Chapter 2

The End Points Effect

2.1 Two different problems

As we have noticed, the theorems related with direct and converse results for trigonometric approximation can not be translated word by word to the case of algebraic approximation. Thus we have two different questions:

1. Given a modulus of smoothness, how can the associated (generalized) Lipschitz classes be characterized with the help of approximation by means of algebraic polynomials?
2. How can the class of functions with a given rate of algebraic polynomial approximation (say $E_n(f) \leq M/n^{\alpha}$) be characterized in terms of properties related with smoothness and/or differentiability?

Some solutions to the first problem were given by Nikolskii, Timan and Dzyadyk. They considered the space of continuous functions and uniform approximation. In this approach the use of the quantity

$$\Delta_n(x) = \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \tag{2.1}$$

was essential. Fuksman presented the first results related with solutions of the second problem. Fuksman obtained a characterization of functions $f \in C[-1,1]$ for which $E_n(f) \leq M/n^{\alpha}$ $(0 < \alpha < 1)$ with the help of a local modulus of continuity.

After the works of Timan some different problems were considered:

(i) When can $\Delta_n(x)$ be changed by

$$\delta_n(x) = \frac{\sqrt{1-x^2}}{n}? \tag{2.2}$$

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(ii) When can the polynomials used in approximating functions also approximate the derivatives?
(iii) When can the estimates given in terms of the first modulus of continuity be improved by using higher-order moduli?
(iv) Is it possible to obtain estimates which combine (i) and (ii) (or other relations)?

2.2 Nikolskii's discovery

Since the known estimates for trigonometric approximation did not always lead to optimal results in algebraic approximation, some new methods were needed.

In 1946 Nikolskii made an important advance by considering point-wise estimation by means of a sequence of linear operators. Let $\{T_n\}$ be the sequence of Chebyshev polynomials
$$T_n(x) = \cos(n \arccos x).$$
It is known that these polynomials are orthogonal with respect to the measure $2dt/\pi\sqrt{1-t^2}$ on the interval $[-1, 1]$.

As usual, for $f \in C[-1, 1]$ the Fourier-Chebyshev coefficients are defined by
$$a_n(f) = \frac{2}{\pi}\int_{-1}^{1} \frac{f(t)T_n(t)dt}{\sqrt{1-t^2}}.$$

For $f \in C[-1, 1]$ and $x \in [-1, 1]$ define
$$U_n(f, x) = \frac{a_0(f)}{2} + \sum_{k=1}^{n} k\lambda_{n,k}\, a_k(f)\, T_k(x)$$
where
$$\lambda_{n,k} = \frac{\pi}{2n} \cot \frac{k\pi}{2n}.$$

Theorem 2.2.1 (Nikolskii, [271]). *For each $n \in \mathbb{N}$, one has $U_n : C[-1, 1] \to \mathbb{P}_n$. Moreover, if $f \in \mathrm{Lip}_1(M, [-1, 1])$ (see (1.4)) and $n \in \mathbb{N}$, then*
$$|\, f(x) - U_n(f, x)\, | \leq \frac{M\pi}{2} \frac{\sqrt{1-x^2}}{n+1} + |\, x\, |\, \mathcal{O}\left(\frac{\log n}{n^2}\right) \tag{2.3}$$
and $\mathcal{O}$ can not be replaced by o.

Proof. It is sufficient to consider the case $M = 1$. Notice that every function in $\mathrm{Lip}_1[-1, 1]$ is absolutely continuous. Let us write
$$K(t) = \sum_{k=1}^{\infty} \frac{\sin kt}{t}, \qquad I_n = \frac{2}{\pi}\int_0^{\pi} \left| K(t) - \sum_{k=1}^{n-1} \lambda_{n,k} \sin(kt) \right| dt$$

and

$$J_n = \frac{2}{\pi}\int_0^{\pi}\left|K(t) - \sum_{k=1}^{n-1}\lambda_{n,k}\sin(kt)\right|\sin t\,dt.$$

By setting $x = \cos\theta$ and integrating by parts, one has

$$f(\cos\theta) = \frac{a_0}{2} + \frac{1}{\pi}\int_0^{2\pi} K(t)\sin(t+\theta)f'(\cos(t+\theta))dt$$

and

$$U_n(f,\cos\theta) = \frac{a_0}{2} + \frac{1}{\pi}\int_0^{2\pi}\left(\sum_{k=1}^{n-1}\lambda_{n,k}\sin(k+t)\right)\sin(t+\theta)f'(\cos(t+\theta))dt.$$

Since $\mid f'(\cos(t+\theta)) \mid \leq 1$,

$$\mid f(\cos\theta) - U_n(f,\cos\theta) \mid \leq \frac{1}{\pi}\int_0^{2\pi}\left|K(t) - \sum_{k=1}^{n-1}\sin(k+t)\right| \mid \sin(t+\theta) \mid dt.$$

$$\leq I_n \mid \sin\theta \mid + J_n \mid \cos\theta \mid \leq I_n\sqrt{1-x^2} + J_n \mid x \mid .$$

Then Nikolskii proved that $I_n = \pi/(2n)$ and $J_n = \mathcal{O}(\ln n/n^2)$. $\square$

There is a great difference with Jackson's theorem: the position of x on the interval $[-1,1]$ is taken into account in the factor $\sqrt{1-x^2}$.

Timan and Dzyadik [380] proved that, if $f \in C^r[a,b]$ and $f^{(r)}$ is quasi-smooth (Zygmund), then $E_n(f) = \mathcal{O}(n^{-(r+1)})$. The sentence improves a result of Montel for $E_n(f)$ which gave an estimate only inside of the interval.

2.3 Problems connected with Nikolskii's result

Nikolskii's result motivated several investigations on the possibility of approximation (including the asymptotically best approximation) of functions of various classes by algebraic polynomials and many results concerning the improvement of approximation at the endpoints of the segment $[-1,1]$.

In 1958 Lebed [226] gave an extension of Nikolskii's theorem by considering functions in the Zygmund class:

$$Z[-1,1] = \{f : C[-1,1] \to \mathbb{R} \ : \mid \Delta_h^2 f(x) \mid \leq Mh,\ h \in (0,1]\}.$$

Theorem 2.3.1 (Lebed, [226]). *If $f \in C^m[-1,1]$ and $f^{(m)} \in Z[-1,1]$ (with constant M), then there exists a sequence $\{P_n\}$ $(P_n \in \mathbb{P}_n)$ such that*

$$\mid f(x) - P_n(x) \mid \leq C(m)\,M\,(\Delta_n(x))^m\left(\Delta_n(x) + \frac{\log n}{n^2}\right),$$

where

$$\Delta_n(x) = \left(\sqrt{1-x^2} + \mid x \mid /n\right). \tag{2.4}$$

The factor $\pi/2$ in (2.3) cannot be replaced by a smaller one (see (4.2)). Temlyakov proved the existence of a sequence $\{P_n\}$ for which an estimate with specific constants in both terms holds. His construction was not obtained by means of a sequence of linear operators, but he strengthened inequality (2.3) by omitting $\log n$ in the remainder.

Theorem 2.3.2 (Temlyakov, [372]). *Assume $f \in \mathrm{Lip}_1(1, [-1, 1])$. For any natural number n there exists an algebraic polynomial P_n of degree n such that*

$$| f(x) - P_n(x) | \leq \frac{\pi\sqrt{1-x^2}}{2n} + \frac{\pi^2 \mid x \mid}{8n^2}. \tag{2.5}$$

Proof. The proof is based on an inequality for the best trigonometric approximation of a differentiable function:

$$E_n(h) \leq \frac{K_r}{(n+1)^r} E_n(h^{(r)}), \qquad r \in \mathbb{N}, \tag{2.6}$$

where K_r is the Favard constant. Since $K_1 = \pi/2$, what we need is a good representation of the function $g(t) = f(\cos t)$. If

$$-f'(\cos t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kt) \qquad \text{and} \qquad \varphi(t) = \sum_{k=1}^{\infty} \frac{a_k}{k} \sin(kt),$$

then g can be written as

$$g(t) = -\frac{a_0}{2} \cos t + \varphi(t) \sin t + \sigma(t) \cos t + G(t),$$

where

$$\sigma(t) = \int_0^t \varphi(s) ds \qquad \text{and} \qquad G'(t) = -\sigma(t) \sin t.$$

Now, let u_{n-1} and v_{n-1} be the trigonometric polynomial of best approximation of order $n-1$ of the functions φ and σ respectively and define $P_n(\cos t) = u_{n-1}(t) \sin t$ and $Q_n(\cos t) = -v_{n-1}(t) \cos t$. Since $E_0(\varphi') \leq 1$, it follows from (2.6) that

$$\begin{aligned} | \varphi(t) - P_n(\cos t) | &\leq | \varphi(t) - u_{n-1}(t) || \sin t | \leq E_{n-1}(\varphi) | \sin t | \\ &\leq \frac{\pi}{2n} E_{n-1}(\varphi') | \sin t | \leq \frac{\pi}{2n} | \sin t | \end{aligned}$$

and

$$\begin{aligned} | \sigma(t) \cos t - Q_n(\cos t) | &\leq | \sigma(t) - v_{n-1}(t) || \cos t | \leq E_{n-1}(\sigma) | \cos t | \\ &\leq \frac{K_2}{n^2} E_{n-1}(\varphi') | \cos t | \leq \frac{K_2}{n^2} | \cos t | . \end{aligned}$$

Finally, since

$$\frac{d}{dx} G(\arccos x) = \frac{-G'(\arccos x)}{\sqrt{1-x^2}} = \frac{-G'(t)}{\sin t} = \sigma(t) = \sigma(\arccos x),$$

there exists $S_n \in \mathbb{P}_n$ such that

$$| (G(\arccos x))' - S_n(x) | \leq E_n(\sigma) \leq \frac{K_2}{(n+1)^2} E_n(\varphi') \leq \frac{K_2}{(n+1)^2}.$$

Thus, if we define

$$R_n(x) = G(\pi/2) + \int_0^x S_n(y)dy$$

then

$$| G(\arccos x) - R_n(x) |= \left| \int_0^x ((G(\arccos y))' - S_n(y))dy \right| \leq \frac{K_2 \mid x \mid}{(n+1)^2}.$$

The proof finishes by considering the polynomial

$$P(x) = -a_0 x/2 + P_n(x) + Q_n(x) + R_n(x). \qquad \square$$

The second term in (2.5) was written as $\mathcal{O}(\mid x \mid /n^2)$ in the original statement, but we prefer to present here what was really proved. If we want to compare (2.5) with (2.3), for $f \in \mathrm{Lip}_1(M, [-1,1])$ the last term in (2.5) should be multiplied by M. It could be avoided, if such a term can be replaced by zero. But Temlyakov did not know whether such a term can be removed. However, in the same paper he proved the following assertion. For each natural number n one can find a function $f_n \in \mathrm{Lip}_1(1, [-1,1]$ for which there exists no polynomial $P_n \in \mathbb{P}_n$ such that

$$| f_n(x) - P_n(x) | \leq \frac{\pi\sqrt{1-x^2}}{2(n+1)}.$$

From Theorem 2.3.2, making use of some arguments of Teliakovskii [371], Temliakov obtained the following theorem (the constant in $\mathcal{O}(1/n)$ was not given).

Theorem 2.3.3 (Temlyakov, [372]). *Let $f \in \mathrm{Lip}_1(1, [-1,1])$. For any natural number n there exists an algebraic polynomial P_n of degree n such that*

$$| f(x) - P_n(x) | \leq \frac{\pi\sqrt{1-x^2}}{2(n+1)} \left(1 + \mathcal{O}(1/n)\right).$$

In the chapter devoted to asymptotics we will include some other results. For differentiable functions Ligun presented in 1980 a version which provides some information concerning the constants.

Theorem 2.3.4 (Ligun, [236]). *Let r be an odd number. Then for any function $f \in C^r[-1,1]$ there exists a sequence of algebraic polynomials $\{Q_{n,r}(x)\}$ of degree not greater than $n \geq r$ such that, uniformly with respect to $x \in [-1,1]$,*

$$| f(x) - Q_{n-1,r}(x) | \leq \frac{K_r(\delta_n(x))^r}{2} \omega\left(f^{(r)}, \pi\delta_n(x)\right) + \mathcal{O}\left(\frac{1}{n^r}\omega\left(f^{(r)}, \frac{1}{n}\right)\right).$$

The proof of the last theorem is very long and technical, so it will not be included here. But we notice that the construction was obtained by means of linear operators.

For a given modulus of continuity ω and $r \in \mathbb{N}$, the associated Lipschitz class is defined by

$$H_\omega^k = \{f \in C[a,b] : \omega_k(f,t) \le C(f)\omega(t)\}.$$

Theorem 2.3.5 (Polovina, [286]). *Let w be a modulus of continuity. For each function $f \in C^1[-1,1]$ such that $f' \in H_w$, there exists a sequence of polynomials $\{P_n(f,x)\}$ ($P_n \in \mathbb{P}_n$) such that*

$$| f(x) - P_{n-1}(x) | \le \frac{1}{2}\int_0^{\pi\sqrt{1-x^2}/n} w(t)dt + o\left(\frac{1}{n}\omega\left(\frac{1}{n}\right)\right).$$

Moreover, if w is a concave modulus of continuity, $1/2$ can be changed to $1/4$. The constant $1/4$ cannot be made smaller.

In [18] Bashmakova presented a similar result for functions f such that $f' \in H_w$, for a continuous and concave modulus of continuity.

2.4 Timan-type estimates

In [373] Timan proved that, if $f \in \mathrm{Lip}_\alpha(M,[-1,1])$ and $S_n(f,x)$ is the nth partial sum of the Fourier-Chebyshev series of f, then

$$| f(x) - S_n(f,x) | \le \frac{2^{\alpha+1}M(1-x^2)^{\alpha/2}}{\pi^2}\frac{\log n}{n^2}\int_0^{\pi/2} t^\alpha \sin t dt + \mathcal{O}\left(\frac{1}{n^\alpha}\right).$$

Later, in 1951, he [374] improved the Nikolskii estimate as follows: for

$$f \in \mathrm{Lip}_\alpha([-1,1]) \quad (0 < \alpha \le 1)$$

one can find a sequence $\{P_n\}$ ($P_n \in \mathbb{P}_n$) such that

$$| f(x) - P_n(x) | \le \frac{C}{n^\alpha}\left((\sqrt{1-x^2})^\alpha + \left(\frac{| x |}{n}\right)^\alpha\right). \tag{2.7}$$

In the same year he generalized this result. For the proof we need the Jackson (also called Jackson-Matsuoka) kernels. [250]

$$K_{n,s}(t) = c_{n,s}\left(\frac{\sin(nt/2)}{\sin(t/2)}\right)^{2s}, \tag{2.8}$$

where $c_{n,s}$ is chosen from the condition $\pi^{-1}\int_{-\pi}^{\pi} K_{n,s}(t)dt = 1$.

Theorem 2.4.1 (Timan, [375]). *For $r \in \mathbb{N}_0$ there exists a constant C_r such that, for each $f \in C^r[-1,1]$ and $n \in \mathbb{N}$, one can find a polynomial $P_n(f) \in \mathbb{P}_n$ satisfying*

$$| f(x) - P_n(f,x) | \leq C_r \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \right)^r \omega \left(f^{(r)}, \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \right). \tag{2.9}$$

Proof. The proof presented in [379] (p. 262–266) follows by an inductive argument with respect to r. Here we show the case $r = 0$. Define

$$Q_{2n-2}(f,x) = \frac{1}{\pi} \int_0^\pi f(\cos t) \left[K_{n,2}(t+y) + K_{n,2}(t-y)\right] dt,$$

where $x = \cos y$. It can be proved that $Q_{2n-2}(f) \in \mathbb{P}_{2n-2}$. On the other hand,

$$\begin{aligned} | f(x) - Q_{2n-2}(f,x) | &= \left| \frac{1}{\pi} \int_0^\pi [f(\cos y) - f(\cos t)] \left[K_{n,2}(t+y) + K_{n,2}(t-y)\right] dt \right| \\ &\leq \frac{1}{\pi} \int_0^\pi \omega(f, | \cos y - \cos t |) \left[K_{n,2}(t+y) + K_{n,2}(t-y)\right] dt \\ &= \frac{1}{2\pi} \int_{-\pi}^\pi \omega \left(f, 2 \left| \sin \frac{t+y}{2} \sin \frac{t-y}{2} \right| \right) \left[K_{n,2}(t+y) + K_{n,2}(t-y)\right] dt. \end{aligned}$$

Let us estimate the integral corresponding to $K_{n,2}(t+y)$ (the other one can be estimated with similar arguments).

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\pi}^\pi \omega \left(f, 2 \left| \sin \frac{t+y}{2} \sin \frac{t-y}{2} \right| \right) K_{n,2}(t+y) dt \\ &\qquad = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \omega \left(f, 2 \left| \sin(t) \sin(t+y) \right| \right) K_{n,2}(2t) dt \\ &\qquad \leq \frac{1}{\pi} \int_0^{\pi/2} \left[\omega(f, t^2) + \omega(f, t \mid \sin y \mid)\right] K_{n,2}(2t) dt. \end{aligned}$$

If we consider that

$$\omega(f, t^2) \leq (1 + n^2 t^2)\, \omega(f, 1/n^2)$$

and

$$\omega(f, t \mid \sin y \mid) \leq (1 + nt)\, \omega(f, \sqrt{1-x^2}/n),$$

it is sufficient to verify that there exists a constant C (independent of n) such that

$$\int_0^{\pi/2} \left[1 + nt + (nt)^2\right] K_{n,2}(2t) dt \leq C. \qquad \square$$

In particular, if $\|f^{(r)}\| \leq M$,

$$| f(x) - P_n(x) | \leq \frac{M C_r}{n^r} \left(\sqrt{1-x^2} + \frac{| x |}{n} \right)^r. \tag{2.10}$$

Theorem 2.4.2 (Timan [376]). *If w is a modulus of continuity for which*

$$\sum_{n=1}^{\infty} \frac{1}{n} w\left(\frac{1}{n}\right) < \infty$$

and if, for $f \in C[-1,1]$ and algebraic polynomials P_n of degree at most n, $n = 1, 2, 3, \ldots$,

$$| f(x) - P_n(x) | \leq \Delta_n(x) w(\Delta_n(x)),$$

then $f \in C^1[-1,1]$.

Some years later Hasson showed that this last theorem cannot be improved.

Theorem 2.4.3 (Hasson, [157]). *Let $\{a_n\}$ be an increasing sequence of positive numbers such that $\sum_{n=1}^{\infty} 1/na_n = \infty$, then there exists a function f defined on $[0,1]$ and not continuously differentiable on that interval such that $E_n(f) = \mathcal{O}(1/na_n)$.*

Theorem 2.4.4 ([157]). *Let $f \in C[0,1]$. If $\sum_{n=1}^{\infty} n^{2r-1} E_n(f) < \infty$, then $f \in C^r[0,1]$, $r \in N$.*

Theorem 2.4.5 ([157]). *For every positive integer k and for every $0 < \alpha < 1$, there exists a function $f \in C[0,1]$ such that, for $n \in \mathbb{N}$, $E_n(f) \leq C_1 n^{-2(k+\alpha)}$ and such that $C_2 n^{-\alpha} \leq \omega(f^{(k)}, 1/n) \leq C_2 n^{-\alpha}$.*

Corollary 2.4.6 ([157]). *For every positive integer r and for every $0 < \beta < 1$, there exists a function $f \in C[0,1]$, $f \notin C^r[0,1]$ and $E_n(f) = \mathcal{O}(n^{2r-\beta})$.*

In 1958 Timan noticed that some asymptotics can be improved, if we take into account the position of the point x on the interval $[-1,1]$. From (2.10) we know that, if $f \in W^r(M, [-1,1])$ (see (1.14)) and $x \in [-1,1]$, then

$$\lim_{n\to\infty} \sup n^r \mid f(x) - P_n(f,x) \mid \leq M\, C_r(\sqrt{1-x^2})^r.$$

As Timan proved in [378], instead of C_r we can take Favard's constant

$$\lim_{n\to\infty} \sup n^r \mid f(x) - P_n(f,x) \mid \leq M\, K_r(\sqrt{1-x^2})^r \tag{2.11}$$

and K_r is the best constant for this kind of inequality. The construction of Timan was connected with the asymptotic best linear method of approximation in the class $W^r(M, [-1,1])$. He suggested that the same idea can be used to construct other asymptotic best linear methods and he considered some method of summation of Fourier series.

