Chapter 2
Lie Groups and Homogeneous Spaces

In this chapter we will give a brief introduction to Lie groups and homogeneous spaces. Today, Lie theory is an important field of mathematics with so many topics that it is impossible to explain the details in such a short survey. So we will omit many proofs of the results in this chapter. Readers who are familiar with Lie theory may skip this chapter and go directly to the following chapters.

2.1 Lie Groups and Lie Algebras

We first give the definition of Lie groups.

Definition 2.1. Let G be a smooth manifold with the structure of an abstract group. If the map

$$\rho : G \times G \to G, \quad \rho(g_1, g_2) = g_1 g_2^{-1}, \tag{2.1}$$

is smooth, then G is called a Lie group.

In the literature, it is generally required that G be an analytic manifold and the map in (2.1) be analytic. However, it can be proved that these two definitions are equivalent. More precisely, if G is a Lie group in the sense of Definition 2.1, then there exists a unique analytic structure on G such that the group operations are analytic. Setting $g_1 = e$ in (2.1), we see that the map $g \to g^{-1}$ is smooth. This then implies that the map $G \times G \to G : (g_1, g_2) \mapsto g_1 g_2$ is also smooth.

Let G be a Lie group. Given $g \in G$, it is easily seen that the left translation $L_g : G \to G$, $L_g(h) = gh$, for $h \in G$, is a diffeomorphism of G onto itself. Similarly, the right translation $R_g(h) = hg$, $h \in G$, is also a diffeomorphism. A (smooth) vector field X on G is called left-invariant if for any $g \in G$ we have $\mathrm{d}L_g(X) = X$. Similarly, we can define right-invariant vector fields. Let $\mathfrak{g}$ denote the set of all left-invariant vector fields of G. Then the map $X \to X_e$, where e is the identity element of G, is a linear isomorphism from $\mathfrak{g}$ onto $T_e(M)$. On the other hand, if X, Y are two left-invariant vector fields, then the Lie bracket $[X,Y]$ is also left-invariant. This implies

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that $\mathfrak{g}$ has the structure of a Lie algebra over the field of real numbers. Namely, there is a binary operation $[\,,]$ on $\mathfrak{g}$ that satisfies the following conditions

1. $[\,,]$ is bilinear in each of its two entries.
2. $[\,,]$ is skew-symmetric: $[X,X]=0$, for any $X \in \mathfrak{g}$.
3. $[\,,]$ satisfies the Jacobi identity: for any $X,Y,Z \in \mathfrak{g}$,

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

In general, let $\mathbb{F}$ be a field and V a vector space over $\mathbb{F}$. If there is a binary operation $[\,,]$ on V satisfying the conditions (1)–(3) above, then V is called a Lie algebra over $\mathbb{F}$. Hence in this sense $\mathfrak{g}$ is a real Lie algebra.

Since as real vector spaces, $\mathfrak{g} \simeq T_e(M)$, we have $\dim \mathfrak{g} = \dim G$. We call $\mathfrak{g}$ the Lie algebra of G.

Sometimes the Lie algebra of G will be denoted by $\mathrm{Lie}\, G$.

Definition 2.2. Let G be a Lie group. A submanifold H of G is called a Lie subgroup of G if

1. H is an abstract subgroup of G.
2. H is a topological group.

A Lie subgroup H is called a topological Lie subgroup if the topology of H is equal to the induced topology.

Note that there are examples of Lie subgroups that are not topological Lie subgroups; see for example [83].

Definition 2.3. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. A subspace $\mathfrak{h}$ is called a subalgebra of $\mathfrak{g}$ if for any $X,Y \in \mathfrak{h}$, we have $[X,Y] \in \mathfrak{h}$. If in addition $[X,Y] \in \mathfrak{h}$, for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, then $\mathfrak{h}$ is called an ideal of $\mathfrak{g}$.

Example 2.1. Let M be a vector subspace of a Lie algebra $\mathfrak{g}$. Define

$$Z_{\mathfrak{g}}(M) = \{X \in \mathfrak{g} \,|\, [X,Y]=0,\ \forall Y \in M\}.$$

Then it is easily seen that $Z_{\mathfrak{g}}(M)$ is a subalgebra of $\mathfrak{g}$. The subalgebra $Z_{\mathfrak{g}}(M)$ is called the centralizer of M in $\mathfrak{g}$. In the special case that $M=\mathfrak{g}$, $Z_{\mathfrak{g}}(\mathfrak{g})$ is an ideal of $\mathfrak{g}$. The ideal $Z_{\mathfrak{g}}(\mathfrak{g})$ is usually denoted simply by $Z(\mathfrak{g})$ and is called the center of $\mathfrak{g}$. Similarly, for a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, define

$$N_{\mathfrak{g}}(\mathfrak{h}) = \{X \in \mathfrak{g} \,|\, [X,Y] \in \mathfrak{h},\ \forall Y \in \mathfrak{h}\}.$$

Then it is easy to check that $N_{\mathfrak{g}}(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. This subalgebra is called the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$.

Now we state an important theorem on Lie subgroups and Lie subalgebras.

Theorem 2.1. *Let G be a Lie group and H a Lie subgroup of G. Then the Lie algebra $\mathfrak{h}$ of H is a subalgebra of the Lie algebra $\mathfrak{g}$ of G. Conversely, given any subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there exists a unique connected Lie subgroup H of G whose Lie algebra is $\mathfrak{h}$.*

Definition 2.4. Let G and H be two Lie groups. A map φ from G to H is called a homomorphism if φ is an abstract group homomorphism and it is continuous with respect to the topology of the groups. A homomorphism φ is called an isomorphism if φ is a homeomorphism. In case there exists an isomorphism from G onto H, we say that G is isomorphic to H.

Similarly, we can define the notion of a homomorphism between Lie algebras. Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two Lie algebras over a field F. A linear map ϕ from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ is called a homomorphism if $\phi([X,Y]) = [\phi(X), \phi(Y)]$. A homomorphism between two Lie algebras is called an isomorphism of Lie algebras if it is also a linear isomorphism. In this case, we say that the two Lie algebras are isomorphic.

It is easy to prove that if φ is a Lie group homomorphism, then its differential at the unit element, $\mathrm{d}\varphi|_e$, is a homomorphism of the Lie algebras. Usually, we denote $\mathrm{d}\varphi|_e$ simply by $\mathrm{d}\varphi$.

Now we introduce the notion of the exponential map of a Lie group. For this we first define the one-parameter subgroups of a Lie group.

Definition 2.5. A one-parameter subgroup of a Lie group G is a homomorphism from the additive group $\mathbb{R}$ to G.

The next theorem gives the existence and classification of one-parameter subgroups.

Theorem 2.2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then for any $X \in \mathfrak{g}$ there exists a unique one-parameter subgroup φ with $\dot{\varphi}(0) = X_e$.*

Proof. We first prove the existence. Since X is a vector field on G, there exists $\varepsilon > 0$ such that the maximal integral curve $\varphi(t)$ through e is defined on $(-\varepsilon, \varepsilon)$, i.e., $\varphi(t)$ is a smooth curve with $\varphi(0) = e$ and $\dot{\varphi}(t) = X_{\varphi(t)}$. Moreover, such a curve is unique where it is defined. Now we show that if (a,b) is an open interval containing 0 such that φ is defined on (a,b), and $s, t, s+t \in (a,b)$, then

$$\varphi(s)\varphi(t) = \varphi(s+t).$$

To see this, we just fix s and consider $\varphi_1(t) = \varphi(s)\varphi(t)$ and $\varphi_2(t) = \varphi(s+t)$ as two curves in t. Then $\varphi_1(0) = \varphi_2(0) = \varphi(s)$. On the other hand, we have

$$\dot{\varphi}_1(t) = \mathrm{d}L_{\varphi(s)}\dot{\varphi}(t) = \mathrm{d}L_{\varphi(s)}X_{\varphi(t)}$$

and

$$\dot{\varphi}_2(t) = X_{\varphi(s+t)}. \tag{2.2}$$

Since X is left-invariant, we have

$$\mathrm{d}L_{\varphi(s)}X_{\varphi(t)} = X_{\varphi(s)\varphi(t)}.$$

This means that both the curves φ_1 and φ_2 are integral curves of the vector field X. Since they coincide at 0, they must coincide wherever they are defined. This proves the assertion.

Now we prove that the integral curve φ is defined on the whole space $\mathbb{R}$. By the above argument, if we define $\gamma(t) = \varphi(\frac{\varepsilon}{2})\varphi(t - \frac{\varepsilon}{2})$, then $\gamma(t)$ is again an integral curve of X and it is defined on $(-\frac{3}{2}\varepsilon, \frac{3}{2}\varepsilon)$. Therefore φ can be extended to $(-\frac{3}{2}\varepsilon, \frac{3}{2}\varepsilon)$. This then implies that φ can be defined on the whole real line $\mathbb{R}$. On the other hand, (2.2) shows that φ is the required one-parameter subgroup. $\square$

Now we can define the notion of exponential map of a Lie group.

Definition 2.6. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The exponential map from $\mathfrak{g}$ to G is defined by

$$\exp(X) = \varphi_X(1),$$

where φ_X is the one-parameter subgroup determined by X.

From the definition, it is easily seen that the exponential map is smooth. In fact, it has better properties. Recall that the tangent space $T_e(G)$ is spanned by X_e, $X \in \mathfrak{g}$. Since the curve $\exp tX$ is the integral curve of X through e, we easily see that $\mathrm{d}\exp|_e = \mathrm{id}$. By the inverse function theorem, we have the following result.

Theorem 2.3. *There exists a neighborhood U of the zero vector 0 in $\mathfrak{g}$ such that $\exp$ is a diffeomorphism from U onto $\exp(U)$.*

We now collect some properties of the exponential map.

Proposition 2.1. *For $X, Y \in \mathfrak{g}$, we have*

$$\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \frac{t^2}{2}[X,Y] + \mathrm{O}(t^3)\right),$$

$$\exp(tX)\exp(tY)\exp(-tX) = \exp(tY + t^2[X,Y] + \mathrm{O}(t^3)),$$

$$\exp(tX)\exp(tY)\exp(-tX)\exp(-tY) = \exp(t^2[X,Y] + \mathrm{O}(t^3)),$$

where $\mathrm{O}(t^3)$ means a vector in $\mathfrak{g}$ satisfying the condition that there exists $\varepsilon > 0$ such that $\frac{1}{t^3}\mathrm{O}(t^3)$ is bounded and smooth for $|t| < \varepsilon$.

For the proof, see [83].

Using the exponential map, we can determine the Lie algebra of a Lie subgroup, especially when the subgroup is closed. We first recall the following result.

