Chapter II

Frames and Locales. Spectra

Is it worthwhile to try to base topology on the algebra of open sets (and to neglect points)? We hope that this question has been answered in the previous chapter satisfactorily by showing that there are two large classes of spaces in which no information is lost, the class of sober spaces, and the class of T_D-ones.

The point-free topology as we will present it will lean slightly to the sober side. There are various reasons. One of them is that for reconstructing continuous maps we have a very simple condition. Not only simple, also a very natural one: preserving all joins and all finite meets by the maps on the algebraic side – and there is no doubt that in topology one likes to think in terms of general unions and finite intersections. And here is another reason: although this class does not go as far down in separation as the other one (not even all the T_1-spaces are covered), the Scott spaces (appearing in many modern applications) are typically (not always, but typically) sober, and seldom T_D. The T_D axiom will, nevertheless, also play a role later.

In this chapter we will introduce the basic algebraic background we will work with: frames and locales representing spaces, and frame homomorphisms resp. localic morphisms representing continuous maps. Further, we will discuss in some detail the reconstruction of points (spectra of frames) and some aspects of the spectrum adjunction.

1. Frames

1.1. The category of topological spaces and continuous maps will be denoted by

$$\mathbf{Top}$$

and we will write

$$\mathbf{Sob}$$

for its full subcategory generated by sober spaces.

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1.2. The complete lattice of open sets satisfies the distributive law

$$\Big(\bigvee_{i\in J} U_i\Big) \wedge V = \bigvee_{i\in J} (U_i \wedge V)$$

since an arbitrary join coincides with the union, and the meet $U \wedge V$ is the intersection $U \cap V$.

(Note that the general meet in $\Omega(X)$ typically *does not* coincide with the intersection. Hence there is no surprise that we generally do not have $(\bigwedge_{i\in J} U_i) \vee V = \bigwedge_{i\in J}(U_i \vee V)$.)

Thus, aiming for "generalized spaces" represented as special complete lattices (mimicking the lattices of open sets) we are led to the following definition.

A *frame* is a complete lattice L satisfying the distributivity law

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\} \tag{frm}$$

for any subset $A \subseteq L$ and any $b \in L$.

Recalling I.1.6 we define *frame homomorphisms* $h\colon L \to M$ between frames L, M as maps $L \to M$ preserving all joins (including the bottom 0) and all finite meets (including the top 1). The resulting category will be denoted by

$$\mathbf{Frm}.$$

1.2.1. The lattice of open sets of the void space is the one-element lattice $\{0 = 1\}$. It will be denoted by

$$\mathsf{O}.$$

The lattice of open sets of the one-point space is the two-element Boolean algebra $\{0 < 1\}$. It will be denoted by

$$\mathbf{2} \quad \text{or sometimes by} \quad \mathsf{P}.$$

1.2.2. Later on (for instance for the structure of generalized subspaces of a frame) we will also need the dual notion of a *co-frame*, that is of a complete lattice satisfying

$$(\bigwedge A) \vee b = \bigwedge\{a \vee b \mid a \in A\}. \tag{co-frm}$$

1.3. Recall the symbol Ω introduced in the preamble to Chapter I and extended for continuous maps in I.1.6.3. We have frames $\Omega(X)$ and frame homomorphisms $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ for continuous maps $f\colon X \to Y$ resulting in a contravariant functor

$$\Omega\colon \mathbf{Top} \to \mathbf{Frm}.$$

1.3.1. A frame L is said to be *spatial* if it is isomorphic to an $\Omega(X)$.

There exist non-spatial frames (we will present some soon) and hence we have "more spaces than before". This may not seem a priori desirable; it has turned out, however, that it makes the theory, in some respects, more satisfactory.

Note. Nevertheless, the question naturally arises whether one can further narrow down the class of lattices by algebraic means so as to obtain just spatial frames, or at least much fewer non-spatial ones. There is a stricter distributivity rule (still holding in all the $\Omega(X)$) under which it is really hard to find a non-spatial example. See V.4.

1.4. The rule (frm) says that the mappings $- \wedge b = (x \mapsto x \wedge b)$ preserve all suprema. Hence, they are left Galois adjoints, and the corresponding right adjoints $b \to -$ produce a Heyting operation in L (see Appendix I). Thus,

a frame is automatically a complete Heyting algebra.

The Heyting operation will be frequently used. The reader has to have in mind, though, that this operation is not generally preserved by frame homomorphisms.

If a frame happens to be a Boolean algebra we speak of a *Boolean frame*. Note that frame homomorphisms between Boolean frames are complete Boolean homomorphisms (because lattice homomorphisms preserve complements – see AI.7.4.5), and that a Boolean frame is a co-frame as well.

2. Locales and localic maps

2.1. The contravariant functor Ω from 1.3, restricted to the category **Sob** of sober spaces is, by I.1.6.2, a full embedding (for any two sober spaces X, Y we have a one-one correspondence between the continuous maps $f\colon X \to Y$ and the frame homomorphisms $h\colon \Omega(Y) \to \Omega(X)$ provided by sending f to $\Omega(f)$). That is, it would be a full embedding if only it were covariant which it is not. This can be easily helped by introducing the *category of locales*

Loc

as the dual category $\mathbf{Frm}^{\mathrm{op}}$, and this category can be viewed as an extension of the category of sober spaces (and the frames – now referred to as *locales* – as generalized spaces).

