

Chapter 2

Flow and Heat Transfer: Formulation

We present the full statement of the governing equations, boundary conditions and dimensionless groups for a film falling down a heated plate and exposed to ambient air. Due to the relatively low dynamic (shear) viscosity of the air the mechanical influence of the air motion on the liquid can be considered negligible, allowing us to simplify the boundary conditions on the open surface and thus consider it a “free” surface. We also ignore the heat transfer in the air by using an ad hoc boundary condition for the temperature on the free surface so that only the heat transfer process in the liquid is taken into account.

Our formulation considers in parallel an alternative with two possible thermal boundary conditions on the plate, namely a specified temperature (Dirichlet condition) and a heat flux distribution (Robin/mixed condition). For the latter to be realistic, heat losses between the wall and the surrounding medium must be included, which implies one additional empirical parameter in the equations, the heat transfer coefficient of the wall–air interface.

2.1 Governing Equations and Boundary Conditions

We consider the evolution of a viscous *thin film* flowing down a heated plate as illustrated in Fig. 2.1. The plate forms an angle β with the horizontal direction ($\beta = 0$, horizontal film; $\beta = \pi/2$, vertical film). For the isothermal case, the flow is then driven by the streamwise gravitational acceleration $g \sin \beta$. We introduce a Cartesian coordinate system (x, y, z) where x is the streamwise coordinate in the direction of flow, y is the outward-pointing coordinate normal to the plate and z is the spanwise coordinate. The plate is then located at $y = 0$ and the interface at $y = h(x, z, t)$, a single-valued function of x , z and time t . The main hypotheses are:

H1: The density ρ of the liquid is constant, or, equivalently the liquid remains incompressible. This assumption is valid for thin films ($\bar{h}_N < 1$ mm) where buoyancy can be neglected, i.e., the effect of thermal expansion upon density in the buoyancy force is negligible; see, e.g., [27, 58]. We examine this assumption

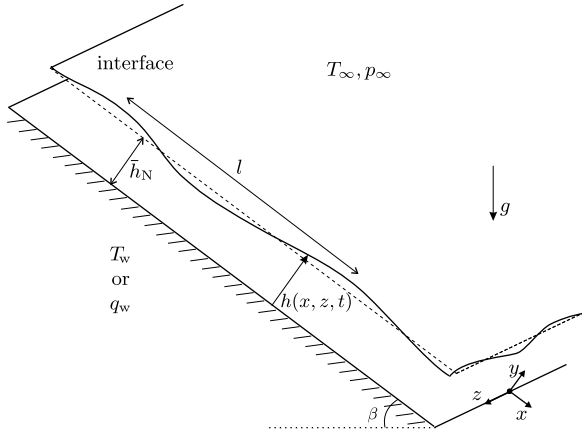


Fig. 2.1 Sketch of the geometry for a viscous thin film flowing down an inclined plate forming an angle β with the horizontal direction. The system is unbounded in the x and z directions. $h(x, z, t)$ is the local film thickness, \bar{h}_N is the mean film thickness, typically < 1 mm, and g is gravitational acceleration. The surrounding gas phase is air maintained at temperature and pressure T_∞ and p_∞ , respectively. The wall is heated either by a specified temperature distribution T_w or a given heat flux distribution q_w . l , a typical wavelength of the interfacial waves in the x direction, is much longer than \bar{h}_N

in more detail in Appendix D.1. We shall not, however, consider liquid thicknesses in the 100 nm range where intermolecular interactions become significant.

- H2: The liquid is Newtonian and hence it obeys a linear stress-strain relationship whose proportionality coefficient is the dynamic (shear) viscosity μ . The kinematic viscosity is given by $\nu = \mu/\rho$.
- H3: The plate is rigid and hence a no-slip (stick) and no-penetration condition for the velocity field applies on the plate (thus excluding the possibility of porous walls).
- H4: The liquid is assumed to be nonvolatile so that in the range of temperatures we shall consider evaporation effects can be neglected.
- H5: The air acts as a reservoir of infinite heat capacity and hence is maintained at the constant temperature T_∞ . It is also maintained at the constant pressure p_∞ . In addition, the air is assumed to be mechanically “passive” in the sense that the viscous stress from the air side is negligible compared to that from the liquid side (e.g., $\mu_{\text{air}}/\mu_{\text{water}} \approx 10^{-2}$). This “one-fluid” approach enables us to consider the momentum equation for the liquid without considering the momentum equation for the air. Note that despite the smallness of the dynamic viscosity ratio between air and liquid, the opposite is true for the kinematic viscosity, e.g., $\nu_{\text{air}}/\nu_{\text{water}} \approx 10$. This can make the one-fluid approach at times problematic when dealing with turbulent flows. However, a discussion of such flows is beyond the scope of this monograph.
- H6: To obtain the constitutive equation for the surface tension σ , let us expand $\sigma(T_s)$, with T_s the interfacial temperature in a Taylor series at a reference tem-

perature taken to be the temperature of the surrounding gas phase, T_∞ . Taking the first two terms in the expansion leads to the linear decrease with temperature,

$$\sigma = \sigma_\infty - \gamma(T_s - T_\infty), \quad (2.1)$$

where σ_∞ is the surface tension at the gas temperature and $\gamma = -(d\sigma/dT_s)_{T_\infty}$ is positive for most liquids. An alternative for the reference temperature would be the interfacial temperature for a flat film. However, in this case we would have to assume that the interfacial temperature remains close to its flat film value while the ultimate aim of this monograph is to examine the nonlinear flow regimes where the film thickness departs significantly from its initial constant value. Further, unlike T_∞ , the flat film temperature is not a control parameter in experiments. The above constitutive relation is further discussed in Appendix B.1.

- H7: Any boundary between two phases, such as the liquid–gas or the liquid–solid interface, has typically a nonnegligible thermal resistance that leads to a difference in temperature across the boundary. This temperature difference is balanced by the heat flux normal to the boundary so that the following Robin/mixed condition applies there:

$$-\lambda \nabla T \cdot \mathbf{n} = \alpha(T - T_0), \quad (2.2)$$

where \mathbf{n} is the outward-pointing (from the phase under consideration to the other side of the boundary) unit vector normal to the boundary, the dot is used to denote the dot product either of two vectors or of a tensor with a vector, λ is the thermal conductivity of the phase under consideration, T is the temperature at the boundary of the phase under consideration, T_0 is the temperature away from the boundary and α is the heat transfer coefficient that describes the rate of heat transport from the phase under consideration to the other phase across the boundary.

