Chapter 2

Smooth Functions on Subsets of $\mathbb{R}^n$

This chapter is an introduction to the final part of the book (Chapters 9–11) that studies the problems of a field which may be regarded as a part of Global Differential Analysis on subsets of $\mathbb{R}^n$. Such subsets may be of a very complicated geometrical nature (i.e., fractals) and, in general, have no naturally associated differential structure. The presence of such different structures – rigid in the sense of Differential Calculus and possibly chaotic in terms of the underlying set – makes the extension–trace problems for smooth functions one of the most difficult problems of Analysis. Therefore the Whitney solution to the problem for univariate C^k functions appears as a miracle. The title of Whitney's paper [Wh-1934b] includes number I which, probably, indicates his intention to proceed further with this study. However, significant multivariate results appeared only about fifty years after Whitney's seminal papers [Wh-1934a] and [Wh-1934b]. The extension theorem from the first paper is one of the basic results of Global Differential Analysis; its generalizations and ramifications have been presented in many papers and books, see, e.g., the books [Hor-1983], [Mal-1966] and [Ste-1970]. Therefore it would be natural to present this material in as brief a manner as possible. However, Whitney's extension method plays an important role in almost all approaches to the problems under consideration. In Section 2.2 we will give it a rather detailed exposition. The reader is referred to the aforementioned books for further details.

On the other hand, Whitney's beautiful result [Wh-1934b] has never appeared in book form. We present (in Section 2.4) its complete proof but not only for this reason. In fact, an appropriate reformulation of this result leads to several general conjectures for multivariate functions whose study appears to be one of the main trends in the field. We discuss the results obtained in this direction within the last two decades in the final part of the book.

A. Brudnyi and Y. Brudnyi, *Methods of Geometric Analysis in Extension and Trace Problems: Volume 1*, Monographs in Mathematics 102, DOI 10.1007/978-3-0348-0209-3_2,

Difference characteristics of C^k functions in the univariate case and local approximation by Taylor polynomials in the multivariate case play an essential role in Whitney's proofs. The results of this kind which are not commonly known can be found in Section 2.3.

Turning to the extension and trace problems for Lipschitz functions of higher order (see Section 2.1 for the corresponding definitions) one immediately notices that Taylor approximation is inefficient in this setting. For instance, the Weierstrass nowhere differentiable function is a Lipschitz function of second order. A new tool, *local polynomial best approximation*, which will be used for this case, will first appear in Section 2.3 in connection with S. Bernstein's classical theorem [Ber-1940]. This result gives a complete description of the space $C^k(\mathbb{R})$ via local best approximation.

In Chapters 9 and 10 (Volume II), we will show that local polynomial approximation theory gives powerful tools for the study of extension–trace problems for Lipschitz functions of higher order. In fact, the range of its applications is much broader and includes the study of basic properties of the classical spaces of smooth functions.

The results obtained are then applied in Section 2.5 to solve the restricted Main Problem for several classes of domains in $\mathbb{R}^n$ (quasiconvex and Lipschitz).

The final section presents selected trace and extension problems for weakly differentiable functions, in particular, the P. Jones extension theorem [Jon-1981] and the Peetre theorem [Peet-1979] on the nonexistence of a simultaneous extension for the Sobolev space $W_1^1(\mathbb{R}^n_+)$.

2.1 Classical function spaces: notation and definitions

2.1.1 Differentiable functions

Throughout the book $C^k(\mathbb{R}^n)$ denotes the space of k-times continuously differentiable functions on $\mathbb{R}^n$ equipped with the topology of uniform convergence of the functions and their derivatives on compact subsets of $\mathbb{R}^n$; here k may be infinity. Hence, $C^k(\mathbb{R}^n)$ is a Fréchet space with topology determined by the collection of seminorms

$$|f|_m^K := \sum_{|\alpha|\le m} \sup_K |D^\alpha f|, \tag{2.1}$$

where $K \subset \mathbb{R}^n$ is compact and $0 \le m \le k$.

Hereafter we will use the standard notation of Differential Analysis. In particular, $\alpha = (\alpha_1, \dots, \alpha_n), \beta, \dots$ are multi-indices, i.e., vectors from $\mathbb{Z}^n_+$, and $|\alpha| := \sum_{i=1}^n \alpha_i$. Moreover, for $x \in \mathbb{R}^n$,

$$x^\alpha := \prod_{i=1}^n x_i^{\alpha_i} \quad \text{and} \quad D^\alpha := \prod_{i=1}^n D_i^{\alpha_i}, \tag{2.2}$$

where $D_i := \frac{\partial}{\partial x_i}$.

