

# Preface

In recent years, Finsler geometry has been developing rapidly. There are two main reasons for this development. First, the influence of S.S. Chern in the late twentieth century was enormous; his guidelines rapidly accelerated both interest and results, first in the United States, including David Bao, Zhongmin Shen, as well as other mathematicians, and then in China. Today, a large group of young and talented mathematical researchers all over the world are working on specific issues in this field. Second, new areas of application arose in the 1980s (R.S. Ingarden, P.L. Antonelli, among others) in biology, optics, quantum physics, and also in psychology, geosciences, geodesy, and other fields. The importance of the field also grew in the 1990s, when its strong relationship with complex analysis became clear. A new field, called complex Finsler geometry, was launched. In the past 20 years, much significant progress has been made in this field, and work in this subject has resulted in a number of substantial books and monographs that have greatly enhanced the study of Finsler geometry.

This monograph contains a series of results obtained by the author and collaborators in the last decade. Our main idea was to show that one can use Lie theory to study Finsler geometry, and our research shows that this method is actually applicable to many problems. Moreover, results obtained on related topics are generally simpler in this form than similar results in general Finsler geometry. In fact, in some special cases (e.g., Randers spaces), we can express curvatures using only the algebraic structures and the metric, without local coordinate systems. Such approaches are particularly welcome, since Finsler geometry has a reputation for complexity due to the extensive use of tensors and indices.

The field of Finsler geometry originated from Riemann's celebrated habilitation lecture, "On the Hypotheses, Which Lie at the Foundation of Geometry," given on June 10, 1854. A translation of this lecture can be found in M. Spivak's book *A Comprehensive Introduction to Differential Geometry*, Volume II, Chap. 4, Publish or Perish Inc., 1970. In this lecture, Riemann introduced the notion of a manifold and metric structures on a manifold. In the special case that the manifold is smooth and the metric is a quadratic differential form, Riemann successfully introduced the notions of curvature tensor and sectional curvature, and provided a complete

treatment of this complicated quantity. This setting had an important impact on Einstein's general theory of relativity, and has received a great deal of attention from both mathematicians and physicists. The subject is now called Riemannian geometry in the literature.

The restriction to a quadratic form constitutes only a special case. Riemann did not regard this restriction as necessary. However, for the general case, he wrote the following: "The next simplest case would perhaps include the manifolds in which the line element can be expressed as the fourth root of a differential expression of the fourth degree. Investigation of this more general class would actually require no essentially different principles, but it would be rather time-consuming and throw proportionally little new light on the study of space." This commentary rendered the general cases dormant for a rather long period. In 1918, Paul Finsler initiated the study of variational problems in the spaces where the metric is defined by Minkowski norms. This setting was developed into a new field, which was much more complicated and difficult compared to Riemannian geometry and was eventually named Finsler geometry.

In 1926, L. Berwald introduced the notion of flag curvature, which is the natural generalization of sectional curvature in Riemannian geometry. Berwald found an important connection, called the Berwald connection, which is a very important connection in Finsler geometry. He also studied the geometric properties of a special kind of Finsler spaces—Berwald spaces. The work of Berwald has had a great impact in the study of Finsler geometry. The Berwald connection is torsion-free but not metric-compatible. In 1934, É. Cartan found another connection—the Cartan connection—which is metric-compatible but has torsion. Great progress was also made in 1943 by S.S. Chern, who found a connection that is torsion-free and almost metric-compatible. The Chern connection is the simplest in form and now appears very frequently in the literature.

As a branch of geometry, Finsler geometry has inevitably been influenced by group theory. The celebrated Erlangen program of F. Klein, posed in 1872, greatly influenced the development of geometry. Klein proposed to categorize the new geometries by their characteristic groups of transformations. To this day, this program has been extremely successful. Nowadays, every geometer expects to find an effective method for studying geometry by applying group theory. The theory of homogeneous/symmetric Riemannian spaces provides a sample for this program. Lie theory was developed in the late nineteenth century, first in the local fashion, by S. Lie, W. Killing, and É. Cartan. Global Lie groups were emphasized through the work of H. Weyl, É. Cartan, and O. Schreier during the 1920s. One of the most important applications of Lie theory to Riemannian geometry was Cartan's work on the classification of globally symmetric Riemannian spaces. The Myers–Steendrod theorem, published in 1939, extended the application of Lie theory to the scope of all homogeneous Riemannian manifolds. By now, the theory of homogeneous/symmetric Riemannian manifolds has become the basis of many branches of mathematics, including group and geometric analysis, and representation theory. The Myers–Steenrod theorem was generalized to the Finslerian case by the author and Z. Hou in 2002. This result opened a door to using

Lie theory to study Finsler geometry. In the last decade, the author and collaborators have successfully developed the theory of homogeneous/symmetric Finsler spaces. Meanwhile, there has appeared in the literature some work of other mathematicians on Finsler geometry that is closely related to Lie theory, including some classical results of H.C. Wang and Z.I. Szabó's on Berwald spaces.

The purpose of this book is to introduce the major part of the aspect of Finsler geometry that has a close relationship with Lie theory, and to bring the reader to the frontiers of the active research on related topics. The book consists of seven chapters. Chapters 1 and 2 are an introduction to Finsler geometry and Lie theory. Chapter 1 can also be used as a textbook for a course in Finsler geometry, provided the lecturer adds all the necessary details. Chapter 3 is focused on the study of isometries of Finsler space. The main result of this chapter is the generalized Myers–Steenrod theorem. In Chap. 4, the theory of general homogeneous Finsler spaces is developed. Chapter 5 deals with symmetric Finsler spaces. The main features of É. Cartan's theory on Riemannian symmetric spaces are generalized to the Finslerian case. In Chap. 6, we develop a theory of weakly symmetric Finsler spaces. Chapter 7 is devoted entirely to homogeneous Randers spaces.

Apart from the first two chapters, each chapter begins with a brief introduction to the background and the motivation of the topics under consideration. We believe that some historical elements will help readers grasp the main ideas more easily. Moreover, we discuss possible further development of the fields discussed in the chapter. We hope that this will enhance the study of homogeneous Finsler spaces to some extent.

There are some aspects of the field of Finsler geometry that have not been involved in this book. First, although we present some results on invariant complex structures on homogeneous/symmetric Finsler spaces in Sect. 5.5, a complete theory of homogeneous complex Finsler spaces has not been established, which in my opinion will be a main focus in the near future. Second, this book does not deal with any explicit applications of the theory to the real world, although we have provided a large number of carefully selected examples, which definitely have promising applications to other scientific fields.

Finally, although the author has done his best to make everything as accurate as possible, errors or mistakes, minor or major, are very likely to exist in the book. Comments and suggestions from readers, either for the improvement of the book or pointing out mistakes to the author, are very welcome.

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