Theorem 2.4.1 involves the first modulus of continuity. In the note [377] of 1957, Timan also extended the Nikoslkii estimate (2.3) to the case of functions in the Zygmund class. For $f \in Z[-1,1]$ he constructed a sequence $\{P_n\}$ such that

$$| f(x) - P_n(x) | \leq C\, \Delta_n(x).$$

The results of Timan motivated investigations in several directions that we will present below. Some authors tried to change the first modulus of continuity in (2.9) by moduli of higher order. Brudnyi showed that such a change is possible. Others looked for results in which the function $\Delta_n(x)$ is replaced by $\delta_n(x)$. This approach began with the works of Teliakovskii and Gopengauz. On the other hand, Trigub noticed that the results of Timan can be generalized to include simultaneous approximation. That is, the same polynomials are used to approximate the functions and several derivatives. Finally, we can consider combinations of these ideas.

2.5 Estimates with higher-order moduli

In 1958 Dzyadyk [109] constructed some kernels which allowed him to give a new (and simpler) proof of Theorem 2.4.1 For a fixed $k \in \mathbb{N}$, $x \in [-\sqrt{2}, \sqrt{2}]$ and $n \in \mathbb{N}$, he defined

$$D_{n,k}(x) = \frac{1}{\gamma_{n,k}} \left(\frac{\sin \frac{1}{2} n \arccos\left(1 - x^2/2\right)}{\sin \frac{1}{2} \arccos\left(1 - x^2/2\right)} \right)^{2k}, \tag{2.12}$$

where $\gamma_{n,k}$ is taken from the condition $\int_{-1}^{1} D_{n,k}(x)dx = 1$. It is an even positive (in $[-\sqrt{2}, \sqrt{2}]$) algebraic polynomial of degree $2k(n-1)$ which can be written in terms of the Chebyshev polynomials T_n in the form

$$(D_{n,k}(x))^{1/k} = \gamma_{n,1} D_{n,1}(x) = 2\frac{1 - T_n(1 - x^2/2)}{x^2}.$$

Using these kernels, one can transform different approximation results related with the Féjer kernels to results concerning approximation by algebraic polynomials. In fact, with the substitution $x = 2\sin(t/2)$ we obtain from an even trigonometric kernel an algebraic kernel with similar properties in the neighborhood of the origin, and vice versa.

Dzyadyk not only presented a new proof of the direct result, he also improved (2.9) by using the second-order modulus instead of the first one. In fact, the kernel constructed in [109] is only used to provided a new proof of Theorem 2.4.1. The assertion relative to the second-order modulus appears (without proof) as a footnote on p. 343.

Theorem 2.5.1 (Dzyakyk, [109]). *For each $r \in \mathbb{N}_0$, there exists a constant C_r such that, for each $f \in C^r[a,b]$ and $n \in \mathbb{N}$ we can construct a polynomial $P_n(f) \in \mathbb{P}_n$ such that*

$$| f(x) - P_n(f,x) | \leq C_r \left(\rho_n(x)\right)^r \omega_2\left(f^{(r)}, \rho_n(x)\right), \tag{2.13}$$

where

$$\rho_n(x) = \frac{\sqrt{(b-x)(x-a)}}{n} + \frac{1}{n^2}.$$

Proof. The proof is presented for the interval $[0,1]$ and $r=1$.

1) We can assume that $\omega_2(f,t)>0$ for $t>0$ and $f(0)=f(1)$. It can be proved that there is a constant A such that

$$\frac{1}{n^2}\leq A\omega_2\left(f,\frac{1}{n}\right).$$

2) Set $g(x)=f(1-x)$, $x\in[0,1]$. It can be proved that there are extensions F and G of the functions f and g to the interval $[0,4]$ such that

$$\omega_2(F,t)\leq 25\omega_2(f,t)\quad\text{and}\quad\omega_2(G,t)\leq 25\omega_2(g,t).$$

3) Define

$$\varphi(x)=2\int_0^{1/3}F(x^2+9u^2)D_{n,3}(u)du$$

and

$$\psi(x)=2\int_0^{1/3}F(x^2+\frac{9}{2}u^2)D_{n,3}(u)du.$$

Since

$$\begin{aligned}
&\mid f(x^2)-2\psi(x)+\varphi(x)\mid\\
&=\left|f(x^2)\int_{1/3}^{1}D_{n,3}(u)du\right.\\
&\quad\left.+\int_0^{1/3}\left(F(x^2)-2F\left(x^2+\frac{9}{2}u^2\right)+F\left(x^2+9u^2\right)\right)D_{n,3}(u)du\right|\\
&\leq\frac{C_1}{n^5}\|f\|+2\int_0^{1/3}\omega_2\left(f,\frac{9}{2}u^2\right)K_{n,3}(u)du\\
&\leq\frac{C_1}{n^5}\|F\|+2\omega_2\left(f,\frac{1}{n^2}\right)\int_0^{1/3}\left(1+\frac{9}{2}(nu)^2\right)^2K_{n,3}(u)du\leq C_2\,\omega_2\left(f,\frac{1}{n^2}\right),
\end{aligned}$$

it is sufficient to approximate φ and ψ with polynomials in x^2.

4) Set

$$\begin{aligned}
P_1(x^2)&=\frac{1}{6}\int_{-2}^{2}F(u^2)\left[K_{n,3}\left(\frac{u+x}{3}\right)+K_{n,3}\left(\frac{u-x}{3}\right)\right]du\\
&=\frac{1}{2}\int_{(-2+x)/3}^{(2+x)/3}F((3u-x)^2)K_{n,3}(u)du+\frac{1}{2}\int_{(-2-x)/3}^{(2-x)/3}F((3u+x)^2)K_{n,3}(u)du
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_{-1/3}^{1/3} [F(x^2-6xu+9u^2)+F(x^2+6xu+9u^2)K_{n,3}(u)du \\
&\quad + \frac{1}{2}\left(\int_{(-2+x)/3}^{-1/3} + \int_{1/3}^{(2+x)/3}\right) F((3u-x)^2)K_{n,3}(u)du \\
&\quad + \frac{1}{2}\left(\int_{(-2-x)/3}^{-1/3} + \int_{1/3}^{(2-x)/3}\right) F((3u-x)^2)K_{n,3}(u)du \\
&= \frac{1}{2}\int_{-1/3}^{1/3} [F(x^2-6xu+9u^2)+F(x^2+6xu+9u^2)K_{n,3}(u)du + \|F\|\mathcal{O}(n^{-5}).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mid \varphi(x) - P_1(x^2) \mid &\le \int_0^{1/3} \omega_2(F, 6xu)K_{n,3}(u)du + C_3\|F\|\frac{1}{n^4} \\
&\le \omega_2\left(F, \frac{x}{n}\right)\int_0^{1/3}(1+6un)^2K_{n,3}(u)du + C_3\|F\|\frac{1}{n^4} \\
&\le C_4\left(\omega_2\left(F, \frac{x}{n}\right) + \omega_2\left(F, \frac{1}{n^2}\right)\right).
\end{aligned}
$$

The analogous construction for ψ is obtained by setting

$$
P_2(x^2) = \frac{\sqrt{2}}{6}\int_{-2}^{2} F(u^2)\left[K_{n,3}\left(\sqrt{2}\frac{u+x}{3}\right) + K_{n,3}\left(\sqrt{2}\frac{u-x}{3}\right)\right]du.
$$

Thus, if we define

$$
P_n(f, x^2) = 2P_2(x^2) - P_1(x^2),
$$

then

$$
\mid f(x) - P_n(f,x) \mid \le C_5\left(\omega_2\left(F, \frac{\sqrt{x}}{n}\right) + \omega_2\left(F, \frac{1}{n^2}\right)\right)
$$

for $x \in [0,1]$.

If we realize the analogous construction for the function G, then

$$
\begin{aligned}
\mid f(x) - P_n(G, 1-x) \mid &= \mid G(1-x) - P_n(G, 1-x) \mid \\
&\le C_5\left(\omega_2\left(F, \frac{\sqrt{1-x}}{n}\right) + \omega_2\left(F, \frac{1}{n^2}\right)\right).
\end{aligned}
$$

For the final construction take $m = [n/3]$ and define

$$
P_m(x^2) = (1-x)P_m(F,x) + xP_m(G,x).
$$

We know that $P_m \in \mathbb{P}_n$ and

$$\begin{aligned}|\, f(x) - P_n(x) \,| &\leq (1-x) \,|\, f(x) - P_n(f, 1-x) \,| + x \,|\, f(x) - P_n(G, 1-x) \,| \\ &\leq C_6 \left((1-x)\omega_2 \left(F, \frac{\sqrt{x}}{n} \right) + x\omega_2 \left(F, \frac{\sqrt{1-x}}{n} \right) + \omega_2 \left(F, \frac{1}{n^2} \right) \right) \\ &\leq C_7 \left(\omega_2 \left(F, \frac{\sqrt{x(1-x)}}{n} \right) + \omega_2 \left(F, \frac{1}{n^2} \right) \right). \qquad \square\end{aligned}$$

Dzyadyk also presented the following result.

Theorem 2.5.2 (Dzyakyk, [109]). *Assume that $f : [-1, 1] \to \mathbb{R}$ and $r \in \mathbb{N}_0$. One has $f \in C^r[-1, 1]$ and $f^{(r)} \in \mathrm{Lip}_1[-1, 1]$ if and only if there exists a sequence of polynomials $\{P_n\}$ $(P_n \in \mathbb{P}_n)$, such that*

$$|\, f(x) - P_n(x) \,| = o \left\{ \left(\frac{\Delta_n(x)}{n} \right)^{r+1} \right\}.$$

In 1959 Freud [123] (independently of Dzyadyk) constructed another sequence of polynomials for which a Timan result holds in terms of the second-order modulus. Freud used the method of intermediate spaces. That is, he first approximates the function f by an adequate piecewise linear function g and then approximates g by polynomials. Freud said that the construction of polynomial kernels (such as the one used by Timan) is not a simple task and he stated the problem of obtaining similar results using differences of higher order and good estimations for the constants. The extension to moduli of smoothness of arbitrary order was given by Brudnyi in 1963.

Theorem 2.5.3 (Brudnyi, [38]). *Given $r \in \mathbb{N}$, there exists a constant C_r such that, for each $f \in C[-1, 1]$ and $n \in \mathbb{N}$ $(n \geq r-1)$, there exists a polynomial $P_n(f) \in \mathbb{P}_n$ such that*

$$|\, f(x) - P_n(f, x) \,| \leq C_r \omega_r \left(f, \Delta_n(x) \right). \tag{2.14}$$

Let Φ^k denote the class of all non-decreasing continuous functions φ such that, $\varphi(0) = 0$ and $\varphi(t)/t^k$ is non-increasing. Sometimes this last condition is changed by the weaker one: $\varphi(t)/t^k \leq C\varphi(s)/s^k$, for $0 < s < t$. Functions of these classes are said to be of the type of the kth order modulus of continuity. It is known that if $\omega_k(f, t) \neq 0$, then $\omega_k(f, t) \in \Phi^k$ (see [247] and [347]).

For $\varphi \in \Phi^k$ and fixed constant M, set

$$H_k^\varphi(M, [-1, 1]) = \{ f : [-1, 1] \to \mathbb{R} : \omega_k(f, h) \leq M\varphi(h), h \in (0, 1/k] \}$$

and

$$W^r H_k^\varphi(M, [-1, 1]) = \{ f; f^{(k)} \in H_k^\varphi(M, [-1, 1]) \}$$

$(W^0(M, [-1, 1]) = H_k^\varphi(M, [-1, 1]))$. Moreover set

$$W^r H_k[\varphi] = \bigcup_{M>0} W^r H_k^\varphi(M, [-1, 1]).$$

Theorem 2.5.4. *Let k and r be natural numbers and assume*

$$\int_0^{b-a} \frac{\omega_{k+r}(f,u)}{u^{r+1}} du < \infty.$$

Then, for $0 < t \le b-a$,

$$\omega_k(f^{(r)},t) \le C(r,k) \int_0^t \frac{\omega_{k+r}(f,u)}{u^{r+1}} du.$$

This theorem was proved by Marchaud in [247] for $k=1$. For $k \ge 2$ the result was also proved by Brudnyi and Gopengauz in [41].

Guseinov [155] considered the problem of obtaining necessary and sufficient conditions on $\varphi \in \Phi^k$ and $w \in \Phi^{k+r}$ under which the equality $H_w^{k+r} = W^r H_\varphi^k$ holds.

Theorem 2.5.5 (Guseinov, [155]). *Let k, r be natural numbers and $w \in \Phi^r$ and set $\varphi(t) = w(t)/t^r$. Then $H_w^{k+r} = W^r H_\varphi^k$ if and only if*

$$\int_0^t \frac{\omega(u)}{u^{r+1}} du \le C(r,k) \frac{w(t)}{t^r}.$$

Theorem 2.5.6 (Guseinov, [155]). *Let k, r be natural numbers, $w \in \Phi^{r+k}$ and set $\varphi(t) \in \Phi^k$. Then $H_w^{k+r} = W^r H_\varphi^k$ if and only if $\varphi(t) \le Cw(t)/t^r$ and*

$$\int_0^t \frac{\omega(u)}{u^{r+1}} du \le C\varphi(t).$$

For the classes Φ^k the Brudnyi theorem yields

Theorem 2.5.7. *If $\varphi \in \Phi^k$, $r \in \mathbb{N}_0$ and $k \in \mathbb{N}$, there exists a constant $C = (r,k)$ such that, if $f \in W^r H_k^\varphi(1,[-1,1])$, then for any natural number $n \ge r+k-1$ there is an algebraic polynomial P_n of degree not greater than n, such that*

$$| f(x) - P_n(x) | \le C(r,k) \, (\Delta_n(x))^r \;\; \varphi(\Delta_n(x)). \tag{2.15}$$

2.6 Gopengauz-Teliakovskii-type estimates

In a Timan-type estimate the term $\Delta_n(x) = \sqrt{1-x^2}/n + 1/n^2$ appears. It is natural to ask whether such a term can be replaced by the simpler one $\delta_n(x) = \sqrt{1-x^2}/n$. A positive answer follows from the works of Teliakovskii (1966) using the first modulus of continuity

Theorem 2.6.1 (Teliakovskii, [371]). *Let r be a non-negative integer. There exists a constant C_r such that, for each $f \in C^r[-1,1]$ and $n > r$ one can find a polynomial $Q_n(f) \in \mathbb{P}_n$ such that*

$$| f(x) - Q_n(f,x) | \le C_r \; (\delta_n(x))^r \, \omega\left(f^{(r)}, \delta_n(x)\right). \tag{2.16}$$

Proof. We will consider only the case $r > 0$. Fix polynomials $P_n(f)$ such that the estimates (2.9) in Timan's Theorem 2.4.1 holds and define

$$Q_n(x) = P_n(f,x) + R_n(f,x)$$

where $R_n(f) \in \mathbb{P}_q$ is the polynomial which interpolates the function $f(x) - P_n(f,x)$ and its derivatives up to the order $q = [r/2]$ at the points ± 1. Teliakovskii proved that Q_n satisfies (2.16). The proof also uses some ideas of simultaneous approximation which will be presented in another section. In particular it follows from Theorem 2.8.10 that

$$\mid f^{(k)}(\pm 1) - P_n(f)^{(k)}(\pm 1) \mid \leq \frac{R}{n^{2(r-k)}} \omega \left(f^{(r)}, \frac{1}{n^2} \right),$$

where the constant R is independent of f and n. Now using the formula of interpolation of Hermite one has

$$\mid R_n(f,x) \mid \leq R \sum_{j=0}^{q} \frac{(1-x^2)^j}{n^{2(r-j)}} \, \omega \left(f^{(r)}, \frac{1}{n^2} \right) \tag{2.17}$$

and for $0 \leq k \leq r$ and $x \in [-1,1]$,

$$\mid R_n^{(k)}(f,x) \mid \leq \frac{C}{n^{2(r-q)}} \, \omega \left(f^{(r)}, \frac{1}{n^2} \right). \tag{2.18}$$

If $1/n \leq \sqrt{1-x^2}$, then from (2.9) and (2.17) one has

$$\begin{aligned} \mid f(x) - Q_n(x) \mid &\leq C_r \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \right)^r \omega \left(f^{(r)}, \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \right) \\ &\quad + R \left(\frac{\sqrt{1-x^2}}{n} \right)^r \omega \left(f^{(r)}, \frac{1}{n^2} \right) \sum_{j=0}^{q} \frac{1}{(n\sqrt{1-x^2})^{r-2k}} \\ &\leq C \, (\delta_n(x))^r \, \omega \left(f^{(r)}, \delta_n(x) \right). \end{aligned}$$

Now assume $1/n \geq \sqrt{1-x^2}$ and suppose $x > 0$. If $r > 0$ $(q+1 \leq r)$, then it follows from Theorem 2.8.10 and (2.18) that

$$\begin{aligned} &\mid f(x) - Q_n(x) \mid \\ &= \left| (-1)^{q+1} \int_x^1 \int_{u_1}^1 \cdots \int_{u_q}^1 [f^{(q+1)}(u) - P_n^{(q+1)}(f,u) - R_n^{(q+1)}(f,u)] du du_q \cdots du_1 \right| \\ &\leq \frac{R}{n^{2(r-q-1)}} \omega \left(f^{(r)}, \frac{1}{n^2} \right) \int_x^1 \int_{u_1}^1 \cdots \int_{u_q}^1 du du_q \cdots du_1 \\ &\leq \frac{R}{n^{2(r-q-1)}} \omega \left(f^{(r)}, \frac{1}{n^2} \right) (1-x^2)^{q+1}. \end{aligned}$$

Since $1/n \geq \sqrt{1-x^2}$,

$$\omega\left(f^{(r)}, \frac{1}{n^2}\right) \leq \frac{2}{n\sqrt{1-x^2}}\omega\left(f^{(r)}, \frac{\sqrt{1-x^2}}{n}\right),$$

we obtain

$$\begin{aligned} |\, f(x) - Q_n(x) \,| &\leq \frac{R}{n^{2r-2q-1}}(1-x^2)^{q+1/2}\omega\left(f^{(r)}, \frac{\sqrt{1-x^2}}{n}\right) \\ &= R\left(\frac{\sqrt{1-x^2}}{n}\right)^r \omega\left(f^{(r)}, \frac{\sqrt{1-x^2}}{n}\right)(n\sqrt{1-x^2})^{2q+1-r} \\ &\leq R\left(\frac{\sqrt{1-x^2}}{n}\right)^r \omega\left(f^{(r)}, \frac{\sqrt{1-x^2}}{n}\right). \end{aligned}$$

For $x < 0$ the result follows analogously. □

In 1967 Gopengauz obtained a similar result in terms of the modulus of continuity of second order.

Theorem 2.6.2 (Gopengauz, [151]). *For each $f \in C[-1,1]$ and $n \geq 2$, there exists $P_n \in \mathbb{P}_n$ such that*

$$|\, f(x) - P_n(x) \,| \leq C\omega_2\left(f, \delta_n(x)\right),$$

where the constant C does not depend on f or n.

Bashmakova and Malozemov gave an estimate including interpolation.

Theorem 2.6.3 (Bashmakova and Malozemov, [17]). *For $f \in C[-1,1]$ and $-1 = x_0 < x_1 < \cdots < x_n = 1$ $(n > 1)$ there exists $P_n(f) \in \mathbb{P}_n$ such that*

$$|\, f(x) - P_n(f,x) \,| \leq A\omega\left(f, \delta_n(x)\right)$$

and, for $x \in [-1,1]$ and $k = 1, \ldots, m-1$,

$$|\, f(x) - P_n(f,x) \,| \leq A\omega\left(f, |\, x - x_k \,|, \sqrt{|\, x - x_k \,|}/n\right).$$

There is also a more complicated version of Theorem 2.6.2 for fractional derivatives.

Theorem 2.6.4 (Shalashova, [336]). *If $f \in C[0,1]$ has continuous derivatives of fractional order r ($r = r' + \alpha$, with r' integer and $\alpha \in (0,1)$), there there exists for any $n \geq r-1$ a polynomial $P_n \in \mathbb{P}_n$ such that*

$$|\, f(x) - P_n(x) - Ax^r \,| \leq C_r\left(\frac{\sqrt{x(1-x)}}{n}\right)^r \omega\left(f^{(r)}, \frac{\sqrt{x(1-x)}}{n}\right),$$

where C_r does not depend on f or n and A depends on f and r. For fractional r, the term Ax^r can not be omitted.