Theorem 2.4. *Let G be a Lie group and H an abstract subgroup of G. If H is a closed subset, then there exists a unique differential structure on H such that H is a topological subgroup of G.*

We refer the proof of this theorem to [83]. The following theorem gives an effective method to compute the Lie algebra of a Lie subgroup.

Theorem 2.5. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and H a Lie subgroup of G with Lie algebra $\mathfrak{h}$. If H is a topological Lie subgroup, or H has at most countably many connected components, then*

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp(tX) \in H, \ \forall t \in \mathbb{R}\}.$$

Example 2.2. Let $G = \mathrm{GL}(n,\mathbb{R})$ be the set of all $n \times n$ nonsingular matrices. Obviously G has the structure of an abstract group under the usual matrix multiplication. Note that G can also be viewed as an open subset of the Euclidean space $\mathbb{R}^{n^2}$, and hence it has the structure of a smooth manifold. It can be easily checked that the group operations are smooth with respect to the manifold structure; hence G is a Lie group, called the *real general linear group*. It is not hard to compute its Lie algebra:

$$\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R}) = \mathbb{R}^{n \times n},$$

with Lie brackets

$$[X,Y] = XY - YX.$$

This Lie algebra is called the *real general linear Lie algebra*. The exponential map of G is

$$\exp(X) = \mathrm{e}^X = I_n + X + \frac{1}{2}X^2 + \cdots + \frac{1}{n!}X^n + \cdots.$$

In general, let V be an n-dimensional real vector space. Then the set of all invertible linear transformations forms a Lie group isomorphic to $\mathrm{GL}(n,\mathbb{R})$. In this case, we usually denote the Lie group by $\mathrm{GL}(V)$.

Now we consider the following subgroups of G:

1. The special linear group $\mathrm{SL}(n,\mathbb{R})$ consisting of all elements in G with determinant 1.
2. The orthogonal group $\mathrm{O}(n)$ consisting of the orthogonal matrices in G.
3. The special orthogonal group $\mathrm{SO}(n)$ consisting of the orthogonal matrices with determinant 1.

As in the case of general Lie groups, we sometimes write the special linear group as $\mathrm{SL}(V)$, where V is a real vector space. Moreover, if V is an n-dimensional Euclidean space, then the set of all orthogonal transformations of V form a Lie group isomorphic to $\mathrm{O}(n)$. This group will usually be denoted by $\mathrm{O}(V)$. Similarly, we have the notation $\mathrm{SO}(V)$. The Lie algebras of these Lie groups will be denoted by corresponding notation.

By Theorem 2.5, we easily get the Lie algebras of these Lie subgroups:

1. The Lie algebra of $\mathrm{SL}(n,\mathbb{R})$ is $\mathfrak{sl}(n,\mathbb{R})$, consisting of all the traceless matrices in $\mathfrak{g}$. This Lie algebra is called the special linear Lie algebra.
2. The Lie algebras of $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are equal. It is $\mathfrak{so}(n)$, consisting of all the skew-symmetric matrices in $\mathfrak{g}$. This Lie algebra is called the orthogonal Lie algebra.

Example 2.3. The complex version of Example 2.2 is more interesting and useful. Let $\mathrm{GL}(n,\mathbb{C})$ denote the set of all invertible complex $n\times n$ matrices. It is a complex manifold. Moreover, it is easily seen that the map $(g_1,g_2)\mapsto g_1g_2^{-1}$ is holomorphic. In general, if G is a complex manifold as well as an abstract group such that the map $G\times G\to G$, $(g_1,g_2)\mapsto g_1g_2^{-1}$ is holomorphic, then G is called a complex Lie group. A complex Lie group is automatically a (real) Lie group. Hence $\mathrm{GL}(n,\mathbb{C})$ is a Lie group.

The Lie algebra of $\mathrm{GL}(n,\mathbb{C})$ is $\mathfrak{gl}(n,\mathbb{C})$, the complex general linear Lie algebra, consisting of all complex $n\times n$ matrices, with Lie brackets $[A,B]=AB-BA$. One can also define the similar Lie subgroups and determine their Lie algebras as in the real case. We list some of the examples below:

1. The complex special linear group $\mathrm{SL}(n,\mathbb{C})$. It is the subgroup of $\mathrm{GL}(n,\mathbb{C})$ consisting of the matrices with determinant 1. Its Lie algebra is $\mathfrak{sl}(n,\mathbb{C})$, consisting of all the complex $n\times n$ matrices with zero trace.
2. The unitary group $\mathrm{U}(n)$. It is the subgroup of $\mathrm{GL}(n,\mathbb{C})$ consisting of all unitary matrices. Its Lie algebra is $\mathfrak{u}(n)$, consisting of all the skew-Hermitian matrices.
3. The special unitary group $\mathrm{SU}(n)=\mathrm{SL}(n,\mathbb{C})\cap\mathrm{U}(n)$. Its Lie algebra is $\mathfrak{su}(n)$, consisting of all the traceless skew-Hermitian matrices.
4. The complex symplectic group $\mathrm{Sp}(n,\mathbb{C})$. It is the group of matrices g in $\mathrm{GL}(2n,\mathbb{C})$ satisfying the condition

$$g^tJ_ng=J_n,$$

where

$$J_n=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}.$$

Its Lie algebra is

$$\mathfrak{sp}(n,\mathbb{C})=\left\{\begin{pmatrix}Z_1 & Z_2\\ Z_3 & -Z_1^t\end{pmatrix}\,\middle|\,Z_1,Z_2,Z_3\,\text{complex}\,n\times n\,\text{matrices},Z_2,Z_3\,\text{symmetric}\right\}.$$

We put $\mathrm{Sp}(n)=\mathrm{Sp}(n,\mathbb{C})\cap\mathrm{U}(2n)$ and call it the symplectic group. Note that in some books the definition of symplectic group is different from the above. We leave as an exercise for the reader to determine the Lie algebra of $\mathrm{Sp}(n)$.

Note that the Lie groups $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are not simply connected. Their universal covering groups are denoted by $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$ (called the spin group), respectively. We now recall briefly the construction of these two groups. For the details, we refer the reader to [34, Sect. 1.6].

Example 2.4 (See [12]). Let V be a real inner product space. The Clifford algebra $\mathrm{Cliff}(V)$ is the associative algebra freely generated by V modulo the relations

$$vw+wv=-2(v,w),\quad v,w\in V.$$

If $V = \mathbb{R}^n$ is the standard Euclidean space, then $\mathrm{Cliff}(V)$ is generally denoted by $\mathrm{Cliff}(n)$. For example, $\mathrm{Cliff}(0) = \mathbb{R}$, $\mathrm{Cliff}(1) = \mathbb{C}$, $\mathrm{Cliff}(2) = \mathbb{H}$ (the quaternions). Now consider $\mathrm{Cliff}(n)$. Let $\mathrm{Pin}(n)$ be the group sitting inside $\mathrm{Cliff}(n)$ consisting of all the products of the unit elements in $\mathbb{R}^n$. Recall that each element in $\mathrm{O}(n)$ can be written as a finite product of reflections. Since each unit element v in $\mathbb{R}^n$ induces a reflection transformation τ_v of $\mathbb{R}^n$, and $\pm v$ induce the same transformation, the group $\mathrm{Pin}(V)$ is a double covering of $\mathrm{O}(n)$. Let $\mathrm{Spin}(n) \subset \mathrm{Pin}(n)$ be the subgroup of the elements that are products of an even number of unit elements in $\mathbb{R}^n$. Then $\mathrm{Spin}(n)$ is a double covering of $\mathrm{SO}(n)$. It is not difficult to prove that $\mathrm{Spin}(n)$ is simply connected.

In the final part of this section, we introduce some terminology in representation theory. Since in this book we will use only some results on finite-dimensional representations, we shall consider only the finite-dimensional cases. Let G be a Lie group and V a finite-dimensional real vector space. A representation of G on V is a continuous homomorphism ρ from G to the general linear group $\mathrm{GL}(V)$. We say that (V,ρ) is a representation of G, or a G-module. A representation (V,ρ) of G is called faithful if the homomorphism ρ has trivial kernel.

Similarly, let $\mathfrak{g}$ be a Lie algebra over the field F and let V be a vector space over F. A Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{gl}(V)$ is called a representation of $\mathfrak{g}$.

It is clear that if ρ is a representation of a Lie group G on the vector space V, then its differential $\mathrm{d}\rho$ is a representation of its Lie algebra $\mathfrak{g}$. The converse is generically not true. But if G is a connected simply connected Lie group, then every representation of $\mathfrak{g}$ can be lifted to a representation of G.

For a Lie group G with Lie algebra $\mathfrak{g}$, we have a natural representation of G on $\mathfrak{g}$. Note that for $X \in \mathfrak{g}$, $\exp(tX)$ is a one-parameter subgroup of G. Given $g \in G$, it is easy to check that $g(\exp(tX))g^{-1}$ is again a one-parameter subgroup. Hence there exists a unique $\widetilde{X}_g \in \mathfrak{g}$ such that $g(\exp(tX))g^{-1} = \exp(t\widetilde{X}_g)$. We denote $\widetilde{X}_g$ by $\mathrm{Ad}(g)(X)$. It is easy to check that the map $G \to \mathrm{GL}(\mathfrak{g})$, $g \to \mathrm{Ad}(g)$ is a representation of G on $\mathfrak{g}$. This representation is called the adjoint representation of G. The differential of the adjoint representation of the Lie group is called the adjoint representation of the Lie algebra.

Let G be a Lie group and (V,ρ) a real representation of G. Suppose V is endowed with an inner product $\langle\, ,\rangle$. If

$$\langle \rho(g)(X), \rho(g)(Y)\rangle = \langle X, Y\rangle, \quad \forall X, Y \in V, \quad g \in G,$$

then we say that $\langle\, ,\rangle$ is an invariant inner product, and $(V,\rho,\langle\, ,\rangle)$ is an orthogonal representation of G.

A Lie group G is called compact if G is a compact manifold. The following theorem is called Weyl's unitary trick.

Theorem 2.6. *Let G be a connected compact Lie group and (V,ρ) a representation of G. Then there exists an invariant inner product on V.*

2.2 Lie Transformation Groups and Coset Spaces

In this section we will introduce the fundamental properties of Lie group actions on smooth manifolds. The general idea is to consider this problem originating from Felix Klein's Erlangen Program proposed in 1872. Klein's proposal was to categorize the new geometries by their characteristic groups of transformations. This program has been extremely successful. Today, it is the expectation of each geometer to find an effective method to study geometry by applying group theory.

We first define the notion of the action of a Lie group on a manifold. Let G be a Lie group and M a smooth manifold. A smooth action of G on M is a map ρ from $G \times M$ onto M satisfying the following two conditions:

1. ρ is a smooth map.
2. $\rho(g_2, \rho(g_1, x)) = \rho(g_2 g_1, x)$, $\forall g_1, g_2 \in G, x \in M$.

