Chapter 2

The Morse Indices in Lagrangian Dynamics

In this chapter we introduce the Morse index and nullity for periodic orbits of Euler-Lagrange systems of Tonelli type, an orbit being regarded as an extremal point of the action functional. These indices were first introduced by Morse [Mor96] in the study of closed geodesics. As we will see in the forthcoming chapters, they play a crucial role in the proof of existence and multiplicity results of periodic orbits. In Section 2.1 we give the definition of Morse index and nullity of a periodic orbit, and we prove that they are always finite. In Section 2.2 we outline the beautiful iteration theory of Bott, which studies the behavior of the Morse indices as a periodic orbit is iterated. Finally, in Section 2.3 we describe the relation between the Morse index and the Maslov index from symplectic geometry, which is an index for periodic orbits of general Hamiltonian systems.

2.1 The Morse index and nullity

Let us consider a closed manifold M of dimension $m \geq 1$ with a fixed Riemannian metric $\langle \cdot, \cdot \rangle_\cdot$, and 1-periodic Tonelli Lagrangian $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$. We denote by $\mathscr{A}$ the associated action defined on the space of 1-periodic curves $\gamma : \mathbb{R}/\mathbb{Z} \to M$ of class C^2, given by

$$\mathscr{A}(\gamma) = \int_0^1 \mathscr{L}(t, \gamma(t), \dot\gamma(t))\,\mathrm{d}t, \qquad \forall \gamma \in C^2(\mathbb{R}/\mathbb{Z}; M).$$

The extremal points of this functional are precisely the 1-periodic solutions of the Euler-Lagrange system of $\mathscr{L}$ (cf. Section 1.1). We want to compute the second

M. Mazzucchelli, *Critical Point Theory for Lagrangian Systems*, Progress in Mathematics 293,
DOI 10.1007/978-3-0348-0163-8_2, © Springer Basel AG 2012

variation of the functional $\mathscr{A}$ at an extremal γ. Hence, let us consider a smooth section σ of the pull-back bundle $\gamma^*\mathrm{T}M$. Notice that σ is 1-periodic, being a map of the form $\sigma : \mathbb{R}/\mathbb{Z} \to \gamma^*\mathrm{T}M$. We can use this section to define a homotopy $\Sigma : (-\varepsilon, \varepsilon) \times \mathbb{R}/\mathbb{Z} \to M$ of γ as

$$\Sigma(s,t) := \exp_{\gamma(t)}(s\,\sigma(t)), \qquad \forall (s,t) \in (-\varepsilon,\varepsilon) \times \mathbb{R}/\mathbb{Z},$$

where $\varepsilon > 0$ is a sufficiently small real constant and exp is the exponential map associated to a Riemannian metric on M. The section σ can be reobtained by differentiating the homotopy Σ in the s direction at $s = 0$, for

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \Sigma(s,t) = \mathrm{d}\exp_{\gamma(t)}(0)(\sigma(t)) = \sigma(t) \qquad \forall t \in \mathbb{R}/\mathbb{Z}.$$

If we consider a finite atlas $\mathfrak{U} = \{\phi_\alpha : U_\alpha \to \mathbb{R}^m \mid \alpha = 0, \dots, u\}$ for M, as in section 1.1, we can find a subdivision $0 = r_0 < r_1 < \dots < r_n = 1$ such that the support $\gamma([r_k, r_{k+1}])$ is contained in some coordinate domain U_{α_k}, for each $k = 0, \dots, n-1$. We define

$$\begin{aligned}
\mathscr{B}_\gamma(\sigma) &:= \left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_{s=0} \mathscr{A}(\Sigma(s,t)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \sum_{k=0}^{n-1} \sum_{j=1}^{m} \int_{r_k}^{r_{k+1}} \left[\frac{\partial \mathscr{L}}{\partial v^j_{\alpha_k}} \left(t, \Sigma(s,t), \frac{\partial \Sigma}{\partial t}(s,t) \right) \frac{\partial^2 \Sigma^j_{\alpha_k}}{\partial s\, \partial t}(s,t) \right. \\
&\quad \left. + \frac{\partial \mathscr{L}}{\partial q^j_{\alpha_k}} \left(t, \Sigma(s,t), \frac{\partial \Sigma}{\partial t}(s,t) \right) \frac{\partial \Sigma^j_{\alpha_k}}{\partial s}(s,t) \right] \mathrm{d}t \\
&= \sum_{k=0}^{n-1} \sum_{j,h=1}^{m} \int_{r_k}^{r_{k+1}} \left[\frac{\partial^2 \mathscr{L}}{\partial v^h_{\alpha_k} \partial v^j_{\alpha_k}}(t,\gamma,\dot\gamma) \dot\sigma^j_{\alpha_k} \dot\sigma^h_{\alpha_k} \right. \\
&\quad \left. + 2 \frac{\partial^2 \mathscr{L}}{\partial v^h_{\alpha_k} \partial q^j_{\alpha_k}}(t,\gamma,\dot\gamma) \sigma^j_{\alpha_k} \dot\sigma^h_{\alpha_k} + \frac{\partial^2 \mathscr{L}}{\partial q^h_{\alpha_k} \partial q^j_{\alpha_k}}(t,\gamma,\dot\gamma) \sigma^j_{\alpha_k} \sigma^h_{\alpha_k} \right] \mathrm{d}t \in \mathbb{R}.
\end{aligned}$$

Notice that $\mathscr{B}_\gamma(\sigma)$ is independent of the particular choice of the homotopy Σ, and for each $r \in \mathbb{R}$ we have $\mathscr{B}_\gamma(r\sigma) = r^2 \mathscr{B}_\gamma(\sigma)$. This shows that $\mathscr{B}_\gamma$ is a well-defined quadratic form, and by polarization we can define the symmetric bilinear form

$$\begin{aligned}
\mathscr{B}_\gamma(\sigma,\xi) &:= \frac{1}{4}\Big[\mathscr{B}_\gamma(\sigma+\xi) - \mathscr{B}_\gamma(\sigma-\xi) \Big] \qquad (2.1) \\
&= \sum_{k=0}^{n-1} \sum_{j,h=1}^{m} \int_{r_k}^{r_{k+1}} \left[\frac{\partial^2 \mathscr{L}}{\partial v^h_{\alpha_k} \partial v^j_{\alpha_k}}(t,\gamma,\dot\gamma) \dot\sigma^j_{\alpha_k} \dot\xi^h_{\alpha_k} + \frac{\partial^2 \mathscr{L}}{\partial v^h_{\alpha_k} \partial q^j_{\alpha_k}}(t,\gamma,\dot\gamma) \sigma^j_{\alpha_k} \dot\xi^h_{\alpha_k} \right. \\
&\quad \left. + \frac{\partial^2 \mathscr{L}}{\partial q^h_{\alpha_k} \partial v^j_{\alpha_k}}(t,\gamma,\dot\gamma) \dot\sigma^j_{\alpha_k} \xi^h_{\alpha_k} + \frac{\partial^2 \mathscr{L}}{\partial q^h_{\alpha_k} \partial q^j_{\alpha_k}}(t,\gamma,\dot\gamma) \sigma^j_{\alpha_k} \xi^h_{\alpha_k} \right] \mathrm{d}t,
\end{aligned}$$

where σ and ξ are smooth sections of $\gamma^*\mathrm{T}M$. The above expression still makes sense if we only require that σ and ξ have $W^{1,2}$ regularity[1], and actually $\mathscr{B}_\gamma$ extends to a bounded symmetric bilinear form

$$\mathscr{B}_\gamma : W^{1,2}(\gamma^*\mathrm{T}M) \times W^{1,2}(\gamma^*\mathrm{T}M) \to \mathbb{R}.$$

Here we have denoted by $W^{1,2}(\gamma^*\mathrm{T}M)$ the space of $W^{1,2}$ sections of $\gamma^*\mathrm{T}M$. This is a Hilbert space with inner product

$$\langle\!\langle \xi, \zeta \rangle\!\rangle_{W^{1,2}} := \int_0^1 \Big[\langle \xi(t), \zeta(t) \rangle_{\gamma(t)} + \langle \nabla_t \xi, \nabla_t \zeta \rangle_{\gamma(t)} \Big] \, \mathrm{d}t, \qquad \forall \xi, \zeta \in W^{1,2}(\gamma^*\mathrm{T}M),$$

where ∇_t denotes the covariant derivative of the Riemannian manifold $(M, \langle \cdot, \cdot \rangle.)$.

Before studying the properties of $\mathscr{B}_\gamma$, let us recall some definitions concerning symmetric bilinear forms over a Hilbert space. Let $\boldsymbol{E}$ be a real separable Hilbert space with inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{E}}$ and norm $\| \cdot \|_{\boldsymbol{E}}$. We denote by $\mathrm{Bil}(\boldsymbol{E})$ the set of bounded (or, equivalently, continuous) bilinear forms on $\boldsymbol{E}$, which is a Banach space with norm

$$\|\mathscr{B}\|_{\mathrm{Bil}(\boldsymbol{E})} = \max \left\{ \mathscr{B}(\boldsymbol{v}, \boldsymbol{w}) \mid \|\boldsymbol{v}\|_{\boldsymbol{E}} = \|\boldsymbol{w}\|_{\boldsymbol{E}} = 1 \right\}, \qquad \forall \mathscr{B} \in \mathrm{Bil}(\boldsymbol{E}).$$

A bounded symmetric bilinear form $\mathscr{B} : \boldsymbol{E} \times \boldsymbol{E} \to \mathbb{R}$ is called **Fredholm** when the unique bounded self-adjoint linear operator B on $\boldsymbol{E}$, given by

$$\mathscr{B}(\boldsymbol{v}, \boldsymbol{w}) = \langle\!\langle B\boldsymbol{v}, \boldsymbol{w} \rangle\!\rangle_{\boldsymbol{E}}, \qquad \forall \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{E}, \tag{2.2}$$

is Fredholm (see section A.2). We define the **Morse index** $\mathrm{ind}(\mathscr{B})$ as the supremum of the dimension of the vector subspaces of $\boldsymbol{E}$ on which $\mathscr{B}$ is negative-definite. Analogously, we define the **nullity** $\mathrm{nul}(\mathscr{B})$ as the dimension of the kernel of the associated operator B, and the **large Morse index** $\mathrm{ind}^*(\mathscr{B})$ as $\mathrm{ind}(\mathscr{B}) + \mathrm{nul}(\mathscr{B})$.

