

Chapter 2

Connectedness

Abstract A fundamental question about inverse limits with set-valued bonding functions relates to the connectedness of the inverse limit. For inverse limits on compact, connected factor spaces with bonding functions that are mappings, the inverse limit is always connected. However, for inverse limits with set-valued functions as bonding functions, the inverse limit is rarely connected. One might suspect that this is due to the fact that the graph of an upper semicontinuous function on a compact, connected space can fail to be connected, but the reasons go much deeper. In this chapter we study connectedness of inverse limits on $[0, 1]$ with set-valued functions.

2.1 Introduction

All but one of the examples from Chap. 1 were not connected, and after examining those examples, one might conjecture that if the graphs of the bonding functions are connected, then the inverse limit is connected. This is not the case, and in Example 2.1 of Sect. 2.2, we provide the first of several examples of functions having connected graphs and a nonconnected inverse limit. In fact, there are examples of inverse limit sequences with only one bonding function such that the inverse limit is totally disconnected even though the graph of the bonding function is connected. Although their example is beyond the scope of this book, it has been shown by Sina Greenwood and Judy Kennedy that there exists an inverse limit on $[0, 1]$ with a single surjective set-valued bonding function having a connected graph such that the inverse limit is a Cantor set. They go on to show that in some sense most inverse limits with upper semicontinuous bonding functions are not connected. In this chapter we discuss connectedness of the inverse limit with upper semicontinuous bonding functions. Much of our discussion relates, in one way or another, the unsolved problem of characterizing connectedness of inverse limits with upper semicontinuous bonding functions in terms of the bonding functions.

One of the major problems in the theory of inverse limits with set-valued functions is the question of under what conditions is the inverse limit connected (see Problem 6.1 in Chap. 6). A word about a solution to the problem of characterizing connectedness is in order. In the next section, we actually present a solution to the problem. However, the solution is not very satisfying because it is not given specifically in terms of the nature of the bonding functions. Ideally, a solution would allow us to determine the connectedness of the inverse limit by an examination of the bonding functions. From this perspective, the problem remains unsolved even in the case that each factor space is the interval $[0, 1]$.

2.2 A Characterization of Connectedness

We begin our discussion of connectedness with a theorem that characterizes this property for inverse limits. Unfortunately, when the bonding functions are set-valued, it is rarely easy to verify that the hypothesis of Theorem 2.1 is satisfied, so the theorem is not very useful except under special circumstances. As with many theorems in this book, the following theorem holds in a much more general setting than we state. For a more general theorem, see [6, Theorem 116, p. 85] where it is shown that the connectedness of the inverse limit follows from the connectedness of the terms of the sequence \mathbf{G} . Recall from Chap. 1 that, for a sequence \mathbf{X} of closed subsets of $[0, 1]$ and a sequence \mathbf{f} of upper semicontinuous functions such that $f_i : X_{i+1} \rightarrow 2^{X_i}$ for each positive integer i , G_n is defined to be $\{\mathbf{x} \in \mathcal{Q} \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}$.

Theorem 2.1. *Suppose \mathbf{X} is a sequence of closed subsets of $[0, 1]$ and \mathbf{f} is a sequence of upper semicontinuous functions such that $f_i : X_{i+1} \rightarrow 2^{X_i}$ for each positive integer i . Then, $\varprojlim \mathbf{f}$ is a continuum if and only if G_n is connected for each positive integer n .*

Proof. Let $M = \varprojlim \mathbf{f}$. We showed in Theorem 1.5 that G_n is compact for each positive integer n and $M = \bigcap_{n>0} G_n$. Thus, if G_n is connected for each $n \in \mathbb{N}$, then M is a continuum, being the intersection of a nested sequence of subcontinua of \mathcal{Q} .

On the other hand, if M is connected, then $\pi_{\{1,2,\dots,n+1\}}(M)$ is connected for each positive integer n because $\pi_{\{1,2,\dots,n+1\}}$ is a mapping. However, $G_n = \pi_{\{1,2,\dots,n+1\}}(M) \times \prod_{i>n+1} X_i$, a connected set because it is a product of two connected sets. \square

The proof of Theorem 2.1 makes use of the continuity of π_A where $A \subseteq \mathbb{N}$ to conclude that the image of a connected set under π_A is connected. This leads to our next theorem that was first observed by Van Nall in [10]. One can often obtain information about inverse limits with mappings by examining composites of the bonding maps, especially if the inverse limit in question is produced by a single bonding map. As we shall see later, this is rarely the case when the bonding

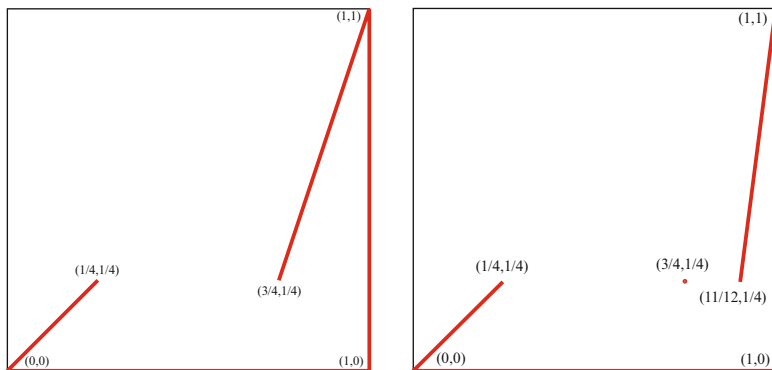


Fig. 2.1 Graph of the bonding function f (left) in Example 2.1 and f^2 (right)

functions are set-valued. However, Theorem 2.2 provides one of the few cases in the theory of inverse limits where composites provide some insight into the nature of the inverse limit when the bonding functions are set-valued.

Theorem 2.2 (Nall). *Suppose X is a sequence of closed subsets of $[0, 1]$ and f is a sequence of surjective set-valued functions such that $f_i : X_{i+1} \rightarrow 2^{X_i}$ is upper semicontinuous for each positive integer i . If m and n are positive integers such that $m < n$ and $G(f_{mn})$ is not connected, then $\varprojlim f$ is not connected.*

Proof. If $\varprojlim f$ is connected and $m, n \in \mathbb{N}$ with $m < n$, then $\pi_{\{m,n\}}(\varprojlim f)$ is connected. However, by using Theorem 1.8, we see that $\pi_{\{m,n\}}(\varprojlim f) = (G(f_{mn}))^{-1}$, so $G(f_{mn})$ is connected, a contradiction. \square

One consequence of Theorem 2.2 is that if an inverse limit with surjective bonding functions is connected, then each of the bonding functions (including compositions) has a connected graph. In Example 1.8, we saw that an inverse limit using nonsurjective bonding functions with graphs that are not connected can be connected. Our first example of this chapter is a surjective upper semicontinuous function having a connected graph and an inverse limit that is not connected. It was first published in [5] but with a different proof that it is not connected.

Example 2.1. Let $f : [0, 1] \rightarrow 2^{[0,1]}$ be given by $f(t) = \{0, t\}$ for $0 \leq t \leq 1/4$, $f(t) = 0$ for $1/4 < t < 3/4$, $f(t) = \{3t-2, 0\}$ for $3/4 \leq t < 1$, and $f(1) = [0, 1]$. Then, $G(f)$ is connected, but $\varprojlim f$ is not connected. (See the graph on the left in Fig. 2.1 for $G(f)$.)

Proof. It is not difficult to show that $f^2(t) = \{t, 0\}$ for $0 \leq t \leq 1/4$, $f^2(t) = 0$ for $1/4 < t < 3/4$, $f^2(3/4) = \{1/4, 0\}$, $f^2(t) = 0$ for $3/4 < t < 11/12$, and $f^2(t) = \{9t-8, 0\}$ for $11/12 \leq t \leq 1$ ($G(f^2)$ are pictured on the right in Fig. 2.1). Because $(3/4, 1/4)$ is an isolated point of $G(f^2)$, it follows from Theorem 2.2 that $\varprojlim f$ is not connected. \square

To conclude this section, we present a second simple theorem, this one characterizing connectedness of an inverse limit on intervals with a single bonding function. Nall included this theorem for inverse limits on Hausdorff continua in [10, Theorem 3.3, p. 171]. The proof given here is based on his proof.

Theorem 2.3. *Suppose $f : [0, 1] \rightarrow 2^{[0,1]}$ is an upper semicontinuous function that is surjective. Then, $\varprojlim f$ is connected if and only if $\varprojlim f^{-1}$ is connected.*

Proof. Suppose $n \in \mathbb{N}$. Observe that $h : \mathcal{Q} \rightarrow \mathcal{Q}$ given by $h(\mathbf{x}) = (x_{n+1}, x_n, \dots, x_1, x_{n+2}, x_{n+3}, \dots)$ is a homeomorphism. Let $g = f^{-1}$. If $G_n = \{\mathbf{x} \in \mathcal{Q} \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}$ and $H_n = \{\mathbf{x} \in \mathcal{Q} \mid x_i \in g(x_{i+1}) \text{ for } 1 \leq i \leq n\}$, then $h(G_n) = H_n$, and thus H_n is connected if and only if G_n is connected. The theorem now follows from Theorem 2.1. \square

In Theorem 2.10 below, we show that a theorem similar to Theorem 2.3 holds if the constant sequence f is replaced by a sequence of mappings. It would be interesting to know if the constant sequence in Theorem 2.3 can be replaced by a sequence of surjective upper semicontinuous functions (see Problem 6.7).

2.3 A Basic Connectedness Theorem

Theorem 2.7 as stated below for sequences of upper semicontinuous functions on subintervals of $[0, 1]$ appeared in Mahavier's original paper [7] on inverse limits with subsets of $[0, 1] \times [0, 1]$. It was generalized in [5] to inverse limits of sequences of compact, connected Hausdorff spaces with upper semicontinuous bonding functions and in [6] to consistent inverse limit systems on compact, connected Hausdorff spaces with upper semicontinuous bonding functions over directed sets. Although we state and prove it for inverse limits on subintervals of $[0, 1]$, we give a different proof of the theorem based on the following theorems not found in those references.

Suppose that each of X and Y is a continuum. A mapping $f : X \rightarrow Y$ is said to be *monotone* provided $f^{-1}(y)$ is connected for each $y \in f(X)$. In the case that f^{-1} is a surjective mapping, Theorem 2.4 follows from the well-known theorem that a surjective map of continua is monotone if and only if the preimage of each subcontinuum of the range is a subcontinuum of the domain [9, Exercise 8.46, p. 137]. We present a proof of a slightly different theorem based on a direct use of upper semicontinuity. Recall our notation that if $f : X \rightarrow 2^Y$ and $A \subseteq X$, by $f(A)$ we mean $\{y \in Y \mid \text{there is a point } x \in A \text{ such that } y \in f(x)\}$.

Theorem 2.4. *Suppose X is a continuum, Y is a compact metric space, and $f : X \rightarrow C(Y)$ is an upper semicontinuous function. Then, $f(X)$ is a continuum.*

Proof. Because $G(f)$ is compact and $f(X) = p(G(f))$ where p is the projection of $X \times Y$ onto Y , $f(X)$ is compact. If $f(X)$ is not connected, then there are two mutually exclusive compact sets H and K such that $f(X) = H \cup K$. The normality of metric spaces provides two mutually exclusive open sets U and V

such that $H \subseteq U$ and $K \subseteq V$. If $x \in X$, $f(x)$ is connected so $f(x) \subseteq H$ or $f(x) \subseteq K$. Let $X_H = \{x \in X \mid f(x) \subseteq H\}$ and $X_K = \{x \in X \mid f(x) \subseteq K\}$. Then, $X = X_H \cup X_K$, and no point of X belongs to both X_H and X_K . If $t \in X_H$, then $f(t) \subseteq U$. Because f is upper semicontinuous at t , there is an open set W containing t such that if $s \in W$, then $f(s) \subseteq U$. Then, $W \subseteq X_H$, and it follows that X_H is open. Therefore, X_K is closed. A similar argument yields that X_H is closed. This involves a contradiction because the continuum X is not the union of two mutually exclusive closed sets. \square

Theorem 2.5. *Suppose X is a continuum and Y is a compact metric space. If $f : X \rightarrow C(Y)$ is upper semicontinuous, then $G(f)$ is a continuum.*

Proof. Let $\varphi : X \rightarrow X \times Y$ be function given by $\varphi(x) = \{x\} \times f(x)$; φ is upper semicontinuous by Theorem 1.3. Then $G(f) = \varphi(X)$ and $\varphi(X)$ is a continuum by Theorem 2.4. \square

Next we extend our notion of the graph of an upper semicontinuous function in the following way. Suppose $\{X_1, X_2, \dots, X_{n+1}\}$ is a finite collection of metric spaces and $\{f_1, f_2, \dots, f_n\}$ is a finite collection of functions such that $f_i : X_{i+1} \rightarrow 2^{X_i}$ for $1 \leq i \leq n$. Define $G'(f_1, f_2, \dots, f_n) = \{\mathbf{x} \in \prod_{i=1}^{n+1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}$. Note that if $f : X \rightarrow 2^Y$ is a function, $G'(f) = (G(f))^{-1} = G(f^{-1})$.

Lemma 2.1. *Suppose $\{X_1, X_2, \dots, X_{n+1}\}$ is a finite collection of closed subsets of $[0, 1]$ and $\{f_1, f_2, \dots, f_n\}$ is a finite collection of upper semicontinuous functions such that $f_i : X_{i+1} \rightarrow 2^{X_i}$ for $1 \leq i \leq n$. Then, $G'(f_1, f_2, \dots, f_n)$ is compact.*

Proof. For $i > n + 1$, let $X_i = X_{n+1}$ and $f_i = Id_{X_{n+1}}$. By Theorem 1.5, $G_n = \{\mathbf{x} \in \prod_{i \geq 0} X_i \mid x_i \in f_i(x_{i+1})\}$ is nonempty and compact. Because $G'(f_1, f_2, \dots, f_n) = \pi_{\{1, 2, \dots, n+1\}}(G_n)$, the conclusion follows. \square

Lemma 2.2. *Suppose $\{X_1, X_2, \dots, X_{n+1}\}$ is a finite collection of continua, $\{f_1, f_2, \dots, f_n\}$ is a finite collection of upper semicontinuous functions such that $f_i : X_{i+1} \rightarrow 2^{X_i}$ for $2 \leq i \leq n$, and $f_1 : X_2 \rightarrow C(X_1)$. If $G'(f_2, f_3, \dots, f_n)$ is connected, then $G'(f_1, f_2, \dots, f_n)$ is a continuum.*

Proof. In light of Lemma 2.1, we only need to show that $G'(f_1, f_2, \dots, f_n)$ is connected. Let p denote the projection of $\prod_{i=2}^{n+1} X_i$ to its first factor space X_2 . Because p is a mapping, $f_1 \circ p$ is upper semicontinuous. The function $\psi : G'(f_2, f_3, \dots, f_n) \rightarrow G'(f_1, f_2, \dots, f_n)$ given by $\psi(\mathbf{x}) = f_1(p(\mathbf{x})) \times \{\mathbf{x}\}$ is upper semicontinuous. It follows that $G'(f_1, f_2, \dots, f_n)$ is connected by Theorem 2.4 being the image of $G'(f_2, f_3, \dots, f_n)$ under the upper semicontinuous function ψ . \square

Theorem 2.6. *Suppose $\{X_1, X_2, \dots, X_{n+1}\}$ is a finite collection of continua and $\{f_1, f_2, \dots, f_n\}$ is a finite collection of upper semicontinuous functions such that $f_i : X_{i+1} \rightarrow C(X_i)$ for $1 \leq i \leq n$. Then, $G'(f_1, f_2, \dots, f_n)$ is a continuum.*

Proof. If the finite collection of continua contains only one function f_1 , then $G'(f_1) = (G(f_1))^{-1}$ is a continuum by Theorem 2.5.