The main assumption here is that all the resistance to heat transfer (via conduction and convection) happens in a thin layer of the order λ/α in the immediate vicinity of the boundary, i.e., a significant temperature gradient exists over a small distance from the boundary to the other side. T_0 is then the temperature right outside this “thermal resistance layer.” The larger the thickness of the layer the stronger the resistance to heat transfer. In general, the thickness of this layer is not known, although for high Reynolds number flows, e.g., in tube reactors, taking the thermal resistance layer to be the thermal boundary layer and T_0 to be the nearly constant temperature in the bulk outside the thermal boundary layer leads to a good approximation for the heat transfer process. Equation (2.2) is usually quoted as *Newton’s law of cooling*. It is further discussed in Appendix B.2.

- H8: The contribution of viscous dissipation in the heat equation is omitted. This is a reasonable assumption for thin liquid films and the thermal gradients we shall consider here (see Appendix D.1 for details).

H9: In addition to density (H1), all other fluid parameters, i.e., dynamic viscosity μ (and so kinematic viscosity ν), thermal conductivity λ and thermal diffusivity $\chi = \lambda/\rho c_p$ with c_p the constant pressure heat capacity, are not altered significantly by the action of the relatively small thermal gradients in the problem and are taken to be constant. Tacitly, this corresponds well to our situation of heating thin films only.

In the following we shall be making use of basic knowledge of fluid mechanics and interfacial phenomena as well as vector/tensor calculus. The reader should refer to some of the many textbooks available. For example, fundamental principles and derivations of the basic equations for fluid flow and heat transport are given in [26, 159, 160] while an interesting derivation of these equations using elements of nonequilibrium thermodynamics can be found in [69]. Reference [17] focuses on heat transfer while [158, 159] derive in detail the interfacial boundary conditions. References [12], Appendix A in [26] and [109] cover in detail vector/tensor calculus.

With hypotheses H1, H2, H8 and H9, the governing equations, namely continuity, momentum (Navier–Stokes) and energy (Fourier) equations, can be written as:

$$\nabla \cdot \mathbf{v} = 0, \quad (2.3)$$

$$\frac{D\mathbf{v}}{Dt} = -\rho^{-1}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{F}, \quad (2.4)$$

$$\frac{DT}{Dt} = \chi\nabla^2 T, \quad (2.5)$$

where $D/Dt \equiv \partial_t + \mathbf{v} \cdot \nabla$ is the “material derivative” (it is the “derivative following the motion,” also called the “Lagrangian,” “substantial” or “convective derivative”), $\nabla \equiv (\partial_x, \partial_y, \partial_z)$ is the gradient operator and the subscripts denote differentiation with respect to the corresponding variables. $\mathbf{v} = (u, v, w)$ is the fluid velocity vector with components u , v and w in the x , y and z directions, respectively. T and p are the temperature and total pressure of the fluid (including both dynamic and hydrostatic contributions), respectively, and $\mathbf{F} = (g \sin \beta, -g \cos \beta, 0)$ is the body force with g the gravitational acceleration.

Equations (2.3)–(2.5) are subject to the following boundary conditions:

At the Plate $y = 0$:

- The no-slip and no-penetration boundary condition (H3):

$$\mathbf{v} = \mathbf{0}. \quad (2.6)$$

- The wall heating, e.g., an electric heating device embedded in the wall, generates a temperature distribution in the film. We impose two types of boundary conditions for the temperature field in the film, namely a Dirichlet or *specified temperature* (ST) and a Robin/mixed or *heat flux* (HF). The ST thermal boundary condition is

$$T = T_w, \quad (2.7)$$

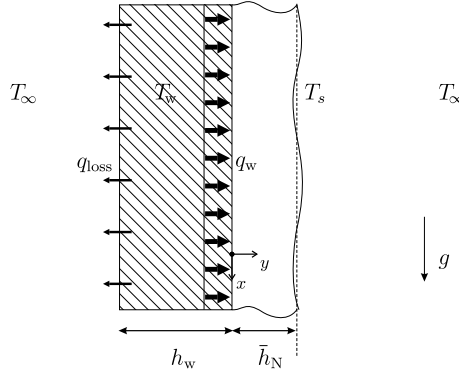


Fig. 2.2 Sketch of the heat fluxes at the wall for the HF case. q_w is the heat flux generated by the heater embedded in the wall, q_{loss} is the heat loss from the wall to the ambient gas phase and T_w , h_w , are the wall temperature and thickness, respectively. T_s denotes the interfacial temperature. Both q_w and q_{loss} contribute to the liquid temperature gradient at $y = 0$

i.e., the heater maintains the wall temperature at the value $T_w (> T_\infty)$. The HF thermal boundary condition can be obtained by solving the energy equation in the wall. The derivation is given in Appendix C.1. The result is

$$\lambda \partial_y T = -q_w + \alpha_w (T - T_\infty), \quad (2.8)$$

where q_w is the heat flux generated by the heater and supplied by the plate to the liquid (see Fig. 2.2) and α_w is the heat transfer coefficient of the wall-air interface. The term $\alpha_w (T - T_\infty)$ is a measure of the heat losses to the gas phase in contact with the wall. Indeed, formally $q_{\text{loss}} = \lambda_w \partial_y T_w|_{y=-h_w}$. From (C.3a) in Appendix C.1 we find $\lambda_w \partial_y T_w|_{y=-h_w} = q_w + \lambda_w A$. But for $h_w \rightarrow 0$ as we did in Appendix C.1, $\lambda_w A = -q_w + \alpha_w (T - T_\infty)$ so that $q_{\text{loss}} = \alpha_w (T - T_\infty)$. Hence the mixed boundary condition in (2.8) expresses the simple physical fact that both the flux supplied by the plate to the liquid, q_w , and the heat losses, q_{loss} , to the gas phase in contact with the plate contribute to the temperature gradient at $y = 0$. In the particular case that this gradient vanishes, $q_w = \alpha_w (T - T_\infty)$, so that all the heat generated by the heater is lost to the gas phase in contact with the wall and we have a specified temperature boundary condition (Dirichlet condition). If, on the other hand, the wall is perfectly insulated from the air, i.e., $\alpha_w = 0$, we have a specified heat flux boundary condition (Newmann condition).

At the Free Surface $y = h(x, z, t)$:

- Provided H4, the *kinematic boundary condition* is obtained by differentiating $y - h(x, z, t) = 0$ with respect to t : $y - h$ is a scalar function which vanishes on the liquid interface so that its time derivative following any material point on the interface (which has velocity \mathbf{v}) also vanishes, that is,

$$\frac{D}{Dt}(y - h) = 0$$

at all points on the interface. Using the definition of the material derivative, this condition provides a relationship between the film thickness and the normal velocity component $v = Dy/Dt$ on the free surface:

$$v = \partial_t h + \mathbf{v} \cdot \nabla h. \quad (2.9)$$

It is a constraint on the material velocities in terms of the shape of the interface: A fluid particle on the free surface will remain there at all times and move with the velocity of the surface. We shall use extensively the kinematic boundary condition in this form throughout the monograph.

