Chapter 2
Properties of Complex Numbers of a Real Argument and Real Functions of a Complex Argument

Important results can be obtained if we apply simple complex-value models in economic modeling – complex functions of a real argument and real functions of a complex argument. This chapter focuses on the properties of these models and the possibility of using them in economic practice.

Complex models of a real argument represent the dependence of a complex variable on a real argument. This dependence can be obtained only if one uses a function that transforms real variables into complex ones. The Laplace transform is a well-known transformation method; however, this chapter focuses on other methods widely applied in economics. Real models of complex argument solve another problem – the transformation of a complex variable into a real one. The properties of the simplest models of this type are considered in this chapter with respect to economic modeling.

2.1 General Problem of Conformal Mapping in Complex-Valued Economics

Before using a tool of the theory of functions of a complex variable (TFCV) in economics it is necessary to study the properties of this tool [1]. One of the methods for understanding these properties is provided by conformal mapping of points from one complex plane to another. With reference to various cases of the TFCV, conformal mapping provides problems of varying degrees of complexity. We will consider the simplest cases since an understanding of conformal mappings of elementary complex-valued functions will allow researchers to choose the proper complex-valued function for modeling.

Thus, we can say that conformal mapping is a convenient graphical method for understanding how, by means of a given function, one complex variable in a complex plane of an argument is mapped to another complex variable modeling the value of the variable of the complex result.

S. Svetunkov, *Complex-Valued Modeling in Economics and Finance*,
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Since we work in the sphere of complex numbers, any real number may be represented as a complex number with zero imaginary part. Then we obtain three types of functions to be used in economic modeling.

The first type represents the relationship between a complex variable and a real argument:

$$y_r + iy_i = F(x_r + i0) = f(x_r) + if(x_r). \tag{2.1}$$

It is a complex function of a real variable.

The second type comes up when a complex argument is associated with a real result:

$$y_r + i0 = F(x_r + ix_i) = f_r(x_r, x_i) + if_i(x_r, x_i) \Leftrightarrow \begin{cases} y_r = f_r(x_r, x_i), \\ f_i(x_r, x_i) = 0. \end{cases} \tag{2.2}$$

It is a function of a complex argument.

The third type is the relationship between a complex variable and a complex result:

$$y_r + iy_i = F(x_r + ix_i) = f_r(x_r, x_i) + if_i(x_r, x_i). \tag{2.3}$$

It is a complex function (a function with complex values).

The TFCV considers mainly conformal mappings of the third type. However, in economics we can use all three as models of a complex-valued economy. This is why it is essential to examine in depth the properties of all three types of functions. This chapter will focus on the properties of the first two types.

2.2 Complex Functions of a Real Argument

The complex functions of a real argument represent a certain "mapping" of a set of real numbers on a numerical axis to the plane of complex variables:

$$y_r + iy_i = F(x + i0) = f(x) + if(x). \tag{2.4}$$

This function transforms real variables and the respective functions to complex variables and the respective functions.

Situations where one variable influences two others are quite frequent in economics. For example, in marketing, consumers are grouped in particular categories – segments where the basic indicator is a similar reaction of all consumers of this segment to a product and its marketing support. This means that consumers with similar levels of income (if we categorize by income) will react similarly to a given price and buy the same quantity of the product at that price. This in turn means that the price y_r and the consumption volume y_i depend on the level of income x. With this knowledge, one can look at the reaction to goods by consumers from various segments as being subject to an increase in income of each segment and model this reaction by a function of a real argument (2.4).

The variety of possible functions of a real argument that may be put forth to model the aforementioned economic processes is limited only by the imagination of the researcher creating the model. This is why in this section we deal only with the simplest functions and their properties.

A linear model of a real argument,

$$y_r + iy_i = (a_0 + ia_1) + (b_0 + ib_1)(x + i0) = (a_0 + b_0 x) + i(a_1 + b_1 x), \qquad (2.5)$$

is of little interest because any change in the argument entails a directly proportional change in the real and imaginary parts of the complex result. This means that for any change in the real argument – linear or nonlinear – we have a line in a complex plane whose slope and position thereon is completely determined by the values of a complex proportionality coefficient.

Nonlinear transformations of a real variable to a complex plane are of practical interest. The first of these methods is the complex involution of a real argument:

$$y_r + iy_i = (a_0 + ia_1)x^{(b_0 + ib_1)}. \qquad (2.6)$$

The proportionality coefficient that can be placed before the argument that is subject to involution can be not only complex, but also real or imaginary. Let us represent the complex function of a real argument in exponential form, then (2.6) will be written as

$$y_r + iy_i = \sqrt{a_0^2 + a_1^2}\, x^{b_0} e^{i\left(arctg\frac{a_1}{a_0} + b_1 \ln x\right)}. \qquad (2.7)$$

To simplify the notation of the complex proportionality coefficient, let us write it as $A = \sqrt{a_0^2 + a_1^2}$ and the polar angle as $\alpha = arctg\frac{a_1}{a_0}$.

The equality of the real and imaginary parts of this equation may be represented as the following system:

$$\begin{cases} y_r = Ax^{b_0}\cos(\alpha + b_1 \ln x), \\ y_i = Ax^{b_0}\sin(\alpha + b_1 \ln x). \end{cases} \qquad (2.8)$$

It is clear that both the real and imaginary parts of this complex real variable function change with an increase in the argument according to the cosine (real part) and sine (imaginary part) law. Taking into account the fact that the real argument in these trigonometric functions is not direct but a logarithm, with a uniform increase in the real argument, periods of oscillation of both the real and imaginary parts of the function under consideration will increase. A logarithm limits the function domain;

since a logarithm of zero does not exist, the zero point is not included in the function domain.

If we consider the result (2.7) in the complex plane, the points of this function will be located as follows. The module of this function

$$r = Ax^{b_0} \tag{2.9}$$

will increase with an increase in the argument $x > 0$ for $b_0 > 0$ and decrease for $b_0 < 0$, and the polar angle will increase,

$$\varphi = \alpha + b_1 \ln x, \tag{2.10}$$

if $b_1 > 0$ and decrease (move in a clockwise direction) if $b_1 < 0$.

Hence, it is easy to see that in the complex plane function (2.6) is mapped subject to the values of the complex exponent in the form of a convergent or divergent spiral.

Let us consider a special case of function (2.6), where time t acts as the argument:

$$y_r + iy_i = (a_0 + ia_1)t^{(b_0+ib_1)}. \tag{2.11}$$

This function represents a complex trend and may be used in practice in certain economic situations.

As follows from the aforementioned properties of the function under consideration, the character of a complex trend will be fully determined by its coefficients. Here are some interesting types of such trends.

Thus, if we use the trend

$$y_{rt} + iy_{it} = t^{(-0,5+i10)}, \tag{2.12}$$

then each of the components of the complex-valued trend will look like Figs. 2.1 and 2.2.

The same form of the trend but with other coefficients

$$y_{rt} + iy_{it} = t^{(0,25+i0,35)} \tag{2.13}$$

models completely different dynamics (Figs. 2.3 and 2.4).

Trends like those shown in Figs. 2.3. and 2.4 are quite frequent in the domain of real variables; however, models describing the dynamics of trends like 2.1 and 2.2 are quite rare in studies on socioeconomic processes, except for stock markets.

The next model of a real argument may be a complex exponential function of the real argument. It may be presented as follows:

$$y_r + iy_i = (a_0 + ia_1)e^{(b_0+ib_1)x}. \tag{2.14}$$

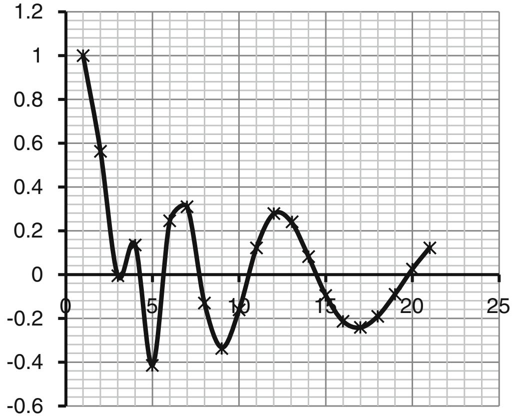


Fig. 2.1 Dynamics of real part of complex trend (2.12)

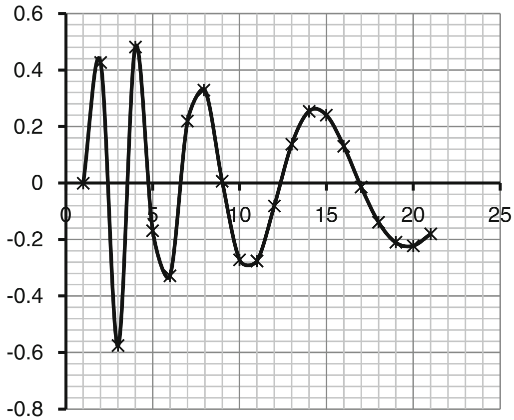


Fig. 2.2 Dynamics of imaginary part of complex trend (2.12)

