $C, M \in \mathbb{R}^{>0}$ such that $|f(x)| \leq C|x|^r$ for all $x \in A$ with $|x| > M$. Recall the notation for fibers of sets from Sect. 2.

4.3 (Uniform Bounds on Orders of Vanishing). *Let $A \subseteq \mathbb{R}^{m+n}$ and $f: A \rightarrow \mathbb{R}$ be definable. Then there exists $r \in K^{>0}$ such that for all $(x, y) \in \mathbb{R}^{m+n}$, if $y \in \text{cl}(A_x)$ and $f(x, z) = O(|y - z|^r)$ as $z \rightarrow y$, then $f(x, z) = 0$ for all $z \in A_x$ near y .*

Proof. For all $(x, y, t) \in \mathbb{R}^{m+n} \times (0, \infty)$, put $A(x, y, t) = \{z \in A_x : |y - z| = t\}$, and put

$$X = \left\{ (x, y, t) \in \mathbb{R}^{m+n} \times (0, \infty) : A(x, y, t) \neq \emptyset \text{ \& \; } \sup_{z \in A(x, y, t)} |f(x, z)| < +\infty \right\}.$$

Define $F: \mathbb{R}^{m+n} \times (0, \infty) \rightarrow \mathbb{R}$ by

$$F(x, y, t) = \begin{cases} \sup\{|f(x, z)| : z \in A(x, y, t)\}, & \text{if } (x, y, t) \in X \\ 0, & \text{otherwise.} \end{cases}$$

By 4.2, there exists $r \in K$ such that for all $(x, y) \in \mathbb{R}^{m+n}$ if $F(x, y, t) = O(t^r)$ as $t \rightarrow 0^+$, then $F(x, y, t) = 0$ for all sufficiently small positive t (depending on (x, y)).

Now suppose that $(x, y) \in \mathbb{R}^{m+n}$, $y \in \text{cl}(A_x)$ and $f(x, z) = O(|y - z|^r)$ as $z \rightarrow y$. Then $f(x, -)$ is bounded near y , so $F(x, y, t) = \sup_{z \in A(x, y, t)} |f(x, z)|$ for all sufficiently small $t > 0$. Then $F(x, y, t) = O(t^r)$ as $t \rightarrow 0^+$, and thus $f(x, z) = 0$ for all $z \in A_x$ sufficiently close to y . $\square$

Remark. The “ O -constant” depends on x and y .

4.4. *Let $U \subseteq \mathbb{R}^n$ be open and $f: U \rightarrow \mathbb{R}$ be definable. Then there exists $N \in \mathbb{N}$ such that for all $x \in U$, if f is C^N in an open neighborhood of x and all partial derivatives of f of order at most N vanish at x , then f vanishes identically on some open $V \subseteq U$ with $x \in V$.*

Proof. By 4.3 there exists $N \in \mathbb{N}$ such that for all $x \in U$, if $f(x) = O(|x - y|^N)$ as $|x - y| \rightarrow 0^+$, then f vanishes identically in a neighborhood of x . Now apply Taylor’s formula. $\square$

Quasianalyticity. For $U \subseteq \mathbb{R}^n$ a definable open set, let $C_{\text{df}}^\infty(U)$ denote the ring of definable C^∞ functions $f: U \rightarrow \mathbb{R}$.

4.5. Let $U \subseteq \mathbb{R}^n$ be definable, open and connected.

- (1) If $f \in C_{\text{df}}^\infty(U)$ and all partials of f vanish at some $x_0 \in U$, then $f = 0$.
 (2) $C_{\text{df}}^\infty(U)$ is an integral domain.

Proof. (1). Consider the definable open set A consisting of all $x \in U$ such that $f \upharpoonright V = 0$ for some open $V \subseteq U$ with $x \in V$. By 4.4, $x_0 \in A$. Let $x \in \text{cl}(A) \cap U$. All partials of f are continuous on U and vanish identically on A . Then all partials of f vanish at x . By 4.4, $x \in A$. Thus, A is both open and closed in U , so $A = U$.

- (2). Let $f, g \in C_{\text{df}}^\infty(U)$ with $fg = 0$. Suppose that $g(x_0) \neq 0$ for some $x_0 \in U$. By continuity, g has no zeros in some open neighborhood of x_0 . Then f vanishes identically on this neighborhood, so all partials of f vanish at x_0 . Hence, $f = 0$ by (1). $\square$

Failure of “Naive QE”

4.6. Let $\mathfrak{R}_0$ be the reduct of $\mathfrak{R}$ generated over $\overline{\mathbb{R}}$ by $f \in \bigcup_{n \in \mathbb{N}} C_{\text{df}}^\infty(\mathbb{R}^n)$. Then QE fails for $\mathfrak{R}_0$ unless every $f \in \bigcup_{n \in \mathbb{N}} C_{\text{df}}^\infty(\mathbb{R}^n)$ is definable in $\overline{\mathbb{R}}$.

Proof. Suppose that $\mathfrak{R}_0$ has QE. Let $h: \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous and definable in $\mathfrak{R}_0$. We show that h is definable in $\overline{\mathbb{R}}$. Define $g: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(x, y) = \begin{cases} yh(x/y), & \text{if } y \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

By QE, there is some $F \in C^\infty(\mathbb{R}^{n+2})$, definable in $\mathfrak{R}_0$ and not identically equal to 0, such that the graph of g is contained in the zero set of F . By quasianalyticity (and because g vanishes at the origin), there is a nontrivial homogeneous polynomial P of degree $d > 0$ and $G \in C^\infty(\mathbb{R}^{n+2})$ definable in $\mathfrak{R}$ such that $F = P + G$ and $\lim_{|v| \rightarrow 0^+} G(v) |v|^{-d} = 0$. Let $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. We have

$$0 = F(tx, t, g(tx, t)) = F(tx, t, tg(x, 1)) = t^d P(x, 1, g(x, 1)) + G(tx, t, tg(x, 1)).$$