An alternative form combines the definition of the material derivative with that of a unit vector \mathbf{n} normal to the surface that points into the surrounding gas as shown in Fig. 2.3,

$$\mathbf{n} = \frac{1}{n}(-\partial_x h, 1, -\partial_z h) \equiv \frac{1}{n} \nabla(y - h),$$

with $n = (1 + (\partial_x h)^2 + (\partial_z h)^2)^{1/2}$, i.e., the vector $(-\partial_x h, 1, -\partial_z h)$ is appropriately normalized so that its modulus is unity. Since, $v - \mathbf{v} \cdot \nabla h = \mathbf{v} \cdot \nabla(y - h)$, the kinematic boundary condition in (2.9) can be written as,

$$\frac{1}{n} \partial_t h = \mathbf{v} \cdot \mathbf{n},$$

where $\mathbf{v} \cdot \mathbf{n}$ is the component of \mathbf{v} normal to the interface. In this form, the component of the velocity normal to the interface is balanced by the time variation of the interface; after all the interface changes for normal motions only (the tangential component causes motion on the interface but it does not change the material location of the interface).

- The *stress balance* or “momentum jump” on the free surface,

$$(\mathbf{T} - \mathbf{T}_\infty) \cdot \mathbf{n} = 2\sigma K(h)\mathbf{n} + \nabla_s \sigma, \quad (2.10)$$

where ∇_s is the surface gradient operator (see Appendix C.2) and $K(h)$ is the mean curvature of the free surface given by the average, K , of the two principal curvatures k_1, k_2 , $K = (1/2)(k_1 + k_2) = (1/2)(1/R_1 + 1/R_2) = -(1/2)\nabla_s \cdot \mathbf{n}$ where R_1, R_2 are the principal radii of curvature. Hence:

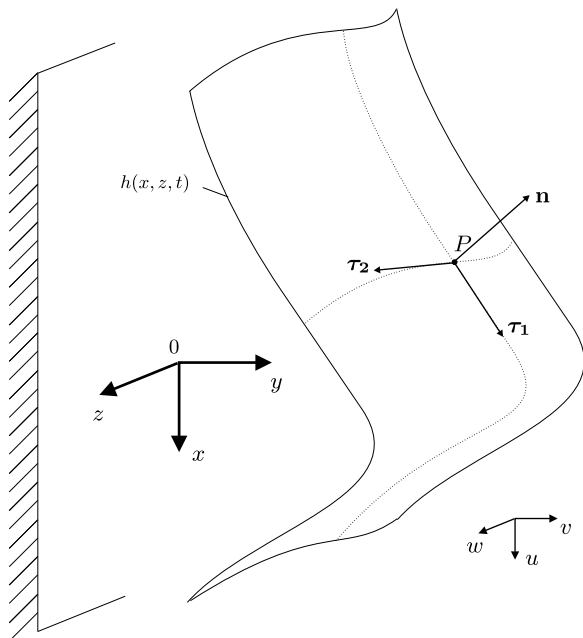
$$K(h) = \frac{1}{2} \frac{\partial_{xx} h [1 + (\partial_z h)^2] + \partial_{zz} h [1 + (\partial_x h)^2] - 2\partial_x h \partial_z h \partial_{xz} h}{[1 + (\partial_x h)^2 + (\partial_z h)^2]^{3/2}}.$$

The quantity $\mathbf{T} = -p\mathbf{I} + \mathbf{P}$ is the stress tensor for the liquid with \mathbf{I} the identity matrix, $\mathbf{P} = 2\mu\mathbf{E}$ (using H2) is the deviatoric stress tensor and $\mathbf{E} = (1/2)(\nabla\mathbf{v} + (\nabla\mathbf{v})^t)$ is the rate-of-strain tensor. For the gas $\mathbf{T}_\infty = -p_\infty\mathbf{I}$ (see H5). We then have:

$$(p_\infty - p)\mathbf{n} + \mathbf{P} \cdot \mathbf{n} = 2\sigma K(h)\mathbf{n} + \nabla_s \sigma. \quad (2.11)$$

Fig. 2.3 Definition of normal \mathbf{n} and tangential unit vectors $\boldsymbol{\tau}_{1,2}$ at point $P(x, y, z)$ on the liquid film free surface.

$h(x, z, t)$ denotes the location of the free surface and u, v, w the streamwise, cross-stream and spanwise components of the velocity field, respectively



We now identify the physical meaning of the different terms in (2.10). The term $\mathbf{T} \cdot \mathbf{n}$ is the force per unit area acting on the liquid side of the interface and in the direction normal to the interface, the term $2\sigma K(h)\mathbf{n}$ is the Laplace surface tension term also acting in the normal direction, while the term $\nabla_s \sigma$ gives rise to a force tangent to the interface. The condition (2.10) then simply states that all forces acting on the interface must balance. Note that for rigid bodies, this condition is usually ignored as the stress in such materials is indeterminate and the condition does not provide a useful constraint on the stress.

To obtain the *normal stress balance* or *normal stress boundary condition*, we take the dot product of (2.11) with \mathbf{n} . This gives:

$$p_\infty - p + (\mathbf{P} \cdot \mathbf{n}) \cdot \mathbf{n} = 2\sigma K(h). \quad (2.12)$$

Hence, in crossing the interface, the normal component of the total stress undergoes a “jump” equal to $\sigma(\nabla_s \cdot \mathbf{n})$. In the limiting case of no motion in the fluid, $p_\infty - p = -\sigma(\nabla_s \cdot \mathbf{n})$. Therefore, as Laplace noted, the pressure inside the convex region of a curved interface at equilibrium is larger than the outside by an amount that depends on both the curvature and σ . In the limiting case of a sphere, $R_1 = R_2 = -R$, the sphere’s radius. Thus, for a spherical bubble or drop, $p = p_\infty + 2\sigma/R$ and the internal pressure exceeds the external one by $2\sigma/R$. For a planar geometry the factor of 2 does not appear.