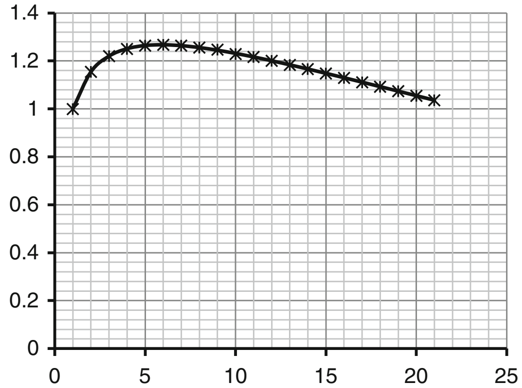


Fig. 2.3 Dynamics of real part of complex trend (2.13)

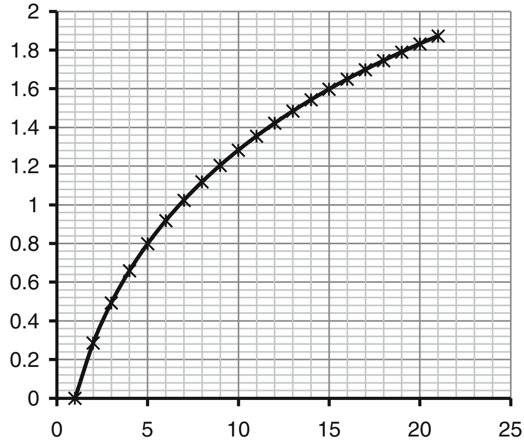


Fig. 2.4 Dynamics of imaginary part of complex trend (2.13)

The base of an exponential function may also be different, for example, a complex number, but we will not consider these variants.

In the exponential form function (2.14) may look like this:

$$y_r + iy_i = Ae^{b_0 x} e^{i(\alpha + b_1 x)}. \tag{2.15}$$

As we see, the module of this function varies according to the exponential law and variations in the polar angle are directly proportional to variations in the argument. Since the complex coefficient of an exponent can take various values, a modeled function can describe different variants of the dynamic whose details differ from function (2.7), but, similarly to that function, its mapping to the complex plane is spiral.

If we consider the real and imaginary parts of this complex function separately, we will have a system of equations:

$$\begin{cases} y_r = Ae^{b_0 x} \cos(\alpha + b_1 x), \\ y_i = Ae^{b_0 x} \sin(\alpha + b_1 x). \end{cases} \tag{2.16}$$

Now the differences between a complex exponential function of a real argument and a complex power function of a real argument are evident. The real and imaginary parts of an exponential function vary according to the cosine and sine laws with a constant period of oscillations, the oscillation range varying with the change in the argument. If $b_0 > 0$, then the oscillation range increases with the growth of the argument; if $b_0 < 0$, then the oscillation range decreases.

A complex trend model is a simple variant of this model.

For example, for low positive values of coefficients of a complex exponent like

$$y_{rt} + iy_{it} = e^{(0{,}15 + i0{,}05)t} \tag{2.17}$$

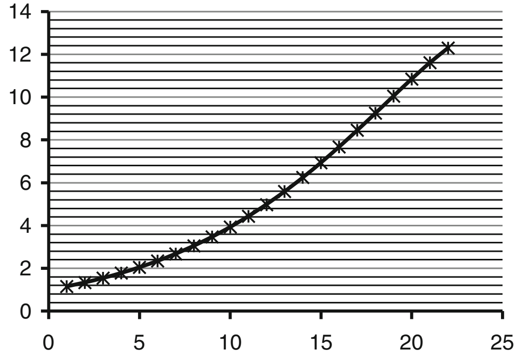


Fig. 2.5 Dynamics of real part of trend (2.17)

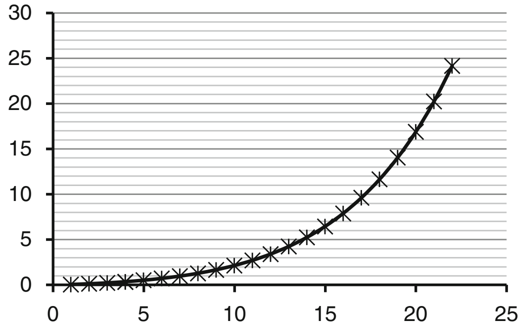


Fig. 2.6 Dynamics of imaginary part of trend (2.17)

each of the components is described for $t = 1,2,\ldots 22$ with an increasing area (Figs. 2.5 and 2.6).

For other coefficients a considerably more complex cyclical dynamics can be modeled, for example, if the model has the form

$$y_{rt} + iy_{it} = e^{(0{,}05+i60)t} \tag{2.18}$$

then the dynamics of the real and imaginary constituents of the complex-valued trend takes the form shown in Figs. 2.7 and 2.8.

It is seen from these figures that the function models the oscillation process with increasing amplitude at a constant oscillation frequency.

We could continue looking at similar elementary complex functions of real arguments, but this goes beyond the problems covered by our study. Thus, we will consider several special nonstandard functions that are subspecies of those mentioned previously.

First is an exponential-power function with an imaginary exponent:

$$y_r + iy_i = x^{ix}. \tag{2.19}$$

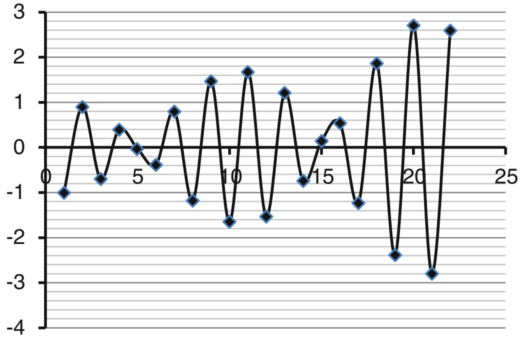


Fig. 2.7 Dynamics of real part of trend (2.18)

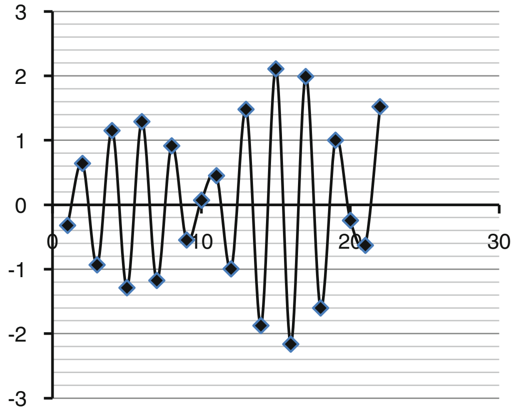


Fig. 2.8 Dynamics of imaginary part of trend (2.19)

Let us present this function in exponential form:

$$y_r + iy_i = e^{ix \ln x}. \tag{2.20}$$

This helps us to determine a change in the real and imaginary parts with the growth of the argument:

$$\begin{cases} y_r = \cos(x \ln x), \\ y_i = \sin((x \ln x). \end{cases} \tag{2.21}$$

They change according to the cosine and sine laws with an increasing oscillation period. A logarithm limits the function domain; since a logarithm of zero does not exist, the zero point is not included in the function domain.

Since the module of this function is equal to one, in a complex plane the function represents a unit circumference.

An exponential-power function with a complex exponent is an expected development of this function:

$$y_r + iy_i = x^{(x+ix)} = x^{(1+i)x}. \tag{2.22}$$

In this case, the right-hand side of the equality is easily represented in exponential form:

$$y_r + iy_i = x^x e^{ix \ln x}. \tag{2.23}$$

This model has a domain in the positive part of real numbers since a logarithm of a negative number, as well as a logarithm of zero, does not exist.

Let us present the real and imaginary parts of this complex function separately:

$$\begin{cases} y_r = x^x \cos(x \ln x), \\ y_i = x^x \sin(x \ln x). \end{cases} \tag{2.24}$$

The module of this complex function increases sharply with an increase in the argument; this is why the real and imaginary parts of the function represent an oscillatory function with increasing oscillation period and sharply growing oscillation range. In a complex plane this function is shown as a sharply diverging spiral. This feature gives the function little applicability in economic modeling, though the initial part of the function could be of interest. The module of a function in the positive neighborhood of the zero point is close to one (any number to the zero power is equal to one); however, with an increase in the argument it will first decrease and then increase. The module of the complex function reaches its minimum value at the point where the first derivative is equal to zero:

$$\frac{dr}{dx} = (x^x)' = 0$$

After solving this equation and using the Leibniz-Bernoulli formula we have

$$r' = x^x(1 + \ln x) = 0.$$

Since $|x| > 0$, the module of the complex function reaches its minimum value at the point $x = e^{-1}$.

The dynamics of the polar angle u with changes in the argument within the interval [0;1) is complicated since it is determined by the following equality:

$$\theta = x \ln x.$$

The first derivative of this relationship with respect to the argument will have the following form:

$$\frac{d\theta}{dx} = (x)' \ln x + x(\ln x)' = \ln x + 1,$$

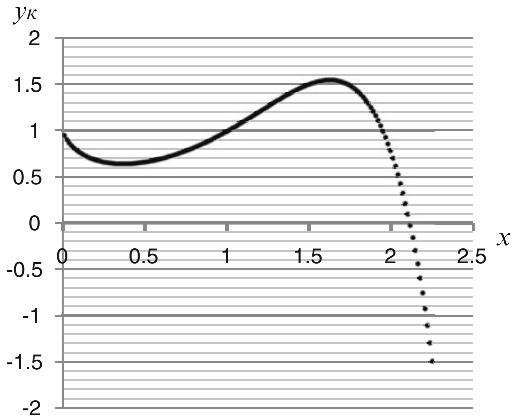


Fig. 2.9 Real parts of complex function (2.22)

which means that the polar angle reaches its minimum value at the same point as the module of the complex point $x = e^{-1}$.