Lemma 2.1.1. *The Morse index* [*resp. the large Morse index*] *is lower semi-continuous* [*resp. upper semi-continuous*] *over the space of Fredholm symmetric bilinear forms on* $\boldsymbol{E}$.

Proof. Let $\mathscr{B}$ be a Fredholm symmetric bilinear form on $\boldsymbol{E}$, with associated operator B as in (2.2). By the spectral theorem, this operator induces an orthogonal splitting $\boldsymbol{E} = \boldsymbol{E}^0 \oplus \boldsymbol{E}^- \oplus \boldsymbol{E}^+$, where $\boldsymbol{E}^0$ is the kernel of B, and $\boldsymbol{E}^-$ [resp. $\boldsymbol{E}^+$] is a subspace of $\boldsymbol{E}$ over which B is negative-definite [resp. positive-definite]. Notice that the dimension of $\boldsymbol{E}^0$, $\boldsymbol{E}^-$ and $\boldsymbol{E}^+$ is respectively $\mathrm{nul}(\mathscr{B})$, $\mathrm{ind}(\mathscr{B})$ and $\mathrm{ind}(-\mathscr{B})$.

Since B is a Fredholm operator, $0 \in \mathbb{R}$ is an isolated point in the spectrum of B. In fact, the subspace $\boldsymbol{E}^0$ is finite-dimensional, so that the quotient $\boldsymbol{E}/\boldsymbol{E}^0$ is still a Hilbert space, and B induces a bounded self-adjoint linear operator $\bar{B}$ on $\boldsymbol{E}/\boldsymbol{E}^0$ that is bijective. By the inverse mapping theorem, $\bar{B}$ is an isomorphism.

[1]See Section 3.1 for the background on $W^{1,2}$ sections of the pull-back bundle $\gamma^*\mathrm{T}M$.

Therefore, since the spectrum of B is the disjoint union of the spectrum of $\bar{B}$ and $\{0\}$, the claim follows. This implies that there exists a constant $c_B > 0$ such that

$$\begin{aligned} \mathscr{B}(\boldsymbol{v},\boldsymbol{v}) &\geq c_B \|\boldsymbol{v}\|^2_{\boldsymbol{E}}, && \forall \boldsymbol{v} \in \boldsymbol{E}^+, \\ \mathscr{B}(\boldsymbol{v},\boldsymbol{v}) &\leq -c_B \|\boldsymbol{v}\|^2_{\boldsymbol{E}}, && \forall \boldsymbol{v} \in \boldsymbol{E}^-. \end{aligned}$$

Now, consider an arbitrary Fredholm symmetric bilinear form $\mathscr{B}'$ such that $\|\mathscr{B}' - \mathscr{B}\|_{\mathrm{Bil}(\boldsymbol{E})} < c_B$. For each $\boldsymbol{v} \in \boldsymbol{E}^- \setminus \{\boldsymbol{0}\}$ and $\boldsymbol{w} \in \boldsymbol{E}^+ \setminus \{\boldsymbol{0}\}$, we have

$$\begin{aligned} \mathscr{B}'(\boldsymbol{v},\boldsymbol{v}) &\leq \mathscr{B}(\boldsymbol{v},\boldsymbol{v}) + \|\mathscr{B}' - \mathscr{B}\|_{\mathrm{Bil}(\boldsymbol{E})}\, \|\boldsymbol{v}\|^2_{\boldsymbol{E}} < (-c_B + c_B)\|\boldsymbol{v}\|^2_{\boldsymbol{E}} = 0, \\ \mathscr{B}'(\boldsymbol{w},\boldsymbol{w}) &\geq \mathscr{B}(\boldsymbol{w},\boldsymbol{w}) - \|\mathscr{B}' - \mathscr{B}\|_{\mathrm{Bil}(\boldsymbol{E})}\, \|\boldsymbol{w}\|^2_{\boldsymbol{E}} > (c_B - c_B)\|\boldsymbol{w}\|^2_{\boldsymbol{E}} = 0. \end{aligned}$$

Therefore, $\mathscr{B}'$ is still negative-definite on $\boldsymbol{E}^-$ and positive-definite on $\boldsymbol{E}^+$. This implies that $\mathrm{ind}(\mathscr{B}') \geq \mathrm{ind}(\mathscr{B})$ and $\mathrm{ind}(-\mathscr{B}') \geq \mathrm{ind}(-\mathscr{B})$, which forces $\mathrm{ind}^*(\mathscr{B}') \leq \mathrm{ind}^*(\mathscr{B})$. □

Lemma 2.1.2. *Consider a bounded symmetric bilinear form $\mathscr{B} = \mathscr{P} + \mathscr{K}$, where:*

- $\mathscr{P} = \langle\langle P\cdot,\cdot\rangle\rangle_{\boldsymbol{E}} : \boldsymbol{E} \times \boldsymbol{E} \to \mathbb{R}$ *is a* **semipositive-definite** *symmetric bilinear Fredholm form, i.e., $\mathscr{P}(\boldsymbol{v},\boldsymbol{v}) \geq 0$ for each $\boldsymbol{v} \in \boldsymbol{E}$,*
- $\mathscr{K} = \langle\langle K\cdot,\cdot\rangle\rangle_{\boldsymbol{E}} : \boldsymbol{E} \times \boldsymbol{E} \to \mathbb{R}$ *is a* **compact** *symmetric bilinear form, i.e., its associated self-adjoint operator K is compact.*

Then $\mathscr{B}$ is a Fredholm bilinear form and its Morse index $\mathrm{ind}(\mathscr{B})$ *is finite.*

Proof. First of all, notice that the self-adjoint bounded operator B associated to $\mathscr{B}$ is given by $P + K$. In particular, B is a compact perturbation of the Fredholm operator P, and therefore it is Fredholm (see, e.g., [Arv02, Section 3.3]). This proves that $\mathscr{B}$ is a Fredholm form. Now, consider the orthogonal splittings of $\boldsymbol{E}$ induced by $\mathscr{B}$, $\mathscr{P}$ and $\mathscr{K}$, i.e.,

$$\boldsymbol{E} = \boldsymbol{E}^0_{\mathscr{B}} \oplus \boldsymbol{E}^+_{\mathscr{B}} \oplus \boldsymbol{E}^-_{\mathscr{B}} = \boldsymbol{E}^0_{\mathscr{P}} \oplus \boldsymbol{E}^+_{\mathscr{P}} = \boldsymbol{E}^0_{\mathscr{K}} \oplus \boldsymbol{E}^+_{\mathscr{K}} \oplus \boldsymbol{E}^-_{\mathscr{K}}.$$

Here, $\boldsymbol{E}^0_{\mathscr{B}}$, $\boldsymbol{E}^0_{\mathscr{P}}$ and $\boldsymbol{E}^0_{\mathscr{K}}$ are the kernels of B, P and K respectively, and

$$\begin{aligned} &\pm \mathscr{B}(\boldsymbol{v},\boldsymbol{v}) > 0 && \forall \boldsymbol{v} \in \boldsymbol{E}^\pm_{\mathscr{B}} \setminus \{\boldsymbol{0}\}, \\ &\pm \mathscr{K}(\boldsymbol{v},\boldsymbol{v}) > 0 && \forall \boldsymbol{v} \in \boldsymbol{E}^\pm_{\mathscr{K}} \setminus \{\boldsymbol{0}\}, \\ &\mathscr{P}(\boldsymbol{v},\boldsymbol{v}) \geq c_P\, \|\boldsymbol{v}\|^2_{\boldsymbol{E}} && \forall \boldsymbol{v} \in \boldsymbol{E}^+_{\mathscr{P}}, \end{aligned}$$

where $c_P > 0$ is a constant determined by the spectrum of the Fredholm operator P (see the proof of Lemma 2.1.1). We can further express the negative eigenspace $\boldsymbol{E}^-_{\mathscr{B}}$ as the orthogonal direct sum

$$\boldsymbol{E}^-_{\mathscr{B}} = (\boldsymbol{E}^-_{\mathscr{B}} \cap \boldsymbol{E}^0_{\mathscr{P}}) \oplus (\boldsymbol{E}^-_{\mathscr{B}} \cap \boldsymbol{E}^+_{\mathscr{P}}),$$

where the first direct summand $\boldsymbol{E}^-_{\mathscr{B}} \cap \boldsymbol{E}^0_{\mathscr{P}}$ is finite-dimensional (since $\mathscr{P}$ is a Fredholm form). Hence, in order to conclude we only need to show that the second summand $\boldsymbol{E}^-_{\mathscr{B}} \cap \boldsymbol{E}^+_{\mathscr{P}}$ is finite-dimensional as well. This is easily proved as follows. For each $\boldsymbol{v} \in (\boldsymbol{E}^-_{\mathscr{B}} \cap \boldsymbol{E}^+_{\mathscr{P}})$, we have

$$0 \geq \mathscr{B}(\boldsymbol{v},\boldsymbol{v}) = \mathscr{P}(\boldsymbol{v},\boldsymbol{v}) + \mathscr{K}(\boldsymbol{v},\boldsymbol{v}) \geq c_P \, \|\boldsymbol{v}\|^2_{\boldsymbol{E}} + \mathscr{K}(\boldsymbol{v},\boldsymbol{v}),$$

and therefore

$$\boldsymbol{E}^-_{\mathscr{B}} \cap \boldsymbol{E}^+_{\mathscr{P}} \subseteq \boldsymbol{E}^{\leq -c_P}_{\mathscr{K}} := \{\boldsymbol{v} \in \boldsymbol{E} \,|\, \mathscr{K}(\boldsymbol{v},\boldsymbol{v}) \leq -c_P \, \|\boldsymbol{v}\|^2_{\boldsymbol{E}}\}.$$

Notice that $\boldsymbol{E}^{\leq -c_P}_{\mathscr{K}}$ is the direct sum of the eigenspaces of K corresponding to the eigenvalues less than or equal to $-c_P$. Since K is a compact operator, each of these eigenspaces is finite-dimensional. Moreover, by the spectral theorem for compact self-adjoint operators (see, e.g., [Mac09, Section 4.3]), K admits only a finite number of distinct eigenvalues less than or equal to $-c_P$. This proves that $\boldsymbol{E}^{\leq -c_P}_{\mathscr{K}}$ is finite-dimensional, which in turn implies that $\boldsymbol{E}^-_{\mathscr{B}} \cap \boldsymbol{E}^+_{\mathscr{P}}$ is finite-dimensional. □

After these preliminaries, let us go back to consider the form $\mathscr{B}_\gamma$ defined in (2.1). In order to simplify the notation, we denote its Morse index $\mathrm{ind}(\mathscr{B}_\gamma)$ by $\mathrm{ind}(\gamma)$ and its nullity $\mathrm{nul}(\mathscr{B}_\gamma)$ by $\mathrm{nul}(\gamma)$. In the following chapters, we will also write them as $\mathrm{ind}(\mathscr{A},\gamma)$ and $\mathrm{nul}(\mathscr{A},\gamma)$ whenever we need to keep track of the action functional with respect to which γ is an extremal.