2.2. It is not very intuitive to have morphisms $A \to B$ in a category represented as mappings $A \leftarrow B$. But this can be mended:

frame homomorphisms $h\colon M \to L$, since they preserve all suprema, have uniquely defined right Galois adjoints $h_*\colon L \to M$.

Thus, we can represent the morphisms in **Loc** as

the infima-preserving $f\colon L \to M$ such that the corresponding left adjoints $f^*\colon M \to L$ preserve finite meets.

Such f will be referred to as *localic maps* and we will think of the category **Loc** as having these for morphisms.

2.2.1. Note. We have not introduced localic maps this way only for the sake of having actual maps where we had, in the formal categorical construction, in essence only symbols. It will turn out later that it often really helps understanding further notions and constructions.

2.2.2. The locale O from 1.2.1 behaves indeed as the void (generalized) space should. There is precisely one localic map $\mathsf{O} \to L$ for any locale L, and none $L \to \mathsf{O}$ unless L is void (that is, O itself).

Also, the locale P behaves like a point. So far we immediately see that for any locale L there is precisely one localic map $L \to \mathsf{P}$ (since there is precisely one frame homomorphism $\mathsf{P} \to L$, namely $(0 \mapsto 0, 1 \mapsto 1)$). As for the localic maps $\mathsf{P} \to L$ we will see shortly (as soon as we will introduce points in the next section) that they can be viewed as embeddings of points.

2.3. The following characteristics of localic maps will be sometimes useful.

Proposition. *Let $f\colon L \to M$ have a left adjoint f^*. Then*

(a) $f^*(1) = 1$ *iff* $f[L \smallsetminus \{1\}] \subseteq M \smallsetminus \{1\}$, *and*

(b) f^* *preserves binary meets iff*

$$f(f^*(a) \to b) = a \to f(b).$$

Proof. (a): If $f^*(1) = 1$ and $1 \le f(x)$ then $1 = f^*(1) \le x$; on the other hand, if the inclusion holds, use the inequality $f(f^*(1)) \ge 1$.

(b) $\Rightarrow$: $x \le f(f^*(a) \to b)$ iff $f^*(x \wedge a) = f^*(x) \wedge f^*(a) \le b$ iff $x \wedge a \le f(b)$ iff $x \le a \to f(b)$.

$\Leftarrow$: $f^*(a \wedge b) \le x$ iff $a \le b \to f(x) = f(f^*(b) \to x)$ iff $f^*(a) \le f^*(b) \to x$ iff $f^*(a) \wedge f^*(b) \le x$. $\square$

2.4. Now, when the category **Loc** has this concrete representation, it is worthwhile to modify the functor Ω accordingly. We will make it to a functor

$$\mathsf{Lc}\colon \mathbf{Top} \to \mathbf{Loc}$$

by setting

$$\mathsf{Lc}(X) = \Omega(X), \quad \mathsf{Lc}(f) = \Omega(f)_*.$$

From the standard adjunction

$$f^{-1}[B] \subseteq A \quad \text{iff} \quad B \subseteq Y \smallsetminus f[X \smallsetminus A]$$

of preimage we immediately learn that

$$\mathsf{Lc}(f)(U) = Y \smallsetminus \overline{f[X \smallsetminus U]}.$$

This does not seem to be a very transparent formula, but it becomes more acceptable after the following observation.

Represent a point $x \in X$ by the element

$$\widetilde{x} = X \smallsetminus \overline{\{x\}}$$

of $\mathsf{Lc}(X)$. Then we have

$$\mathsf{Lc}(f)(\widetilde{x}) = \widetilde{f(x)}$$

(indeed, $\mathsf{Lc}(f)(\widetilde{x}) = Y \smallsetminus \overline{f[X \smallsetminus (X \smallsetminus \overline{\{x\}})]} = Y \smallsetminus \overline{f[\overline{\{x\}}]} = Y \smallsetminus \overline{\{f(x)\}} = \widetilde{f(x)}$).

3. Points

3.1. The intuitive concept of a diminishing system of spots. We believe that this is what most people do when visualizing a point: one thinks of a very small spot; then admits it may be still too large, thinks of a smaller one inside, etc. It should be noted that this view of a point was around decades before there was anything like a point-free topology (for instance we can go as far back as to Carathéodory 1913 [59]).

Thus we can think of a point in a locale as a specific filter. We are still in trouble in telling what "diminishing size" of spots should mean (later on, in Chapter X we will be able to do it in various ways, with various resulting definitions of points, but now we do not want to employ any extra structure). A general filter, for instance $\{x \in L \mid x \geq a\}$ will hardly do. But take the hint from sober spaces: in I.1.3 we have seen that in this fairly broad class of spaces the points can be detected (represented by their sets of neighbourhoods $\mathcal{U}(x)$) as the completely prime filters. This leads to the following definition:

(P1) a *point* in a frame (locale) L is a completely prime filter $F \subseteq L$.