Since $g(x, 1) = h(x)$, we have $P(x, 1, h(x)) = -t^{-d} G(tx, t, th(x))$. Keeping x fixed and letting $t \rightarrow 0$, we have $P(x, 1, h(x)) = 0$. As x is arbitrary, the graph of h is contained in the zero set of the polynomial $Q(x, z) := P(x, 1, z)$, which is nontrivial by homogeneity of P . Thus, the zero set of Q is a finite union of nonopen cells definable in $\overline{\mathbb{R}}$. By continuity, both the set zero set of Q and the graph of h are closed. By cell decomposition in $\mathfrak{R}_0$, the graph of h is a finite union of cells definable in $\overline{\mathbb{R}}$. $\square$

The above is a minor variant of a result due to van den Dries (generalizing the “Osgood example”):

4.7 ([26]). *The conclusion of 4.6 holds without assuming polynomial bounds or o-minimality if every $f \in \bigcup_{n \in \mathbb{N}} C_{\text{df}}^\infty(\mathbb{R}^n)$ is real analytic.*

We shall close this section with a result to be used in the proofs of Theorems 1.4 and 1.7; some preliminary work is needed.

4.8. *Every power function of $\mathfrak{R}$ is $\emptyset$ -definable.*

Proof. Let $r \in K$, and suppose that the function x^r is defined by $\varphi(c, x, y)$ for some $(m + 2)$ -ary formula φ (in the language of $\mathfrak{R}$) and $c \in \mathbb{R}^m$. The set B of all $b \in \mathbb{R}^m$ such that $\varphi(b, x, y)$ defines a solution on $\mathbb{R}^{>0}$ to the initial value problem $xy' = ry$, $y(1) = 1$ is $\{r\}$ -definable. Hence, by uniqueness of solutions to ODEs, it suffices to show that r is $\emptyset$ -definable. Put

$$C = \{c \in \mathbb{R}^m : \varphi(c, x, y) \text{ defines a function } f_c : \mathbb{R} \rightarrow \mathbb{R}\}.$$

Note that C is $\emptyset$ -definable. Let $f : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}$ be the function given by $f(c, x) = f_c(x)$ if $c \in C$ and $f(c, x) = 0$ otherwise. Note that f is $\emptyset$ -definable. Let $S \subseteq K$ be as in the statement of Piecewise Uniform Asymptotics applied to f . The proof of Piecewise Uniform Asymptotics shows that S is $\emptyset$ -definable. As S is also finite, every $s \in S$ is $\emptyset$ -definable. Hence, r is $\emptyset$ -definable, and we are done. $\square$

4.9 (exercise). Let A be (the underlying set of) an ordered field.

- Define an equivalence relation on A^* by identifying $x, y \in A^*$ iff there exists $n \in \mathbb{N}$ such that $2^{-n} < |y/x| < 2^n$. Denote the natural quotient map by v (or v_A if needed); it is called the **archimedean valuation** on A . The image $v(A^*)$ is an ordered abelian group by setting $v(x) + v(y) = v(xy)$ and $v(x) > 0$ iff $|x| < 2^{-n}$ for all $n \in \mathbb{N}$.
- If $x, y \in A^*$ and $|x| \geq |y|$, then $v(y) \leq v(x)$. The converse fails.
- If $x, y \in A^*$ and $x \neq -y$, then $v(x + y) \geq \min\{v(x), v(y)\}$, with equality if $v(x) \neq v(y)$.
- If A is a Hardy field, then $v = v$ (as defined in Sect. 3).
- If A can be expanded to a model $\mathfrak{A}$ of $\text{Th}(\overline{\mathbb{R}}^K)$, then the map

$$(r, v(a)) \mapsto v(|a|^r) : K \times v(A^*) \rightarrow v(A^*)$$

is well defined (where x^r is interpreted in $\mathfrak{A}$), and $v(A^*)$ is an ordered K -vector space with this map as scalar multiplication. We let $r.v(a)$ denote the image of $(r, v(a))$ under this map.

4.10. *Let $\mathfrak{A} \models \mathfrak{B} \equiv \mathfrak{R}$ and $\mathfrak{A} \subseteq \mathfrak{B}$. If $Y \subseteq B^*$ and $v(Y)$ is K -linearly independent over $v(A^*)$, then Y is definably independent with respect to $\text{Th}(\mathfrak{R})$ over A .*

This is a special case of a result due to van den Dries [30, Theorem C] that depends on earlier work with Lewenberg [32] as well as 4.8. A complete and self-contained (modulo 4.8) proof appears in [24].

5 Expanding by Powers: Proof of Theorem 1.4

Throughout this section, we assume that $\mathfrak{R}$ is o-minimal and polynomially bounded with field of exponents K . By 4.8, we suppose that $\mathfrak{R}$ is an expansion in the syntactic sense of $\mathbb{R}^K$, so that the notation x^r makes sense in arbitrary models of $\text{Th}(\mathfrak{R})$. Let $S \subseteq \mathbb{R}$ be such that for every $r \in S$ the restriction $x^r \upharpoonright [1, 2]$ is $\emptyset$ -definable. Put $J = K(S)$. Note that $x^r \upharpoonright [1, 2]$ is $\emptyset$ -definable in $\mathfrak{R}$ for every $r \in J$. Let L be the language of $\mathfrak{R}$, L^J be the extension of L by pairwise distinct new unary function symbols f_r for the $r \in J$, and T be the extension of $\text{Th}(\mathfrak{R})$ by the universal closures of the following formulas:

P1. For each $r \in J$,

$$[(x = 0 \vee x < 0) \rightarrow f_r(x) = 0] \wedge [(0 < x \wedge 0 < y) \rightarrow f_r(xy) = f_r(x) \cdot f_r(y)],$$

P2. For each $r \in J^{>0}$, $1 < x \rightarrow 1 < f_r(x)$.