Remark 2.1.1. In the next chapter we will see that, for a certain subclass of Tonelli Lagrangians $\mathscr{L}$ and for a suitable functional setting for the Lagrangian action $\mathscr{A}$, the bilinear form $\mathscr{B}_\gamma$ is the Hessian (in the Gâteaux or Fréchet sense, depending on the properties of $\mathscr{L}$) of $\mathscr{A}$ at γ. Hence, in that case, the indices $\mathrm{ind}(\gamma)$ and $\mathrm{nul}(\gamma)$ are respectively the Morse index and the nullity of the functional $\mathscr{A}$ at the critical point γ (see Section A.1). By abuse of terminology, for any Tonelli Lagrangian $\mathscr{L}$, from now on we will always call $\mathrm{ind}(\gamma)$ and $\mathrm{nul}(\gamma)$ the Morse index and nullity of γ. □

Proposition 2.1.3. *The symmetric bilinear form $\mathscr{B}_\gamma$ is Fredholm and the Morse index $\mathrm{ind}(\gamma) = \mathrm{ind}(\mathscr{B}_\gamma)$ is finite.*

Proof. Consider the bounded bilinear forms

$$\begin{aligned}
\mathscr{C} &: W^{1,2}(\gamma^*\mathrm{T}M) \times W^{1,2}(\gamma^*\mathrm{T}M) \to \mathbb{R},\\
\mathscr{K}_0 &: L^2(\gamma^*\mathrm{T}M) \times L^2(\gamma^*\mathrm{T}M) \to \mathbb{R},\\
\mathscr{K}' &: L^2(\gamma^*\mathrm{T}M) \times W^{1,2}(\gamma^*\mathrm{T}M) \to \mathbb{R},\\
\mathscr{K}'' &: W^{1,2}(\gamma^*\mathrm{T}M) \times L^2(\gamma^*\mathrm{T}M) \to \mathbb{R},\\
\mathscr{K}''' &: L^2(\gamma^*\mathrm{T}M) \times L^2(\gamma^*\mathrm{T}M) \to \mathbb{R},
\end{aligned}$$

given by

$$\mathscr{C}(\sigma,\xi) = \sum_{k=0}^{n-1} \sum_{j,h=1}^{m} \int_{r_k}^{r_{k+1}} \frac{\partial^2 \mathscr{L}}{\partial v^h_{\alpha_k} \partial v^j_{\alpha_k}} (t,\gamma(t),\dot\gamma(t))\, \dot\sigma^j_{\alpha_k}(t)\, \dot\xi^h_{\alpha_k}(t)\, \mathrm{d}t,$$

$$\begin{aligned}
\mathscr{K}_0(\zeta,\chi) &= \int_0^1 \langle \zeta(t), \chi(t)\rangle_{\gamma(t)}\,\mathrm{d}t,\\
\mathscr{K}'(\zeta,\sigma) &= \sum_{k=0}^{n-1}\sum_{j,h=1}^{m}\int_{r_k}^{r_{k+1}} \frac{\partial^2\mathscr{L}}{\partial v^h_{\alpha_k}\,\partial q^j_{\alpha_k}}(t,\gamma(t),\dot\gamma(t))\,\zeta^j_{\alpha_k}(t)\,\dot\sigma^h_{\alpha_k}(t)\,\mathrm{d}t,\\
\mathscr{K}''(\sigma,\zeta) &= \sum_{k=0}^{n-1}\sum_{j,h=1}^{m}\int_{r_k}^{r_{k+1}} \frac{\partial^2\mathscr{L}}{\partial q^h_{\alpha_k}\,\partial v^j_{\alpha_k}}(t,\gamma(t),\dot\gamma(t))\,\dot\sigma^j_{\alpha_k}(t)\,\zeta^h_{\alpha_k}(t)\,\mathrm{d}t,\\
\mathscr{K}'''(\zeta,\chi) &= \sum_{k=0}^{n-1}\sum_{j,h=1}^{m}\int_{r_k}^{r_{k+1}} \frac{\partial^2\mathscr{L}}{\partial q^h_{\alpha_k}\,\partial q^j_{\alpha_k}}(t,\gamma(t),\dot\gamma(t))\,\zeta^j_{\alpha_k}(t)\,\chi^h_{\alpha_k}(t)\,\mathrm{d}t,
\end{aligned}$$

for each $\sigma,\xi \in W^{1,2}(\gamma^*\mathrm{T}M)$ and $\zeta,\chi \in L^2(\gamma^*\mathrm{T}M)$. By the Rellich-Kondrachov theorem (see [AF03, page 168]), the embedding $W^{1,2}(\gamma^*\mathrm{T}M) \hookrightarrow L^2(\gamma^*\mathrm{T}M)$ is compact. This implies that $\mathscr{K}_0$, $\mathscr{K}'$, $\mathscr{K}''$ and $\mathscr{K}'''$, restricted as bilinear forms

$$W^{1,2}(\gamma^*\mathrm{T}M) \times W^{1,2}(\gamma^*\mathrm{T}M) \to \mathbb{R},$$

are compact (meaning that their associated bounded operator on $W^{1,2}(\gamma^*\mathrm{T}M)$ is compact). The symmetric bilinear form $\mathscr{C} + \mathscr{K}_0$ is Fredholm with

$$\mathrm{ind}(\mathscr{C} + \mathscr{K}_0) = \mathrm{nul}(\mathscr{C} + \mathscr{K}_0) = 0.$$

In fact, it is easy to see that $\mathscr{C} + \mathscr{K}_0$ is a positive-definite bounded symmetric bilinear form. Therefore its associated operator on $W^{1,2}(\gamma^*\mathrm{T}M)$ is an isomorphism, and in particular it is Fredholm. Notice that the forms $\mathscr{K}_0$ and $\mathscr{K}' + \mathscr{K}'' + \mathscr{K}'''$ are both symmetric and compact, which implies that $\mathscr{B}_\gamma$ is a symmetric compact perturbation of a positive-definite symmetric Fredholm form, for

$$\mathscr{B}_\gamma = (\mathscr{C} + \mathscr{K}_0) + (\mathscr{K}' + \mathscr{K}'' + \mathscr{K}''' - \mathscr{K}_0).$$

Applying the abstract Lemma 2.1.2, with $\boldsymbol{E} = W^{1,2}(\gamma^*\mathrm{T}M)$, $\mathscr{P} = \mathscr{C} + \mathscr{K}_0$ and $\mathscr{K} = \mathscr{K}' + \mathscr{K}'' + \mathscr{K}''' - \mathscr{K}_0$, we obtain the claim. □

The Morse indices are often used to distinguish among periodic orbits that have been detected by abstract methods. To this purpose, it is useful to have some a priori estimates for them, such as the one given by the following statement due to Abbondandolo and Figalli [AF07, Lemma 2.2].

Proposition 2.1.4. *Assume that the Euler-Lagrange flow $\Phi_{\mathscr{L}}$ of $\mathscr{L}$ is global (see Section 1.1). Then for each $a \in \mathbb{R}$ there exists $j = j(\mathscr{L}, a) \in \mathbb{N}$ such that, for each 1-periodic solution γ of the Euler-Lagrange system of $\mathscr{L}$ with $\mathscr{A}(\gamma) \leq a$, we have $\mathrm{ind}(\gamma) + \mathrm{nul}(\gamma) \leq j$.*

Proof. For each $\zeta \in C^1(\mathbb{R}/\mathbb{Z}; M)$, we can define a bilinear form $\mathscr{B}_\zeta$ as in equation (2.1). We introduce the smooth infinite-dimensional vector bundle

$$\pi : \mathscr{E} \to C^1(\mathbb{R}/\mathbb{Z}; M)$$

such that, for each $\zeta \in C^1(\mathbb{R}/\mathbb{Z}; M)$, the fiber $\pi^{-1}(\zeta)$ is the space of Fredholm symmetric bilinear forms on $W^{1,2}(\zeta^*\mathrm{T}M)$. By the dominated convergence theorem, the map $\zeta \mapsto \mathscr{B}_\zeta$ is a continuous section of this vector bundle. Now, we claim that the set of 1-periodic orbits with action less than or equal to a are compact in the C^1 topology. By the upper semi-continuity of the large Morse index (Lemma 2.1.1), the thesis of the proposition follows. All we need to do in order to conclude the proof is to establish the claim.