For example, the Taylor polynomial of degree k at a point $y \in \mathbb{R}^n$ may be written as

$$T_y^k F(x) := \sum_{|\alpha| \le k} D^\alpha F(y) \frac{(x-y)^\alpha}{\alpha!}, \tag{2.3}$$

where $\alpha! := \prod_{i=1}^{n} (\alpha_i!)$.

If now G is a *domain* (connected open subset) of $\mathbb{R}^n$, then the Fréchet space $C^k(G)$ is defined similarly to the case $G = \mathbb{R}^n$ by the collection of seminorms (2.1), but with the subsets K being compactly embedded in G. Such an embedding is denoted by $K \Subset G$.

Definition 2.1. $C_b^k(G)$ is a Banach subspace of $C^k(G)$ defined by the norm

$$\|f\|_{C_b^k(G)} := \sum_{|\alpha| \le k} \sup_G |D^\alpha f|. \tag{2.4}$$

A "homogeneous" C^k space is defined by the seminorm

$$|f|_{C_b^k(G)} := \sum_{|\alpha| = k} \sup_G |D^\alpha f| \tag{2.5}$$

and is denoted by $\dot{C}_b^k(G)$ (with a dot!).

Another subspace of $C^k(G)$ consists of functions with prescribed behavior of the moduli of continuity for their higher derivatives; in its definition presented now k is finite and ω denotes a nonnegative function on $\mathbb{R}_+ := [0, +\infty)$ satisfying the following conditions:

(i) $\omega(t)$ and $\frac{t}{\omega(t)}$ are nondecreasing as t increases to $+\infty$;

(ii) $\omega(0+) = 0$.

Definition 2.2. $C_b^{k,\omega}(G)$ is a Banach subspace of $C^k(G)$ defined by the norm

$$\|f\|_{C_b^{k,\omega}(G)} := \|f\|_{C_b^k(G)} + |f|_{C_b^{k,\omega}(G)}, \tag{2.6}$$

where the seminorm on the right-hand side is given by

$$|f|_{C_b^{k,\omega}(G)} := \sum_{|\alpha|=k} \sup_{[x,y] \subset G} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\omega(\|x-y\|)}. \tag{2.7}$$

Here $[x, y]$ is the closed segment with the endpoints x and y, and $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^n$.

As before, $\dot{C}^{k,\omega}(G)$ will denote the homogeneous $C_b^{k,\omega}$-space defined by seminorm (2.7).

For $G = \mathbb{R}^n$, in particular, we have the spaces $C_b^k(\mathbb{R}^n)$, $\dot{C}_b^k(\mathbb{R}^n)$ and so on.

Remark 2.3. If the function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the above condition (i), then ω is *subadditive*, i.e., for $0 < t_1,\ t_2 < \infty$,

$$\omega(t_1 + t_2) \le \omega(t_1) + \omega(t_2). \tag{2.8}$$

This and (ii) imply that $d_\omega(x, y) := \omega\big(\|x - y\|\big),\ x, y \in \mathbb{R}^n$, is a metric on $\mathbb{R}^n$ and $\dot{C}^{0,\omega}(\mathbb{R}^n) = \mathrm{Lip}(\mathbb{R}^n, d_\omega)$.

Finally, $\dot{C}^k_u(G)$ is the closed linear subspace of $\dot{C}^k_b(G)$ consisting of functions with *uniformly continuous* higher derivatives:

$$\dot{C}^k_u(G) := \big\{f \in \dot{C}^k_b(G);\quad D^\alpha f,\ |\alpha| = k,\ \ \text{are uniformly continuous on}\ \ G\big\}. \tag{2.9}$$

2.1.2 k-jets

Let F be a linear space of k-times differentiable functions on a *domain* $G \subset \mathbb{R}^n$. One defines a space of *k-jets* $J^k F$ as the space of all vector functions (k-jets) $\vec{f} := \{f_\alpha\}_{|\alpha| \le k}$ on G with values in $\mathbb{R}^N$ such that for some $f \in F$ and all $|\alpha| \le k$,

$$D^\alpha f = f_\alpha.$$

Here $N = N(k, n) := \mathrm{card}\{\alpha \in Z^n_+\ ;\ |\alpha| \le k\}$.

