

## Preface

Are you ready for an experiment? Then please fix the first thought that comes to your mind when you are asked “What is a sine function?” Once you have done that compare your answer with the four options on page [xiii](#).

Did you chose (b)? Indeed most of us memorize pictures more easily than formulas, and an overwhelming majority of people associate with the sine function the wavy curve of its graph.

However, this is a book on complex functions. So let me ask again and more precisely, “What is a *complex sine function*?” Probably this time your inner eye refuses to show you a picture, and this happens for a simple reason – our brain is trained to visualize objects in three spatial dimensions, while the graphs of complex functions live in a four-dimensional space. Hence most of us are unable to imagine such an object.

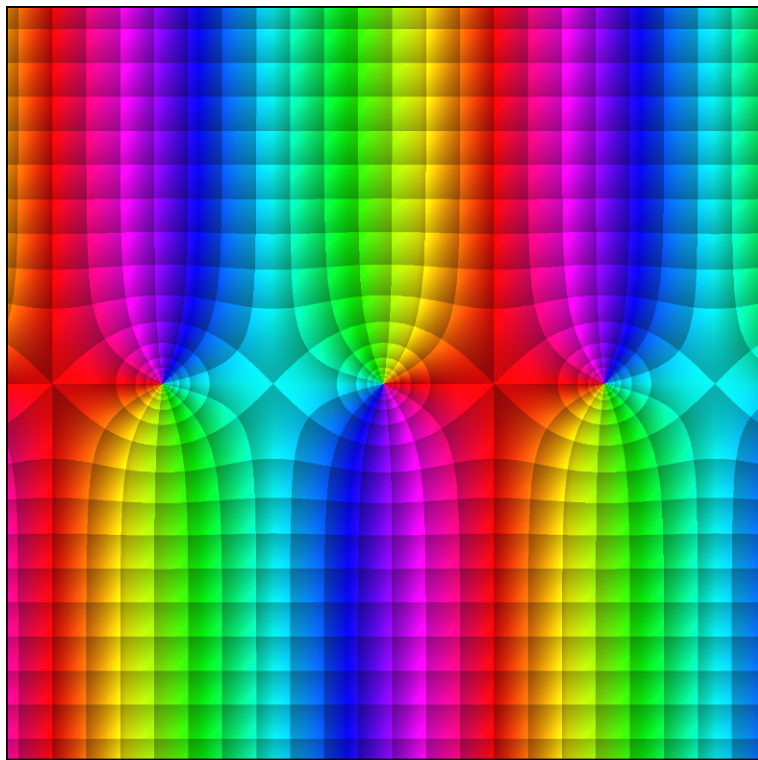


Figure 1: An (enhanced) phase portrait of the complex sine function

But the situation is not completely hopeless since the missing spatial dimension can be added using *color*. The figure above shows an image of the complex sine

function, and though you probably cannot interpret this cryptic picture yet, this will change when you have read this book.

**What it is about.** This textbook is an introduction to theory and applications of complex functions. The presentation is intuitive and requires only basic knowledge of calculus. Volume 1 covers the standard topics of a first course in complex analysis. With a few exceptions all results are provided with proofs. The forthcoming Volume 2 will be devoted to selected topics and various applications of complex methods, like integral transforms, boundary value problems, and signal analysis.

What distinguishes this book from other texts in the first instance is the systematic use of *phase portraits*, a special coloring technique which visualizes functions as images. Readers will learn not only how properties of a function are reflected in and can be read off from its phase portrait, but also how experiments with phase portraits can be designed and adapted to answer specific questions.

Secondly, I have gone about building the theory of complex functions in a manner that is somewhat out of fashion these days. It is well known that the three prominent protagonists of complex function theory in the 19th century, Augustin Louis Cauchy, Bernhard Riemann and Karl Weierstrass, considered the subject from differing points of view. Most contemporary textbooks<sup>1</sup> develop the theory of analytic functions along the ideas of Cauchy and Riemann, with complex differentiability as the entry point, followed by complex integrals. Here we adopt Karl Weierstrass' *constructive* approach via power series, which also best exemplifies the *intrinsic evolution* of mathematical concepts: once the complex number system is established, the interplay between asking questions and looking for answers guides us from one step to the next till we *almost inevitably* end up with Riemann surfaces.