Consider γ as in the statement. Since $\mathscr{A}(\gamma) \leq a$, there exists $t_0 \in [0,1]$ such that

$$\mathscr{L}(t_0, \gamma(t_0), \dot\gamma(t_0)) \leq a. \tag{2.3}$$

By the uniform fiberwise superlinearity of $\mathscr{L}$ (see Remark 1.2.1) there exists a real constant $C(\mathscr{L}) > 0$ such that

$$\mathscr{L}(t,q,v) \geq |v|_q - C(\mathscr{L}), \qquad \forall (t,q,v) \in \mathbb{R}/\mathbb{Z} \times \mathrm{T}M. \tag{2.4}$$

Since we are assuming that the Euler-Lagrange flow $\Phi_{\mathscr{L}}$ is global, we can define a compact subset of $\mathrm{T}M$ by

$$K = K(a) = \left\{ \Phi_{\mathscr{L}}^{t,s}(q,v) \in \mathrm{T}M \,\middle|\, t,s \in [0,1],\ (q,v) \in \mathrm{T}M,\ |v|_q \leq a + C(\mathscr{L}) \right\}.$$

By (2.3) and (2.4), $|\dot\gamma(t_0)|_{\gamma(t_0)} \leq a + C(\mathscr{L})$ and therefore, for each $t \in [0,1]$, we have that $(\gamma(t), \dot\gamma(t)) = \Phi_{\mathscr{L}}^{t,t_0}(\gamma(t_0), \dot\gamma(t_0))$ belongs to the compact set K. By the compactness of K and the arbitrariness of the choice of γ, we obtain a uniform C^1 bound for the 1-periodic orbits with action less than or equal to a. This immediately implies also a uniform C^2 bound, since, for each γ as above, the lifted orbit $(\gamma, \dot\gamma)$ is a 1-periodic integral curve of the (smooth and 1-periodic in time) Euler-Lagrange vector field $X_{\mathscr{L}}$, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma(t), \dot\gamma(t)) = X_{\mathscr{L}}(t, \gamma(t), \dot\gamma(t)), \qquad \forall t \in \mathbb{R}/\mathbb{Z}.$$

This uniform C^2 bound, together with the Arzelà-Ascoli theorem, implies that the set of 1-periodic orbits with action less than or equal to a is compact in the C^1 topology. □

2.2 Bott's iteration theory

Consider again the 1-periodic solution γ of the Euler-Lagrange system of $\mathscr{L}$. For each $n \in \mathbb{N}$, we denote by $\gamma^{[n]} : \mathbb{R}/n\mathbb{Z} \to M$ its nth **iteration** defined as the composition of γ with the n-fold covering map of the circle $\mathbb{R}/n\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, i.e.,

$$\gamma^{[n]}(k+t) = \gamma(t), \qquad \forall k \in \{0, \ldots, n-1\},\ t \in [0,1].$$

Geometrically, γ and $\gamma^{[n]}$ are the same curve. However, at a formal level they are different objects, and indeed $\gamma^{[n]}$ is an extremal point of the Lagrangian action $\mathscr{A}^{[n]}$ in period n, given by

$$\mathscr{A}^{[n]}(\zeta) = \frac{1}{n}\int_0^n \mathscr{L}(t,\zeta(t),\dot\zeta(t))\,\mathrm{d}t.$$

Here, we consider $\mathscr{A}^{[n]}$ as a functional defined on a space of n-periodic curves $\zeta : \mathbb{R}/n\mathbb{Z} \to M$ with suitable regularity (for instance C^2). As we did in the previous section, we can introduce the bilinear form $\mathscr{B}_{\gamma^{[n]}}$ representing the second variation of $\mathscr{A}^{[n]}$ at $\gamma^{[n]}$, the Morse index $\mathrm{ind}(\gamma^{[n]})$ and the nullity $\mathrm{nul}(\gamma^{[n]})$. In the 1950s, Bott [Bot56] studied the behavior of these indices as the period n varies. Here, we wish to outline the part of his beautiful iteration theory that is relevant in Lagrangian dynamics, more specifically to the problem of the multiplicity of periodic orbits with unprescribed period (see Section 6.5)

For each integer $n \in \mathbb{N}$, consider the Hilbert space $\boldsymbol{E}^{[n]} := W^{1,2}((\gamma^{[n]})^*\mathrm{T}M)$ with inner product

$$\langle\!\langle \xi,\zeta\rangle\!\rangle_{\boldsymbol{E}^{[n]}} := \frac{1}{n}\int_0^n \Big[\langle \xi(t),\zeta(t)\rangle_{\gamma(t)} + \langle \nabla_t\xi,\nabla_t\zeta\rangle_{\gamma(t)}\Big]\,\mathrm{d}t, \qquad \forall \xi,\zeta\in\boldsymbol{E}^{[n]},$$

where ∇_t denotes the covariant derivative of the Riemannian manifold $(M,\langle\cdot,\cdot\rangle_\cdot)$. Let $B^{[n]}$ be the self-adjoint bounded operator on $\boldsymbol{E}^{[n]}$ associated to the bilinear form $\mathscr{B}_{\gamma^{[n]}}$, i.e.,

$$\mathscr{B}_{\gamma^{[n]}}(\sigma,\xi) = \langle\!\langle B^{[n]}\sigma,\xi\rangle\!\rangle_{\boldsymbol{E}^{[n]}}, \qquad \forall \sigma,\xi\in\boldsymbol{E}^{[n]}.$$

Notice that, for each 1-periodic section $\sigma \in \boldsymbol{E}^{[1]}$, the nth iteration of the curve $B^{[1]}\sigma$ is precisely $B^{[n]}\sigma^{[n]}$, i.e.,

$$(B^{[1]}\sigma)^{[n]} = B^{[n]}\sigma^{[n]}. \tag{2.5}$$

Consider the orthogonal spectral decomposition of $\boldsymbol{E}^{[n]}$ associated to the operator $B^{[n]}$. The Morse index $\mathrm{ind}(\gamma^{[n]})$ and the nullity $\mathrm{nul}(\gamma^{[n]})$ are equal to the dimension of the negative eigenspace and of the kernel of $B^{[n]}$ respectively. Bott's idea was to study the eigenvalue problems associated to $B^{[n]}$ in the complexified setting $\boldsymbol{E}^{[n]}\otimes\mathbb{C}$: since $B^{[n]}$ is a real self-adjoint operator, its eigenvalues are all real and the dimension of its eigenspaces in $\boldsymbol{E}^{[n]}$ is the same as the complex dimension of its corresponding eigenspaces in $\boldsymbol{E}^{[n]}\otimes\mathbb{C}$. As we will see in a moment, in the complexified setting, the Fourier decomposition allows to reduce the analysis in period n to the one in period 1.

Let $S^1 \subset \mathbb{C}$ be the unit circle in the complex plane. For each $z\in S^1$ we define the space of complex sections

$$\boldsymbol{E}^{[n]}_z := \Big\{\sigma\in W^{1,2}(\mathbb{R};\mathrm{T}M\otimes\mathbb{C})\ \Big|\ \sigma(t)\in\mathrm{T}_{\gamma(t)}M\otimes\mathbb{C},\ \sigma(t+n)=z\sigma(t)\ \forall t\in\mathbb{R}\Big\}.$$

This is a complex Hilbert space with Hermitian inner product

$$\langle\!\langle \xi, \zeta \rangle\!\rangle_{\boldsymbol{E}_z^{[n]}} := \frac{1}{n} \int_0^n \left[\langle \xi(t), \overline{\zeta(t)} \rangle_{\gamma(t)} + \langle \nabla_t \xi, \overline{\nabla_t \zeta} \rangle_{\gamma(t)} \right] \mathrm{d}t, \qquad \forall \xi, \zeta \in \boldsymbol{E}_z^{[n]}.$$

In particular $\boldsymbol{E}_1^{[n]} = \boldsymbol{E}^{[n]} \otimes \mathbb{C}$, and the symmetric bilinear form $\mathscr{B}_{\gamma^{[n]}}$ can be extended as a Hermitian form on $\boldsymbol{E}_1^{[n]}$. By the same expression, for each $z \in S^1$ we can define the Hermitian bilinear form $\mathscr{B}_{\gamma^{[n]}}$ and the associated operator $B^{[n]}$ on $\boldsymbol{E}_z^{[n]}$.

Remark 2.2.1. Notice that $\boldsymbol{E}_z^{[1]}$ is a vector subspace of $\boldsymbol{E}_{z^n}^{[n]}$, hence the restriction of the operator $B^{[n]} : \boldsymbol{E}_{z^n}^{[n]} \to \boldsymbol{E}_{z^n}^{[n]}$ to $\boldsymbol{E}_z^{[1]}$ is precisely $B^{[1]} : \boldsymbol{E}_z^{[1]} \to \boldsymbol{E}_z^{[1]}$ (cf. equation (2.5)). □

For each $\lambda \in \mathbb{R}$ we consider the eigenvalue problem

$$\begin{cases} B^{[n]}\sigma = \lambda\sigma, \\ \sigma \in \boldsymbol{E}_z^{[n]}. \end{cases}$$

We denote by $\mathrm{ind}_{z,\lambda}(\gamma^{[n]})$ the complex dimension of the vector space of solutions of this eigenvalue problem, and we set

$$\begin{aligned} \mathrm{ind}_z(\gamma^{[n]}) &:= \sum_{\lambda<0} \mathrm{ind}_{z,\lambda}(\gamma^{[n]}), \\ \mathrm{nul}_z(\gamma^{[n]}) &:= \mathrm{ind}_{z,0}(\gamma^{[n]}). \end{aligned}$$

As for the standard Morse index, for each $z \in S^1$ and $\lambda \in \mathbb{R}$, the indices $\mathrm{ind}_{z,\lambda}(\gamma^{[n]})$ are finite. Moreover, for each $z \in S^1$, the set of those $\lambda \leq 0$ for which $\mathrm{ind}_{z,\lambda}(\gamma^{[n]}) \neq 0$ is finite (in order to verify this, the reader can simply check that the proof of Proposition 2.1.3 goes through in the complexified setting $\boldsymbol{E}_z^{[n]}$). Notice that the standard Morse index and nullity are a particular case of these indices, as

$$\begin{aligned} \mathrm{ind}(\gamma^{[n]}) &= \mathrm{ind}_1(\gamma^{[n]}), \\ \mathrm{nul}(\gamma^{[n]}) &= \mathrm{nul}_1(\gamma^{[n]}). \end{aligned}$$

The advantage of working with these more general indices is that the function $z \mapsto \mathrm{ind}_{z,\lambda}(\gamma)$ completely determines the indices $\mathrm{ind}_{z,\lambda}(\gamma^{[n]})$ in any period n. The precise relation is described in the following statement.