P3. For each $r, s \in J$, $f_{rs}(x) = f_r(f_s(x)) \wedge f_{r+s}(x) = f_r(x) \cdot f_s(x)$.

P4. For each $r \in J$, a formula $\varphi_r(x, y) \rightarrow y = f_r(x)$, where φ_r is an L -formula that defines the graph of $x^r \upharpoonright [1, 2]$ in $\mathfrak{R}$.

P5. $0 < x \rightarrow f_0(x) = 1 \wedge f_1(x) = x$.

Evidently, $\mathfrak{R}^J \models T$ by interpreting f_r as x^r for $r \in J$. We are going to show that T axiomatizes $\mathfrak{R}^J$ by showing that T is complete. Our first goal is to show that T admits QE relative to $\text{Th}(\mathfrak{R})$. Next is the key technical lemma.

5.1. Suppose that $\mathfrak{R}$ admits QE and is $\forall$ -axiomatizable. Let $\mathfrak{A} \subsetneq \mathfrak{B} \models T$ and $\mathfrak{B}' \supseteq \mathfrak{A}$ be $|\mathfrak{B}|^+$ -saturated. Then there exists $\mathfrak{D} \models T$ with $\mathfrak{A} \subsetneq \mathfrak{D} \subseteq \mathfrak{B}$ such that $\mathfrak{D}$ embeds over $\mathfrak{A}$ into $\mathfrak{B}'$.

We have some preliminary work to do, beginning with some easy (proofs are left to the reader) consequences of ordered field properties and P1–P5.

5.2. Let $\mathfrak{A} \models T$.

- For all $r \in K$, $f_r \upharpoonright A^{>0} = x^r \upharpoonright A^{>0}$.
- For all $r \in J^*$, $f_r \upharpoonright A^{>0}$ is an automorphism of the multiplicative group $A^{>0}$, with compositional inverse $f_{1/r} \upharpoonright A^{>0}$. If $r > 0$, then f_r is strictly increasing. If $r < 0$, then f_r is strictly decreasing.
- If $a \in A^{>0}$ and $s = \sum_{i=1}^n r_i s_i$ with $r_i \in K$ and $s_i \in J$ for $i = 1, \dots, n$, then $f_s(a) = \prod_{i=1}^n f_{r_i}(f_{s_i}(a))$.
- If $0 < r < s$ and $a > 1$, then $f_r(a) < f_s(a)$.
- If $r \neq s$ and $a > 0$, then $f_r(a) = f_s(a)$ iff $a = 1$.
- The map $r.v(a) := (r, v(a)) \mapsto v(f_r(|a|))$: $J \times v(A^*) \rightarrow v(A^*)$ is well defined, and $v(A^*)$ is an ordered J -vector space with this map as scalar multiplication.

We may write a^r instead of $f_r(a)$ without ambiguity whenever $r \in J$ and $a > 0$. Both notations will be used according to convenience.

Note that, as K is a subfield of J , we can also regard $v(A^*)$ as an ordered K -vector space; we shall have occasion to do so.

Put $\text{Un}^+(A) = A^{>0} \cap v^{-1}(0)$. Note that if $u \in \text{Un}^+(A)$ then there is some nonzero integer k such that $1 \leq u^{1/k} \leq 2$.

Given $\mathfrak{B} \models T$, $X \subseteq B$ and $S \subseteq J$, we say that X is **closed under powers from S** if $x^s \in X$ for all $x \in X$ and $s \in S$.

Given an L^J -structure $\mathfrak{A}$, let $\mathfrak{A}_L$ denote the reduct in the syntactic sense of $\mathfrak{A}$ to L . Note that if $\mathfrak{A} \models T$, then $\mathfrak{A}_L \equiv \mathfrak{A}$.

5.3. Let $\mathfrak{B} \models T$ and $\mathfrak{A} \subseteq \mathfrak{B}_L$. If there exists $P \subseteq A^{>0}$ such that $v(P) = v(A)$ and $p^r \in A$ for all $p \in P$ and $r \in J$, then A is closed under powers from J and $(\mathfrak{A}, (x^r \upharpoonright A)_{r \in J}) \models T$.

Proof. Since T has a universal axiomatization, it suffices to show that A is closed under powers from J . Let $a \in A^{>0}$. (The result is trivial for $a \leq 0$ by definition of f_r .) Then there exists $u \in \text{Un}^+(A)$ and $p \in P$ with $a = up$. Let $r \in J$. Since $p^r \in A$, it suffices to show that $u^r \in A$. We have some $k \in \mathbb{Z}^*$ such that $1 \leq u^{1/k} \leq 2$. Since $\mathfrak{A} \subseteq \mathfrak{B}_L \models \text{Th}(\mathfrak{A})$, we have $(u^{1/k})^r \in A$ by P4. Thus, $u^r = ((u^{1/k})^r)^k \in A$. $\square$

5.4. Let $\mathfrak{A}, \mathfrak{B} \models T$ and $\phi: \mathfrak{A}_L \rightarrow \mathfrak{B}_L$ be an embedding. If there exists $P \subseteq A^{>0}$ such that $v(P) = v(A)$ and $\phi(p^r) = \phi(p)^r$ for all $r \in J$ and $p \in P$, then ϕ is an embedding $\mathfrak{A} \rightarrow \mathfrak{B}$.