Inductively, suppose $k \geq 2$ is an integer such that if $\{X_1, X_2, \dots, X_k\}$ is a collection of k continua and $\{g_1, g_2, \dots, g_{k-1}\}$ is a collection of $k - 1$ upper semicontinuous functions such that $g_i : X_{i+1} \rightarrow C(X_i)$ for $1 \leq i \leq k - 1$, then $G'(g_1, g_2, \dots, g_{k-1})$ is a continuum. Let $\{X_1, X_2, \dots, X_{k+1}\}$ be a collection of $k + 1$ continua and let $\{f_1, f_2, \dots, f_k\}$ be a collection of k upper semicontinuous functions such that $f_i : X_{i+1} \rightarrow C(X_i)$ for $1 \leq i \leq k$. By the inductive hypothesis, $G'(f_2, f_3, \dots, f_k)$ is a continuum. It follows from Lemma 2.2 that $G'(f_1, f_2, \dots, f_k)$ is a continuum. \square

We now prove the main theorem of this section.

Theorem 2.7. *Suppose X is a sequence of subintervals of $[0, 1]$ and f is a sequence of upper semicontinuous functions such that $f_i : X_{i+1} \rightarrow C(X_i)$. Then, $\varprojlim f$ is a continuum.*

Proof. Suppose $n \in \mathbb{N}$ and note that $G_n = G'(f_1, f_2, \dots, f_n) \times \prod_{i>n+1} X_i$. By Theorem 2.6, $G'(f_1, f_2, \dots, f_n)$ is a continuum, so G_n is a continuum being the product of two continua. That $\varprojlim f$ is a continuum now follows from Theorem 2.1. \square

A stronger statement than that of Theorem 2.7 is true. We leave its proof to the interested reader. A proof in a very general setting may be found in [6, Theorem 125, p. 89].

Theorem 2.8. *Suppose X is a sequence of subintervals of $[0, 1]$ and f is a sequence of upper semicontinuous functions such that $f_i : X_{i+1} \rightarrow 2^{X_i}$. Suppose further that for each positive integer i such that f_i does not have connected values, $f_i(X_{i+1})$ is connected and $f_i^{-1}(x)$ is an interval for each $x \in f_i(X_{i+1})$. Then, $\varprojlim f$ is a continuum.*

By way of contrast to Theorem 2.7, in Example 2.7 below we see that an inverse limit can be connected even if most of the values of the bonding function are totally disconnected. This is only one of many such examples to be found in this book.

2.4 Examples

We consider some examples having connected inverse limits in which the connectedness is a consequence of Theorem 2.7. However, in these examples, we show more than that continua are produced in the inverse limit. We are able to say something about the nature of the inverse limit; in fact, in some cases, we identify what the inverse limit is and provide a model for it. Such is the case in our next example.

Example 2.2 (An arc). Let $f : [0, 1] \rightarrow C([0, 1])$ be given by $f(t) = 0$ for $0 \leq t < 1$ and $f(1) = [0, 1]$. Then $\varprojlim f$ is an arc (see the graph on the left side of Fig. 2.2 for $G(f)$ and Fig. 2.3 for a model of the inverse limit).

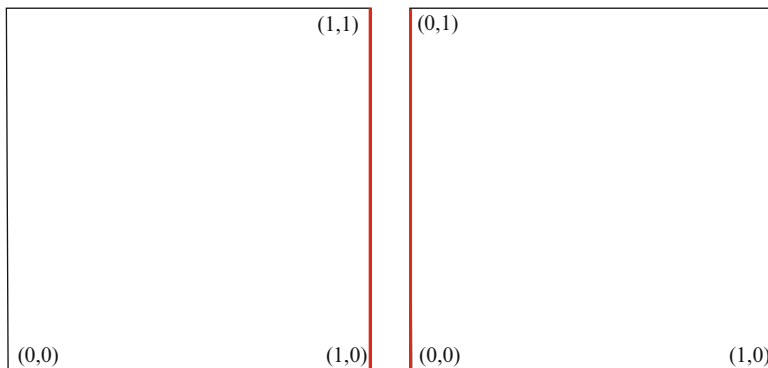
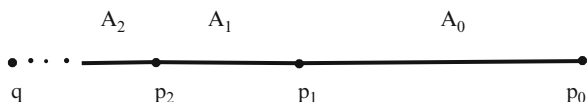


Fig. 2.2 Graphs of the bonding functions in Examples 2.2 (left) and 2.3 (right)

Fig. 2.3 A model of the arc that is the inverse limit in Example 2.2



Proof. Let $M = \varprojlim f$. The connectedness of M follows from Theorem 2.7, but we wish to conclude that the inverse limit is an arc. To that end, let $q = (0, 0, 0, \dots)$, $p_0 = (1, 1, 1, \dots)$, and $A_0 = \{x \in M \mid x_i = 1 \text{ for } i > 1\}$. For each positive integer n , let p_n be the point whose first n coordinates are 0 and all remaining coordinates are 1 and let $A_n = \{x \in M \mid x_i = 0 \text{ for } i \leq n \text{ and } x_i = 1 \text{ for } i > n + 1\}$. Then, for each $n \geq 0$, A_n is an arc and $A_n \cap A_{n+1} = \{p_{n+1}\}$. It is not difficult to see that $M = (\bigcup_{n \geq 0} A_n) \cup \{(0, 0, 0, \dots)\}$, and if $x_i \in A_i$ for each positive integer i , then x_1, x_2, x_3, \dots converges to q . It follows that M is an arc. \square

If we move the vertical line to the left side of $[0, 1]^2$ in Example 2.2, we get an entirely different inverse limit.

Example 2.3 (An infinite-dimensional continuum). Let $f : [0, 1] \rightarrow C([0, 1])$ be given by $f(0) = [0, 1]$ and $f(t) = 0$ for $0 < t \leq 1$. Then $\varprojlim f$ is an infinite-dimensional continuum (see the graph on the right side of Fig. 2.2 for $G(f)$).

Proof. The inverse limit is a continuum by Theorem 2.7, and it contains a continuum homeomorphic to \mathcal{Q} , namely, $([0, 1] \times \{0\})^\infty$. \square

Placing the vertical line at $1/2$ yields yet a third continuum as we see in Example 2.4.

Example 2.4 (A “comb”). Let $f : [0, 1] \rightarrow C([0, 1])$ be given by $f(t) = 0$ for $t \neq 1/2$ and $f(1/2) = [0, 1]$. Then $\varprojlim f$ is the union of a sequence of arcs A_1, A_2, A_3, \dots and the point $(0, 0, 0, \dots)$ such that, for each positive integer i , $A_{i+1} \cap A_i$ is a single point that is an endpoint of A_i and an interior point of A_{i+1} (see Fig. 2.4 for the graph of the bonding function and a model of the inverse limit).

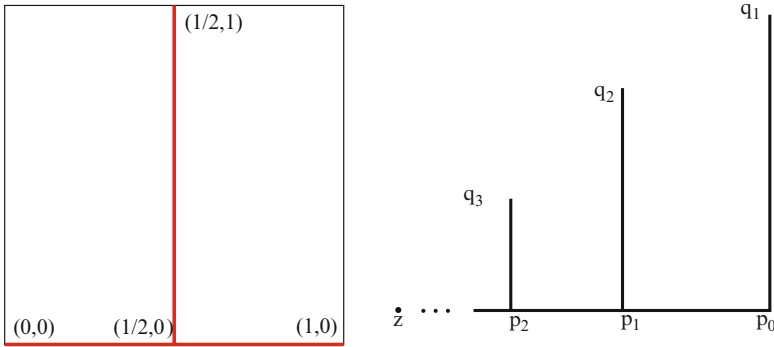


Fig. 2.4 The graph of the bonding function and a model of the inverse limit in Example 2.4

Proof. Let $M = \varprojlim \mathbf{f}$ and, for each $i \in \mathbb{N}$, let $A_i = \{\mathbf{x} \in M \mid x_i \in [0, 1], x_j = 1/2 \text{ for } j > i, \text{ and if } i > 1, \text{ then } x_j = 0 \text{ for } j < i\}$. For each $i \in \mathbb{N}$, let \mathbf{p}_i be the point of A_i such that $\pi_i(\mathbf{p}_i) = 0$ and \mathbf{q}_i be the point of A_i such that $\pi_i(\mathbf{q}_i) = 1$. Then A_i is an arc with endpoints \mathbf{p}_i and \mathbf{q}_i . Letting $\mathbf{z} = (0, 0, 0, \dots)$, we have $\varprojlim \mathbf{f} = \{\mathbf{z}\} \cup (\bigcup_{n \geq 1} A_n)$. If $i \in \mathbb{N}$, then $A_i \cap A_{i+1} = \{\mathbf{p}_i\}$. Let \mathbf{p}_0 denote the point of M having every coordinate $1/2$. Note that \mathbf{p}_i is an interior point of A_{i+1} for each nonnegative integer i . \square

2.5 Topological Conjugacy

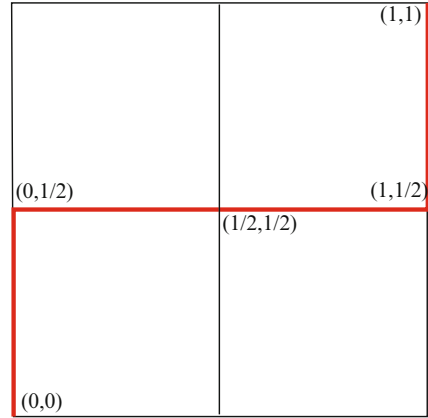
Placing the vertical line in Example 2.4 above any number other than 0 and 1 yields a continuum homeomorphic to the continuum of Example 2.4. To show this, we introduce the notion of topological conjugacy. Functions $f : [0, 1] \rightarrow 2^{[0,1]}$ and $g : [0, 1] \rightarrow 2^{[0,1]}$ are said to be *topologically conjugate* provided there is a surjective homeomorphism $h : [0, 1] \rightarrow [0, 1]$ such that $fh = hg$.

The following theorem may be found in [5, Theorem 5.3, p. 126] in a more general setting.

Theorem 2.9. *If $f : [0, 1] \rightarrow 2^{[0,1]}$ and $g : [0, 1] \rightarrow 2^{[0,1]}$ are topologically conjugate upper semicontinuous functions, then $\varprojlim \mathbf{f}$ and $\varprojlim \mathbf{g}$ are homeomorphic.*

Proof. Let $M = \varprojlim \mathbf{f}$ and $N = \varprojlim \mathbf{g}$. If $h : [0, 1] \rightarrow [0, 1]$ is a surjective homeomorphism such that $fh = hg$, then $H : N \rightarrow M$ given by $H(\mathbf{x}) = (h(x_1), h(x_2), h(x_3), \dots)$ is a homeomorphism from N onto M . To see this, note that the function H is continuous because it is coordinatewise continuous and it is 1–1 because h is 1–1. That $H(\mathbf{x}) \in M$ for each $\mathbf{x} \in N$ is a consequence of $fh = hg$ for $h(x_i) \in h(g(x_{i+1})) = f(h(x_{i+1}))$.

Fig. 2.5 The graph of the bonding function in Example 2.6



If $y \in M$, then $x = (h^{-1}(y_1), h^{-1}(y_2), h^{-1}(y_3), \dots)$ is a point of \mathcal{Q} such that $H(x) = y$. That $x \in N$ may be seen as follows. Because $f = hgh^{-1}$ and $y \in M$, for each positive integer i , $y_i \in hgh^{-1}(y_{i+1})$. Therefore, $h^{-1}(y_i) \in g(h^{-1}(y_{i+1}))$, i.e., $x_i \in g(x_{i+1})$. \square

Example 2.5. If $0 < c < 1$ and $f : [0, 1] \rightarrow C([0, 1])$ is the function given by $f(t) = 0$ for $t \neq c$ and $f(c) = [0, 1]$, then $\varprojlim f$ is homeomorphic to the inverse limit from Example 2.4.

Proof. Denote by g the function from Example 2.4. Using Theorem 2.9, it can be seen that $\varprojlim f$ is homeomorphic to $\varprojlim g$. Indeed, the piecewise linear homeomorphism h whose graph is the union of two straight line intervals, one from $(0, 0)$ to $(1/2, c)$ and the other from $(1/2, c)$ to $(1, 1)$, is a conjugacy because $fh = hg$. \square

For $c \in [0, 1]$, let $g_c : [0, 1] \rightarrow C([0, 1])$ denote the function given by $g_c(t) = 1$ for $t \neq c$ and $g_c(c) = [0, 1]$. Because the function f from Example 2.5 is conjugate to the function g_{1-c} under the homeomorphism $1 - Id$, $\varprojlim f$ is homeomorphic to $\varprojlim g_{1-c}$. For $c = 0$, the function g_c is conjugate under $1 - Id$ to the bonding function in Example 2.2 so $\varprojlim g_c$ is an arc; for $c = 1$, the function g_c is conjugate under $1 - Id$ to the bonding function in Example 2.3, so $\varprojlim g_c$ is infinite dimensional.

We end this section with an additional application of Theorem 2.9. Our proof also uses Theorem 1.9.

Example 2.6 (An arc). Let $f : [0, 1] \rightarrow C([0, 1])$ be given by $f(0) = [0, 1/2]$, $f(t) = 1/2$ for $0 < t < 1$, and $f(1) = [1/2, 1]$ (see Fig. 2.5 for the graph of f). Then, $\varprojlim f$ is an arc.

Proof. Let $f_1 = f|_{[0, 1/2]}$ and $f_2 = f|_{[1/2, 1]}$. Because $G(f_i) \subseteq G(f)$, $\varprojlim f_i \subseteq \varprojlim f$ for $i = 1, 2$. Suppose $x \in \varprojlim f$ and $x \neq (1/2, 1/2, 1/2, \dots)$. Let k be the least integer j such that $x_j \neq 1/2$. If $x_k < 1/2$, then $x_j = 0$ for

$j \geq k$, and, in case $k > 1$, $x_j = 1/2$ for $j < k$. If $x_k > 1/2$, then $x_k = 1$ for $j \geq k$, and, in case $k > 1$, $x_j = 1/2$ for $j < k$. Thus, $\mathbf{x} \in \varprojlim \mathbf{f}_1$ when $x_k < 1/2$ and $\mathbf{x} \in \varprojlim \mathbf{f}_2$ when $x_k > 1/2$. Therefore, $\varprojlim \mathbf{f} = \varprojlim \mathbf{f}_1 \cup \varprojlim \mathbf{f}_2$. Let $g : [0, 1] \rightarrow C([0, 1])$ be the function given by $g(t) = 0$ for $0 \leq t < 1$ and $g(1) = [0, 1]$ (i.e., g is the function from Example 2.2 having an arc as its inverse limit). Then f_1 and g are conjugate under the homeomorphism h_1 given by $h_1(t) = 1/2 - t/2$ for $0 \leq t \leq 1$, while f_2 and g are conjugate under the homeomorphism h_2 given by $h_2(t) = 1/2 + t/2$ for $0 \leq t \leq 1$. Thus, $\varprojlim \mathbf{f}$ is the union of two arcs intersecting only at $\{(1/2, 1/2, 1/2, \dots)\}$ and is therefore an arc. \square

2.6 Connectedness from Bonding Functions Without Connected Values

An inverse limit with a single surjective bonding function can be connected even if the function does not have all of its values connected. We shall see many such examples in this book. Example 2.7 below is an interesting one. The inverse limit is the cone over the Cantor set, often called the *Cantor fan*. There are a Cantor set C lying in the inverse limit and a point \mathbf{v} of the inverse limit such that the inverse limit is the union of a collection of arcs each having \mathbf{v} as one of its endpoints with its other endpoint in C and such that the only point common to any two of the arcs is \mathbf{v} .