Thus, for the argument $x{=}e^{-1}$ the complex function under consideration reaches its minimum values of both the module and polar angle. In the complex plane this will be shown with an increase in the argument as follows. The curve starts its movement in the clockwise direction from the neighborhood of a point with the coordinates $xr = 1, xi = 0$ until it reaches the point where both the module and the argument take their minimum values. The module then is equal to

$$r_{\min} = x^x = \left(e^{-1}\right)^{e^{-1}} = \left(\frac{1}{e}\right)^{\frac{1}{e}}$$

and the polar angle to

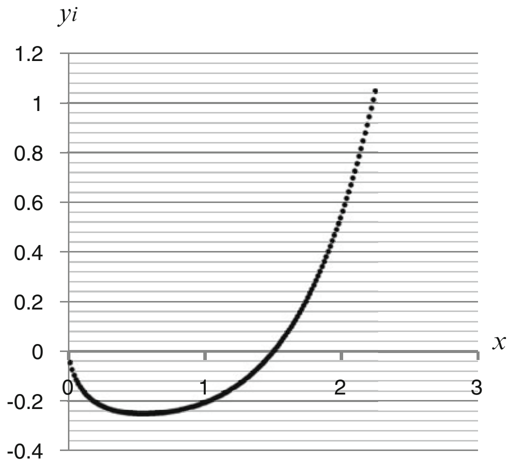
$$\theta_{\min} = x \ln x = -\frac{1}{e}.$$

Then, with growth of the x module of the complex function, its polar angle starts growing, too. In the complex plane this growth is revealed in movement along the same line but in a counterclockwise direction.

Then the function module starts growing sharply, which leads to an increase in the values of the real and imaginary parts of the function under consideration. The separate dynamics of the real and imaginary parts of this complex function at low values of the real argument $x = [0;2)$ are of more interest. This dynamics is given in Fig. 2.9.

It is possible to narrow the spiral span and increase or reduce the rotation frequency by involution of the complex proportionality coefficient, which is different from the complex unit:

Fig. 2.10 Imaginary parts of complex function (2.22)



$$y_r + iy_i = x^{(b_0+ib_1)x}. \tag{2.25}$$

For different values of the real and imaginary parts of this coefficient, for the function under consideration there one obtains a great variety of spirals in the complex plane, as well as various types of dynamics of the real and imaginary parts of the complex function (Fig. 2.10).

We can continue the logic of the real argument transformation to the complex plane by suggesting a complex exponential-power function with a complex base:

$$y_r + iy_i = (x + ix)^x = (x(1 + i))^x. \tag{2.26}$$

Its exponential form will look like this:

$$y_r + iy_i = (\sqrt{2}x)^x e^{i\frac{\pi}{4}x}. \tag{2.27}$$

Then for the real and imaginary parts of this complex function we have

$$\begin{cases} y_r = (\sqrt{2}x)^x \cos\left(\dfrac{\pi}{4}x\right), \\ y_i = (\sqrt{2}x)^x \sin\left(\dfrac{\pi}{4}x\right). \end{cases}$$

The zero point of the real argument is included in the function domain. In general this function looks like a spiral; however, at low values its behavior is complicated as its polar angle increases with the growth of the real argument and its module first decreases, reaches its minimum, and starts increasing again.

The first derivative of the module is

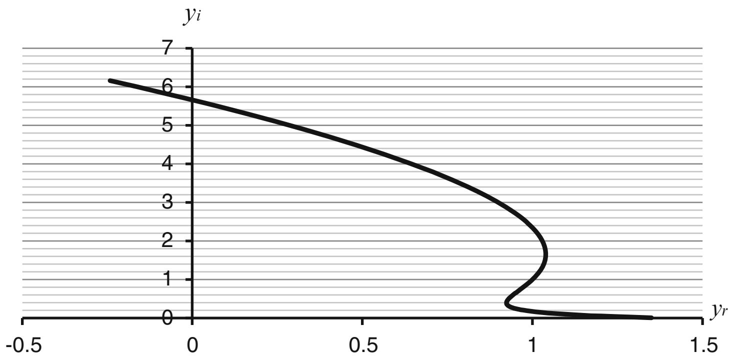


Fig. 2.11 Function (2.26) in complex plane at low values of real argument

$$r' = (\sqrt{2}x)^x \left(\frac{x}{\sqrt{2}x} + \ln(\sqrt{2}x) \right).$$

If we set it equal to zero we obtain a point where the module takes minimum values:

$$x = \frac{e^{-\frac{1}{\sqrt{2}}}}{\sqrt{2}} = 0.348652215$$

Taking into account these specifics, for initial values of the real argument the complex function will have nonlinear dynamics (Fig. 2.11).

The last elementary function of a real argument is a complex exponential-power function with a complex base and complex exponent:

$$y_r + iy_i = (x + ix)^{(x+ix)} = (x(1+i))^{(1+i)x}. \tag{2.28}$$

In exponential form it looks like this:

$$y_r + iy_i = (\sqrt{2}x)^{(1+i)x} e^{i\frac{\pi}{4}(1+i)x} = [(\sqrt{2}x)^x e^{-\frac{\pi}{4}x}][(\sqrt{2}x)^{ix} e^{i\frac{\pi}{4}x}]. \tag{2.29}$$

The module of this complex function of a real argument will be

$$R = (\sqrt{2}x)^x e^{-\frac{\pi}{4}x}. \tag{2.30}$$

And its polar angle will have the following form:

$$\theta = x\left[\ln(\sqrt{2}x) + \frac{\pi}{4} \right]. \tag{2.31}$$

It is easy to see that the zero value of the real argument is not included in the function domain.

The behavior of the module of this function is more complicated than that of the previous ones. For an argument close to zero the module will be close to one, then it gets lower up to a certain value, after which it starts increasing again, but not sharply as in the case of the previous function.

To determine the point where the module of complex function (2.30) takes its minimum value, we should find its first derivative:

$$R' = [(\sqrt{2}x)^x]' e^{-\frac{\pi}{4}x} + (\sqrt{2}x)^x (e^{-\frac{\pi}{4}x})' = (\sqrt{2}x)^x \left(\frac{1}{\sqrt{2}} + \ln(\sqrt{2}x) \right) e^{-\frac{\pi}{4}x} - \frac{\pi}{4} (\sqrt{2}x)^x e^{-\frac{\pi}{4}x},$$

which should be equal to zero. Then, solving the equation we find the point at which the module is at its minimum:

$$x = \frac{e^{\frac{\pi - 2^{\frac{3}{2}}}{4}}}{\sqrt{2}}.$$

The polar angle also varies nonlinearly – it decreases from values close but not equal to zero (the points are in the fourth quadrant of the complex value) and then grows. To determine the minimum value of the polar angle, let us find its first derivative of the real argument:

$$\frac{d\theta}{dx} = (x)' \left(\ln(\sqrt{2}x) + \frac{\pi}{4} \right) + x \left(\ln(\sqrt{2}x) + \frac{\pi}{4} \right)' = \ln(\sqrt{2}x) + \frac{\pi + 2^{\frac{3}{2}}}{4}.$$

After setting it equal to zero and solving the equation we find the value of the real argument for which the polar angle reaches its minimum value:

$$x = \frac{e^{-\frac{\pi + 2^{\frac{3}{2}}}{4}}}{\sqrt{2}}$$

With an increase in the argument, the conformal mapping of the function under consideration in a complex plane takes place in a spiral moving in a clockwise direction, as shown in Fig. 2.12.

With the further growth of the argument, the function module increases sharply, as does the polar angle, the function itself continuing its spiral movement in the clockwise direction.

The elementary complex exponential-power function with complex base and complex exponent (2.28) can be represented in a form more applicable to practical purposes, namely:

$$y_r + iy_i = (x(a_0 + ia_1))^{(b_0 + ib_1)x}. \tag{2.32}$$

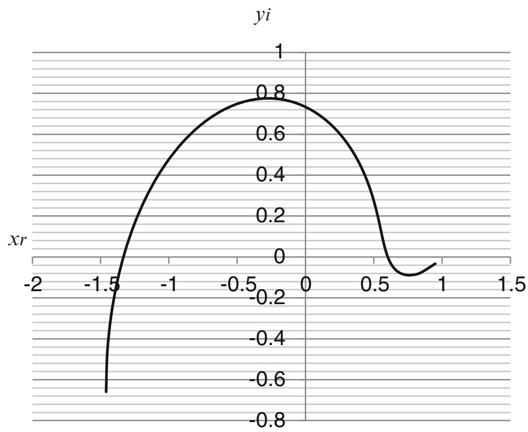


Fig. 2.12 Part of conformal mapping of function (2.28) for low value of argument

Its exponential form will have the form

$$y_r + iy_i = \left[(Ax)^{b_0 x} e^{-b_1 x\alpha}\right]\left[(Ax)^{ib_1 x} e^{ib_0 x\alpha}\right]. \tag{2.33}$$

Hence, for the module of this complex function of a real argument

$$R = (Ax)^{b_0 x} e^{-b_1 x\alpha} \tag{2.34}$$

and for the polar angle

$$\theta = x[b_1 \ln(Ax) + b_0\alpha]. \tag{2.35}$$

The function domain lies in the area of positive arguments, which clearly follows from (2.35).