Proof. Let $a \in A^{>0}$ and $r \in J$. We must show that $\phi(a^r) = \phi(a)^r$. Since $a = up$ as in 5.3, it suffices consider the case that $v(a) = 0$. As before, we then have $1 \leq a^{1/k} \leq 2$ for some $k \in \mathbb{Z}^*$. By P4, we have $\phi(a^r) = \phi((a^{1/k})^{rk}) = \phi(a^{1/k})^{rk} = \phi(a)^{rk(1/k)} = \phi(a)^r$ as required. $\square$

Given $\mathfrak{M} \equiv \mathfrak{A}$ and $X \subseteq M$, let $\mathfrak{M}\langle X \rangle$ denote the substructure of $\mathfrak{M}$ generated by X .

5.5. Suppose that:

- $\mathfrak{M}, \mathfrak{M}', \mathfrak{N}, \mathfrak{N}' \models \text{Th}(\mathfrak{A})$;
- $\mathfrak{M} \leq \mathfrak{N}$ and $\mathfrak{M}' \leq \mathfrak{N}'$;
- $\phi_0: \mathfrak{M} \rightarrow \mathfrak{M}'$ is an isomorphism;
- $X \subseteq N$ and $X' \subseteq N'$ are such that ϕ_0 extends to an order-preserving bijection $\phi_1: M \cup X \rightarrow M' \cup X'$.

Then there is an isomorphism $\phi: \mathfrak{M}\langle X \rangle \rightarrow \mathfrak{M}'\langle X' \rangle$ extending ϕ_1 .

Proof. By o-minimality, the type of an element $x \in N$ over $\mathfrak{M}$ is determined by the cut that it realizes in M , and similarly for $x' \in N'$ over $\mathfrak{M}'$. $\square$

Proof of 5.1. In order to reduce clutter, we shall delete the superscripted $*$ from expressions like $v(A^*)$ for ordered fields A .

Suppose that $\mathfrak{A}$ admits QE and is $\forall$ -axiomatizable. Note that substructures of models of $\text{Th}(\mathfrak{A})$ are elementary substructures. Let $\mathfrak{A} \subsetneq \mathfrak{B} \models T$ and $\mathfrak{B}' \geq \mathfrak{A}$ be

$|B|^+$ -saturated. We must find $\mathfrak{D} \models T$ with $\mathfrak{A} \subsetneq \mathfrak{D} \subseteq \mathfrak{B}$ such that $\mathfrak{D}$ embeds over $\mathfrak{A}$ into $\mathfrak{B}'$. We proceed by a trivial case distinction: $v_B(A) = v_B(B)$ or $v_B(A) \neq v_B(B)$. The overall strategy is the same in both cases, but the tactics are more involved for the latter.

Suppose that $v_B(A) = v_B(B)$. Let $x \in B \setminus A$. By saturation, choose $y \in B' \setminus A$ realizing the same cut in A as x . Put $\mathfrak{C} = \mathfrak{A}_L \langle x \rangle$ and $\mathfrak{C}' = \mathfrak{A}_L \langle y \rangle$. Note that $\mathfrak{C} \preceq \mathfrak{B}_L$ and $\mathfrak{C}' \preceq \mathfrak{B}'_L$. By 5.5, there is an isomorphism $\phi: \mathfrak{C} \rightarrow \mathfrak{C}'$ fixing A pointwise and sending x to y . Trivially, $v_B(C) = v_B(A) = v_A(A) = v_{B'}(C')$. By 5.3, $\mathfrak{C}$ and $\mathfrak{C}'$ expand to submodels $\mathfrak{D}$ and $\mathfrak{D}'$ of $\mathfrak{B}$ and $\mathfrak{B}'$. By 5.4, ϕ is an L^J -embedding of $\mathfrak{D}$ into $\mathfrak{B}'$ over $\mathfrak{A}$ as desired.

Suppose that $v_B(B) \neq v_B(A)$. Take $x \in B^{>0}$ with $v_B(x) \notin v_B(A)$. Choose $y \in B'$ realizing the same cut in A as x . Fix a K -basis E for J such that $1 \in E$. Put $\mathfrak{C} = \mathfrak{A}_L \langle x^e : e \in E \rangle$ and $\mathfrak{C}' = \mathfrak{A}_L \langle y^e : e \in E \rangle$. Put $P = \{ax^r : a \in A^{>0} \text{ \& } r \in J\}$ and $P' = \{ay^r : a \in A^{>0} \text{ \& } r \in J\}$. As A is closed under powers from J , so are P and P' . Observe that $v_B(P) = v_B(A) + J.v_B(x)$ and $v_{B'}(P') = v_B(A) + J.v_B(y)$. Thus, it suffices now by 5.3 and 5.4 (and basic algebra) to show that $v_B(C) = v_B(A) + \sum_{e \in E} K.v_B(x^e)$, $v_{B'}(C') = v_{B'}(A) + \sum_{e \in E} K.v_{B'}(y^e)$, and there is an isomorphism $\phi: \mathfrak{C} \rightarrow \mathfrak{C}'$ fixing A pointwise and sending x^e to y^e for all $e \in E$. Recall that y satisfies the same cut in A as does x . Hence, the same is true of the pairs (x^e, y^e) with $e \in E$, because A is closed under powers from J . It is easy to see that $\{v_B(x^e) : e \in E\}$ is K -linearly independent over $v_B(A)$ and $\{v_{B'}(y^e) : e \in E\}$ is K -linearly independent over $v_{B'}(A) = v_B(A)$. The rest of the argument is now routine via 4.10 and 5.5. $\square$

A useful corollary of the proof of 5.1:

5.6. *Let $\mathfrak{A} \equiv \mathfrak{B} \equiv \mathfrak{R}^J$ and $\mathfrak{A} \subseteq \mathfrak{B}$. If $Y \subseteq B^*$ and $v(Y)$ is J -linearly independent over $v(A^*)$, then Y is definably independent (with respect to $\text{Th}(\mathfrak{R}^J)$) over A .*

(In other words, the conclusion of 4.10 holds with $\mathfrak{R}^J$ in place of $\mathfrak{R}$ and J in place of K .)