Example 2.7 (The Cantor fan). Let $f : [0, 1] \rightarrow 2^{[0,1]}$ be given by $f(t) = \{t, 1-t\}$ for $0 \leq t \leq 1$ (i.e., $G(f)$ is the union of Id and $1 - Id$). Then $\varprojlim \mathbf{f}$ is the Cantor fan with vertex $\mathbf{v} = (1/2, 1/2, 1/2, \dots)$. (See Fig. 2.6.)

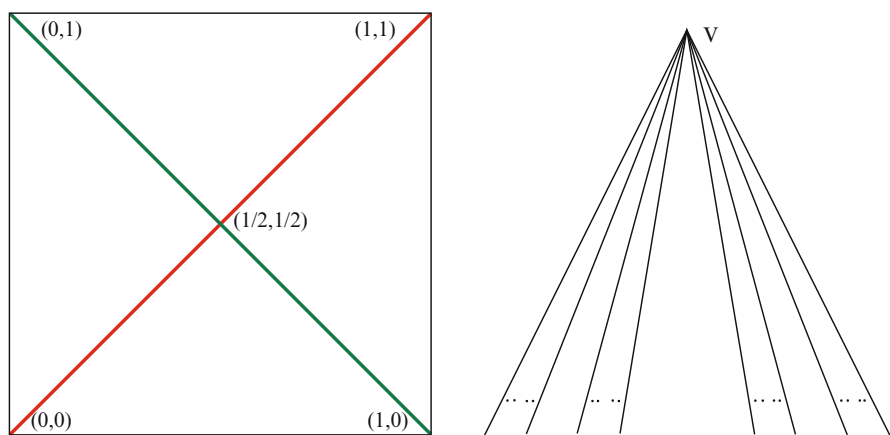


Fig. 2.6 The graph of the bonding function and a model of the inverse limit in Example 2.7

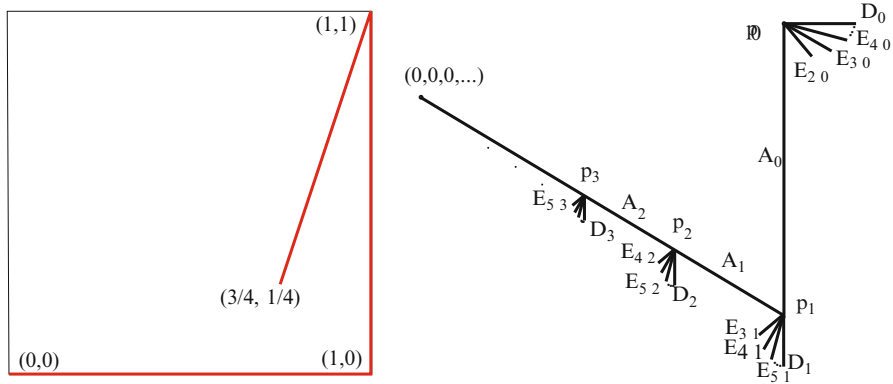


Fig. 2.7 The graph of the bonding function and a model of the inverse limit in Example 2.8

Proof. There are four homeomorphisms whose union is f . They are $g_1 : [0, 1/2] \rightarrow [0, 1/2]$ given by $g_1(t) = t$, $g_2 : [0, 1/2] \rightarrow [1/2, 1]$ given by $g_2(t) = 1 - t$, $g_3 : [1/2, 1] \rightarrow [0, 1/2]$ given by $g_3(t) = 1 - t$, and $g_4 : [1/2, 1] \rightarrow [1/2, 1]$ given by $g_4(t) = t$. A point x is in $\varprojlim f$ if and only if there is a sequence h_1, h_2, h_3, \dots such that $h_i \in \{g_1, g_2, g_3, g_4\}$ for each i and $x_i = h_i(x_{i+1})$ for each positive integer i . Each such inverse limit is an arc having $v = (1/2, 1/2, 1/2, \dots)$ as one endpoint and the other endpoint in the Cantor set $\{0, 1\}^\infty$. Moreover, if $p \in \{0, 1\}^\infty$, there is a sequence h such $h_i \in \{g_1, g_2, g_3, g_4\}$ for each i and $p \in \varprojlim h$. \square

The connectedness of Example 2.7 is demonstrated here in an ad hoc manner, although it is a consequence of a theorem that we prove later (see Theorem 2.11). We now include some additional examples of inverse limits that are continua even though in each case, the bonding function does not have all of its values connected. The connectedness of each of the next two examples is also demonstrated in an ad hoc manner. Unlike the previous example, however, we do not have a subsequent theorem from which the connectedness follows (see Problem 6.5). We make additional use of these examples later.

Example 2.8. Let $f : [0, 1] \rightarrow 2^{[0,1]}$ be the function given by $f(t) = 0$ for $0 \leq t < 3/4$, $f(t) = \{0, 3t - 2\}$ for $3/4 \leq t < 1$, and $f(1) = [0, 1]$. Then $\varprojlim f$ is a continuum. (See Fig. 2.7 for the graph of f and a model of its inverse limit.)

Proof. Let $M = \varprojlim f$. Let g be the bonding function from Example 2.2, i.e., $g : [0, 1] \rightarrow C([0, 1])$ is given by $g(t) = 0$ for $0 \leq t < 1$ and $g(1) = [0, 1]$, and let $A = \varprojlim g$. Then A is an arc and $A \subseteq M$ because $G(g) \subseteq G(f)$. Let p_0 be the point $(1, \overleftarrow{1}, 1, \dots)$ and, for each positive integer j , let p_j be the point of M whose first j coordinates are 0 and all other coordinates are 1. Each point of the sequence p_0, p_1, p_2, \dots is a point of A . For $j \geq 0$, let $D_j = \{x \in M \mid 1/4 \leq x_{j+1} \leq 1, x_{k+1} = (x_k + 2)/3 \text{ for } k > j, \text{ and, if } j > 0, x_k = 0 \text{ for } 1 \leq k \leq j\}$. For each

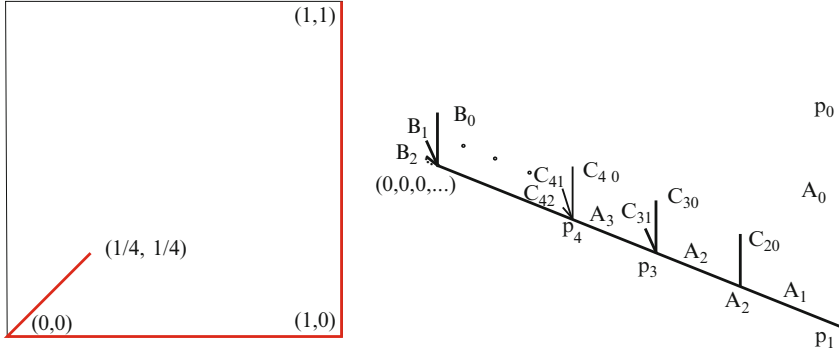


Fig. 2.8 The graph of the bonding function and a model of the inverse limit in Example 2.9

integer $j \geq 0$ and each integer i such that $i \geq j + 2$, let $E_{ij} = \{x \in M \mid 1/4 \leq x_{j+1} \leq 1, x_{k+1} = (x_k + 2)/3 \text{ for } j + 1 \leq k < i, x_k = 1 \text{ for } k > i \text{ and, if } j > 0, x_k = 0 \text{ for } 1 \leq k \leq j\}$. For each $n \geq 0$, $F_n = D_n \cup (\bigcup_{k \geq n+1} E_{kn})$ is a fan with vertex p_n . Note that $M = A \cup (\bigcup_{n \geq 0} F_n)$, so M is connected. \square

Example 2.9. Let $f : [0, 1] \rightarrow 2^{[0,1]}$ be the function given by $f(t) = \{0, t\}$ for $0 \leq t \leq 1/4$, $f(t) = 0$ for $1/4 < t < 1$, and $f(1) = [0, 1]$. Then, $\varprojlim f$ is a continuum. (See Fig. 2.8 for the graph of f and a model of its inverse limit.)

Proof. Let $M = \varprojlim f$. As in the previous example, let g be the bonding function from Example 2.2, i.e., $g : [0, 1] \rightarrow C([0, 1])$ is given by $g(t) = 0$ for $0 \leq t < 1$ and $g(1) = [0, 1]$, and let $A = \varprojlim g$. Then A is an arc and $A \subseteq M$ because $G(g) \subseteq G(f)$. Let i and j be integers with $i \geq 2$ and $0 \leq j < i - 1$. Let $C_{ij} = \{x \in M \mid x_i \in [0, 1/4], x_k = x_i \text{ for } j < k \leq i, x_k = 1 \text{ for } k > i, \text{ and if } j > 0, x_k = 0 \text{ for } 1 \leq k \leq j\}$. Let $B_0 = \{x \in M \mid x_k \in [0, 1/4] \text{ and } x_{k+1} = x_k \text{ for each positive integer } k\}$ and, for each positive integer i , let $B_i = \{x \in M \mid x_{i+1} \in [0, 1/4] \text{ and } x_k = x_{i+1} \text{ for } k \geq i + 1 \text{ and } x_k = 0 \text{ for } k \leq i\}$. Note that $F = \bigcup_{i \geq 0} B_i$ is a fan with vertex $(0, 0, 0, \dots)$, its point of intersection with A . Let $p_0 = (1, 1, 1, \dots)$ and, for each positive integer i , let p_i be the point such that the first i coordinates of p_i are 0 and the remaining coordinates are 1. If i and j are integers with $i \geq 2$ and $0 \leq j < i - 1$, then C_{ij} intersects A at the point p_i . To see that M is connected, one only need observe that if $x \in M - (A \cup F)$, then x is in C_{ij} for some i, j . \square

We close this section with a simple theorem that is easy to prove. Except for the case that some of the terms of the sequence of mappings in the hypothesis of Theorem 2.10 are homeomorphisms, the bonding functions do not have all of their values connected.

Theorem 2.10. *If g is a sequence of surjective mappings of $[0, 1]$ onto $[0, 1]$ and $f_i = g_i^{-1}$ for each $i \in \mathbb{N}$, then $\varprojlim f$ is an arc.*

Proof. Because g_i is surjective, $f_i : [0, 1] \rightarrow 2^{[0,1]}$ is an upper semicontinuous function for each $i \in \mathbb{N}$. Then, $h : [0, 1] \rightarrow \varprojlim f$ given by $h(t) = (t, g_1(t), g_2(g_1(t)), g_3(g_2(g_1(t))), \dots)$ is a homeomorphism of $[0, 1]$ onto $\varprojlim f$. \square

2.7 Union Theorems

One method of obtaining connected inverse limits with set-valued functions is to use upper semicontinuous bonding functions having graphs that are set-theoretic unions of the graphs of upper semicontinuous functions with connected values as shown below in Theorem 2.11. Because mappings (continuous functions) on $[0, 1]$ have connected values, set-valued functions that are unions of mappings often (but not always) produce connected inverse limits. That some restrictions along the lines of those in Theorem 2.11 must be imposed can be seen from Example 1.2 where the bonding function is the union of two constant maps and the inverse limit is a Cantor set.

Our first theorem in this section is due to Nall [Theorem 3.1, 10], although we have cast it in slightly different language from his original statement. Theorem 2.11 generalizes a theorem on unions of upper semicontinuous continuum-valued functions published earlier [3, Theorem 2.12, p. 363] (at least in the metric setting), and its hypothesis is perhaps somewhat easier to verify. After proving this theorem, we provide examples that can be shown to be connected using it. Although Nall proves this theorem for compact metric spaces, we state and prove it on $[0, 1]$. Recall that if $\{X_1, X_2, \dots, X_{n+1}\}$ is a finite collection of closed subsets of $[0, 1]$ and $\{f_1, f_2, \dots, f_n\}$ is a finite collection of functions such that $f_i : X_{i+1} \rightarrow 2^{X_i}$ for $1 \leq i \leq n$ and $G'(f_1, f_2, \dots, f_n) = \{\mathbf{x} \in \prod_{i=1}^{n+1} X_i \mid x_i \in f_i(x_{i+1}) \text{ for } 1 \leq i \leq n\}$.

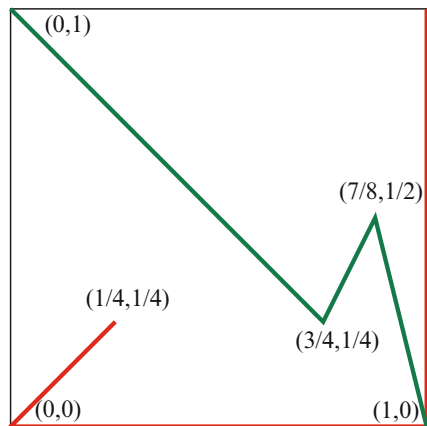
Theorem 2.11 (Nall). *Suppose \mathcal{F} is a collection of upper semi-continuous functions such that if $g \in \mathcal{F}$, then $g : [0, 1] \rightarrow C([0, 1])$, and f is the function whose graph is the set-theoretic union of all of the graphs of the functions in \mathcal{F} . If f is surjective and $G(f)$ is a continuum, then $\varprojlim f$ is a continuum.*

Proof. Because $G(f)$ is a continuum, f is upper semicontinuous. By Theorem 2.1, showing that $G_n = \{\mathbf{x} \in \mathcal{Q} \mid x_i \in f(x_{i+1}) \text{ for } 1 \leq i \leq n\}$ is connected for each $n \in \mathbb{N}$ is sufficient to prove the theorem. To that end, we proceed by induction.

Note that $G_1 = G(f^{-1}) \times \mathcal{Q}$ is connected, being the product of two connected sets, so G_1 is connected.

Suppose k is a positive integer such that G_k is connected. We adopt the following notation. If j is a positive integer, let $G'_j = G'(f_1, f_2, \dots, f_j)$ where $f_i = f$ for $1 \leq i \leq j$. Then, $G'_k = p_{\{1,2,\dots,k+1\}}(G_k)$ is connected. Suppose G'_{k+1} is the union of two closed sets H and K . Then, $G(f^{-1}) = p(G'_{k+1}) = p(H) \cup p(K)$ where $p : [0, 1]^{k+2} \rightarrow [0, 1]^2$ is the mapping given by $p(\mathbf{x}) = (x_1, x_2)$. Because $G(f)$ is connected, $G(f^{-1})$ is connected, so there is a point $(c, d) \in p(H) \cap p(K)$.