A more general version of Theorem 2.6.1 was given by Stens in 1980. He proved the following theorem. The case $\alpha = 0$ and $\alpha = \sigma$ are very illustrative.

Theorem 2.6.5 (Stens, [348], [349]). *Let $s \in \mathbb{N}$ and $0 \le \alpha \le \sigma < s$. For $f \in C[-1,1]$ the following assertions are equivalent:*

(i) *There exists a sequence $\{P_n\}$, $P_n \in \mathbb{P}_n$ such that, for $x \in [-1,1]$,*

$$| f(x) - P_n(x) | \le C \left(\frac{(\sqrt{1-x^2})^{\sigma-\alpha}}{n^\sigma} \right).$$

(ii) *For $\varphi(x) = \sqrt{1-x^2}$,*

$$\sup_{|h| \le t} ||\varphi^\alpha \Delta_h^s f||_{C[-1,1]} \;\le\; C t^\sigma.$$

In [150] Gopengauz analyzed the following questions. Is it possible to obtain a Timan-type estimate but improving the rate of approximation in a certain interior point? Is it possible to obtain a speed of approximating at the ends of the segment greater than the one in Timan's theorem? He considered that the rate of approximation on the whole segment is retained. Both questions were answered in the negative. For instance, he proved that an estimation of the form

$$| f(x) - p_n(x) | \le C\omega \left(f, \frac{\psi_1(| x - a |) + \psi_2(1/n)}{n} \right),$$

is not possible for all $f \in C[-1,1]$, where $| a |< 1$ and ψ_i is an increasing function satisfying $\psi_i(t) \to 0$ as $t \to 0$ $(i = 1, 2)$. Moreover, we can not replace the expression $\Delta_n(x)$ by $o(\Delta_n(x))$.

In 1985 Yu showed some inequalities which are not possible. In particular, the following is proved.

Theorem 2.6.6 (Yu, [410]). *Let $r \in \mathbb{N} \cup \{0\}$ and $C > 0$. Then there exists a function $f \in C^r[-1,1]$ such that there exists no polynomial $P_n \in \Pi_n$ satisfying*

$$|f(x) - P_n(x)| \le C(\sqrt{1-x^2}/n)^r \omega_{r+3}(f^{(r)}, \sqrt{1-x^2}/n).$$

An analogous result is stated for the case in which the quantity $\sqrt{1-x^2}/n$ is replaced by $\sqrt{1-x^2}/n + \epsilon_n/n^2, \epsilon_n$ a positive number null sequence.

In 2000 Gonska, Leviatan, Shevchuk and Wenz presented a result in the following form.

Theorem 2.6.7 (Gonska, Leviatan, Shevchuk and Wenz, [148]). *Let $k \le r + 2$ and assume that $f \in C^r[-1,1]$. Then there is a polynomial $p \in \Pi_{2[(r+k+1)/2]-1}$ for which*

$$| f(x) - p(x) | \le C_r(\sqrt{1-x^2})^r \omega_k(f^{(r)}, \sqrt{1-x^2}), \quad -1 \le x \le 1, \qquad (2.19)$$

where C_r depends only on r. Moreover, for each $f \in C^r[-1,1]$ and $n \geq 2[(r+k+1)/2]-1$, there is a polynomial $P_n(f) \in \mathbb{P}_n$, such that

$$| f(x) - P_n(f,x) | \leq C(r) \left(\delta_n(x)\right)^r \omega_k \left(f^{(r)}, \delta_n(x)\right) \tag{2.20}$$

holds with a constant C_r depending only on r.

It should be noted that it is impossible to replace $2[(r+k+1)/2]-1$ by any lower figure. It has been shown by Yu [410] (see also Li [235]), that (2.20) is not valid if $k > r+2$: assume that $k > r+2 \geq 2$, then for each n and every constant $A > 0$, there exists a function $f = f_{r,k,n,A} \in C^r[-1,1]$ such that, for any polynomial $p_n \in \mathbb{P}_n$, there is a point $x \in [-1,1]$ for which

$$| f(x) - p_n(x) | > A(\delta_n(x))^r \omega_k(f^{(r)}, \delta_n(x))$$

holds. One also has

Theorem 2.6.8 ([148]). *Given $r \geq 0$, there exists a function $f \in C^r[-1,1]$ such that, for any algebraic polynomial p,*

$$\lim_{x \to -1} \sup \frac{| f(x) - p(x) |}{(\sqrt{1-x^2})^r \omega_{r+3}(f^{(r)}, \sqrt{1-x^2})} = \infty.$$

It is possible to construct a function which exhibits this phenomenon at both endpoints.

2.7 Characterization of some classes of functions

As we recall, if $f \in C[0,2\pi]$, $r \in \mathbb{N}$ and $\alpha \in (0,1)$, then $f \in C^r[0,2\pi]$ and $f^{(r)} \in \mathrm{Lip}_\alpha$ if and only if $E_n(f)^* = \mathcal{O}(n^{-(r+\alpha)})$. The same result is not true in the non-periodical case. Some characterizations appeared in works of Timan and Dzyadyk.

Theorem 2.7.1. *Let $f \in C[-1,1]$, r a positive integer and $\alpha \in (0,1)$. The following assertions are equivalent:*

i) *$f \in C^r[-1,1]$ and for each $x \in [-1,1]$,*

$$\sup_{\{h:\, |h| \leq \delta,\, |x+h| \leq 1\}} | f^{(r)}(x) - f^{(r)}(x+h) | \leq C\,\delta,$$

where C is a positive constant which does not depend on x or δ;

ii) *For each $n \in \mathbb{N}$ there exists $P_n \in \mathbb{P}_n$ such that, for each $x \in [-1,1]$,*

$$| f(x) - P_n(x) | \leq \frac{D}{n^{r+\alpha}} \left(\sqrt{1-x^2} + \frac{1}{n}\right)^{r+\alpha},$$

where D is a positive constant which does not depend on x or δ.

In 1957 Timan presented a converse result assuming that an estimate in terms of the function $\Delta_n(x)$ is known. For the definition of modulus of continuity see (1.7).

Theorem 2.7.2 (Timan, [376]). *Let ω be a modulus of continuity and $f : [-1,1] \to \mathbb{R}$ be a function and suppose there exists a sequence $\{P_n\}$ $(P_n \in \mathbb{P}_n)$ such that*

$$| f(x) - P_n(x) | \le \omega\left(\frac{1}{n}\left(\sqrt{1-x^2} + \frac{|x|}{n}\right)\right), \qquad x \in [-1,1].$$

Then

$$\omega(f,t) \le C\, t \int_t^1 \frac{\omega(s)}{s^2} ds, \qquad 0 < t \le \frac{1}{2},$$

where C is a fixed constant. Moreover, assume that $\int_0^1 (\omega(s)/s) ds < \infty$ and there exists a sequence $\{P_n\}$ $(P_n \in \mathbb{P}_n)$ such that, for $x \in [-1,1]$,

$$| f(x) - P_n(x) | \le \frac{1}{n^r}\left(\sqrt{1-x^2} + \frac{|x|}{n}\right)^r \omega\left(\frac{1}{n}\left(\sqrt{1-x^2} + \frac{|x|}{n}\right)\right). \tag{2.21}$$

Then $f \in C^r[-1,1]$ and

$$\omega(f^{(r)},t) \le C\left(\int_0^t \frac{\omega(s)}{s} ds + t\int_t^1 \frac{\omega(s)}{s^2} ds\right), \qquad 0 < t \le \frac{1}{2}.$$

Also a more general assertion can be proved:

$$\omega_r(f,t) \le C\, t^r \int_t^1 \frac{\omega(s)}{s^{r+1}} ds, \qquad 0 < t \le \frac{1}{2}.$$

Theorem 2.7.3 (Timan, [376]). *Let ω be a modulus of continuity such that*

$$\int_0^t \frac{\omega(s)}{s} ds \le C\omega(t), \quad \text{and} \quad t\int_t^1 \frac{\omega(s)}{s^2} ds \le C\omega(t) \tag{2.22}$$

and let $f : [-1,1] \to \mathbb{R}$ be a function. One has $f \in C^r[-1,1]$ and $\omega(f^{(r)},t) \le C\omega(t)$ if and only if there exists a sequence $\{P_n\}$ $(P_n \in \mathbb{P}_n)$ satisfying (2.21).

In order to obtain the converse result, different variants of the Bernstein inequality are needed. That is we should estimate the derivatives of an algebraic polynomial in terms of the polynomial.

Theorem 2.7.4. *Assume that $r, n \in \mathbb{N}$ and let $\|\cdot\|$ denote the uniform norm on $[-1,1]$.*

(i) (Markov, [248]) *If $P_n \in \mathbb{P}_n$, then*

$$\| P_n^{(r)} \| \le n^{2r} \| P_n \|.$$

(ii) (Bernstein, [27]) *If* $P_n \in \mathbb{P}_n$ *and* $x \in [-1,1]$, *then*

$$\left| \sqrt{1-x^2}P_n'(x) \right| \leq n \parallel P_n \parallel.$$

(iii) *There exists a constant* C_r *such that (see* [379], p. 227), *if* $P_n \in \mathbb{P}_n$ *and* $x \in [-1,1]$; *then*

$$\left| (\Delta_n(x))^r P_n^{(r)}(x) \right| \leq C_r \parallel P_n \parallel.$$

(iv) (Potapov, [288]) *If* $P_n \in \mathbb{P}_n$ *and* $x \in [-1,1]$, *then*

$$\left\| (\sqrt{1-x^2})^r P_n^{(r)}(x) \right\|_p \leq Cn^r \parallel P_n \parallel.$$

(v) *If* $p \geq 0$, $q \geq 0$ *and* $p+q = l$, *then there exists a constant* C_l *such that, if* $P_n \in \mathbb{P}_n$ *and* $x \in [-1,1]$, *then*

$$\left| (\Delta_n(x))^{q/2} P_n^{(l)}(x) \right| \leq C_l\, n^{l+p} \parallel P_n \parallel. \tag{2.23}$$

Theorem 2.7.5. *Fix positive constants* L *and* ρ.

(i) (Dzyadyk 1956, [107]) *If for* $x \in [-1,1]$ *a polynomial* $P_n \in \mathbb{P}_n$ *satisfies the inequality*

$$| P_n(x) | \leq L \left[(\sqrt{1-x^2})^\rho + \frac{1}{n^\rho} \right],$$

then there exists a constant C *(which depends only on* ρ*) such that, for* $x \in (-1,1)$ *one has*

$$| P_n'(x) | \leq C\, n\, L \min\left\{ (\sqrt{1-x^2})^{\rho-1}, \frac{1}{n^{\rho-1}} \right\}, \qquad if \quad \rho \leq 1$$

and

$$| P_n'(x) | \leq C\, n\, L \left[(\sqrt{1-x^2})^{\rho-1} + \frac{1}{n^{\rho-1}} \right], \qquad if \quad \rho \geq 1.$$

(ii) (Potapov 1960, [288]) *If* $\rho, \gamma \in \mathbb{R}$, *there exists a constant* C *such that, if for* $x \in [-1,1]$ *a polynomial* $P_n \in \mathbb{P}_n$ *satisfies the inequality*

$$| P_n(x) | \leq L\, (n+1)^{\gamma+\rho}\, (\Delta_{n+1}(x))^\rho\,,$$

then for $x \in (-1,1)$ *one has*

$$| P_n'(x) | \leq C\, L\, (n+1)^{\gamma+\rho}\, (\Delta_{n+1}(x))^{\rho-1}\,.$$

In (2.21) only integer values of r are involved. The characterization of Lipschitz functions was done by Dzyadyk.

Theorem 2.7.6 (Dzyadyk, [107]). *Assume that* $r \in \mathbb{N}_0$, $0 < \alpha < 1$ *and* C *is a positive constant. For a function* $f : [-1,1] \to \mathbb{R}$ *the following assertions are equivalent*:

(i) *For each* $n \in \mathbb{N}$, *there exists* $P_n \in \mathbb{P}_n$ *such that*

$$| f(x) - P_n(x) | \leq \frac{C}{n^{r+\alpha}} \left((\sqrt{1-x^2})^{r+\alpha} + \frac{1}{n^{r+\alpha}} \right). \tag{2.24}$$

(ii) $f \in C^r[-1,1]$ *and* $f^{(r)} \in \mathrm{Lip}_\alpha[-1,1]$.

Proof. (ii) $\Longrightarrow$ (i) follows from Timan's Theorem 2.4.1.

(i) $\Longrightarrow$ (ii). Let us first consider the case $r = 0$. Fix $h < 0$ and $x \in [0,1)$ such that $x + h \in (0,1]$. Let us write

$$U_{2^{i+1}}(x) = P_{2^{i+1}}(x) - P_{2^i}(x), \qquad i = 0, 1, \ldots.$$

Notice that $f(x) = P_1(x) + \sum_{i=1}^{\infty} U_{2^i}(x)$.

For any $k \in \mathbb{N}$ one has

$$\begin{aligned} | f(x+h) - f(x) | \leq{}& | P_1(x+h) - P_1(x) | + \sum_{i=0}^{k-1} | U_{2^{i+1}}(x+h) - U_{2^{i+1}}(x) | \\ &+ \sum_{i=k}^{\infty} | U_{2^{i+1}}(x+h) | + \sum_{i=k}^{\infty} | U_{2^{i+1}}(x) | . \end{aligned}$$

From (i) we obtain

$$\begin{aligned} | U_{2^{i+1}}(x) | &\leq | P_{2^{i+1}}(x) - f(x) | + | f(x) - P_{2^i}(x) | \\ &\leq \frac{2^{1+\alpha} C_1}{2^{i\alpha}} \left[(\sqrt{1-x^2})^{\alpha} + \frac{1}{2^{(i+1)\alpha}} \right]. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i=k}^{\infty} | U_{2^{i+1}}(x) | &\leq 2C_1 (\sqrt{1-x^2})^{\alpha} \frac{2^{2\alpha}}{2^{\alpha}-1} \frac{1}{2^{k\alpha}} + 2C_1 \frac{4^{\alpha}}{4^{\alpha}-1} \frac{1}{2^{k\alpha}} \\ &= \frac{2^{1+3\alpha} C_1}{2^{\alpha}-1} \frac{(\sqrt{1-x^2})^{\alpha}}{2^{(k+1)\alpha}} + \frac{4^{2\alpha+1/2}}{4^{\alpha}-1} \frac{C_1}{2^{2(k+1)\alpha}}. \end{aligned}$$

Now we consider two cases.

Case 1. Assume first that $x \geq 0$ and $x \in [1-2h, 1]$. Fix k such that

$$2^k \leq \frac{1}{\sqrt{h}} < 2^{k+1}.$$

From the arguments given above we know that, if $\xi \in [1-2h, 1]$, then

$$\sum_{i=k}^{\infty} \mid U_{2^{i+1}}(\xi) \mid \leq \frac{2^{1+3\alpha}C_1}{2^\alpha - 1} \frac{(\sqrt{1-(1-2h)^2})^\alpha}{2^{(k+1)\alpha}} + \frac{4^{2\alpha+1/2}}{4^\alpha - 1} \frac{C_1}{2^{2(k+1)\alpha}}$$

$$\leq \frac{2^{1+4\alpha}C_1}{2^\alpha - 1} h^{\alpha/2+\alpha/2} + \frac{4^{2\alpha+1/2}}{4^\alpha - 1} h^\alpha < \frac{64C_1}{2^\alpha - 1} h^\alpha.$$

On the other hand, taking into account Theorem 2.7.5 we obtain

$$\sum_{i=0}^{k-1} \mid U_{2^{i+1}}(x+h) - U_{2^{i+1}}(x) \mid \leq h \sum_{i=0}^{k-1} \mid U'_{2^{i+1}}(x + h\theta_i) \mid$$

$$\leq C_2 2^{1+\alpha} h \sum_{i=0}^{k-1} \frac{1}{2^{i\alpha}} 2^{(i+1)(2-\alpha)} = C_3 h \sum_{i=0}^{k-1} 2^{i(1-\alpha)} \leq C_4 h\, 2^{k(1-\alpha)}.$$

Thus, for $x \in [1-2h, 1-h]$ we have proved that there exists a constant K such that

$$\mid f(x+h) - f(x) \mid \leq Kh^\alpha.$$

Case 2. Assume that $x \geq 0$ and $x \in [0, 1-2h]$. Fix k such that

$$2^k \leq \frac{\sqrt{1-x^2}}{h} < 2^{k+1}.$$

Notice that

$$\frac{1}{2^{k+1}} < \frac{h}{\sqrt{1-x^2}} \leq \frac{h}{\sqrt{1-(1-2h)^2}} = \frac{h}{\sqrt{4h(1-h)}} \leq \sqrt{\frac{h}{2}}.$$

In this case, if $\xi \in [x, 1-h]$, then

$$\sum_{i=k}^{\infty} \mid U_{2^{i+1}}(\xi) \mid \leq \frac{2^{1+3\alpha}C_1}{2^\alpha - 1} (\sqrt{1-\xi^2})^\alpha \frac{h}{(\sqrt{1-\xi^2})^\alpha} + \frac{4^{2\alpha+1/2}C_1}{4^\alpha - 1} \left(\sqrt{\frac{h}{2}}\right)^{2\alpha} \leq C_5 h^\alpha.$$

For the estimate of the sum for $0 \leq i \leq k$, notice that for $x \in [0, 1-2h]$ and $0 < \theta < 1$

$$\frac{1-x^2}{2} \leq (1-(x+h)^2 \leq (1-(x+h\theta)^2$$

and

$$2h \leq 4h(1-h) = 1-(1-2h)^2 \leq 1-x^2.$$

Hence

$$\sum_{i=0}^{k-1} \mid U_{2^{i+1}}(x+h) - U_{2^{i+1}}(x) \mid \leq h \sum_{i=0}^{k-1} \mid U'_{2^{i+1}}(x+h\theta_i) \mid$$

$$\leq C_6\, h \sum_{i=0}^{k-1} \frac{1}{(\sqrt{1-(x+h\theta_i)^2})^{1-\alpha}} \frac{2^{i+1}}{2^{i\alpha}} \leq C_7\, h \sum_{i=0}^{k-1} 2^{i(1-\alpha)} \left(\sqrt{\frac{1-x^2}{2}}\right)^{\alpha-1}$$

$$\leq C_8\, h \frac{2^{k(1-\alpha)}}{(\sqrt{1-x^2})^{1-\alpha}} \leq C_9\, h \frac{1}{(\sqrt{1-x^2})^{1-\alpha}} \frac{(\sqrt{1-x^2})^{1-\alpha}}{h^{1-\alpha}} = C_9\, h^{\alpha}.$$

The theorem is proved for the case $r = 0$.

For $r > 0$ we differentiate the representation of f to obtain $f^{(r)}(x) = P_1^{(r)}(x) + \sum_{i=1}^{\infty} U_{2^i}^{(r)}(x)$. Then use Theorem 2.7.5 to obtain the inequality

$$\mid U_{2^{i+1}}^{(r)}(x) \mid \leq \frac{C}{2^{i\alpha}} \left[(\sqrt{1-x^2})^{\alpha} + \frac{1}{2^{(i+1)\alpha}} \right].$$

Then we can use arguments similar to ones for the case $r = 0$. □

With respect to the Zygmund class, Dzyadyk proved the following:

Theorem 2.7.7 (Dzyadyik, [107]). *Assume that $r \in \mathbb{N}_0$, $0 < \alpha < 1$. If for a function $f : [-1,1] \to \mathbb{R}$ there exists a sequence $\{P_n\}$ $(P_n \in \mathbb{P}_n)$ such that*

$$\mid f(x) - P_n(x) \mid \leq \frac{C}{n^{r+1}} \left((\sqrt{1-x^2})^{r+1} + \frac{1}{n^{r+1}} \right)$$

where C does not depend on n, then $f \in C^r[-1,1]$ and $f^{(r)} \in Z[-1,1]$.

In 1960 Potapov obtained a characterization related with the first modulus.

Theorem 2.7.8 (Potapov, [289]). *For $f \in C[-1,1]$ one has $E_n(f) = \mathcal{O}(n^{-\alpha})$ if and only if*

$$\mid f(\cos(\theta+t)) - f(\cos\theta) \mid \leq C \mid t \mid^{\alpha},$$

where C is a positive constant which does not depend on θ or t.

This result clearly shows that if for $f \in C[-1,1]$ one has $E_n(f) = \mathcal{O}(n^{-\alpha})$, then inside the segment f satisfies a Lipschitz condition of order α and in the end of the segment a Lipschitz condition of order $\alpha/2$.