If we write $\rho(g,x)$ as $g \cdot x$, then condition (2) can be rewritten as $g_2 \cdot (g_1 \cdot x) = (g_2 g_1) \cdot x$. Meanwhile, this condition implies that $e \cdot x = x$ (note that ρ is assumed to be onto). This then implies that for each $g \in G$, the map $x \to g \cdot x$ is a diffeomorphism of M.

If G has a smooth action on M, then G is called a Lie transformation group of M. The action is called effective (resp. almost effective) if e is the only element in G such that $\rho(e)$ is the identity map (resp. if the subgroup $\rho^{-1}(\mathrm{id})$ of G is discrete). It is called free if for any $g \neq e$ in G, the map $\rho(g)$ has no fixed point.

The most important action is the action of a Lie group on the coset spaces. We have the following theorem.

Theorem 2.7. *Let G be a Lie group and H a closed subgroup of G. Then there exists a unique differentiable structure on the left coset space G/H with the induced topology that turns G/H into a smooth manifold such that G is a Lie transformation group of G/H.*

For the proof, see [83]. Note that the induced topology on G/H is the unique topology such that the natural projection π from G onto G/H is both continuous and open. More precisely, a subset U in G/H is open if and only if the preimage $\pi^{-1}(U)$ is an open subset of G.

Suppose G is a Lie group acting transitively on a smooth manifold M. Given $p \in M$, denote by G_p the set of elements of G that keep p fixed. Then G_p is a closed subgroup of G. If we endow G/G_p with the induced topology, then Theorem 2.7 asserts that there exists a unique differentiable structure such that G is a Lie transformation group of G/G_p. Let α be the map from G onto M defined by $\alpha(g) = g \cdot p$. Then α is a diffeomorphism if it is a homeomorphism (see [83]). On the other hand, if G has a countable base (this is the case if G has countably many connected components), then α must be a homeomorphism. Hence in this case G/G_p is diffeomorphic to M. The following proposition is very useful in practice.

Proposition 2.2. *Let G be a Lie group acting transitively on a connected manifold M. Fix $p \in M$ and let α be the map from G/G_p onto M defined by $\alpha(g) = g \cdot p$. If α*

is a homeomorphism, then the unit connected component G_0 is also transitive on M. In particular, if G has countably many connected components, then G_0 is transitive on M.

Let G be a Lie group and H a closed subgroup of G. In the following we will always endow G/H with the differentiable structure so that G is a Lie transformation group of G/H. In general, H will be called the isotropy subgroup. The tangent space of G/H at the origin $eH = H$ of G/H can be identified with the quotient vector space $\mathfrak{g}/\mathfrak{h}$. Since the adjoint action of the group H keeps $\mathfrak{h}$ invariant, H has an action on $\mathfrak{g}/\mathfrak{h}$ defined by

$$\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)(X+\mathfrak{h}) = \mathrm{Ad}(h)(X)+\mathfrak{h},$$

where Ad is the adjoint action of G. This action defines a representation of H on $\mathfrak{g}/\mathfrak{h}$ and is called the linear isotropy representation.

In some special cases there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h}+\mathfrak{m} \quad \text{(direct sum of subspaces)} \tag{2.3}$$

and

$$\mathrm{Ad}(h)(\mathfrak{m}) \subset \mathfrak{m}, \quad \forall h \in H.$$

Then the coset space G/H is called a reductive homogeneous manifold and (2.3) is usually called a reductive decomposition of $\mathfrak{g}$. In this case, the tangent space $T_{eH}(G/H)$ of G/H at the origin eH can be identified with $\mathfrak{m}$ through the map

$$X \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\exp(tX)H\Big|_{t=0}, \quad X \in \mathfrak{m}.$$

Then the linear isotropy representation corresponds to the adjoint action of H on $\mathfrak{m}$. For the geometry of reductive homogeneous space, we refer the reader to [102].

Next we introduce some notions about the adjoint group of a Lie algebra. Let $\mathfrak{g}$ be a real Lie algebra. Since $\mathfrak{g}$ is a real vector space, we have the general linear group $\mathrm{GL}(\mathfrak{g})$. Note that a priori, $\mathrm{GL}(\mathfrak{g})$ has no relation with the Lie algebra structure of $\mathfrak{g}$. As pointed out in Sect. 2.1, the Lie algebra of the Lie group $\mathrm{GL}(\mathfrak{g})$ is $\mathfrak{gl}(\mathfrak{g})$, consisting of all the linear endomorphisms of $\mathfrak{g}$ with Lie brackets $[A,B] = AB-BA$. Now for each $X \in \mathfrak{g}$, we can define a linear endomorphism $\mathrm{ad}(X)$ of $\mathfrak{g}$ by

$$\mathrm{ad}(X)(Y) = [X,Y], \quad Y \in \mathfrak{g}.$$

It is obvious that the endomorphisms $\mathrm{ad}(X)$, $X \in \mathfrak{g}$, form a subalgebra of $\mathfrak{gl}(\mathfrak{g})$. We denote this subalgebra by $\mathrm{ad}(\mathfrak{g})$. By Theorem 2.1, there exists a unique connected Lie subgroup of $\mathrm{GL}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$. This Lie group is called the adjoint group of $\mathfrak{g}$ and will be denoted by $\mathrm{Int}\,\mathfrak{g}$.

There is another Lie group that is closely related to the Lie algebra structure of $\mathfrak{g}$. A linear isomorphism σ of $\mathfrak{g}$ is called an automorphism of $\mathfrak{g}$ if

$$\sigma([X,Y]) = [\sigma(X),\sigma(Y)], \quad \forall X,Y \in \mathfrak{g}.$$

It is clear that the set of all the automorphisms of $\mathfrak{g}$ is a closed subgroup of $GL(\mathfrak{g})$. Hence it is a topological Lie subgroup of $GL(\mathfrak{g})$. This group is called the automorphism group of $\mathfrak{g}$ and will be denoted by $\mathrm{Aut}(\mathfrak{g})$. The Lie algebra of $\mathrm{Aut}(\mathfrak{g})$ is denoted by $\partial(\mathfrak{g})$. It is easy to prove that $\partial(\mathfrak{g})$ consists of all the linear endomorphisms of $\mathfrak{g}$ satisfying

$$D([X,Y]) = [D(X),Y] + [X,D(Y)], \quad \forall X,Y \in \mathfrak{g}. \tag{2.4}$$

A linear endomorphism satisfying (2.4) is called a derivation. It is easily seen that for a derivation D, e^{tD} is an automorphism of $\mathfrak{g}$. Hence $\partial(\mathfrak{g})$ consists of all the derivations of $\mathfrak{g}$. By the Jacobi identity we see that for $X \in \mathfrak{g}$, $\mathrm{ad}(X)$ is a derivation. Thus $\mathrm{ad}(\mathfrak{g})$ is a subalgebra of $\partial(\mathfrak{g})$ and $\mathrm{Int}(\mathfrak{g})$ is a subgroup of $\mathrm{Aut}\,\mathfrak{g}$. Further, one easily checks that for any automorphism σ of $\mathfrak{g}$ and any $X \in \mathfrak{g}$ we have, as endomorphisms of $\mathfrak{g}$,

$$\sigma e^{\mathrm{ad(X)}} \sigma^{-1} = e^{\mathrm{ad}(\sigma(\mathrm{X}))}.$$

Since $\mathrm{Int}\,\mathfrak{g}$ is connected, it is generated by the elements of the form $e^{\mathrm{ad(X)}}$, $X \in \mathfrak{g}$. The above identity then implies that $\mathrm{Int}(\mathfrak{g})$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{g})$.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. For $g \in G$, $\mathrm{Ad}(g)$ is an automorphism of $\mathfrak{g}$. We assert that $\mathrm{Ad}(g)$ lies in $\mathrm{Int}(\mathfrak{g})$. In fact, this follows from the facts that G is generated by the elements of the form $\exp(X)$, $X \in \mathfrak{g}$, and that

$$\mathrm{Ad}(\exp(X)) = e^{\mathrm{ad}(X)}.$$

(The proof of the above identity is left to the reader). This means that Ad defines a map from G onto $\mathrm{Int}(\mathfrak{g})$. In fact we have the following.

Proposition 2.3. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then*

1. *The map $g \to \mathrm{Ad}(g)$ is a homomorphism from G onto $\mathrm{Int}(\mathfrak{g})$ with kernel $Z(G)$.*
2. *The map $gZ(G) \to \mathrm{Ad}(g)$ is an isomorphism from $G/Z(G)$ onto $\mathrm{Int}(\mathfrak{g})$.*

Now we define the notions of compactly embedded Lie algebras and compact Lie algebras. This notion is very useful in the study of homogeneous Riemannian and Finsler manifolds.

Definition 2.7. Let $\mathfrak{g}$ be a real Lie algebra. A subalgebra $\mathfrak{k}$ is called a compactly embedded subalgebra of $\mathfrak{g}$ if the connected Lie subgroup K^* of $\mathrm{Int}(\mathfrak{g})$, corresponding to the subalgebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$ of $\mathrm{ad}(\mathfrak{g})$, is a compact Lie group. A real Lie algebra is called compact if it is a compactly embedded subalgebra of itself.

As an example, any abelian real Lie algebra $\mathfrak{g}$ is compact, since in this case its adjoint Lie group $\mathrm{Int}(\mathfrak{g})$ consists of the single element $\{e\}$, which is compact. It can be proved that a real Lie algebra $\mathfrak{g}$ is compact if and only if there is a compact Lie group G whose Lie algebra is $\mathfrak{g}$; see [83].

At the final part of this section, we introduce some terminology concerning the orbits of general Lie group actions on smooth manifolds, which need not be transitive. In particular, we will recall the definition of principal orbits. Let M be a (connected) smooth n-dimensional manifold and G a Lie group acting smoothly on M. For $x \in M$, the orbit $G \cdot x = \{g(x) \mid g \in G\}$ is a submanifold of M and if the action is proper, namely, the inverse image of every compact subset of $M \times M$ under the map

$$G \times M \to M \times M : \quad (g, p) \mapsto (p, g(p))$$

is compact, then every orbit is a closed submanifold of M. It is easily seen that if G is a compact Lie group, then the action is proper. If the action is proper, then the space of orbits is a Hausdorff space and can be endowed with a differentiable structure.

If G acts transitively on M, then M is diffeomorphic to G/G_x, $\forall x \in M$. In this case, the action has only one orbit. When the action is nontransitive, the orbit structure is in general very complicated. One important tool for studying the orbit structure is the introduction of the notion of a slice of a proper action.