Fig. 2.9 The graph of bonding the function in Example 2.10



There are points $\mathbf{x} \in H$ and $\mathbf{y} \in K$ such that $p(\mathbf{x}) = (c, d) = p(\mathbf{y})$. There is a function $g \in \mathcal{F}$ such that $(d, c) \in g$. By Lemma 2.2, $G'(g_1, g_2, \dots, g_{k+1})$ is a connected subset of G'_{k+1} where $g_1 = g$ and $g_i = f$ for $2 \leq i \leq k+1$ and contains both \mathbf{x} and \mathbf{y} . Thus, $H \cap K \neq \emptyset$, and it follows that G'_{k+1} is connected. Because $G_{k+1} = G'_{k+1} \times \mathcal{Q}$, G_{k+1} is connected. \square

A major reason for at least some of the initial interest in inverse limits with upper semicontinuous functions of the type satisfying the hypothesis of Theorem 2.11 is a potential application to economics. Models in backward economics can involve two mappings, and it is important to understand the potential outcomes of the models no matter which of the mappings is used at each stage. Thus, we are led to consider inverse limits with set-valued functions having graphs that are the union of two mappings. However, this topic is of interest in its own right because of theorems like Theorem 2.11. Indeed, most of the research on set-valued functions that are unions of mappings has been concentrated on determining when the inverse limit is a continuum, and this is the case for two maps with a coincidence point and a surjective union. It would be of interest to conduct a study of inverse limits of upper semicontinuous functions that are the union of two maps of $[0, 1]$ that do not have a coincidence point (see Problem 6.6) even though such inverse limits are not connected.

Our next example demonstrates that we cannot weaken the hypothesis in Theorem 2.11 that the elements of \mathcal{F} have connected values to require simply that the elements of \mathcal{F} have connected inverse limits.

Example 2.10. Let $f_1 : [0, 1] \rightarrow 2^{[0,1]}$ be given by $f_1(t) = \{0, t\}$ for $0 \leq t \leq 1/4$, $f_1(t) = 0$ for $1/4 < t < 1$, and $f_1(1) = [0, 1]$. Let $g : [0, 1] \rightrightarrows [0, 1]$ be the mapping given by $g(t) = 1 - t$ for $0 \leq t \leq 3/4$, $g(t) = 2t - 5/4$ for $3/4 < t \leq 7/8$, and $g(t) = 4 - 4t$ for $7/8 < t \leq 1$. Let f be the upper semicontinuous function whose graph is $G(f_1) \cup g$. Then $\varprojlim f_1$ and $\varprojlim g$ are connected, but $\varprojlim f$ is not connected. (See Fig. 2.9 for a graph of f .)

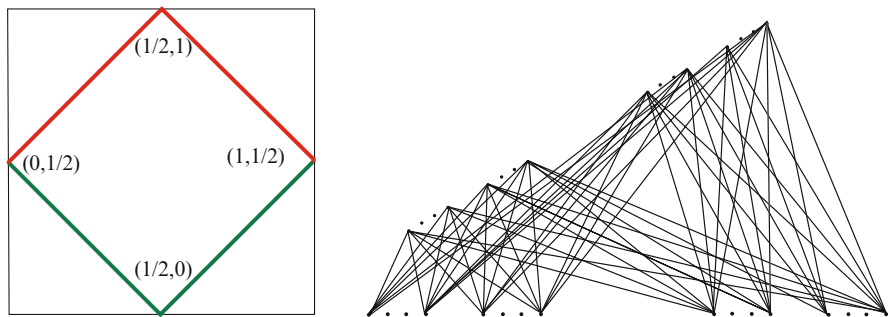


Fig. 2.10 The graph of the bonding function and a model of the inverse limit in Example 2.11

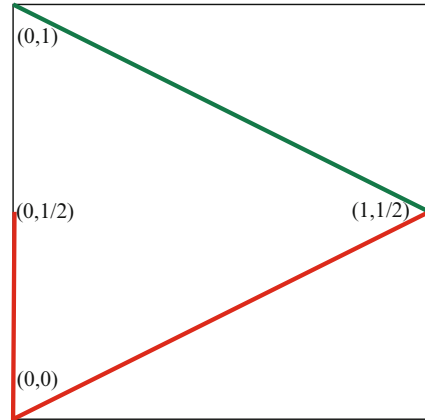
Proof. Let $M = \varprojlim f$. Because f_1 is the bonding function from Example 2.9, its inverse limit is connected. Because g is a mapping, its inverse limit is connected. Let $N = \{x \in M \mid x_1 = x_2 = 1/4 \text{ and } x_3 = 3/4\}$ and note that N is closed. However, because $N = M \cap ((1/8, 3/8) \times (1/8, 3/8) \times (5/8, 7/8) \times \mathcal{Q})$, N is also open in M . Thus, M is not connected. \square

Example 2.11. Let $g_1 : [0, 1] \rightarrow [0, 1]$ be the mapping given by $g_1(t) = t + 1/2$ for $0 \leq t \leq 1/2$ and $g_1(t) = 3/2 - t$ for $1/2 \leq t \leq 1$. Let $g_2 : [0, 1] \rightarrow [0, 1]$ be the mapping given by $g_2(t) = 1/2 - t$ for $0 \leq t \leq 1/2$ and $g_2(t) = t - 1/2$ for $1/2 \leq t \leq 1$. Let $\mathcal{F} = \{g_1, g_2\}$ and $f : [0, 1] \rightarrow 2^{[0,1]}$ be the upper semicontinuous function whose graph is the set-theoretic union of g_1 and g_2 . Then, $\varprojlim f$ is a nonplanar continuum. (See Fig. 2.10 for the graph of f and a model of the inverse limit.)

Proof. Let $M = \varprojlim f$. Because g_1 and g_2 are mappings, $G(f) = g_1 \cup g_2$ is connected, and \bar{f} is surjective; the proof that M is a continuum is a simple application of Theorem 2.11.

To obtain a model for the inverse limit, we view it in a slightly different way. There are two intervals $J_1 = [0, 1/2]$ and $J_2 = [1/2, 1]$ and four mappings $f_1 : J_1 \rightarrow J_1$, $f_2 : J_2 \rightarrow J_1$, $f_3 : J_2 \rightarrow J_2$, and $f_4 : J_1 \rightarrow J_2$ such that $G(f) = f_1 \cup f_2 \cup f_3 \cup f_4$. The continuum M contains two Cantor sets: C_1 containing all the points p of M with all odd coordinates $1/2$ and all even coordinates in $\{0, 1\}$ and C_2 containing all the points p of M with all even coordinates $1/2$ and all odd coordinates in $\{0, 1\}$. The continuum M consists of all arcs of the form $\varprojlim g$ where, for each $i \in \mathbb{N}$, $g_i \in \{f_1, f_2, f_3, f_4\}$ and the domain of g_i is the range of g_{i+1} . Each such arc joins a point of C_1 with a point of C_2 , and furthermore, if p is point of C_1 and q is a point of C_2 , there is such an arc having endpoints p and q . Moreover, each two such arcs that intersect do so at only one point belonging to $C_1 \cup C_2$. The reader should note that M contains numerous simple closed curves. Because each point of C_1 is a vertex of a Cantor fan over C_2 and each two such Cantor fans contain mutually exclusive fans, M contains uncountably many mutually exclusive triods

Fig. 2.11 The graph of the bonding function in Example 2.12



and so is nonplanar [8, Theorem 84, p. 222] (a *triod* is a continuum that contains a subcontinuum having a complement with at least three components).

The continuum can also be seen to be nonplanar because it contains a Kuratowski complete bipartite graph $K_{3,3}$ that consists of six vertices, three from C_1 and three from C_2 , and edges joining each vertex of the three in C_1 with each vertex of the three in C_2 .

To depict our model, choose two skew lines in three-dimensional Euclidean space and embed C_1 in one of these lines and C_2 in the other. By joining each point of C_1 with each point of C_2 by a straight line interval, we obtain a model of M . \square

The inverse limit M in Example 2.11 is the well-known *Hurewicz continuum* having the property that if C is a continuum, then there exist a subcontinuum H of M and a monotone mapping of H onto C [2].

We end this section with one more example of an inverse limit that we show is a continuum using Theorem 2.11. The function in this example does not satisfy the hypothesis of Theorem 2.12 of [3].

Example 2.12. Let T denote the full tent map, $T(t) = 2t$ for $0 \leq t \leq 1/2$ and $T(t) = 2 - 2t$ for $1/2 \leq t \leq 1$. Let $f : [0, 1] \rightarrow 2^{[0,1]}$ be the upper semicontinuous function whose graph $G(f) = T^{-1} \cup (\{0\} \times [0, 1/2])$. Then $\varprojlim f$ is a continuum. (See Fig. 2.11 for the graph of f and Fig. 2.12 for an indication of a model for the inverse limit.)

Proof. Let $g_1 : [0, 1] \rightarrow [0, 1]$ be given by $g_1(t) = 1 - t/2$ for $0 \leq t \leq 1$ and $g_2 : [0, 1] \rightarrow C([0, 1])$ be given by $g_2(0) = [0, 1/2]$ and $g_2(t) = t/2$ for $0 < t \leq 1$. Because $G(f) = G(g_1) \cup G(g_2)$ is connected, the proof that $M = \varprojlim f$ is a continuum is a simple application of Theorem 2.11.

We now construct a model for this inverse limit. Let $\varphi : [0, 1] \rightarrow [0, 1]$ be the homeomorphism given by $\varphi(t) = t/2$ and denote by $\hat{\varphi}$ the mapping of M given by $\varphi(\mathbf{x}) = (\varphi(x_1), x_1, x_2, \dots)$. Because φ is a homeomorphism and $\varphi \subseteq G(f)$, $\hat{\varphi}$ is

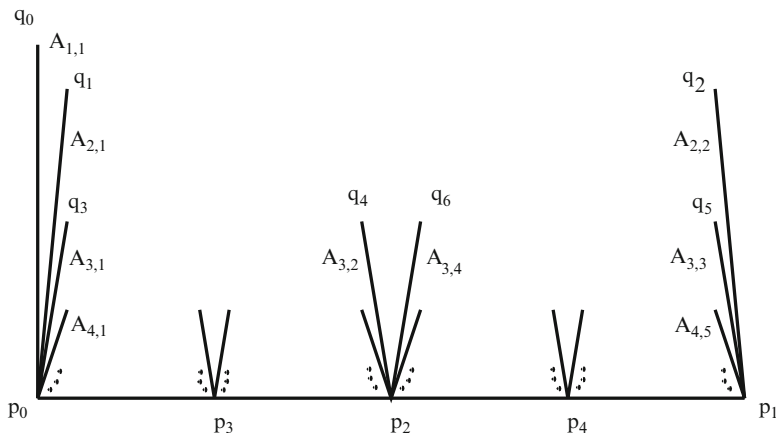


Fig. 2.12 A model of the inverse limit in Example 2.12

a homeomorphism of M into M . Let $\psi : [0, 1] \rightarrow [0, 1]$ be the homeomorphism given by $\psi(t) = 1 - t/2$ and denote by $\hat{\psi}$ the homeomorphism of M into M given by $\psi(x) = (\psi(x_1), x_1, x_2, \dots)$.

Let $A = \varprojlim T^{-1}$. By Theorem 1.9, A is a subset of M because $G(T^{-1}) \subseteq G(f)$. By Theorem 2.10, A is an arc. The endpoints of A are $p_0 = (0, 0, 0, \dots)$ and $p_1 = (1, 0, 0, \dots)$.

Suppose $x \in M$ and $x \notin A$. There is a positive integer n such that $x_n \in (0, 1/2]$ and $x_j = 0$ for $j > n$. Let $M_n = \{x \in M \mid x_n \in [0, 1/2] \text{ and } x_j = 0 \text{ for } j > n\}$. It follows that $M = A \cup (\bigcup_{i \geq 0} M_i)$.

As we proceed, we identify arcs and points that are shown in our model of M depicted in Fig. 2.12. With that in mind, note that M_1 is an arc with endpoints p_0 and $q_0 = (1/2, 0, 0, 0, \dots)$. Letting $A_{1,1} = M_1$, we see that $M_2 = \hat{\psi}(A_{1,1}) \cup \hat{\psi}(A_{1,1})$. Let $A_{2,1} = \hat{\psi}(A_{1,1})$ and $A_{2,2} = \hat{\psi}(A_{1,1})$. So, $A_{2,1}$ is an arc with endpoints $p_0 = (0, 0, 0, \dots)$ and $q_1 = (1/4, 1/2, 0, 0, \dots)$, while $A_{2,2}$ is an arc with endpoints $p_1 = (1, 0, 0, 0, \dots)$ and $q_2 = (3/4, 1/2, 0, 0, \dots)$. Further, M_3 is the union of four arcs $A_{3,1} = \hat{\psi}(A_{2,1})$, $A_{3,2} = \hat{\psi}(A_{2,2})$, $A_{3,3} = \hat{\psi}(A_{2,1})$, and $A_{3,4} = \hat{\psi}(A_{2,2})$. Note that the endpoints of $A_{3,1}$ are p_0 and $q_3 = (1/8, 1/4, 1/2, 0, 0, \dots)$; the endpoints of $A_{3,2}$ are $p_2 = (1/2, 1, 0, 0, 0, \dots)$ and $q_4 = (3/8, 3/4, 1/2, 0, 0, \dots)$; $A_{3,3}$ has endpoints p_1 and $q_5 = (7/8, 1/4, 1/2, 0, 0, \dots)$; and $A_{3,4}$ has endpoints p_2 and $q_6 = (5/8, 3/4, 1/2, 0, 0, \dots)$. Continuing inductively, we observe that, for each positive integer n , M_{n+1} is the union of 2^n arcs that are, respectively, the images under $\hat{\psi}$ of the arcs that comprise the components of M_n together with the arcs that are the images under $\hat{\psi}$ of the arcs comprising the components of M_n . \square

We revisit Example 2.12 in Chap. 4 (Example 4.3).

2.8 Examples from Eight Similar Functions

In this section we consider inverse limits produced by eight similar graphs of upper semicontinuous functions that one obtains by the following process: choose one of the corners of $[0, 1] \times [0, 1]$ and take the union of the diagonal of the square emanating from that point and either the horizontal or the vertical side of the square that emanates from that point. Use that union as the graph of an upper semicontinuous set-valued function. Let \mathcal{E} denote the collection of these eight inverse limits. Due to the fact that these eight graphs consist of the graphs of four topologically conjugate pairs of upper semicontinuous functions, we may examine four such graphs and through Theorem 2.9 know all the elements of \mathcal{E} . Interestingly enough, \mathcal{E} contains four quite different continua even though the graphs that produce them are very similar. We begin with perhaps the simplest of these examples by choosing the graph that is the union of the diagonal and the horizontal side of the square lying on the bottom of the square.

Example 2.13 (A simple fan). Let $f : [0, 1] \rightarrow C([0, 1])$ be given by $f(t) = \{0, t\}$ for $0 \leq t \leq 1$. Then, $\varprojlim f$ is a fan with vertex $v = (0, 0, 0, \dots)$. (See Fig. 2.13 for the graph of f and a model of its inverse limit.)

Proof. Let $M = \varprojlim f$. That M is a continuum is a consequence of Theorem 2.8 (or of Theorem 2.11 by observing that $G(f)$ is a union of two mappings). Let $A_0 = \{x \in M \mid x_j = x_1 \text{ for each positive integer } j\}$. If $i \in \mathbb{N}$, let $A_i = \{x \in M \mid x_j = 0 \text{ for } j \leq i \text{ and } x_j = x_{i+1} \text{ for } j > i + 1\}$. Note that A_i is an arc containing $v = (0, 0, 0, \dots)$ for each nonnegative integer i . Moreover, $\varprojlim f = \bigcup_{k \geq 0} A_k$. \square

The function in Example 2.13 is conjugate to the function $g : [0, 1] \rightarrow 2^{[0,1]}$ given by $g(t) = \{t, 1\}$ for $0 \leq t \leq 1$ under the homeomorphism $h = 1 - Id$. Consequently, $\varprojlim g$ is homeomorphic to the inverse limit from Example 2.13.