**Why to read it.** Complex functions are everywhere. Besides their relevance in most fields of mathematics, they have also become indispensable tools in the natural sciences and engineering. It was Carl Friedrich Gauss, an outstanding pioneer of complex analysis, who already wrote in the first half the 19th century:<sup>2</sup>

*“Die vollständige Erkenntniss der Natur einer analytischen Function muss auch die Einsicht in ihr Verhalten bei den imaginären Werthen des Arguments in sich schließen und oft ist sogar letztere unentbehrlich zu einer richtigen Beurtheilung der Gebarung der Function im Gebiete der reellen Argumente.”*

*“Complete knowledge of the nature of an analytic function must also include insight into its behavior for imaginary values of the arguments, and often the latter is indispensable for a proper appreciation of the behavior of the function for real arguments.”*

Complex functions sometimes have the reputation of being mysterious entities; *seeing* these alien objects may help to overcome the awe one might feel while dealing with them.

<sup>1</sup>One of the rare exceptions is Noguchi [46].

<sup>2</sup>Gauss Werke Vol. 10, No 1, p. 405, quoted after [56]

Phase portraits provide functions with an individual face and deepen our intuitive understanding of basic and advanced concepts in complex analysis. They reveal intrinsic structures behind the formulas, literally open our eyes to the wonderful realm of complex functions, and may serve students, teachers, scientists, and engineers as simple and efficient tools in their work.

It is my conviction that even experienced mathematicians can benefit from this visual approach as it might help them to get a fresh perspective on old problems and inspire them to ask new and more challenging questions.<sup>3</sup>

**How to read it.** We assume that readers have some knowledge of real numbers, calculus, and two-dimensional Euclidean geometry. Though the text is self-contained, it may also serve as a companion to one of the many excellent textbooks on complex analysis; this is certainly an option for those who are interested in phase portraits but do not like Weierstrass' approach. Readers who are already familiar with complex functions may not need to do more than simply browse through the book, look at the illustrations, and pick out themes they find interesting to investigate further.

A number of worked out examples demonstrating the use of phase portraits are scattered throughout the text. More importantly, readers are invited to participate actively and create their own experiments. With a computer and some basic software it is quite easy to generate nice and interesting pictures. For example, making a phase portrait of an elementary function requires less than ten lines of MATLAB<sup>®</sup> code.<sup>4</sup>

**What it contains.** After some general comments on the visualization of complex functions and an informal introduction to phase portraits in Chapter 1, the systematic exposition begins in Chapter 2. Having established the system of complex numbers and their arithmetic operations, we investigate general properties of functions and discuss various options for their pictorial representation.

Equipped with the toolkit of phase portraits, we explore and discover analytic functions in Chapter 3, following some 'natural' line of development. We begin with those functions that can be formed using only the four basic arithmetic operations: polynomials and rational functions. In the next step, limit processes lead to power series, which in turn are the local building blocks of general analytic functions. Weierstrass' disk chain method of analytic continuation facilitates the transition from local to global entities. The functions resulting from this procedure however, may be 'multiple-valued', which does not fit into the usual concept of a function. A satisfactory solution, functions on Riemann surfaces, is technically more demanding and will therefore be postponed till Chapter 7. Those unhappy with this proposal may continue with Chapter 7 directly after Chapter 3.

In Chapter 4 we bring in the concept of complex differentiability. It allows

---

<sup>3</sup>Why do the orbits of the mathematical pendulum show up in the phase portrait of the sine function depicted in Figure 1?

<sup>4</sup>MATLAB is a registered trademark of the MathWorks Inc. Most images in this book have been created using MATLAB.

us to push the theory much further and helps us to avoid cumbersome manipulations with power series. Here we develop the powerful machinery of complex calculus, including series expansions, path (contour) integrals, integral formulas, and residues.

