Chapter 2
Statistical Description of Interacting Particle Systems

Abstract In a system which consists of many interacting particles, the statistical mechanism of "mixing" in phase space works and makes description of the system's behavior *on average* more simple. A state and evolution of the system is described by a *statistically smoothed* distribution function and the averaged Liouville equation, i.e. the kinetic equation, with attached chain of the equations for correlation functions.

2.1 The Averaging of Liouville's Equation

2.1.1 Averaging Over Phase Space

As was shown in the first chapter, the exact state of a system consisting of N interacting particles can be given by the *exact* distribution function (see definition (1.20)) in the six-dimensional (6D) phase space $X = \{\mathbf{r}, \mathbf{v}\}$. This function is defined as the sum of δ-functions at N points of the phase space:

$$\hat{f}(\mathbf{r}, \mathbf{v}, t) = \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i(t))\, \delta(\mathbf{v} - \mathbf{v}_i(t))\,. \tag{2.1}$$

Instead of the equations of motion, we use Liouville's equation to describe the change of the system state (Sect. 1.1.5):

$$\frac{\partial \hat{f}}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}}\, \hat{f} + \frac{\hat{\mathbf{F}}}{m} \cdot \nabla_{\mathbf{v}}\, \hat{f} = 0\,. \tag{2.2}$$

Once the exact initial state of all the particles is known, it can be represented by N points in the phase space X (Fig. 2.1). The motion of these points is described by Liouville's equation or by the $6N$ equations of motion (1.25).

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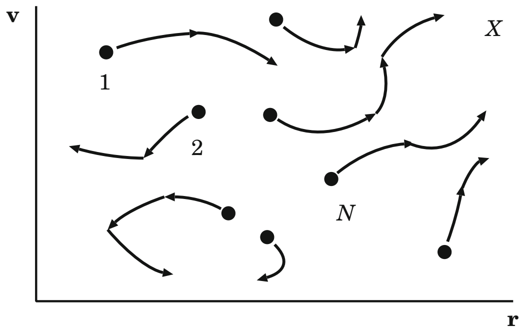


Fig. 2.1 Particle trajectories in the 6D phase space X

In fact we usually know only some average characteristics of the system's state, such as the temperature, density, etc. Moreover the behavior of each single particle is in general of no interest. For this reason, instead of the exact distribution function (2.1), let us introduce the distribution function *averaged* over a small volume ΔX of phase space, i.e. over a small interval of coordinates $\Delta \mathbf{r}$ and velocities $\Delta \mathbf{v}$ centered at the point $(\mathbf{r}, \mathbf{v})$, at a moment of time t:

$$\langle \hat{f}(\mathbf{r}, \mathbf{v}, t) \rangle_X = \frac{1}{\Delta X} \int\limits_{\Delta X} \hat{f}(X, t)\, dX =$$

$$= \frac{1}{\Delta \mathbf{r}\, \Delta \mathbf{v}} \int\limits_{\Delta \mathbf{r}\, \Delta \mathbf{v}} \hat{f}(\mathbf{r}, \mathbf{v}, t)\, d^{\,3}\mathbf{r}\, d^{\,3}\mathbf{v}\,. \tag{2.3}$$

Here $d^{\,3}\mathbf{r} = dx\,dy\,dz$ and $d^{\,3}\mathbf{v} = dv_x\,dv_y\,dv_z$, if use is made of Cartesian coordinates.

To put the same in another way, the *mean* number of particles that present at a moment of time t in an element of phase volume ΔX is

$$\langle \hat{f}(\mathbf{r}, \mathbf{v}, t) \rangle_X \cdot \Delta X = \int\limits_{\Delta X} \hat{f}(\mathbf{r}, \mathbf{v}, t)\, dX\,. \tag{2.4}$$

The total number N of particles in the system is the integral over the whole phase space X.

Obviously the distribution function averaged over phase volume differs from the exact one as shown in Fig. 2.2.

2.1.2 Two Statistical Postulates

Let us average the same exact distribution function (2.1) over a small time interval Δt centered at a moment of time t:

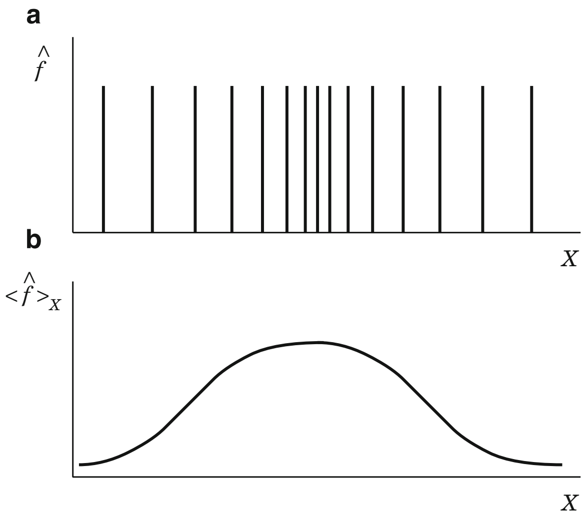


Fig. 2.2 The one-dimensional analogy of the distribution function averaging over phase space X: (**a**) the exact distribution function (2.1), (**b**) the averaged function (2.3)

$$\langle\, \hat{f}(\mathbf{r}, \mathbf{v}, t)\, \rangle_t = \frac{1}{\Delta t} \int_{\Delta t} \hat{f}(\mathbf{r}, \mathbf{v}, t)\, dt\, . \tag{2.5}$$

Here Δt is small in comparison with the characteristic time of the system's evolution:

$$\Delta t \ll \tau_{ev}\, . \tag{2.6}$$

We assume that the following *two statistical postulates* concerning systems containing a large number of particles are applicable to the system considered.

The first postulate:

> The mean values $\langle\, \hat{f}\, \rangle_X$ and $\langle\, \hat{f}\, \rangle_t$ exist for sufficiently small ΔX and Δt and are *independent* of the averaging scales ΔX and Δt.