Example 2.14. Let $f : [0, 1] \rightarrow C([0, 1])$ be given by $f(t) = t$ for $0 \leq t < 1$ and $f(1) = [0, 1]$. Then the inverse limit is a fan with vertex $v = (1, 1, 1, \dots)$. (See Fig. 2.14 for the graph of f and a model of its inverse limit.)

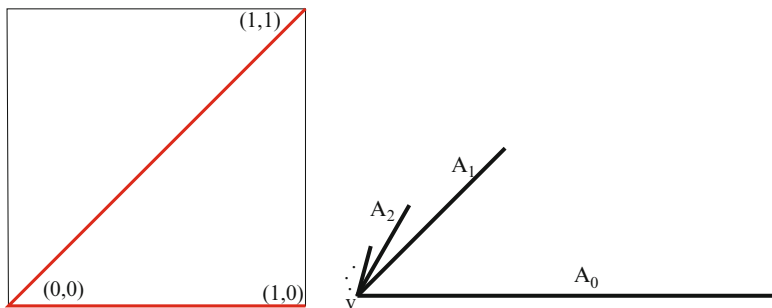


Fig. 2.13 The graph of the bonding function and a model of the inverse limit in Example 2.13

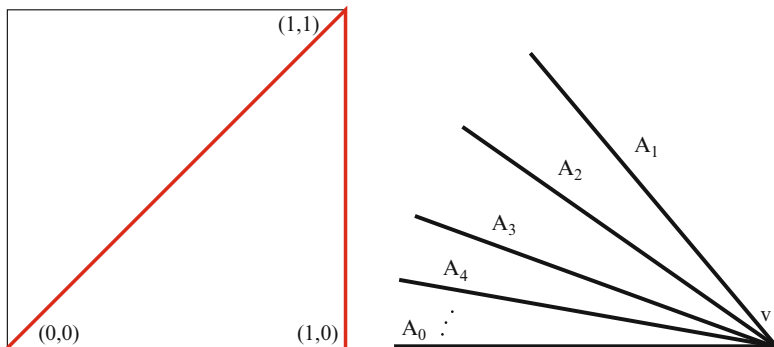


Fig. 2.14 The graph of the bonding function and a model of the inverse limit in Example 2.14

Proof. Let $M = \varprojlim f$ and let $A_0 = \{x \in M \mid x_j = x_1 \text{ for each } j\}$. That M is a continuum is a consequence of Theorem 2.7. Let i be a positive integer and let $A_i = \{x \in M \mid x_j = 1 \text{ for } j > i\}$. Note that A_k is an arc containing $v = (1, 1, 1, \dots)$ for each positive integer k . Moreover, $\varprojlim f = \bigcup_{k>0} A_k$. \square

The function in Example 2.14 is conjugate to the function $g : [0, 1] \rightarrow C([0, 1])$ given by $g(0) = [0, 1]$ and $g(t) = t$ for $0 < t \leq 1$ under the homeomorphism $h = 1 - Id$. Consequently, $\varprojlim g$ is homeomorphic to the inverse limit from Example 2.14.

It is known that if $0 < c < 1$ and $f_c : [0, 1] \rightrightarrows [0, 1]$ is the mapping whose graph is the union of two straight line intervals one from $(0, 0)$ to $(c, 1)$ and the other from $(c, 1)$ to $(1, 0)$, then $\varprojlim f_c$ is homeomorphic to the BJK horseshoe, $\varprojlim f_c$ for $c = 1/2$ (i.e., f_c is the full tent map). In Example 2.14, we examined the corresponding set-valued function for $c = 1$. Although we do not get a fan as an inverse limit of the upper semicontinuous set-valued function for $c = 0$, we next look at the surprisingly complicated inverse limit for this function.

Example 2.15. Let $f : [0, 1] \rightarrow C([0, 1])$ be given by $f(0) = [0, 1]$ and $f(t) = 1 - t$ for $0 < t \leq 1$. The complicated inverse limit $\varprojlim f$ is a nonplanar continuum that contains numerous $\sin(1/x)$ -curves, two copies of the inverse limit from Example 2.14 attached along the limit arc, and many mutually exclusive n -ods for each positive integer n . (See Fig. 2.15 for the graph of f and Fig. 2.16 for a model of the inverse limit.)

Proof. Let $M = \varprojlim f$; M is a continuum by Theorem 2.7. This continuum is reasonably simple to describe, although it is rather complicated in its nature. Let $A = \{x \in M \mid x_{j+1} = 1 - x_j \text{ for each positive integer } j\}$. For each positive integer n , let $B_n = \{x \in M \mid x_{n+1} = 0\}$. Then, A is an arc, and each B_n is a product of an arc with a Cantor set. Note that $M = A \cup (\bigcup_{i>0} B_i)$. Denote by \mathcal{B}_n the collection of arcs that are the components of B_n .

Fig. 2.15 The graph of the bonding function in Example 2.15

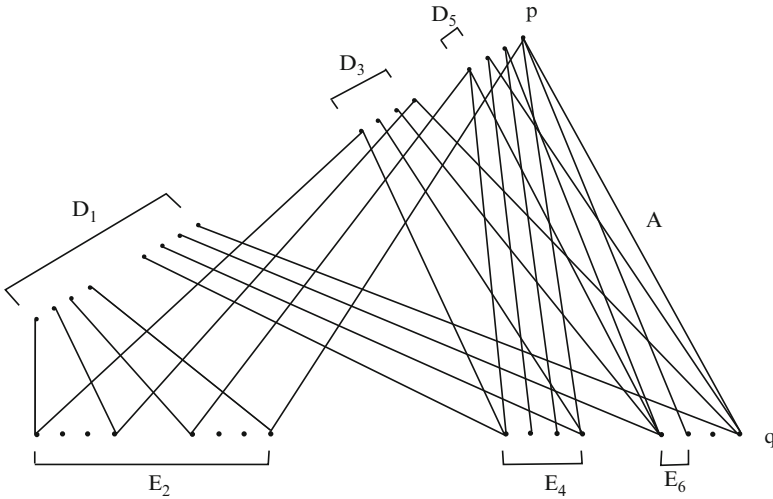
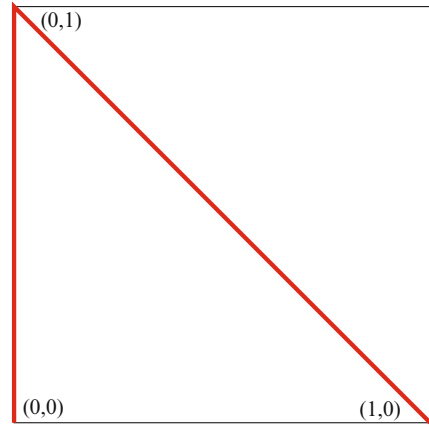


Fig. 2.16 A model of the inverse limit in Example 2.15

To obtain a model for M , we provide the following description of M that also allows us to indicate the properties listed for it. There is a Cantor set C lying in M that results from $\varprojlim g$ where $g(0) = \{0, 1\}$ and $g(1) = 0$. Let $C_0 = \{x \in C \mid x_1 = 0\}$, $C_1 = \{x \in C \mid x_1 = 1\}$. Then, $C = C_0 \cup C_1$. We partition C_0 and C_1 in the following way. For each positive integer n , let p_n be the point of $\{0, 1\}^n$ such that $\pi_1(p_n) = 0$ and $\pi_{i+1}(p_n) = 1 - \pi_i(p_n)$ for $1 \leq i < n$ and q_n be the point of $\{0, 1\}^n$ such that $\pi_1(q_n) = 1$ and $\pi_{i+1}(q_n) = 1 - \pi_i(q_n)$ for $1 \leq i < n$. For each positive integer n , let $D_n = \{p_n\} \times C_0$ and $E_n = \{q_n\} \times C_0$. Observe that $C_0 = D_1 \cup D_2$ and $C_1 = E_1$. For each positive integer j , $D_{2j} = D_{2j+1} \cup D_{2j+2}$ and $E_{2j-1} = E_{2j} \cup E_{2j+1}$. Thus, if we let $p = (0, 1, 0, 1, \dots)$ and $q = (1, 0, 1, 0, \dots)$, then $C_0 = D_1 \cup D_3 \cup D_5 \cup \dots \cup \{p\}$ where $D_i \cap D_j = \emptyset$ if i and j are odd,

$i \neq j$. Also, $C_1 = E_2 \cup E_4 \cup E_6 \cup \cdots \cup \{q\}$ where $E_i \cap E_j = \emptyset$ if i and j are even, $i \neq j$. Furthermore, $D_2 \supseteq D_4 \supseteq D_6 \supseteq \cdots$, while $E_1 \supseteq E_3 \supseteq E_5 \supseteq \cdots$. If n is an odd positive integer, each element of \mathcal{B}_n is an arc having one endpoint in D_n and the other endpoint in $E_n = E_{n+1} \cup E_{n+2}$, while if n is even, then each element of \mathcal{B}_n is an arc having one endpoint in E_n and the other in $D_n = D_{n+1} \cup D_{n+2}$. Moreover, if $n \in \mathbb{N}$ and $x \in D_n \cup E_n$, then x is an endpoint of some arc in \mathcal{B}_n .

Choose two skew lines in three-dimensional Euclidean space and embed C_0 in one of these lines and C_1 in the other; see Fig. 2.16 where we have also shown the partitions of $C_0 = D_1 \cup D_3 \cup D_5 \cup \cdots \cup \{p\}$ and $C_1 = E_2 \cup E_4 \cup E_6 \cup \cdots \cup \{q\}$. From each point of D_1 , draw a straight line interval representing an arc in \mathcal{B}_1 that joins it to a point of E_1 , from each point of E_2 , draw a straight line interval representing an arc in \mathcal{B}_2 that joins it to a point of D_2 , and continue this process. Finally, connect the points representing p and q with a straight line interval.

A double fan. The points p and q are the endpoints of the arc A . For each n , let A_n denote the element of \mathcal{B}_n having p or q as one of its endpoints. One fan is $F = A \cup (\bigcup_{i>0} A_{2i})$ with vertex q and the other is $A \cup (\bigcup_{i>0} A_{2i-1})$ having vertex p .

Triods and nonplanarity. Let v be a point of E_5 and let α be an arc in \mathcal{B}_5 from v to a point of D_5 . Because $E_5 \subseteq E_3 \subseteq E_1$, there are arcs β and γ in \mathcal{B}_3 and \mathcal{B}_1 , respectively, having v as an endpoint. Let $T_v = \alpha \cup \beta \cup \gamma$. Because D_1 , D_3 , and D_5 are pairwise mutually exclusive, T_v is a triod. If v and w are two different points of E_5 , $T_v \cap T_w = \emptyset$. Because E_5 is uncountable, M contains uncountably many mutually exclusive triods, so M is a nonplanar continuum [8, Theorem 84, p. 222].

n -ods. To obtain a 4-od lying in M , start with a point x of E_{13} . Choose four arcs containing x , one from each of \mathcal{B}_{13} , \mathcal{B}_{11} , \mathcal{B}_9 , and \mathcal{B}_7 , respectively. The union of these four arcs is a 4-od. In a similar manner, we can see that for each positive integer n , there are n -ods in M for each $n \in \mathbb{N}$.

A $\sin(1/x)$ -curve. There is an arc in \mathcal{B}_1 from the point $(0, 0, 0, \dots)$ of D_1 to the point $(1, 0, 0, \dots)$ of E_1 . In \mathcal{B}_2 , there is an arc from the point $(1, 0, 0, \dots)$ of E_1 to the point $(0, 1, 0, 0, \dots)$ of D_3 . In \mathcal{B}_3 , there is an arc from $(0, 1, 0, 0, \dots)$ to $(1, 0, 1, 0, 0, \dots)$ of E_3 , and in \mathcal{B}_4 , there is an arc from $(1, 0, 1, 0, 0, \dots)$ to $(0, 1, 0, 1, 0, 0, \dots)$ of D_5 . Continuing in this way, we obtain a $\sin(1/x)$ -curve having limit bar the arc A from p to q . There are other $\sin(1/x)$ -curves in M . For example, instead of starting from the point with all coordinates 0, start from the point of D_1 whose coordinates are 0 except for the $4j - 1$ coordinates for $j \in \mathbb{N}$ where the coordinates are 1 and use a procedure similar to the one above. Here, the first arc would be chosen in \mathcal{B}_1 to a point of E_4 , the second arc would be chosen from \mathcal{B}_4 to a point of D_5 , the third arc would be chosen from \mathcal{B}_5 to a point of E_8 , and so on. This yields a $\sin(1/x)$ -curve whose intersection with the one above is the limit bar, A . \square

The function in Example 2.15 is conjugate to the function $g : [0, 1] \rightarrow C([0, 1])$ given by $g(t) = 1 - t$ for $0 \leq t < 1$ and $g(1) = [0, 1]$. Consequently, $\varprojlim g$ is homeomorphic to the inverse limit from Example 2.15.

Fig. 2.17 The graph of the bonding function in Example 2.16

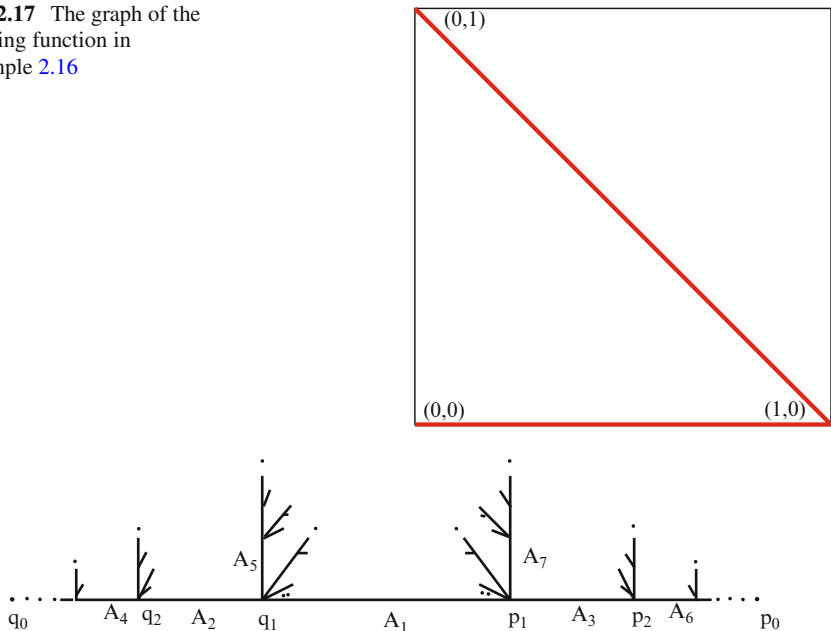


Fig. 2.18 A model of the inverse limit in Example 2.16

Example 2.16. Let $f : [0, 1] \rightarrow 2^{[0,1]}$ be given by $f(t) = \{0, 1 - t\}$ for $0 \leq t \leq 1$. The inverse limit is an arcwise connected continuum that contains an arc $A = \varprojlim g$ where $g = 1 - Id$ and a Cantor set $\mathcal{C} = \{\mathbf{x} \in \{0, 1\}^\infty \mid \text{if } x_i = 1, \text{ then } x_{i+1} = 0\}$. Each point of \mathcal{C} lies in an arc that intersects A . (See Fig. 2.17 for the graph of f and Fig. 2.18 for a model of $\varprojlim f$.)