Definition 2.8 (see [122]). Let G be a Lie group acting smoothly and properly on the manifold M and $p \in M$. Then a slice of the action at p is a submanifold Σ satisfying the conditions

1. $p \in \Sigma$.
2. $G \cdot \Sigma = \{g(q) \mid g \in G, q \in \Sigma\}$ is an open submanifold of M.
3. $G_p \cdot \Sigma = \Sigma$.
4. The action of G_p on Σ is isomorphic to an orthogonal linear action of G_p on an open ball of a Euclidean space.
5. Let $(G \times \Sigma)/G_p$ be the orbit space of the action of G_p on $G \times \Sigma$ given by $k((g,q)) = (gk^{-1}, k(q))$, $k \in G_p$, $g \in G$, $q \in \Sigma$. Then the map

$$(G \times \Sigma)/G_p \to M : \quad G_p \cdot (g,q) \mapsto g(q)$$

is a diffeomorphism onto $G \cdot \Sigma$.

It was proved by Montgomery and Yang that every proper action admits a slice at each point [122]. This fact enables us to define a partial ordering on the orbit space. Given $p, q \in M$, we say that the two orbits $G \cdot p$ and $G \cdot q$ have the same orbit type if the isotropic subgroups G_p and G_q are conjugate in G. This defines an equivalence relation among the orbits of G. Denote the orbit type of $G \cdot p$ by $[G \cdot p]$. Then we can introduce a partial ordering on the set of orbit types by saying that $[G \cdot p] \leq [G \cdot q]$ if and only if G_q is conjugate in G to some subgroup of G_q. If the orbit space is connected, then there exists an orbit type that is the largest among all the orbit types. Each representative of the largest orbit type will be called a principal orbit. Each principal orbit has the maximal dimension among all the orbits. Note that there may be some orbit that is of maximal dimension but not principal. The following result is useful.

Proposition 2.4 (see [122]). *Let G be a connected Lie group acting smoothly and properly on a connected manifold M. Then the union of the principal orbits is an open and dense subset of M.*

2.3 Semisimple Lie Algebras

In this section we will present a survey of the main results on the structure and classification of real and complex semisimple Lie algebras. This is the basis for the classification of Riemannian symmetric spaces. We first recall some notions on the solvability and nilpotency of Lie algebras.

Definition 2.9. A Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is called solvable if there exists a positive integer m such that $\mathfrak{g}^{(m)} = 0$, where $\mathfrak{g}^{(1)} = \mathfrak{g}$ and $\mathfrak{g}^{(l+1)} = [\mathfrak{g}^{(l)}, \mathfrak{g}^{(l)}]$, for $l \geq 1$.

Definition 2.10. A Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is called nilpotent if there exists a positive integer m such that $\mathfrak{g}^m = 0$, where $\mathfrak{g}^1 = \mathfrak{g}$ and $\mathfrak{g}^{l+1} = [\mathfrak{g}^l, \mathfrak{g}]$, for $l \geq 1$. The minimal integer m satisfying the condition $\mathfrak{g}^m = 0$ is called the nilpotent index of $\mathfrak{g}$.

Note that here $\mathbb{F}$ can be any field. It is obvious that any nilpotent Lie algebra must be solvable. The converse is not true, as can be seen from the 2-dimensional solvable Lie algebra. The main tools to study solvable and nilpotent Lie algebras are the following Engel's theorem and Lie's theorem.

Recall that an element x in a Lie algebra $\mathfrak{g}$ is called ad-nilpotent if there exists a natural number n such that $[\mathrm{ad}(x)]^n = 0$. The Lie algebra $\mathfrak{g}$ is called ad-nilpotent if all its elements are ad-nilpotent.

Theorem 2.8 (Engel's theorem). *A finite-dimensional Lie algebra is nilpotent if and only if it is* ad*-nilpotent.*

The following is an equivalent version of this result.

Theorem 2.9 (Engel's theorem). *Let V be a nonzero finite-dimensional vector space and L a subalgebra of $\mathfrak{gl}(V)$. If L consists of nilpotent endomorphisms of V, then there exists a nonzero element v in V such that $A(v) = 0$, for all $A \in L$.*

Theorem 2.10 (Lie's theorem). *Let V be a nonzero finite-dimensional vector space over an algebraic closed field $\mathbb{F}$ of characteristic 0. Suppose L is a solvable subalgebra of $\mathfrak{gl}(V)$. Then there exists a common eigenvector for all the endomorphisms in L.*

Theorem 2.11 (Cartan's criterion). *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over an algebraically closed field of characteristic 0. Then $\mathfrak{g}$ is solvable if and only if for every $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$, $\mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y)) = 0$.*

The Killing form of the Lie algebra $\mathfrak{g}$ is the bilinear function

$$B(x,y) = \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y)), \quad x, y \in \mathfrak{g}.$$

Then Cartan's criterion can be restated that $\mathfrak{g}$ is solvable if and only if

$$B([\mathfrak{g}, \mathfrak{g}], \mathfrak{g}) = 0.$$

The Killing form is symmetric with respect to its two entries and it is easy to check (using the Jacobi identity) that the following identity holds:

$$B([x,y],z)+B(y,[x,z])=0, \quad \forall x,y,z \in \mathfrak{g}.$$

Definition 2.11. A Lie algebra $\mathfrak{g}$ is called semisimple if it has no nonzero abelian ideal.

We can also define a semisimple Lie algebra as the following. Note that the sum of two solvable ideals of a Lie algebra is still a solvable ideal. This fact enables us to define the radical of a Lie algebra as the maximal solvable ideal of it. Then a Lie algebra is semisimple if and only if its radical is zero.

From now on we will consider only real or complex Lie algebras.

Theorem 2.12. *Let $\mathfrak{g}$ be a real or complex Lie algebra. Then $\mathfrak{g}$ is semisimple if and only if the Killing form is nondegenerate.*

Theorem 2.13. *Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra. Then each derivation of $\mathfrak{g}$ is inner.*

Recall that a derivation of a Lie algebra $\mathfrak{g}$ is a linear endomorphism D of $\mathfrak{g}$ such that

$$D([x,y])=[D(x),y]+[x,D(y)], \quad \forall x,y \in \mathfrak{g}.$$

It is called inner if there exists $z \in \mathfrak{g}$ such that $D=\mathrm{ad}(z)$.

A Lie algebra $\mathfrak{g}$ is called simple if $\mathfrak{g}$ has no ideal other than $\{0\}$ and $\mathfrak{g}$ itself. It can be easily proved that a real or complex simple Lie algebra must be semisimple. We have the following result.

Theorem 2.14. *Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra. Then $\mathfrak{g}$ has a decomposition*

$$\mathfrak{g}=\mathfrak{g}_1+\mathfrak{g}_2+\cdots+\mathfrak{g}_k,$$

where $\mathfrak{g}_i$, $1 \leq i \leq k$, are simple ideals of $\mathfrak{g}$. Moreover, the decomposition is unique up to an adjustment of the order of the simple ideals.

This theorem reduces the classification of semisimple Lie algebras to that of the simple ones. In the following we will consider the classification of complex simple Lie algebras. We begin with the structure theorems for semisimple Lie algebras.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. An element $x \in \mathfrak{g}$ is called semisimple if $\mathrm{ad}(x)$ is a semisimple endomorphism of $\mathfrak{g}$ (i.e., it has a diagonal matrix under a certain basis of $\mathfrak{g}$). It is called nilpotent if $\mathrm{ad}(x)$ is a nilpotent endomorphism. The Jordan–Chevalley decomposition theorem asserts that each endomorphism can be uniquely written as the sum of a semisimple and a nilpotent endomorphism that commute with each other. Now given $x \in \mathfrak{g}$, the endomorphism $\mathrm{ad}(x)$ can be decomposed into the sum of two endomorphisms: $\mathrm{ad}(x)=s+n$, where s is semisimple and n is nilpotent with $[s,n]=sn-ns=0$. It can be easily checked that s and n are also derivatives of the Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, each derivative is inner. Thus there exist a semisimple element x_s and a nilpotent element x_n such that $s = \mathrm{ad}(x_s)$ and $n = \mathrm{ad}(x_n)$, with $[\mathrm{ad}(x_s), \mathrm{ad}(x_n)] = 0$. Then we have $x = x_s + x_n$ and $[x_s, x_n] = 0$. We usually call this decomposition the Jordan–Chevalley decomposition of x.

A subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is called a toral subalgebra of $\mathfrak{g}$ if $\mathfrak{t}$ consists of semisimple elements. It is easily seen that a toral subalgebra of a complex semisimple Lie algebra must be abelian.

Theorem 2.15. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}$ a maximal toral subalgebra of $\mathfrak{g}$. Then $\mathfrak{g}$ has a decomposition*

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha,$$

where Δ is a subset of $\mathfrak{h}^*\setminus\{0\}$ and

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h,x] = \alpha(h)x,\ \forall h \in \mathfrak{h}\}.$$

The decomposition has the following properties:

1. *The span of Δ is $\mathfrak{h}^*$.*
2. *For any $\alpha \in \Delta$,* $\dim \mathfrak{g}_\alpha = 1$.
3. *If $\alpha \in \Delta$, then $-\alpha \in \Delta$. Moreover, if $\alpha \in \Delta$ and c is a nonzero number such that $c\alpha \in \Delta$, then $c = \pm 1$.*
4. *If $\alpha, \beta \in \Delta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ (here $\mathfrak{g}_\gamma = 0$ if $\gamma \notin \Delta \cup \{0\}$).*
5. *The restriction of the Killing form B to $\mathfrak{h}$ is nondegenerate.*
6. *Fix $\alpha \in \Delta$ and let $t_\alpha \in \mathfrak{h}$ be the unique element satisfying $\alpha(h) = B(t_\alpha, h)$, $\forall h \in \mathfrak{h}$. Then $B(t_\alpha, t_\alpha) \neq 0$, and for $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$, $[x_\alpha, y_\alpha] = B(x_\alpha, y_\alpha)t_\alpha$.*
7. *Fix $\alpha \in \Delta$. For any nonzero $x_\alpha \in \mathfrak{g}_\alpha$, there exists $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $x_\alpha, y_\alpha, h_\alpha = [x_\alpha, y_\alpha]$ span a 3-dimensional Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. Moreover, h_α is independent of the choice of x_α and y_α, and $h_\alpha = \frac{2t_\alpha}{B(t_\alpha,t_\alpha)}$, $h_\alpha = -h_{-\alpha}$.*

The subset Δ of $\mathfrak{h}^*$ in the above theorem is called the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. An element $\alpha \in \Delta$ is called a root. For $\alpha, \beta \in \Delta$, we define

$$(\alpha, \beta) = B(t_\alpha, t_\beta).$$

Then we have the following theorem.