Clearly the first postulate implies that the number of particles should be large. For a small number of particles the mean value depends upon the averaging scale: if, for instance, $N = 1$ then the exact distribution function (2.1) is simply a δ-function, and the average over the variable X is $\langle\, \hat{f}\, \rangle_X = 1/\Delta X$. For illustration, the case $(\Delta X)_1 > \Delta X$ is shown in Fig. 2.3.

The second postulate is

$$\langle\, \hat{f}(X, t)\, \rangle_X = \langle\, \hat{f}(X, t)\, \rangle_t = f(X, t)\, . \tag{2.7}$$

In other words, the averaging of the distribution function over phase space is *equivalent* to the averaging over time.

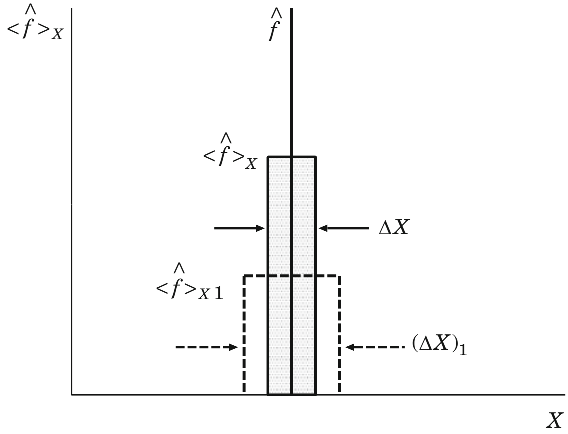


Fig. 2.3 Averaging of the exact distribution function $\hat{f}$ which is equal to a δ-function. ΔX is a small volume of the phase space X

While speaking of the small ΔX and Δt, we assume that they are not too small: ΔX must contain a reasonably large number of particles while Δt must be large in comparison with the duration of drastic changes of the exact distribution function, such as the duration of the particle Coulomb collisions: $\Delta t \gg \tau_c$. Thus we assume that

$$\tau_c \ll \Delta t \ll \tau_{ev} \tag{2.8}$$

and

$$\langle\, l\, \rangle \ll \Delta X \ll L\,. \tag{2.9}$$

Here $\langle\, l\, \rangle \approx n^{-1/3}$ is a mean distance between the particles, n is a number of the particles in a unit volume; L is a distance over which the macroscopic quantities of the system change considerably. In this case the statistical mechanism of particle "*mixing*" in phase space works, and

> the averaging of the exact distribution function over the time Δt is equivalent to the averaging over the phase volume ΔX.

2.1.3 A Statistical Mechanism of Mixing in Phase Space

Let us understand qualitatively how the mixing mechanism works in phase space. We start from the dynamical description of the N-particle system in $6N$-dimensional phase space in which

$$\Gamma = \{\, \mathbf{r}_i\,,\, \mathbf{v}_i \,\}\,, \qquad i = 1,\, 2, \ldots\, N, \tag{2.10}$$

a point is determined ($t = 0$ in Fig. 2.4) by the initial conditions of all the particles. The motion of this point, that is the dynamical evolution of the system, can be

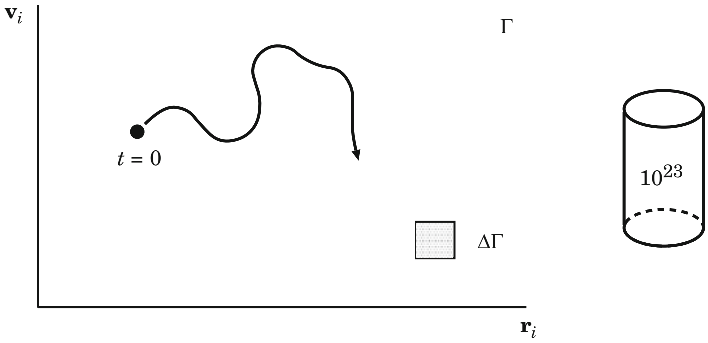


Fig. 2.4 The dynamical trajectory of a system of N interacting particles in the $6N$-dimensional phase space Γ

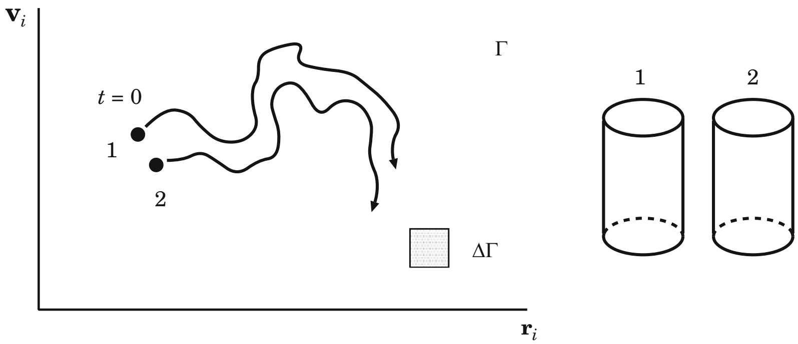


Fig. 2.5 The dynamical trajectories of two systems never cross each other

described by Liouville's equation or equations of motion. The point moves along a complicated *dynamical trajectory* because the interactions in a many-particle system are extremely intricate and complicated.

The dynamical trajectory has a remarkable property which we shall illustrate by the following example. Imagine a glass vessel containing a gas consisting of a large number N of particles (e.g., 10^{23} molecules or charged particles). The state of this gas at any moment of time is depicted by a single point in the phase space Γ.

Let us imagine another vessel which is identical to the first one, with one exception, being that at any moment of time the gas state in the second vessel is different from that in the first one. These states are depicted by two different points in the space Γ. For example, at $t = 0$, they are points 1 and 2 in Fig. 2.5.