The results of Timan and Dzyadyk seems to be of a point-wise nature. Some authors tried to put them as estimates in norm, but they used varying weights. For instance, Scherer-Wagner [330] defined the weighted best approximation by

$$E_n^{(r,\alpha)}(f) = \inf_{p\in\mathbb{P}_n} \left\| \frac{f(x)-p(x)}{(n\Delta_n(x))^{r+\alpha}} \right\|_C \tag{2.25}$$

and proved that (see Golitschek [143] for similar results concerning $L_p(-1,1)$)

$$E_n^{(r,\alpha)}(f) = \mathcal{O}(n^{-(r+\alpha)}) \Leftrightarrow f^{(r)} \in C[-1,1] \quad \text{and} \quad \omega_1(f^{(r)}, 1/n) \leq Cn^{-\alpha}.$$

Teliakovskii used Theorem 2.6.1 to obtain a characterization theorem using the function $\delta_n(x)$.

Theorem 2.7.9 (Teliakovskii, [371]). *Let r be a non-negative integer and $f:[-1,1]\to\mathbb{R}$ a function.*

(i) *Let w be a modulus of continuity satisfying* (2.22). *One has $f \in C^r[-1,1]$ and $\omega(f^{(r)}, t) \leq C(f)w(t)$ if and only if, for each $n > r$ there exists $P_n \in \mathbb{P}_n$ such that*

$$| f(x) - P_n(f,x) | \leq C(f) \, (\delta_n(x))^r \, \omega\left(\delta_n(x)\right).$$

(ii) *If $\alpha \in (0,1)$, one has $f \in C^r[-1,1]$ and $f^{(r)} \in \mathrm{Lip}_\alpha[1,1]$ if and only if, for each $n > r$ there exists $P_n \in \mathbb{P}_n$ such that*

$$| f(x) - P_n(f,x) | \leq C(f) \, (\delta_n(x))^{r+\alpha}.$$

Other classes can be characterized. Let us consider functions ψ satisfying the following condition:

$$\int_0^t \frac{\psi(u)}{u} du + t^k \int_t^1 \frac{\psi(u)}{u^{k+1}} du \leq C\psi(u). \tag{2.26}$$

This kind of function has been used in the works of Stechkin [347], Lozinskii [241] and Bari and Stechkin [15] to present results for the approximation of periodic functions.

Let us write

$$W^r H_k[\psi] = \{ f \, : \, \omega(f^{(r)}, t) \leq C(f)\psi(t) \}.$$

Theorem 2.7.10. *Fix ψ such that* (2.26) *holds. If for a function f and every $n \in N$ there exists a polynomial P_n such that* (2.15) *is satisfied, then $f \in W^r H_k[\psi]$.*

Thus, if $E_n(f) = \mathcal{O}(n^{-2r}\varphi(n^{-2}))$, then $f \in W^r H_k[\psi]$.

Notice that for $\varphi(t) = t^\alpha$ one has Dzyadyk's theorem. As Shevchuk showed, the converse of the last result is not true.

Theorem 2.7.11 (Shevchuk, [338]). *Suppose that ψ does not satisfy* (2.26).

- *There exists a function f for which $E_n(f)=\mathcal{O}(n^{-2r}\psi(n^2))$ and $f \notin W^r H_k[\psi]$.*
- *There exists a function $f \notin W^r H_k[\varphi]$ and a sequence $\{P_n\}$ of polynomials such that the Timan estimate holds.*

For $r = 1$, a significantly stronger result (which, in particular, implies Theorem 2.7.11 for $k = 1$, $r = 0$) was obtained earlier by Dolzhenko and Sevastyanov [103].

Theorem 2.7.12 (Shevchuk, [338]). *For any function $\varphi \in \Phi^k$, there is a function $f \in W^r H_k^{\varphi}$ such that*

(i) *For all $n \in N$, $E_n(f) \le n^{-2r}\varphi(n^{-2})$,*

(ii) *$\omega_k(f^{(r)}, t) \ge c\varphi(t)$, $t \in [0, 1/k]$, $c = c(r, k) > 0$.*

Theorem 2.7.13 ([338]). *Let a_n be an increasing sequence of natural numbers, such that $\sum_{n=1}^{\infty}(na_n)^{-1} = \infty$. There exists a function $f \in C[0,1]$ for which $f \notin C^r[0,1]$ and $E_n(f) = \mathcal{O}(n^{-2r}/a_n)$ $(r \ge 2)$.*

Theorem 2.7.13 was proved by another method in a paper of Xie [412]. The theorem was stated as a conjecture in the work [157] of Hasson. For the case $\varphi(t) = t^{\alpha}$, $0 < \alpha < k$, $\alpha \notin N$, Theorem 2.7.12 follows from results of Bernstein on the approximation of the function $(1-x)^{\alpha}$ on $[-1, 1]$. A proof of Theorem 2.7.12 for the indicated case can also be found in [157]. For $\varphi(t) = t^{\alpha}$, $0 < \alpha < k$, $\alpha \in N$ this theorem follows from results of Ibragimov. In connection with Theorem 2.7.12 we note the following example of Brudnyi [38]. The continuous function $f_{a,b} : [0,1] \to \mathbb{R}$, defined on $(0,1]$ by the formula $f_{a,b}(x) = x^a \sin x^{-b}$, $a, b > 0$, has for $k > a/(1+b)$ the modulus of continuity $\omega(f_{a,b})(t) = t^{a/(1+b)}$, whereas $E_n(f) \sim n^{-2a/(2b+1)}$. Theorems 2.7.11 and 2.7.12 show, in particular, that the assertion of the inverse theorem cannot be sharpened for any of the classes $W^r H_k[\varphi]$ if the rate of approximation is characterized not by the quantities $\rho_n(x)$ but rather by n^{-2}.

2.8 Simultaneous approximation

First, let us recall some facts related with trigonometric approximation. The approximation of the derivatives of a function by the derivatives of the polynomial which approximate the function was considered by Freud [122]. He proved that, for any polynomial T_n,

$$\|f^{(r)} - T_n^{(r)}\| \ \le \ C_r \ \{n^r \, \|f - T_n\| + E_n^*(f^{(r)})\}, \tag{2.27}$$

where C_r is a constant which depends only on r. A related inequality was given by Czipszer and Freud in [78].

Theorem 2.8.1. *Fix $k \in \mathbb{N}$.*

(i) *There exists a constant K such that, if $f \in C^k[0, 2\pi]$ and $T_n \in \mathbb{T}_n$, then*

$$\|f^{(k)} - T_n^{(k)}\| \le K \log\left(1 + \min\{k, n\}\right) \{n^r \, \|f - T_n\| + E_n^*(f^{(r)})\}.$$

Moreover, if $\|f - T_n\| \le CE_n^*(f)$, then

$$\|f^{(k)} - T_n^{(k)}\| \le KCE_n^*(f^{(r)}). \tag{2.28}$$

(ii) *There exists a constant K such that, if $f, f^{(k)} \in L_1[0,2\pi]$ and $T_n \in \mathbb{T}_n$ satisfies $\|f - T_n\|_1 \le C E_n^*(f)_1$, then*

$$\|f^{(k)} - T_n^{(k)}\|_1 \le CK \log(1 + \min\{k,n\})\, E_n^*(f^{(r)})_1.$$

(iii) *For each $p \in (1,\infty)$, there exists a constant B_p such that, if $f, f^{(k)} \in L_p[0,2\pi]$ and $T_n \in \mathbb{T}_n$ satisfies $\|f - T_n\|_p \le C E_n^*(f)_p$, then*

$$\|f^{(k)} - T_n^{(k)}\|_p \le C B_p E_n^*(f^{(r)})_p.$$

The inequality (2.27) was improved by Garkavi [133]. Set

$$C_{n,r}(f) = \inf_{T_n \in \mathbb{T}_n} \max_{1 \le k \le r} \frac{\|f^{(k)} - T_n^{(k)}\|}{E_n^*(f^{(k)})} \quad \text{and} \quad C_{n,r} = \sup_{f \in W^r(1,[0,2\pi])} C_{n,r}(f).$$

Garkavi proved that

$$C_{n,r} = \frac{4}{\pi^2}(\ln(p+1)) + \mathcal{O}(\ln\ln\ln p)),$$

where $p = \min\{n,r\}$ and

$$\|f^{(r)} - T_n^{(r)}\| \;\le\; n^r \, \|f - T_n\| + \left(1 + \frac{\pi}{2}\right) C_{n,r} E_n(f^{(r)}).$$

One of the first results on simultaneous approximation is due to Gelfond in 1955.

Theorem 2.8.2 (Gelfond, [141]). *If $f \in C^m[a,b]$, for $n \ge n_0$, there exists $P_n \in \mathbb{P}_n$ such that*

$$\|f^{(k)} - P_n^{(k)}\| \le C \frac{1}{n^{m-k}} \omega\left(f^{(m)}, \frac{1}{n}\right), \qquad (0 \le k \le m).$$

Theorem 2.8.3 (Feinerman and Newman, [117]). *There exists a constant K such that, if $f \in C^1[a,b]$, then*

$$E_n(f) \;\le\; \frac{K}{n} E_{n-1}(f') \qquad n \ge 1. \tag{2.29}$$

Hasson found estimates in norms in the spirit of Garkavi's results.

Proposition 2.8.4 (Hasson, [156]). *There exists a constant M with the following property: Let $f \in C[a,b]$ be such that, for some λ, $E_n(f) \le \lambda/n$, $n \ge 1$, $E_n(f) \le \lambda$. Then, if P_n is the polynomial of best approximation to f, one has*

$$\|P_n'\| \le \; M\lambda\, n, \qquad n \ge 1.$$

Proof. Fix k such that $2^k \le n < 2^k$. By differentiating the identity

$$P = P_n - P_{2^k} + \sum_{i=1}^{k}(P_{2^i} - P_{2^{i-1}}) + (P_1 - P_0) + P_0$$

and applying the Markov inequality we obtain

$$\begin{aligned}
\|P'\| &\le K\left(n^2\|P_n - P_{2^k}\| + \sum_{i=1}^{k} 2^{2i}\|P_{2^i} - P_{2^{i-1}}\| + (P_1 - P_0)\right)\\
&\le K\left(2n^2 E_{2^k}(f) + \sum_{i=1}^{k} 2^{2i+1} E_{2^{i-1}}(f) + 2E_0(f))\right)\\
&\le K\left(2\frac{2^{2(k+1)}}{2^k}\lambda + \sum_{i=1}^{k} 2^{2i+1}\frac{\lambda}{2^{i-1}} + 2\lambda)\right)\\
&\le K\,\lambda\left(82^k + 4\sum_{i=1}^{k} 2^i + 2)\right) \le\ M\,\lambda\, n. \qquad \square
\end{aligned}$$

Theorem 2.8.5 (Hasson, [156]). *Let k and r be integers. For $f \in C^r[a,b]$, let $P_n(f) \in \mathbb{P}_n$ be the polynomial of best approximation for f. There exist constants M, S and T depending on r such that*

$$\begin{aligned}
\|f^{(k)} - P_n^{(k)}(f)\| &\le M\, n^k\, E_{n-k}(f^{(k)}), & 0 \le k \le r,\quad n \ge k,\\
\|P_n^{(k)}(f)\| &\le \|f^{(k)}\| +\ M\, n^k\, E_{n-k}(f^{(k)}), & 0 \le k \le r,\quad n \ge k,
\end{aligned} \tag{2.30}$$

and

$$\|f^{(k)} - P_n^{(k)}(f)\| \le SE_{n-2k}(f^{(2k)}) \le TE_{n-r}(f)\frac{1}{n^{r-2k}}E_{n-r}(f^{(r)}),$$

for $0 \le k \le m/2$ and $n \ge m$.

Moreover (Roulier, [314])

$$\|f^{(k)} - P_n^{(k)}(f)\| \le\ M\frac{1}{n^{r-2k}}\omega\left(f^{(r)}\frac{1}{n}\right), \qquad n > r.$$

Proof. It is clear that (2.30) holds for $k = 0$. Assume that (2.30) holds for r. By induction, for $f \in C^{r+1}[0,1]$ one has

$$\|f^{(k+1)} - Q_{n-1}^{(k)}\| \le M\, n^k\, E_{n-1-k}(f^{(k+1)}), \qquad 0 \le k \le r,\quad n \ge k,$$

where Q_{n-1} is the polynomial of best approximation to f'. If we set

$$g(x) = f(x) - f(a) - \int_a^x Q_{n-1}(t)dt, \qquad x \in [a,b],$$

then, for $a \le x < y \le b$,

$$| g(x) - g(y) | \le \int_x^y | f'(t) - Q_{n-1}(t) | \, dt \le E_{n-1}(f') \; | x - y | \, .$$

Let R_n be the polynomials of best approximation to g. From the direct estimate and Proposition 2.8.4 we know that

$$\|R_n'\| \le K_1 n \, E_{n-1}(f'), \qquad n \ge 1,$$

and, using Markov inequality and (2.29), we obtain

$$\begin{aligned} \|R_n^{(k)}\| &\le K_k \, n \, n^{2(k-1)} \, E_{n-1}(f') \\ &\le K_k^* \frac{n^{2k-1}}{(n-1)(n-2)\cdots(n-(k-1))} E_{n-k}(f^{(k)} \\ &\le K_k' \, n^k \, E_{n-k}(f^{(k)}, \qquad 0 \le k \le r+1, \quad n \ge k. \end{aligned}$$

Therefore

$$\begin{aligned} \|f^{(k)} - Q_{n-1}^{(k-1)} - R_n^{(k)}\| \le \; & K_k' \, n^k E_{n-k}(f^{(k)}) + M_r n^{k-1} \, E_{n-k}(f^{(k)}) \\ \le \; & M_{r+1} \, n^k E_{n-k}(f^{(k)}), \qquad 0 \le k \le r+1, \quad n \ge k. \end{aligned}$$

The result follows because $-f(a) + \int_a^x Q_{n-1}(t)dt + R_n(x)$ is the polynomial of best approximation to f.

The last assertion follows from Jackson's theorem. In fact

$$E_{n-r}(f^{(r)}) \le \; C \, \omega \left(f^{(r)} \frac{1}{n-r} \right) \le C \left(1 + \frac{1}{n-r} \right) \omega \left(f^{(r)} \frac{1}{n} \right). \qquad \square$$

Theorem 2.8.6 ([156]). *Let $a < c < d < b$ and let m and k be integers with $0 \le k \le m$. There exists a constant C, which depends on m, c and d such that, if P_n is the polynomial of best approximation to $f \in C^m[a,b]$, then*

$$\|f^{(k)} - P_n^{(k)}\|_{[c,d]} \le C E_{n-k}(f^{(k)}), \qquad n \ge k.$$

Theorem 2.8.7 ([156]). *Let k and r be integers, $k > r \ge 0$. Fix $f \in C^r[a,b]$ and, for each $n \in \mathbb{N}$, let $P_n(f) \in \mathbb{P}_n$ be the polynomial of best approximation for f. If f is not a polynomial, there exist constants $M(f,k)$, such that*

$$\|P_n^{(k)}(f)\| \; \le \; M(f,r) \, n^{2k-r} \, \omega \left(f^{(r)}, \frac{1}{n} \right) \quad n \ge 1.$$

Proof. The proof of this theorem is based on an extension of f. Fix two reals c and d ($c < a$ and $b > d$) and assume that f has been extended to a function $F \in C^r[c,d]$ in such a way that $\omega(F^{(r)}, h) \le C\omega(f^{(r)}, h)$.

Fix a sequence $\{Q_n\}$ of polynomials such that

$$\|Q_n^{(k)} - F^{(k)}\|_{[c,d]} \leq \frac{K}{n^{r-k}}\omega\left(F^{(r)}, \frac{1}{n}\right) \leq \frac{CK}{n^{r-k}}\omega\left(f^{(r)}, \frac{1}{n}\right),$$

for $k \leq r$ and $n \geq k+1$. One has $\|Q_n^{(k)}\|_{[c,d]} \leq C_k$, for $0 \leq k \leq r$ and (by Bernstein's inequality) $\|Q_n^{(k)}\|_{[c,d]} \leq K_k n^{k-r}$, for $k > r$. Since, for $k \geq 0$,

$$\|P_n^{(k)}\|_{[a,b]} \leq \|P_n^{(k)} - Q_n^{(k)}\|_{[a,b]} + \|Q_n^{(k)}\|_{[a,b]}$$

and

$$\|P_n^{(k)} - Q_n^{(k)}\|_{[a,b]} \leq S_k\, n^{2k}[\|P_n - Q_n\|_{[a,b]} \leq N_k n^{2k}\left(E_n(f) + \frac{Kl}{n^r}\omega(f^{(r)}, \frac{1}{n}\right),$$

Jackson's inequality yields

$$\|P_n^{(k)}\|_{[a,b]} \leq C_5\, n^{2k-r}\omega\left(f^{(r)}, \frac{1}{n}\right) + \max\left(K_k, K_k n^{k-r}\right).$$

Thus the proof finishes by proving that the second term can be estimated with the first one. □

Trigub was one of the first in considering a point-wise estimate for simultaneous approximation by algebraic polynomials. He also noticed that we can use the second-order modulus, instead of the first one, and provided some inequalities for the derivatives of the polynomials. In 1968 Malozemov [246] proved that the constant in the corresponding estimates of Gelfond and Trigub do not depend on the functions. We present the assertion as it appeared in a paper of Malosemov [245].

Theorem 2.8.8 (Trigub, [388]). *If $f \in C^r[-1,1]$, then for each $n \in \mathbb{N}$ there exists a polynomial $P_n \in \mathbb{P}_n$ such that, for all $x \in [-1,1]$ and $k = 0, 1, \dots, r$,*

$$|\, f^{(k)}(x) - P_n^{(k)}(x)\,| \leq C_r\,(\Delta_n(x))^{r-k}\, w\left(f^{(r)}, \Delta_n(x)\right) \tag{2.31}$$

where C_r does not depend upon n or f.

Is the last result a consequence of the particular polynomials used in the approximation? In 1966 Teliakovskii showed that, for a differentiable function f, the derivatives of any sequence of the polynomials which approximate f with the rate given in Timan's theorem, approximate f' with a similar rate.

We need an estimate for the derivatives of polynomials.

Proposition 2.8.9. *There exists a constant R with the following property: let $a \geq 0$ be a real number, $r \geq 1$ an integer and ω a modulus of continuity. If a polynomial P_n satisfies the inequality*

$$|\, P_n(x)\,| \leq\, (\Delta_n(x))^r\, \omega(\Delta_n(x)) + a, \qquad x \in [-1,1],$$

then

$$|\, P_n'(x)\,| \leq\, R\left((\Delta_n(x))^{r-1}\, \omega(\Delta_n(x)) + a(\Delta_n(x))^{-1}\right), \qquad x \in [-1,1].$$

The last result was proved by Lebed [225] in the case $a = 0$. Other proofs were given in [107] and [379] (p. 219–226). According to Teliakovskii [371], for $a > 0$ the proof can be obtained with arguments similar to the one used in [379].

Theorem 2.8.10 (Teliakovskii, [371]). *Assume $r \in \mathbb{N}_0$ and $f \in C^r[-1,1]$. If $\{P_n(f,x)\}$ is a sequence of polynomials satisfying* (2.9)*, then for $k = 1, \dots, r$,*

$$| f^{(k)}(x) - P_n^{(k)}(f,x) | \leq C_{r,k} \left(\Delta_n(x)\right)^{r-k} \omega\left(f^{(r)}, \Delta_n(x)\right),$$

where the constant $C_{r,k}$ does not depend upon f or n.

Proof. We only present the main ideas of the proof.

Let $\{P_n\}$ be a sequence of polynomials for which the Timan estimate (2.9) holds. For $s \in \mathbb{N}_0$, write $n_s = 2^s n$, $p_0 = P_n$ and $p_s = P_{n_s}$. From the identity

$$f(x) - p_0(x) = \sum_{s=1}^{\infty} [p_s(x) - p_{s-1}(x)]$$

we obtain

$$\begin{aligned}
| f^{(k)}(x) - p_0^{(k)}(x) | &= \left| \sum_{s=1}^{\infty} [p_s^{(k)}(x) - p_{s-1}^{(k)}(x)] \right| \\
&\leq (RA + R) \sum_{s=1}^{\infty} \left(\frac{\sqrt{1-x^2}}{n_s} + \frac{1}{n_s} \right)^{r-k} \omega\left(f^{(r)}, \frac{\sqrt{1-x^2}}{n_s} + \frac{1}{n_s} \right) \\
&\leq (RA + R) \sum_{s=1}^{\infty} \left(\frac{\sqrt{1-x^2}}{2^s n} + \frac{1}{4^s n^2} \right)^{r-k} \omega\left(f^{(r)}, \frac{\sqrt{1-x^2}}{2^s n} + \frac{1}{4^s n^2} \right).
\end{aligned}$$

If $k < r$, then

$$| f^{(k)}(x) - P_n^{(k)}(x) | \leq C \left(\Delta_n(x)\right)^{r-k} \omega\left(f^{(r)}, \Delta_n(x)\right) \sum_{s=1}^{\infty} \frac{1}{2^{s(r-k)}}.$$

The theorem is proved for $k < r$.