3.2. Another, but practically the same representation. The points x in a space X are in a natural one-one correspondence with the (continuous) mappings $f = (0 \mapsto x)\colon \{0\} \to X$ and hence, in the sober case, with the frame homomorphisms $h\colon \Omega(X) \to \mathbf{2}$. Thus,

(P2) a *point* in L can be viewed as a frame homomorphism $h\colon L \to \mathbf{2}$

(or as a *localic map* $\mathsf{P} \to L$, with the "one-point locale" $\mathsf{P} = \mathbf{2}$).

The translation between (P1) and (P2) is obvious: given a completely prime filter $F \subseteq L$ define $h\colon L \to \mathbf{2}$ by setting $h(x) = 1$ iff $x \in F$, and given an $h\colon L \to \mathbf{2}$ we have the completely prime filter $F = \{x \mid h(x) = 1\}$.

3.3. Yet another, this time not so obvious representation of points. Recall that an element $p \neq 1$ in a lattice L is *meet-irreducible* (also, *prime*) if for any $a, b \in L$,

$a \wedge b \leq p$ implies that either $a \leq p$ or $b \leq p$. (This means that the principal ideal ${\downarrow}p$ is a prime ideal – see AI.6.4.1).

Given a completely prime filter $F \subseteq L$ define

$$p_F = \bigvee\{x \mid x \notin F\}.$$

Then

p_F is a meet-irreducible element.

(Indeed, by the complete primeness, $p_F \notin F$ and hence $p_F \neq 1$, and if $a \wedge b \leq p_F$ then we cannot have both $a, b \in F$ and hence, say, $a \notin F$ and $a \leq p_F$.)

On the other hand, if $p \in L$ is meet-irreducible, set

$$F_p = \{x \mid x \nleq p\}.$$

Then

F_p is a completely prime filter.

(In fact, if $x, y \in F_p$ then $x \wedge y \in F_p$ by meet-irreducibility; obviously $y \geq x \in F_p$ implies $y \in F_p$, and if $\bigvee_{i \in J} x_i \nleq p$ then $x_i \nleq p$ for some i; finally, $1 \nleq p$ and $0 \leq p$ so that F_p is non-void and proper.)

Finally, $p_{F_p} = \bigvee\{x \mid x \leq p\} = p$ and $x \in F_{p_F}$ iff $x \nleq \bigvee\{y \mid y \notin F\}$ iff $x \in F$.

(If $x \in F$ we cannot have $x \leq \bigvee\{y \mid y \notin F\}$ by complete primeness, and if $x \notin F$ then $x \leq \bigvee\{y \mid y \notin F\}$ trivially.)

Thus,

(P3) a *point* in L can also be viewed as a meet-irreducible element $p \in L$.

3.4. Localic maps preserve points in the sense of (P3). We have

Lemma. *Localic maps send meet-irreducible elements to meet-irreducible ones again.*

Proof. First, localic maps reflect tops, that is, if $f(a) = 1$ then $a = 1$: indeed, if $1 \leq f(a)$ then $1 = f^*(1) \leq a$.

Now let $f\colon L \to M$ be a localic map and let a be meet-irreducible in L. Let $x \wedge y \leq f(a)$. Then $f^*(x) \wedge f^*(y) = f^*(x \wedge y) \leq a$ so that, say, $f^*(x) \leq a$ and $x \leq f(a)$ (which cannot be the top, by the preceding paragraph). $\square$

3.5. Now we can better explain what we said in 2.2.2 about the behaviour of the locale P as a one-point space. Namely, the localic maps $f\colon \mathsf{P} \to L$ are in a natural one-one correspondence with the meet-irreducibles in L:

- $\mathsf{P} = \mathbf{2}$ has precisely one meet-irreducible element, namely 0; associate with f the meet-irreducible $f(0)$;
- if $p \in L$ is meet-irreducible we have the localic map $(0 \mapsto a, 1 \mapsto 1)$, adjoint to the frame homomorphism $h(x) = 0$ iff $x \leq p$.

4. Spectra

Adopting the idea of a point as in 3.1 we implicitly think of the filter F also as of the system of neighbourhoods of the "ideal point in the center". This makes the system of points of L to a space, the *spectrum* of L. It will be discussed in the remaining sections of this chapter.

We will abbreviate "completely prime filter" to "c.p. filter", and, as before, we will write

$$f^* \quad \text{for the left Galois adjoint of} \quad f.$$

4.1. For an element a of a locale L set

$$\Sigma_a = \{F \text{ c.p. filter in } L \mid a \in F\}.$$

We easily check the

Observation. *$\Sigma_0 = \emptyset$ and $\Sigma_1 = \{$all c.p. filters$\}$ (note that our filters are non-void and proper), $\Sigma_{a\wedge b} = \Sigma_a \cap \Sigma_b$, and $\Sigma_{\bigvee_{i\in J} a_i} = \bigcup_{i\in J} \Sigma_{a_i}$.*

4.2. Consequently we have a topological space

$$\mathsf{Sp}(L) = \Big(\{\text{all c.p. filters in } L\}, \{\Sigma_a \mid a \in L\}\Big).$$

This space is called the *spectrum* of L.