With the passage of time, the gas states in both vessels change, whereas the two points in the space Γ draw two *different* dynamical trajectories (Fig. 2.5). These trajectories do *not* intersect. If they had intersected at just one point (see point A in Fig. 2.6a), then the state of the first gas, determined by $6N$ numbers $(\mathbf{r}_i, \mathbf{v}_i)$, would have coincided with the state of the second gas. These numbers could have

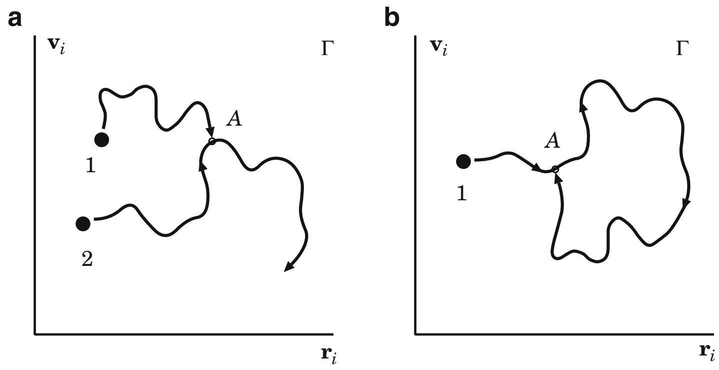


Fig. 2.6 Two types of unbelievable dynamical trajectories

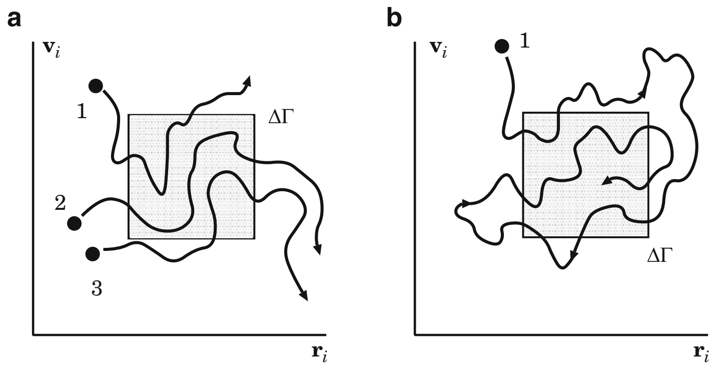


Fig. 2.7 Two types of statistical averaging

been taken as the initial conditions which, in turn, would have uniquely determined the motion. The two trajectories would have merged into one. For the same reason the trajectory of a system cannot intersect itself (Fig. 2.6b). Thus we come to the conclusion that

> only one dynamical trajectory of a many particle system passes through each point of the phase space Γ.

Since the trajectories differ in initial conditions, we can introduce an infinite *ensemble of systems* (glass vessels) corresponding to the different initial conditions. In a finite time the ensemble of dynamical trajectories will *closely* fill the phase space Γ, without intersections. By averaging over the ensemble we can answer the question of what the probability is that, at a moment of time t, the system will be found in an element $\Delta\Gamma = \Delta\mathbf{r}_i\ \Delta\mathbf{v}_i$ of the phase space Γ (see Fig. 2.7a):

$$dw = \langle\, \hat{f}(\mathbf{r}_i, \mathbf{v}_i)\,\rangle_\Gamma\ d\Gamma. \tag{2.11}$$

Here $\langle\, \hat{f}(\mathbf{r}_i, \mathbf{v}_i)\,\rangle_\Gamma$ is a function of all the coordinates and velocities. It plays the role of the *probability distribution density* in the phase space Γ and is called the statistical distribution function or simply the distribution function. It is obtained by way of statistical averaging over the ensemble and evidently corresponds to definition (2.3).

* * *

It is rather obvious that the same *probability density* can be obtained in another way—through the averaging over time. The dynamical trajectory of a system, given a sufficient time Δt, will closely cover the phase space Γ. There will be no self-intersections; but since the trajectory is very intricate it will repeatedly pass through the phase space element $\Delta\Gamma$ (Fig. 2.7b).

Let $(\Delta t\,)_\Gamma$ be the time during which the system locates in $\Delta\Gamma$. For a sufficiently large Δt, which is formally restricted by the characteristic time of a relatively slow evolution of the system as a whole, the ratio $(\Delta t\,)_\Gamma / \Delta t$ tends to the limit

$$\lim_{\Delta t \to \infty} \frac{(\,\Delta t\,)_\Gamma}{\Delta t} = \frac{dw}{d\,\Gamma} = \langle\, \hat{f}(\mathbf{r}_i, \mathbf{v}_i, t)\,\rangle_t\,. \tag{2.12}$$

By virtue of the role of the probability density, it is clear that

> the statistical averaging over the ensemble (2.11) is equivalent to the averaging over time (2.12) as well as to the definition (2.5).

2.1.4 The Derivation of a General Kinetic Equation

Now we have everything what we need to average the exact Liouville equation (2.2). Since the equation contains the derivatives with respect to time t and phase-space coordinates $(\mathbf{r}, \mathbf{v})$, the procedure of averaging over the small interval $\Delta X\, \Delta t$ is defined as follows:

$$f(X,t) = \frac{1}{\Delta X\, \Delta t} \int\limits_{\Delta X} \int\limits_{\Delta t} \hat{f}(X,t)\, dX\, dt\,. \tag{2.13}$$

Averaging the first term of the Liouville equation gives

$$\begin{aligned} \frac{1}{\Delta X\, \Delta t} \int\limits_{\Delta X} \int\limits_{\Delta t} \frac{\partial \hat{f}}{\partial t}\, dX\, dt &= \frac{1}{\Delta t} \int\limits_{\Delta t} \frac{\partial}{\partial t} \left[\frac{1}{\Delta X} \int\limits_{\Delta X} \hat{f}\, dX \right] dt = \\ &= \frac{1}{\Delta t} \int\limits_{\Delta t} \frac{\partial}{\partial t}\, f\, dt = \frac{\partial f}{\partial t}\,. \end{aligned} \tag{2.14}$$

In the last equality the use is made of the fact that, by virtue of the second postulate of statistics (2.7), the averaging of the smooth averaged function does not change it.