Let us now define two unit vectors tangential to the free surface (see Fig. 2.3),

$$\boldsymbol{\tau}_1 = \frac{1}{\tau_1}(1, \partial_x h, 0) \quad \text{and} \quad \boldsymbol{\tau}_2 = \frac{1}{\tau_2}(0, \partial_z h, 1),$$

with $\tau_1 = (1 + (\partial_x h)^2)^{1/2}$ and $\tau_2 = (1 + (\partial_z h)^2)^{1/2}$, i.e., the vectors $(1, \partial_x h, 0)$ and $(0, \partial_z h, 1)$ are appropriately normalized so that their modulus is unity. The choice of these vectors is discussed in Appendix C.3. The projection of (2.11) on these vectors then gives the components of the *tangential stress balance* or *tangential stress boundary conditions*:

$$(\mathbf{P} \cdot \mathbf{n}) \cdot \boldsymbol{\tau}_i = \nabla_s \sigma \cdot \boldsymbol{\tau}_i, \quad i = 1, 2. \quad (2.13)$$

The evaluation of the right hand side is discussed in Appendix C.4. An important consequence of this condition is that systems with $\nabla_s \sigma \neq \mathbf{0}$, such as those considered here, must undergo motion even in the absence of an externally imposed flow: If $\nabla_s \sigma \neq \mathbf{0}$, then $\mathbf{P} \neq \mathbf{0}$ and thus, any mechanism which maintains $\nabla_s \sigma \neq \mathbf{0}$ will necessarily drive motion in the fluid or alter an existing one. This is the Marangoni effect defined in the Introduction. We refer to such motions as *Marangoni driven flows*.

- Newton's law of cooling (see H7),

$$-\lambda \nabla T \cdot \mathbf{n} = \alpha (T - T_\infty), \quad (2.14)$$

where α is the heat transfer coefficient that measures the rate of heat transport between the liquid film and the ambient air.

2.2 Dimensionless Equations, Scalings and Parameters

The system in (2.3)–(2.5) with corresponding plate and free-surface boundary conditions (2.6)–(2.8), (2.9) and (2.12)–(2.14) has a trivial or base solution corresponding to the plane-parallel base state with thickness $h = \bar{h}_N$,

$$U(y) = \frac{g \sin \beta}{2\nu} y(2\bar{h}_N - y), \quad (2.15a)$$

$$V(y) = 0, \quad (2.15b)$$

$$W(y) = 0, \quad (2.15c)$$

$$P(y) = p_\infty + \rho g \cos \beta (\bar{h}_N - y), \quad (2.15d)$$

$$\text{ST: } \Theta(y) = T_w - \frac{\alpha(T_w - T_\infty)}{\lambda + \alpha \bar{h}_N} y, \quad (2.15e)$$

or

$$\text{HF: } \Theta(y) = T_\infty + \frac{q_w(\lambda + \alpha(\bar{h}_N - y))}{\lambda(\alpha + \alpha_w) + \alpha\alpha_w \bar{h}_N}. \quad (2.15f)$$

The streamwise gravitational acceleration balances viscous forces giving rise to a semiparabolic velocity profile while heat propagates by pure conduction, giving rise

to a linear temperature distribution. The semiparabolic velocity profile was first obtained by Nusselt [194] and we shall refer to the trivial solution in (2.15a)–(2.15f) as the *Nusselt flat film solution*. Note that the temperature distributions for the two thermal wall boundary conditions can also be written in the unified form,

$$\Theta(y) = T_\infty + \beta_T [\alpha(\bar{h}_N - y) + \lambda],$$

where

$$\text{ST: } \beta_T = \frac{T_w - T_\infty}{\alpha \bar{h}_N + \lambda}$$

or

$$\text{HF: } \beta_T = \frac{q_w}{\lambda(\alpha_w + \alpha) + \alpha_w \alpha \bar{h}_N}.$$

We now utilize the Nusselt flat film solution to introduce the nondimensionalization (details are given in Appendix D.1),

$$(x, y, z) \rightarrow \bar{h}_N(x, y, z), \quad h \rightarrow \bar{h}_N h, \quad t \rightarrow \frac{t_v l_v}{\bar{h}_N} t, \quad (2.16a)$$

$$(u, v, w) \rightarrow \frac{\bar{h}_N^2}{t_v l_v}(u, v, w), \quad p \rightarrow p_\infty + \rho \frac{l_v \bar{h}_N}{t_v^2} p, \quad (2.16b)$$

$$\text{ST: } T \rightarrow T_\infty + T \Delta T \quad (2.16c)$$

or

$$\text{HF: } T \rightarrow T_\infty + T \Delta T_N, \quad (2.16d)$$

where the temperature scale ΔT is chosen as

$$\text{ST: } \Delta T = T_w - T_\infty, \quad (2.16e)$$

or

$$\text{HF: } \Delta T_N = \frac{q_w \bar{h}_N}{\lambda} = \Delta T h_N, \quad (2.16f)$$

and

$$l_v = \left(\frac{v^2}{g \sin \beta} \right)^{1/3} \quad \text{and} \quad t_v = \left(\frac{v}{(g \sin \beta)^2} \right)^{1/3}$$

are the *viscous-gravity* length and time scales built from the streamwise gravity acceleration and the kinematic viscosity. These scales make explicit the balance between gravity and viscous forces giving rise to the Nusselt flat film solution in (2.15a)–(2.15f) and are discussed in Appendix D.1. The subscript N in (2.16f) is used to denote that the corresponding temperature scale for HF is based on \bar{h}_N .

$\Delta T = q_w l_v / \lambda$ is a temperature scale for HF based on l_v and $h_N = \bar{h}_N / l_v$ is the dimensionless Nusselt flat film thickness based on l_v . Note that the temperature scales for both ST and HF cases are natural control parameters in experiments. Note also that utilizing the streamwise gravitational acceleration in the scaling forbids us from taking the limit of an horizontal plane. As a matter of fact, for slightly inclined planes the dominant hydrodynamic mode is a shear mode associated with the semi-parabolic Nusselt profile [23, 94] and not the long-wave interfacial H-mode whose interaction with the long-wave thermocapillary S-mode is one of the key points in this monograph (see Introduction).