Lemma 2.2.1. *For each $n \in \mathbb{N}$, $z \in S^1$ and $\lambda \leq 0$ we have*

$$\mathrm{ind}_{z,\lambda}(\gamma^{[n]}) = \sum_{w \in \sqrt[n]{z}} \mathrm{ind}_{w,\lambda}(\gamma).$$

Proof. Every section $\sigma \in \boldsymbol{E}_z^{[n]}$ admits a unique Fourier expansion

$$\sigma = \sum_{w \in \sqrt[n]{z}} \sigma_w,$$

where each σ_w belongs to the vector space $\boldsymbol{E}_w^{[1]}$ and can be computed as

$$\sigma_w(t) = \frac{1}{n} \sum_{j=0}^{n-1} w^{1-j}\, \sigma(t+j), \qquad \forall t \in \mathbb{R}.$$

Now, as we already pointed out in Remark 2.2.1, $\boldsymbol{E}_w^{[1]}$ is a vector subspace of $\boldsymbol{E}_z^{[n]}$ and

$$B^{[n]}\sigma = \sum_{w \in \sqrt[n]{z}} B^{[n]}\sigma_w = \sum_{w \in \sqrt[n]{z}} B^{[1]}\sigma_w.$$

Since $B^{[n]}\sigma$ belongs to $\boldsymbol{E}_z^{[n]}$ and each $B^{[1]}\sigma_w$ belongs to $\boldsymbol{E}_w^{[1]}$, the above expression gives the unique Fourier expansion of $B^{[n]}\sigma$. From this, we conclude that σ satisfies $B^{[n]}\sigma = \lambda\sigma$ is and only if each σ_w satisfies $B^{[1]}\sigma_w = \lambda\sigma_w$, and the lemma follows. $\square$

Thanks to this lemma, we can now restrict ourselves to study the indices $\mathrm{ind}_{z,\lambda}(\gamma^{[n]})$ for $n = 1$. To start with, let us investigate the properties of $\mathrm{nul}_z(\gamma) = \mathrm{ind}_{z,0}(\gamma)$. Following Bott's terminology, we call $z \in S^1$ a **Poincaré point** of γ when $\mathrm{nul}_z(\gamma) \neq 0$.

Lemma 2.2.2. *The curve γ has only finitely many Poincaré points, and we have*

$$\sum_{z \in S^1} \mathrm{nul}_z(\gamma) \leq 2m,$$

where $m = \dim(M)$.

Proof. By the definition of $B^{[1]}$ and after an integration by parts in (2.1), we infer that a curve $\sigma \in \boldsymbol{E}_n^{[1]}$ is contained in the kernel of the operator $B^{[1]}$ if and only if it satisfies the linear second-order differential system that can by written in local coordinates as

$$\sum_{h=1}^{m} \Big[a_{hj}(t)\, \ddot{\sigma}^h(t) + \big(b_{hj}(t) - b_{jh}(t) + \dot{a}_{hj}(t)\big)\, \dot{\sigma}^h(t) + \big(\dot{b}_{hj}(t) - c_{hj}(t)\big)\, \sigma^h(t) \Big] = 0,$$

$$j = 1, \ldots, m,$$

where

$$a_{hj}(t) = \frac{\partial^2 \mathscr{L}}{\partial v^h \partial v^j}(t, \gamma(t), \dot\gamma(t)),$$
$$b_{hj}(t) = \frac{\partial^2 \mathscr{L}}{\partial v^h \partial q^j}(t, \gamma(t), \dot\gamma(t)),$$
$$c_{hj}(t) = \frac{\partial^2 \mathscr{L}}{\partial q^h \partial q^j}(t, \gamma(t), \dot\gamma(t)).$$

Since $\mathscr{L}$ is a Tonelli Lagrangian, by condition **(T1)** in Section 1.2 the $m \times m$ matrix $a(t) = [a_{hj}(t)]$ is invertible. Hence we can put the above differential system in normal form as

$$\ddot\sigma^j(t) = \sum_{h,k=1}^{m} -a(t)^{-1}_{kj} \Big[\big(b_{hk}(t) - b_{kh}(t) + \dot a_{hk}(t) \big)\, \dot\sigma^h(t) + \big(\dot b_{hk}(t) - c_{hk}(t) \big)\, \sigma^h(t) \Big],$$
$$j = 1, \dots, m,$$

Now, consider a curve $\sigma : \mathbb{R} \to \mathrm{T}M$ with $\sigma(t) \in \mathrm{T}_{\gamma(t)}M$ for each $t \in \mathbb{R}$, and assume that σ is a solution of this system. Then $\sigma(t)$ depends linearly on the initial conditions $(\sigma(0), \dot\sigma(0))$. Notice that the initial conditions belong to a vector space that we can identify with $\mathrm{T}_{\gamma(0)}M \oplus \mathrm{T}_{\gamma(0)}M$. Therefore the vector space of solutions of the above system has dimension at most $2m = \dim(\mathrm{T}_{\gamma(0)}M \oplus \mathrm{T}_{\gamma(0)}M)$. This proves the lemma. $\square$

Now, let us discuss the properties of the index $\mathrm{ind}_z(\gamma)$.

Lemma 2.2.3. *Let $z_1, \dots, z_r \in S^1$ be the Poincaré points of γ. Then the function $z \mapsto \mathrm{ind}_z(\gamma)$ is locally constant on $S^1 \setminus \{z_1, \dots, z_r\}$ and lower semi-continuous on the whole of S^1. Moreover, the jump of this function at any Poincaré point z_j is bounded in absolute value by $\mathrm{nul}_{z_j}(\gamma)$, i.e.,*

$$\mathrm{ind}_{z_j}(\gamma) \leq \lim_{z \to z_j^{\pm}} \mathrm{ind}_z(\gamma) \leq \mathrm{ind}_{z_j}(\gamma) + \mathrm{nul}_{z_j}(\gamma).$$

Proof. For each $z \in S^1$, the real self-adjoint linear operator $B^{[1]} : \boldsymbol{E}_z^{[1]} \to \boldsymbol{E}_z^{[1]}$ is Fredholm (this can be proved by the same argument as in Proposition 2.1.3). We denote by σ_z its spectrum. By continuity, for each interval $(\alpha, \beta] \subset \mathbb{R}$ such that α and β do not belong to σ_z, there is a neighborhood of z in S^1 such that, for each z' in this neighborhood, α and β do not belong to $\sigma_{z'}$ and moreover

$$\sum_{\lambda \in (\alpha,\beta)} \mathrm{ind}_{z,\lambda}(\gamma) = \sum_{\lambda \in (\alpha,\beta)} \mathrm{ind}_{z',\lambda}(\gamma).$$

Assume that $z \in S^1$ is not a Poincaré point of γ, namely 0 does not belong to σ_z. By the above continuity property, 0 does not belong to $\sigma_{z'}$ for each z' in a sufficiently small neighborhood of z, and therefore

$$\mathrm{ind}_z(\gamma) = \sum_{\lambda<0} \mathrm{ind}_{z,\lambda}(\gamma) = \sum_{\lambda<0} \mathrm{ind}_{z',\lambda}(\gamma) = \mathrm{ind}_{z'}(\gamma).$$

Now, assume that $z \in S^1$ is a Poincaré point of γ. Since $B^{[1]} : \boldsymbol{E}_z^{[1]} \to \boldsymbol{E}_z^{[1]}$ is a Fredholm operator, 0 is an isolated point in the spectrum σ_z (see the second paragraph in the proof of Lemma 2.1.1). Therefore, we can fix a sufficiently small $\varepsilon > 0$ such that $[-\varepsilon, \varepsilon] \cap \sigma_z = \{0\}$. By applying once more the above continuity property, $-\varepsilon$ and ε do not belong to $\sigma_{z'}$ for each z' in a sufficiently small neighborhood of z, and we have

$$\begin{aligned} \mathrm{ind}_{z'}(\gamma) &= \sum_{\lambda<-\varepsilon} \mathrm{ind}_{z',\lambda}(\gamma) + \sum_{\lambda\in(-\varepsilon,0)} \mathrm{ind}_{z',\lambda}(\gamma) \\ &= \sum_{\lambda<-\varepsilon} \mathrm{ind}_{z,\lambda}(\gamma) + \sum_{\lambda\in(-\varepsilon,0)} \mathrm{ind}_{z',\lambda}(\gamma) \\ &= \mathrm{ind}_z(\gamma) + \sum_{\lambda\in(-\varepsilon,0)} \mathrm{ind}_{z',\lambda}(\gamma). \end{aligned}$$

This proves that $\mathrm{ind}_z(\gamma) \le \mathrm{ind}_{z'}(\gamma)$. Finally

$$\begin{aligned} \mathrm{ind}_{z'}(\gamma) &= \mathrm{ind}_z(\gamma) + \sum_{\lambda\in(-\varepsilon,0)} \mathrm{ind}_{z',\lambda}(\gamma) \\ &\le \mathrm{ind}_z(\gamma) + \sum_{\lambda\in(-\varepsilon,\varepsilon)} \mathrm{ind}_{z',\lambda}(\gamma) \\ &= \mathrm{ind}_z(\gamma) + \sum_{\lambda\in(-\varepsilon,\varepsilon)} \mathrm{ind}_{z,\lambda}(\gamma) \\ &= \mathrm{ind}_z(\gamma) + \mathrm{nul}_z(\gamma). \end{aligned}$$

□

Now, we introduce the index $\overline{\mathrm{ind}}(\gamma)$ defined by

$$\overline{\mathrm{ind}}(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{ind}_{e^{i\vartheta}}(\gamma)\, \mathrm{d}\vartheta.$$

We call $\overline{\mathrm{ind}}(\gamma)$ the **mean Morse index** of γ, as it is obtained by averaging the indices $\mathrm{ind}_z(\gamma)$ on S^1. Notice that, by Lemma 2.2.3, $\overline{\mathrm{ind}}(\gamma)$ is always finite (it is a non-negative real number). We conclude the section by showing that the Morse index of $\gamma^{[n]}$ grows linearly in the period n, and the asymptotic slope is exactly given by the mean Morse index. For a stronger result we refer the reader to Liu and Long [LL98, LL00] (see also [Lon02, page 213]).

Proposition 2.2.4. *For each periodic solution γ of the Euler-Lagrange system of a Tonelli Lagrangian $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$, the following claims hold.*

(i) *The nullity of* $\gamma^{[n]}$ *is uniformly bounded by* $2m = 2\dim(M)$*, i.e.,*

$$\mathrm{nul}(\gamma^{[n]}) \leq 2m.$$

(ii) *We have* $\overline{\mathrm{ind}}(\gamma) = 0$ *if and only if* $\mathrm{ind}(\gamma^{[n]}) = 0$ *for every* $n \in \mathbb{N}$.