Let us average the second term in (2.2):

$$\frac{1}{\Delta X\,\Delta t}\int\limits_{\Delta X}\int\limits_{\Delta t} v_\alpha\,\frac{\partial \hat{f}}{\partial r_\alpha}\,dX\,dt = \frac{1}{\Delta X}\int\limits_{\Delta X} v_\alpha\,\frac{\partial}{\partial r_\alpha}\left[\frac{1}{\Delta t}\int\limits_{\Delta t}\hat{f}\,dt\right]dX = \\ = \frac{1}{\Delta X}\int\limits_{\Delta X} v_\alpha\,\frac{\partial}{\partial r_\alpha} f\,dX = v_\alpha\,\frac{\partial f}{\partial r_\alpha}\,. \tag{2.15}$$

Here the index $\alpha = 1, 2, 3$.

In order to average the term containing the force $\hat{\mathbf{F}}$, let us represent this exact force as a sum of a *mean force* $\langle\,\mathbf{F}\,\rangle$ and the force due to the difference of the real force field from the mean (statistically smoothed) one:

$$\hat{\mathbf{F}} = \langle\,\mathbf{F}\,\rangle + \mathbf{F}'. \tag{2.16}$$

Substituting definition (2.16) in the third term in (2.2) and averaging this term, we have

$$\frac{1}{\Delta X\,\Delta t}\int\limits_{\Delta X}\int\limits_{\Delta t}\frac{\hat{F}_\alpha}{m}\,\frac{\partial \hat{f}}{\partial v_\alpha}\,dX\,dt = \\ = \frac{\langle\,F_\alpha\,\rangle}{m}\,\frac{1}{\Delta X}\int\limits_{\Delta X}\frac{\partial}{\partial v_\alpha}\left[\frac{1}{\Delta t}\int\limits_{\Delta t}\hat{f}\,dt\right]dX + \frac{1}{\Delta X\,\Delta t}\int\limits_{\Delta X}\int\limits_{\Delta t}\frac{F'_\alpha}{m}\,\frac{\partial \hat{f}}{\partial v_\alpha}\,dX\,dt = \\ = \frac{\langle\,F_\alpha\,\rangle}{m}\,\frac{\partial f}{\partial v_\alpha} + \frac{1}{\Delta X\,\Delta t}\int\limits_{\Delta X}\int\limits_{\Delta t}\frac{F'_\alpha}{m}\,\frac{\partial \hat{f}}{\partial v_\alpha}\,dX\,dt\,. \tag{2.17}$$

Gathering all three terms together, we write the averaged Liouville equation in the form

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}\,f + \frac{\langle\,\mathbf{F}\,\rangle}{m}\cdot\nabla_{\mathbf{v}}\,f = \left(\frac{\partial \hat{f}}{\partial t}\right)_{\mathrm{c}},} \tag{2.18}$$

where

$$\boxed{\left(\frac{\partial \hat{f}}{\partial t}\right)_{\mathrm{c}} = -\,\frac{1}{\Delta X\,\Delta t}\int\limits_{\Delta X}\int\limits_{\Delta t}\frac{F'_\alpha}{m}\,\frac{\partial \hat{f}}{\partial v_\alpha}\,dX\,dt\,.} \tag{2.19}$$

Equation (2.18) and its right-hand side (2.19) are called the *kinetic equation* and the *collisional integral* (cf. definition (1.19)), respectively.

Therefore we have found the *most general* form of the kinetic equation with a collisional integral, which is nice but cannot be directly used in plasma astrophysics, without making some additional simplifying assumptions. The main assumption, the binary character of collisions, will be taken into account in the next section, see also Sect. 3.3.

2.2 A Collisional Integral and Correlation Functions

2.2.1 Binary Interactions

The statistical mechanism of mixing in phase space makes particles have no individuality. However we have to distinguish *different kinds* of particles, for example, electrons and protons, because their behaviors differ. Let $\hat{f}_k\,(\mathbf{r},\mathbf{v},t)$ be the exact distribution function (2.1) of particles of the *kind* k, i.e.

$$\hat{f}_k\,(\mathbf{r},\mathbf{v},t) = \sum_{i=1}^{N_k} \delta\,(\mathbf{r}-\mathbf{r}_{ki}(t))\;\delta\,(\mathbf{v}-\mathbf{v}_{ki}(t))\,, \tag{2.20}$$

the index i denoting the ith particle of kind k, N_k being the number of particles of kind k. The Liouville equation (2.2) for the particles of kind k takes a view

$$\frac{\partial \hat{f}_k}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{r}}\,\hat{f}_k + \frac{\hat{\mathbf{F}}_k}{m_k}\cdot\nabla_{\mathbf{v}}\,\hat{f}_k = 0\,, \tag{2.21}$$

m_k is the mass of a particle of kind k.

The force acting on a particle of kind k at a point $(\mathbf{r},\mathbf{v})$ of the phase space X at a moment of time t, $\hat{F}_{k,\alpha}\,(\mathbf{r},\mathbf{v},t)$, is the sum of forces acting on this particle from all other particles (see Fig. 2.8):

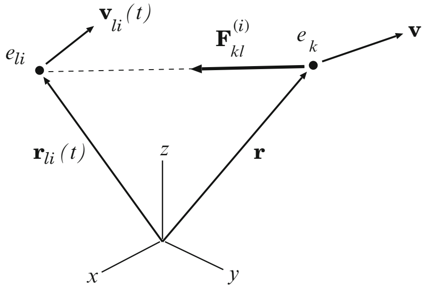
$$\hat{F}_{k,\alpha}\,(\mathbf{r},\mathbf{v},t) = \sum_{l}\sum_{i=1}^{N_l} \hat{F}^{(i)}_{kl,\alpha}\,(\mathbf{r},\mathbf{v},\mathbf{r}_{li}(t),\mathbf{v}_{li}(t)). \tag{2.22}$$

So the total force $\hat{F}_{k,\alpha}\,(\mathbf{r},\mathbf{v},t)$ depends upon the instant positions and velocities of all the particles generally with the time delay taken into account (see Landau and Lifshitz 1975, Chap. 8, Sect. 63).