4.3. The functor Sp. If $f\colon L \to M$ is a localic map we define

$$\mathsf{Sp}(f)\colon \mathsf{Sp}(L) \to \mathsf{Sp}(M)$$

by setting

$$\mathsf{Sp}(f)(F) = (f^*)^{-1}[F]$$

(since f^* is a frame homomorphism the preimage is a c.p. filter).

4.3.1. Lemma. $(\mathsf{Sp}(f))^{-1}[\Sigma_a] = \Sigma_{f^*(a)}$. Consequently, $\mathsf{Sp}(f)$ is a continuous map.

Proof. $(\mathsf{Sp}(f))^{-1}[\Sigma_a] = \{F \mid \mathsf{Sp}(f)(F) = (f^*)^{-1}[F] \in \Sigma_a\} = \{F \mid a \in (f^*)^{-1}[F]\}$ $= \{F \mid f^*(a) \in F\} = \Sigma_{f^*(a)}$. □

4.3.2. Obviously $\mathsf{Sp}(\mathrm{id}_L) = \mathrm{id}_{\mathsf{Sp}(L)}$ and $\mathsf{Sp}(g \cdot f) = \mathsf{Sp}(g) \cdot \mathsf{Sp}(f)$. Hence we have

Fact. *The formulas for $\mathsf{Sp}(L)$ and $\mathsf{Sp}(f)$ define a functor*

$$\mathsf{Sp}\colon \mathbf{Loc} \to \mathbf{Top}.$$

One speaks of the *spectrum functor*.

4.4. Note. Usually one considers as the spectrum functor the contravariant

$$\Sigma\colon \mathbf{Frm} \to \mathbf{Top}, \quad \Sigma L = \mathsf{Sp}(L), \quad \Sigma h(F) = h^{-1}[F],$$

as a counterpart to the $\Omega\colon \mathbf{Top} \to \mathbf{Frm}$. We have modified the notation to emphasize the covariance. Shortly we will show that Sp is a right adjoint to Lc. Of course, the adjunction can also be described in terms of an adjunction of Ω and Σ; but which of the functors is to the left and which is to the right is much more transparent for covariant functors.

4.5. The spectrum in terms of meet-irreducibles. If we represent the points by meet-irreducible elements as in 3.3 we obtain the spectrum functor in a particularly simple form. We set, of course,

$$\mathsf{Sp}'(L) = \Big(\{\text{all meet-irreducible elements in } L\}, \{\Sigma'_a \mid a \in L\}\Big)$$

where $\Sigma'_a = \{p \mid a \not\leq p\}$ (in the translation from 3.2: $F_p \in \Sigma_a$ iff $a \in F_p$ iff $a \not\leq p$ iff $p \in \Sigma'_a$) and using 3.4 we can define, for a localic map $f\colon L \to M$,

$$\mathsf{Sp}'(f)\colon \mathsf{Sp}'(L) \to \mathsf{Sp}'(M)$$

simply as the restriction, that is, by the formula $\mathsf{Sp}'(f)(q) = f(q)$.

> (This is indeed what the $\mathsf{Sp}(f)$ above translates to. Let $q \in \mathsf{Sp}'(L)$. By the formulas from 3.3 we obtain $(f^*)^{-1}[F_q] = \{a \mid f^*(a) \not\leq q\} = \{a \mid a \not\leq f(q)\}$ and hence $p_{\mathsf{Sp}(f)(F_q)} = p_{\{a \,\mid\, a \not\leq f(q)\}} = \bigvee\{a \mid a \leq f(q)\} = f(q)$.)

The reader may rightfully ask why we have not preferred this description of spectrum to the one above. The reason is that we want to keep the intuition of a point as a "limit of a diminishing system of spots", and this is, of course, fairly explicitly visible when using the filters, and not at all when using the meet-irreducibles. But we will use the construction Sp' as well.

4.6. The spectrum adjunction. (Recall AII.6) The functors Lc and Sp are not inverse to each other (and certainly have not been expected to be). But we have the next best: they are adjoint to each other, and as we will see in the following sections, the adjunction units have very nice properties.

4.6.1. By 4.1 we have frame homomorphisms

$$\phi_L\colon L \to \mathsf{Lc}(\mathsf{Sp}(L))$$

given by $\phi_L(a) = \Sigma_a$. Consider their right Galois adjoints, the localic maps

$$\sigma_L = (\phi_L)_*\colon \mathsf{Lc}(\mathsf{Sp}(L)) \to L.$$

Lemma. (1) *The system $(\sigma_L)_L$ constitutes a natural transformation $\mathsf{LcSp} \xrightarrow{\cdot} \mathrm{Id}$.*

(2) *Each σ_L is one-one.*

Proof. (1) Let $f\colon L \to M$ be a localic map. We have

$$\begin{aligned}(\sigma_M{\cdot}\mathsf{LcSp}(f))^*(a) &= \mathsf{LcSp}(f)^*(\phi_M(a)) = \Omega(\mathsf{Sp}(f))(\Sigma_a)\\ &= \{F \mid \mathsf{Sp}(f)(F) \ni a\} = \{F \mid (f^*)^{-1}[F] \ni a\} = \{F \mid f^*(a) \in F\}\\ &= \Sigma_{f^*(a)} = \phi_L(f^*(a)) = (\sigma_L)^*(f^*(a) = (f \cdot \sigma_L)^*(a).\end{aligned}$$