Changing the values of function coefficients (2.32) we can obtain a great diversity of conformal mappings and variations of the real and imaginary parts of this function that have an oscillatory character.

As we see from (2.35), the polar angle of this complex function of a real argument depends largely on the constant b_1. The higher the values of this constant, the more rapid is the increase in the polar angle with the increase of the argument, and the faster is the turnover of the function values in the complex plane. This coefficient also influences the change in the module of the function under consideration, but for a low value of a_1 and high value of a_0 this influence decreases.

The coefficient b_0 is responsible for the growth in the function module. At its positive values the module increases sharply.

For various coefficient values the function behaves in a different way – it converges to zero and diverges, changes values around some circumference, changes chaotically, etc.

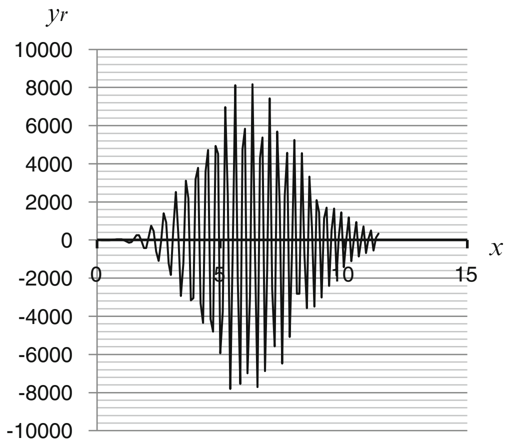


Fig. 2.13 Dynamics of real part of function (2.36)

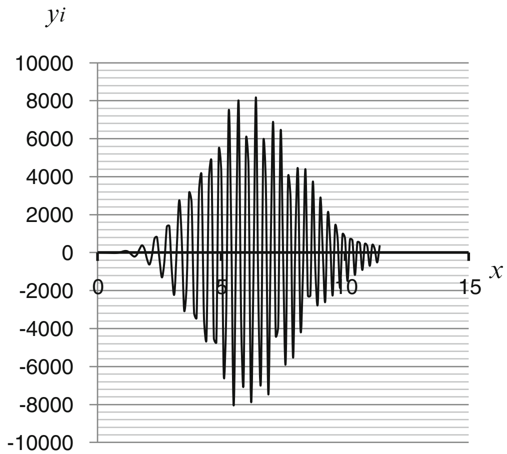


Fig. 2.14 Dynamics of imaginary part of function (2.36)

It is interesting that the function can also model the process of reaction of some system to an external influence with further stabilization at the previous level. This function behaves in this way, for example, for the following coefficients:

$$y_r + iy_i = (x(1 - i))^{(-1{,}5+i6)x}. \tag{2.36}$$

Subsequent change in the real and imaginary parts of this function with the growth of the argument within $0 < x \leq 10$ subject to the argument is shown in Figs. 2.13 and 2.14.

According to the results of this section we can draw the conclusion that complex functions of a real argument model a great diversity of cyclical dynamics.

Numerous functions of complex arguments are not limited at all to only the aforementioned types. However, it is not possible to consider all functions within the framework of this study; this would be at odds with the purpose of the present study, where we state only the basics of the application of the TFCV to solutions of economic problems.

The superposition of elementary complex functions provides vast possibilities for the generation of new functions. A simple example is the case where the complex power function of a real argument

$$z_r + iz_i = (c_0 + ic_1)x^{d_0+id_1}$$

is added to by a complex function of a real argument (2.32)

$$z_r + iz_i = (c_0 + ic_1)x^{d_0+id_1} + (x(a_0 + ia_1))^{(b_0+ib_1)x}. \tag{2.37}$$

If, for example, for the second term of this function we use coefficients like those proposed in (2.36), the resulting model will describe the dynamics of some nonlinear process, which may chaotically deviate from its previous trajectory on a certain segment under a certain external influence, but due to the stability of the object it returns to its former trajectory. It is evident that instead of the power function, the first term may be represented by other forms, for example, by a step function. With the proper selection of parameters, with the assistance of such a superposition, we model the transition from one stationary state to another.

The real argument itself can be presented in complex functions as a real function of a real argument, for example, *sinx* or *cosx*.

It is evident that the variety of complex functions of a real argument is enormous, and it is impossible to cover them in one section or chapter.

2.3 Functions of a Complex Argument: Linear Function

Since it is possible to transform a real argument to a complex plane using particular functions, a reverse transformation procedure is also possible – from the field of complex variables to the numerical axis of real variables. The relationship between a complex argument and a real result will represent a function of a complex argument:

$$y = f(x_r + ix_i) = f_r(x_r, x_i) + if_i(x_r, x_i). \tag{2.38}$$

Since there is a complex number in the right-hand side of this equality and a real one in the left-hand side, the function of the complex argument may be written as follows:

$$y + i0 = f_r(x_r, x_i) + if_i(x_r, x_i). \tag{2.39}$$

Hence we have a system of two real equations:

$$\begin{cases} y = f_r(x_r, x_i), \\ 0 = f_i(x_r, x_i). \end{cases} \tag{2.40}$$

The first equality of the system represents an equation of some surface in a three-dimensional space, the second one a line in the argument's plane. Since the problem is considered in a three-dimensional coordinate system, for the second equation of system (2.40) the equality is valid for any y value. This means a surface in three-dimensional space that is not crossed by the y-axis, i.e., the y-axis is parallel to this surface, the surface itself being perpendicular to the complex plane of the argument.

Since these two equations are combined in a system, they are simultaneously satisfied. Geometrically this means that system (2.40) and the initial function (2.38) represent an intersection of two planes in three-dimensional space – the first and second equations of system (2.40). The perpendicular nature of the second equation of system (2.40) means that the aggregate of the points lying on the surface of the first equation of system (2.40.) must be projected onto the plane of the complex argument as a line described by the second equation of system (2.41).

Let us sequentially consider the main functions of a complex argument and their graphical interpretation in order of increasing complexity, bringing each of them to the form (2.40).

The first such model to be used in economics is a linear function of a complex argument with a zero free term:

$$y = (b_0 + ib_1)(x_r + ix_i). \tag{2.41}$$

If we single out the real and imaginary parts of this function and group them, we have

$$\begin{cases} y = b_0 x_r - b_1 x_i, \\ 0 = b_1 x_r + b_0 x_i. \end{cases} \tag{2.42}$$

The first Eq. (2.42) is that of a plane in space passing through a zero point. The slope angle and position of the plane in space is fully determined by the signs and values of the coefficients of the complex proportionality coefficient.

The second equation of the system under consideration represents an equation of a line in the plane of the argument:

$$x_r = -\frac{b_0}{b_1} x_i. \tag{2.43}$$

This straight line originates from the zero point, and its location in the particular quadrant of the complex plane is determined by the values of the real and imaginary parts of the complex proportionality coefficient. Since the second equation of system (2.42) should be considered in space, it represents a plane parallel to the y-axis and perpendicular to the complex plane.

The two crossing planes form a line, meaning that the linear function of complex argument (2.41) represents a line in three-dimensional space ($0y$;$0x_r$;$0x_i$) passing through the zero point.

If we now consider a linear function of a complex argument with a free complex coefficient:

$$y = (a_0 + ia_1) + (b_0 + ib_1)(x_r + ix_i). \tag{2.44}$$

then singling out the real and imaginary parts of this function and grouping them as in the previous case, we have

$$\begin{cases} y = a_0 + b_0 x_r - b_1 x_i, \\ 0 = a_1 + b_1 x_r + b_0 x_i. \end{cases} \tag{2.45}$$

It is clear that the nature of the function has not changed – both the first and the second equations are planar equations – only the location of the planes in space has changed, as has the location of the line resulting from the planes' intersecting. It follows from the first equation that the plane does not pass in space through the zero point and crosses the y-axis at the point $y = a_0$. The second equation shows that the line in space of the complex argument does not pass through the zero point either since

$$x_r = -\frac{a_1}{b_1} - \frac{b_0}{b_1} x_i. \tag{2.46}$$

And on the axis of real values of the complex argument this line passes through the point $x_r = -\frac{a_1}{b_1}$.

Thus, the linear function of a complex argument with free complex coefficient (2.44) represents the equation of a line in three-dimensional space "a complex plane of an argument – an axis of a real number." Or put another way, any line in three-dimensional space may be described by a linear function of complex argument (2.44).