In terms of these dimensionless variables, the equations of motion and energy (2.3)–(2.5) become,

$$\partial_x u + \partial_y v + \partial_z w = 0, \quad (2.17)$$

$$3Re(\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u) = -\partial_x p + \partial_{xx} u + \partial_{yy} u + \partial_{zz} u + 1, \quad (2.18)$$

$$3Re(\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v) = -\partial_y p + \partial_{xx} v + \partial_{yy} v + \partial_{zz} v - Ct, \quad (2.19)$$

$$3Re(\partial_t w + u\partial_x w + v\partial_y w + w\partial_z w) = -\partial_z p + \partial_{xx} w + \partial_{yy} w + \partial_{zz} w, \quad (2.20)$$

$$3Pe(\partial_t T + u\partial_x T + v\partial_y T + w\partial_z T) = \partial_{xx} T + \partial_{yy} T + \partial_{zz} T, \quad (2.21)$$

where the dimensionless parameters Re and Pe will be defined shortly. These equations are subject to the dimensionless versions of the boundary conditions (2.6)–(2.9) and (2.12)–(2.14):

- At the plate $y = 0$:

$$u = v = w = 0, \quad (2.22)$$

$$\text{ST: } T = 1 \quad (2.23a)$$

or

$$\text{HF: } \partial_y T = -1 + B_w T. \quad (2.23b)$$

- On the film surface $y = h(x, z, t)$:

$$v = \partial_t h + u\partial_x h + w\partial_z h, \quad (2.24)$$

$$\begin{aligned} p = \frac{2}{n^2} & \left[(\partial_x h)^2 \partial_x u + (\partial_z h)^2 \partial_z w + \partial_x h \partial_z h (\partial_z u + \partial_x w) - \partial_x h (\partial_y u + \partial_x v) \right. \\ & \left. - \partial_z h (\partial_z v + \partial_y w) + \partial_y v \right] - \frac{1}{n^3} (We - MT) [\partial_{xx} h (1 + (\partial_z h)^2) \\ & + \partial_{zz} h (1 + (\partial_x h)^2) - 2\partial_x h \partial_z h \partial_{xz} h], \end{aligned} \quad (2.25)$$

$$\begin{aligned} 0 = \frac{1}{n} & \left[2\partial_x h (\partial_y v - \partial_x u) + (1 - (\partial_x h)^2) (\partial_y u + \partial_x v) - \partial_z h (\partial_z u + \partial_x w) \right. \\ & \left. - \partial_x h \partial_z h (\partial_z v + \partial_y w) \right] + M(\partial_x T + \partial_x h \partial_y T), \end{aligned} \quad (2.26)$$

$$0 = \frac{1}{n} [2\partial_z h(\partial_y v - \partial_z w) + (1 - (\partial_z h)^2)(\partial_y w + \partial_z v) - \partial_x h(\partial_z u + \partial_x w) - \partial_x h \partial_z h(\partial_y u + \partial_x v)] + M(\partial_z T + \partial_z h \partial_y T), \quad (2.27)$$

$$BT = \frac{1}{n} (\partial_x h \partial_x T + \partial_z h \partial_z T - \partial_y T). \quad (2.28)$$

Let us now introduce the following dimensionless groups and parameters:

- The *inclination number*

$$Ct = \cot \beta, \quad (2.29)$$

which compares the cross-stream component of the gravitational force to its streamwise component. It quantifies the contribution of the hydrostatic pressure that vanishes for a film falling down a vertical wall.

- The *Prandtl number*

$$Pr = \frac{\nu}{\chi}, \quad (2.30)$$

which compares the momentum diffusivity to the thermal diffusivity.

- The *Kapitza number*

$$\Gamma = \frac{\sigma_\infty l_v}{\rho v^2} = \frac{\sigma_\infty}{\rho (g \sin \beta)^{1/3} v^{4/3}}, \quad (2.31)$$

which compares the surface tension force $\sigma_\infty l_v$ to the force of inertia $\rho(u_v l_v)^2 = \rho v^2$, which is independent of the flow rate. The Kapitza number is a function of the liquid properties and β . For a vertical falling film the Kapitza number becomes $\Gamma_\perp = \sigma_\infty / \rho g^{1/3} v^{4/3}$, a *vertical Kapitza number*, and depends on the liquid properties only. It is thus fixed once the liquid is selected.

- The *Marangoni number*

$$Ma = \frac{\gamma \Delta T l_v}{\rho v^2} = \frac{\gamma \Delta T}{\rho (g \sin \beta)^{1/3} v^{4/3}} = \Gamma \frac{\gamma \Delta T}{\sigma_\infty}, \quad (2.32)$$

which for both ST and HF compares the force induced by the surface tension gradient $\gamma \Delta T l_v$ to ρv^2 (recall that for HF $\Delta T = q_w l_v / \lambda$, but we use $\Delta T_N = q_w \bar{h}_N / \lambda$ to nondimensionalize the temperature).

- The *free-surface Biot number*

$$Bi = \frac{\alpha l_v}{\lambda} = \frac{\alpha v^{2/3}}{\lambda (g \sin \beta)^{1/3}}, \quad (2.33)$$

a dimensionless heat transfer coefficient describing the rate of heat transport from the liquid to the ambient gas.

- The *wall Biot number*

$$Bi_w = \frac{\alpha_w l_v}{\lambda} = \frac{\alpha_w v^{2/3}}{\lambda (g \sin \beta)^{1/3}}, \quad (2.34)$$

a dimensionless heat transfer coefficient describing the rate of heat transport from the wall to the ambient gas. This dimensionless group appears only in the HF problem. Note that since the heat transfer coefficient of the liquid–gas interface is in general smaller to that of the solid–gas interface as pointed out in Appendix B.2, and Bi and Bi_w both scale with the thermal conductivity of the liquid, in general $Bi < Bi_w$. As also pointed out in Appendix B.2 this situation can be reversed by insulating the solid, in which case $Bi > Bi_w$ (increasing the thermal conductivity of the liquid does increase Bi but it cannot really lead to $Bi > Bi_w$).

We now define the dimensionless groups in the system (2.17)–(2.28) and we write these groups in terms of h_N and the dimensionless groups in (2.30)–(2.34):

– The *Reynolds number*

$$Re = \frac{\bar{u}_N \bar{h}_N}{\nu} = \frac{\bar{q}_N}{\nu} = \frac{g \sin \beta \bar{h}_N^3}{3\nu^2} \quad (2.35)$$

compares inertia to viscous forces with \bar{u}_N ,¹ the average velocity of the Nusselt flat film solution (2.15a), $\bar{u}_N = g \sin \beta \bar{h}_N^2 / (3\nu)$, and \bar{q}_N , the specific volumetric flow rate (flow rate per unit width of wall in the transverse direction), defined as

$$\bar{q}_N = \int_0^{\bar{h}_N} U(y) dy = \frac{g \sin \beta \bar{h}_N^3}{3\nu}, \quad (2.36)$$

the control parameter that determines the Nusselt flat film thickness \bar{h}_N in experiments. Hence, the Reynolds number in (2.35) is merely the dimensionless flow rate based on the viscous gravity scales, $q_N = \bar{u}_N \bar{h}_N / [(l_v/t_v)l_v] \equiv \bar{q}_N/\nu$. From (2.35) we can also directly relate the dimensionless Nusselt flat film thickness based on l_v to the Reynolds number:

$$h_N = \frac{\bar{h}_N}{l_v} = (3Re)^{1/3}. \quad (2.37)$$

Clearly the definition of the Reynolds number can vary depending on the chosen velocity scale, i.e., one can use the average velocity \bar{u}_N , the velocity at the interface $3\bar{u}_N/2$ (see (2.15a)) or the speed of linear waves $3\bar{u}_N$ (see Chap. 3). In this monograph, we choose the definition (2.35).