(2) The mapping ϕ_L is onto. Hence, from the standard Galois equation (see AI.5.3.1) $\phi_L\sigma_L\phi_L = \phi_L$ we obtain $\phi_L\sigma_L = \mathrm{id}$ which makes σ_L one-one. □

4.6.2. For a space X consider the mapping

$$\lambda_X\colon X \to \mathsf{Sp}(\mathsf{Lc}(X))$$

defined by

$$\lambda_X(x) = \mathcal{U}(x) = \{U \mid x \in U\}.$$

Lemma. (1) λ_X *is a continuous map.*

(2) *The system* $(\lambda_X)_X$ *constitutes a natural transformation* $\mathrm{Id} \stackrel{\cdot}{\to} \mathsf{SpLc}$.

Proof. (1) Each open set in $\mathsf{SpLc}(X)$ is of the form Σ_U with U open in X. We have $\lambda_X^{-1}[\Sigma_U] = \{x \mid \mathcal{U}(x) \in \Sigma_U\} = \{x \mid U \in \mathcal{U}(x)\} = \{x \mid x \in U\} = U$.

(2) Let $f\colon X \to Y$ be continuous. Since $\Omega(f)$ is the left adjoint to $\mathsf{Lc}(f)$ we have

$$\begin{aligned}(\mathsf{SpLc}(f) \cdot \lambda_X)(x) &= \mathsf{SpLc}(f)(\mathcal{U}(x)) = (\Omega(f))^{-1}(\mathcal{U}(x))\\ &= \{U \mid f^{-1}[U] \in \mathcal{U}(x)\} = \{U \mid x \in f^{-1}[U]\}\\ &= \{U \mid f(x) \in U\} = \lambda_Y(f(x)).\end{aligned}$$ □

4.6.3. Theorem. *The functors* Lc *and* Sp *are adjoint,* Lc *to the left and* Sp *to the right, with units* $\lambda\colon \mathrm{Id} \stackrel{\cdot}{\to} \mathsf{SpLc}$ *and* $\sigma\colon \mathsf{LcSp} \stackrel{\cdot}{\to} \mathrm{Id}$.

Proof. Consider

$$\mathsf{Sp}(\sigma_L) \cdot \lambda_{\mathsf{Sp}L}\colon \mathsf{Sp}L \to \mathsf{Sp}(\mathsf{Lc}(\mathsf{Sp}L)) \to \mathsf{Sp}L.$$

Since $(\sigma_L)^* = \phi_L$ we have

$$\begin{aligned}(\mathsf{Sp}(\sigma_L){\cdot}\lambda_{\mathsf{Sp}L})(F) &= \mathsf{Sp}(\sigma_L))\mathcal{U}(F)) = \phi_L^{-1}[U(F)]\\ &= \{U \mid \Sigma_U \in \mathcal{U}(F)\} = \{U \mid F \in \Sigma_U\} = \{U \mid U \in F\} = F.\end{aligned}$$

Consider

$$\sigma_{\mathsf{Lc}(X)} \cdot \mathsf{Lc}(\lambda_X)\colon \mathsf{Lc}X \to \mathsf{Lc}(\mathsf{Sp}(\mathsf{Lc}X)) \to \mathsf{Lc}X.$$

We have

$$\begin{aligned}(\sigma_{\mathsf{Lc}X}{\cdot}\mathsf{Lc}(\lambda_X))^*(U) &= (\mathsf{Lc}(\lambda_X)^* \cdot \phi_{\mathsf{Lc}(X)})(U) = (\mathsf{Lc}(\lambda_X))^*(\Sigma_U)\\ &= \Omega(\lambda_X)(\Sigma_U) = \lambda_X^{-1}[\Sigma_U] = \{x \mid \mathcal{U}(x) \ni U\} = \{x \mid x \in U\} = U.\end{aligned}$$ □

4.6.4. Observation. We already know that each σ_L is one-one. Now we have also learned that

$$\sigma_{\mathsf{Lc}(X)} \cdot \mathsf{Lc}(\lambda_X) = \mathrm{id}_{\mathsf{Lc}(X)}$$

which makes $\sigma_{\mathsf{Lc}(X)}$ an onto map and we conclude that

for each X, the localic map $\sigma_{\mathsf{Lc}(X)} \colon \mathsf{Lc}(\mathsf{Sp}(\mathsf{Lc}X)) \to \mathsf{Lc}X$ is an isomorphism.