For the case $k = r$, it is sufficient to consider the case $r = 1$.

Assume $r = 1$ and fix a point x_0 and set $h = \Delta_n(x_0)$. There exists a function $F_h(f) \in C^1[-1,1]$ such that

$$| f(x) - F_h(f,x) | \leq \frac{1}{2} h\, \omega(f', h), \tag{2.32}$$

$$| f'(x) - F_h'(f,x) | \leq \omega(f', h) \tag{2.33}$$

and

$$\omega(F_h', t) \leq \begin{cases} \delta\, \omega(f', h)/h, & \text{if} \quad \delta \leq h, \\ 3\omega(f', h), & \text{if} \quad h < \delta. \end{cases} \tag{2.34}$$

Then

$$\begin{aligned}\mid f'(x)-p_n'(x)\mid &\le\mid f'(x)-F_h'(x)\mid+\mid F_h'(x)-p_n'(x)\mid\\ &\le \omega(f',h)+\mid F_h'(x)-p_n'(x)\mid .\end{aligned}\tag{2.35}$$

There exist polynomials Q_m such that

$$\mid F_h'(x)-Q_m(x)\mid\le C\,\Delta_m(x)\,\omega(F_h',\Delta_m(x)).\tag{2.36}$$

Now, we use the representation

$$F_h(x)-p_n(x)=\sum_{s=1}^{\infty}[Q_{n_s}(x)-Q_{n_{s-1}}(x)]+Q_n(x)-p_n(x).\tag{2.37}$$

In this case we have

$$\mid Q_{n_s}(x)-Q_{n_{s-1}}(x)\mid\le C\,\Delta_{n_s}(x)\,\omega\left(F_h',\Delta_{n_s}(x)\right).$$

From Proposition 2.8.9 (with $a=0$) we obtain

$$\mid Q_{n_s}'(x)-Q_{n_{s-1}}'(x)\mid\le C_1\ \omega\left(F_h',\Delta_{n_s}(x)\right).$$

On the other hand, we can use (2.36), (2.32), the hypothesis (2.9) and (2.34) to estimate the difference Q_n-p_n. In fact

$$\begin{aligned}\mid Q_n(x)-p_n(x)\mid &\le\mid Q_n(x)-F_h(x)\mid+\mid F_h(x)-f(x)\mid+\mid f(x)-p_n(x)\mid\\ &\le C\,\Delta_n(x)\,\omega(F_h',\Delta_n(x))+\frac{1}{2}h\omega(f',h)+A\Delta_n(x)\omega(f',\Delta_n(x))\\ &\le C_2\,\Delta_n(x)\,\omega(f',\Delta_n(x))+\frac{1}{2}h\omega(f',h).\end{aligned}$$

From the last estimate and Proposition 2.8.9 (with $a=h\omega(f',h)/2$) we obtain

$$\mid Q_n'(x)-p_n'(x)\mid\le C_3\ \omega(f',\Delta_n(x))+C_4(\Delta_n(x))^{-1}h\omega(f',h).$$

Therefore the series in (2.37) converges uniformly and we can differentiate term by term. That is

$$\mid F_h'(x)-p_n'(x)\mid\le C_1\left(\sum_{s=1}^{\infty}\omega(F_h',\Delta_{n_s}(x))+\omega(f',\Delta_n(x))+(\Delta_n(x))^{-1}h\omega(f',h)\right).$$

Finally, for $x=x_0$ and $h=\Delta_n(x_0)$, from the last inequality and (2.35) one has

$$\begin{aligned}\mid f'(x_0)-p_n'(x_0)\mid &\le C\left(\sum_{s=1}^{\infty}\frac{\Delta_{n_s}(x_0)\omega(f',h)}{h}+\omega(f',\Delta_n(x_0))+\frac{h\omega(f',h)}{\Delta_n(x_0)}\right)\\ &\le C\left(\omega(f',\Delta_n(x_0))+\frac{\omega(f',h)}{h}\sum_{s=1}^{\infty}\left(\frac{\sqrt{1-x_0^2}}{2^s n}+\frac{1}{4^s n^2}\right)\right)\\ &\le C\omega(f',\Delta_n(x_0)).\end{aligned}$$

□

In particular, from Timan's theorem Teliakovskii derived a new proof of the Trigub result presented above. On the other hand, Theorem 2.8.10 can be obtained from Theorem 2.8.8 and Proposition 2.8.9.

An analogue of Theorem 2.8.10, with $\delta_n(x)$ instead of $\Delta_n(x)$ is due to Gopengauz. He constructed linear polynomial operators $L_{n,r} : C^r[-1,1] \to \mathbb{P}_n$ for each fixed $r \geq 0$, such that the following theorem holds:

Theorem 2.8.11 (Gopengauz, [150]). *For each $r \geq 0$ there exists a sequence of linear operators $L_{n,r} : C^r[-1,1] \to \mathbb{P}_n$ $(n \geq 4r+5)$ such that, for $f \in C^r[-1,1]$ for $0 \leq k \leq r$,*

$$| f^{(k)}(x) - L_{n,r}^{(k)}(x) | \leq C_r \left(\delta_n(x)\right)^{r-k} \omega\left(f^{(r)}, \delta_n(x)\right), \tag{2.38}$$

where the constant C_r does not depend on f, n and x.

In 1978 Vértesi noticed that, under additional assumptions, one can replace $\Delta_n(x)$ by $\delta_n(x)$.

Theorem 2.8.12 (Vértesi, [399]). *Assume $r \in \mathbb{N}_0$ and $f \in C^r[-1,1]$. If $\{P_n(f,x)\}$ is a sequence of polynomials satisfying (2.9) and*

$$P_n^{(k)}(f, \pm 1) = f^{(k)}(\pm 1), \qquad (k = 0, 1, \dots, r),$$

then for $k = 0, 1, \dots, r$,

$$| f^{(k)}(x) - P_n^{(k)}(f,x) | \leq C_{r,k} \left(\delta_n(x)\right)^{r-k} \omega\left(f^{(r)}, \delta_n(x)\right),$$

where the constant $C_{r,k}$ does not depend on f or n.

There are other similar inequalities due to Gonska and Hinnemann.

Theorem 2.8.13 (Gonska and Hinnemann, [147]). *Fix an integer $r \geq 0$, a constant C_r and let $L_n : C[-1,1] \to \mathbb{P}_n$ $(n \geq r)$ be a sequence of linear operators such that, for every $x \in [-1,1]$ and $f \in C^r[-1,1]$,*

(i) $$\|L_n(f)\| \leq C_r \|f\|, \ f \in C[-1,1],$$

(ii) $$| f(x) - L_n(f,x) | \leq C_r \left(\Delta_n(x)\right)^r \|f^{(r)}\|.$$

Then, there exists a constant D_r such that, for each $0 \leq k \leq r$ and $f \in C^r[-1,1]$,

$$\|L_n^{(k)}(f)\| \leq C_r \|f^{(k)}\|.$$

Theorem 2.8.14 ([147]). *Fix $r \geq 0$, $s \geq 1$ and let C_r and $C_{r,s}$ be constants.*

(i) *There exists a constant D_r such that, if $f \in C^r[-1,1]$ and $P_n \in \mathbb{P}_n$ $(n \geq r)$ satisfies*

$$| f(x) - P_n(x) | \leq C_r \left(\Delta_n(x)\right)^r \|f^{(r)}\|,$$

then for $0 \leq k \leq r$

$$| f^{(k)}(x) - P_n^{(k)}(x) | \leq D_r \left(\Delta_n(x)\right)^{r-k} \|f^{(r)}\|.$$

(ii) *There exists a constant $M_{r,s}$ such that, if $f \in C^r[-1,1]$ and for $P_n \in \mathbb{P}_n$ $(n \geq r+s)$ one has*

$$| f(x) - P_n(x) | \leq C_{r,s} (\Delta_n(x))^r \omega_s(f^{(r)}, \Delta_n(x)),$$

then, for $0 \leq k \leq r$,

$$| f^{(k)}(x) - P_n^{(k)}(x) | \leq M_{r,s} (\Delta_n(x))^{r-k} \omega_s(f^{(r)}, \Delta_n(x)).$$

The Hasson results (Theorem 2.8.5 and 2.8.6) involve estimates in norm. In [234] Leviatan found point-wise estimates in the spirit of the results of Timan and Trigub, but considering the best approximation instead of the modulus of smoothness of the derivatives.

Theorem 2.8.15 (Leviatan, [234]). *For $r \geq 0$ let $f \in C^r[-1,1]$ and let $P_n \in \mathbb{P}_n$ denote its polynomial of best approximation on $[-1,1]$. Then for each $0 \leq k \leq r$ and every $-1 \leq x \leq 1$,*

$$| f^{(k)}(x) - P_n^{(k)}(x) | \leq \frac{C_r}{n^r} [\Delta_n(x)]^{-k} E_{n-k}(f^{(k)}), \quad n \geq k,$$

and

$$| f^{(k)}(x) - P_n^{(k)}(x) | \leq \frac{C_r}{n^r} [\Delta_n(x)]^{-k} E_{n-r}(f^{(r)}), \quad n \geq k,$$

where C_r is an absolute constant which depends only on r.

Proof. For $k = 0$ the result is evident. Assume that it is true for r. By induction, for $f \in C^{r+1}[0,1]$ one has

$$| f^{(k+1)}(x) - Q_{n-1}^{(k)}(x) | \leq \frac{M}{n^k} (\Delta_n(x))^{-k} E_{n-1-k}(f^{(k+1)}), \qquad 0 \leq k \leq r, \quad n \geq k,$$

where Q_{n-1} is the polynomial of best approximation to f'. If we set

$$g(x) = f(x) - \int_{-1}^{x} Q_{n-1}(t)dt = f(x) - Q_n(x), \qquad x \in [-1,1],$$

then, $| g'(x)) | \leq C E_{n-1}(f')$.

There exists a polynomial S_n such that

$$\|g - S_n\| \leq \frac{C}{n} E_{n-1}(f'), \qquad \text{and} \qquad \|S_n'\| \leq C\, E_{n-1}(f').$$

Thus, from (iii) of Theorem 2.7.4 one has

$$| S_n^{(k)}(x) | \leq C(\Delta_n(x))^{1-k} \|S_n'\| \leq C_1(\Delta_n(x))^{1-k} E_{n-1}(f').$$

Let R_n be the polynomial of best approximation to g. Using again (iii) of Theorem 2.7.4 and taking into account that $E_n(g) = E_n(f)$, one has

$$\begin{aligned}| R_n^{(k)}(x) - S_n^{(k)}(x) | &\le C(\Delta_n(x))^{-k}\|R_n - S_n\| \le C(\Delta_n(x))^{-k}\left(E_n(g) + \|g - S_n\|\right)\\ &\le C_1 \frac{(\Delta_n(x))^{-k}}{n} E_{n-1}(f').\end{aligned}$$

Therefore

$$| R_n^{(k)}(x) | \le C_2 \frac{(\Delta_n(x))^{-k}}{n} E_{n-1}(f') \le C_3 \frac{(\Delta_n(x))^{-k}}{n^k} E_{n-k}(f^{(k)}).$$

Since $P_n = Q_n + R_n$ is the polynomial of the best approximation of f we have the result. □

For the last theorem some of the results of Hasson are easily derived.

Theorem 2.8.16 ([234]). *For $r \ge 0$ let $f \in C^r[-1,1]$ and let $n \ge r$. Then there exists a polynomial $P_n \in \mathbb{P}_n$ such that*

$$| f^{(k)}(x) - P_n^{(k)}(x) | \le C_r \, [\Delta_n(x)]^{r-k} \, E_{n-r}(f^{(r)}), \quad n \ge k, \tag{2.39}$$

for $k = 0, 1, \ldots, r$ and $-1 \le x \le 1$.

Kilgore combined the estimates of Gopengauz and Leviatan.

Theorem 2.8.17 (Kilgore, [191]). *If $f \in C^m[-1,1]$, for each $n > 2m$, there exists a polynomial $P_n \in \mathbb{P}_n$ such that, for $k = 0, 1, \ldots, m$,*

$$| f^{(k)}(x) - P_n^{(k)}(x) | \le C(m,k) \left(\frac{\sqrt{1-x^2}}{n} \right)^{m-k} E_{n-m}(f^{(m)}), \tag{2.40}$$

where the constants $C(m,k)$ depend only on m and k.

An algebraic analog of the result of Czipszer and Freud in [78] is the following.

Theorem 2.8.18 (Kilgore and Szabados, [196]). *Let $g \in C^q[-1,1]$ be such that $g^{(k)}(\pm 1) = 0$ for $k \le q-1$. Let $\varepsilon > 0$ and assume there is a sequence $\{P_{n+q}\}$ $(P_{n+q} \in \mathbb{P}_{n+q})$ such that*

$$\left| \frac{g(x) - P_{n+q}(x)}{(\sqrt{1-x^2})^q} \right| \le \frac{\varepsilon}{n^q}.$$

Then, for $|x| \le 1$ and $k \le q$,

$$\left| \big(g(x) - p_n(x)\big)^{(k)} \right| \le \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \right)^{q-k} \left(\delta_{k,q} \inf_{p_n} \left\| (g - p_n)^{(q)} \right\| + \gamma_{k,q}\varepsilon \right),$$

where $\delta_{k,q}$ and $\gamma_{k,q}\varepsilon$ depend on k and q.

2.9 Zamansky-type estimates

As we see in Theorem 1.2.1 concerning trigonometric approximation, for $\sigma < s$, the conditions $E_n^*(f) = \|f - T_n\| = \mathcal{O}(n^{-\sigma})$ and $\|T_n^{(s)}\| = \mathcal{O}(n^{-(\sigma-s)})$ are equivalent. As Hasson showed there is not a direct analogue in the algebraic case.

Theorem 2.9.1 (Hasson, [156]). *There exists a function $f \in C[-1,1]$ such that $E_n(f) \leq K/n$ and, if P_n is the polynomial of best approximation to f on $[-1,1]$, $\|P_n'\|_{[a,b]} > K \log n$, $n \in \mathbb{N}$, whenever $-1 < a < b < 1$.*

Leviatan also studied the growth of the sequence $\{P_n^{(k)}\}$. His proof is based in a theorem of Runck.

Theorem 2.9.2 (Runck, [315]). *For $r \geq 0$ let $f \in C^r[-1,1]$ and let $n \geq r$. Then there exists a polynomial $P_n \in \mathbb{P}_n$ such that*

$$| f^{(k)}(x) - P_n^{(k)}(x) | \leq C_k \, [\Delta_n(x)]^{r-k} \, \omega(f^{(r)}, \Delta_n(x)), \qquad 0 \leq k \leq r$$

and

$$| P_n^{(k)}(x) | \leq C_r \, [\Delta_n(x)]^{r-k} \omega \left(f^{(r)}, \Delta_n(x) \right), \quad k \geq r+1,$$

with constant independent of f.

Theorem 2.9.3 (Leviatan, [234]). *For $r \geq 0$ let $f \in C^r[-1,1]$ and let $P_n \in \mathbb{P}_n$, denote its polynomial of best approximation on $[-1,1]$. Then for each $k > r$ there exists a constant K, depending only on k, such that, for every $-1 \leq x \leq 1$,*

$$| P_n^{(k)}(x) | \leq \frac{K}{n^r} \, [\Delta_n(x)]^{-k} \, \omega \left(f^{(r)}, \frac{1}{n} \right), \quad n \in \mathbb{N}.$$

This improves some results of Hasson. In particular, for $k > r$.

$$\|P_n^{(k)}\| \leq K \, n^{2k-r} \omega \left(f^{(r)}, \frac{1}{n} \right),$$

where the constant K depends only on k. An extension to an estimate with higher-order moduli is given as follows.

Theorem 2.9.4 ([234]). *For $r \geq 1$, let $f \in C[-1,1]$ and let $P_n \in \mathbb{P}_n$, denote its polynomial of best approximation on $[-1,1]$. Then for each $k \geq r$ there exists a constant K depending on k and r, such that for every $-1 \leq x \leq 1$,*

$$| P_n^{(k)}(x) | \leq K \, [\Delta_n(x)]^{-k} \, \omega_r \left(f, \frac{1}{n} \right) \quad n \in \mathbb{N}. \tag{2.41}$$

There is also a nice remark of Leviatan in the paper quoted above: the upper bound of the K-functional in the characterization of the usual modulus of continuity can be given by polynomials. That is, for every $f \in C[-1,1]$ and $n \in \mathbb{N}$

there is a polynomial $P_n \in \mathbb{P}_n$ such that

$$\|f - P_n\| \le C\,\omega_r\left(f, \frac{1}{n}\right) \quad \text{and} \quad \|P_n^{(r)}\| \le C\,n^r\,\omega_r\left(f, \frac{1}{n}\right).$$

In 1985 Ditzian [95] improved (2.41) by proving a similar inequality but in terms of so-called Ditzian-Totik moduli (the definition will be given below). That is

$$|\,P_n^{(k)}(x)\,| \le K\,[\Delta_n(x)]^{-k}\,\omega_r^{\varphi}\left(f, \frac{1}{n}\right).$$

The results are the best possible. If $|\,P_n^{(k)}(x)\,| \le K\,[\Delta_n(x)]^{-k}\,\psi(n)$ where $\psi(n)$ is decreasing, $\psi(n) = o(1)$, and satisfies some additional conditions, then $\omega_r^{\varphi}(f, 1/n) \le M\psi(n)$. This provides the analogue to the Sunouchi-Zamanski theorem.

Theorem 2.9.5 (Ditzian, [95]). *If for some integer r and decreasing sequence $\psi(n)$,*

$$\sum_{k=1}^{l} 2^{kr}\psi(2^k) \le M\,2^{lr}\psi(2^l) \quad \textit{and} \quad E_n(f) \le \psi(n),$$

then for P_n, the polynomial satisfying $\|f - P_n\| = E_n(f)$, one has

$$|\,P_n^{(k)}(x)\,| \le K\,[\Delta_n(x)]^{-k}\,\psi(n).$$

In particular, if for some r,

$$\sum_{k=1}^{l} 2^{kr}E_{2^k}(f) \le M\,2^{lr}E_{2^l}(f)$$

then

$$|\,P_n^{(k)}(x)\,| \le K\,[\Delta_n(x)]^{-k}\,E_n(f).$$

Another extension is due to Shevchuk.

Theorem 2.9.6 (Shevchuk, [338]). *If $f \in C^r[-1,1]$ and $\omega_k(f^{(r)}, t) \le \omega(t)$ $(0 < t \le 1/k)$, then for any $n \ge r+k-1$ there exists $P_n \in \mathbb{P}_n$ such that, for all $x \in [-1,1]$,*

$$|\,f^{(j)}(x) - P_n^{(j)}(x)\,| \le C(\Delta_n(x))^{r-j}\,\omega(\Delta_n(x)), \qquad 0 \le j \le r,$$

and

$$|\,P_n^{(j)}(x)\,| \le C(\Delta_n(x))^{r-j}\,\omega(\Delta_n(x)) + C(r+k-j)(\Delta_n(x))^{-j}\,\|f\|_{x,n},$$

for $0 \le j \le r+k$, where

$$\|f\|_{x,n} = \max\{|\,f(u)\,| : u \in [x - \Delta_n(x), x + \Delta_n(x)] \cap [-1,1]\}.$$

2.10 Fuksman-Potapov solution to the second problem

In the last sections we have seen theorems which provide characterization for certain classes of functions. For instance, Theorem 2.7.3 characterizes functions satisfying $\omega(f^{(r)}, t) \leq C\omega(t)$ by means of its approximations for algebraic polynomials. In Theorems 2.7.5 and 2.7.7 similar results were presented for functions satisfying $f^{(r)} \in \mathrm{Lip}_\alpha[-1,1]$ or in the Zygmund class respectively. These theorems provide the analogue of the first interpretation after (1.11). That is, we have a characterization of functions satisfying a classical Lipschitz condition in terms of the rate of pointwise approximation by algebraic polynomials. Let us consider the problem of characterization of other classes of functions.