With the help of the exact distribution function, we can rewrite formula (2.22) as follows:

$$\hat{F}_{k,\alpha}\,(\mathbf{r},\mathbf{v},t) = \sum_{l}\int\limits_{X_1} F_{kl,\alpha}\,(X,X_1)\;\hat{f}_l\,(X_1,t)\;dX_1\,. \tag{2.23}$$

Fig. 2.8 An action of a particle e_{li} located at the point $\mathbf{r}_{li}$ on a particle of kind k at a point $\mathbf{r}$ at a moment of time t



Here we assume that an interaction law $F_{kl,\alpha}(X, X_1)$ is explicitly independent of time t;

$$\hat{f}_l(X,t) = \sum_{i=1}^{N_l} \delta\left(X - X_{li}(t)\right)$$

is the exact distribution function of particles of kind l, the variable of integration is designated as $X_1 = \{\mathbf{r}_1, \mathbf{v}_1\}$ and $dX_1 = d^3\mathbf{r}_1\, d^3\mathbf{v}_1$.

Formula (2.23) takes into account that the forces considered are *binary* ones, i.e. they can be represented as a sum of interactions between *two* particles.

Making use of the representation (2.23), let us average the force term in the Liouville equation (2.2), as this has been done in formula (2.17). We have

$$\begin{aligned}
&\frac{1}{\Delta X\, \Delta t} \int\limits_{\Delta X} \int\limits_{\Delta t} \frac{1}{m_k} \hat{F}_{k,\alpha}(\mathbf{r}, \mathbf{v}, t)\, \frac{\partial \hat{f}_k}{\partial v_\alpha}\, dX\, dt = \\
&= \frac{1}{\Delta X\, \Delta t} \int\limits_{\Delta X} \int\limits_{\Delta t} \sum_l \int\limits_{X_1} \frac{1}{m_k} F_{kl,\alpha}(X, X_1)\, \hat{f}_l(X_1, t) \frac{\partial}{\partial v_\alpha}\, \hat{f}_k(X, t)\, dX\, dX_1\, dt = \\
&= \frac{1}{\Delta X} \int\limits_{\Delta X} \sum_l \int\limits_{X_1} \frac{1}{m_k} F_{kl,\alpha}(X, X_1) \times \\
&\quad \times \frac{\partial}{\partial v_\alpha} \left[\frac{1}{\Delta t} \int\limits_{\Delta t} \hat{f}_k(X, t)\, \hat{f}_l(X_1, t)\, dt \right] dX dX_1 .
\end{aligned} \tag{2.24}$$

Here we have taken into account that the exact distribution function $\hat{f}_l(X_1, t)$ is independent of the velocity $\mathbf{v}$, which is a part of the variable $X = \{\mathbf{r}, \mathbf{v}\}$ related to the particles of the kind k.

Formula (2.24) contains the *pair products* of the exact distribution functions of different particle kinds, as is natural for the case of *binary interactions*.

2.2.2 Binary Correlation

Let us represent the exact distribution function $\hat{f}_k$ as

$$\hat{f}_k\,(X,t) = f_k\,(X,t) + \hat{\varphi}_k\,(X,t)\,, \tag{2.25}$$

where $f_k\,(X,t)$ is the *statistically averaged* distribution function, $\hat{\varphi}_k\,(X,t)$ is the deviation of the exact distribution function from the averaged one. In general the deviation is not small, of course. It is obvious that, according to definition (2.25),

$$\hat{\varphi}_k\,(X,t) = \hat{f}_k\,(X,t) - f_k\,(X,t)\,;$$

hence

$$\langle\, \hat{\varphi}_k\,(X,t)\,\rangle = 0\,. \tag{2.26}$$

Let us consider the integrals of pair products, appearing in the averaged force term (2.24). In view of definition (2.25), they can be rewritten as

$$\frac{1}{\Delta t}\int\limits_{\Delta t} \hat{f}_k\,(X,t)\,\hat{f}_l\,(X_1,t)\,dt = f_k\,(X,t)\,f_l\,(X_1,t) + f_{kl}\,(X,X_1,t)\,, \tag{2.27}$$

where

$$f_{kl}\,(X,X_1,t) = \frac{1}{\Delta t}\int\limits_{\Delta t} \hat{\varphi}_k\,(X,t)\,\hat{\varphi}_l\,(X_1,t)\,dt\,. \tag{2.28}$$

The function f_{kl} is referred to as the *correlation function* or, more exactly, the *binary* correlation function.

The physical meaning of the correlation function is clear from (2.27). The left-hand side of (2.27) means the probability to find a particle of kind k at a point X of the phase space at a moment of time t *under condition* that a particle of kind l places at a point X_1 at the same time. By definition this is a *conditional probability* (e.g., Gnedenko 1965, Sects. 23 and 52). In the right-hand side of (2.27) the distribution function $f_k\,(X,t)$ characterizes the probability that a particle of kind k stays at a point X at a moment of time t. The function $f_l\,(X_1,t)$ plays the analogous role for the particles of kind l.

If the particles of kind k did not interact with those of kind l, then their distributions would be independent, i.e. probability densities would simply multiply:

$$\langle \hat{f}_k(X,t)\ \hat{f}_l(X_1,t)\rangle = f_k(X,t)\ f_l(X_1,t)\,. \tag{2.29}$$

So in the right-hand side of (2.27) there should be

$$f_{kl}(X, X_1, t) = 0\,. \tag{2.30}$$

In other words, there would be no correlation in the particle distribution.