– The *Péclet number*

$$Pe = PrRe \quad (2.38)$$

expresses the relative importance of convection and heat conduction. This number is also referred to as the “heat transport Péclet number” in combined heat-mass transport problems to distinguish it from the “mass transport Péclet number”, $PrSc$, where $Sc = \nu/D$ is the “Schmidt number” and D is the molecular diffusivity.

¹Bars are used to distinguish dimensional from dimensionless quantities unless the distinction is unnecessary.

– The *Weber number*

$$We = \frac{\sigma_\infty}{\rho g \bar{h}_N^2 \sin \beta} = \frac{\Gamma}{h_N^2} \quad (2.39)$$

compares the surface tension pressure σ_∞/\bar{h}_N to the viscous normal stress generated by gravity at the film surface, $\mu \bar{u}_N/\bar{h}_N = \rho g \bar{h}_N \sin \beta$. For large We , the fluid behavior is mainly determined by surface tension (e.g., at small \bar{h}_N), while gravity dominates for small We . The Weber number allows us to access surface deformability due to the flow. High values of We mean that the viscous forces due to the flow fail to generate pressure capable of deforming the surface.

– The *film Marangoni number*

$$\text{ST: } M = \frac{\gamma \Delta T}{\rho g \bar{h}_N^2 \sin \beta} = \frac{Ma}{h_N^2} \quad (2.40a)$$

or

$$\text{HF: } M = \frac{\gamma \Delta T_N}{\rho g \bar{h}_N^2 \sin \beta} = \frac{Ma}{h_N} \quad (2.40b)$$

expresses the relative importance of the thermocapillary stress induced by the surface tension gradient, $\gamma \Delta T/\bar{h}_N$ for ST or $\gamma \Delta T_N/\bar{h}_N$ for HF, to the viscous normal stress generated by gravity at the film surface, $\mu \bar{u}_N/\bar{h}_N = \rho g \bar{h}_N \sin \beta$.

– The *free-surface and wall film Biot numbers*

$$B = \frac{\alpha \bar{h}_N}{\lambda} = Bi h_N \quad \text{and} \quad B_w = \frac{\alpha_w \bar{h}_N}{\lambda} = Bi_w h_N. \quad (2.41)$$

As pointed out in Appendix D.1, since all dimensionless parameters in (2.35)–(2.41) depend on the Nusselt flat film solution, which is controlled by the flow rate, they vary when the flow rate is varied. Nevertheless, the parametrization in (2.35)–(2.41) demarcates clearly the dependence of the problem on the flow rate and the properties of the gas–liquid–solid system (physical properties of the gas–liquid system² and wall temperature/heat flux supplied by the heater) and inclination angle. As also pointed out earlier the nondimensionalization in (2.16a)–(2.16f) is based on the Nusselt flat film solution, here the term *Nusselt scaling* adopted in Appendix D.1. This is the most widely used scaling in the literature.

The definitions (2.39) and (2.40a), (2.40b) show that $M, We \rightarrow \infty$ as $h_N \rightarrow 0$, while $Re \rightarrow 0$ from (2.37). Hence, for very thin films, interfacial forces, i.e., Marangoni and capillary forces, dominate over inertia. On the other hand, with $h_N \rightarrow \infty$, $M, We \rightarrow 0$, so that interfacial forces are not important in the region of large film thicknesses and inertia forces dominate over interfacial ones. This point will be discussed further in Chap. 9 when we analyze the relative influence of inertia,

²As discussed in Appendix B.2, the liquid–gas heat transfer coefficient is practically independent of what is happening in the liquid and only dependent on the physical properties of the gas.

Marangoni and capillary forces on the bifurcation diagrams for single-hump solitary pulses.

Noteworthy is that in the literature one frequently encounters a “static Bond number” and a “dynamic Bond number.” They can be written in terms of the conventional Weber number and film Marangoni number for the ST case defined above as: $Bo = \rho g \bar{h}_N^2 \sin \beta / \sigma_\infty \equiv We^{-1}$ and $Bo_d = \rho g \bar{h}_N^2 \sin \beta / (\gamma \Delta T) \equiv M^{-1}$. Strictly speaking, the static Bond number is appropriate for static problems, e.g., a motionless horizontal liquid layer or a drop at equilibrium. This group compares the role of gravity trying to make the free surface leveled to an equipotential, with surface tension trying to make the interface spherical. Setting $Bo = 1$ gives that \bar{h}_N is equal to the capillary length, $l_\sigma = (\sigma_\infty / \rho g \sin \beta)^{1/2}$, which on Earth is about 2.5 mm for water at room temperature. Our case, however, is such that the flow is driven by $g \sin \beta$ and hence Bo as defined earlier is in fact related to the Kapitza number Γ through (2.39). But Γ is a direct consequence of $g \sin \beta$, and so flow; g here is a dynamic quantity “creating flow” and not a passive body force whose role is simply restricted to creating equipotential surfaces, like in static problems. Hence, due to the nature of our problem Bo appears as a flow-related parameter and compares viscous normal stresses with surface tension. This is the main reason we prefer to use the Weber number defined by (2.39) instead.

We close this section with comments on the different limits for the Reynolds and Péclet numbers. The equations of motion and energy in (2.18)–(2.21) describe the competition between two agents: inertia, which gives rise to the H-mode of instability for a falling film, and the long-wave thermocapillary S-mode (see Introduction). This competition is expressed by two dimensionless groups: the Reynolds number in (2.18)–(2.20) and the Péclet number in (2.21), which is a “thermal Reynolds number”, as (2.18)–(2.20) and (2.21) are formally equivalent.

For $Pe \rightarrow 0$ (because $Pr \rightarrow 0$ like in liquid metals), the temperature field is slaved to the velocity field and we can drop the left hand side of (2.21). This then “freezes” the Marangoni mode, the time evolution is set by the velocity field and the system is driven by inertia (appropriately modified of course by the Marangoni effect). On the other hand, for $Re \rightarrow 0$ (while Pe remains finite), the velocity field is slaved to the temperature field and we can drop the left hand side of (2.18)–(2.20). This then “freezes” inertia, i.e., the H-mode, the time evolution is set by the temperature field and the system is driven by the Marangoni forces.