Theorem 2.16. *Let $\alpha, \beta \in \Delta$. Then*

1. *The number $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ is an integer.*
2. $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Delta$.
3. *The elements of the form $\beta + l\alpha$, $l \in \mathbb{Z}$, contained in Δ constitute a continuous string. Let q, r be the largest integers such that $\beta + q\alpha \in \Delta$ and $\beta - r\alpha \in \Delta$. Then $q - r = -\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.*

The set of the roots $\beta + l\alpha$, $-r \le l \le q$, is called the α-string through β. Select a basis $\alpha_1, \alpha_2, \ldots, \alpha_m$ of $\mathfrak{h}^*$ such that $\alpha_i \in \Delta$. Then the bilinear form $(\alpha, \beta) = B(t_\alpha, t_\beta)$ defined above can be extended to an inner product of the real vector space

$$E = \sum_{i=1}^{m} \mathbb{R}\alpha_i.$$

Then Δ is a root system in the Euclidean space E in the following sense.

Definition 2.12. Let E be a finite-dimensional Euclidean space, with inner product $(\,,\,)$. A subset Φ of E is called a root system if the following axioms are satisfied:

1. Φ is a finite subset spanning E and it does not contain 0.
2. $\alpha \in \Phi$ and $c\alpha \in \Phi$, $c \in \mathbb{R}$, implies that $c = \pm 1$.
3. For any $\alpha \in \Phi$, the reflection σ_α defined by α leaves the set Φ invariant.
4. For any $\alpha, \beta \in \Phi$, the number

$$\langle \alpha, \beta \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

   is an integer.

Let Φ be a root system in E. The group generated by all the reflections σ_α, $\alpha \in \Phi$, is a finite group of isometric transformations of E. This group is called the Weyl group of Φ and will be denoted by W. Using the axioms of a root system, we can easily deduce the following.

Proposition 2.5. *Let Φ be a root system in E. Then*

1. *Suppose $\alpha, \beta \in \Phi$, $\alpha \neq \pm\beta$. If $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi$. If $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Phi$.*
2. *For $\alpha, \beta \in \Phi$, the set of roots of the form $\beta + i\alpha$ is a unbroken string, called the α-string through β.*
3. *Let q, r be respectively the largest integers such that $\beta + q\alpha \in \Phi$ and $\beta - r\alpha \in \Phi$. Then $r - q = -\langle \beta, \alpha \rangle = -\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.*

A subset Π of Φ is called a base if Π is a basis of the vector space E and each root β of Φ can be uniquely written as the sum $\beta = \sum_{\alpha \in \pi} k_\alpha \alpha$, where k_α are integers that are either all nonnegative or all nonpositive. An element in Π is called a simple root. It is a fundamental result that every root system has a base. Now we introduce a method to construct a base. Given $\alpha \in \Phi$, set

$$P_\alpha = \{x \in E | \alpha(x) = 0\}.$$

An element $\gamma \in E \setminus \cup_{\alpha \in \Phi} P_\alpha$ is called regular. For a regular element γ, define

$$\Phi^+(\gamma) = \{\alpha \in \Phi \mid (\gamma, \alpha) > 0\},\ \Phi^-(\gamma) = \{\alpha | (\gamma, \alpha) < 0\}.$$

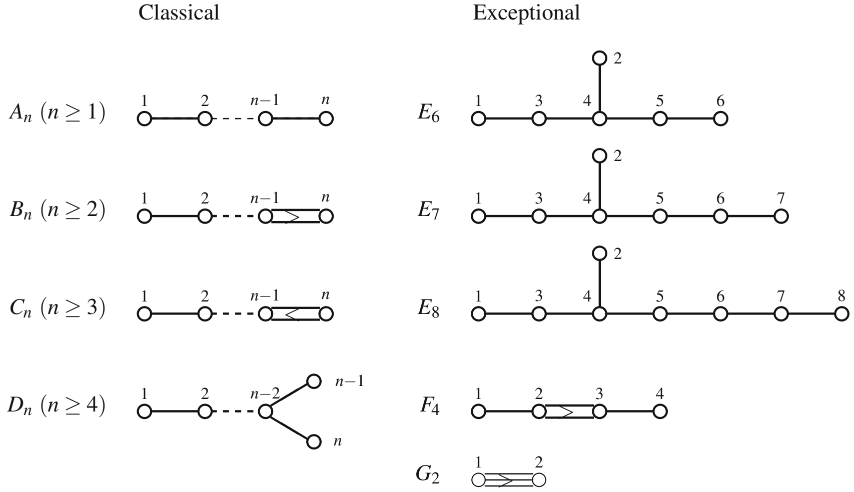

Fig. 2.1 Dynkin diagrams

A root in $\Phi^+(\gamma)$ is called indecomposable if it cannot be written as the sum of two roots in $\Phi^+(\gamma)$.

Theorem 2.17. *Let γ be a regular element in E. Then the set of all indecomposable roots in $\Phi^+(\gamma)$ is a base of Φ. Moreover, every base of Φ can be obtained in this manner.*

A connected component of $E \setminus \cup_{\alpha\in\Phi} P_\alpha$ is called a Weyl chamber. It is clear that each regular element of E lies in exactly one Weyl chamber. There is a one-to-one correspondence between the Weyl chambers and the bases of Φ. The Weyl group can be viewed as a permutation group of the set of Weyl chambers. In fact, the action of the Weyl group on the set of Weyl chambers is simply transitive.

Let Φ be a root system and $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ a base of Φ. The Dynkin diagram of Π is a diagram with l vertices such that the ith is joined to the jth by $\langle\alpha_j, \alpha_i\rangle\langle\alpha_i, \alpha_j\rangle$ edges. Moreover, if the lengths $|\alpha_i|$ and $|\alpha_j|$ are not equal, then we add an arrow pointing to the shorter one.

A root system Φ is called irreducible if it cannot be partitioned into two proper and orthogonal subsets. It is easily seen that Φ is irreducible if and only if its Dynkin diagram is connected. Two root systems Φ_1, Φ_2 in E_1, E_2 respectively are called isomorphic if there exists a linear isomorphism σ from E_1 onto E_2 such that $\sigma(\Phi_1) = \Phi_2$ and for any $\alpha, \beta \in \Phi_1$ we have $\langle\alpha, \beta\rangle_1 = \langle\sigma(\alpha), \sigma(\beta)\rangle_2$. It is clear that two irreducible root systems are isomorphic if and only if they have the same Dynkin diagram. The following theorem gives a complete classification of irreducible root systems.

Theorem 2.18. *Let Φ be an irreducible root system. Then the Dynkin diagram of Φ must be one of the diagrams in Fig. 2.1. Moreover, any of the diagrams in Fig. 2.1 is the Dynkin diagram of a root system.*

As we have seen above, for a complex semisimple Lie algebra $\mathfrak{g}$ and a maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, the root system Δ of $\mathfrak{g}$ with respect to $\mathfrak{h}$ is a root system in the sense of Definition 2.12. If $\mathfrak{h}_1$ is another maximal toral subalgebra of $\mathfrak{g}$, then we have another root system Δ_1, the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}_1$. It is a very nice fact that different maximal toral subalgebras are conjugate to each other. In other words, there exists an automorphism τ of $\mathfrak{g}$ such that $\tau(\mathfrak{h}) = \mathfrak{h}_1$ (see Sect. 2.5 below). It is easily seen that this automorphism induces an isomorphism between the root systems Δ and Δ_1. On the other hand, it is clear that $\mathfrak{g}$ is simple if and only if Δ is irreducible. Moreover, it can be proved that for each Dynkin diagram in Fig. 2.1, there exists a complex simple Lie algebra whose Dynkin diagram is exactly the given one (see [83]). Thus we have the following theorem.

Theorem 2.19. *Up to isomorphism, there are nine types of complex simple Lie algebras, whose Dynkin diagram are respectively A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2.*

This theorem gives a complete classification of complex simple Lie algebras, as well as semisimple Lie algebras. Usually, we use the symbols of the Dynkin diagram to represent the corresponding Lie algebra. The first four classes are called classical simple Lie algebras and the other five types are called exceptional ones.

In the final part of this section we introduce some results on real semisimple Lie algebras. We first recall the notion of the complexification of a real Lie algebra.

Suppose $\mathfrak{g}_0$ is a real Lie algebra. The complexification of $\mathfrak{g}_0$ is the complex vector space $\mathfrak{g}_0 \otimes \mathbb{C}$ endowed with the Lie brackets

$$\left[x_1 + \sqrt{-1}y_1, x_2 + \sqrt{-1}y_2\right] = [x_1, x_2] - [y_1, y_2] + \sqrt{-1}\left([x_1, y_2] + [y_1, x_2]\right),$$

where $x_i, y_i \in \mathfrak{g}_0$, $i = 1, 2$. It is easy to check that the above brackets make $\mathfrak{g}_0 \otimes \mathbb{C}$ into a complex Lie algebra. We will denote this Lie algebra by $\mathfrak{g}_0^C$. For convenience, we sometimes use $\mathfrak{g}$ to denote the complexification of $\mathfrak{g}_0$.

Conversely, suppose $\mathfrak{g}$ is a complex Lie algebra. Then $\mathfrak{g}$ can also be viewed as a real Lie algebra: just view the complex vector space $\mathfrak{g}$ as a real vector space and define the same Lie brackets. This real Lie algebra will be denoted by $\mathfrak{g}^{\mathrm{R}}$. Note that set theoretically $\mathfrak{g}^{\mathrm{R}}$ and $\mathfrak{g}$ are the same thing and that $\dim_{\mathbb{R}} \mathfrak{g}^{\mathrm{R}} = 2\dim_{\mathbb{C}} \mathfrak{g}$. The following proposition can be proved by a direct computation.

Proposition 2.6. *Let $\mathfrak{g}_0$ be a real lie algebra and $\mathfrak{g}$ its complexification. Denote the Killing forms of $\mathfrak{g}_0$, $\mathfrak{g}$, and $\mathfrak{g}^{\mathrm{R}}$ by K_0, K, and K_{R}, respectively. Then we have*

$$K_0(X,Y) = K(X,Y), \quad \textit{for } X, Y \in \mathfrak{g}_0,$$
$$K_{\mathrm{R}}(X,Y) = 2\mathrm{Re}(K(X,Y)), \quad \textit{for } X, Y \in \mathfrak{g}^{\mathrm{R}},$$

In view of Theorem 2.12, we have the following.

Proposition 2.7. *A real Lie algebra is semisimple if and only if its complexification is semisimple.*

Let $\mathfrak{g}$ be a complex Lie algebra. A real form of $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_0$ of the real Lie algebra $\mathfrak{g}^{\mathrm{R}}$ such that

$$\mathfrak{g}^{\mathrm{R}} = \mathfrak{g}_0 + \sqrt{-1}\mathfrak{g}_0 \quad \text{(direct sum of vector spaces).}$$

In this case $\mathfrak{g}$ is isomorphic to the complexification of $\mathfrak{g}_0$. The following theorem is very important for the classification of real semisimple Lie algebras.