It is appropriate to recall that in the Cartesian coordinate system the equation of a line is also determined by the intersection of two planes and may be written as follows:

$$\begin{cases} a_1 y + b_1 x_r + c_1 x_i + d_1 = 0, \\ a_2 y + b_2 x_r + c_2 x_i + d_2 = 0. \end{cases} \tag{2.47}$$

As follows from (2.47), a line in a Cartesian coordinate system is defined by eight coefficients; the same line in the form of a linear function of a complex argument, as follows from (2.44), is defined by only four coefficients and is represented in the form of a linear equation. We can again see that to actions with complex numbers correspond actions with real numbers, and functions of complex variables often represent a more convenient form of notation than those of real numbers.

It should be noted that in the Cartesian coordinate system the equation of a line passing through two different points $P_1(y_1,x_{r1},x_{i1})$ and $P_2(y_2,x_{r2},x_{i2})$ will be written as follows:

$$\frac{y - y_1}{y_2 - y_1} = \frac{x_r - x_{r1}}{x_{r2} - x_{r1}} = \frac{x_i - x_{i1}}{x_{i2} - x_{i1}}. \tag{2.48}$$

With reference to the line described by the function of the complex argument (2.44), the equation of the line for these two points will be written as follows:

$$\frac{y - y_1}{y_2 - y_1} = \frac{(x_r + ix_i) - (x_{r1} + ix_{i1})}{(x_{r2} + ix_{i2}) - (x_{r1} + ix_{i1})}. \tag{2.49}$$

The specifics of a linear function of a complex argument with reference to some economic problems will be considered in other chapters of this book.

2.4 Power Function of a Complex Argument with a Real Exponent

The linear function of a complex argument can be applied in many cases of economic modeling since in accordance with the general scientific principle “from the simple to the complex,” to study some complex object, first simple models including linear ones are used, after which models become increasingly complex as the object’s properties become clearer for a more adequate description of complex processes.

The power function of a complex argument is more complex than a linear one, its general form being

$$y = (a_0 + ia_1)(x_r + ix_i)^{(b_0+ib_1)}. \tag{2.50}$$

It is easy to see that if the exponent of the model is equal to one, it is turned into an elementary linear model of the complex argument (2.41).

Let us consider function (2.50) sequentially in order of increasing complexity depending on the exponent – real, imaginary, or complex.

The first of the possible models determined by the equality (2.50) is one with a real exponent:

$$y = (a_0 + ia_1)(x_r + ix_i)^{b_0}. \tag{2.51}$$

To understand the properties of this function, let us represent the complex proportionality coefficient and complex resource variable in exponential form. Then we have

$$y = ae^{i\alpha}\left(re^{i\varphi}\right)^{b_0} = ar^{b_0}e^{i(\alpha+b_0\varphi)}, \tag{2.52}$$

$$a = \sqrt{a_0^2 + a_1^2}, \alpha = arctg\frac{a_1}{a_0}, r = \sqrt{x_x^2 + x_i^2}, \varphi = arctg\frac{x_i}{x_r}.$$

Hence we have a system of equations for the real and imaginary parts of the function under consideration:

$$\begin{cases} y = ar^{b_0}\cos(\alpha + b_0\varphi), \\ 0 = ar^{b_0}\sin(\alpha + b_0\varphi). \end{cases} \tag{2.53}$$

It follows from the last equality that it holds for the following conditions:

$$\sin(\alpha + b_0\varphi) = 0 \rightarrow \alpha + b_0\varphi = \pm\pi k, \tag{2.54}$$

where k is a whole number.

It should be noted that for values of the polar angle of the function determined by these conditions, its cosine takes the following values:

$$\begin{cases} \cos(\alpha + b_0\varphi) = 1, \quad \forall\ k = 0, \pm 2, \pm 4, \ldots \\ \cos(\alpha + b_0\varphi) = -1, \forall\ k = \pm 1, \pm 3, \pm 5, \ldots \end{cases} \tag{2.55}$$

If, for example, we consider the polar angle in the complex plane of the argument from 0 to 2π, at $\sigma = 0$ and $b_0 = 1$, we have that y is positive for $\varphi = 0$ and $\varphi = 2\pi$ and negative for $\varphi = \pi$. For any α and for $b_0 \neq 0$ (with exponent $b_0 = 0$, the function is turned into the point $y = a\cos\alpha = a_0$) function (2.51), subject to the values of coefficients α and b_0 and polar angle φ, takes both positive and negative values.

Since it follows from the second equation of system (2.53) that the relationship between the real and imaginary parts of the complex argument in this function is a constant value regardless of the values of y, in three-dimensional space this equation is represented by a plane parallel to the axis of the real variable $0y$ and perpendicular to the plane of the complex argument.

The first Eq. (2.53) determines the change in the y depending on the change in the two factors x_r and x_i, which may be represented in three-dimensional space in the form of some surface.

The complex proportionality coefficient changes the surface scale and slopes, which is why its influence on the result is negligible and we may consider this coefficient to be equal to one. For this reason let us consider a simplified analog of function (2.51):

$$y = (x_r + ix_i)^{b_0}. \tag{2.56}$$

Then the real and imaginary parts of this function will have the form

$$\begin{cases} y = r^{b_0}\cos b_0\varphi, \\ 0 = r^{b_0}\sin b_0\varphi. \end{cases} \tag{2.57}$$

Let us consider the influence on the real part of model (2.56) of exponent b_0, i.e., equation of the first function of system (2.57), since the second equation describes the linear relationship between the real and imaginary parts of the complex argument, as

$$0 = r^{b_0}\sin b_0\varphi \rightarrow \sin b_0\varphi = \pi k \rightarrow \varphi = arctg\frac{x_i}{x_r} = const.$$

Here three variants of the function's behavior are possible:

1. When the exponent is negative, $b_0 < 0$;
2. When the exponent lies within the range: $0 < b_0 < \mathit{1}$.;
3. When the exponent is higher than 1, $b_0 > 1$.

Let us consider the first case where the exponent of the power function of complex argument b_0 is less than zero. For convenience, let us give the full form of the first equation of the system:

$$y = \left(\sqrt{x_r^2 + x_i^2}\right)^{b_0}\cos\left(b_0 arctg\frac{x_i}{x_r}\right). \tag{2.58}$$

Since the second equation of system (2.57) states that the function under consideration is projected in the complex plane of the argument to a line passing through the zero point and having a constant value of the polar angle, let us consider first what values the factor containing this constant polar angle can take. Cosine is a periodic function, but the argument of this function does not change; it is always constant due to the constant nature of the polar angle. The surface described in three-dimensional space by the first equation of system (2.57) is nonlinear and its character is determined by coefficient b_0, since it characterizes the frequency of oscillations – the higher it is by module, the more uneven ("corrugated") is the surface.

Since subsequently we will not need the type of this surface, we will consider only the nature of the lines in the space made by system (2.57).

Let us first consider the situation where the polar angle of the argument in the complex plane is equal to zero. This is possible when $x_r > 0$, $x_i = 0$. Since the cosine of zero is one, (2.58) will look as follows:

$$y = x_r^{b_0}. \tag{2.59}$$

In the (y,x_r) plane, this function will represent a hyperbola that decreases from plus infinity to zero with the growth of the module of the argument.

Now let us suppose that the polar angle of the complex argument is $\pi/4$, i.e., have the following form:

$$y = \left(x_r\sqrt{2}\right)^{b_0}\cos\left(b_0\frac{\pi}{4}\right). \tag{2.60}$$

The function is positive for $-2 < b_0 < 0$, equal to zero for $b_0 = -2$, negative for $-6 < b_0 < -2$, etc. Absolute values of the function for this case also decrease with the growth of the module of the complex argument, as previously, but in the case of negative values of the function, it decreases from minus infinity to zero.

Let us consider another case where the real part of the complex argument is equal to zero: $x_r = 0$, and its imaginary part is positive: $x_i > 0$. The polar angle of the complex argument is equal to $\pi/2$ and the function looks like this:

$$y = x_i^{b_0}\cos\left(b_0\frac{\pi}{2}\right). \tag{2.61}$$

If the coefficient b_0 lies within the range $-1 < b_0 < 0$, then the cosine of function (2.61) will be positive, which means the function has a positive character. If the exponent is equal to $b_0 = -1$, then cosine becomes zero and the function also becomes equal to zero. If the values of this coefficient are within the range $-3 < b_0 < -1$, then the cosine of the function becomes negative like the function itself. Since the cosine is a periodical function, then with the subsequent increase of the module of the values of the exponent b_0, the function becomes both positive and negative. With the growth in the values of the function argument (2.61), the absolute values of the function also decrease with the hyperbola.

Continuing on, it is clear that the power function of a complex argument with negative real exponent decreases in its absolute values according to the hyperbolic law with an increase in the argument's module. With this, the sign of the function is determined by the result of multiplying the exponent by the polar angle. In some cases the function is equal to zero.

In the second case, where the exponent of the function of the complex argument lies within the range $0 < b_0 < 1$, the function behaves slightly differently:

$$y = x_r^{b_0}. \tag{2.62}$$

However, since the exponent is positive and not greater than one, with the growth of the module, the function increases nonlinearly from zero to plus infinity according to the exponential law with a negative second derivative.

