Preface

This monograph is devoted to presenting in detail a few selected applications of critical point theory, and in particular Morse theory, to Lagrangian dynamics. A Lagrangian system is defined by a configuration space M, which has the structure of a smooth manifold, and a Lagrangian function $\mathscr{L}$ defined on the tangent bundle of M and, in the non-conservative case, depending upon time as well. In Joseph-Louis Lagrange's reformulation of classical mechanics, the Lagrangian function is given by the difference of kinetic and potential energy. The motion of the associated mechanical system is described by the Euler-Lagrange equations, a system of second-order ordinary differential equations that involves the Lagrangian. The principle of least action, which in different settings is even anterior to Lagrange's work, states that the curves that are solutions of the Euler-Lagrange equations admit a variational characterization: they are extremal points of a functional, the action, associated to the Lagrangian. The development of critical point theory in the nineteenth and twentieth century provided a powerful machinery to investigate dynamical questions in Lagrangian systems, such as existence, multiplicity or uniqueness of solutions of the Euler-Lagrange equations with prescribed boundary conditions.

In this monograph, we will consider closed configuration spaces M and we will focus on the class of so-called Tonelli Lagrangians: these are smooth Lagrangian functions $\mathscr{L} : \mathbb{R} \times \mathrm{T}M \to \mathbb{R}$ that, when restricted to the fibers of $\mathrm{T}M$, have positive definite Hessian and superlinear growth. We will normally restrict ourselves to those Tonelli Lagrangians $\mathscr{L}$ whose solutions of the Euler-Lagrange equations are defined for all times, a condition that is always fulfilled when the time-derivative of the Lagrangian is suitably bounded. The importance of the Tonelli class is twofold. From the variational point of view, the Tonelli assumptions imply existence and regularity of action minimizing curves joining given points in the configuration space. From a symplectic point of view, Tonelli Lagrangians constitute the broadest family of fiberwise convex Lagrangian functions for which the Lagrangian-Hamiltonian duality holds. These generalities on Tonelli Lagrangians, together with a brief introduction to the Lagrangian and Hamiltonian formalism, will be the subject of **Chapter 1**.

If a Tonelli Lagrangian is 1-periodic in time, namely if it is a function of the form $\mathscr{L} : \mathbb{R}/\mathbb{Z} \times \mathrm{T}M \to \mathbb{R}$, then one can look for periodic solutions (with integer period) of the associated Euler-Lagrange equations. Finding a lower bound for the number of periodic orbits with prescribed period is one of the main themes in Lagrangian dynamics. The least action principle characterizes the n-periodic orbits as the extremal points of the Lagrangian action functional defined on the space of smooth (say C^2) n-periodic curves. In view of this, one is tempted to study the multiplicity of n-periodic orbits by means of critical point theory: the "richer" the topology of the space of n-periodic curves, the larger the minimal number of n-periodic orbits. More precisely, one expects a lower bound for the number of n-periodic orbits to be given by the cup-length of the space of n-periodic curves. Another celebrated question is whether the Euler-Lagrange systems admit infinitely many periodic solutions (without prescribing their period). The main difficulty here is to recognize when an n-periodic orbit found by abstract methods is the iteration of another periodic orbit of lower period. The Morse index of periodic orbits helps us with this: a periodic orbit that is obtained by homological techniques in a certain degree d will have Morse index close to d. In the 1950s, Bott investigated the behavior of the Morse index of periodic orbits under iteration. In **Chapter 2**, we will introduce the notion of Morse index and we will present in detail Bott's iteration theory. We will also mention a symplectic interpretation of the Morse index as a Maslov index. This latter index, which can be associated to periodic orbits of more general Hamiltonian systems, was independently introduced and investigated by many people among whom are Gel'fand, Lidskiĭ, Maslov, Conley, Zehnder, Long, Robbin and Salamon.

In order to be able to apply the abstract results of critical point theory to the Lagrangian action, we need a suitable functional setting: a space of sufficiently smooth n-periodic curves with the structure of a (possibly infinite-dimensional) manifold, over which the action is regular, say at least C^1, and has sublevels that are "sufficiently compact". For the subclass of Tonelli Lagrangians with quadratic growth (which we will synthetically call "convex quadratic-growth"), a suitable choice is given by the Hilbert manifold of n-periodic curves with $W^{1,2}$ regularity. As proved by Benci, with this functional setting the action is $C^{1,1}$ and satisfies the Palais-Smale condition, a "weak compactness" condition on its sublevels that is enough for critical point theory. In **Chapter 3**, we will introduce this functional setting for the case of periodic curves and of curves with prescribed endpoints. After studying the properties of the action of convex quadratic-growth Lagrangians in this setting, we will derive a few elementary results on the existence of action minimizing orbits and of periodic orbits with prescribed period.