Theorem 2.20. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then there exists a real form $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $\mathfrak{g}_0$ is a compact Lie algebra.*

Definition 2.13. Let $\mathfrak{g}_0$ be a real semisimple Lie algebra and $\mathfrak{g}$ its complexification. Denote the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$ by σ. A decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ of $\mathfrak{g}_0$ into the direct sum of a subalgebra $\mathfrak{k}_0$ and a vector subspace $\mathfrak{p}_0$ is called a Cartan decomposition if there exists a compact real form $\mathfrak{g}_k$ of $\mathfrak{g}$ such that

$$\sigma(\mathfrak{g}_k) \subset \mathfrak{g}_k, \quad \mathfrak{k}_0 = \mathfrak{g}_0 \cap \mathfrak{g}_k, \quad \mathfrak{p}_0 = \mathfrak{g}_0 \cap (\sqrt{-1}\mathfrak{g}_k).$$

Theorem 2.21. *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra. Then there exists a Cartan decomposition of $\mathfrak{g}_0$. Moreover, if $\mathfrak{g}_0 = \mathfrak{k}_1 + \mathfrak{p}_1$ and $\mathfrak{g}_0 = \mathfrak{k}_2 + \mathfrak{p}_2$ are two Cartan decompositions of $\mathfrak{g}_0$, then there exists $g \in \mathrm{Int}(\mathfrak{g}_0)$ such that $g(\mathfrak{k}_1) = \mathfrak{k}_2$ and $g(\mathfrak{p}_1) = \mathfrak{p}_2$.*

Theorem 2.22. *Let $\mathfrak{g}_0$ be a real semisimple Lie algebra. Then a decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ of $\mathfrak{g}_0$ into a subalgebra $\mathfrak{k}_0$ and a vector space $\mathfrak{p}_0$ is a Cartan decomposition if and only if the map $s : X + T \to X - T$, $X \in \mathfrak{k}_0$, $T \in \mathfrak{p}_0$, of $\mathfrak{g}_0$ is an automorphism and the Killing form B is positive definite on $\mathfrak{p}_0$ and negative definite on $\mathfrak{k}_0$.*

2.4 Homogeneous Riemannian Manifolds

In this section we will collect some results on homogeneous Riemannian manifolds. For a connected Riemannian manifold (M,Q), we have two ways to define an isometry. On the one hand, we call a diffeomorphism σ of M onto itself an isometry if for $x \in M$ and any tangent vectors $X, Y \in T_x(M)$, we have $Q(\mathrm{d}\sigma|_x(X), \mathrm{d}\sigma|_x(Y)) = Q(X,Y)$. On the other hand, we can also define an isometry of (M,Q) to be a map τ of M onto itself such that $d(\tau(x), \tau(y)) = d(x,y)$, for any $x, y \in M$. The essence of the celebrated Myers–Steenrod theorem is that the above two definitions are equivalent. In the literature, the following corollary of the above result is often referred to as the Myers–Steenord theorem [123].

Theorem 2.23 (Myers–Steenrod). *Let (M,Q) be a connected Riemannian manifold. Then the group of isometries $I(M,Q)$ of (M,Q) admits a differentiable structure such that $I(M,Q)$ is a Lie transformation group of (M,Q).*

Definition 2.14. A connected Riemannian manifold (M,Q) is called homogeneous if the group of isometries of (M,Q) is transitive on M.

If (M,Q) is a homogeneous Riemannian manifold, then $I(M,Q)$ has at most countably many connected components. Therefore by Proposition 2.2, the identity component $I_0(M,G)$ is also transitive on M. Fix $x \in M$ and denote by K the isotropy subgroup of $I_0(M,Q)$ at x, Then K is a compact subgroup of $I_0(M,Q)$ and M is diffeomorphic to $I_0(M,Q)/K$. The Riemannian metric Q can be viewed as an $I_0(M,Q)$-invariant Riemannian metric on M. This means that any homogeneous Riemannian manifold can be written as a coset space of a connected Lie group with an invariant Riemannian metric.

Now we consider the more general case. Suppose G is a connected Lie group and H is a closed subgroup of G such that G acts almost effectively on G/H. We study the invariant Riemannian metrics on the coset space G/H. Let $\mathfrak{g}, \mathfrak{h}$ be respectively the Lie algebras of G and H. Denote the adjoint representation of G on $\mathfrak{g}$ by Ad. Note that for $h \in H$, $\mathrm{Ad}(h)$ keeps the subalgebra $\mathfrak{h}$ invariant. Hence $\mathrm{Ad}(h)$ induces a linear map on the quotient space $\mathfrak{g}/\mathfrak{h}$. We denote this map by $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(h)$. The differential of this action, which is a representation of $\mathfrak{h}$ on $\mathfrak{g}/\mathfrak{h}$, will be denoted by $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}$.

Proposition 2.8. *There is a one-to-one correspondence between the G-invariant Riemannian metrics on G/H and the $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(H)$-invariant inner products on $\mathfrak{g}/\mathfrak{h}$.*

For the proof, see [102].

Given $X \in \mathfrak{h}$, $\exp tX$ is a one-parameter subgroup of H. With respect to an $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(H)$-invariant inner product $\langle\, , \rangle$ on $\mathfrak{g}/\mathfrak{h}$, $\exp(tX)$ are orthogonal transformations. Taking the derivation with respect to t, we see that $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(X)$ is skew-symmetric with respect to $\langle\, , \rangle$. This observation leads to the following assertion.

Proposition 2.9. *Let G/H be a coset space. Suppose G/H admits a G-invariant Riemannian metric. If the action of G on G/H is almost effective, i.e., if $\mathfrak{h}$ contains no ideal of $\mathfrak{g}$ other than $\{0\}$, then the following assertions hold:*

1. *The Killing form of $\mathfrak{h}$ is negative semidefinite.*
2. *The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$ is negative definite.*

The following proposition gives a necessary and sufficient condition for a coset space to admit an invariant Riemannian metric.

Proposition 2.10. *Suppose the action of G on the coset space G/H is almost effective. Then G/H admits an invariant Riemannian metric if and only if there is an inner product $\langle\, , \rangle$ on the Lie algebra $\mathfrak{g}$ of G such that for every $X,Y \in \mathfrak{g}$ and $h \in H$,*

$$\langle \mathrm{Ad}(h)(X), \mathrm{Ad}(h)(Y) \rangle = \langle X, Y \rangle.$$

Next we will consider the Levi-Civita connection and the curvatures of homogeneous Riemannian manifolds. Let (M,Q) be a homogeneous Riemannian manifold. A vector field X on M is called a Killing vector field if every local one-parameter transformation group generated by X consists of local isometries of (M,Q).

Theorem 2.24. *Let (M,Q) be a homogeneous Riemannian manifold with the Levi-Civita connection ∇. Suppose X,Y,Z are Killing vector fields on M. Then we have*

$$\nabla_X Y = \frac{1}{2}([X,Y] + U(X,Y)), \tag{2.5}$$

where $U(X,Y)$ is a symmetric tensor field of type $(2,1)$ on M determined by

$$Q(U(X,Y),Z) = \frac{1}{2}\left(Q([X,Z],Y) + Q(X,[Y,Z])\right).$$

If we write (M,Q) as a coset space G/H with H compact, then the Lie algebra $\mathfrak{g}$ has a decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} \quad \text{(direct sum of subspaces)}, \tag{2.6}$$

where $\mathfrak{m}$ is a subspace of $\mathfrak{g}$ such that $\mathrm{Ad}(h)(\mathfrak{m}) \subset \mathfrak{m}$, for every $h \in H$. In fact, we just need to fix an inner product as in Proposition 2.10 and take $\mathfrak{m}$ to be the orthogonal complement of $\mathfrak{h}$ with respect to this inner product. Thus in this case, the coset space G/H is reductive.

The tangent space $T_o(G/H)$, where $o = H$ is the origin, can be identified with $\mathfrak{m}$ through the map

$$X \mapsto \frac{\mathrm{d}}{\mathrm{d}t}(\exp(tX)\cdot o)|_{t=0}.$$

Note that for any $Y \in \mathfrak{g}$, the vector field $\widetilde{Y}|_{gH} = \frac{\mathrm{d}}{\mathrm{d}t}(g\exp(tY)H)|_{t=0}$ is a Killing vector field (the fundamental Killing vector field generated by Y). If $Y \in \mathfrak{m}$, then $\widetilde{Y}_o = Y$. This fact combined with the homogeneity of M implies that formula (2.5) completely determines the Levi-Civita connection of (M,Q). This also leads to the following formulas of the sectional curvature and Ricci curvature of (M,Q).

Theorem 2.25. *Let $(G/H,Q)$ be a homogeneous Riemannian manifold with a reductive decomposition (2.6). Suppose X, Y are two orthogonal vectors in $\mathfrak{m}$ of unit length with respect to Q. Then the sectional curvature of the tangent plane spanned by X, Y is given by*

$$\begin{aligned} K(X,Y) = &-\frac{3}{4}|[X,Y]_{\mathfrak{m}}|^2 - \frac{1}{2}Q([X,[X,Y]]_{\mathfrak{m}},Y) \\ &-\frac{1}{2}Q([Y,[Y,X]]_{\mathfrak{m}},X) + |U(X,Y)|^2 - Q(U(X,X),U(Y,Y)), \end{aligned}$$

where $Z_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of Z and $|\cdot|$ is the length with respect to Q.

The proof can be obtained through a direct computation using (2.5); see [28].

Let $X_1, X_2, \ldots, X_n$ be an orthonormal basis of $\mathfrak{m}$ with respect to Q. Set

$$Z = \sum_{i=1}^{n} U(X_i, X_i).$$

It is easy to check that Z is the unique element in $\mathfrak{m}$ such that

$$Q(Z,X) = \mathrm{tr}(\mathrm{ad}X), \quad \forall X \in \mathfrak{m}.$$

Corollary 2.1. *The Ricci curvature is given by*

$$\begin{aligned}\mathrm{Ric}(X,X) = & -\frac{1}{2}\sum_i |[X,X_i]_\mathfrak{m}|^2 - \frac{1}{2}\sum_i Q([X,[X,X_i]_\mathfrak{m}]_\mathfrak{m},X_i) \\ & -\sum_i Q([X,[X,X_i]_\mathfrak{h}]_\mathfrak{m}) + \frac{1}{4}\sum_{i,j}(Q([X_i,X_j]_\mathfrak{m},X))^2 \\ & -Q([Z,X]_\mathfrak{m},X).\end{aligned}$$

Finally, we recall some results on homogeneous Einstein manifolds. The following definition is a special case of the Finslerian notion of Einstein metrics.