For $Pe \rightarrow 0$ and $Re \rightarrow \infty$, we expect the usual “Kolmogorov–Reynolds inertial turbulence”, which is characterized by an energy transfer from long to short scales mostly dissipation-free (within the inertia interval; dissipation occurs only at the end of the cascade where viscosity kills the small eddies). In this case one can neglect the viscous terms in (2.18)–(2.20). Note that the “Tollmien–Schlichting instability” and transition to fully developed turbulence usually occurs for very large Re , in the region 1000–2000 [56]. This turbulent regime is beyond the scope of the monograph, which focuses on low and moderate Reynolds numbers (in the region 0–50).

On the other hand, for $Re \rightarrow 0$ and $Pe \rightarrow \infty$ (because $Pr \rightarrow \infty$, as with some silicone oils), one expects turbulence with strong dissipation [265] (the velocity scale is now set by the Marangoni effect which appears in the tangential boundary condition). In this case one can neglect the thermal boundary condition in (2.21). This

limit should lead to an inverse cascade from small to large eddies (opposite to the one for Kolmogorov–Reynolds turbulence), e.g., for Bénard–Marangoni convection increasing the Péclet number increases the Bénard–Marangoni cells: dissipation increases and the cells grow to accommodate the large dissipation [291]. This dissipative turbulent regime is also beyond the scope of the monograph.

As a consequence of the above observations, different liquids should exhibit different behavior with respect to the instabilities considered here, depending on their Prandtl number. For falling films with liquids having low Prandtl numbers, e.g., liquid metals, the waves on the films should be controlled by the H-mode. On the other hand, for liquids having large Prandtl numbers, e.g., silicone oils, the waves should be controlled by the thermocapillary S-mode. The competition between the two modes will be discussed in detail in Chaps. 3 and 9.

2.3 On the Development of the Nusselt Flat Film Solution

For the isothermal case, the Nusselt flat film flow in (2.15a)–(2.15f) will in general develop very rapidly after the inlet at $x = 0$: once this happens the flow is “fully developed” (of course this flow will subsequently develop an instability). Clearly the location where the Nusselt flat film flow develops depends on the initial film thickness h_i provided by the manifold. From the integral analysis of the momentum boundary layer in the monograph by Alekseenko et al. [3], we find that for $h_i/\bar{h}_N = 3$, the distance x_h necessary for the film to reach the Nusselt flat film solution with an accuracy of 10^{-4} , is $x_h/\bar{h}_N \approx 1.2Re$ (which also seems to agree with experiments). For instance, a water film with $\bar{h}_N = 0.15$ mm and $Re = 11$ gives $x_h \sim 2$ mm.

For the nonisothermal case, simple scaling arguments can be used to show that the thickness δ_T of the thermal boundary grows proportionally with the Péclet number immediately downstream from the inlet at $x = 0$ where $\delta_T \ll \bar{h}_N$; at $y = \delta_T$, the terms $u\partial_x T$ and $\chi\partial_{yy} T$ of the energy equation in (2.5) scale as $u|_{y=\delta_T} T|_{y=\delta_T}/x$ and $\chi T|_{y=\delta_T}/\delta_T^2$, respectively. Balancing these two terms yields, $u|_{y=\delta_T}/x \sim \chi/\delta_T^2$. Note that from the continuity equation in (2.3), $u|_{y=\delta_T}/x \sim v|_{y=\delta_T}/\delta_T$ and hence the terms $u\partial_x T$ and $v\partial_y T$ of the energy equation balance automatically. But $u|_{y=\delta_T} \equiv U|_{y=\delta_T}$ where U is given in (2.15a)–(2.15f) or $u|_{y=\delta_T} = (g \sin \beta/2\nu)\delta_T(2\bar{h}_N - \delta_T) \sim g \sin \beta \bar{h}_N \delta_T/\nu \sim \bar{u}_N \delta_T/\bar{h}_N$ or $\delta_T \sim (\chi \bar{h}_N x/\bar{u}_N)^{1/3}$, i.e., $x/\bar{h}_N \sim (\delta_T/\bar{h}_N)^3 Pe$ with \bar{u}_N the average velocity of the Nusselt flat film solution and Pe the Péclet number defined in (2.38).

This estimate shows that the development of the thermal boundary layer for moderate Péclet numbers (i.e., for liquids of moderate Prandtl number and flows of moderate Reynolds number) occurs close to the inlet. As an example, for a water film with $\bar{h}_N = 0.15$ mm, $Re = 11$ and $Pr = 7$, the location at which $\delta_T = \bar{h}_N/3$ is $x \approx 0.5$ mm. On the other hand, for a water film with $\bar{h}_N = 0.25$ mm, $Re = 50$ and $Pr = 7$, the location at which $\delta_T = \bar{h}_N/3$ is $x \approx 3.2$ mm. An accurate estimate of the length at which entrance effects associated with the inlet region are neglected would require a detailed integral analysis of the thermal boundary layer which is beyond the scope of this monograph.

Consequently, for liquids of moderate Prandtl number and flows at moderate Reynolds number, the semiparabolic velocity profile and linear temperature distribution can be assumed soon after the inlet so that the film reaches the state governed by (2.15a)–(2.15f), i.e., both flow and heat transfer are fully developed before they undergo any instability.

2.4 On the Two Wall Thermal Boundary Conditions: Retrieving ST from HF

For HF the temperature field has been nondimensionalized with $\Delta T_N = q_w \bar{h}_N / \lambda$. An alternative scaling could have been $T^* = (T - T_\infty) / (q_w / \alpha_w)$, which with $y \rightarrow \bar{h}_N y$ would convert (2.8) to

$$\partial_y T = B_w (-1 + T^*). \quad (2.42)$$

In the limit $B_w \rightarrow \infty$, (2.42) yields $T^* \rightarrow 1$, thus retrieving the boundary condition for ST (2.23a); but (2.23a) is obtained by scaling the temperature field with $\Delta T = T_w - T_\infty$. This scaling must be related to that used to obtain (2.42). Converting $T^* = 1$ to dimensional variables and setting $T = T_w$, yields $q_w = \alpha_w (T_w - T_\infty)$: all the heat generated by the solid is now removed to the gas below so that the temperature T at $y = 0$ is kept constant at $T = T_w = T_\infty + q_w / \alpha_w$. Note that although T refers to the temperature in the liquid, in the limit $B_w \rightarrow \infty$, we lose the term $(1/B_w) \partial_y T$ in (2.42), hence the communication between the wall and the liquid, and so we must set $T = T_w$. In other words, the wall is effectively removed from the problem and we are only concerned with the heat transfer between the liquid and the gas.