In Chapter 5 we have collected various techniques for constructing analytic functions: sequences, series, products and integrals. A central theme is the notion of normal convergence, which in connection with Montel's theorem provides us with an efficient tool for verifying the existence of solutions to extremal problems.

The focus of Chapter 6 is on geometric aspects. Here we adopt Riemann's view of complex functions as mappings between (domains of) two complex planes. In this geometric setting, analytic functions are characterized as angle preserving mappings. A large part of the chapter is devoted to bijective conformal mappings between specific domains, in particular to Möbius transformations, elliptic integrals, and Schwarz–Christoffel mappings. Highlights are the Riemann mapping theorem and the Carathéodory–Osgood theorem on boundary correspondence.

Chapter 7 is devoted to Riemann surfaces, and we will look at them at varying levels of abstraction. Concrete Riemann surfaces are constructed from a patchwork of function elements, abstract Riemann surfaces are modelled as manifolds on a topological space. Since phase portraits are images, functions on Riemann surfaces can be depicted directly on the familiar models of such surfaces embedded in three-dimensional space. In the alternative representation of a function on the (flat) sheets of its Riemann surface, phase portraits help us to locate the branch cuts and to explore in which way the different sheets are glued along their edges to form the global surface.

**Literature.** It is impossible to list the vast existing literature of introductory texts in complex analysis. Within the last ten years alone, more than 25 textbooks in English<sup>5</sup> have been published in this field. In writing this text I benefitted especially from Ablowitz and Fokas [1], Donaldson [9], Freitag [20], Freitag and Busam [21], Henrici [26], Krantz [31], [33], Lin [35], Lorenz [36], Marsden and Hoffman [41], Needham [44], Norton [47], Palka [52], Remmert [56], Schabat [62], and Ullrich [67]. Since many results, proofs and examples are standard, references are not often explicitly mentioned. The list of references at the end is based on the literature which was available and useful to me, and which I can recommend to students. However, this list is by no means complete and could be supplemented by a great number of other excellent texts. An interested reader will certainly have no problems in finding appropriate material for further reading which suits his or her taste better.

**A plea for phase.** Since this is a book involving phase portraits, let us spend a few moments exploring the role of phase in general. Basically there are two representations of complex numbers, the cartesian form  $z = x + iy$  and the polar form  $z = r(\cos \varphi + i \sin \varphi)$ . While the first one uses the real part  $x$  and the imaginary part  $y$ , the polar form employs the modulus  $r$  and the argument  $\varphi$  of  $z$ .

---

<sup>5</sup>According to Zentralblatt Mathematik.

Among these quantities  $x$ ,  $y$  and  $r$  are uniquely determined, but this is not so for  $\varphi$ . This multi-valuedness makes the argument somewhat unwieldy, and students often try to avoid using it.

A simple trick may help to resolve this complication: substitute the argument  $\varphi$  by the phase  $\cos \varphi + i \sin \varphi$  whenever possible. Since the phase of  $z$  is just  $z/|z|$ , it is well defined for all complex numbers except zero.

Compared to the modulus, the importance of phase is often underestimated. But the truth is, due to a subtle asymmetry between modulus and phase, the phase of a function sometimes delivers even *more information* than the modulus.<sup>6</sup> Since phase portraits depict the color-coded values of the phase on the domain of the function, it is my hope that their use may contribute to give phase the attention it deserves.

My acquaintance with complex functions dates back almost forty years, but it took a long time until I could begin to *see* my friends. I love them even more ever since I know their phases. This book has been written to let you share my joy.