If $\mathcal{F}$ is a (semi-) normed space, $J^k F$ is equipped with the (semi-) norm

$$\|\vec{f}\|_{J^k F} := \inf\big\{\|f\|_F\, ;\, D^\alpha f = f_\alpha,\ |\alpha| \le k\big\}.$$

Specifically, choosing F being equal to $C^k(G), C^k_b(G)$, etc., we will simply write $J^k(G), J^k_b(G), \ldots$ instead of $J^k F$. Since G is open, all f_α with $\alpha \ne 0$ are uniquely defined by f_0, e.g., the norm of $\vec{f} \in J^k_b(G)$ satisfies

$$\|\vec{f}\|_{J^k_b(G)} = \|f_0\|_{C^k_b(G)} := \sup_{|\alpha| \le k} \|f_\alpha\|_{C_b(G)}. \tag{2.10}$$

In particular, the linear projection $\vec{f} \mapsto f_0$ maps isometrically $J^k_b(G)$ onto $C^k_b(G)$; sometimes these spaces will be identified in the sequel.

In turn, the Fréchet topology on $J^k(G)$ (k may be infinity) is defined by the uniform convergence on compact subsets of G, i.e., this topology is determined by the family of seminorms

$$|f|_{m,C} := \sup_{|\alpha| \le m}\ \sup_{x \in C} |f_\alpha(x)|,$$

where $0 \le m \le k$ and $C \subset G$ runs over all compact subsets of G.

2.1.3 Lipschitz functions of higher order

The definition of this family is based on the notion of *k-modulus of continuity* characterizing smoothness of functions in more detail than do derivatives and Taylor polynomials.

We first recall the definition of the *k-th difference*. Given $h \in \mathbb{R}^n$ and $k \in \mathbb{N}$, the k-th difference is a linear operator on functions on $\mathbb{R}^n$ given by

$$\Delta_h^k := (\tau_h - 1)^k,$$

where $\tau_h : f \mapsto f(\cdot + h)$ is the shift operator.

Hence, for $f : \mathbb{R}^n \to \mathbb{R}$,

$$\Delta_h^k f(x) := \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh), \quad x \in \mathbb{R}^n. \tag{2.11}$$

Definition 2.4. k-modulus of continuity is the function on $\ell_\infty^{\mathrm{loc}}(\mathbb{R}^n) \times (0, +\infty)$ with range in $\mathbb{R}_+ \cup \{+\infty\}$ given by

$$\omega_k(t\,; f) := \sup_{\|h\| \le t} \left\| \Delta_h^k f \right\|_{\ell_\infty(\mathbb{R}^n)}. \tag{2.12}$$

Here $\ell_\infty^{\mathrm{loc}}(\mathbb{R}^n)$ is the (Fréchet) space of locally bounded functions f on $\mathbb{R}^n$ equipped with the collection of seminorms $\left\{ \sup\limits_C |f| \right\}$ where C runs over the family of compact subsets of $\mathbb{R}^n$.

The following fact whose proof may be found in Appendix F to this chapter explains the necessity of local boundedness in this definition.

Theorem 2.5. (a) *Let f be bounded on a compact subset of $\mathbb{R}^n$. Then*

$$\omega_k(\cdot\,; f) = 0$$

if and only if f is a polynomial of degree $k - 1$.

(b) *Let $f \in \ell_\infty^{\mathrm{loc}}(\mathbb{R}^n)$ and $\omega_k(t\,; f) < \infty$ for some (and therefore for all) $t > 0$. Then $f = f_0 + p$ where f_0 is bounded on $\mathbb{R}^n$ and p is a polynomial of degree $k - 1$.*

Remark 2.6. In assertion (a), boundedness may be replaced by measurability on $\mathbb{R}^n$. The classical Hamel example of a function H which is nonmeasurable and unbounded on every open subset and satisfies

$$\Delta_h^k H = 0 \quad \text{for all} \quad h \in \mathbb{R}^n \quad \text{and} \quad k \ge 2$$

explains the necessity of having at most one of these assumptions.

In the next theorem we formulate the basic properties of k-modulus of continuity and outline proofs for some of them; see any book on Approximation Theory, e.g., [DeVL-1993] or [Tim-1963] for the proofs of the remaining facts.

Theorem 2.7. *Let $\omega_k(t_0\,;\,f)<\infty$ for some $t_0>0$. Then the following is true:*

(a) *ω_k is nondecreasing and continuous in $t\in(0,+\infty)$ and, moreover,*

$$\omega_k(0+\,;f)=0$$

if and only if f is uniformly continuous on $\mathbb{R}^n$.