(iii) *The Morse index of* $\gamma^{[n]}$ *verifies the inequality*

$$n\,\overline{\mathrm{ind}}(\gamma^{[n]}) - 2m \leq \mathrm{ind}(\gamma^{[n]}) \leq n\,\overline{\mathrm{ind}}(\gamma^{[n]}) + 2m - \mathrm{nul}(\gamma^{[n]}).$$

In particular the mean Morse index $\overline{\mathrm{ind}}(\gamma)$ *can be computed as*

$$\overline{\mathrm{ind}}(\gamma) = \lim_{n\to\infty} \frac{1}{n}\mathrm{ind}(\gamma^{[n]}).$$

Proof. Point (i) follows from Lemma 2.2.2 together with Lemma 2.2.1. By the definition of mean Morse index, $\overline{\mathrm{ind}}(\gamma) = 0$ if and only if the function $z \mapsto \mathrm{ind}_z(\gamma)$ is zero almost everywhere. By Lemma 2.2.3, this function is locally constant outside the Poincaré points and lower semi-continuous on the whole S^1. Hence $\overline{\mathrm{ind}}(\gamma) = 0$ if and only if $\mathrm{ind}_z(\gamma) = 0$ for every $z \in S^1$. This, together with Lemma 2.2.1, proves point (ii). As for point (iii), let us denote by $z_1, \ldots, z_r \in S^1$ the Poincaré points of γ. By Lemma 2.2.1, the Poincaré points of the nth iteration $\gamma^{[n]}$ are precisely $z_1^n, \ldots, z_r^n$. By Lemma 2.2.3, for each $w, z \in S^1$ we have

$$\mathrm{ind}_w(\gamma^{[n]}) + \mathrm{nul}_w(\gamma^{[n]}) \leq \mathrm{ind}_z(\gamma^{[n]}) + \sum_{j=1}^{r} \mathrm{nul}_{z_j^n}(\gamma^{[n]}). \tag{2.6}$$

By Lemmas 2.2.1 and 2.2.2 we have

$$\sum_{j=1}^{r} \mathrm{nul}_{z_j^n}(\gamma^{[n]}) = \sum_{j=1}^{r} \mathrm{nul}_{z_j}(\gamma) \leq 2m,$$

and, together with (2.6), we obtain

$$\mathrm{ind}_w(\gamma^{[n]}) + \mathrm{nul}_w(\gamma^{[n]}) \leq \mathrm{ind}_z(\gamma^{[n]}) + 2m. \tag{2.7}$$

Notice that, by Lemma 2.2.2, the mean nullity of γ and of $\gamma^{[n]}$ is zero, i.e.,

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{nul}_{e^{i\vartheta}}(\gamma)\,\mathrm{d}\vartheta = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{nul}_{e^{i\vartheta}}(\gamma^{[n]})\,\mathrm{d}\vartheta = 0,$$

whereas, by Lemma 2.2.1,

$$\overline{\mathrm{ind}}(\gamma^{[n]}) := \frac{1}{2\pi}\int_0^{2\pi} \mathrm{ind}_{e^{i\vartheta}}(\gamma^{[n]})\,\mathrm{d}\vartheta = n\,\overline{\mathrm{ind}}(\gamma).$$

Now, by setting $z = 1$ and integrating w on S^1 in (2.7), we get

$$n\,\overline{\mathrm{ind}}(\gamma) \leq \mathrm{ind}(\gamma^{[n]}) + 2m.$$

By setting $w = 1$ and integrating z on S^1 in (2.7), we get

$$\mathrm{ind}(\gamma^{[n]}) + \mathrm{nul}(\gamma^{[n]}) \leq n\,\overline{\mathrm{ind}}(\gamma) + 2m.$$

□

2.3 A symplectic excursion: the Maslov index

In this section we wish to provide a symplectic interpretation of the Morse index for Tonelli Lagrangian systems: the Morse index of a periodic orbit coincides with the Maslov index of the associated periodic orbit of the Legendre-dual Hamiltonian system (see section 1.1). The Maslov index can actually be defined for periodic orbits of more general Hamiltonian systems, and indeed it can be regarded as a generalization of the Morse index.

We begin by recalling some background from linear symplectic geometry (we refer the reader to [MS98, Chapter 2] for more details). The **standard symplectic structure** on $\mathbb{R}^{2m}$ is given by the skew-symmetric bilinear form $\omega_0 : \mathbb{R}^{2m} \wedge \mathbb{R}^{2m} \to \mathbb{R}$ defined by

$$\omega_0(v, w) = \langle \mathrm{J}_0 v, w \rangle , \qquad \forall v, w \in \mathbb{R}^{2m}.$$

Here, $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product of $\mathbb{R}^{2m}$, and J_0 denotes the **standard complex structure** on $\mathbb{R}^{2m}$ given in matrix form by

$$\mathrm{J}_0 = \begin{bmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix}.$$

The **symplectic group** $\mathrm{Sp}(2m)$ is defined as the subgroup of $\mathrm{GL}(2m)$ given by the matrices A such that $A^*\omega = \omega$, or equivalently

$$\mathrm{Sp}(2m) = \left\{ A \in \mathrm{GL}(2m) \mid A^{\mathrm{T}} \mathrm{J}_0 A = \mathrm{J}_0 \right\},$$

where A^{T} denotes the transpose of A. We denote by $\mathrm{Sp}^0(2m)$ the subset of $\mathrm{Sp}(2m)$ consisting of matrices having 1 as an eigenvalue, and by $\mathrm{Sp}^*(2m)$ the complementary subspace of $\mathrm{Sp}(2m)$, i.e.,

$$\begin{aligned} \mathrm{Sp}^0(2m) &= \{ A \in \mathrm{Sp}(2m) \mid \det(A - \mathrm{I}) = 0 \}, \\ \mathrm{Sp}^*(2m) &= \mathrm{Sp}(2m) \setminus \mathrm{Sp}^0(2m). \end{aligned}$$

The space $\mathrm{Sp}^0(2m)$ is a singular hypersurface of $\mathrm{Sp}(2m)$ that separates $\mathrm{Sp}^*(2m)$ in two connected components $\mathrm{Sp}^+(2m)$ and $\mathrm{Sp}^-(2m)$ given by

$$\mathrm{Sp}^{\pm}(2m) = \{ A \in \mathrm{Sp}(2m) \mid \pm \det(A - I) > 0 \}.$$

Every symplectic matrix A admits a unique **polar decomposition**

$$A = \underbrace{(AA^{\mathrm{T}})^{1/2}}_{P} \underbrace{(AA^{\mathrm{T}})^{-1/2} A}_{Q},$$

where $P \in \mathrm{Sp}(2m)$ is symmetric and positive-definite, while $Q \in \mathrm{Sp}(2m) \cap \mathrm{O}(2m)$. The map $r(A) = (AA^{\mathrm{T}})^{-1/2} A$ is a retraction

$$r : \mathrm{Sp}(2m) \to \mathrm{Sp}(2m) \cap \mathrm{O}(2m)$$

coming from the deformation retraction $r_t : \mathrm{Sp}(2m) \to \mathrm{Sp}(2m)$ given by

$$r_t(A) = (AA^{\mathrm{T}})^{-t/2}A. \tag{2.8}$$

In particular r is a homotopy equivalence. Now, if we consider $\mathrm{GL}(m,\mathbb{C})$ to be a subgroup of $\mathrm{GL}(2m)$ via the embedding

$$X + iY \longmapsto \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}, \qquad \forall X + iY \in \mathrm{GL}(m,\mathbb{C}),$$

then $\mathrm{Sp}(2m) \cap \mathrm{O}(2m)$ is identified with the unitary group $\mathrm{U}(m)$. We denote by $\det_{\mathbb{C}}(M) : \mathrm{Sp}(2m) \cap \mathrm{O}(2m) \to S^1 \subset \mathbb{C}$ the complex determinant function, that is

$$\det_{\mathbb{C}}\left(\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}\right) := \det(X + iY).$$

We define the **rotation function** as the composition

$$\rho := \det_{\mathbb{C}} \circ r : \mathrm{Sp}(2m) \to S^1.$$

From our discussion above and the properties of the unitary group, it readily follows that this map induces an isomorphism $\pi_1(\rho)$ between the fundamental groups of $\mathrm{Sp}(2m)$ and S^1.

Let $\mathscr{P}$ be the space of continuous paths $\Psi : [0,1] \to \mathrm{Sp}(2m)$ such that $\Psi(0) = \mathrm{I}$. This space is the disjoint union of the subspaces $\mathscr{P}^*$ and $\mathscr{P}^0$ given by those Ψ such that $\Psi(1) \in \mathrm{Sp}^*(2m)$ and $\Psi(1) \in \mathrm{Sp}^0(2m)$ respectively. We fix the $2m \times 2m$ diagonal symplectic matrices

$$\begin{aligned} W' &:= \mathrm{diag}(2, 1/2, -1, -1, \dots, -1), \\ W'' &:= -\mathrm{I} = \mathrm{diag}(-1, -1, \dots, -1). \end{aligned}$$

Notice that

$$\rho(W') = (-1)^{m-1} = -\rho(W''), \tag{2.9}$$

and therefore W' and W'' belong to different connected components of $\mathrm{Sp}^*(2m)$. Now, consider a path $\Psi \in \mathscr{P}^*$ and fix arbitrarily an auxiliary continuous path $\tilde{\Psi} : [1,2] \to \mathrm{Sp}^*(2m)$ such that $\tilde{\Psi}(1) = \Psi(1)$ and $\tilde{\Psi}(2) \in \{W', W''\}$. We denote by $\Psi \bullet \tilde{\Psi} : [0,2] \to \mathrm{Sp}(2m)$ the concatenation of the paths Ψ and $\tilde{\Psi}$, i.e.,

$$(\Psi \bullet \tilde{\Psi})(t) = \begin{cases} \Psi(t), & t \in [0,1], \\ \tilde{\Psi}(t), & t \in [1,2]. \end{cases}$$

We define the **Maslov index** $\iota(\Psi)$ as

$$\iota(\Psi) := \frac{\vartheta(2) - \vartheta(0)}{\pi}, \tag{2.10}$$

where $\vartheta : [0,2] \to \mathbb{R}$ is a continuous function such that $\rho \circ (\Psi \bullet \tilde{\Psi})(t) = e^{i\vartheta(t)}$. It turns out that this index is well defined (i.e., it does not depend on the choice of the auxiliary path $\tilde{\Psi}$) and, due to (2.9), it is an integer. This notion of Maslov index is due to Conley and Zehnder [CZ84] (indeed, some authors refer to it as the **Conley-Zehnder index**), and it is related to other notions of Maslov index previously defined by Gel'fand and Lidskiĭ [GL58] and Maslov [Mas72]. The original definition of Conley and Zehnder, although equivalent, is different from the one given in (2.10), which is due to Long and Zehnder [LZ90].