For $r \in \mathbb{N}$ and $\alpha \in (0,1)$, let

$$K(r,\alpha) \;=\; \{f \in C[-1,1] : \; E_n(f) \leq M(f) n^{-r-\alpha}\}.$$

Classes $K(r,\alpha)$ are defined in terms of the rate of convergence of the best approximation. The classes $C^{r,\alpha}[-1,1]$ and $K(r,\alpha)$ are different. For instance, for $f(x) = \sqrt{1-x^2}$ one has, $f \in K(0,1)$ but, for any $\delta > 1/2$, $f \notin C^{0,\delta}[-1,1]$.

It was an interesting question to describe classes $K(r,\alpha)$ without any reference to approximation by polynomials. One of the first results in this direction is due to Fuksman [129]. For $f \in C^r(-1,1)$ and $0 \leq k \leq r/2$, let $\psi_k(x) = f^{(r-k)}(x)(1-x^2)^{r/2-k}$ and consider the condition

$$\sup_{h \in \Lambda(x,\delta)} \mid \psi_k(x) - \psi_k(x+h) \mid \leq C \left(\frac{\delta}{\sqrt{1-x^2} + \sqrt{\delta}} \right)^{\alpha}, \tag{2.42}$$

where $\Lambda(x,\delta) = \{h : \mid h \mid \leq \delta, \mid x+h \mid \leq 1\}$. We assume $\psi_1(1) = \psi_1(-1) = 0$ for odd k. Let

$$S(r,\alpha) \;=\; \{f \in C^r(-1,1) \,:\, \psi_k \in C[-1,1] \; (0 \leq k \leq r/2) \quad \text{and} \quad (2.42) \quad \text{holds}\}.$$

Theorem 2.10.1 (Fuksman, [129]). *For each $r \in \mathbb{N}_0$ and $0 < \alpha < 1$, one has $K(r,\alpha) \;=\; S(r,\alpha)$.*

Proof. In order to verify the inclusion $S(r,\alpha) \subset K(r,\alpha)$, for $f \in S(r,\alpha)$, define $F(t) = f(\cos(t))$.

If $r = 0$, then $\psi_0(x) = f(x)$ and (2.42) yields

$$\mid f(x+h) - f(x) \mid \leq C \left(\frac{\delta}{\sqrt{1-x^2} + \sqrt{\delta}} \right)^{\alpha} \leq C \min\left\{ \left(\frac{\delta}{\sqrt{1-x^2}} \right)^{\alpha}, \left(\sqrt{\delta} \right)^{\alpha} \right\}$$

for $h \in \lambda(x,\delta)$. Set $h = \cos(t+h) - \cos t$ and $\delta = \mid h \sin t \mid + h^2$. Since

$$\mid \cos(t+h) - \cos t \mid = \mid \cos t(1-\cos h) + \sin t \sin h \mid \leq \delta,$$

one has

$$| F(t+h) - F(t) | \leq C \min\left\{ \left(\frac{\delta}{\sqrt{1-\cos^2 t}} \right)^{\alpha}, \left(\sqrt{\delta} \right)^{\alpha} \right\}. \qquad (2.43)$$

We should consider two cases.

Case 1. If $| h | \leq | \sin t |$, then

$$\frac{\delta}{| \sin t |} = \frac{| h \sin t | + h^2}{| \sin t |} \leq 2 | h | .$$

Case 2. If $| h | > | \sin t |$, then

$$\sqrt{\delta} = \sqrt{| h \sin t | + h^2} \leq | h | \sqrt{2} \leq 2 | h | .$$

Therefore

$$| F(t+h) - F(t) | \leq C_1 | h |^{\alpha} .$$

Now we consider that $r > 0$. By induction with respect to r it can be proved that there exists trigonometric polynomials $\varphi_{i,r} \in \mathbb{T}_{r-i}$ such that

$$F^{(r)}(t) = \sum_{i=0}^{[r/2]} f^{(r-i)}(\cos t) \sin^{r-2i}(t) \varphi_{i,k}(t) + \sum_{i=[r/2]+1}^{r} f^{(r-i)}(\cos t) \varphi_{i,k}(t).$$

But

$$f^{(r-i)}(\cos t) \ | \sin^{r-2i}(t) | = \psi_i(\cos t),$$

then we can write

$$F^{(r)}(t) = \sum_{i=0}^{[r/2]} \Psi_i(t) \varphi_{i,k}(t) + \sum_{i=[r/2]+1}^{r} f^{(r-i)}(\cos t) \varphi_{i,k}(t),$$

where $\Psi_i(t) = f^{(r-i)}(\cos t)\mathrm{sign}(\sin t)^k$. It can be proved that these functions are continuous. Moreover, as in the proof of the case $r = 0$, each function Ψ_i satisfies a Lipschitz condition of order α. Therefore, there exist a constant C and a sequence $\{T_n\}$ of even trigonometric polynomials such that $| F(t) - T_n(t) | \leq Cn^{-(k+\alpha)}$. By taking $P_n(x) = T_n(\arccos x)$ we conclude that $f \in K(r, \alpha)$.

Let us consider the relation $K(r, \alpha) \subset S(r, \alpha)$. Fix $f \in K(r, \alpha)$ and a sequence $\{P_n\}$ of polynomials such that $\|f - P_n\| \leq Cn^{-(k+\alpha)}$. If we set

$$Q_n = P_{2^n} - P_{2^{n-1}} \qquad (n \geq 1), \qquad (2.44)$$

then

$$\|Q_n\| \leq C 2^{-n(r+\alpha)} \qquad (2.45)$$

and take into account that

$$f(x) = \sum_{n=1}^{\infty} Q_n(x),$$

then

$$f^{(i)}(x) = \sum_{n=1}^{\infty} Q_n^{(i)}(x), \qquad (i = 0, 1, \ldots, r).$$

Set $\psi_j(x) = f^{(r-j)}(x)(1-x^2)^{r/2-j}$, then

$$\psi_j(x) = \sum_{n=0}^{\infty} Q_n^{(r-j)}(x)(1-x^2)^{r/2-j} = \sum_{n=0}^{m} + \sum_{n=m+1}^{\infty} = L_m(x) + L_m^*(x), \quad (2.46)$$

where m will chosen later.

We will estimate the modulus of continuity of L_m and L_m^*. First

$$| L_m(x+h) - L_m(x) | \leq | h | \sum_{n=0}^{m} \left| \frac{d}{du} \left[(1-u^2)^{r/2-j} Q_n^{r-j}(u) \right] \right|_{u=x+h\theta}$$

$$\leq | h | \sum_{n=0}^{m} \left\{ \left| 2u(1-u^2)^{r/2-j-1} Q_n^{r-j}(u) \right| + \left| (1-u^2)^{r/2-j} Q_n^{r-j-1}(u) \right| \right\}_{u=x+h\theta}.$$

Now we have two different estimates: taking into account (2.45) and (2.23) (with $l = r-j$, $q = r-2j-1$ $p = j+1$ $(p+q = l)$) one has

$$\sum_{n=0}^{m} 2u(1-u^2)^{r/2-j-1} Q_n^{r-j}(u) \Big|_{u=x+h\theta}$$

$$\leq C_1 \sum_{n=0}^{m} 2^{(r+1)n} \|Q_n\| (1-u^2)^{r/2-j-1} (1-u^2)^{-r/2+j+1/2} \Big|_{u=x+h\theta}$$

$$\leq C_2 \, (1-(x+h\theta)^2))^{-1/2} \sum_{n=0}^{m} 2^{(r+1)n} 2^{-n(r+\alpha)} \leq C_3 \, \frac{2^{(1-\alpha)m}}{(1-(x+h\theta)^2))^{1/2}}.$$

On the other hand, (2.23) (with $l = r-j$, $q = r-2j-2$ $p = j+2$ $(p+q = l)$) one has

$$\sum_{n=0}^{m} 2u(1-u^2)^{r/2-j-1} Q_n^{r-j}(u) \Big|_{u=x+h\theta}$$

$$\leq C_1 \sum_{n=0}^{m} 2^{(r+2)n} \|Q_n\| (1-u^2)^{r/2-j-1} (1-u^2)^{-r/2+j+1} \Big|_{u=x+h\theta}$$

$$\leq C_2 \sum_{n=0}^{m} 2^{(r+2)n} 2^{-n(r+\alpha)} \leq C_3 \, 2^{(2-\alpha)m}.$$

Since for the other term in the estimate of $| L_m(x+h) - L_m(x) |$ we can obtain similar inequalities, we have proved that

$$| L_m(x+h) - L_m(x) | \le C_4 \frac{2^{(1-\alpha)m}}{(1-(x+h\theta)^2))^{1/2}} \tag{2.47}$$

and

$$| L_m(x+h) - L_m(x) | \le C_5 \, 2^{(2-\alpha)m}. \tag{2.48}$$

With similar arguments we also prove that

$$| L_m^*(x+h) - L_m^*(x) |\le C_6 \, 2^{-\alpha m}. \tag{2.49}$$

If $\varepsilon > 0$, $| h |\le \varepsilon$ and $| x |\le 1-\varepsilon$, we take m such that $2^m < \sqrt{(1-\varepsilon)^2 - x^2} = r \le 2^{m+1}$. Then from (2.46), (2.47) and (2.49) we obtain

$$| \psi_j(x+h) - \psi_j(x) |\le C \left(| h | \left(\frac{| h |}{r} \right)^{1-\alpha} \frac{1}{r} + \left(\frac{r}{| h |} \right)^{\alpha} \right) = 2C \left(\frac{r}{| h |} \right)^{\alpha}.$$

If we take m such that $2^m < (| h |)^{-1/2} \le 2^{m+1}$, then from (2.46), (2.48) and (2.49) we obtain

$$| \psi_j(x+h) - \psi_j(x) |\le C \left(| h | \left(\frac{1}{\sqrt{| h |}} \right)^{2-\alpha} + \left(\frac{1}{\sqrt{| h |}} \right)^{\alpha} \right) = 2C \, (| h |)^{\alpha/2}.$$

Thus, if $h \in \Lambda(x, \varepsilon)$, then

$$\begin{aligned} | \psi_j(x+h) - \psi_j(x) | &\le C \min \left((| h | /r)^{\alpha}, | h |^{\alpha/2} \right) \\ &= C \, | h |^{\alpha} \min \left((1/r), | h |^{-1/2} \right)^{\alpha} \\ &\le \frac{C_1 \, | h |^{\alpha}}{(r + \sqrt{| h |})^{\alpha}} = C_1 \left(\frac{| h |}{r + \sqrt{| h |}} \right)^{\alpha} \\ &\le C_1 \left(\frac{\varepsilon}{r + \sqrt{\varepsilon}} \right)^{\alpha} \le C_2 \left(\frac{\varepsilon}{\sqrt{1-x^2} + \sqrt{\varepsilon}} \right)^{\alpha}, \end{aligned}$$

since

$$\frac{1}{r + \sqrt{\varepsilon}} = \frac{1}{\sqrt{(1-\varepsilon)^2 - x^2} + \sqrt{\varepsilon}} \le \frac{6}{\sqrt{1-x^2} + \sqrt{\varepsilon}}.$$

Finally, if r is odd, from (2.49) we know that the series (2.46) converges uniformly on $[-1, 1]$. Moreover, for $j < [r/2]$, one has $k > 2j$, thus $\psi_j(\pm 1) = 0$. We have proved that $f \in S(r, \alpha)$. □

The result also can be extended to the case when

$$E_n(f) \le \frac{M(f)}{n^k} \omega \left(\frac{1}{n} \right)$$

where ω is a modulus of continuity satisfying conditions (2.22). In this case, the definition of the class $S(r,\omega)$ is similar to the one of $S(r,k)$, but condition (2.42) is replaced by

$$\sup_{(h,x)\in\Lambda(\delta)} |\ \psi_k(x)-\psi_k(x+h)\ | \le C\,\omega\left(\frac{\delta}{\sqrt{1-x^2}+\sqrt{\delta}}\right).$$

Theorem 2.10.2 (Fuksman, [129]). *Let $f\in C[-1,1]$, r a positive integer and $\alpha\in(0,1)$. The following assertions are equivalent:*

(i) *$f\in C^{2r}(-1,1)$ and, for $0\le k\le r$ and $\psi_k(x)=f^{(r-k)}(x)(1-x^2)^{r-k}$, one has $\psi_k\in C[-1,1]$ and (2.42) holds.*

(ii) *For each $n\in\mathbb{N}$ there exists an algebraic polynomial $P_n\in\Pi_{n-1}$ such that, for each $x\in[-1,1]$,*

$$|\ f(x)-P_n(x)\ | \le \frac{D}{n^{2r+\alpha}}$$

where D is a positive constant which does not depend on x or n.

In 1980 Potapov [295] unified the results of Dzyadyk and Fuksman. He proved an analogue to Theorem 2.10.3, but with the condition $\alpha+\beta<1$ instead of $\alpha+\beta/2<1$. Notice that, by taking $\beta=0$ we obtain the Dzyadyk characterization and for $\beta=-\alpha$ the Fuksman result. The results we present here were proved by Potapov in 2005 [303].

Theorem 2.10.3 (Potapov, [303]). *Fix reals α and β such that $\alpha\in(0,1)$, $\alpha+\beta\ge 0$ and $\alpha+\beta/2<1$. For $f\in C[-1,1]$ the following assertions are equivalent:*

(i) *For each $x\in[-1,1]$, one has*

$$\sup_{\{h\,:\,|h|\le\delta,\,|x+h|\le 1\}} |\ f(x+h)-f(x)\ | \le C_1\,\delta^\alpha\left(\sqrt{1-x^2}+\sqrt{\delta}\right)^\beta,$$

where C_1 is a positive constant which does not depend on δ or x;

(ii) *For each $n\in\mathbb{N}$ there exists $P_{n-1}\in\Pi_{n-1}$ such that, for each $x\in[-1,1]$,*

$$|\ f(x)-P_{n-1}(x)\ | \le C_2\,n^\beta\,(\Delta_n(x))^{\alpha+\beta}$$

where C_2 is a positive constant which does not depend on x or n.

Proof. (i) $\Longrightarrow$ (ii). Fix $m,s\in\mathbb{N}$ such that $(n-1)/s<m\le 1+(n-1)/s$ and define

$$Q(x)=\int_{-\pi}^{\pi} f(\cos(t+y))K_{m,s}(t)dt,$$

where $x=\cos y$ and where K_{2q} is given by (2.8) with $s=q$. There exist positive constant C_1 and C_2 such that

$$C_1 m^{2s-1}\le c_{m,s}\le C_2 m^{2s-1}\quad\text{and}\quad C_1 m^\beta\le\int_{-\pi}^{\pi}|\ t\ |^\beta\,K_{m,s}(t)dt\le C_2 m^{-\beta}.$$

Moreover, $Q_m\in\mathbb{P}_{n-1}$.

Since for $| t |\leq \pi$, one has

$$\gamma =| t | \sqrt{1-x^2}+t^2)^\alpha \leq 3(| t | \sqrt{1-x^2}+t^2),$$

as in the proof of (2.43) from (i) we have

$$\begin{aligned} | f(\cos(y+t)) - f(\cos y) | &\leq C\,(| t | \sqrt{1-x^2}+t^2)^\alpha\,(\gamma)^\beta \\ &\leq 3C\ \ | t |^\alpha\,(\sqrt{1-x^2}+t^2)^{\alpha+\beta} \\ &\leq C_1\,(| t |^\alpha\,(\sqrt{1-x^2})^{\alpha+\beta}+ | t |^{2\alpha+\beta}). \end{aligned}$$

Now one has

$$\begin{aligned} &| f(x) - Q(x) |\leq \int_{-\pi}^{\pi} | f(\cos(t+y)) - f(\cos y) | \,K_{m,s}(t)dt \\ &\leq C\left((\sqrt{1-x^2})^{\alpha+\beta}\int_{-\pi}^{\pi} | t |^\alpha\, K_{m,s}(t)dt + \int_{-\pi}^{\pi} | t |^{2\alpha+\beta}\, K_{m,s}(t)dt \right) \\ &\leq C_1\left(\frac{(\sqrt{1-x^2})^{\alpha+\beta}}{m^\alpha} + \frac{1}{m^{2\alpha+\beta}} \right) \leq C_2\, n^\beta (\Delta_n(x))^{\alpha+\beta}. \end{aligned}$$

We have proved (ii).

(ii) $\Longrightarrow$ (i). We should modify the arguments of the proof of Theorem 2.10.1. If Q_n be defined by (2.44), then

$$| Q_k(x) |\leq C 2^{k\,\beta}(\Delta_{2^k}(x))^{\alpha+\beta}$$

and from Theorem 2.7.5 we obtain

$$| Q_k(x+h) - Q_k(x) |\leq C_1\ \ | h |\ \ 2^{k\beta}(\,\Delta_{2^k}(x+h\theta)\,)^{\alpha+\beta-1}.$$

Fix $x \in [-1,1]$ and $| x+h |\leq 1$. Fix $N \in \mathbb{N}$ which will be chosen later. Notice that

$$\begin{aligned} &| \Delta_h f(x) | \\ &\leq | f(x) - P_{2^N}(x) | + | f(x+h) - P_{2^N}(x+h) | + \sum_{k=0}^{N} | Q_k(x+h) - Q_k(x) | \\ &\leq C_3\left(2^{N\beta}((\Delta_{2^N}(x))^{\alpha+\beta} + (\Delta_{2^N}(x+h))^{\alpha+\beta}) + | h | \sum_{k=0}^{N} \frac{(\,\Delta_{2^k}(x+h\theta)\,)^{\alpha+\beta-1}}{2^{-k\beta}} \right) \\ &\leq C_3 2^{N\beta}\left((\Delta_{2^N}(x))^{\alpha+\beta} + (\Delta_{2^N}(x+h))^{\alpha+\beta}) + | h |\ (\Delta_{2^N}(x+h\theta))^{\alpha+\beta-1} \right), \end{aligned}$$

where the sum is estimated as follows. If $\alpha+\beta\leq 1$ and $\alpha<1$, then

$$2^{k(\alpha+\beta-1)}(\Delta_{2^k}(x+h\theta))^{\alpha+\beta-1} \leq 2^{N(\alpha+\beta-1)}(\Delta_{2^N}(x+h\theta))^{\alpha+\beta-1}.$$

Hence

$$\sum_{k=0}^{N} \frac{(\Delta_{2^k}(x+h\theta))^{\alpha+\beta-1}}{2^{-k\beta}} \le 2^{N(\alpha+\beta-1)}(\Delta_{2^N}(x+h\theta))^{\alpha+\beta-1} \sum_{k=0}^{N} 2^{k(1-\alpha)}$$
$$\le C\, 2^{N\beta}(\Delta_{2^N}(x+h\theta))^{\alpha+\beta-1}.$$

On the other hand, if $\alpha+\beta>1$ and $\alpha+\beta/2<1$, then

$$2^{k(\alpha+\beta-1)}(\Delta_{2^k}(x+h\theta))^{\alpha+\beta-1} \le \left(\frac{2^N}{2^k}\right)^{\alpha+\beta-1} 2^{N(\alpha+\beta-1)}(\Delta_{2^N}(x+h\theta))^{\alpha+\beta-1}.$$

Hence

$$\sum_{k=0}^{N} \frac{(\Delta_{2^k}(x+h\theta))^{\alpha+\beta-1}}{2^{-k\beta}} \le 2^{2N(\alpha+\beta-1)}(\Delta_{2^N}(x+h\theta))^{\alpha+\beta-1} \sum_{k=0}^{N} 2^{k(2-2\alpha-\beta)}$$
$$\le C\, 2^{2N(\alpha+\beta-1)}(\Delta_{2^N}(x+h\theta))^{\alpha+\beta-1} 2^{N(2-2\alpha-\beta)}$$
$$= C\, 2^{N\beta}(\Delta_{2^N}(x+h\theta))^{\alpha+\beta-1}.$$

To finish the proof we should choose N.