Even though the $C^{1,1}$ regularity of the action of convex quadratic-growth Lagrangians is sufficient for most of the results of critical point theory, all the abstract statements involving the Morse index require at least the C^2 regularity. However, this requirement turns out to be not necessary for the Lagrangian action functional. Indeed, one can equivalently work in a finite-dimensional functional setting in which the action is C^∞ and has compact sublevels. In this setting, one considers the space of continuous and piecewise broken n-periodic solutions of the Euler-Lagrange equations. This space turns out to be a smooth finite-dimensional submanifold of the $W^{1,2}$ loop space, and it may be regarded as a homotopic approximation of this latter space. In particular, all the indices and invariants coming from critical point theory are the same in the $W^{1,2}$ setting and in the finite-dimensional one: Morse index and nullity, local homology of periodic orbits, relative homology of sublevels of the action and so forth. **Chapter 4** will be devoted to introducing this finite-dimensional functional setting, and to proving a few multiplicity results for periodic orbits with prescribed period.

Bott's iteration theory can be pushed one step further by investigating the behavior of the local homology of periodic orbits under iterations. This problem was first studied by Gromoll and Meyer, and further by Long. It turns out that the behavior of local homology is sometimes dictated by the Morse index and nullity: if these indices do not change by iteration, then the local homology does not change as well. This seemingly technical result turns out to be crucial in the study of the multiplicity of periodic orbits with unprescribed period. In **Chapter 5**, we will deduce this theorem from an analogous abstract result: the local homology of a critical point of a function does not change when restricted to a submanifold which is invariant under the gradient flow of the function, provided the Morse index and nullity do not change as well.

For a general Tonelli Lagrangian with global Euler-Lagrange flow, a functional setting in which the action is both regular and satisfies the Palais-Smale condition is not known. However, one can still apply abstract results from critical point theory to a suitable modification of the Lagrangian, that coincides with the original one in a neighborhood of the zero section of the tangent bundle and it is fiberwise quadratic at infinity. This idea is due to Abbondandolo and Figalli, who showed that, for a fixed period n and a fixed action value a, all the n-periodic orbits of the modified Lagrangian with action less than a are also periodic orbits of the original Tonelli Lagrangian, provided the modification was performed sufficiently far from the zero section. In **Chapter 6** we will discuss this idea, and we will use it to extend the validity of the multiplicity results of Chapter 4 to the Tonelli case. The second part of the chapter will be devoted to proving that Tonelli Lagrangians with global Euler-Lagrange flow always admit infinitely many periodic orbits. This result, first established by Long for mechanical Lagrangians on the torus and further extended by Lu and by the author, is based upon the iteration theory for periodic orbits discussed in Chapters 2 and 5, and upon an important technique developed by Bangert and Klingenberg in the setting of closed geodesics.

We have tried to make this monograph accessible even to non-specialists, and in particular to students from the graduate level onward. The **Appendix** collects all the background from Morse theory that is needed for our applications. The interested reader can find more material together with the proofs in [Cha93]. As we have already remarked, we have only presented a few selected applications of critical point theory to Lagrangian dynamics. A topic which is very close to ours and that we did not touch at all is the existence and multiplicity of periodic orbits with prescribed energy in autonomous Lagrangian systems. For a summary of the recent state of the art of this problem we refer the reader to [Con06] and the bibliography therein. It is impossible to mention here all the other applications of critical point theory to Lagrangian and Hamiltonian dynamics. The interested reader can find an account of some of these topics in the excellent textbooks [MW89, Eke90, HZ94, CI99, Abb01, Lon02, Fat08].

Acknowledgment

I wish to thank Alberto Abbondandolo for introducing me to the realm of critical point theory a few years ago, when I was a graduate student in Pisa. I owe a lot to Gonzalo Contreras, Albert Fathi and Eduard Zehnder who, with their classes, advice or indirectly with their books, taught me the formalism of Lagrangian and Hamiltonian dynamics. Many people provided me with feedback on preliminary versions of the manuscript. I thank Will Merry for his detailed review, for his encouragement and for his valuable comments. I thank the anonymous referee for pointing out several inaccuracies in the text. Among the others who contributed to this monograph, special thanks go to Ana Lecuona, Barney Bramham, Joana Santos, José Fernando and Pierre Pageault. Finally, it is my pleasure to acknowledge the University of Pisa, the Max Planck Institute for Mathematics in the Sciences of Leipzig, and the École Normale Supérieure of Lyon for their support during the preparation of this work.

Lyon, April 2011 Marco Mazzucchelli

http://www.springer.com/978-3-0348-0162-1

Critical Point Theory for Lagrangian Systems
Mazzucchelli, M.
2012, XII, 188 p., Hardcover
ISBN: 978-3-0348-0162-1
A product of Birkhäuser Basel