4.7. The units in meet-irreducibles. In the spectrum description Sp' the formulas for the units appear as

$$\lambda_X(x) = X \smallsetminus \overline{\{x\}} \quad \text{and} \quad (\sigma_X)^*(a) = \Sigma'_a = \{p \mid a \nleq p\}.$$

5. The unit σ and spatiality

5.1. Proposition. *The following statements on a locale L are equivalent.*

(1) *L is spatial,*
(2) *$\sigma_L \colon \mathsf{Lc}(\mathsf{Sp}(L)) \to L$ is a complete lattice isomorphism,*
(3) *$\sigma_L^* \colon L \to \mathsf{Lc}(\mathsf{Sp}(L))$ is a complete lattice isomorphism,*
(4) *σ_L is onto,*
(5) *σ_L^* is one-one.*

Proof. The implications (2)⇒(1), (2)⇒(4) and (3)⇒(5) are obvious. By 4.6.1 any of (4), (5) implies any of (2), (3), (4), (5) (σ_L and σ_L^* are monotone maps and any of (4), (5) makes them inverse to each other, and hence complete lattice isomorphisms).

It remains to prove that (1)⇒(2). Recall 4.6.4. If $h\colon L \to \mathsf{Lc}(X)$ is an isomorphism we have the isomorphism $\sigma_L = h^{-1} \cdot \sigma_{\mathsf{Lc}(X)} \cdot \mathsf{LcSp}(h)$. □

5.2. Thus the question whether L is spatial or not reduces to checking whether it is (isomorphic to) the topology of its own spectrum. If it is not, it is not isomorphic to the topology of any space.

5.3. Consider the reformulation of the spectrum in terms of meet-irreducibles. Then the equivalence (1)⇔(5) in 5.1 concerns the implication

$$\Sigma'_a = \{p \mid a \nleq p\} \neq \Sigma'_b = \{p \mid b \nleq p\} \quad \Rightarrow \quad a \neq b. \tag{5.3.1}$$

We obtain the following criterion.

Proposition. *A locale L is spatial if and only if each element $a \in L$ is a meet of meet-irreducible ones.*

Proof. 5.1(5) can be reformulated as

$$b \nleq a \;\Rightarrow\; \Sigma'_b \nleq \Sigma'_a.$$

Thus, if $b \nleq a$ there is a point $p \in \Sigma'_b \smallsetminus \Sigma'_a$, that is, $a \leq p$ and $b \nleq p$. Consequently, $a = \bigwedge\{p \text{ points} \mid a \leq p\}$.

Conversely, if $a = \bigwedge\{p \text{ points} \mid a \leq p\}$ then for $b \nleq a$ there is a point p such that $a \leq p$ and $b \nleq p$, hence $p \in \Sigma'_b \smallsetminus \Sigma'_a$. □

5.4. Aside: spatial Boolean algebras. Recall that an *atom* (resp. a *co-atom*) in L is an element $a > 0$ (resp. $a < 1$) such that for each x, $a \geq x > 0$ implies that $x = a$ (resp. $a \leq x < 1$ implies that $x = a$).

The following is a trivial

5.4.1. Observation. *In a Boolean algebra, a is an atom iff its complement a^{c} is a co-atom.*

5.4.2. A Boolean algebra is said to be *atomic* if each of its elements is a join of atoms. The following is a well-known fact.

Proposition. *The following statements on a Boolean algebra B are equivalent.*

(1) *B is atomic,*
(2) *each element of B is a meet of co-atoms,*
(3) *B is isomorphic to the Boolean algebra $\mathfrak{P}(X)$ of all subsets of a set X.*

Proof. (1) and (2) are equivalent by De Morgan formulas. Now if B is atomic take X the set of all atoms and represent $b \in B$ by the set $A(b) = \{x \in X \mid x \leq b\}$. This is obviously an isomorphism (the only fact to realize is that $A(a) = A(b)$ implies that $a = b$ and this is obtained as follows: if $x \in X$ and $x \leq a = \bigvee A(a)$ then $0 \neq x = \bigvee\{x \wedge y \mid y \in A(a)\}$ and hence, by atomicity, $x = y$ for some $y \in A(b)$). □

5.4.3. Proposition. *Every meet-irreducible element in a Boolean algebra is a co-atom.*

Proof. Let p be meet-irreducible in a Boolean algebra B. Let $p < x$. If y is the complement of x we have $0 = x \wedge y \leq p$ and by meet-irreducibility $y \leq p$ but then $y \leq x$ and $1 = x \vee y = x$. □

5.4.4. Now by 5.3, every element of a spatial Boolean locale is a meet of co-atoms. Consequently, by 5.4.2 a Boolean locale is spatial only if it is atomic; also, only if it is (isomorphic to) the discrete topology of a space. Hence,

the spatial Boolean locales correspond to the discrete spaces.

The non-atomic complete Boolean algebras are, hence, examples of non-spatial locales. To present a concrete one, take the real line and consider the Boolean algebra B of all the open subsets U such that $U = \operatorname{int}\overline{U}$. Here we have no atoms at all (each non-void element contains a strictly smaller non-void one) so that $\mathsf{Sp}(B) = \emptyset$ although B is fairly large.

Frames of this type play a prominent role as we will see later (III.8.3, VI.6).

6. The unit λ and sobriety

6.1. Proposition. *The space* $\mathsf{Sp}(L)$ *is always sober.*

Proof. It is T_0 because if c.p. filters F, G are distinct there is, say, an $a \in F \smallsetminus G$ and then $G \notin \Sigma_a \ni F$.