Case 1. Suppose that $0<h<1/4$ and $x \in [-1,-1+2h] \cup [1-2h, 1-h]$. Chose N such that $2^{-2N-1} \le h < 2^{-2N}$. Then

$$1-x^2 \le 1-(1-2h)^2 \le 4h \le 4\, 2^{-2N}$$

and

$$1-(x+h\theta)^2 \le 1-x^2+2h \le 6h \le 6\, 2^{-2N}.$$

Hence

$$2^{-N} \le \sqrt{1-x^2}+2^{-N} \le 3 2^{-N},$$
$$2^{-N} \le \sqrt{1-(x+h)^2}+2^{-N} \le 4 2^{-N},$$
$$2^{-N} \le \sqrt{1-(x+h\theta)^2}+2^{-N} \le 4\, 2^{-N},$$

and

$$\frac{1}{3}\left(\sqrt{h}+\sqrt{1-x^2}\right) \le \sqrt{h} \le \sqrt{h}+\sqrt{1-x^2},$$

and we obtain

$$|\, f(x+h)-f(x)\,| \le \frac{C}{2^{N\alpha}}(2^{-N(\alpha+\beta)}+h2^N 2^{-N(\alpha+\beta-1)})$$
$$\le C_1\, 2^{-N(2\alpha+\beta)} \le C_2\, h^{\alpha+\beta/2} \le C_3\, h^{\alpha}(\sqrt{1-x^2}+\sqrt{h})^{\beta}.$$

Case 2. Suppose that $0<h<1/4$ and $x \in [-1+2h, 1-2h]$. Choose N such that

$$\frac{\sqrt{1-x^2}}{2^{N+1}} < h \le \frac{\sqrt{1-x^2}}{2^N}.$$

Now we consider the inequalities

$$2\sqrt{h} \le \sqrt{1-(1-2h)^2} \le \sqrt{1-x^2},$$

$$2\sqrt{1-x^2} \le \sqrt{1-x^2} + \frac{2h}{\sqrt{1-x^2}} \le \sqrt{1-x^2} + \frac{1}{2^N} < \sqrt{1-x^2},$$

$$1-(x+h\theta)^2 \le 1-x^2+4h \le 2(1-x^2),$$

and

$$1-x^2 = 1-(x+h\theta)^2+h\theta(2x+h\theta) \le 1-(x+h\theta)^2+2h \le 1-(x+h\theta)^2+2\frac{\sqrt{1-x^2}}{2^N}.$$

Then

$$\begin{aligned}\left(\sqrt{1-x^2}-\frac{1}{2^N}\right)^2 &= 1-x^2-2\frac{\sqrt{1-x^2}}{2^N}\\ &\le 1-(x+h\theta)^2+\frac{1}{2^N} \le \left(\sqrt{1-(x+h\theta)^2}+\frac{1}{2^N}\right)^2.\end{aligned}$$

Therefore, if $2^{-N} \le \sqrt{1-x^2}$, then

$$\sqrt{1-x^2}-\frac{1}{2^N} \le \sqrt{1-(x+h\theta)^2}+\frac{1}{2^N},$$

and

$$\sqrt{1-x^2} \le 2(\sqrt{1-(x+h\theta)^2}+\frac{1}{2^N}.$$

On the other hand, if $2^{-N} > \sqrt{1-x^2}$, then

$$\sqrt{1-x^2} \le \sqrt{1-(x+h\theta)^2}+\frac{1}{2^N}.$$

Hence, in this case

$$\frac{1}{2}\sqrt{1-x^2} \le \sqrt{1-(x+h\theta)^2}+\frac{1}{2^N} \le 2\left(\sqrt{1-x^2}+\frac{1}{2^N}\right) \le 4\sqrt{1-x^2}.$$

With these inequalities we obtain

$$\begin{aligned}|\, f(x+h)-f(x)\,| &\le \frac{C}{2^{N\alpha}}\left((\sqrt{1-x^2})^{\alpha+\beta}+h2^N(\sqrt{1-x^2})^{\alpha+\beta-1)}\right)\\ &\le \frac{C_2}{2^{N\alpha}}(\sqrt{1-x^2})^{\alpha+\beta} \le C_3h^\alpha(\sqrt{1-x^2}+\sqrt{h})^\beta.\end{aligned}$$

Case 3. The case $h \in (-1/4, 0)$ can be treated as the case $h \in (0, 1/4)$.

Case 4. For $\delta < 1/4$ the proof follows from the arguments given above. For $\delta \ge 1/4$ the proof is simple. □

The Chebyshev differential operator is defined by

$$D(f,x) = (1-x^2)f''(x) - xf'(x).$$

Moreover, set $D^{(1)} = D$ and $D^{(r)} = D(D^{(r-1)})$, for $r \geq 2$.

Theorem 2.10.4 (Potapov, [303]). *Fix real numbers $\sigma \geq 0$ and $\gamma > 0$. For $f \in C[-1,1]$, the following assertions are equivalent:*

(i) *For each $n \in \mathbb{N}$ $(n \geq 2)$ there exists $P_{n-1} \in \Pi_{n-1}$ such that, for each $x \in [-1,1]$,*

$$| f(x) - P_{n-1}(x) | \leq \frac{C_1}{n^{2+\gamma-\sigma}} \left(\Delta_n(x)\right)^{\sigma}$$

where C_1 is a positive constant which does not depend on x or n.

(ii) *For any interval $[a,b] \subset (-1,1)$, $f \in C^2[a,b]$, $Df \in C[-1,1]$ and for each $n \in \mathbb{N}$ $(n \geq 2)$ there exists $R_{n-1} \in \Pi_{n-1}$ such that, for each $x \in [-1,1]$,*

$$| Df(x) - R_{n-1}(x) | \leq \frac{C_2}{n^{\gamma-\sigma}} \left(\Delta_n(x)\right)^{\sigma}$$

where C_3 is a positive constant which does not depend on x or n.

(iii) *For any interval $[a,b] \subset (-1,1)$, $f \in C^2[a,b]$, $f'(x), (1-x^2)f''(x) \in C[-1,1]$ and for each $n \in \mathbb{N}$ $(n \geq 2)$ there exists $Q_{n-1,1}, Q_{n-1,2} \in \Pi_{n-1}$ such that, for each $x \in [-1,1]$,*

$$| f'(x) - Q_{n-1,1}(x) | \leq C_3\, n^{\sigma-\gamma} \left(\Delta_n(x)\right)^{\sigma}$$

and

$$| (1-x^2)f''(x) - Q_{n-1,2}(x) | \leq C_1\, n^{\sigma-\gamma} \left(\Delta_n(x)\right)^{\sigma}$$

where C_3 is a positive constant which does not depend on x or n.

Theorem 2.10.5 ([303]). *Fix real numbers $\sigma \geq 0$ and $\gamma > 0$. For $f \in C[-1,1]$, the following assertions are equivalent:*

(i) *For each $n \in \mathbb{N}$ $(n \geq 2)$ there exists $P_{n-1} \in \Pi_{n-1}$ such that, for each $x \in [-1,1]$,*

$$| f(x) - P_{n-1}(x) | \leq \frac{C_1}{n^{\gamma-\sigma}} \left(\Delta_n(x)\right)^{\sigma+1}$$

where C_3 is a positive constant which does not depend on x or n.

(ii) *$f \in C^1[-1,1]$ and for each $n \in \mathbb{N}$ $(n \geq 2)$ there exists $R_{n-1} \in \Pi_{n-1}$ such that, for each $x \in [-1,1]$,*

$$| f'(x) - R_{n-1}(x) | \leq \frac{C_1}{n^{\gamma-\sigma}} \left(\Delta_n(x)\right)^{\sigma}$$

where C_3 is a positive constant which does not depend on x or n.

Theorem 2.10.6 ([303]). *Let $f \in C[-1,1]$, r and ρ be non-negative integers and fix α and β such that $\alpha \in (0,1)$, $\alpha+\beta \geq 0$ and $\alpha+\beta/2 < 1$. The following assertions are equivalent:*

(i) *$f \in C^{2\rho+r}(-1,1)$, $\psi(x) = D^{(\rho)} f^{(r)}(x) \in C[-1,1]$ and*

$$\sup_{\{ h : |h| \leq \delta, |x+h| \leq 1\}} | \psi(x) - \psi(x+h) | \leq C_1 \, \delta^{\alpha} \left(\sqrt{1-x^2} + \sqrt{\delta} \right)^{\beta},$$

where C_1 is a positive constant which does not depend on δ or x;

(ii) *$f \in C^{2\rho+r}(-1,1)$ and, for $0 \leq k \leq \rho$ and $\psi_k(x) = f^{(2\rho+r-k)}(x)(1-x^2)^{\rho-k}$, one has $\psi_k \in C[-1,1]$ and*

$$\sup_{\{ h : |h| \leq \delta, |x+h| \leq 1\}} | \psi_k(x) - \psi_k(x+h) | \leq C_2 \, \delta^{\alpha} \left(\sqrt{1-x^2} + \sqrt{\delta} \right)^{\beta},$$

where C_2 is a positive constant which does not depend on δ or x;

(iii) *For each $n \in \mathbb{N}$ there exists an algebraic polynomial $P_{n-1} \in \Pi_{n-1}$ such that, for each $x \in [-1,1]$,*

$$| f(x) - P_{n-1}(x) | \leq \frac{C_3}{n^{2\rho-\beta}} \left(\Delta_n(x)\right)^{r+\alpha+\beta}$$

where C_3 is a positive constant which does not depend on x or n.

Proof. Assume condition (iii) holds. Then there exists a sequence $\{P_n\}$ of algebraic polynomials for which

$$| f(x) - P_{n-1}(x) | \leq \frac{C}{n^{2\rho-\beta}} (\Delta_n(x))^{r+\alpha+\beta}.$$

By applying ρ-times Theorem 2.10.5 we obtain that condition (iii) is equivalent to the following condition A: there exists a sequence $\{R_n\}$ $(n \geq 2)$ of algebraic polynomials $R_n \in \Pi_n$ such that, for each $x \in [-1,1]$,

$$| f^{(r)}(x) - R_n(x) | \leq \frac{C}{n^{2\rho-\beta}} (\Delta_n(x))^{\alpha+\beta}.$$

By applying ρ-times Theorem 2.10.4 (which is equivalent to condition (i) and (ii)) we obtain that condition A is equivalent to the following condition B: there exists a sequence $\{Q_n\}$ $(n \geq 2)$ of algebraic polynomials $Q_n \in \Pi_n$ such that, for each $x \in [-1,1]$,

$$| D^{(\rho)}(f^{(r)}(x)) - Q_n(x) | \leq C n^{\beta} (\Delta_n(x))^{\alpha+\beta}.$$

Applying Theorem 2.10.3 we obtain that condition B is equivalent to condition (i). Thus we have proved that (i) and (ii) are equivalent.

Let us prove that (ii) and (iii) are equivalent. By applying r-times we obtain that condition (iii) is equivalent to condition A. From Theorem 2.10.4 (condition (i) and (ii) are equivalent) we obtain that condition A is equivalent to condition B: for each $n \in \mathbb{N}$ $(n \geq 2)$ there exist algebraic polynomials $T_{n,1}, T_{n,2} \in \Pi_n$ such that, for each $x \in [-1,1]$,

$$| f^{(r+1)}(x)(1-x^2)^{i-1} - T_{n,i}(x) | \leq \frac{C\,(\Delta_n(x))^{\alpha+\beta}}{n^{2(\rho-1)-\beta}}, \qquad i = 1,2.$$

If $\rho > 1$, then from Theorem 2.10.4 we obtain that condition B is equivalent to the following condition C: for each $n \in \mathbb{N}$ $(n \geq 2)$ there exist algebraic polynomials $H_{n,i} \in \Pi_n$ $(i \in \{1,2,3,4\})$, such that, for each $x \in [-1,1]$,

$$| f^{(r+1+i)}(x)(1-x^2)^{i-1} - T_{n,i}(x) | \leq \frac{C\,(\Delta_n(x))^{\alpha+\beta}}{n^{2(\rho-2)-\beta}}, \qquad i = 1,2$$

and

$$| f^{(r+1+i)}(x)(1-x^2)^{i-3}(1-x^2)^{(i-2)} - T_{n,i}(x) | \leq \frac{C\,(\Delta_n(x))^{\alpha+\beta}}{n^{2(\rho-2)-\beta}}, i = 3,4.$$

It can be proved that condition C is equivalent to the condition D: for each $n \in \mathbb{N}$ $(n \geq 2)$ there exist algebraic polynomials $L_{n,i} \in \Pi_n$ $(i \in \{1,2,3\})$, such that, for each $x \in [-1,1]$,

$$| f^{(r+1+i)}(x)(1-x^2)^{i-1} - L_{n,i}(x) | \leq \frac{C\,(\Delta_n(x))^{\alpha+\beta}}{n^{2(\rho-2)-\beta}}, \qquad i = 1,2,3.$$

If $\rho > 2$, we repeat $\rho - 2$-times the arguments given above to obtain that condition D is equivalent to the following condition E: for each $n \in \mathbb{N}$ $(n \geq 2)$ there exist algebraic polynomials $S_{n,i} \in \Pi_n$ $(i \in \{0,1,2,\ldots,\rho\})$, such that, for each $x \in [-1,1]$,

$$| f^{(r+2\rho-i)}(x)(1-x^2)^{\rho-i} - S_{n,i}(x) | \leq C\,n^{\beta}\,(\Delta_n(x))^{\alpha+\beta}, \qquad i = 0,2,\ldots,\rho.$$

From Theorem 2.10.3 we obtain that condition E is equivalent to condition (ii). Thus we have proved that conditions (ii) and (iii) are equivalent. □

2.11 Integral metrics

In the works of Timan and Dzyadyk the best approximation by algebraic polynomials was well studied in the case of the uniform norm. Several authors considered that extension of the Timan-type estimates the spaces of integrable functions. The problem of characterization for some classes of functions was considered by Potapov and Lebed.

In 1956 Potapov [287] extended the Timan theorem by considering functions with derivative of order r ($r > 0$, integer) is in $\mathrm{Lip}(p, \alpha)$, $p > 1$, $0 < \alpha \leq 1$. He also studied functions satisfying the condition

$$\left(\int_c^d \mid f^{(r)}(x+h) - f^{(r)}(x) \mid^p \frac{dx}{\sqrt{(x-a)(b-x)}} \right)^{1/p} \leq M(f) \mid h \mid^\alpha,$$

where $a \leq c < d \leq b$.

In 1958 Lebed obtained a direct result. He considered the term $\Delta_n(x)$ as a varying weight.

Theorem 2.11.1 (Lebed, [226]). *Assume that $p \geq 1$ and $1 - s - 1/p \geq 0$. If $f \in C^m[-1,1]$ and $\|(\sqrt{1-x^2})^s f^{(m)}(x)\|_p \leq M$, then there exists a sequence $\{P_n\}$ ($P_n \in \mathbb{P}_n$) such that*

$$\left\| \frac{f(x) - P_n(x)}{(\Delta_n(x))^{m-s}} \right\|_p \leq C(m) \frac{M}{n^s}.$$

Denote by $W^{(r)} H_p^w$ the class of functions given on the interval $[-1,1]$ and having an rth derivative $f^{(r)}$ whose pth power is integrable, and for which the inequality

$$\|f^{(r)}(x+h) - f^{(r)}(x)\|_{L_p[-1,1-h]} \leq w(h), \qquad 0 < h < 1,$$

hods, where w is a fixed modulus of continuity. The class $W^{(r)} A_p^w$ is defined analogously, but with the condition

$$\left\| \frac{f^{(r)}(x + \sqrt{1-h^2} - h\sqrt{1-x^2}) - f^{(r)}(x)}{w(h\sqrt{1-x^2} + h^2)} \right\|_p \leq C.$$

For $w(t) = t^\alpha$ we shall denote these classes by $H_p^{(r+\alpha)}$ ($A_p^{(r+\alpha)}$ respectively).

The classes $W^{(r)} H_p^w$ were introduced by Lebed and Potapov (see [290]). They proved that $W^{(r)} H_\infty^w = W^{(r)} A_\infty^w$ (uniform norm). It is also obvious that the intersection of these classes is not empty, for $1 \leq p < \infty$.

Potapov also used classes defined by two parameters. For $1 \leq p < \infty$, $r \in \mathbb{N}_0$, $0 \leq \beta \leq 1$ and $0 < \alpha \leq 1$, $f \in H_p^{(r)} A_\beta^\alpha$ if $f^{(r)} \in L_p[-1,1]$ if

$$\left(\int_{-1}^1 \left| \frac{f^{(r)}(x\sqrt{1-h^2} - h\sqrt{1-x^2}) - f^{(r)}(x)}{\sqrt{1-x^2} + \mid h \mid^\beta} \right|^p dx \right)^{1/p} \leq \mid h \mid^\alpha$$

in the case $0 < \alpha < 1$ and

$$\int_{-1}^1 \left| \frac{f^{(r)}(\lambda(x,h)x - \lambda(h,x)) - 2f^{(r)}(x) + f^{(r)}(\lambda(x,h) + \lambda(h,x))}{\sqrt{1-x^2} + \mid h \mid^\beta} \right|^p dx \leq \mid h \mid^p$$

in the case $\alpha = 1$, where $\lambda(x,h) = x\sqrt{1-h^2}$. Here is a typical result.

Theorem 2.11.2 (Potapov, [289]). *For a function f one has $f \in H_p^{(r)} A_\beta^\alpha$ if and only if, for each $n \geq r+2$, there exists a polynomial $P_n \in \mathbb{P}_n$, such that*

$$\left(\int_{-1}^{1} \left| \frac{f(x) - P_n(x)}{(\sqrt{1-x^2} + 1/n)^{r+\beta}} \right|^p dx \right)^{1/p} \leq \frac{C}{n^{r+\alpha}},$$

where the constant C does not depend on n or f.

The paper of Potapov also contains analogous results when the Lebesgue measure is changed by the Chebyshev one. Some other results were presented in [290]. The following result follows from the works of Lebed and Potapov.

Theorem 2.11.3 (Lebed-Potapov). *For $\alpha \in (0,1)$ and a function f, one has $f \in A^{(r+\alpha)}$ if and only if for each $n \geq r$ there exists a polynomial $P_n \in \mathbb{P}_n$, such that*

$$\left(\int_{-1}^{1} \left| \frac{f(x) - P_n(x)}{(\sqrt{1-x^2} + 1/n)^{r+\alpha}} \right|^p dx \right)^{1/p} \leq \frac{C}{n^{r+\alpha}},$$

where the constant C does not depend on n or f.

Taking into account (2.25), it was natural to look for weighted spaces. In this way some class of functions can be studied, but the original problems (characterization of classical Lipschitz spaces in terms of the best algebraic approximation or a characterization of a class of functions with a given rate for the best algebraic approximation in terms of the classical Lipschitz classes) was not solved. Since weighted approximation will not be discussed here in detail, we have included only a few remarks.

The characterization of the class $H_p^{(r+\alpha)}$ was also considered by Motornyi in 1971. He verified that the quantity

$$\lambda_n(f) = \inf_{P \in \mathbb{P}} \left\| \frac{f(x) - P(x)}{(\Delta_n(x))^\alpha} \right\|_{L_p}$$

are unbounded in the class $H_p^{(\alpha)}$ and established that classes $H_p^{(r+\alpha)}$ and $A_p^{(r+\alpha)}$ are different for $0 < \alpha < 1$ and coincide for $\alpha = 1$. He also characterized some functions, but not in terms of approximation by polynomials (see Theorem 11 of [255]) on the whole interval. Oswald [277] extended some of the results of Motornyi to the case of moduli of smoothness of higher order.

In 1978 DeVore [89] showed that we can not obtain a result similar to Theorem 2.7.7, if in (2.25) we replace the uniform norm by the $L_p[-1,1]$ norm, $1 \leq p < \infty$. That is, by considering

$$F_n(f, r, \alpha)_p = \inf_{p \in \mathbb{P}_n} \left\| \frac{f(x) - p(x)}{\Delta_n^{r+\alpha}(x)} \right\|_p . \tag{2.50}$$

In fact DeVore showed (with an incomplete proof) that for $0 < \alpha < 1$ and $1 \leq p < \infty$,

$$\omega_1(f,t)_p = \mathcal{O}(t^\alpha) \qquad \Longrightarrow \qquad F_n(f,0,\alpha)_p = \mathcal{O}(\log n).$$

Moreover, for each $0 < \alpha < 1$ there exists $f \in L_p[-1,1]$ such that $\omega_1(f,t)_p = \mathcal{O}(t^\alpha)$ and $F_n(f,0,\alpha)_p \geq C \log n$ for infinitely many n. As we remarked above, Motornyi proved that these quantities are not bounded when f varies on the class $H_p^{(\alpha)}$.