We consider a system of interacting particles. With the proviso that the parameter characterizing the binary interaction, namely Coulomb collision considered below,

$$\zeta_{\mathrm{i}} \approx \frac{e^2}{\langle l\rangle} \Big/ \left\langle \frac{mv^2}{2} \right\rangle , \tag{2.31}$$

is small under conditions in a wide range, the correlation function must be *relatively small*:

if the interaction is weak, the second term in the right-hand side of (2.27) must be small in comparison with the first one.

We shall come back to the discussion of this property in Sect. 3.1. This fundamental property allows us to construct a theory of plasma in many cases of astrophysical interest.

2.2.3 *The Collisional Integral and Binary Correlation*

Now let us substitute (2.27) in formula (2.24) for the statistically averaged force term in the kinetic equation:

$$\frac{1}{\Delta X\,\Delta t} \int\limits_{\Delta X} \int\limits_{\Delta t} \frac{1}{m_k}\, \hat{F}_{k,\alpha}(X,t)\, \frac{\partial \hat{f}_k}{\partial v_\alpha}\, dX\, dt =$$

$$= \frac{1}{\Delta X} \int\limits_{\Delta X} \sum_l \int\limits_{X_1} \frac{1}{m_k}\, F_{kl,\alpha}(X, X_1)\, \frac{\partial}{\partial v_\alpha}\, [\, f_k(X,t)\ f_l(X_1,t) +$$

$$+\, f_{kl}(X, X_1, t)\,]\, dX\, dX_1 \ =$$

since: (a) the averaging of smooth functions does not change them, (b) the function $f_k(X,t)$ does not depend of X_1, and (c) the function $f_l(X_1,t)$ does not depend of X, we can proceed as follows

$$= \frac{1}{m_k} \left[\frac{\partial}{\partial v_\alpha} f_k(X,t) \right] \left\{ \sum_l \int_{X_1} F_{kl,\alpha}(X,X_1)\; f_l(X_1,t)\, dX_1 \right\} +$$

$$+ \sum_l \int_{X_1} \frac{1}{m_k} F_{kl,\alpha}(X,X_1)\, \frac{\partial}{\partial v_\alpha} f_{kl}(X,X_1,t)\, dX_1 =$$

$$= \frac{1}{m_k} F_{k,\alpha}(X,t)\, \frac{\partial f_k(X,t)}{\partial v_\alpha} +$$

$$+ \sum_l \int_{X_1} \frac{1}{m_k} F_{kl,\alpha}(X,X_1)\, \frac{\partial f_{kl}\ (X,X_1,t)}{\partial v_\alpha}\, dX_1\,. \tag{2.32}$$

Here we have taken into account the following definition of the *averaged force*

$$F_{k,\alpha}(X,t) = \sum_l \int_{X_1} F_{kl,\alpha}(X,X_1)\; f_l(X_1,t)\, dX_1\,. \tag{2.33}$$

This definition follows from averaging the definition (2.23) of the exact force $\hat{\mathbf{F}}_k$ and coincides with the previous definition (2.17) of the *mean* (smoothed or averaged) force, since

> all the deviations of the real force $\hat{\mathbf{F}}_k$ from the mean (smooth) force $\mathbf{F}_k$ are taken care of in the deviations $\hat{\varphi}_k$ and $\hat{\varphi}_l$ of the real distribution functions $\hat{f}_k$ and $\hat{f}_l$ from their mean values f_k and f_l.

Thus the collisional integral can be represented in the form

$$\left(\frac{\partial \hat{f}_k}{\partial t} \right)_{\mathrm{c}} = - \sum_l \int_{X_1} \frac{1}{m_k} F_{kl,\alpha}(X,X_1)\, \frac{\partial f_{kl}(X,X_1,t)}{\partial v_\alpha}\, dX_1\,. \tag{2.34}$$

Moreover, if in the last term of (2.32) the binary interactions can be represented by smooth functions of the type $e_k e_l\, (|\,\mathbf{r}_k - \mathbf{r}_l\,|)^{-2}$ with or without account of the Debye-Hückel shielding (see Sects. 3.2.3 and 8.2.1), then formally the velocity dependence may be neglected.

Let us recall an important particular case considered in Sect. 1.1. For the Lorentz force (1.15) as well as for the gravitational one (1.43), the condition (1.8) is satisfied. Let us require that in formula (2.34)

$$\frac{\partial}{\partial v_\alpha} F_{kl,\alpha}(X,X_1) = 0\,. \tag{2.35}$$

In fact this condition was tacitly assumed from the early beginning, from (2.2). Anyway, in the case (2.35), we obtain from formula (2.34) the following expression

$$\left(\frac{\partial \hat{f}_k}{\partial t} \right)_{\rm c} = - \frac{\partial}{\partial v_\alpha} \sum_l \int_{X_1} \frac{1}{m_k} F_{kl,\alpha} \left(X, X_1 \right) \; f_{kl} \left(X, X_1, t \right) \; dX_1 \, . \tag{2.36}$$

Hence

> the collisional integral, at least, for the Coulomb and gravity forces can be written in the *divergent* form in the *velocity* space $\mathbf{v}$:

$$\boxed{\left(\frac{\partial \hat{f}_k}{\partial t} \right)_c = - \frac{\partial}{\partial v_\alpha} J_{k,\alpha} \, ,} \tag{2.37}$$

where the flux of particles of kind k in the velocity space (cf. Fig. 1.3b) is

$$J_{k,\alpha} \left(X, t \right) = \sum_l \int_{X_1} \frac{1}{m_k} F_{kl,\alpha} \left(X, X_1 \right) \; f_{kl} \left(X, X_1, t \right) dX_1 \, . \tag{2.38}$$

Therefore we arrive to conclusion that the averaged Liouville equation or *the kinetic equation for particles of kind k*

$$\begin{aligned} \frac{\partial f_k \left(X, t \right)}{\partial t} + v_\alpha \frac{\partial f_k \left(X, t \right)}{\partial r_\alpha} + \frac{F_{k,\alpha} \left(X, t \right)}{m_k} \frac{\partial f_k \left(X, t \right)}{\partial v_\alpha} = \\ = - \frac{\partial}{\partial v_\alpha} \sum_l \int_{X_1} \frac{1}{m_k} F_{kl,\alpha} \left(X, X_1 \right) \; f_{kl} \left(X, X_1, t \right) dX_1 \end{aligned} \tag{2.39}$$

contains the *unknown* function f_{kl}. Hence the kinetic equation (2.39) for distribution function f_k is not closed. We have to find the equation for the correlation function f_{kl} . This will be done in the next section.