Proof of 1.4

- (1) Suppose that $\mathfrak{R}$ has QE and is $\forall$ -axiomatizable. We must show that the same is true of $\mathfrak{R}^J$. By 5.1 and general model theory (say, 2.3.9 and 4.3.28 of [7]), T has QE. Let $\mathfrak{P}$ be the prime submodel of $\mathfrak{R}$ (recall that $\mathfrak{R}$ has definable Skolem functions) and P be its underlying set. As $(P^{>0}, \cdot)$ is archimedean as an ordered group, we have $a^r \subseteq P$ for every $a \in P^{>0}$ and $r \in J$. (For every $a \in P^{>0}$ there exist $b \in [1, 2) \cap P$ and $k \in \mathbb{Z}$ such that $a^r = (b^r)^k$.) Thus, $\mathfrak{P}$ expands naturally to an L^J -structure that embeds into every model of T . Hence, T is complete, so $\mathfrak{R}^J$ has QE and is axiomatized by T . Since $\mathfrak{R}$ has QE, the formulas φ_r in P4 can be taken to be quantifier free, so that all of the formulas of P1–P5 are quantifier free. Since $\mathfrak{R}$ is $\forall$ -axiomatizable, so is T , hence also $\mathfrak{R}^J$.
- (2) Suppose that $\mathfrak{R}$ is model complete. We must show that the same is true of $\mathfrak{R}^S$. Let $\tilde{\mathfrak{R}}$ be the expansion of $\mathfrak{R}$ by all $\emptyset$ -definable Skolem functions. Then $\tilde{\mathfrak{R}}$ has QE and is $\forall$ -axiomatizable. By Theorem 1.4.1, $\tilde{\mathfrak{R}}^J$ has QE. Since $\mathfrak{R}$ is

model complete, its $\emptyset$ -definable Skolem functions are both existentially and universally definable, so a routine syntactic argument involving “de-nesting” of terms shows that $\mathfrak{R}^J$ is model complete. For every $r \in J$, both r and x^r are existentially and universally definable in $\overline{\mathbb{R}}^S$. (There exist $m \in \mathbb{N}$, polynomials $p, q \in K[x_1, \dots, x_m]$ and $s \in S^m$ such that $p(s) = rq(s)$.) Hence, $\mathfrak{R}^S$ is model complete.

- (3) Let $\mathfrak{R}$ be as in the previous paragraph. Since $\widetilde{\mathfrak{R}}$ has QE and is $\forall$ -axiomatizable, the same is true of $(\mathfrak{R}, (r)_{r \in \mathbb{R}})$. (Each real number is determined by its position in $(\mathbb{R}, <)$ relative to $\mathbb{Q}$, and each $q \in \mathbb{Q}$ is $\emptyset$ -definable in $\overline{\mathbb{R}}$.) By Theorem 1.4.1, $(\mathfrak{R}, (r)_{r \in \mathbb{R}})^J$ has QE and is $\forall$ -axiomatizable. By general model theory (or see [38, 2.15]), every function $\emptyset$ -definable in $(\mathfrak{R}, (r)_{r \in \mathbb{R}})^J$ is given piecewise by terms.⁸ Hence, every function definable in $\mathfrak{R}^J$ is given piecewise by (iterated) compositions of functions definable in $\mathfrak{R}$ and powers from J .
- (4) We must show that $\mathfrak{R}^J$ is polynomially bounded with field of exponents J . Recall that $\mathfrak{R}^J$ is o-minimal. After passing to an extension by definitions, we reduce to the case that $\mathfrak{R}$ has QE and is $\forall$ -axiomatizable.

Suppose that $\mathfrak{R}^J$ is not polynomially bounded; then it is exponential by Growth Dichotomy. Let $\mathfrak{B}$ be a proper elementary extension of $\mathfrak{R}^J$. There exists $b \in B$ such that $b > \mathbb{R}$. As b and e^b are definably dependent (where e^x is interpreted in $\mathfrak{B}$), there exists by 5.6 some $r \in K$ and $N \in \mathbb{N}$ such that $e^b \leq Nb^r$, contradicting that there exists $k \in \mathbb{N}$ such that $e^x > Nx^r$ for all $x > k$.

Now we show that $\mathfrak{R}^J$ has field of exponents J . Let $s \in \mathbb{R}$ be such that x^s is definable in $\mathfrak{R}^J$. Let $\mathfrak{B}$ and b be as before. By arguing as above, we have $v(b^s) = v(b^r)$ for some $r \in K$. Then $v(b^{s-r}) = 0$, so there exists $N \in \mathbb{N}$ such that $b^{|s-r|} \leq N$. Thus, $s \in J$, for if not, there exists $k \in \mathbb{N}$ such that $x^{|s-r|} > N$ for all $x > k$, contradicting that $b^{|s-r|} \leq N$. $\square$

6 Proof of Theorem 1.7

Let $\mathfrak{R}$ be o-minimal and polynomially bounded with field of exponents K . Let $N \in \mathbb{N}$, $G: \mathbb{R}^{N+2} \rightarrow \mathbb{R}$ be definable in $\mathfrak{R}$, and g be a solution on some ray (a, ∞) to the differential equation $y^{(N+1)} = G(x, y, y', \dots, y^{(N)})$. Suppose that g is definable in an o-minimal expansion $\widetilde{\mathfrak{R}}$ of $\mathfrak{R}$. We must show that there exists $r \in K$ such that $|g(x)| / \exp_{N+1}(x^r)$ is ultimately bounded. (Of course, we can take $r \in \mathbb{N}$ if desired since we are working over $\mathbb{R}$.) This is obvious if $\widetilde{\mathfrak{R}}$ is polynomially bounded, so assume otherwise; then it is exponential by Growth Dichotomy. Thus, working in the Hardy field $\widetilde{\mathcal{H}}$ of $\widetilde{\mathfrak{R}}$ (recall Sect. 3), it suffices to take g infinitely increasing and show that $v(g) \geq v(\exp_{N+1}(x^r))$ for some $r \in K$.