If the polar angle of the complex argument is $\pi/4$, then when the variables are in the first quadrant of the complex plane and $x_r = x_i$, then function (2.58) takes the following value:

$$y = \left(x_r\sqrt{2}\right)^{b_0} \cos\left(b_0 \frac{\pi}{4}\right). \tag{2.63}$$

The function will be positive for an exponent lying within $0 < b_0 < 2$ and equal to zero at $b_0 = 2$. It will be negative for $2 < b_0 < 6$, etc. With the growth of the module of the complex argument, absolute values will behave similarly to (2.62).

It makes no sense to examine this case further since it is clear that the function will behave just like this – its absolute values will increase nonlinearly from zero, and the function sign will be determined by the exponent of the function.

In the third case of the power function with a real exponent, the exponent $b_0 > 1$, function (2.58) will take negative or positive values, as well as values equal to zero, depending on the result of multiplying the exponent by the polar angle since the cosine of an angle may be both positive and negative and be equal to zero. However, by its absolute value, with the growth of the complex argument, the function will tend from zero to infinity according to the exponential law with positive second derivative.

We can now go back to the model with a complex proportionality coefficient under consideration (2.51):

$$y = (a_0 + ia_1)(x_r + ix_i)^{b_0}$$

and pay more attention to the influence of this coefficient on the function behavior.

Values of this complex proportionality coefficient influence both the module of the function and the polar angle.

For various values of the proportionality coefficient, the module of the function of the complex argument equal to $R = ar^{b_0}$ is presented on various scales.

When the values of this proportionality coefficient vary, the polar angle also turns in the plane of the complex argument:

$$\varphi = \frac{\pm \pi k}{b_0} - \alpha.$$

This is why variations in the proportionality coefficient move the power function curve in various parts of space symmetrically to the y-axis and change the row curve scale.

2.5 Exponential Function of Complex Argument with Imaginary Exponent

Having considered the case where the power function of complex argument (2.50) is represented by a real exponent and the function represents a line of the power function in three-dimensional space, let us move on to a more complicated case where the real part of this function is equal to zero and the exponent is imaginary:

$$y = (a_0 + ia_1)(x_r + ix_i)^{ib_1}. \tag{2.64}$$

In this case another relationship besides the power one will be modeled, though the complex argument is subject to involution.

Since the influence of the proportionality coefficient on the result in this case will remain the same, let us consider it to be equal to one:

$$a_0 + ia_1 = 1.$$

Then the function in exponential form will look like this:

$$y = \left(re^{i\varphi}\right)^{ib_1} = e^{-\varphi b_1} e^{ib_1 \ln r}, \tag{2.65}$$

where $r = \sqrt{x_x^2 + x_i^2}, \varphi = arctg \frac{x_i}{x_r}$.

From this follows a system of equations for the real and imaginary parts of the function under consideration:

$$\begin{cases} y = e^{-b_1 arctg \frac{x_i}{x_r}} \cos\left(b_1 \ln \sqrt{x_r^2 + x_i^2}\right), \\ 0 = e^{-b_1 arctg \frac{x_i}{x_r}} \sin\left(b_1 \ln \sqrt{x_r^2 + x_i^2}\right). \end{cases} \tag{2.66}$$

The first equation of this system represents a description of some nonlinear function in three-dimensional space, to be discussed later.

The second equation of this system represents a nonlinear surface perpendicular to the plane of a complex argument, where all the lines lying in that plane are parallel to the 0y-axis.

The intersection of these two surfaces is simply a graphical interpretation of function (2.64) in the space.

Let us consider what the second equation looks like in the plane of a complex argument:

$$0 = e^{-b_1 arctg \frac{x_i}{x_r}} \sin\left(b_1 \ln \sqrt{x_r^2 + x_i^2}\right). \tag{2.67}$$

The first factor can be equal to zero only if its exponent is equal to infinity. Variants when b_1 is equal to infinity are not considered here because they are meaningless. The arctangent is known to lie within a range of $-\pi/2$ to $+\pi/2$. Therefore, the first factor (2.67) will never be equal to zero and the equality holds when the second factor is equal to zero:

$$\sin\left(b_1 \ln \sqrt{x_r^2 + x_i^2}\right) = 0. \tag{2.68}$$

This equality holds when

$$b_1 \ln \sqrt{x_r^2 + x_i^2} = \pi k. \tag{2.69}$$

Hence,

$$x_r^2 + x_i^2 = e^{2\frac{\pi k}{b_1}}. \tag{2.70}$$

This means that the imaginary part (2.64) is equal to zero when the values of the complex argument in the complex plane lie on a circumference with radius $e^{2\frac{\pi k}{b_1}}$. In particular, if $k = 0$, then the equality holds when

$$x_r^2 + x_i^2 = 1, \tag{2.71}$$

i.e., when the points in the complex plane lie on a one-unit circumference.

k may take any whole values, which means a family of circumferences in a complex plane of arguments. In the three-dimensional space it is a cylindrical surface perpendicular to the plane of the complex argument.

Now, let us consider the first equation of system (2.66) referring to the real part of the function:

$$y = e^{-b_1 arctg\frac{x_i}{x_r}} \cos\left(b_1 \ln \sqrt{x_r^2 + x_i^2}\right). \tag{2.72}$$

This equation describes a nonlinear surface in space, but since the type of this surface will not be used subsequently, we should consider what line on this surface is cut off by the cylinder, since the function of complex argument (2.64) is an intersection of two nonlinear surfaces one of which is a cylinder (2.70). Thus, let us consider the behavior of (2.72) in the case where the variables x_r and x_i lie on some circumference.

In this case the logarithm argument is a constant, which is why the nature of this curve is completely determined by the first factor (2.72), which represents the exponent.

Since a complex argument changes its values in a circumference, in the initial point where the minimal component is equal to zero, the first factor is equal to one since any number to the zero power equals one. Then, for a polar angle equal to zero, the function will take the following values:

$$y(0) = \cos(b_1 \ln r). \tag{2.73}$$

With the growth of the values of the imaginary component x_i on the circumference and the respective decrease in the real component x_r (increasing polar angle), the polar angle of the complex argument tends from zero to π/2. In the extreme point, when the real component of the complex argument is equal to zero, the function will have the following form:

$$y\left(\frac{\pi}{2}\right) = e^{-b_1\frac{\pi}{2}} \cos(b_1 \ln r). \tag{2.74}$$

In the interval between these two points the function will vary exponentially from points with coordinates determined by (2.73) to points determined by (2.74).

Further movement of the complex argument on the circumference corresponds to a variation in the polar angle from π/2 to π. In the extreme point, when the polar angle is equal to π, which means that the imaginary component is equal to zero and the real constituent $x_r = -r$, the function will take the following values:

$$y(\pi) = e^{-b_1\pi} \cos(b_1 \ln r). \tag{2.75}$$

Continuing along the circumference and arriving at the point where the real part is equal to zero and the imaginary $x_i = -r$, we obtain a function value equal to

$$y\left(\frac{3}{2}\pi\right) = e^{-b_1\frac{3\pi}{2}} \cos(b_1 \ln r). \tag{2.76}$$

Completing the movement along the circumference in the point where the imaginary part is equal to zero and the real one is equal to the radius, the function takes the following value:

$$y(2\pi) = e^{-b_1 2\pi} \cos(b_1 \ln r). \tag{2.77}$$

Now it is clear what the power function of a complex argument represents if the exponent is an imaginary number – this exponent is located on the cylinder surface. Completing the full circle equal to 2π we see that the function differs from its initial point by $e^{-b_1 2\pi}$ times.

If the coefficient b_1 is positive, then the function tends to zero; if the exponent is negative, then the function tends to infinity, making turn after turn on the cylinder surface, if the complex argument makes rotational movements in the complex plane. However, since in economics we do not observe such rotational movements, meaning variations in the polar angle of the complex argument within a range $0 \leq \varphi \leq 2\pi$, function (2.64) should be considered an exponent on the cylinder surface making one complete turn thereon.

2.6 Power Function of Complex Argument with Complex Exponent

When a complex argument of a power function is raised to a real power, in three-dimensional space this represents a curve described by an exponential function and lying in space in a plane perpendicular to the plane of the complex argument.

If the exponent of this function is an imaginary number, then it represents an exponent varying with the increase in the polar angle of the complex variable and lying in the space on the cylinder surface perpendicular to the complex argument plane.

Now let us consider the nature of a power function of a complex argument where the exponent is complex:

$$y = (a_0 + ia_1)(x_r + ix_i)^{(b_0+ib_1)}. \tag{2.78}$$

This function, taking into account previously introduced designations and the assumption that the complex proportionality coefficient is equal to one, may be written in exponential form as follows:

$$y = r^{b_0} e^{ib_0\varphi} e^{-\varphi b_1} e^{ib_1 \ln r} = r^{b_0} e^{-\varphi b_1} e^{i(b_0\varphi + b_1 \ln r)}. \tag{2.79}$$

The real and imaginary parts of this function may be written as a system of equations:

$$\begin{cases} y = r^{b_0} e^{-b_1\varphi} \cos(b_0\varphi + b_1 \ln r), \\ 0 = r^{b_0} e^{-b_1\varphi} \sin(b_0\varphi + b_1 \ln r). \end{cases} \tag{2.80}$$

Again, we have equations of complex nonlinear surfaces in the space, the second equation describing the surface perpendicular to the complex argument plane. As in the previous cases, let us examine the properties of function (2.78) with the condition that the imaginary part of the complex argument function is equal to zero.