Now let $\mathcal{F}$ be a completely prime filter in $\mathsf{Sp}(L)$. Set $F = \{a \mid \Sigma_a \in \mathcal{F}\}$. By Observation 4.1, F is a c.p. filter in L. If $\Sigma_a \in \mathcal{F}$ then $a \in F$ and $F \in \Sigma_a$; if $F \in \Sigma_a$ then $a \in F$ and $\Sigma_a \in \mathcal{F}$. Thus, $\mathcal{F} = \{\Sigma_a \mid F \in \Sigma_a\} = \mathcal{U}(F)$. $\square$

6.2. Proposition. *The following statements about a space X are equivalent.*

(1) *X is sober,*
(2) $\lambda_X \colon X \to \mathsf{SpLc}(X)$ *is one-one onto,*
(3) $\lambda_X \colon X \to \mathsf{SpLc}(X)$ *is a homeomorphism.*

Proof. (1)⇒(2) is in I.1.3.1.

(2)⇒(3): We have to prove that $\lambda_X[U]$ is open whenever U is. But $\lambda_X[U]$ is the set $\{\mathcal{U}(x) \mid x \in U\} = \{\mathcal{U}(x) \mid U \in \mathcal{U}(X)\} = \{F \text{ c.p. filter} \mid U \in F\} = \Sigma_U$.

(3)⇒(1) is in 6.1. $\square$

6.3. Hence, $\mathsf{SpLc}(X)$ is sober, and for a sober X we have $\mathsf{SpLc}(X) \cong X$. Thus, $\mathsf{SpLc}[\mathbf{Top}] = \mathbf{Sob}$ and we can decompose

$$\mathbf{Top} \xrightarrow{\mathsf{Lc}} \mathbf{Loc} \xrightarrow{\mathsf{Sp}} \mathbf{Top} \qquad \text{as} \qquad \mathbf{Top} \xrightarrow{R} \mathbf{Sob} \xrightarrow{J=\subseteq} \mathbf{Top}$$

with R identical on **Sob**.

This makes **Sob** a reflective subcategory in **Top** (see AII.8) with the reflection

$$\lambda_X \colon X \to JR(X).$$

6.3.1. Remarks. (1) Note that the reflection in 6.3 (the *sobrification*, as it is often called) does the same as the completion of metric spaces: it fills in the special filters that have not been, so far, neighbourhood systems as new points; they become neighbourhood systems "of themselves".

(2) From 4.6.4 we see that for a spatial frame L, the X such that $L \cong \Omega(X)$ is not uniquely determined: the lattice of open sets of a space and that of its sobrification are isomorphic. But nothing worse can happen: if there is an isomorphism $\alpha \colon \mathsf{Lc}(X) \to \mathsf{Lc}(Y)$ we have an isomorphism $\mathsf{Sp}(\alpha) \colon \mathsf{SpLc}(X) \to \mathsf{SpLc}(Y)$. Thus, the sobrifications are isomorphic. This is, of course, no surprise. We know since I.1.4 that a sober space can be reconstructed from its lattice of open sets.

(3) The fact that the open set lattice of a space is isomorphic with that of its sobrification shows that the sobriety cannot be expressed as an algebraic property of a frame. Other "completeness properties" can, however, as we will see later.

6.4. Aside: sober posets. We have already said in I.3.4 that the Alexandroff space

$$(X, \mathfrak{Up}(X, \leq))$$

(here $\mathfrak{Up}(X, \leq)$ is the set of all up-sets in $(X, \leq)$) is not very often sober. We will now present a characteristics of those that are. Furthermore, we will see that the sobrification of an Alexandroff space is its set of ideals endowed with the *Scott topology* (recall that a subset S of a poset is *Scott-open* if it is an upper set and all directed sets D that have a supremum in S have non-empty intersection with S).

6.4.1. We have already had three representations of points. For $\mathfrak{Up}(X, \leq)$ there is yet another, and a very handy one.

For a completely prime filter $\mathcal{F}$ in $\mathfrak{Up}(X, \leq)$ define

$$J_{\mathcal{F}} = \{x \in X \mid \uparrow x \in \mathcal{F}\}.$$

Fact. *$J_{\mathcal{F}}$ is an ideal in $(X, \leq)$.*

(Indeed, if $\uparrow x \in \mathcal{F}$ and $y \leq x$ then $\uparrow x \subseteq \uparrow y$. If $\uparrow x, \uparrow y \in \mathcal{F}$ then $\emptyset \neq \uparrow x \cap \uparrow y \in \mathcal{F}$ and we have a $z \geq x, y$. Further, $J_{\mathcal{F}}$ is non-empty by the complete primeness: take any $U = \bigcup\{\uparrow x \mid x \in U\} \in \mathcal{F}$.)