⁸That is, if $g: \mathbb{R}^n \rightarrow \mathbb{R}$ is definable, then there is a finite $\mathcal{F} \subseteq \mathcal{T}_n$ such that the graph g is contained in the union of the graphs of the $f \in \mathcal{F}$.

As this statement is definability-theoretic, we make some convenient assumptions about the languages L and $\widetilde{L}$ of $\mathfrak{R}$ and $\widetilde{\mathfrak{R}}$: (i) $\{<, +, -, \cdot\} \subseteq L \subseteq \widetilde{L}$; (ii) L has constants for all real numbers; (iii) $\widetilde{L}$ has no relation symbols other than $<$; and (iv) both $\mathfrak{R}$ and $\widetilde{\mathfrak{R}}$ have QE and are $\forall$ -axiomatizable (after expanding by all definable Skolem functions). Let $\mathcal{H}$ be the Hardy field of $\mathfrak{R}$. Note that $\widetilde{\mathcal{H}}$ is an extension of $\mathcal{H}$ as an ordered differential field. For an n -ary function symbol F of $\widetilde{L}$ and $f_1, \dots, f_n \in \widetilde{\mathcal{H}}$, let $F(f_1, \dots, f_n)$ denote the germ of the function $F \circ (f_1, \dots, f_n)$; then $F(f_1, \dots, f_n) \in \widetilde{\mathcal{H}}$. It follows that $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ (as ordered rings) expand naturally to L -structures, with $\mathfrak{R} \subseteq \mathcal{H} \subseteq \widetilde{\mathcal{H}}$. Let $\mathfrak{H}$ be the L -substructure of $\widetilde{\mathcal{H}}$ generated by $\mathcal{H} \cup \{g^{(k)} : k \leq N\}$; then $g \in \mathfrak{H}^*$ and $\mathfrak{R} \subseteq \mathcal{H} \subseteq \mathfrak{H} \subseteq \widetilde{\mathcal{H}}$. We will show that for every $f \in \mathfrak{H}^*$ there exists $r \in K$ such that $v(f) \geq v(\exp_{N+1}(x^r))$ to finish.

We now show that the underlying ring of $\mathfrak{H}$ is a Hardy field. Let $f \in \mathfrak{H}^*$. We must show that both $1/f$ and f' belong to $\mathfrak{H}$. There exist: $n \in \mathbb{N}$; $a_1, \dots, a_n \in \mathcal{H}$; and $F: \mathbb{R}^{n+1+N} \rightarrow \mathbb{R}$ definable in $\mathfrak{R}$ such that $f = F(a_1, \dots, a_n, g^{(0)}, \dots, g^{(N)})$. Since $a_1, \dots, a_n$ are germs of functions definable in $\mathfrak{R}$, we may take $n = 1$ and $a_1 = x$ by replacing F with

$$(y_1, \dots, y_{N+2}) \mapsto F(a_1(y_1), \dots, a_n(y_1), y_2, \dots, y_{N+2}).$$

The function

$$(y_1, \dots, y_{N+2}) \mapsto \begin{cases} 1/F(y_1, \dots, y_{N+2}), & F(y_1, \dots, y_{N+2}) \neq 0 \\ 0, & F(y_1, \dots, y_{N+2}) = 0 \end{cases}$$

is definable in $\mathfrak{R}$, so $1/f \in \mathfrak{H}$. By cell decomposition, there is a C^1 -cell $C \subseteq \mathbb{R}^{2+N}$ such that $F \upharpoonright C$ is C^1 and the curve $(x, g^{(0)}, \dots, g^{(N)})$ ultimately lies in C . If C is open, then ultimately $f' = H(x, g^{(0)}, \dots, g^{(N)})$, where

$$H(y_1, \dots, y_{N+2}) = \nabla F(y_1, \dots, y_{N+2}) \cdot (1, y_2, \dots, y_{N+2}, G(y_1, \dots, y_{N+2})).$$

Since H is definable in $\mathfrak{R}$, we have $f' \in \mathfrak{H}$. The case that C is not open is left as an exercise. (*Hint*: Every C^1 -cell is definably C^1 -diffeomorphic to an open C^1 -cell.)

We now show that $\mathcal{H}$, $\mathfrak{H}$ and $\widetilde{\mathcal{H}}$ are models of $\text{Th}(\mathfrak{R})$. By QE and $\forall$ -axiomatizability of $\mathfrak{R}$, substructures of models of $\text{Th}(\mathfrak{R})$ are elementary substructures, so it suffices to show that $\widetilde{\mathcal{H}} \models \text{Th}(\mathfrak{R})$; for this, it suffices by $\forall$ -axiomatizability of $\mathfrak{R}$ to show that every universal $\widetilde{L}$ -sentence true in $\mathfrak{R}$ is true in $\widetilde{\mathcal{H}}$. Let $\varphi(v_1, \dots, v_n)$ be a quantifier-free $\widetilde{L}$ -formula such that $\mathfrak{R} \models \forall v_1 \dots v_n \varphi$. Let $f_1, \dots, f_n$ be unary functions definable in $\mathfrak{R}$. Then $\mathfrak{R} \models \forall x \varphi(f_1(x), \dots, f_n(x))$, so $\mathcal{H} \models \varphi(f_1, \dots, f_n)$. Hence, $\widetilde{\mathcal{H}} \models \forall v_1 \dots v_n \varphi$.

Now, $\mathfrak{H}$ is closed under powers from K , so $v(\mathfrak{H}^*)$ is a K -linear space. As we already know that $v(\mathcal{H}^*) = K.v(x)$ by Growth Dichotomy, it follows from 4.10 and the previous paragraph that the K -linear dimension of $v(\mathfrak{H}^*)$ is at least 1 and at most $N + 2$.