The second equation will be equal to zero when the argument is equal to zero and when the sine is equal to zero:

$$\sin(b_0\varphi + b_1 \ln r) = 0. \tag{2.81}$$

In the zero point the function itself is equal to zero, which is why (2.81) is of interest and may be written as follows:

$$b_0\varphi + b_1 r = \pi k. \tag{2.82}$$

Here, as in the previous case, k is a whole number.

Let us take $k = 0$. Then (2.82) can be written as follows:

$$r = \frac{-b_0}{b_1}\varphi. \tag{2.83}$$

It is evident that we have obtained Archimedes' spiral, where with the change in the polar angle within the range $0 \le \varphi \le 2\pi$ the coefficient before the polar angle should always be positive since the module of the complex argument cannot be negative. This means that the signs of the real and imaginary parts of the complex exponent should be different.

We do not consider rotational processes that practically do not exist in economics, which is why in the space under consideration the second equation of system (2.80) indicates one turn of Archimedes' spiral, that is, a nonlinear surface in Archimedes' spiral perpendicular to the complex argument plane. This surface "cuts off" a nonlinear curve on the other surface represented by the first equation of system (2.80).

We are not interested in the type of the surface described in the space by the first equation of system (2.80), but in the line on this surface that is cut off by Archimedes' spiral.

To understand this, we substitute (2.83) into the first equation of system (2.80) and get

$$y = \left(\frac{-b_0}{b_1}\varphi\right)^{b_0} e^{-b_1\varphi} \cos\left(b_0\varphi + b_1 \ln\left(\frac{-b_0}{b_1}\varphi\right)\right). \tag{2.84}$$

Since it was shown previously that b_0 and b_1 have different signs, let us first take $b_0 > 0$ and $b_1 > 0$. For this case with a growing polar angle:

- The first factor $\left(\frac{-b_0}{b_1}\varphi\right)^{b_0}$ increases according to the power law;
- The second factor $e^{-b_1\varphi}$ increases according to the exponential law;
- The third factor $\cos\left(b_0\varphi + b_1 \ln\left(\frac{-b_0}{b_1}\varphi\right)\right)$ varies nonlinearly depending on the modules of the values of coefficients b_0 and b_1. If the module of the complex cosine argument increases with the growth of the polar angle, this factor decreases up to zero, after which it becomes negative.

Thus, on the whole, (2.84) describes a function increasing up to a certain limit with its subsequent decrease to zero and further to the negative range. This line is located in the space on the nonlinear surface of Archimedes' spiral.

If we now change signs of the coefficients to the opposite ones and set $b_0 < 0$, $b_1 > 0$, then the picture will look as follows:

- The first factor $\left(\frac{-b_0}{b_1}\varphi\right)^{b_0}$ decreases according to the power law;
- The second factor $e^{-b_1\varphi}$ decreases according to the exponential law;

– The third factor $\cos(b_0\varphi + b_1 \ln(\frac{-b_0}{b_1}\varphi))$ behaves in the same way as in the first case as the cosine is a symmetrical function.

On the whole, with such signs of the coefficients, the function decreases with the growth of the argument and becomes negative as it travels along Archimedes' spiral.

Various combinations of coefficients give various forms of a curve in space. If the imaginary part of a complex exponent is equal to zero, then the function represents points lying on a line of the exponential function in space in a plane perpendicular to the plane of the complex argument. When the real part is close to zero, then the curve represents an exponent lying on the cylinder surface. If the exponent is equal to one, then we have a linear function of a complex argument.

To conclude our study of the properties of this function, it should be noted that the coefficients of a function may be easily estimated by two points.

Let there be two points (x_{r1},x_{i1},y_1) and (x_{r2},x_{i2},y_2) available to an economist disposaltwo , and she thinks that there is a relationship between these variables that may be described by a model in the form of a power function of complex argument (2.78). Substituting these values into the function and dividing the left- and right-hand sides by each other we obtain the following equation:

$$\frac{y_1}{y_2} = \frac{x_{r1} + ix_{i1}}{x_{r2} + ix_{i2}}^{(b_0+ib_1)}. \tag{2.85}$$

Here we can derive the exponent

$$b_0 + ib_1 = \frac{\ln \frac{y_1}{y_2}}{\ln \frac{x_{r1}+ix_{i1}}{x_{r2}+ix_{i2}}}. \tag{2.86}$$

Knowing this value we can easily find the value of the proportionality coefficient $(a_0 + ia_1)$.

Thus, for example, if an economist wants to build a model in the form of a power function of a complex argument and she has two points at her disposal in three-dimensional space – (2; 3; 5) and (2.5; 4.7; 15), then she can easily do this using (2.85) and (2.86), and model (2.78) passing through these two points in three-dimensional space will have the following form:

$$y = (-0,014 - i0,082)(x_r + ix_i)^{(2,648-i0,674)}.$$

2.7 Exponential Function of a Complex Argument

It is clear from the aforementioned properties of the power function of a complex argument that it can be used for modeling various complex nonlinear relationships in three-dimensional space. But this model hardly covers the entire possible variety of functions of a complex argument. One of the simple nonlinear functions of a complex argument that differ in their properties from the power function is the exponential function

$$y = (a_0 + ia_1)e^{(b_0+ib_1)(x_r+ix_i)}. \tag{2.87}$$

To study its properties let us first consider a situation where the proportionality coefficient is represented as a real coefficient, then when it is the imaginary part, we can consider the whole function (2.87).

The exponential function of a real argument with a real exponent coefficient will look as follows:

$$y = (a_0 + ia_1)e^{b_0(x_r+ix_i)}. \tag{2.88}$$

In exponential form it will be written as follows:

$$y = ae^{b_0x_r}e^{i(\alpha+b_0x_i)}, \tag{2.89}$$

where the following equalities hold for the real and imaginary parts:

$$\begin{cases} y = ae^{b_0x_r}\cos(\alpha + b_0x_i), \\ 0 = ae^{b_0x_r}\sin(\alpha + b_0x_i). \end{cases} \tag{2.90}$$

The imaginary part can be equal to zero in two cases –with a positive exponent coefficient $x_r \rightarrow -\infty$ and when

$$\alpha + b_0x_i = \pi k. \tag{2.91}$$

Situations where one or all of the factors tend to infinity do not exist in economics, so we will concentrate on equality (2.91).

The last condition represents a combination of equations of a line in a complex plane of the argument parallel to the axis of real values of the complex argument since it follows from (2.91):

$$x_i = \frac{\pi k - \alpha}{b_0}. \tag{2.92}$$

In a simple case where $k = 0$ there is a line

$$x_i = -\frac{\alpha}{b_0} \tag{2.93}$$

in the complex plane.

Since we are looking at the problem of presenting a function in three-dimensional space, (2.93) represents a plane perpendicular to the plane of a complex argument and parallel to the axes of the real part of the complex argument x_r and function y.

The first Eq. (2.90) describes a nonlinear surface in three-dimensional space. Let us consider it.

If the real part of a complex argument is a constant value $x_r = d = const$, then the function varies by the cosine law:

$$y = ae^{b_0 d}\cos(\alpha + b_0 x_i). \tag{2.94}$$

If the imaginary part of the complex argument is constant $x_i = g = const$, then the function varies according to the exponential law:

$$y = ae^{b_0 x_r}\cos(\alpha + b_0 g). \tag{2.95}$$

Since the last condition is a restriction (2.93) that follows from the fact that the imaginary part of the function under consideration is equal to zero, then in three-dimensional space, the exponential function of a complex argument with a real exponent coefficient represents an exponent:

$$y = ae^{b_0 x_r}\cos\left(\alpha + b_0\frac{\pi k - \alpha}{b_0}\right) = ae^{b_0 x_r}. \tag{2.96}$$

Let the exponential function of a real argument have an imaginary coefficient of the exponent:

$$y = (a_0 + ia_1)e^{ib_1(x_r + ix_i)}. \tag{2.97}$$

In exponential form this will look like

$$y = ae^{-b_1 x_i}e^{i(\alpha + b_1 x_r)}, \tag{2.98}$$

where for the real and imaginary parts of the function

$$\begin{cases} y = ae^{-b_1 x_i}\cos(\alpha + b_1 x_r), \\ 0 = ae^{-b_1 x_i}\sin(\alpha + b_1 x_r). \end{cases} \tag{2.99}$$

If the imaginary part of the function under consideration equals zero, then

$$\alpha + b_1 x_r = \pi k \tag{2.100}$$

or

$$x_r = \frac{\pi k - \alpha}{b_1}. \tag{2.101}$$

This means we have again obtained an equation of lines that, in a plane of a complex argument, are parallel to the imaginary axis. We can limit ourselves to the case where $k = 0$. In the space under consideration this equation means a plane perpendicular to the plane of a complex argument and passing through line (2.101).