Taking the limit $B_w \rightarrow \infty$ in (2.23b) yields $T \rightarrow 0$. It would then appear that we cannot retrieve the ST problem from (2.23b) in this limit. However, (2.23b) can be converted to (2.42) by using the transformation:

$$T = \frac{1}{B_w} T^*. \quad (2.43)$$

Thus, in the limit $B_w \rightarrow \infty$, $T^* \rightarrow 1$ becomes $T \rightarrow 0$ and hence, the alternative form of the wall thermal boundary condition in (2.42) is equivalent to (2.23b). The advantage of (2.42) is that it makes the recovery of ST from HF in the limit $B_w \rightarrow \infty$ transparent. On the other hand, the advantage of (2.23b) is that it makes the limit $B_w \rightarrow 0$ more obvious, as in this limit we retrieve the HF case.

2.5 Role of the Biot Number

The role of the Biot number on the Nusselt flat film temperature distribution and how it influences the Marangoni effect is subtle. We discuss separately the ST and HF cases.

Let us nondimensionalize the temperature distributions in (2.15e) and (2.15f) with the gravity-viscous scaling, which then expresses these distributions in terms of h_N and Bi for ST, and h_N , Bi and Bi_w for HF:

$$\text{ST: } \Theta(y) = \frac{1 + Bi(h_N - y)}{1 + Bi h_N} \quad (2.44a)$$

or

$$\text{HF: } \Theta(y) = \frac{1 + Bi(h_N - y)}{Bi + Bi_w(1 + Bi h_N)}. \quad (2.44b)$$

ST

The temperature of the undeformed free surface is obtained from (2.44a) as

$$\Theta_s \equiv \Theta|_{y=h_N} = \frac{1}{1 + Bi h_N}, \quad (2.45)$$

and consequently, the temperature gradient between the surface and the wall is

$$b_s \equiv \frac{\Theta|_{y=0} - \Theta|_{y=h_N}}{h_N} = \frac{Bi}{1 + Bi h_N}. \quad (2.46)$$

Let us now consider the behavior of (2.45) and (2.46) in the limits of $Bi = 0$ and $1/Bi = 0$; the first limit corresponds to very poor heat transfer characteristics of the liquid–gas interface; the second one is not physical but it is mathematically useful.

- With $Bi = 0$, (2.45) shows that $\Theta_s = 1$. This means that the wall and the free surface have the same temperature. In fact, in this case the fluid temperature is uniform and equal to unity.
- In the limit $1/Bi = 0$, (2.45) shows that $\Theta_s = 0$ so that the free surface and the air have the same temperature.

In both cases, the temperature of the free surface is independent of the film thickness so that any perturbation of h does not affect the free-surface temperature distribution and the Marangoni instability (S-mode) does not occur. This can be made explicit by defining a film Marangoni number M^* based on the Nusselt flat film temperature difference between the wall and the free surface,

$$\Delta T_s \equiv (T_w - T_s) = b_s h_N (T_w - T_\infty) = b_s h_N \Delta T, \quad (2.47)$$

$$M^* \equiv \frac{\gamma \Delta T_s}{\rho g \bar{h}_N^2 \sin \beta} = \frac{Ma Bi}{h_N (1 + Bi h_N)} = \frac{BM}{1 + B}, \quad (2.48)$$

referred to hereinafter as the *modified film Marangoni number*, and where the product $Ma Bi$ appears explicitly through (2.46). Therefore, $M^* \rightarrow 0$ if $Bi \rightarrow 0$ so that there is no thermocapillary effect in this limit. Nevertheless, it appears that in the case of a small Biot number, $Bi \ll 1$, which is frequently the case for liquid films

in contact with gases, the base state temperature gradient can be assumed to be independent of the film thickness, $b_s \approx Bi$. In this limit, the base state temperature gradient is uniquely defined by the heat transfer coefficient α and the thermal conductivity λ .

HF

In this case, the temperature of the undeformed free surface is obtained from (2.44b),

$$\Theta_s \equiv \Theta|_{y=h_N} = \frac{1}{Bi + Bi_w(1 + Bi h_N)}, \quad (2.49)$$

and the temperature gradient between the free surface and the wall now reads:

$$b_s \equiv \frac{\Theta|_{y=0} - \Theta|_{y=h_N}}{h_N} = \frac{Bi}{Bi + Bi_w(1 + Bi h_N)}. \quad (2.50)$$

The limit $1/Bi = 0$ leads to the same conclusion with the ST case. However, the limit $Bi = 0$ is now different, as (2.49) shows that the dimensionless temperature on the free surface is $1/Bi_w$, and therefore depends on the heat transfer characteristics of the solid-gas interface. If in addition $Bi_w = 0$, corresponding to a wall perfectly insulated from the gas, the interfacial temperature diverges to infinity as the heat supplied by the wall to the liquid has nowhere else to go.

It should be emphasized that the surface temperature Θ_s depends on the film thickness only through the parameter Bi_w . If the latter vanishes, $\Theta_s = 1/Bi$ is independent of h_N and remains constant (on the other hand for ST, the quantity Θ_s always depends on h_N , provided that $Bi \neq 0$). In other words, the thermocapillary instability is suppressed for a wall perfectly insulated from the gas. In this case, the temperature gradient across the film layer is independent of h_N ($b_s = 1$), which implies that any elevation (depression) of the film thickness will be accompanied by an increase (decrease) of the wall temperature $\Theta|_{y=0} = (1 + Bi h_N)/Bi$ so that the film surface temperature remains constant. Therefore, enabling heat losses at the wall through the Robin/mixed boundary condition (2.8) is the only way to enable the Marangoni instability when a uniform heat flux q_w is applied at the wall. For a nonuniformly heated wall, which is beyond the scope of this monograph, the thermocapillary instability leads to steady state deformations of the liquid-gas interface [138, 239, 254]. This thermocapillary effect is still present for the ST case with $Bi = 0$ and for the HF case with $Bi_w = 0$ [239].

Using the definition (2.47), which remains unaltered for the HF case but now $\Delta T = q_w l_v / \lambda$, the temperature scale for HF based on the viscous-gravity scaling, the modified film Marangoni number for HF has the form:

$$M^* = \frac{MaBi}{h_N(Bi + Bi_w(1 + Bi h_N))} = \frac{BM}{B + B_w(1 + B)}. \quad (2.51)$$

This film Marangoni number M^* based on the Nusselt flat film temperature difference between the wall and the free surface will be useful in Chap. 3, where we examine the linear stability of the Nusselt flat film solution.

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