Let $f_0 \in \mathfrak{H}^*$. We show there exists $r \in K$ such that $v(f_0) \geq v(\exp_{N+1}(x^r))$. (*Aside:* We shall actually need only that $\mathfrak{H}$ is a Hardy field extending $\mathbb{Q}(x)$ that is closed under powers from K and $\dim_K v(\mathfrak{H}^*) \leq N + 2$.) It suffices to consider the case that f_0 is infinitely increasing.

If $v(f'_0/f_0) \geq v(1/x)$, then $v(\log f_0) \geq v(\log x)$ by l'Hôpital, so there exists $r \in K$ such that $v(f_0) \geq v(x^r) = v(\exp_0(x^r))$, and we are done. So assume that $v(f'_0/f_0) < v(1/x)$. By 3.5 and 3.6, there exists $f_1 \in \mathfrak{H}^*$ such that $f'_1 \sim f'_0/f_0$, that is, $f_1 \sim \log f_0$. Observe that $v(f_0)$ and $v(f_1)$ are K -linearly independent (indeed, $\mathbb{R}$ -linearly independent).

If $v(f'_1/f_1) \geq v(1/x)$, then by arguing as before there exists $r \in K$ such that $v(f_1) > v(x^r)$. By increasing r , we have $v(f_0) \geq v(e^{x^r}) = v(\exp_1(x^r))$, and we are done. So assume that $v(f'_1/f_1) < v(1/x)$. By arguing as before, there exists $f_2 \in \mathfrak{H}^*$ such that $f_2 \sim \log f_1 \sim \log \log f_0$; then $\{v(f_0), v(f_1), v(f_2)\}$ is K -linearly independent.

Continuing in this fashion, we obtain $m \leq N + 1$ and $f_1, \dots, f_m \in \mathcal{H}^*$ such that $f_m \sim \log f_{m-1} \sim \dots \sim \log_m f_0$ and $v(f_m) \geq v(x^s)$ for some $s \in K$. Hence, there exists $r \in K$ such that $v(f_0) \geq v(\exp_m(x^r)) \geq v(\exp_{N+1}(x^r))$ as required. (We leave the details to the reader.) $\square$

7 Suggestions for Further Study

We have only scratched the surface in these notes.

For a proper introduction to Hardy fields, I strongly recommend the papers [19–22] by Rosenlicht.

With just a little extra work (primarily, pinning down what should be the definition of power function), the growth dichotomy can be extended to O-minimal expansions of arbitrary ordered fields; see [11] for details. An even further extension to o-minimal expansions of arbitrary ordered groups is due to Miller and Starchenko [15], but its statement and proof are considerably more involved.

The literature on analytic-geometric properties of o-minimal expansions of $\overline{\mathbb{R}}$ is now quite extensive; [34] is a good start.

See van den Dries and Speissegger [37] for a more definability-theoretic approach to (parts of) Theorem 1.4, and [35, 36] for some other interesting examples of polynomially bounded o-minimal expansions of $\overline{\mathbb{R}}$.

See [6] for a more elaborate statement of Theorem 1.7, and [14] for an exponential analogue.

One might wonder about polynomially (or exponentially) bounded expansions of $\overline{\mathbb{R}}$ that are not o-minimal, but it turns out that such structures are “almost o-minimal” in a way that can be made precise; see [3, 13] for details.

We close with just a few open questions.

If $K \neq \mathbb{Q}$ and is a subfield of $\mathbb{R}$, then QE fails for $(\overline{\mathbb{R}}, (r)_{r \in K}, ((1 + x^2)^r)_{r \in K})$ by 4.7. Does QE fail for $\overline{\mathbb{R}}^K$? (Careful!)

Let $\mathfrak{R}$ be o-minimal and polynomially bounded with field of exponents $K \neq \mathbb{R}$.

1. Does $(\mathfrak{R}, x^r \upharpoonright [1, 2])$ have field of exponents K for some $r \in \mathbb{R} \setminus K$?
2. Does $(\mathfrak{R}, x^r \upharpoonright [1, 2])$ have field of exponents K for every $r \in \mathbb{R} \setminus K$?
3. Does $(\mathfrak{R}, (x^r \upharpoonright [1, 2])_{r \in \mathbb{R} \setminus K})$ have field of exponents K ?
4. Does $(\mathfrak{R}, \exp \upharpoonright [0, 1])$ have field of exponents K ?

Evidently, $(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$. Does $(1) \Rightarrow (2)$? Does $(2) \Rightarrow (3)$? Does $(3) \Rightarrow (4)$? Is there any proper subfield K of $\mathbb{R}$ —in particular, $K = \mathbb{Q}$ —for which any of the above can be answered?