On the other hand, for an ideal J in $(X, \leq)$ define

$$\mathcal{F}_J = \{U \in \mathfrak{Up}(X, \leq) \mid U \cap J \neq \emptyset\}.$$

We have

Fact. *$\mathcal{F}_J$ is a completely prime filter in $\mathfrak{Up}(X, \leq)$.*

(Obviously $\emptyset \notin \mathcal{F}_J$, $\bigcup U_i \in \mathcal{F}_J \;\Rightarrow\; \exists i, U_i \in \mathcal{F}_J$, and $V \supseteq U \in \mathcal{F}_J \;\Rightarrow\; V \in \mathcal{F}_J$. If $x \in U \cap J$ and $y \in V \cap J$ choose a $z \in J$ such that $z \geq x, y$; then $z \in (U \cap V) \cap J$.)

6.4.2. Lemma. *$\mathcal{F} \mapsto J_{\mathcal{F}}$ and $J \mapsto \mathcal{F}_J$ are mutually inverse correspondences.*

Proof. If $U \in \mathcal{F}_{J_{\mathcal{F}}}$ then $U \cap J_{\mathcal{F}} \neq \emptyset$ and we have an x with $\uparrow x \in \mathcal{F}$ and $x \in U$. Then $\uparrow x \in U$ and $U \in \mathcal{F}$. Conversely, if $U \in \mathcal{F}$ then $U = \bigcup\{\uparrow x \mid x \in U\} \in \mathcal{F}$ and since $\mathcal{F}$ is completely prime there is an $x \in U$ with $\uparrow x \in \mathcal{F}$; hence $x \in J_{\mathcal{F}}$, $U \cap J_{\mathcal{F}} \neq \emptyset$, and $U \in \mathcal{F}_{J_{\mathcal{F}}}$.

If $x \in J_{\mathcal{F}_J}$ then $\uparrow x \in \mathcal{F}_J$ and we have a $y \in J$ such that $y \in \uparrow x$, that is, $x \leq y$, and $x \in J$. On the other hand, if $x \in J$ then $\uparrow x \in \mathcal{F}_J$ and $x \in J_{\mathcal{F}_J}$. $\square$

6.4.3. Lemma. *A c.p. filter $\mathcal{F}$ is a $\mathcal{U}(x) = \{U \mid x \in U\}$ if and only if x is the maximum of $J_{\mathcal{F}}$.*

Proof. If $\mathcal{F} = \mathcal{U}(x)$ then for all $U \in \mathcal{F}$, $U \supseteq \uparrow x$, and in particular for $y \in J_{\mathcal{F}}$, $x \geq y$. If x is the maximum of $J_{\mathcal{F}}$ then $U \in \mathcal{F}$ iff $\exists y \in J_{\mathcal{F}},\ y \in U$ iff $x \in U$. $\square$

6.4.4. Recall that a poset $(X,\le)$ is *Noetherian* if there is no strictly increasing sequence
$$a_1 < a_2 < \cdots < a_n < \cdots . \qquad (*)$$

Proposition. $\mathfrak{Up}(X,\le)$ *is sober if and only if* $(X,\le)$ *is Noetherian.*

Proof. It immediately follows from 6.4.3. Each ideal has a maximum iff there is no sequence of the form $(*)$: if there were we had the ideal $\{x \mid \exists i,\ x \le a_i\}$. $\square$

6.4.5. Thus, the sober reflection of a poset $(X,\le)$ (more precisely, of its Alexandroff space) can be represented as
$$(\mathsf{Idl}(X,\le),\widetilde{\tau})$$
where $\mathsf{Idl}(X,\le)$ is the set of ideals of $(X,\le)$ and $\widetilde{\tau}$ consists of the $\widetilde{U}=\{J \mid U\cap J \ne \emptyset\}$ with $U \in \mathfrak{Up}(X,\le)$ (use 6.4.2: Σ_U translates to $\{J \mid \mathcal{F}_J \ni U\} = \{J \mid U\cap J \ne \emptyset\}$). Note that $x\in U$ iff ${\downarrow}x \in \widetilde{U}$.

6.4.6. Proposition. $\widetilde{\tau}$ *is the Scott topology of* $(\mathsf{Idl}(X,\le),\subseteq)$.

Proof. Obviously each $\widetilde{U}$ is an up-set in $\subseteq$. Let $J = \bigvee_i^{\uparrow} J_i \in \widetilde{U}$. The directed supremum $\bigvee_i^{\uparrow} J_i$ of ideals is obviously the union $\bigcup_i J_i$, hence $\bigcup_i J_i \cap U \ne \emptyset$ and there is an i such that $J_i \cap U \ne \emptyset$. Thus, $\widetilde{U}$ is Scott open.

Now let $\mathcal{U}$ be Scott open in $(\mathsf{Idl}(X,\le),\subseteq)$. Set
$$U = \{x \mid {\downarrow}x \in \mathcal{U}\}.$$
Let $J \in \mathcal{U}$. We have $J = \bigvee^{\uparrow}\{{\downarrow}x \mid x \in J\}$, hence there is an $x \in J$ such that ${\downarrow}x \in \mathcal{U}$; this x is in $J\cap U$, and ${\downarrow}x \in U$. If J is in $\widetilde{U}$ then $J \cap U \ne \emptyset$ so there is an $x\in J$ with ${\downarrow}x\in\mathcal{U}$; but $J \supseteq {\downarrow}x$, and hence $J \in \mathcal{U}$. $\square$

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