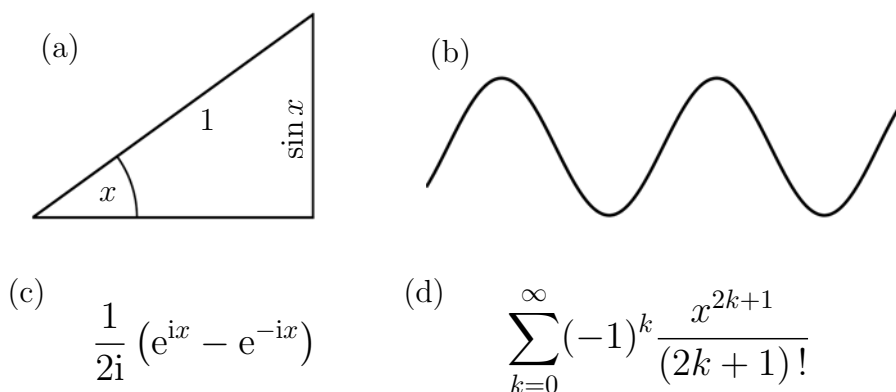


Figure 2: What do you associate with the sine function?

**Acknowledgements.** My first thanks go to Richard Varga. His enthusiastic comments on the usefulness of phase portraits in teaching complex analysis convinced me to undertake this project, the full dimensions of which I could barely foresee at that time.

Several ideas about phase portraits emerged while working on the research project “Nonlinear Riemann-Hilbert Problems, Circle Packing and Conformal Geometry”, supported by the Deutsche Forschungsgemeinschaft (DFG) by grant We1704-8/2.

<sup>6</sup>This will be made precise in Volume 2, see also Wegert and Semmler [70].

The book has benefitted immensely from the work of Gunter Semmler. He read carefully several stages of the complete text, checked many details, and made proposals for improving the presentation. Even though I could not follow all his recommendations, his valuable suggestions and constructive critical remarks are very much appreciated. The blame for the mistakes that remain is mine.

Daniela Kraus and Oliver Roth read an earlier version of the manuscript and corrected a number of errors. Our conversations supported and encouraged me in the process of writing, and I appreciate their useful comments very much.

Special thanks go to my old friend and colleague Albrecht Böttcher, who made several suggestions for improvements, and helped to overcome a critical period when I got really tired. Moral support came also from Nick Trefethen, who expressed his faith in the value of writing a textbook involving phase portraits and stimulated the project in various ways.

The book would not have been finished in due time without the help of Marko Lindner who took over my teaching responsibilities at the TU Bergakademie Freiberg in the summer term 2011, thereby enabling me to concentrate entirely on the writing of the manuscript.

Our students David Hahn and Daniel Lorenz read through large parts of the manuscript, eliminated errors and proposed changes from a student's perspective. My former colleague Günter Schaar discovered several remaining typos. I also would like to thank Jens Wirth, who ventured to use parts of the book in his introductory lecture on complex analysis.

I am greatly indebted to Meenaxi Barkataki-Ruscheweyh who never hesitated to spend her time with the manuscript. With her professional and dedicated work as copy editor she improved many passages and significantly increased the readability of the text.

Thomas Hempfling, the Managing Director of Springer Basel, immediately supported the project when I approached him with the first idea of the book and he gladly agreed to its transmutation from a small booklet of estimated 150 pages into a two-volume text expected to run into more than 600 pages. I am very grateful that he accompanied all stages of the production and accepted my repeated shifts of "deadlines". It is also a pleasure to thank the staff of Springer/Birkhäuser, in particular Reinhold Joest and Sylvia Lotrovsky, for a pleasant and smooth collaboration. Ines Bruhn's professional assistance in preparing some graphic files for publication was also most valuable.

Last but not the least, I would like to express my gratitude to my wife Petra for her infinite patience and for allowing me to ignore almost all duties of daily life while I was writing, as well as to all friends who showed interest in the progress of the ongoing project and gave me emotional support.

<http://www.springer.com/978-3-0348-0179-9>

Visual Complex Functions

An Introduction with Phase Portraits

Wegert, E.

2012, XIV, 360 p. 231 illus., 202 illus. in color.,

Softcover

ISBN: 978-3-0348-0179-9

A product of Birkhäuser Basel