The main properties of the Maslov index are summarized in the following statement. We refer the reader to [SZ92, Section 3] for a proof.

Proposition 2.3.1. *The Maslov index satisfies the following properties:*

(Naturality) *For each $\Psi \in \mathscr{P}^*$ and $A \in \mathrm{Sp}(2m)$ we have $\iota(\Psi) = \iota(A^{-1}\Psi A)$.*

(Homotopy) *For any two paths $\Psi, \Psi' \in \mathscr{P}^*$ which are homotopic with fixed endpoints we have $\iota(\Psi) = \iota(\Psi')$.* □

Notice that the homotopy property stated above can be rephrased in the following way. Let $\widetilde{\mathrm{Sp}}(2m)$ be the universal cover of the symplectic linear group. Here, we regard an element of $\widetilde{\mathrm{Sp}}(2m)$ covering $A \in \mathrm{Sp}(2m)$ as a homotopy class (with fixed endpoints) of paths $\Psi : [0,1] \to \mathrm{Sp}(2m)$ with $\Psi(0) = \mathrm{I}$ and $\Psi(1) = A$. Then the Maslov index descends to a locally-constant integer-valued function on the space of those $[\Psi] \in \widetilde{\mathrm{Sp}}(2m)$ with $\Psi(1) \in \mathrm{Sp}^*(2m)$. This function does not admit a continuous extension to the whole $\widetilde{\mathrm{Sp}}(2m)$. Following Long [Lon90], we define its extension as the maximal lower semi-continuous one[2]. More precisely, for each path $\Psi \in \mathscr{P}^0$ we define the Maslov index $\iota(\Psi)$ as

$$\iota(\Psi) := \liminf_{\substack{\Psi' \to \Psi \\ \Psi' \in \mathscr{P}^*}} \iota(\Psi'), \tag{2.11}$$

where $\Psi' \to \Psi$ denotes pointwise convergence.

For paths $\Psi \in \mathscr{P}^0$ (which might be regarded as "degenerate" paths), it is useful to consider another index $\nu(\Psi)$ that is equal to the multiplicity of 1 as an eigenvalue of $\Psi(1)$, i.e.,

$$\nu(\Psi) := \dim \ker(\Psi(1) - \mathrm{I}).$$

The homotopy invariance property of the Maslov index in the general case is described by the following statement, which is due to Long. We refer the reader to [Lon90] or [Lon02, page 145] for a proof.

Proposition 2.3.2. *Let $\Psi : [0,1] \times [0,1] \to \mathrm{Sp}(2m)$ be a continuous map such that $\Psi(s,\cdot) \in \mathscr{P}$ and $\nu(\Psi(s,\cdot)) = \nu(\Psi(0,\cdot))$ for each $s \in [0,1]$. Then $\iota(\Psi(s,\cdot)) = \iota(\Psi(0,\cdot))$ for each $s \in [0,1]$.* □

[2] Other extensions of the Maslov index on $\widetilde{\mathrm{Sp}}(2m)$ have been considered in the literature. For instance, in [RS93a], Robbin and Salamon consider the average between the maximal lower semi-continuous extension and the minimal upper semi-continuous extension.

Now, we wish to show how to associate a Maslov index to periodic orbits of Hamiltonian systems. This can be done in very general Hamiltonian systems (for instance for contractible periodic orbits of Hamiltonian systems on any symplectic manifold (W, ω) such that the first Chern class $c_1(\mathrm{T}W)$ vanishes over $\pi_2(W)$, see, e.g., [Sal99]). Here, we restrict to the case of Tonelli Hamiltonian systems on cotangent bundles, and we follow Abbondandolo and Schwarz [AS06, Section 1.2]. Let us fix a 1-periodic Tonelli Hamiltonian $\mathscr{H} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}^*M \to \mathbb{R}$ that is Legendre-dual to a Tonelli Lagrangian $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$, and consider a 1-periodic orbit $\Gamma : \mathbb{R}/\mathbb{Z} \to \mathrm{T}^*M$ of the Hamiltonian flow of $\mathscr{H}$. This orbit corresponds to the 1-periodic solution $\gamma = \tau^* \circ \Gamma : \mathbb{R}/\mathbb{Z} \to M$ of the Euler-Lagrange system of $\mathscr{L}$, where $\tau^* : \mathrm{T}^*M \to M$ is the projection onto the base of the cotangent bundle. For simplicity, let us restrict to the case in which the pull-back bundle $\gamma^*\mathrm{T}M$ is trivial. This assumption is always verified if the orbit γ is contractible, or if the manifold M is orientable (in fact, in this latter case, $\gamma^*\mathrm{T}M$ is an oriented vector bundle over the circle and therefore it must be trivial). For the general case, we refer the reader to Weber [Web02].

The Hamiltonian flow of $\mathscr{H}$ defines symplectic transformations of T^*M. In fact, assume that there exists an open set $(t_0, t_1) \times U \subset \mathbb{R} \times \mathrm{T}^*M$ such that $\Phi_{\mathscr{H}}^{t,t'}|_U$ is a well-defined diffeomorphism onto its image for each $t, t' \in (t_0, t_1)$. This diffeomorphism is **symplectic**, meaning

$$(\Phi_{\mathscr{H}}^{t,t'})^*\omega = \omega.$$

In fact, $(\Phi_{\mathscr{H}}^{t',t'})^*\omega = (\mathrm{id}_U)^*\omega = \omega$, and by the Cartan formula

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_{\mathscr{H}}^{t,t'})^*\omega = (\Phi_{\mathscr{H}}^{t,t'})^*\big(\mathrm{d}(\underbrace{X_{\mathscr{H}} \lrcorner\, \omega}_{=\mathrm{d}\mathscr{H}}) + X_{\mathscr{H}} \lrcorner \underbrace{\mathrm{d}\omega}_{=0}\big) = 0.$$

This implies that the differential $\mathrm{d}\Phi_{\mathscr{H}}^{t,0}(\Gamma(0))$ is a symplectic linear map. Therefore, once we fix a trivialization of $\Gamma^*\mathrm{T}\mathrm{T}^*M$, it defines a path in the symplectic group $\mathrm{Sp}(2m)$ and in particular it has a Maslov index $\iota(\Gamma)$. However, in order to produce an index for periodic orbits we have to make sure that $\iota(\Gamma)$ does not depend on the chosen trivialization, at least if we pick it in a certain family of trivializations which are intrinsically associated to the symplectic manifold T^*M.

We denote by $\mathrm{T}^{\mathrm{ver}}\mathrm{T}^*M$ the vertical subbundle of $\mathrm{T}\mathrm{T}^*M$, i.e., $\mathrm{T}^{\mathrm{ver}}\mathrm{T}^*M = \ker(\mathrm{T}\tau^*)$. It is straightforward to verify that this vector bundle is isomorphic to the pull-back of $\mathrm{T}M$ by the map τ^*. Therefore, the pull-back bundle $\Gamma^*\mathrm{T}^{\mathrm{ver}}\mathrm{T}^*M$ is trivial, being

$$\Gamma^*\mathrm{T}^{\mathrm{ver}}\mathrm{T}^*M \simeq \Gamma^*(\tau^*)^*\mathrm{T}M = \gamma^*\mathrm{T}M.$$

Consider an almost complex structure J on T^*M that is **compatible** with the canonical symplectic structure ω of T^*M. This means precisely that $\omega(\cdot, \mathrm{J}\cdot)$ is a Riemannian metric on T^*M. With respect to this metric, we can fix an orthogonal trivialization

$$\tilde{\phi} : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^m \xrightarrow{\simeq} \Gamma^*\mathrm{T}^{\mathrm{ver}}\mathrm{T}^*M,$$

and then we can extend $\tilde{\phi}$ to a trivialization

$$\phi : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2m} \xrightarrow{\simeq} \Gamma^*\mathrm{T}\mathrm{T}^*M,$$

as

$$\phi(t,\cdot) = (-\mathrm{J} \circ \tilde{\phi}(t,\cdot) \circ \mathrm{J}_0) \oplus \tilde{\phi}(t,\cdot), \qquad \forall t \in \mathbb{R}/\mathbb{Z},$$

where J_0 is the standard complex structure on $\mathbb{R}^{2m}$ introduced previously. By construction, the trivialization ϕ sends the vertical Lagrangian subspace

$$\mathbb{V}^m := \{\mathbf{0}\} \times \mathbb{R}^m \subset \mathbb{R}^{2m}$$

diffeomorphically onto the vertical subbundle $\Gamma^*\mathrm{T}^{\mathrm{ver}}\mathrm{T}^*M$, i.e.,

$$\phi|_{\mathbb{R}/\mathbb{Z}\times\mathbb{V}^m} : \mathbb{R}/\mathbb{Z} \times \mathbb{V}^m \xrightarrow{\simeq} \Gamma^*\mathrm{T}^{\mathrm{ver}}\mathrm{T}^*M. \tag{2.12}$$