Proof. Because $f^{-1} : [0, 1] \rightarrow C([0, 1])$, $\varprojlim f^{-1}$ is a continuum. By Theorem 2.3, $M = \varprojlim f$ is a continuum.

Next, we show that if $\mathbf{p} \in \mathcal{C}$, then there is an arc containing \mathbf{p} that intersects A . Let \mathbf{p} be a point of \mathcal{C} . For each positive integer i , let α_i be an arc determined in the following way: if $p_i = 0$, then $\alpha_i = \{\mathbf{x} \in M \mid x_j = p_j \text{ for } 1 \leq j \leq i \text{ and } x_{i+1} \in [0, 1], x_{i+2} = 1 - x_{i+1}, x_{i+3} = x_{i+2}, \dots\}$, while if $p_i = 1$, then $\alpha_i = \{\mathbf{x} \in M \mid x_j = p_j \text{ for } 1 \leq j \leq i \text{ and } x_{i+1} = 0, x_{i+2} \in [0, 1], x_{i+3} = 1 - x_{i+2}, x_{i+4} = x_{i+2}, \dots\}$. Note that in the case that $p_i = 0$, $\alpha_i \cap \alpha_{i+1}$ is a single point, while if $p_i = 1$, $\alpha_i = \alpha_{i+1}$. Because α_1 intersects A at either $(0, 1, 0, \dots)$ or $(1, 0, 1, \dots)$, it follows that $\text{Cl}(\bigcup_{i>0} \alpha_i)$ is an arc containing \mathbf{p} and intersecting A . That the continuum M is arcwise connected now follows.

To describe a model for the inverse limit, we proceed somewhat informally. Because any point of the inverse limit having a 1 as a coordinate must have a 0 in its next coordinate, let $\mathcal{S} = \{s \mid s \text{ is a finite sequence of 0s and 1s such that the final term of } s \text{ is 0 and if a term of } s \text{ is 1, then the next term of } s \text{ is 0}\}$. Then \mathcal{S} is countable.

By using the two symbols 0 and 10 and writing the terms of \mathcal{S} as strings, we may indicate an enumeration of \mathcal{S} by $\{0, 10, 00, 010, 100, 1010, 000, 0010, 0100, 01010, 1000, 10010, 10100, 101010, \dots\}$. Employing this enumeration of \mathcal{S} , define a sequence of arcs in the following way. Let

$$A_1 = A = \{\mathbf{x} \in M \mid x_1 \in [0, 1] \text{ and } x_{i+1} = 1 - x_i \text{ for each positive integer } i\}.$$

Using the first term of \mathcal{S} to determine the first coordinate of all the points of an arc, let

$$A_2 = \{\mathbf{x} \in M \mid x_1 = 0, x_2 \in [0, 1] \text{ and } x_{i+1} = 1 - x_i \text{ for each positive integer } i \geq 2\}.$$

Using the second term of \mathcal{S} to determine the first two coordinates of all the points of an arc, let

$$A_3 = \{\mathbf{x} \in M \mid x_1 = 1, x_2 = 0, x_3 \in [0, 1] \text{ and } x_{i+1} = 1 - x_i \text{ for each positive integer } i \geq 3\}.$$

If $n \geq 1$ and $s^n = s_1, s_2, \dots, s_{k_n}$ is the n th term of \mathcal{S} , let

$$A_{n+1} = \{\mathbf{x} \in M \mid x_i = s_i \text{ for } 1 \leq i \leq k_n, x_{k_n+1} \in [0, 1], \text{ and } x_{i+1} = 1 - x_i \text{ for each positive integer } i \geq k_n + 1\}.$$

Thus,

$$A_4 = \{\mathbf{x} \in M \mid x_1 = 0, x_2 = 0, x_3 \in [0, 1] \text{ and } x_{i+1} = 1 - x_i \text{ for each positive integer } i \geq 3\}.$$

$$A_5 = \{\mathbf{x} \in M \mid x_1 = 0, x_2 = 1, x_3 = 0, x_4 \in [0, 1] \text{ and } x_{i+1} = 1 - x_i \text{ for each positive integer } i \geq 4\}.$$

$$A_6 = \{\mathbf{x} \in M \mid x_1 = 1, x_2 = 0, x_3 = 0, x_4 \in [0, 1] \text{ and } x_{i+1} = 1 - x_i \text{ for each positive integer } i \geq 4\}.$$

$$A_7 = \{\mathbf{x} \in M \mid x_1 = 1, x_2 = 0, x_3 = 1, x_4 = 0, x_5 \in [0, 1] \text{ and } x_{i+1} = 1 - x_i \text{ for each positive integer } i \geq 5\}$$

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For this sequence of arcs, $M = \text{Cl}(\bigcup_{i \geq 0} A_i)$. Let F denote the fan of Example 2.13, i.e., a fan with a sequence of arms of lengths decreasing to 0 emanating from its vertex. At each end of the arc A_1 , there is a copy of the fan F attached at its vertex. At each endpoint of each copy of F , we see a copy of F again attached at its vertex, and this continues on those copies of F , et cetera. The arc A_1 has endpoints $(0, 1, 0, 1, \dots)$ and $(1, 0, 1, 0, \dots)$. The arcs A_3, A_7, A_{15}, \dots comprise the arms of a copy of F attached to A_1 at $(1, 0, 1, 0, \dots)$, and the arcs A_2, A_5, A_{11}, \dots comprise the arms of a copy of F attached to A_1 at $(0, 1, 0, 1, \dots)$. The endpoints of A_2 are $(0, 1, 0, 1, \dots)$ and $(0, 0, 1, 0, 1, \dots)$, and the arcs A_4, A_9, A_{17}, \dots comprise the arms of a copy of F attached to A_2 at the point $(0, 0, 1, 0, 1, \dots)$. The endpoints of A_3 are $(1, 0, 1, 0, \dots)$ and $(1, 0, 0, 1, 0, \dots)$, and the arcs $A_6, A_{13}, A_{27}, \dots$ comprise the arms of a copy of F attached to A_3 at $(1, 0, 0, 1, 0, 1, \dots)$. See Fig. 2.18 for

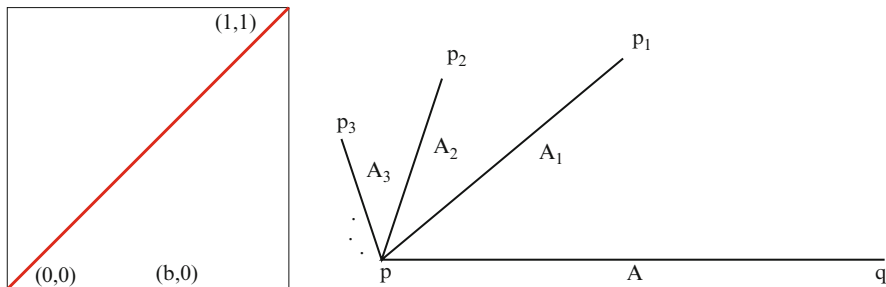


Fig. 2.19 The graph of the bonding function and a model of the inverse limit in Example 2.17

an indication of a model for M based on this informal partial description of M . In the figure, the points shown are $p_0 = (1, 0, 0, 0, \dots)$, $p_1 = (1, 0, 1, 0, \dots)$, $p_2 = (1, 0, 0, 1, 0, \dots)$, $q_0 = (0, 0, 0, \dots)$, $q_1 = (0, 1, 0, 1, \dots)$, and $q_2 = (0, 0, 1, 0, 1, \dots)$. \square

The function from Example 2.16 is conjugate to the function $g : [0, 1] \rightarrow 2^{[0,1]}$ given by $g(t) = \{1, 1 - t\}$ for $0 \leq t \leq 1$. Consequently, $\varprojlim g$ is homeomorphic to the inverse limit from Example 2.16. This completes our look at the collection \mathcal{E} .

2.8.1 Four More Similar Graphs

We now consider what happens if we use only part of the horizontal or vertical line in the construction of the eight functions. Here we only consider the four main graphs and omit reference to the four conjugate graphs. We begin with the function having a portion of the x -axis attached to the diagonal.

Example 2.17. Let b be a number such that $0 < b < 1$ and let $f : [0, 1] \rightarrow 2^{[0,1]}$ be given by $f(t) = \{0, t\}$ for $0 \leq t \leq b$ and $f(t) = t$ for $b < t \leq 1$. Then, $\varprojlim f$ is a fan. (See Fig. 2.19 for the graph of f and a model of its inverse limit.)

Proof. Let $M = \varprojlim f$. Because $f^{-1} : [0, 1] \rightarrow C([0, 1])$, M is a continuum. Let $A = \{x \in M \mid x_1 \in [0, 1] \text{ and } x_j = x_1 \text{ for } j > 1\}$. For each positive integer i , let $A_i = \{x \in M \mid x_j = 0 \text{ for } 1 \leq j \leq i, x_{i+1} \in [0, b], \text{ and } x_j = x_{i+1} \text{ for } j > i + 1\}$. Then $M = A \cup (\bigcup_{i \geq 0} A_i)$. For the purpose of identification in the model, let $p = (0, 0, 0, \dots)$, $q = (1, 1, 1, \dots)$, $p_1 = (0, b, b, b, \dots)$, $p_2 = (0, 0, b, b, \dots)$, and $p_3 = (0, 0, 0, b, b, \dots)$. \square

Example 2.18. Let b be a number such that $0 < b < 1$ and let $f : [0, 1] \rightarrow C([0, 1])$ be given by $f(t) = t$ for $0 \leq t < 1$ and $f(1) = [b, 1]$. Then, $\varprojlim f$ is a fan. (See Fig. 2.20 for the graph of f and a model of its inverse limit.)

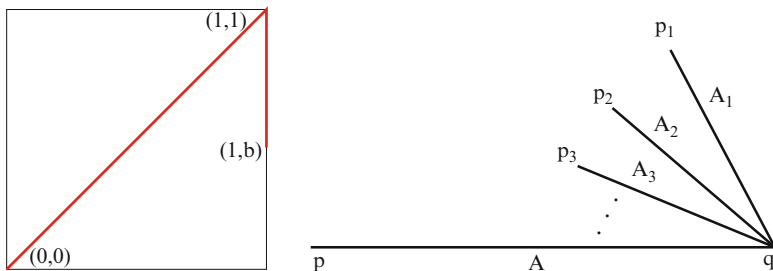


Fig. 2.20 The graph of the bonding function and a model of the inverse limit in Example 2.18

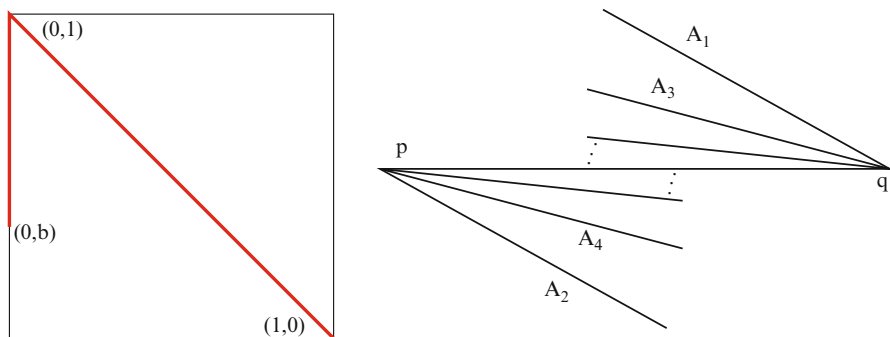


Fig. 2.21 The graph of the bonding function and a model of the inverse limit in Example 2.19

Proof. Let $M = \varprojlim f$. Because $f : [0, 1] \rightarrow C([0, 1])$, M is a continuum. Let $A = \{x \in M \mid x_1 \in [0, 1] \text{ and } x_j = x_1 \text{ for } j > 1\}$. For each positive integer i , let $A_i = \{x \in M \mid x_1 \in [b, 1], x_j = x_1 \text{ for } 1 \leq j \leq i, \text{ and } x_j = 1 \text{ for } j > i\}$. Then $M = A \cup (\bigcup_{i>0} A_i)$. For the purpose of identification in the model, let $p = (0, 0, 0, \dots)$, $q = (1, 1, 1, \dots)$, $p_1 = (b, 1, 1, 1, \dots)$, $p_2 = (b, b, 1, 1, \dots)$, and $p_3 = (b, b, b, 1, 1, \dots)$. \square

Example 2.19. Let b be a number such that $0 < b < 1$ and let $f : [0, 1] \rightarrow C([0, 1])$ be given by $f(0) = [b, 1]$ and $f(t) = 1 - t$ for $0 < t \leq 1$. Then $\varprojlim f$ is the union of two fans that intersect in an arc. (See Fig. 2.21 for the graph of f and a model of its inverse limit.)

Proof. Let $M = \varprojlim f$. Because $f : [0, 1] \rightarrow C([0, 1])$, M is a continuum. Let $A = \{x \in M \mid x_1 \in [0, 1] \text{ and } x_{j+1} = 1 - x_j \text{ for } j \geq 1\}$. If i is a positive integer, let $A_{2i-1} = \{x \in M \mid x_1 \in [b, 1], x_{2i} = 0, \text{ and } x_{j+1} = 1 - x_j \text{ for } j \neq 2i - 1 \text{ and } j \geq 1\}$ and $A_{2i} = \{x \in M \mid x_1 \in [b, 1], x_{2i+1} = 0 \text{ and } x_{j+1} = 1 - x_j \text{ for } j \neq 2i \text{ and } j \geq 1\}$. Then $M = A \cup (\bigcup_{i>0} A_i)$. In the model, the point $p = (0, 1, 0, 1, \dots)$, and the point $q = (1, 0, 1, 0, \dots)$. This inverse limit is homeomorphic to the union of two copies of the inverse limit from Example 2.18 having the arc A in common. In the model, the point p is $(0, 1, 0, 1, \dots)$ and $q = (1, 0, 1, 0, \dots)$. \square

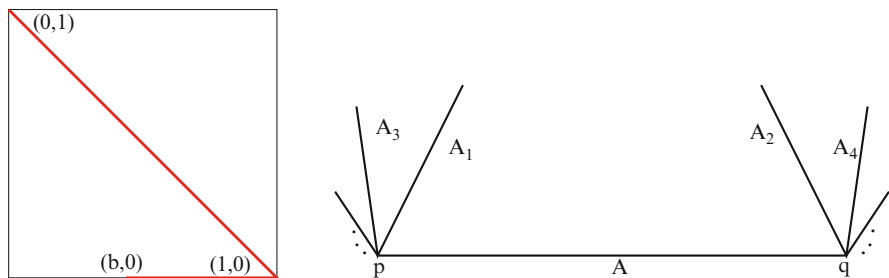


Fig. 2.22 The graph of the bonding function and a model of the inverse limit in Example 2.20

Example 2.20. Let b be a number such that $0 < b < 1$ and let $f : [0, 1] \rightarrow 2^{[0,1]}$ be given by $f(t) = 1 - t$ for $0 \leq t < b$ and $f(t) = \{0, 1 - t\}$ for $b \leq t \leq 1$. Then $\varprojlim f$ is the union of two fans. (See Fig. 2.22 for the graph of f and a model of its inverse limit.)