Definition 2.15. An n-dimensional connected Riemannian manifold (M,Q) is called an Einstein manifold if its Ricci curvature is a multiple of the metric. More precisely, it is called an Einstein manifold if

$$\mathrm{Ric}(X,Y) = \lambda Q(X,Y), \quad \forall X,Y \in T_x(M), x \in M.$$

The number $c = \frac{1}{n}\lambda$ is called the Einstein constant in the literature. If an Einstein manifold is homogeneous as a Riemannian manifold, then it is called a homogeneous Einstein manifold.

There is a great difference between homogeneous Einstein manifolds with different signs of the Einstein constant. For $\lambda = 0$, we have the following result of Alekseevskii and Kimel'fel'd (see [5, 28]):

Theorem 2.26. *A homogeneous Einstein manifold with $c = 0$ is flat; hence it is isometric either to a Euclidean space or to a flat torus.*

By the Bonnect–Myers theorem, a homogeneous Einstein manifold with positive Einstein constant c must be compact. A simple observation leads to the following theorem (see [28]).

Theorem 2.27. *A homogeneous Einstein manifold with negative Einstein constant must be noncompact.*

In recent years, homogeneous Einstein manifolds have been studied extensively. We now survey some important results in this field. It is obvious that the standard Riemannian metric on spheres are Einstein metrics. The first nonstandard homogeneous Einstein metric on spheres was found by Jensen in 1973 [94]. Then Ziller obtained a complete classification of all the homogeneous Einstein Riemannian metrics on spheres in 1982 [182]. D'Atri, Wang, and Ziller made a systematic study of compact homogeneous Einstein manifolds, including the normal homogeneous manifolds and naturally reductive left-invariant metrics on compact Lie groups; see [45, 165]. However, up to now, necessary and sufficient conditions for a compact homogeneous manifold to have invariant Einstein metrics are still unknown; see [166] for some partial results. Meanwhile, there has appeared a series of results on low-dimensional spaces.

In the noncompact case there are also many interesting results. We mention the following conjecture; see [4].

Conjecture 2.1 (D.V. Alekseevskii). Let $M = G/H$ be a noncompact homogeneous Einstein manifold. Then H is a maximal compact subgroup of G.

This conjecture is still open. If it is true, then every noncompact homogeneous Einstein metric can be realized as a left-invariant metric on a solvable Lie group. Based on this point of view, many researchers have studied left-invariant Einstein metrics on solvable Lie groups; see, for example, [75, 80, 160].

2.5 Symmetric Spaces

In this section we recall the main results on the structure and classification of Riemannian symmetric spaces. A connected Riemannian manifold (M,Q) is called locally symmetric if for every $x \in M$ there is a neighborhood U of x such that the geodesic symmetry

$$\exp(tX) \to \exp(-tX), \quad X \in T_x(M),$$

defines a local isometry on U. It is called globally symmetric if for every $x \in M$, there exists an involutive isometry σ (i.e., $\sigma^2 = \mathrm{id}$) such that x is an isolated fixed point of σ. It is obvious that a globally symmetric Riemannian space must be locally symmetric. The results of this section are mainly due to Cartan.

Theorem 2.28. *A connected Riemannian manifold (M,Q) is locally symmetric if and only if its sectional curvature is invariant under parallel displacements. A complete, connected, and simply connected locally symmetric Riemannian manifold must be globally symmetric.*

Let (M,Q) be a globally symmetric Riemannian space and $x \in M$. Then there is an involutive isometry τ_x with x as its isolated fixed point. It is easily seen that $\mathrm{d}\tau_x|_x = -\mathrm{id}|_{T_x(M)}$. Thus τ_x is actually a geodesic symmetry at x. Given $x_1, x_2 \in M$, we can connect x_1 to x_2 using a family of geodesic segments γ_i, $i = 1, 2, \ldots, l$. Let z_i be the midpoint of the geodesic segment γ_i. It is easily seen that the isometry $\tau_{z_l} \circ \tau_{z_{l-1}} \circ \cdots \circ \tau_{z_1}$ sends x_1 to x_2. Thus (M,Q) is homogeneous. Since a homogeneous Riemannian manifold must be complete, (M,Q) is complete. Moreover, it is easy to see that the universal covering Riemannian manifold of (M,Q) is also globally symmetric.

Let $I(M,Q)$ be the full group of isometries of (M,Q). Then $I(M,Q)$ is a Lie transformation group that acts transitively on M. Moreover, $I(M,Q)$ has at most countably many connected components. In view of Proposition 2.2, the unity component of $I(M,Q)$, denoted by G, is also transitive on M. Fix $x \in M$ and denote by K the isotropy subgroup of G at x. Then as a manifold, $M = G/K$ and the Riemannian metric Q can be viewed as a G-invariant metric on the coset space G/K. Now we give a definition:

Definition 2.16. Let G be a connected Lie group and H a closed subgroup of G. If there exists an involutive automorphism σ of G such that $(G_\sigma)_0 \subset H \subset G_\sigma$, where G_σ denotes the set of fixed points of σ and $(G_\sigma)_0$ the unity component of G_σ, then the pair (G,H) is called a symmetric pair. If in addition, the group $\mathrm{Ad}_G(H)$ is a compact Lie group, then (G,H) is called a Riemannian symmetric pair.

Note that $\mathrm{Ad}_G(H)$ is the image of H under the adjoint representation $\mathrm{Ad}_G(G)$ of G. This leads us to our next result.

Theorem 2.29. *If (G,K) is a Riemannian symmetric pair, then there exists a G-invariant Riemannian metric Q on G/K such that $(G/K,Q)$ is a Riemannian globally symmetric space. Conversely, if (M,Q) is a Riemannian globally symmetric space, then there exists a connected Lie group G that acts isometrically and transitively on M such that (G,K) is a Riemannian symmetric pair, where K is the isotropy subgroup of G at certain $x \in M$.*

Note that in the above theorem, G can be taken as the unity component of the full group of isometries. This theorem reduces the research of Riemannian symmetric spaces to coset spaces endowed with an invariant Riemannian metric. In fact, we can reduce the problem further.

Definition 2.17. Let $\mathfrak{g}$ be a real Lie algebra and s an involutive automorphism of $\mathfrak{g}$. If the set of fixed points of s, denoted by $\mathfrak{k}$, is a compactly embedded subalgebra of $\mathfrak{g}$, then $(\mathfrak{g},s)$ is called an orthogonal symmetric Lie algebra. If in addition, $\mathfrak{k} \cap \mathfrak{z} = \{0\}$, where $\mathfrak{z}$ is the center of $\mathfrak{g}$, then the pair is called effective.

Theorem 2.30. *Let (G,K) be a Riemannian symmetric pair and σ an involutive automorphism of G such that $(G_\sigma)_0 \subset K \subset G_\sigma$. Let $\mathfrak{g}$ be the Lie algebra of G and s the differential of σ at $e \in G$. Then $(\mathfrak{g},s)$ is an orthogonal symmetric Lie algebra. If the action of G on G/K is effective, then $(\mathfrak{g},s)$ is effective.*

Theorem 2.31. *Let $(\mathfrak{g},s)$ be an orthogonal symmetric Lie algebra and $\mathfrak{k}$ the set of fixed points of s. Suppose (G,K) is a pair associated with $(\mathfrak{g},s)$ (i.e., G is a connected Lie group with Lie algebra $\mathfrak{g}$ and K is a Lie subgroup of G with Lie algebra $\mathfrak{k}$) such that G is simply connected and K is connected. Then K is closed in G and the pair (G,K) is a Riemannian symmetric pair.*

These theorems reduce the study of symmetric spaces to that of the orthogonal symmetric Lie algebras. Now we consider the structure of such algebras.

Definition 2.18. Let $(\mathfrak{g}, s)$ be an orthogonal symmetric Lie algebra and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the decomposition of $\mathfrak{g}$ into the eigenspaces of s for the eigenvalues $+1$ and -1, respectively. If $\mathfrak{g}$ is a compact semisimple Lie algebra, then $(\mathfrak{g}, s)$ is said to be of compact type; if $\mathfrak{g}$ is noncompact and semisimple, then $(\mathfrak{g}, s)$ is said to be of noncompact type; if $\mathfrak{p}$ is an abelian ideal of $\mathfrak{g}$, then $(\mathfrak{g}, s)$ is said to be of Euclidean type. A pair (G, K) associated with $(\mathfrak{g}, s)$ is said to be of compact, noncompact, or Euclidean type according to the type of $(\mathfrak{g}, s)$. A Riemannian globally symmetric space (M, Q) is said to be of Euclidean, compact, or noncompact type according to the type of the corresponding symmetric pair.

Note that if a connected simply connected globally symmetric space (M, Q) is of Euclidean type, then (M, Q) is the standard Euclidean space.

Theorem 2.32. *Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra. Then $\mathfrak{g}$ has a decomposition into the direct sum of ideals $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_{\mathrm{c}} + \mathfrak{g}_{\mathrm{n}}$, where $\mathfrak{g}_0$, $\mathfrak{g}_{\mathrm{c}}$, and $\mathfrak{g}_{\mathrm{n}}$ are all invariant under s and orthogonal with respect to the Killing form of $\mathfrak{g}$. Moreover, let s_0, s_{c}, and s_{n} be respectively the restrictions of s to $\mathfrak{g}_0$, $\mathfrak{g}_{\mathrm{c}}$, and $\mathfrak{g}_{\mathrm{n}}$. Then $(\mathfrak{g}_0, s_0)$, $(\mathfrak{g}_{\mathrm{c}}, s_{\mathrm{c}})$, and $(\mathfrak{g}_{\mathrm{n}}, s_{\mathrm{n}})$ are orthogonal symmetric Lie algebras of Euclidean, compact, and noncompact type, respectively.*

The global version of the above theorem is the following.

Theorem 2.33. *Let (M, Q) be a connected simply connected Riemannian globally symmetric space. Then (M, Q) can be decomposed as the product $M = M_0 \times M_{\mathrm{c}} \times M_{\mathrm{n}}$, where M_0 is a Euclidean space, M_{c} is a Riemannian globally symmetric space of compact type, and M_{n} is a Riemannian globally symmetric space of noncompact type.*

There is a remarkable duality between Riemannian globally symmetric spaces of compact and noncompact type. We start with orthogonal symmetric Lie algebras. Let $(\mathfrak{g}, s)$ be an orthogonal Lie algebra and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the canonical decomposition. Consider the subalgebra

$$\mathfrak{g}^* = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$$

of the complexification $\mathfrak{g}^{\mathbb{C}}$ and define a linear endomorphism s^* on $\mathfrak{g}^*$ by

$$s^*(X + \sqrt{-1}Y) = X - \sqrt{-1}Y, \quad X \in \mathfrak{k}, Y \in \mathfrak{p}.$$

Then it is easy to check that $(\mathfrak{g}^*, s^*)$ is again an orthogonal Lie algebra. The pair $(\mathfrak{g}^*, s^*)$ is called the dual of $(\mathfrak{g}, s)$. Two orthogonal symmetric Lie algebras $(\mathfrak{g}_1, s_1)$ and $(\mathfrak{g}_2, s_2)$ are called isomorphic if there is a Lie algebra isomorphism φ from $\mathfrak{g}_1$ onto $\mathfrak{g}_2$ such that $\varphi \circ s_1 = s_2 \circ \varphi$.