References

1. R. Bianconi, Nondefinability results for expansions of the field of real numbers by the exponential function and by the restricted sine function. *J. Symb. Log.* **62**, 1173–1178 (1997)
2. J. Denef, L. van den Dries, p -adic and real subanalytic sets. *Ann. Math. (2)* **128**, 79–138 (1988)
3. A. Dolich, C. Miller, C. Steinhorn, Structures having o-minimal open core. *Trans. Am. Math. Soc.* **362**, 1371–1411 (2010)
4. A. Hovanskiĭ, A class of systems of transcendental equations (in Russian). *Dokl. Akad. Nauk SSSR* **255**, 804–807 (1980)
5. J. Knight, A. Pillay, C. Steinhorn, Definable sets in ordered structures. II. *Trans. Am. Math. Soc.* **295**, 593–605 (1986)
6. J.-M. Lion, C. Miller, P. Speissegger, Differential equations over polynomially bounded o-minimal structures. *Proc. Am. Math. Soc.* **131**, 175–183 (2003)
7. D. Marker, *Model Theory*. Graduate Texts in Mathematics, vol. 217 (Springer, New York, 2002)
8. C. Miller, Exponentiation is hard to avoid. *Proc. Am. Math. Soc.* **122**, 257–259 (1994)
9. C. Miller, Expansions of the real field with power functions. *Ann. Pure Appl. Log.* **68**, 79–94 (1994)
10. C. Miller, Infinite differentiability in polynomially bounded o-minimal structures. *Proc. Am. Math. Soc.* **123**, 2551–2555 (1995)
11. C. Miller, A growth dichotomy for o-minimal expansions of ordered fields, in *Logic: From Foundations to Applications*. Oxford Science Publications (Oxford University Press, New York, 1996), pp. 385–399
12. C. Miller, Expansions of o-minimal structures on the real field by trajectories of linear vector fields. *Proc. Am. Math. Soc.* **139**, 319–330 (2011)
13. C. Miller, P. Speissegger, Expansions of the real line by open sets: o-minimality and open cores. *Fund. Math.* **162**, 193–208 (1999)
14. C. Miller, P. Speissegger, Pfaffian differential equations over exponential o-minimal structures. *J. Symb. Log.* **67**, 438–448 (2002)
15. C. Miller, S. Starchenko, A growth dichotomy for o-minimal expansions of ordered groups. *Trans. Am. Math. Soc.* **350**, 3505–3521 (1998)
16. A. Pillay, C. Steinhorn, Definable sets in ordered structures. I. *Trans. Am. Math. Soc.* **295**, 565–592 (1986)
17. J.-P. Rolin, Construction of o-minimal structures from quasianalytic Classes, in *Lecture Notes on O-minimal Structures and Real Analytic Geometry*. Fields Institute Communications, Springer Verlag, **62**, 70–109 (2012)
18. J.-P. Rolin, P. Speissegger, A. Wilkie, Quasianalytic Denjoy-Carleman classes and o-minimality. *J. Am. Math. Soc.* **16**, 751–777 (2003)
19. M. Rosenlicht, Hardy fields. *J. Math. Anal. Appl.* **93**, 297–311 (1983)
20. M. Rosenlicht, The rank of a Hardy field. *Trans. Am. Math. Soc.* **280**, 659–671 (1983)

21. M. Rosenlicht, Rank change on adjoining real powers to Hardy fields. *Trans. Am. Math. Soc.* **284**, 829–836 (1984)
22. M. Rosenlicht, Growth properties of functions in Hardy fields. *Trans. Am. Math. Soc.* **299**, 261–272 (1987)
23. P. Speissegger, The Pfaffian closure of an o-minimal structure. *J. Rein. Angew. Math.* **508**, 189–211 (1999)
24. P. Speissegger, Lectures on o-minimality, in *Lectures on Algebraic Model Theory*. Fields Institute Monographs, vol. 15 (American Mathematical Society, Providence, 2002), pp. 47–65
25. P. Speissegger, Pfaffian sets and o-minimality, in *Lecture Notes on O-minimal Structures and Real Analytic Geometry*. Fields Institute Communications, Springer Verlag, **62**, 178–218 (2012)
26. L. van den Dries, Remarks on Tarski's problem concerning $(\mathbf{R}, +, \cdot, \exp)$, in *Logic Colloquium '82*. Studies in Logic and the Foundations of Mathematics, vol. 112 (North-Holland, Amsterdam, 1984), pp. 97–121
27. L. van den Dries, A generalization of the Tarski-Seidenberg theorem, and some nondefinability results. *Bull. Am. Math. Soc. (N.S.)* **15**, 189–193 (1986)
28. L. van den Dries, On the elementary theory of restricted elementary functions. *J. Symb. Log.* **53**, 796–808 (1988)
29. L. van den Dries, O-minimal structures, in *Logic: From Foundations to Applications*. Oxford Science Publications (Oxford University Press, New York, 1996), pp. 137–185
30. L. van den Dries, T -convexity and tame extensions. II. *J. Symb. Log.* **62**, 14–34 (1997)
31. L. van den Dries, *Tame Topology and O-minimal Structures*. London Mathematical Society Lecture Note Series, vol. 248 (Cambridge University Press, Cambridge, 1998)
32. L. van den Dries, A. Lewenberg, T -convexity and tame extensions. *J. Symb. Log.* **60**, 74–102 (1995)
33. L. van den Dries, C. Miller, On the real exponential field with restricted analytic functions. *Isr. J. Math.* **85**, 19–56 (1994)
34. L. van den Dries, C. Miller, Geometric categories and o-minimal structures. *Duke Math. J.* **84**, 497–540 (1996)
35. L. van den Dries, P. Speissegger, The real field with convergent generalized power series. *Trans. Am. Math. Soc.* **350**, 4377–4421 (1998)
36. L. van den Dries, P. Speissegger, The field of reals with multisummable series and the exponential function. *Proc. Lond. Math. Soc. (3)* **81**, 513–565 (2000)
37. L. van den Dries, P. Speissegger, O-minimal preparation theorems, in *Model Theory and Applications*. Quaderni di Matematica, vol. 11 (Aracne, Rome, 2002), pp. 87–116
38. L. van den Dries, A. Macintyre, D. Marker, The elementary theory of restricted analytic fields with exponentiation. *Ann. Math. (2)* **140**, 183–205 (1994)
39. A. Wilkie, Model completeness results for expansions of the ordered field of real numbers by restricted Pfaffian functions and the exponential function. *J. Am. Math. Soc.* **9**, 1051–1094 (1996)

Lecture Notes on O-Minimal Structures and Real
Analytic Geometry

Miller, C.; Rolin, J.-P.; Speissegger, P. (Eds.)

2012, VIII, 244 p., Hardcover

ISBN: 978-1-4614-4041-3