Proof. Let $M = \varprojlim f$. Because $f^{-1} : [0, 1] \rightarrow C([0, 1])$, M is a continuum. Let $A = \{x \in M \mid x_1 \in [0, 1] \text{ and } x_{j+1} = 1 - x_j \text{ for } j \geq 1\}$. For each positive integer i , let $A_{2i} = \{x \in M \mid x_1 = 0, x_{2i} \in [0, b], \text{ and if } i \neq 2i, \text{ then } x_{j+1} = 1 - x_j \text{ for } j \geq 1\}$ and $A_{2i+1} = \{x \in M \mid x_1 = 1, x_{2i+1} \in [0, b], \text{ and if } i \neq 2i + 1, \text{ then } x_{j+1} = 1 - x_j \text{ for } j \geq 1\}$. This inverse limit is homeomorphic to the union of two copies of the inverse limit from Example 2.17 having the arc A in common. In the model, the point p is $(0, 1, 0, 1, \dots)$, and q is $(1, 0, 1, 0, \dots)$ (Fig. 2.22). \square

2.9 Additional Examples

We examine some additional examples in this section. Our first function yields a fan as its inverse limit. In fact, the inverse limit in Example 2.21 is homeomorphic to the inverse limit in Example 2.14.

Example 2.21. Let $f : [0, 1] \rightarrow C([0, 1])$ be given by $f(t) = t$ for $0 \leq t < 1/2$, $f(1/2) = [1/2, 1]$, and $f(t) = 1 - t$ for $1/2 < t \leq 1$. Then, $\varprojlim f$ is a fan with vertex $v = (1/2, 1/2, 1/2, \dots)$. (See Fig. 2.23 for the graph of f and a model of its inverse limit.)

Proof. Let $M = \varprojlim f$ and let $B_0 = \{x \in M \mid x_1 \in [0, 1/2] \text{ and } x_j = x_1 \text{ for } j > 1\}$. Let i be a positive integer and let $B_i = \{x \in M \mid x_i \in [1/2, 1]\}$. Note that if $x \in B_i$, then $x_j = 1/2$ for $j > i$, and if $i > 1$, then $x_j \in [0, 1/2]$ for $j < i$. Further, if $i > 2$ and $x \in B_i$, then $x_i = 1 - x_{i-1}$ and $x_j = x_1$ for $j \leq i - 1$. There is a natural homeomorphism from $[1/2, 1]$ onto B_i , so B_i is an arc and $v = (1/2, 1/2, 1/2, \dots) \in B_i$ for each $i \geq 0$. Moreover, $M = \bigcup_{i \geq 0} B_i$.

We now show that M is homeomorphic to the inverse limit from Example 2.14. Denote by N that inverse limit. Then $N = \bigcup_{i \geq 0} A_i$ where $A_0 = \{x \in N \mid x_1 \in [0, 1]$

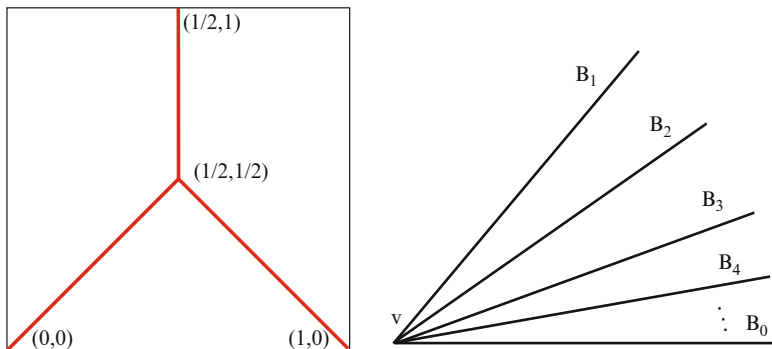
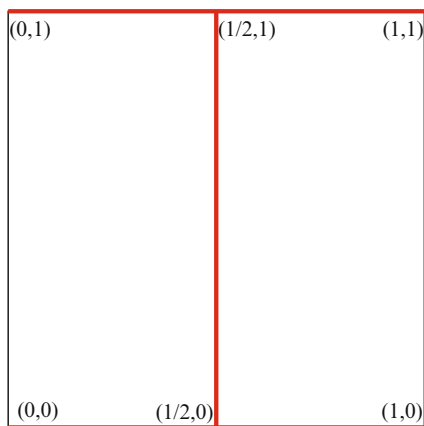


Fig. 2.23 The graph of the bonding function and a model of the inverse limit in Example 2.21

Fig. 2.24 The graph of the bonding function in Example 2.22



and $x_i = x_1$ for $i \geq 1$, while $A_i = \{\mathbf{x} \in N \mid x_1 \in [0, 1], x_j = x_1 \text{ for } 1 \leq j \leq i, \text{ and } x_j = 1 \text{ for } j > i\}$ for $i \geq 1$. If $\mathbf{x} \in B_0$, let $h(\mathbf{x}) = (2x_1, 2x_1, 2x_1, \dots)$, and for $i \geq 1$ and $\mathbf{x} \in B_i$, let $h(\mathbf{x})$ be the element of A_i having i th coordinate $2x_i - 1$. Then, h is a homeomorphism from $\bigcup_{i \geq 0} B_i$ onto $\bigcup_{i \geq 0} A_i$, and we have that M is homeomorphic to the inverse limit from Example 2.14. \square

We describe two additional examples. These examples were worked out by students in Mexico during the two-week short course on which much of the material in this book is based. Both examples grew out of an assignment in which they were to choose an embedding of a letter of the alphabet into $[0, 1]^2$ so that it forms the graph of an upper semicontinuous function and then determine its inverse limit (see Problem 6.61). The inverse limit in Example 2.22 is a familiar dendroid.

Example 2.22 (A dendroid with a Cantor set of endpoints). Let $f : [0, 1] \rightarrow 2^{[0,1]}$ be given by $f(t) = \{0, 1\}$ for $t \neq 1/2$ and $f(1/2) = [0, 1]$. Then, $\varprojlim f$ is a dendroid having a Cantor set of endpoints. (See Fig. 2.24 for the graph of f and Fig. 2.25 for a model of the inverse limit.)

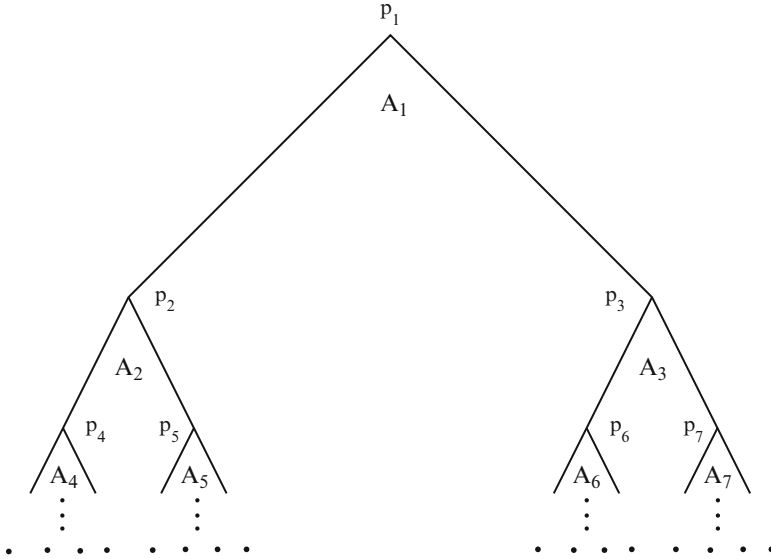


Fig. 2.25 A model of the inverse limit in Example 2.22

Proof. Let $M = \varprojlim f$, a continuum because $f^{-1} : [0, 1] \rightarrow C([0, 1])$. Let $A_1 = \{x \in M \mid x_i = 1/2 \text{ for } i > 1\}$, $A_2 = \{x \in M \mid x_i = 1/2 \text{ for } i > 2 \text{ and } x_1 = 0\}$, and $A_3 = \{x \in M \mid x_i = 1/2 \text{ for } i > 2 \text{ and } x_1 = 1\}$. In general, suppose $n \in \mathbb{N}$. There is a positive integer k such that the base 2 representation of n is $a_k a_{k-1} \cdots a_0$ with $a_k = 1$. For $n \geq 2$, let $A_n = \{x \in M \mid x_{k+1} \in [0, 1], x_i = 1/2 \text{ for } i > k+1 \text{ and } x_i = a_{k-i} \text{ for } 1 \leq i \leq k\}$. Note that $A_1 \cap A_2 = \{(0, 1/2, 1/2, 1/2, \dots)\}$ and $A_1 \cap A_3 = \{(1, 1/2, 1/2, 1/2, \dots)\}$. In fact, letting $p_1 = (1/2, 1/2, 1/2, \dots)$ and, for $n > 1$, $p_n = (a_{k-1}, a_{k-2}, \dots, a_0, 1/2, 1/2, \dots)$, then $A_1 \cap A_2 = \{p_2\}$, $A_1 \cap A_3 = \{p_3\}$, $A_2 \cap A_4 = \{p_4\}$, etc. We see that p_2 is an endpoint of A_1 and an interior point of A_2 . In general, A_n is an arc with endpoints p_{2n} and p_{2n+1} ; the point p_n is an interior point of A_n . Note that $M = \text{Cl}(\bigcup_{i>0} A_i)$ (where Cl denotes the closure). \square

Example 2.23. Let $f : [0, 1] \rightarrow C[0, 1]$ be given by $f(0) = f(1) = [0, 1]$ and $f(t) = \{1/2\}$ for $t \notin \{0, 1\}$. The inverse limit is a continuum that is the union of a sequence B_0, B_1, B_2, \dots of compacta such that $B_{i+1} \cap B_i$ is a Cantor set, B_0 is a single point, and, for each positive integer i , B_i is homeomorphic to a product of a Cantor set and an arc (Figs. 2.26 and 2.27).

Proof. Let $M = \varprojlim f$, a continuum by Theorem 2.7. Let $B_0 = \{(1/2, 1/2, 1/2, \dots)\}$, let $B_1 = \{x \in M \mid x_1 \in [0, 1] \text{ and } x_j \in \{0, 1\} \text{ for } j > 1\}$, and, for each integer $i > 1$, let $B_i = \{x \in M \mid x_j = 1/2 \text{ for } j < i, x_i \in [0, 1], \text{ and } x_j \in \{0, 1\} \text{ for } j > i\}$. Note that $B_i \cap B_{i+1} = \{x \in M \mid x_j = 1/2 \text{ for } j \leq i \text{ and } x_j \in \{0, 1\} \text{ for } j > i\}$. Note that $M = \bigcup_{i \geq 0} B_i$. Because each arc component of B_{i+1} intersects two arc components of B_i and both points of intersection are

Fig. 2.26 The graph of the bonding function in Example 2.23

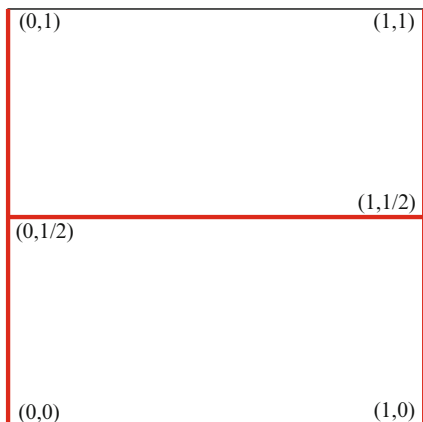
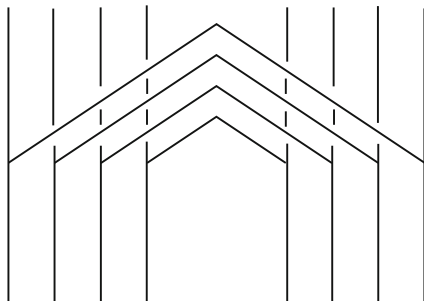


Fig. 2.27 A model of $B_1 \cup B_2$ in Example 2.23, a set homeomorphic to $B_i \cup B_{i+1}$ for each positive integer i



interior to its respective arc component of B_i , M contains uncountably many triods. Consequently, M is a nonplanar continuum, and a model for M is not so simple to depict. In Fig. 2.27, we indicate a typical set $B_i \cup B_{i+1}$ for $i \geq 1$. \square

There is a connection between Examples 2.23 and 2.22. The components of $B_1 \cup B_2$ from Example 2.23 are homeomorphic to $A_1 \cup A_2 \cup A_3$ from Example 2.22 (there are uncountably many of these components in Example 2.23), whereas the components of $B_1 \cup B_2 \cup B_3$ are homeomorphic to $A_1 \cup A_2 \cup \cdots \cup A_7$ (see Fig. 2.25).

2.10 Nonconnected Inverse Limits

In [10, Example 3.4], Nall provides a simple example of a function with a connected graph whose inverse limit is not connected. His proof involves the use of Theorem 2.2. This is our next example.

Example 2.24 (Nall; An inverse that is not connected). Let $f : [0, 1] \rightarrow 2^{[0,1]}$ be given by $f(t) = t/2$ for $0 \leq t < 1/2$ and $f(t) = \{t/2, 2t - 1\}$ for $1/2 \leq t \leq 1$. Then, $G(f)$ is connected, but $\varprojlim f$ is not connected. (See Fig. 2.28 for the graphs of f and f^2 .)

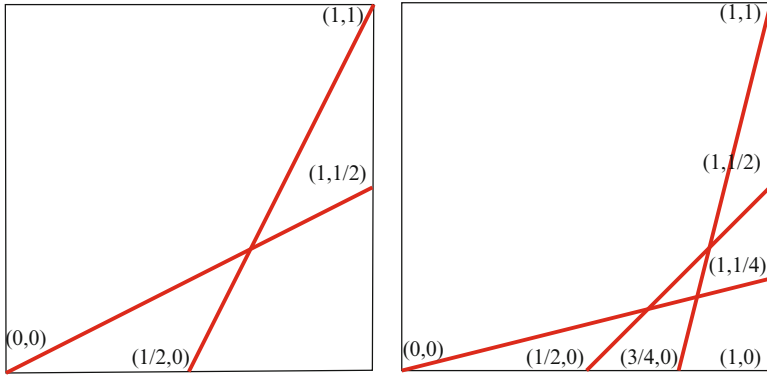


Fig. 2.28 Graphs of the functions f and f^2 in Example 2.24

Proof. The graph of f is easily seen to be connected. The graph of f^2 contains the point $(1, 0)$ as an isolated point. \square

There exist functions for which it is quite difficult to use Theorem 2.2 to determine that the inverse limit is not connected. Our next example provides a sequence of such functions. In this sequence, as n increases, so does the difficulty of making use of Theorem 2.2. Our proof is the same as the one provided in [4]. We begin with a lemma.

Lemma 2.3. *Suppose $f : [0, 1] \rightarrow 2^{[0,1]}$ is an upper semicontinuous function. If k is a positive integer, then $G(f^{k+1}) = \{(x, y) \in [0, 1]^2 \mid \text{there exists a point } t \in [0, 1] \text{ such that } x \in f^{-1}(t) \text{ and } y \in f^k(t)\}$.*

Proof. $y \in f^{k+1}(x)$ if and only if there is a point $t \in [0, 1]$ such that $t \in f(x)$ and $y \in f^k(t)$ therefore, we have that $y \in f^{k+1}(x)$ if and only if there is a point $t \in [0, 1]$ such that $x \in f^{-1}(t)$ and $y \in f^k(t)$. \square

Example 2.25. Let n be an integer greater than 1. Let $f_n : [0, 1] \rightarrow 2^{[0,1]}$ be given by $f_n(t) = t$ for $0 \leq t < 1/n$, $f_n(t) = \{t, 2t - 2/n, t - 1/n\}$ for $1/n \leq t \leq 2/n$, and $f_n(t) = \{t, t - 1/n\}$ for $2/n < t \leq 1$. Then, for $1 \leq k < n$, $G(f_n^k)$ is connected, but $G(f_n^n)$ is not connected (Fig. 2.29).