Proposition 2.11. *The orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ is of compact type if and only if $(\mathfrak{g}^*, s^*)$ is of noncompact type. If $(\mathfrak{g}_1, s_1)$ is isomorphic to $(\mathfrak{g}_2, s_2)$, then $(\mathfrak{g}_1^*, s_1^*)$ is isomorphic to $(\mathfrak{g}_2^*, s_2^*)$.*

An orthogonal symmetric Lie algebra $(\mathfrak{g}, s)$ is called irreducible if in its canonical decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, the adjoint representation of $\mathfrak{k}$ on $\mathfrak{p}$ is irreducible. It can be proved that every effective orthogonal Lie algebra of compact or noncompact type can be decomposed into a direct sum of irreducible orthogonal Lie algebras.

Theorem 2.34. *Let $(\mathfrak{g}, s)$ be an irreducible effective orthogonal Lie algebra. Then $(\mathfrak{g}, s)$ must be isomorphic to one of the following types:*

(CI) $\mathfrak{g}$ *is compact simple and s is an involutive automorphism of* $\mathfrak{g}$.

(CII) $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1$, *where $\mathfrak{g}_1$ is a compact simple Lie algebra, and s is defined by $s(X,Y) = (Y,X)$, $X, Y \in \mathfrak{g}_1$.*

(NI) $\mathfrak{g}$ *is a noncompact simple Lie algebra whose complexification $\mathfrak{g}^C$ is a complex simple Lie algebra, and s is a Cartan involution of* $\mathfrak{g}$.

(NII) $\mathfrak{g}$ *is a noncompact simple Lie algebra whose complexification $\mathfrak{g}^C$ is a complex nonsimple Lie algebra, and s is a Cartan involution of* $\mathfrak{g}$.

Among the above four types of orthogonal symmetric Lie algebras, (CI) *is dual to* (NI)*, and* (CII) *is dual to* (NII).

A connected simply connected Riemannian globally symmetric space is said to be of type (CI), (CII), (NI), or (NII) according to the type of its corresponding orthogonal symmetric Lie algebra. Note that the symmetric spaces of type (CII) are exactly all the compact simple Lie groups endowed with bi-invariant Riemannian metrics. Since each complex simple Lie algebra has a unique (up to an isomorphism) compact real form, each complex simple Lie algebra in Theorem 2.18 corresponds to exactly one symmetric space of type (CII). We will use the same notation as in Theorem 2.18 to denote the corresponding symmetric space.

A complete list of connected simply connected Riemannian globally symmetric spaces of type (CI) can be found in [83].

There is a great difference between Riemannian globally symmetric spaces of compact type and those of noncompact type. In the noncompact case, the structure is very simple:

Theorem 2.35. *Let $(\mathfrak{g}_0, s)$ be an effective orthogonal symmetric Lie algebra of noncompact type and let $\mathfrak{g}_0 = \mathfrak{h}_0 + \mathfrak{m}_0$ be the associated decomposition. Then $\mathfrak{g}_0$ is a noncompact semisimple Lie algebra and the above decomposition is a Cartan decomposition of $\mathfrak{g}_0$. Suppose (G,H) is any pair associated with $(\mathfrak{g}_0, \mathfrak{h}_0)$. Then we have the following:*

1. *H is connected, closed, and contains the center Z of G. Moreover, H is compact if and only if Z is finite. In this case, H is a maximal compact subgroup of G.*
2. *The pair (G,H) is a Riemannian symmetric pair.*
3. *The map $\varphi : (X,h) \to (\exp X)h$ is a diffeomorphism from $\mathfrak{m}_0 \times H$ onto G and the exponential map of the Riemannian manifold $M = G/H$ is a diffeomorphism from $\mathfrak{m}_0$ onto the manifold M.*

This theorem has the following corollary:

Corollary 2.2. *Suppose M and M' are two irreducible Riemannian globally symmetric spaces of noncompact type. If the full groups $I(M)$ and $I(M')$ have the same Lie algebra, then up to a positive scalar, M and M' are isometric.*

From Theorem 2.35 we can also deduce the conjugacy of maximal compact subgroups of noncompact semisimple Lie groups:

Theorem 2.36. *Let (G,H) be a Riemannian symmetric pair of noncompact type. Then we have the following:*

(a) *For any compact subgroup H_1 of G, there exists $x \in G$ such that $x^{-1}H_1 \subset H$.*
(b) *H has a unique compact subgroup H_0 that is a maximal subgroup of G.*
(c) *All maximal compact subgroups of a connected semisimple Lie group are conjugate under an inner automorphism.*

From the above results we see that the structure of noncompact Riemannian symmetric spaces is usually very simple. In particular, if G/H is a connected Riemannian globally symmetric space of noncompact type, and G is a connected Lie group, then H must be a connected compact subgroup of G. However, the above results are usually not true for a Riemannian globally symmetric space of compact type. In fact, one can construct some examples to show the following (see [83, Chap. VII]):

Proposition 2.12. *Let $(\mathfrak{u},s)$ be an orthogonal symmetric Lie algebra of compact type. Let (U,K) be any pair associated with $(\mathfrak{u},s)$. Then*

1. *The center of U need not be contained in K.*
2. *Even if U/K is Riemannian globally symmetric, K is not necessarily connected.*
3. *Even if U/K is Riemannian globally symmetric, s does not necessarily correspond to an automorphism of U.*

This difference causes much difficulty in classifying Riemannian globally symmetric spaces of compact type. We omit the details here; see [83].

Now we recall some results on Hermitian symmetric spaces. Let (M,J) be a complex manifold with complex structure J. A Riemannian metric Q on M is called Hermitian if $Q(J(X),J(Y)) = Q(X,Y)$ for all tangent vectors X,Y on M. If in addition $\nabla_X J = 0$, then the Hermitian metric is called Kählerian.

Definition 2.19. Let M be a connected complex manifold with a Hermitian metric Q. Then (M,Q) is said to be a Hermitian symmetric space if each point $p \in M$ is an isolated fixed point of an involutive holomorphic isometry s_p of M.

It is not hard to prove that a Hermitian symmetric space must be Kählerian. Moreover, a Hermitian symmetric space must be a Riemannian globally symmetric space. Therefore, the classification of Hermitian symmetric spaces is reduced to finding out which Riemannian globally symmetric spaces admit a complex structure that is admissible with the metric structure. We say that a Hermitian symmetric space is of compact or noncompact type according to the type of the corresponding Riemannian symmetric space. It is called irreducible if the corresponding Riemannian symmetric space is irreducible. It can be proved that a Hermitian symmetric space of compact or noncompact type is simply connected.

Theorem 2.37. *Let M be a simply connected Hermitian symmetric space. Then M is a product*

$$M = M_0 \times M_{\mathrm{c}} \times M_{\mathrm{n}},$$

where $M_0, M_{\mathrm{c}}, M_{\mathrm{n}}$ are simply connected Hermitian symmetric spaces, $M_0 = \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}$, and M_{c} and M_{n} are of compact type and noncompact type, respectively.

Theorem 2.38. *Let M be an irreducible Hermitian symmetric space.*

(i) *If M is of noncompact type, then M is the manifold G/K, where G is a connected noncompact simple Lie group with center $\{e\}$ and K is a maximal compact subgroup of G with nondiscrete center.*
(ii) *If M is of compact type, then M is the manifold U/K, where U is a connected compact simple Lie group with center $\{e\}$ and K is a maximal connected proper subgroup with nondiscrete center.*

Finally we recall some results on the rank of symmetric spaces. Let M be a Riemannian globally symmetric space. The rank of M is the maximal dimension of flat, totally geodesic submanifolds of M. Suppose M is of compact or noncompact type and write M as a coset space G/H, where G is the identity component of the full group. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ be the canonical decomposition of the Lie algebra of G. A subspace $\mathfrak{s}$ of $\mathfrak{p}$ is called a Lie triple system if $[[\mathfrak{s},\mathfrak{s}],\mathfrak{s}] \subset \mathfrak{s}$. A result of Cartan says that there is a one-to-one correspondence between totally geodesic submanifolds of M and Lie triple systems in $\mathfrak{p}$ through the map $\mathfrak{s} \to \mathrm{Exp}(\mathfrak{s})$.

Theorem 2.39. *The totally geodesic submanifold* $\mathrm{Exp}(\mathfrak{s})$ *is flat if and only if $\mathfrak{s}$ is abelian.*

Therefore, the rank of a Riemannian globally symmetric space of compact type or noncompact type is equal to the dimension of the maximal abelian subspace of $\mathfrak{p}$.

Theorem 2.40. *Let $\mathfrak{a}$ and $\mathfrak{a}'$ be two maximal abelian subspaces of $\mathfrak{p}$. Then*

1. *There exists an element $u \in \mathfrak{p}$ whose centralizer in $\mathfrak{p}$ is $\mathfrak{a}$.*
2. *There exists an element $k \in H$ such that* $\mathrm{Ad}(k)\mathfrak{a} = \mathfrak{a}'$.
3. $\mathfrak{p} = \bigcup_{k \in H} \mathrm{Ad}(k)\mathfrak{a}$.

This theorem has an important corollary that any two maximal flat totally geodesic submanifolds of M must be congruent under the group of isometries of M. Applying the above results to connected compact Lie groups, we get the following theorem.

Theorem 2.41. *Let G be a connected compact Lie group with Lie algebra $\mathfrak{g}$. Suppose $\mathfrak{t}$ and $\mathfrak{t}'$ are two maximal abelian subalgebras of $\mathfrak{g}$. Then*

1. *There exists an element $u \in \mathfrak{t}$ whose centralizer in $\mathfrak{g}$ is $\mathfrak{t}$.*
2. *There exists an element $g \in G$ such that $\mathrm{Ad}(g)\mathfrak{t} = \mathfrak{t}'$.*
3. $\mathfrak{g} = \bigcup_{g \in G} \mathrm{Ad}(g)\mathfrak{t}$.

From this theorem it follows that two Cartan subalgebras of a complex semisimple Lie algebra are congruent under the group of automorphisms.

The material of this section is taken mainly from [83].

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