2.3 Equations for Correlation Functions

To derive the equations for correlation functions (in the first place for the function of pair correlations f_{kl}), it is not necessary to introduce any new postulates or develop new formalisms. All the necessary equations and averaging procedures are at hand.

Looking at definition (2.28), we see that we need an equation which will describe the deviation of distribution function from its mean value, i.e. the function

$\hat{\varphi}_k = \hat{f}_k - f_k$. In order to derive such equation, we simply have to subtract the averaged representation (2.39) from the exact Liouville equation (2.2). The result is

$$\frac{\partial\,\hat{\varphi}_k\,(X,t)}{\partial t} + v_\alpha\,\frac{\partial\,\hat{\varphi}_k\,(X,t)}{\partial r_\alpha} + \frac{\hat{F}_{k,\alpha}}{m_k}\,\frac{\partial\,\hat{f}_k}{\partial v_\alpha} - \frac{F_{k,\alpha}}{m_k}\,\frac{\partial f_k}{\partial v_\alpha} =$$
$$= \frac{\partial}{\partial v_\alpha}\sum_l \int\limits_{X_1} \frac{1}{m_k}\,F_{kl,\alpha}\,(X,X_1)\ f_{kl}\,(X,X_1)\ dX_1\,. \tag{2.40}$$

Here

$$\hat{F}_{k,\alpha}\,(X,t) = \sum_l \int\limits_{X_1} F_{kl,\alpha}\,(X,X_1)\ \hat{f}_l\,(X_1,t)\ dX_1 \tag{2.41}$$

is the *exact* force (2.23) acting on a particle of the kind k at the point X of phase space, and

$$F_{k,\alpha}\,(X,t) = \sum_l \int\limits_{X_1} F_{kl,\alpha}\,(X,X_1)\ f_l\,(X_1,t)\ dX_1 \tag{2.42}$$

is the statistically *averaged* force (2.33).

Thus the difference between the exact force and the averaged one is

$$\hat{F}_{k,\alpha} - F_{k,\alpha} = \sum_l \int\limits_{X_1} F_{kl,\alpha}\,(X,X_1)\ \hat{\varphi}_l\,(X_1,t)\ dX_1\,. \tag{2.43}$$

We substitute it in (2.40) where, at first, the difference of force terms can be rewritten as follows:

$$\frac{\hat{F}_{k,\alpha}}{m_k}\,\frac{\partial\,\hat{f}_k}{\partial v_\alpha} - \frac{F_{k,\alpha}}{m_k}\,\frac{\partial f_k}{\partial v_\alpha} = \frac{\hat{F}_{k,\alpha}}{m_k}\,\frac{\partial}{\partial v_\alpha}\,(f_k + \hat{\varphi}_k) - \frac{F_{k,\alpha}}{m_k}\,\frac{\partial f_k}{\partial v_\alpha} =$$
$$= \frac{\hat{F}_{k,\alpha} - F_{k,\alpha}}{m_k}\,\frac{\partial f_k}{\partial v_\alpha} + \frac{\hat{F}_{k,\alpha}}{m_k}\,\frac{\partial\,\hat{\varphi}_k}{\partial v_\alpha}\,.$$

The result of the substitution is

$$\frac{\hat{F}_{k,\alpha}}{m_k}\,\frac{\partial\,\hat{f}_k}{\partial v_\alpha} - \frac{F_{k,\alpha}}{m_k}\,\frac{\partial f_k}{\partial v_\alpha} =$$
$$= \sum_l \int\limits_{X_1} \frac{1}{m_k}\,F_{kl,\alpha}\,(X,X_1)\ \hat{\varphi}_l\,(X_1,t)\ dX_1\,\frac{\partial f_k}{\partial v_\alpha} + \frac{F_{k,\alpha}}{m_k}\,\frac{\partial\,\hat{\varphi}_k}{\partial v_\alpha} +$$
$$+ \sum_l \int\limits_{X_1} \frac{1}{m_k}\,F_{kl,\alpha}\,(X,X_1)\ \hat{\varphi}_l\,(X_1,t)\ dX_1\,\frac{\partial\,\hat{\varphi}_k}{\partial v_\alpha}\,. \tag{2.44}$$

On substituting (2.44) in (2.40) we have the equation for the deviation $\hat{\varphi}_k$ of the exact distribution function $\hat{f}_k$ from its mean value f_k:

$$\frac{\partial\,\hat{\varphi}_k\,(X,t)}{\partial t} + v_\alpha\,\frac{\partial\,\hat{\varphi}_k\,(X,t)}{\partial r_\alpha} + \cdots = 0\,. \tag{2.45}$$

Considering that we have to derive the equation for the pair correlation function

$$f_{kl}\,(X_1, X_2, t) = \langle\,\hat{\varphi}_k\,(X_1,t)\;\hat{\varphi}_l\,(X_2,t)\,\rangle\,,$$

let us take two equations:

one for $\hat{\varphi}_k\,(X_1,t)$

$$\frac{\partial\,\hat{\varphi}_k\,(X_1,t)}{\partial t} + v_{1,\alpha}\,\frac{\partial\,\hat{\varphi}_k\,(X_1,t)}{\partial\,r_{1,\alpha}} + \frac{F_{k,\alpha}}{m_k}\,\frac{\partial\,\hat{\varphi}_k\,(X_1,t)}{\partial\,v_{1,\alpha}} + \cdots = 0 \tag{2.46}$$

and another for $\hat{\varphi}_l\,(X_2,t)$

$$\frac{\partial\,\hat{\varphi}_l\,(X_2,t)}{\partial t} + v_{2,\alpha}\,\frac{\partial\,\hat{\varphi}_l\,(X_2,t)}{\partial\,r_{2,\alpha}} + \frac{F_{l,\alpha}}{m_l}\,\frac{\partial\,\hat{\varphi}_l\,(X_2,t)}{\partial\,v_{2,\alpha}} + \cdots = 0\,. \tag{2.47}$$