In 1972 Golischek presented a detailed study of this kind of weighted approximation [143]. For $1 \leq p \leq \infty$ set

$$E_n^{(\lambda)}(f)_p = \inf_{p \in \mathbb{P}_n} \|(\max\{1/n, \sqrt{1-x^2}\})^{-\lambda}(f(x) - p(x)\|_p$$

and consider the following question: under what conditions are the statements

$$E_n^{(\lambda)}(f)_p = \mathcal{O}(n^{-\beta}) \tag{2.51}$$

and

$$\|(\max\{1/n, \sqrt{1-x^2}\})^{r-\lambda} P_n^{(r)}(x)\|_p = \mathcal{O}(n^{r-\beta}) \tag{2.52}$$

equivalent, where $r \in \mathbb{N}$ and β is a real number $0 < \beta < r$? The answer is different for $\lambda \leq 0$ and $\lambda > 0$.

If $\lambda \leq 0$, Golitschek proved that (2.51) and (2.52) are equivalent.

For $\lambda > 0$ the situation is more complicated: if $r > \max\{\beta, (\lambda+\beta)/2\}$, then (2.51) implies (2.52). Moreover, if we assume $E_n^{(\lambda)}(f)_p = 0$, then (2.52) implies (2.51). Golitschek also constructed a class of functions for which both assertions are equivalent.

The following theorem generalizes some results of Motornii in [254].

Theorem 2.11.4 (Shalashova, [337]). *Fix $k \in \mathbb{N}$ and $p \in [1,\infty)$. Suppose $f \in L_p[-1,1]$ and $\omega_k(f,t)_p \leq \Psi(t)$, where $\Psi(t)$ is some positive function satisfying the conditions:*

1) *$\Psi(t)$ does not decrease,*
2) *$\Psi(\lambda t) \leq (\lambda+1)^k \Psi(t)$ for $\lambda > 1$.*

Then for any integer $n > k$, one can find an algebraic polynomial P_n of degree not greater than $(4k+2)n + k - 1$ such that

$$\left\| \frac{f(x) - p_n(x)}{\Psi(\Delta_n(x))} \right\|_p \leq A_k [\log(n+1)]^{1/p},$$

where A_k is a constant depending only on k.

Since A_k does not depend on p, we arrive at the uniform estimate of Brudnyi by letting p tend to ∞ in the last inequality (for a bounded f). If $r = k+1$, and $\omega_1(f^{(k)},t) \leq Ct^\alpha$ $(0 < \alpha \leq 1)$, we obtain from the last result a theorem of Motornyi [254].

Some authors have studied the best approximation of particular classes of functions. For instance, Nasibov considered the approximation by algebraic polynomials of functions of the form

$$f(x) = \int_{-1}^{1} \psi\left(\frac{x-t}{2}\right)\varphi(t)dt \tag{2.53}$$

in the metric of $L_p[-1,1]$.

Theorem 2.11.5 (Nasibov, [266]). *Let $1 \le p, r < \infty$ and assume that $\psi \in L_r[-1,1]$ and $\varphi \in L_p[-1,1]$. If f is defined by* (2.53)*, then*

$$E_n(f)_p \le 2^{2-1/r}\, \|\varphi\|_p\, E_n(\psi)_r.$$

Dynkin used a complex variable method (pseudo-analytical extension of functions) to obtain some results. For $s > 0$ and $1 \le p \le \infty$ he gave a characterization of functions satisfying

$$\left(\sum_{k=1}^{\infty} \frac{1}{n} E_n(f)_{p,\,s}^p\right)^{1/p} < \infty,$$

where

$$E_n(f)_{p,\,s} = \inf_{p\,\in\,\mathbb{P}_n} \left(\int_{-1}^{1} \left|\frac{f(x)-p(x)}{\Delta_n^s(x)}\right| dx\right)^{1/p}.$$

Recall that for $r \in \mathbb{N}$ and $1 < p < \infty$, $W_p^r[-1,1]$ is the class of functions such that $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L_p[-1,1]$.

Oswald [277] considered the classes W_m of increasing functions ω such that $\omega(h) \le 2^m \omega(h/2)$ and $H_{p,m}^{\omega}$ of functions in L_p such that $\omega_m(f,t)_p \le C(f)\omega(t)$.

Theorem 2.11.6 (Oswald, [277]). *Fix $m \in \mathbb{N}$ and $p \in [1,\infty)$. For each $f \in L_p[a,b]$ and $n \ge m-1$,*

$$E_n(f)_p \le C(m)\, \omega_m\left(f, \frac{b-a}{n+1}\right)_p.$$

From the inequality

$$\omega_{m+r}(f,t)_p \le t^r \omega_m(f^{(r)},t)_p,$$

it follows that, for $f \in W_p^r[a,b]$,

$$E_n(f)_p \le C(m+r)\left(\frac{b-a}{n+1}\right)^r \omega_m\left(f^{(r)}, \frac{b-a}{n+1}\right)_p.$$

It can be used to characterize class $H_{p,m}^w$. Given $f \in L_p[a,b]$, $1 \le p < \infty$, and $m \in \mathbb{N}$, it is known that for any interval $[c,d]$, $[a,b] \subset (c,d)$, there exists an extension f^* of f to $[c,d]$ such that,

$$\omega_m(f^*,h)_p \le C\omega_m(f,h)_p.$$

If ω satisfies the condition

$$h^m \int_h^H \frac{\omega(t)}{t^{m+1}} dt \le c\omega(t),$$

then the following conditions $f \in H_{p,m}^{\omega}$ and $E_n(f^*)_p \le C\omega(1/n)$ are equivalent.

Theorem 2.11.7 (Dynkin, [104]). *Fix $1 < p < \infty$ and $r \in \mathbb{N}$. For a function $f : [-1,1] \to \mathbb{R}$ one has $f \in W_p^r[-1,1]$ if and only if, for each $k \in \mathbb{N}_0$, there exists $P_{2^k} \in \Pi_{2^k}$ such that*

$$\int_{-1}^{1} \left(\sum_{k=0}^{\infty} \frac{\mid f(x) - P_{2^k}(x) \mid^2}{(\Delta_{2^k}(x))^{2r}} \right)^{p/2} dx < \infty.$$

It was Operstein, in 1995 [275], who stated the theory as completed as in the uniform norm. Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the condition $\omega(s+t) \le M(\omega(s) + \omega(t))$ and set $\rho_k(x) = 2^{-k}\sqrt{1-x^2} + 2^{-2k}$. We use the customary notation for the mixed norm

$$\|A_k(\cdot)\|_{l_p(L_p)} = \|\{\|A_k(\cdot)\|_{L_p}\}_k\|_{l_p}.$$

That is

$$\|A_k(\cdot)\|_{l_p(L_p)} = \left(\sum_{k=1}^{\infty} \int_{-1}^{1} \mid A_k(x) \mid^p dx \right)^{1/p} = \|\{\|A_k(\cdot)\|_{L_p}\}_k\|_{l_p}.$$

Theorem 2.11.8 (Operstein, [275]). *Fix $p \in [1,\infty]$ and $r \in \mathbb{N}$. There exists a constant $C = C(p,r)$ such that, for each $f \in L_p[-1,1]$ and $k \in \mathbb{N}_0$, there exists an algebraic polynomial $\{P_k\}$ of degree at most $2^k + r - 2$ such that*

$$\left\| \frac{f - P_k}{\omega(\rho_k)} \right\|_{l_p(L_p)} \le C \left\| \frac{\omega_r(f, 2^{-k})}{\omega(2^{-k})} \right\|_{l_p}.$$

Brudnyi's Theorem 2.5.3 follows from this one by setting $\omega(t) = \omega_r(f,t)_p$ and $p = \infty$.

Theorem 2.11.9 ([275]). *Let f be a function defined on $[-1,1]$. If there exists a sequence $\{P_k\}$ of algebraic polynomials of degree at most $2^k - 1$ such that*

$$\left\| \frac{f - P_k}{\omega(\rho_k)} \right\|_{l_p(L_p)} \le 1,$$

then for every $r \in \mathbb{N}$,

$$\omega_r\left(f,t\right)_p \le C\, t^r \left[\int_t^1 \left(\frac{\omega(u)}{u^r} \right)^q \frac{du}{u} \right]^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

where the constant C depends only on r and p.

When $p = \infty$ we obtain the Timan inverse result. With these theorems one has the characterization of $\mathrm{Lip}(\alpha, p)$ spaces.

Theorem 2.11.10 (Operstein, [275]). *A function $f : [-1, 1] \to \mathbb{R}$ belongs to $\mathrm{Lip}(\alpha, p)$ if and only if there exists a sequence $\{P_k\}$ of algebraic polynomials of degree at most 2^k $(k = 0, 1, \dots)$ such that*

$$\|(f - P_k) \min\{1, t/\rho_k\}^s\|_{l_p(L_p)} = \mathcal{O}(t^\alpha), \quad 0 < \alpha < s.$$

The idea of using $\min\{1, t/\rho_k\}$ for a characterization of $\mathrm{Lip}(\alpha, p)$ appears in [89], where it is proved that for each function $f \in \mathrm{Lip}(\alpha, p)$, $0 < \alpha < 1$, there exists a polynomial P_k such that $\|(f - P_k) \min\{1, t/\rho_k\}\|_{l_p(L_p)} = \mathcal{O}(t^\alpha)$. As we remarked above, Motornyi and DeVore showed that the direct analogue $(\|(f - P_n)\rho_n^{-\alpha}\}\|_{L_p} \le C$ does not characterize $\mathrm{Lip}(\alpha, p)$ when $p < \infty$.

2.12 L_p, $0 < p < 1$

The behavior os the best approximation in L_p space, for $0 < p < 1$ is not the same as in the case $p \ge 1$. For instance, the difference $f(x) - P_n(x)$ (where P_n is a polynomial of the best approximation) must not oscillate at least at $n + 1$ point. For studies concerning this problem see [160], [402], [403], [404] and [405].

For $0 < p < 1$ the functional

$$\|f\|_p = \left(\int_a^b | f(x) |^p \, dx \right)^{1/p}$$

is not a norm, but the notation $\|f\|_p$ is used in this case for the sake of convenience.

Some smoothing processes which are usually applied in approximation theory do not work well in L_p spaces $(0 < p < 1)$. Even more, the common definition of Sobolev spaces gives place to spaces with a trivial dual (see [279]). Thus, the ideas associated to K-functionals can not be used. There are also differences with the classical spaces related with the connection between smoothness and the existence of derivatives. In [214] Kortov studied this last topic.

In 1975 Storozhenko, Krotov and Oswald (Osval'd) presented direct and converse results for trigonometric approximation in the space of periodic functions $L_p[0, 2\pi]$, for $0 < p < 1$ [356]. The extension of the classical theory to this setting was motivated by some problems related with embedding theorems (see [352]). In [357] Storonzenko and Oswald presented estimates with the second-order modulus.

Theorem 2.12.1. *If $0 < p < 1$ and $f \in L_p[0, 2\pi]$, then*

$$E_n^*(f)_p \le C_p \, \omega \left(f, \frac{\pi}{n+1} \right)_p ,$$

where

$$E_n^*(f)_p = \inf_{T_n \in \mathbb{T}_n} \|f - T_n\|_p.$$

Moreover, for $n = 0, 1, \ldots,$

$$\omega\left(f, \frac{1}{n+1}\right)_p \le \frac{C_p}{n+1}\left(\sum_{j=0}^{n}(j+1)^{p-1}(E_j(f)_p)^p\right)^{1/p}. \tag{2.54}$$

Similar results were obtained in the same year by V.I. Ivanov [161], but he used moduli of smoothness of higher order:

$$\omega_k\left(f, \frac{1}{n}\right)_p \le \frac{C_{p,k}}{n^k}\left(\sum_{j=0}^{n}(j+1)^{kp-1}(E_j(f)_p)^p\right)^{1/p}.$$

In [356] and [161] a Bernstein inequality was proved for spaces $L_p[0, 2\pi]$, $0 < p < 1$, in the form

$$\|T^{(r)}\|_p \le C(p) n^r \|T\|_p.$$

Other proofs were given by Ivanov [162], Oswald [276], Nevai [269] and Runovskii [320]. The best result was presented by Arestov.

Theorem 2.12.2 (Arestov, [2]). *For* $0 < p < 1$, $n, r \in \mathbb{N}$ *and* $T_n \in \mathbb{T}_n$ *one has*

$$\|T_n^{(r)}\|_p \le n^r \|T_n\|_p.$$

In [320] and [321] Runovskii constructed some linear polynomial operators and obtained direct results in $L_p[0, 2\pi]$ $(0 < p < 1)$ in the periodical case.

For $0 < p < \infty$ and $\mu \ge -1/p$, Khodak considered the spaces $L_{p,\mu}[-1, 1]$ of functions f for which

$$\|f\|_{p,\mu} = \left(\int_{-1}^{1} | f(x)(\sqrt{1-x^2})^{\mu} |^p\right)^{1/p} < \infty.$$

A function $f \in A_{p,\mu}^{\alpha,\beta}$ if

$$\left(\int_0^{\pi}\left|\frac{f(\cos(\gamma+t)) - f(\cos\gamma)}{(\sin\gamma + |\sin t|)^{\beta}}\right|^p (\sin\gamma)^{1+\mu p} d\gamma\right)^{1/p} \le C\,|\sin t|^{\alpha}.$$

A function $f \in \overline{A}_{p,\mu}^{\alpha,\beta}$ if

$$\left(\int_0^{\pi}\left|\frac{f(\cos(\gamma+t_1)) + f(\cos(\gamma+t_2)) - 2f(\cos(\gamma+(t_1+t_2)/2))}{(\sin\gamma + |\sin t|)^{\beta}}\right|^p (\sin\gamma)^{1+\mu p} d\gamma\right)^{1/p}$$
$$\le C\,|\sin t|^{\alpha},$$

where $t = | t_1 | + | t_2 |$. Here β is a real number, for $p \ge 1$ we consider that $0 < \alpha \le 1$ and, for $0 < p < 1$, $0 < \alpha \le 1/p$.

For $p \geq 1$, and $t_1 = -t_2$ these spaces coincide with the one studied by Lebed and Potapov.

Theorem 2.12.3 (Khodak, [189]). *Let $f \in L_{p,\mu}[-1,1]$, $0 < p < 1$, $\mu \geq -1/p$, $0 < \alpha < 2$,*

$$-2/p - 2 - \mu + \alpha \;<\; -\beta \;<\; \alpha - 1/p - \mu.$$

In order that $f \in A^{\alpha,\beta}_{p,\mu}$ for $0 < \alpha < 1$ or $f \in \overline{A}^{\alpha,\beta}_{p,\mu}$ for $0 < \alpha < 2$, it is necessary and sufficient that there exists a sequence $\{P_n\}$, $P_n \in \mathbb{P}_n$ such that

$$\left\| [f(x) - P_n(x)] \left(\sqrt{1-x^2} + \frac{1}{n}\right)^{-\beta} \right\|_{p,\mu} \leq \frac{C}{n^\alpha},$$

where the constant C does not depend on f and n.

As Ditzian showed we can not extend the results related with simultaneous approximation to the case $0 < p < 1$.

Theorem 2.12.4 (Ditzian, [97]). *For each $0 < p < 1$ there exists a function $f \in A.C.[-1,1]$ for which we can not find a sequence $\{p_n\}$, $p_n \in \mathbb{P}_n$ such that*

$$\|f - p_n\|_p \leq C\omega_2(f, 1/n)_p \qquad \text{and} \qquad \|f' - p_n'\|_p \leq C\omega(f', 1/n)_p\,.$$

The same assertion holds if we replace the usual moduli by the Ditzian-Totik one.

2.13 The Whitney theorem

Another form for the direct results in approximation by algebraic polynomials is due to Whitney.

Theorem 2.13.1 (Whitney, [407] and [408]). *For any $n \in \mathbb{N}$ there exists a constant $W_\infty(n)$ such that, for every bounded function $f : [a,b] \to \mathbb{R}$ there exists a polynomial $P_{n-1}(f) \in \mathbb{P}_{n-1}$ satisfying*

$$\|f - P_{n-1}(f)\| \leq W_\infty(n)\,\omega_n\left(f, \frac{b-a}{n}\right).$$

In fact, this was proved by Burkill in 1952 [43] for $n = 1, 2$, who also conjectured that the inequality holds for $n \geq 3$. In 1957 Whitney verified the conjecture for continuous functions and in 1959 for bounded functions. The proof of Whitney, as the one due to Burkill, used the polynomial P which interpolates f over a uniform net

$$P\left(\frac{k}{n-1}\right) = f\left(\frac{k}{n-1}\right), \qquad (k = 0, 1, \ldots, n-1).$$

If we consider the polynomial $Q_{n-1}(f)$ which interpolates f at a uniform net of node, then we can also consider the inequalities

$$\|f - Q_{n-1}(f)\| \leq W'_\infty(n)\,\omega_n\left(f, \frac{b-a}{n}\right).$$

In 1964 Brudnyi found a new proof of the Whitney theorem. He used some smoothing of the function by means of linear combinations of Steklov-type functions. With the new method of proof he was able to extend the result to L_p spaces, with $1 \leq p < \infty$. In 1977 Storozhenko extended the Whitney theorem for algebraic approximation to $L_p[a,b]$ spaces, for $0 < p < 1$ (see also [358]).

Theorem 2.13.2 (Brudnyi, [39] and Storozhenko, [353]). *Suppose* $0 < p < \infty$, $f \in L_p(a,b)$ *and* n *is an arbitrary natural number, then*

$$E_{n-1}(f)_p \leq W_p(n)\,\omega_n\left(f, \frac{b-a}{n}\right)_p,$$

where $W_p(n)$ *depends not on* f.

Another proof was presented in [355] by Storozhenko and Kryakin.

In [354] Storozhenko presented the inequality: for $0 < p < 1$, $f \in L_p[-1,1]$, $k \in \mathbb{N}$ and $n \geq k-1$,

$$E_n(f)_p \leq C_{p,k}\omega_k\left(f, \frac{1}{n+1}\right)_p.$$

A similar inequality appeared in [342] but only for the first modulus. Another proof was given by Khodak in [190].

The proof of Whitney can not be used as the estimate of the constants. Whitney proved that

$$\frac{1}{2} \leq W_\infty(n)$$

and found some bounds for some values of n. For instance

$$1 \leq W_\infty(1) \leq 2, \quad 1 \leq W_\infty(2) \leq 2.$$

The Whitney theorem has been studied by several Bulgarian mathematicians. In 1982 Sendov conjectured that $W_\infty(n) \leq 1$ [331].

This motivated several papers, shown in the table on top of the next page.

The inequality $W_\infty(n) \leq 1$ has been verified only for a few values of n: Whitney $n = 3$ [407], Kryakin $n = 4$ [219] and Zhelnov $k = 5, 6, 7, 8$, [416].

In [221] Kryakin and Takev proved that $W'_\infty(n) \leq 5\omega_n(f, 1/n)$.

Tunc [391] considered Whitney-type theorems in the form

$$E_{k+r+1}(f,[a,b])_\infty \leq W_\infty(k,r)\left(\frac{b-a}{k}\right)^r \omega_k\left(f, \frac{b-a}{k}, [a,b]\right)_\infty.$$

He found upper bounds for $W(k,2)$, $W(1,r)$ and $W(2,r)$.

Year	**Author**	**Reference**	**Estimate**
1952	Burkill	[43]	$W_\infty(2) = 1/2$,
1964	Brudnyi	[39]	$W_\infty(n) \le Cn^{2n}$,
1985	Ivanov-Takev	[174]	$W_\infty(n) \le C(n \ln n)$,
1985	Binev	[31]	$W_\infty(n) \le C\, n$,
1985	Sendov	[332]	$W_\infty(n) \le C$,
1986	Sendov	[333]	$W_\infty(n) \le 6$,
1985	Sendov-Takev	[335]	$W_1 \le 30$,
1989–90	Kryakin	[215], [216]	$W_\infty(n) \le 3$,
1989	Sendov-Popov	[334]	$W_\infty(n) \le 3$,
1990	Kryakin	[216]	$W_p(n) \le 11$,
1992	Kryakin-Kovalenko	[220]	$W_1 \le 6.4$,
1992	Kryakin-Kovalenko	[220]	$W_p \le 9$,
1995	Kryakin	[217], [218]	$W_\infty(n) \le 2$.
2002	Gilewicz-Kryakin-Shevchuk	[142]	$W_\infty(n) \le 2 + e^{-2}$.

2.14 Other classes of functions

Bernstein [30] characterized $C^\infty[a,b]$ as follows: $f \in C^\infty[a,b]$ if and only if each $k \in \mathbb{N}$,

$$\lim_{n\to\infty} n^k E_n(f) \;=\; 0.$$

Some subclasses of functions of $C^\infty[a,b]$ has been studied by Brudnyi-Gopengauz [41] Babenko [4] and Motornyi [256].

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