Proof. Choose a positive integer $n \geq 2$. Observe that f_n is the union of three homeomorphisms:

$$\begin{aligned} g_1 &= Id_{[0,1]}, \\ g_2 &: [1/n, 1] \rightarrow [0, 1-1/n] \text{ where } g_2(x) = x - 1/n, \\ g_3 &: [1/n, 2/n] \rightarrow [0, 2/n] \text{ where } g_3(x) = 2x - 2/n. \end{aligned}$$

It is clear that $G(f_n)$ is connected because $G(g_3)$ intersects both $G(g_1)$ and $G(g_2)$. Note that the points $(0, 0)$ and $(1/n, 0)$ belong to $G(f_n)$, and the entire graph of $G(f_n)$ lies in $[0, 1-1/n]^2$ except for two nonseparating half-open intervals

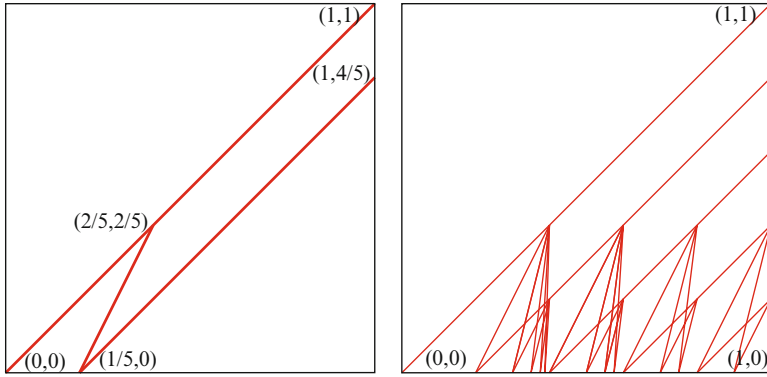


Fig. 2.29 Graphs of the functions f_5 and f_5^5 in Example 2.25

lying in the strip $(1-1/n, 1] \times [0, 1]$. Thus, $G(f_n|[0, 1-1/n])$ is connected. Clearly, $G(f_n|[0, 2/n])$ is connected, whereas $f_n([0, 2/n]) = [0, 2/n]$ and $f_n([0, 1-1/n]) = [0, 1-1/n]$.

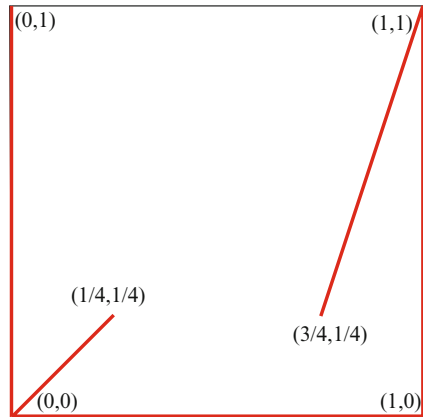
Let $\varphi_1 : [0, 1]^2 \rightarrow [0, 1]^2$ be given by $\varphi_1(x, y) = (x, y)$, let $\varphi_2 : [0, 1-1/n]^2 \rightarrow [1/n, 1] \times [0, 1-1/n]$ be given by $\varphi_2(x, y) = (x + 1/n, y)$, and let $\varphi_3 : [0, 2/n]^2 \rightarrow [1/n, 2/n] \times [0, 2/n]$ be given by $\varphi_3(x, y) = (x/2 + 1/n, y)$. Note that $\varphi_1(x, y) = (g_1^{-1}(x), y)$ for $(x, y) \in [0, 1]^2$; $\varphi_2(x, y) = (g_2^{-1}(x), y)$ for $(x, y) \in [0, 1-1/n]^2$ and $\varphi_3(x, y) = (g_3^{-1}(x), y)$ for $(x, y) \in [0, 2/n]^2$.

We now show that if $1 \leq k \leq n-1$, then $G(f_n^{k+1}) = \varphi_1(G(f_n^k)) \cup \varphi_2(G(f_n^k|[0, 1-1/n])) \cup \varphi_3(G(f_n^k|[0, 2/n]) - \{(2/n, 0)\})$. To see this first, let (x, y) be a point of $G(f_n^{k+1})$. By Lemma 2.3, there is a point $t \in [0, 1]$ such that $x \in f_n^{-1}(t)$ and $y \in f_n^k(t)$. There is an integer i , $1 \leq i \leq 3$, such that $x = g_i^{-1}(t)$, and, for such an i , $(x, y) = \varphi_i(t, y)$ with $(t, y) \in G(f_n^k)$. If $i = 1$, $(x, y) \in \varphi_1(G(f_n^k))$. If $i = 2$, then $0 \leq t \leq 1-1/n$, so $(x, y) \in \varphi_2(G(f_n^k|[0, 1-1/n]))$. If $i = 3$ and $(x, y) \neq (2/n, 0)$, then $t \in [0, 2/n]$ and $(x, y) \in \varphi_3(G(f_n^k|[0, 2/n]) - \{(2/n, 0)\})$. In case $(x, y) = (2/n, 0)$, $(x, y) = \varphi_2(1/n, 0)$, so $(x, y) \in \varphi_2(G(f_n^k|[0, 1-1/n]))$. On the other hand, if $(x, y) \in \varphi_1(G(f_n^k)) \cup \varphi_2(G(f_n^k|[0, 1-1/n])) \cup \varphi_3(G(f_n^k|[0, 2/n]) - \{(2/n, 0)\})$, then for some i , $1 \leq i \leq 3$ and some point $t \in [0, 1]$, $x \in g_i^{-1}(t)$, and $y \in f_n^k(t)$. It follows from Lemma 2.3 that $(x, y) \in G(f_n^{k+1})$.

Next, we proceed inductively to show that $G(f_n^k)$ is a connected set containing $(0, 0)$ and $(m/n, 0)$ for $1 \leq k \leq n-1$ and $1 \leq m \leq k$. We have observed this to be true for $k = 1$ because $G(f_n)$ is connected as are $G(f_n|[0, 1-1/n])$ and $G(f_n|[0, 2/n])$ and $(0, 0)$ and $(1/n, 0)$ are points of $G(f_n)$.

Suppose j is an integer, $1 \leq j < n-1$, such that $G(f_n^j)$ is a connected set as are $G(f_n^j|[0, 1-1/n])$ and $G(f_n^j|[0, 2/n] - \{(2/n, 0)\})$ (we only need to remove the point $(2/n, 0)$ when $j > 1$ because, of course, this point is not in $G(f_n)$). Suppose also that $(0, 0)$ and $(m/n, 0)$ are in $G(f_n^j)$ for $1 \leq m \leq j$. Then, $\varphi_1(G(f_n^j))$ is connected as are $\varphi_2(G(f_n^j|[0, 1-1/n]))$ and $\varphi_3(G(f_n^j|[0, 2/n] - \{(2/n, 0)\}))$.

Fig. 2.30 The graph of the bonding function in Example 2.26



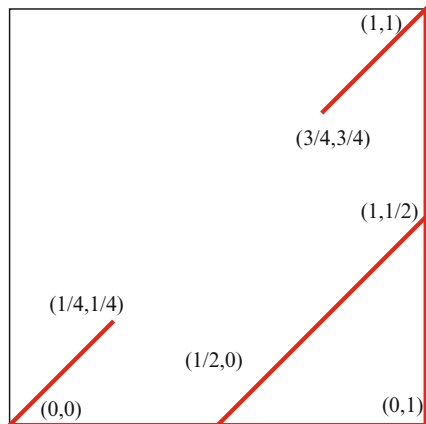
The point $(1/n, 0)$ belongs to all three of these sets because $\varphi_1(1/n, 0) = (1/n, 0)$ and $(1/n, 0) \in G(f_n^j)$, whereas $\varphi_2(0, 0) = \varphi_3(0, 0) = (1/n, 0)$ and $(0, 0)$ belongs to both $G(f_n^j|[0, 1-1/n])$ and $G(f_n^j|[0, 2/n])$. Thus, $G(f_n^{j+1})$ is connected and contains $(0, 0)$ because $\varphi_1(0, 0) = (0, 0)$. Further, the entire graph of f_n^{j+1} lies in $[0, 1-1/n]^2$ except for $j+2$ nonseparating half-open intervals lying in the strip $(1-1/n, 1] \times [0, 1]$ (the extra one that is not part of the graph of f_n^j comes from $\varphi_2(G(f_n^j|[0, 1-1/n]))$), so $G(f_n^{j+1}|[0, 1-1/n])$ is connected. Finally, $G(f_n^{j+1}|[0, 2/n]) - \{(2/n, 0)\}$ is connected. To see this, observe that the portion of $G(f_n^j)$ mapped into $[0, 2/n]^2$ by φ_2 is the union of the straight line interval from $(0, 0)$ to $(1/n, 1/n)$ and the single point $(1/n, 0)$. Thus, $\varphi_2(G(f_n^j|[0, 1-1/n])) \cap [0, 2/n]^2$ is the union of the straight line interval from $(1/n, 0)$ to $(2/n, 1/n)$ and the point $(2/n, 0)$. It follows that $G(f_n^{j+1}|[0, 2/n]) - \{(2/n, 0)\}$ is connected being the union of three connected sets $\varphi_1(G(f_n^j|[0, 2/n]) - \{(2/n, 0)\})$, the straight line interval from $(1/n, 0)$ to $(2/n, 1/n)$, and $\varphi_3(G(f_n^j|[0, 2/n]) - \{(2/n, 0)\})$, all containing $(1/n, 0)$. Because $(m/n, 0)$ is in $G(f_n^j)$ for $1 \leq m \leq j$ and $\varphi_2(i/n, 0) = ((i+1)/n, 0)$ for each i , $1 \leq i \leq j$, $(m/n, 0) \in G(f_n^{j+1})$ for $1 \leq m \leq j+1$.

Therefore, we have that $G(f_n^k)$ is connected for $1 \leq k \leq n-1$ and $(1-1/n, 0) \in G(f_n^{n-1})$. It now follows that $\varphi_2(1-1/n, 0) = (1, 0)$ is in $G(f_n^n)$. However, $(1, 0)$ is an isolated point of $G(f_n^n)$. To see this, observe that $f_n^n(1)$ is a discrete set with minimum 0 and $f_n^{-n}(0)$ is a discrete set with maximum 1. Because $G(f_n^n)$ has an isolated point, it is not connected. \square

One cannot always rely on Theorem 2.2 to detect that an inverse limit is not connected. The following example due to Jonathan Meddaugh demonstrates this. Example 2.26 is a modification of [5, Example 1, p. 266].

Example 2.26 (Meddaugh). Let $f : [0, 1] \rightarrow 2^{[0,1]}$ be given by $f(0) = [0, 1]$, $f(t) = \{t, 0\}$ for $0 < t \leq 1/4$, $f(t) = 0$ for $1/4 < t < 3/4$, $f(t) = \{3t-2, 0\}$ for $3/4 \leq t < 1$, and $f(1) = [0, 1]$. Then $G(f)$ is connected and $G(f^n) = [0, 1]^2$ for $n > 1$, but $\varprojlim f$ is not connected. (See Fig. 2.30 for the graph of f .)

Fig. 2.31 The graph of the bonding function in Example 2.27



Proof. It is clear that $G(f)$ is connected and $G(f^n) = [0, 1]^2$ for $n > 1$. Let $M = \varprojlim f$. The set $N = \{x \in M \mid x_1 = x_2 = 1/4, \text{ and } x_3 = 3/4\}$ is both open and closed in M because N is closed and $N = ((1/8, 3/8) \times (1/8, 3/8) \times (5/8, 7/8) \times \mathcal{Q}) \cap M$. Thus, M is not connected. \square

A second example of an inverse limit that is not connected but the graphs of all composites of the bonding functions are connected is an example in a recent paper by Greenwood and Kennedy in which they showed that the inverse limit is not connected [1, Example 1.4, p. 58].

Example 2.27. Let $f : [0, 1] \rightarrow 2^{[0,1]}$ be given by $f(t) = \{0, t\}$ for $0 \leq t \leq 1/4$, $f(t) = 0$ for $1/4 < t < 1/2$, $f(t) = \{t - 1/2, 0\}$ for $1/2 < t < 3/4$, $f(t) = \{t, t - 1/2, 0\}$ for $3/4 \leq t < 1$, and $f(1) = [0, 1]$. Then, $G(f)$ is connected and $f^n = f$ for each $n \in \mathbb{N}$, but $\varprojlim f$ is not connected. (See Fig. 2.31 for the graph of f .)

Proof. It is clear that $G(f)$ is connected. It is not difficult to show that $f^2 = f$, and therefore, $f^n = f$ for each $n \in \mathbb{N}$. Let $M = \varprojlim f$ and let $N = \{x \in M \mid x_1 = x_2 = 1/4 \text{ and } x_3 = x_4 = 3/4\}$; N is a closed subset of M . Let U be the basic open set $(1/8, 3/8) \times (1/8, 3/8) \times (5/8, 7/8) \times (5/8, 7/8) \times \mathcal{Q}$. Note that $N \subseteq U \cap M$. To see that $N = U \cap M$, suppose $x \in U \cap M$. Because $x_1 \in (1/8, 3/8)$ and $x_2 \in (1/8, 3/8)$, we observe that $x_2 \in (1/8, 1/4]$. From $x_2 \in (1/8, 1/4]$ and $x_3 \in (5/8, 7/8)$, it follows that $x_3 \in (5/8, 3/4]$. However, with $x_3 \in (5/8, 3/4]$ and $x_4 \in (5/8, 7/8)$, it follows that $x_4 = 3/4$. Because $x_4 = 3/4$ and $x_3 \in (5/8, 7/8)$, $x_3 = 3/4$; $x_3 = 3/4$ and $x_2 \in (1/8, 3/8)$ yields $x_2 = 1/4$. Because $x_2 = 1/4$ and $x_1 \in (1/8, 3/8)$, $x_1 = 1/4$. Thus, $x \in N \cap M$. Because M contains a closed set N that is open in M , M is not connected. \square

References

1. Greenwood, S., Kennedy, J.: Connected generalized inverse limits. *Topology Appl.* **159**, 57–68 (2012)
2. Hurewicz, W.: Über oberhalb-stetige Zerlegungen von Punktmengen in Kontinua. *Fund. Math.* **15**, 57–60 (1930)
3. Ingram, W.T.: Inverse limits with upper semi-continuous bonding functions: Problems and partial solutions. *Topology Proc.* **36**, 353–373 (2010)
4. Ingram, W.T.: Concerning nonconnected inverse limits with upper semi-continuous set-valued functions. *Topology Proc.* **40**, 203–214 (2012)
5. Ingram, W.T., Mahavier, W.S.: Inverse limits of upper semi-continuous set valued functions. *Houston J. Math.* **32**, 119–130 (2006)
6. Ingram, W.T., Mahavier, W.S.: Inverse limits: From continua to Chaos. In: *Developments in Mathematics*, vol. 25. Springer, New York (2012)
7. Mahavier, W.S.: Inverse limits with subsets of $[0, 1] \times [0, 1]$. *Topology Appl.* **141**, 225–231 (2004)
8. Moore, R.L.: *Foundations of Point Set Theory*, vol. 13, rev. edn. American Mathematical Society Colloquium Publications, Providence (1962)
9. Nadler, S.B., Jr.: *Continuum Theory*. Marcel-Dekker, New York (1992)
10. Nall, V.: Connected inverse limits with a set-valued function. *Topology Proc.* **40**, 167–177 (2012)

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