Let $e_1,\dots,e_m,f_1,\dots,f_m$ be the standard symplectic basis of $\mathbb{R}^{2m}$, which means that $e_1,\dots,e_m$ is an orthonormal basis of $\mathbb{R}^m \times \{0\}$ and $f_1,\dots,f_m$ is an orthonormal basis of $\mathbb{V}^m = \{0\} \times \mathbb{R}^m$ with $f_j = \mathrm{J}_0 e_j$ for each $j \in \{1,\dots,m\}$. For each $t \in \mathbb{R}/\mathbb{Z}$, if we set

$$\begin{aligned}
\tilde{f}_j &:= \phi(t,f_j) = \tilde{\phi}(t,f_j),\\
\tilde{e}_j &:= \phi(t,e_j) = -J \circ \tilde{\phi}(t,\cdot) \circ J_0 e_j = -J\tilde{f}_j,
\end{aligned}$$

it is straightforward to verify that $\tilde{e}_1,\dots,\tilde{e}_m,\tilde{f}_1,\dots,\tilde{f}_m$ is a symplectic basis of the tangent space $\mathrm{T}_{\Gamma(t)}\mathrm{T}^*M$, which means

$$\begin{aligned}
\omega(\tilde{e}_j,\tilde{e}_h) &= \omega(\tilde{f}_j,\tilde{f}_h) = 0,\\
\omega(\tilde{e}_j,\tilde{f}_h) &= \begin{cases} 1 & j = h,\\ 0 & j \neq h. \end{cases}
\end{aligned}$$

This shows that the trivialization ϕ is **symplectic**, in the sense that

$$\phi(t,\cdot)^*\omega = \omega_0, \qquad \forall t \in \mathbb{R}/\mathbb{Z}. \tag{2.13}$$

Therefore, the differential of the Hamiltonian flow along Γ defines a continuous path $\Gamma_\phi : [0,1] \to \mathrm{Sp}(2m)$ by

$$\Gamma_\phi(t) := \phi(t,\cdot)^{-1} \circ \mathrm{d}\Phi^{t,0}_{\mathscr{H}}(\Gamma(0)) \circ \phi(0,\cdot), \qquad \forall t \in [0,1].$$

Notice that $\Gamma_\phi(0) = \mathrm{I}$, hence $\Gamma_\phi \in \mathscr{P}$. We wish to define the Maslov index of Γ as the Maslov index of this path. Before doing it, however, we need to verify that this index does not depend on the specific choice of the trivialization ϕ.

We denote by $\mathrm{Sp}(2m,\mathbb{V}^m)$ the subgroup of $\mathrm{Sp}(2m)$ consisting of those matrices that preserve the vertical Lagrangian subspace $\mathbb{V}^m \subset \mathbb{R}^{2m}$, i.e.,

$$\begin{aligned}
\mathrm{Sp}(2m,\mathbb{V}^m) &= \{A \in \mathrm{Sp}(2m) \mid A\mathbb{V}^m = \mathbb{V}^m\}\\
&= \left\{ \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix} \,\middle|\, A_1^{\mathrm{T}}A_3 = \mathrm{I},\ A_1^{\mathrm{T}}A_2 = A_2^{\mathrm{T}}A_1 \right\}.
\end{aligned}$$

Lemma 2.3.3. *Consider the inclusion $j : \mathrm{Sp}(2m, \mathbb{V}^m) \hookrightarrow \mathrm{Sp}(2m)$. Then the induced fundamental group homomorphism $\pi_1(j) : \pi_1(\mathrm{Sp}(2m, \mathbb{V}^m)) \to \pi_1(\mathrm{Sp}(2m))$ is the zero homomorphism.*

Proof. The deformation retraction of equation (2.8) restricts to a deformation retraction of $\mathrm{Sp}(2m, \mathbb{V}^m)$ onto

$$\mathrm{Sp}(2m, \mathbb{V}^m) \cap \mathrm{U}(m) = \left\{ \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \;\middle|\; R \in \mathrm{O}(m) \right\}.$$

Therefore, we know that in the following diagram of inclusions

$$\begin{tikzcd} \mathrm{Sp}(2m, \mathbb{V}^m) \arrow[r, hook, "j"] & \mathrm{Sp}(2m) \\ \mathrm{Sp}(2m, \mathbb{V}^m) \cap \mathrm{U}(m) \arrow[u, hook, "\sim"] \arrow[r, hook, "h"] & \mathrm{U}(m) \arrow[u, hook, "\sim"] \end{tikzcd}$$

the vertical arrows are homotopy equivalences, and in order to conclude we just need to show that h induces the zero homomorphism between fundamental groups. The complex determinant $\det_{\mathbb{C}}$ induces a fundamental group isomorphism

$$\pi_1(\det_{\mathbb{C}}) : \pi_1(U(m)) \xrightarrow{\simeq} \pi_1(S^1) \simeq \mathbb{Z}.$$

Consider an arbitrary $[\vartheta] \in \pi_1(\mathrm{Sp}(2m, \mathbb{V}^m) \cap \mathrm{U}(m))$, i.e.,

$$\vartheta : (\mathbb{R}/\mathbb{Z}, 0) \to (\mathrm{Sp}(2m, \mathbb{V}^m) \cap \mathrm{U}(m), \mathrm{I}).$$

Since $\det_{\mathbb{C}} \circ \vartheta \equiv 1$, we have $\pi_1(\det_{\mathbb{C}}) \circ \pi_1(h)[\vartheta] = [\det_{\mathbb{C}} \circ \vartheta] = 0$, and we conclude $\pi_1(h)[\vartheta] = 0$. □

Lemma 2.3.4. *The Maslov index $\iota(\Gamma_\phi)$ is independent of the trivialization ϕ, provided ϕ satisfies* (2.12) *and* (2.13).

Proof. Let $\psi : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^{2m} \xrightarrow{\simeq} \Gamma^*\mathrm{T}\mathrm{T}^*M$ be another symplectic trivialization that satisfies (2.12). Consider the loop $\vartheta : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(2m, \mathbb{V}^m)$ defined by

$$\vartheta(t) = \phi(t, \cdot)^{-1} \circ \psi(t, \cdot), \qquad \forall t \in \mathbb{R}/\mathbb{Z}.$$

Notice that

$$\Gamma_\psi(t) = \vartheta(t)^{-1} \circ \Gamma_\phi(t) \circ \vartheta(0), \qquad \forall t \in \mathbb{R}/\mathbb{Z}.$$

By Lemma 2.3.3 there exists a homotopy $\Theta : [0,1] \times \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(2m)$ such that $\Theta(0, \cdot) = \vartheta$, $\Theta(s, 0) = \vartheta(0)$ for each $s \in [0,1]$, and $\Theta(1, \cdot) \equiv \vartheta(0)$. Hence we can build a homotopy $\Omega : [0,1] \times [0,1] \to \mathrm{Sp}(2m)$ as

$$\Omega(s,t) = \Theta(s,t)^{-1} \circ \Gamma_\phi(t) \circ \vartheta(0), \qquad \forall (s,t) \in [0,1] \times [0,1],$$

such that $\Omega(0,\cdot) = \Gamma_\psi$, $\Omega(1,\cdot) = \vartheta(0)^{-1} \circ \Gamma_\phi \circ \vartheta(0)$, $\Omega(s,0) = \Gamma_\psi(0)$ and $\Omega(s,1) = \Gamma_\psi(1)$ for each $s \in [0,1]$. Since the homotopy Ω fixes the endpoints of the homotoped path, we have that $\nu(\Omega(s,\cdot)) = \nu(\Gamma_\psi)$ for each $s \in [0,1]$. Hence, by the homotopy invariance of the Maslov index (Proposition 2.3.2) we conclude

$$\iota(\Gamma_\psi) = \iota(\Omega(s,\cdot)), \qquad \forall s \in [0,1],$$

and in particular

$$\iota(\Gamma_\psi) = \iota\left(\vartheta(0)^{-1} \circ \Gamma_\phi \circ \vartheta(0)\right).$$

Finally, by the naturality property of the Maslov index, we conclude

$$\iota(\Gamma_\psi) = \iota\left(\vartheta(0)^{-1} \circ \Gamma_\phi \circ \vartheta(0)\right) = \iota(\Gamma_\phi). \qquad \square$$

By the above lemma, we can define the **Maslov index** of the periodic orbit Γ as the integer $\iota(\Gamma) := \iota(\Gamma_\phi)$, where ϕ is any trivialization satisfying (2.12) and (2.13). Similarly we define $\nu(\Gamma) := \nu(\Gamma_\phi)$, that is, the multiplicity of 1 as an eigenvalue of the endomorphism $\mathrm{d}\Phi^{1,0}_{\mathscr{H}}(\Gamma(0))$ of $\mathrm{T}_{\Gamma(0)}\mathrm{T}^*M$. The indices $\iota(\Gamma)$ and $\nu(\Gamma)$ turn out to be equal to the Morse index $\mathrm{ind}(\gamma)$ and the nullity $\mathrm{nul}(\gamma)$, where $\gamma := \tau^* \circ \Gamma$ is the Euler-Lagrange 1-periodic orbit corresponding to Γ. This is an important result first established by Duistermaat [Dui76] in the non-degenerate case $\mathrm{nul}(\gamma) = 0$, and by Viterbo [Vit90] in full generality. Alternative proofs were also given by Long and An [LA98] and Abbondandolo [Abb03]. We refer the reader to these papers or to [Lon02, Section 7.3] for a proof.

Theorem 2.3.5 (Maslov-Morse indices theorem). *Let $\mathscr{H} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}^*M \to \mathbb{R}$ and $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$ be Legendre-dual Tonelli Hamiltonians and Lagrangians, and $\Gamma : \mathbb{R}/\mathbb{Z} \to \mathrm{T}^*M$ a Hamiltonian periodic orbit corresponding to the Euler-Lagrange periodic orbit γ. Assume further that $\gamma^*\mathrm{T}M$ is a trivial bundle. Then the Maslov and Morse indices of these orbits are the same, i.e.,*

$$\begin{aligned} \iota(\Gamma) &= \mathrm{ind}(\gamma), \\ \nu(\Gamma) &= \mathrm{nul}(\gamma). \end{aligned} \qquad \square$$

An extensive study of the Maslov index has been carried out, starting in the 1990s, by Long and his collaborators, who in particular generalized Bott's iteration theory to the Maslov case and provided many applications to the existence of periodic orbits in Hamiltonian systems more general than the Tonelli ones. We refer the reader to Long's monograph [Lon02] for a detailed account of these developments.

http://www.springer.com/978-3-0348-0162-1

Critical Point Theory for Lagrangian Systems
Mazzucchelli, M.
2012, XII, 188 p., Hardcover
ISBN: 978-3-0348-0162-1
A product of Birkhäuser Basel