This means that on a complex nonlinear surface described by the first equality (2.99) there is a curve with a constant value of x_r. It is clear from the first equation of system (2.99) that this curve is described by the exponent

$$y = Ce^{-b_1 x_i}, \quad C = a\cos(\alpha + b_1 x_r), \quad x_r = const.$$

Now it is clear what will represent the full exponential function of a complex argument. Let us present again the complex values of the model – the proportionality coefficient and the complex argument – in exponential form. Grouping the constituents of the module and the polar angle we get

$$y = ae^{b_0 x_r - b_1 x_i} e^{i(\alpha + b_1 x_r + b_0 x_i)}. \tag{2.102}$$

Let us represent this model as an equality system of real and imaginary parts:

$$\begin{cases} y = ae^{b_0 x_r - b_1 x_i}\cos(\alpha + b_1 x_r + b_0 x_i), \\ 0 = ae^{b_0 x_r - b_1 x_i}\sin(\alpha + b_1 x_r + b_0 x_i). \end{cases} \tag{2.103}$$

From the last equality we can easily get

$$x_i = \frac{1}{b_0}(\pi k - \alpha - b_1 x_r). \tag{2.104}$$

This equation describes a family of parallel lines in a complex plane of the argument. In the simple case, where $k = 0$, it is a line in the plane and surface perpendicular to the complex plane of the argument in the space. Both the line and the plane are defined in the whole range of the problem.

This plane cuts off some line on the surface defined by the first equation of system (2.103):

$$y = ae^{b_0 x_r - b_1 x_i}\cos(\alpha + b_1 x_r + b_0 x_i). \tag{2.105}$$

Substituting (2.104) into this equation we get

$$y = ae^{b_0 x_r - \frac{b_1}{b_0}(\pi k - \alpha - b_1 x_r)}\cos(\alpha + b_1 x_r + (\pi k - \alpha - b_1 x_r)) = ae^{\frac{b_1}{b_0}(\alpha - \pi k) + \frac{b_0^2 + b_1^2}{b_0} x_r}. \tag{2.106}$$

This means that we have an exponent in the space located in a plane perpendicular to the complex plane of the argument.

2.8 Logarithmic Function of a Complex Argument

Let us now examine the properties of the logarithmic function of a complex argument. The logarithm of a complex variable is known as a periodical function, which is why when we study it we should specify what part of the function is being studied. It was determined in the first chapter of this book that from the entire combination of logarithmic values we will consider only the main values.

The logarithmic function of a complex argument may be presented in its general form as follows:

$$y = (a_0 + ia_1) + (b_0 + ib_1)\ln(x_r + ix_i). \tag{2.107}$$

If we apply the formula of a logarithm of a complex argument to the model under consideration, we get

$$y = (a_0 + ia_1) + (b_0 + ib_1)\left(\ln\sqrt{x_r^2 + x_i^2} + iarctg\frac{x_i}{x_r}\right). \tag{2.108}$$

Let us consider the variant where the imaginary part of the complex proportionality coefficient is equal to zero:

$$y = (a_0 + ia_1) + b_0\left(\ln\sqrt{x_r^2 + x_i^2} + iarctg\frac{x_i}{x_r}\right). \tag{2.109}$$

Opening the brackets and grouping the real and imaginary parts of this equation we get

$$y = a_0 + b_0\ln\sqrt{x_r^2 + x_i^2} + i\left(a_1 + b_0 arctg\frac{x_i}{x_r}\right). \tag{2.110}$$

Two equalities for the real and imaginary parts follow from the preceding equation:

$$\begin{cases} y = a_0 + b_0\ln\sqrt{x_r^2 + x_i^2}, \\ 0 = a_1 + b_0 arctg\dfrac{x_i}{x_r}. \end{cases} \tag{2.111}$$

The second equation requires a constant polar angle in the plane of the complex argument:

$$arctg\frac{x_i}{x_r} = -\frac{a_1}{b_0}. \tag{2.112}$$

This indicates an equation of the line passing through the neighborhood of the zero point but not including it. The zero point does not exist for the first equation as well since a logarithm of zero does not exist.

The first equation of system (2.111) describes a nonlinear surface in three-dimensional space. We are interested in the location of the line on this surface that satisfies condition (2.112). Thus, let us consider the equation

$$y = a_0 + b_0 \ln\sqrt{x_r^2 + x_i^2}$$

for $x_i = dx_r$.

If we substitute this into the equation, we get

$$y = a_0 + b_0 \ln x_r\sqrt{1 + d^2}. \tag{2.113}$$

Therefore, the logarithmic function of a complex argument with a real proportionality coefficient represents in three-dimensional space a logarithmic function passing through the zero point and lying in a plane perpendicular to the complex plane of the argument.

Now let us consider the second extreme version, when the real part of the complex proportionality coefficient is equal to zero:

$$y = (a_0 + ia_1) + ib_1\left(\ln\sqrt{x_r^2 + x_i^2} + iarctg\frac{x_i}{x_r}\right). \tag{2.114}$$

Grouping the real and imaginary parts of this function we get the following system:

$$\begin{cases} y = a_0 - b_1 arctg\frac{x_i}{x_r}, \\ 0 = a_1 + b_1 \ln\sqrt{x_r^2 + x_i^2}. \end{cases} \tag{2.115}$$

The fact that the imaginary part equals zero means that in the complex plane of the argument the function is determined on a circumference since this equality can easily be made as follows:

$$\sqrt{x_r^2 + x_i^2} = e^{-\frac{a_1}{b_1}} = const = d. \tag{2.116}$$

The zero point does not serve to define the function because a logarithm of zero does not exist. In the space under consideration the second equation of system (2.115) represents a cylinder surface. This cylinder surface cuts off a curve in the plane of the first equation, which we are interested in.

Since the polar angle in the plane of a complex argument varies on a circle, on the surface described by the first equation of system (2.115) a curve is defined that represents an arctangent function lying on the surface of the cylinder perpendicular to the complex plane of the argument.

The general logarithmic function of complex argument (2.107) represents a complex superposition of these two functions. After opening the brackets in the right-hand side of equality (2.107) and grouping the real and imaginary parts we get

$$y = a_0 + b_0 \ln \sqrt{x_r^2 + x_i^2} - b_1 arctg \frac{x_i}{x_r} + i\left(a_1 + b_1 \ln \sqrt{x_r^2 + x_i^2} + b_0 arctg \frac{x_i}{x_r} \right). \tag{2.117}$$

This equality holds only when the real and imaginary parts are equal to each other:

$$\begin{cases} y = a_0 + b_0 \ln \sqrt{x_r^2 + x_i^2} - b_1 arctg \dfrac{x_i}{x_r} \\ 0 = a_1 + b_1 \ln \sqrt{x_r^2 + x_i^2} + b_0 arctg \dfrac{x_i}{x_r} \end{cases}. \tag{2.118}$$

The second equation of the system describes a curve in the plane of the complex argument that does not include the zero point:

$$0 = a_1 + b_1 \ln \sqrt{x_r^2 + x_i^2} + b_0 arctg \frac{x_i}{x_r}. \tag{2.119}$$

The approximate form of the function can be imagined from the location of a function with these coefficients in the plane:

$$0 = -3 \ln \sqrt{x_r^2 + x_i^2} + 0,2 arctg \frac{x_i}{x_r}. \tag{2.120}$$

This function is given in Fig. 2.15.

The first equation of system (2.118) represents a complex surface. Its general form may look like this: in space there are a great number of lines like those shown in Fig. 2.15 that are parallel to each other and increase on the S-axis or decrease with the growth of the argument, depending on the function coefficient. This surface intersects with another one perpendicular to that of the complex argument and passing in the plane of the complex argument through the points determined by

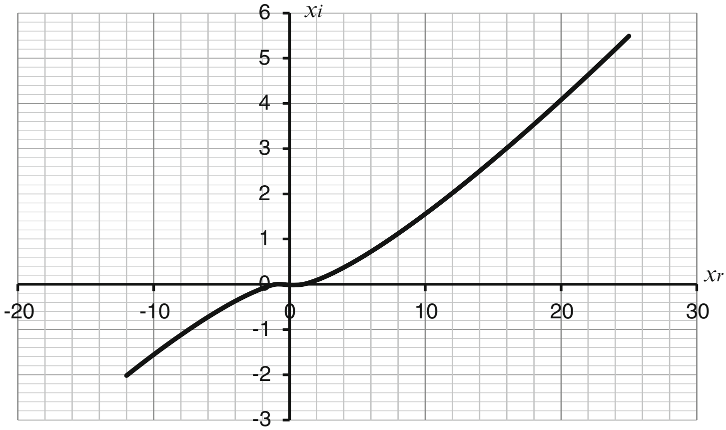


Fig. 2.15 Line (2.120) in plane of complex argument

line (2.119). Intersection of these two planes gives a line in space that has the form of the line in Fig. 2.15.

The functions of a complex argument studied here do not exhaust their full range, but of those that can be used in economic practice, the previously mentioned function are fundamental.

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