Now we add the equations resulting from (2.46) multiplied by $\hat{\varphi}_l$ and (2.47) multiplied by $\hat{\varphi}_k$. We obtain

$$\hat{\varphi}_l\,\frac{\partial\,\hat{\varphi}_k}{\partial t} + \hat{\varphi}_k\,\frac{\partial\,\hat{\varphi}_l}{\partial t} + v_{1,\alpha}\,\frac{\partial\,\hat{\varphi}_k}{\partial\,r_{1,\alpha}}\,\hat{\varphi}_l + v_{2,\alpha}\,\frac{\partial\,\hat{\varphi}_l}{\partial\,r_{2,\alpha}}\,\hat{\varphi}_k + \cdots = 0$$

or

$$\frac{\partial\,(\hat{\varphi}_k\,\hat{\varphi}_l)}{\partial t} + v_{1,\alpha}\,\frac{\partial\,(\hat{\varphi}_k\,\hat{\varphi}_l)}{\partial\,r_{1,\alpha}} + v_{2,\alpha}\,\frac{\partial\,(\hat{\varphi}_k\,\hat{\varphi}_l)}{\partial\,r_{2,\alpha}} + \cdots = 0\,. \tag{2.48}$$

On averaging (2.48) we finally have the equation for the *pair correlation* function in the following form:

$$\begin{aligned}
&\frac{\partial f_{kl}\,(X_1,X_2,t)}{\partial t} + v_{1,\alpha}\,\frac{\partial f_{kl}\,(X_1,X_2,t)}{\partial\,r_{1,\alpha}} + v_{2,\alpha}\,\frac{\partial f_{kl}\,(X_1,X_2,t)}{\partial\,r_{2,\alpha}} + \\
&+ \frac{F_{k,\alpha}\,(X_1,t)}{m_k}\,\frac{\partial f_{kl}\,(X_1,X_2,t)}{\partial\,v_{1,\alpha}} + \frac{F_{l,\alpha}\,(X_2,t)}{m_l}\,\frac{\partial f_{kl}\,(X_1,X_2,t)}{\partial\,v_{2,\alpha}} + \\
&+ \frac{\partial f_k\,(X_1,t)}{\partial\,v_{1,\alpha}}\sum_n \int\limits_{X_3} \frac{1}{m_k}\,F_{kn,\alpha}\,(X_1,X_3)\;f_{nl}\,(X_3,X_2,t)\;dX_3\; + \\
&+ \frac{\partial f_l\,(X_2,t)}{\partial\,v_{2,\alpha}}\sum_n \int\limits_{X_3} \frac{1}{m_l}\,F_{ln,\alpha}\,(X_2,X_3)\;f_{nk}\,(X_3,X_1,t)\;dX_3\; =
\end{aligned}$$

$$= -\frac{\partial}{\partial v_{1,\alpha}} \sum_{n} \int_{X_3} \frac{1}{m_k} F_{kn,\alpha}(X_1, X_3)\ f_{kln}(X_1, X_2, X_3, t)\ dX_3 -$$

$$-\frac{\partial}{\partial v_{2,\alpha}} \sum_{n} \int_{X_3} \frac{1}{m_l} F_{ln,\alpha}(X_2, X_3)\ f_{kln}(X_1, X_2, X_3, t)\ dX_3 . \qquad (2.49)$$

Here

$$f_{kln}(X_1, X_2, X_3, t) = \frac{1}{\Delta t} \int_{\Delta t} \hat{\varphi}_k(X_1, t)\ \hat{\varphi}_l(X_2, t)\ \hat{\varphi}_n(X_3, t)\ dt \qquad (2.50)$$

is the function of *triple correlations* (see also Exercise 2.51).

Thus (2.49) for the pair correlation function contains the *unknown* function of triple correlations. In general,

> the chain of equations for correlation functions can be shown to be *unclosed*: the equation for the correlation function of sth order contains the function of the order $(s+1)$.

2.4 Practice: Exercises and Answers

Exercise 2.1 (Sect. 2.3). By analogy with formula (2.27), show that

$$\begin{aligned} &\langle\, \hat{f}_k(X_1, t)\ \hat{f}_l(X_2, t)\ \hat{f}_n(X_3, t)\,\rangle = \\ &= f_k(X_1, t)\ f_l(X_2, t)\ f_n(X_3, t) + \\ &\quad + f_k(X_1, t)\ f_{ln}(X_2, X_3, t) + f_l(X_2, t)\ f_{kn}(X_1, X_3, t) + \\ &\quad + f_n(X_3, t)\ f_{kl}(X_1, X_2, t) + f_{kln}(X_1, X_2, X_3, t) . \end{aligned} \qquad (2.51)$$

Exercise 2.2. Discuss a similarity and difference between the kinetic theory presented in this chapter and the famous BBGKY hierarchy theory developed by Bogoliubov (1946), Born and Green (1949), Kirkwood (1946), and Yvon (1935).

Hint. Show that essential to both derivations is the weak-coupling assumption, according to which

> grazing encounters, involving small fractional energy and momentum exchange between colliding particles, dominate the evolution of the velocity distribution function.

The *weak-coupling* assumption provides justification of the widely appreciated practice (see Chap. 3) which leads to a very significant simplification of the original collisional integral; for more detail see Klimontovich (1975, 1986).

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