(b) *For every integer $N\ge 1$ and $t>0$,*

$$\omega_k(Nt\,;f)\le N^k\omega_k(t\,;f).$$

(c) *There is a function $\widehat{\omega}:(0,+\infty)\to\mathbb{R}_+$ satisfying the conditions*

(i) *$2^{-k}\widehat{\omega}\le\omega_k(\cdot\,;f)\le\widehat{\omega}$;*

(ii) *$\widehat{\omega}(t)$ and $t^k/\widehat{\omega}(t)$ are nondecreasing as t increases to $+\infty$;*

(d) *For $0\le\ell<k$,*

$$\omega_k(\cdot\,;f)\le 2^{k-\ell}\omega_\ell(\cdot\,;f);$$

here $\omega_0(\cdot\,;f):=\|f\|_{\ell_\infty(\mathbb{R}^n)}$.

Conversely, there is a polynomial p of degree $k-1$ such that for some $c>0$ depending only on k and all $t>0$,

$$\omega_\ell(t\,;f-p)\le ct^\ell\int_t^\infty\frac{\omega_k(s\,;f)}{s^{\ell+1}}\,ds. \tag{2.13}$$

(e) *If f has locally bounded derivatives of order ℓ on $\mathbb{R}^n$ where $\ell<k$, then for some constant $c>0$ and all $t>0$,*

$$\omega_\ell(t\,;f)\le ct^\ell\max_{|\alpha|=\ell}\omega_{k-\ell}(t\,;D^\alpha f);$$

here c depends only on k and n.

Conversely, for some $c=c(k,n)>0$ and all $t>0$,

$$\max_{|\alpha|=\ell}\omega_{k-\ell}(t\,;D^\alpha f)\le c\int_0^t\frac{\omega_k(s\,;f)}{s^{\ell+1}}\,ds. \tag{2.14}$$

Proof (outline). (b) follows from the identity

$$\Delta^k_{Nh}=(\tau_h^N-1)^k=\left(\sum_{j=0}^{N-1}\tau_{jh}\right)^k\Delta_h^k$$

and the equality $\|\tau_h\| = 1$.

Assertions (d) and (e) are usually called Marchaud type inequalities after Marchaud [Mar-1927] who proved these results for continuous univariate functions (in a less precise form for (d)). The multivariate case was proved in the paper [BSha-1973] by Yu. Brudnyi and V. Shalashov. For the reader's convenience we will present the proof of (d) in Appendix F.

Finally, the regularization $\widehat{\omega}$ in (c) may be defined by

$$\widehat{\omega}(t) := t^k \sup_{s\geq t} \frac{\omega_k(s\,;f)}{s^k}\,.$$

Then (ii) is clear while (i) follows from (b) which implies that

$$\frac{\omega_k(s\,;f)}{s^k} \leq 2^k \frac{\omega_k(t\,;f)}{t^k} \quad \text{for} \quad s > t. \qquad \square$$

Property (c) of this theorem motivates

Definition 2.8. A function $\omega : (0, +\infty) \to \mathbb{R}_+$ belongs to the class Ω_k if it satisfies the conditions

(a) ω is nondecreasing, continuous and

$$\omega(0+) = 0;$$

(b) for all $0 < t \leq s$,

$$\frac{\omega(s)}{s^k} \leq \frac{\omega(t)}{t^k}\,.$$

In the sequel the functions of Ω_k will be called *k-majorants*.

Using the notions introduced we now define the desired family of function spaces.

Definition 2.9. Let $\omega \in \Omega_k$. The homogeneous Lipschitz space $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ consists of locally bounded on $\mathbb{R}^n$ functions f satisfying

$$|f|_{\Lambda^{k,\omega}(\mathbb{R}^n)} := \sup_{t>0} \frac{\omega_k(t\,;f)}{\omega(t)} < \infty. \tag{2.15}$$

We also define the Banach space $\Lambda^{k,\omega}(\mathbb{R}^n)$ of Lipschitz functions of order k by

$$|f|_{\Lambda^{k,\omega}(\mathbb{R}^n)} := \sup_{\mathbb{R}^n} |f| + |f|_{\Lambda^{k,\omega}(\mathbb{R}^n)}. \tag{2.16}$$

Finally, we introduce the spaces of *smooth functions* combining definitions from both of the subsections. For instance, the space $C_b^k\Lambda^{s,\omega}(\mathbb{R}^n)$, where $\omega \in \Omega_s$, is the linear subspace of $C_b^k(\mathbb{R}^n)$ given by the norm

$$\|f\|_{C_b^k\Lambda^{s,\omega}(\mathbb{R}^n)} := \|f\|_{C_b^k(\mathbb{R}^n)} + \sup_{|\alpha|=k} |D^\alpha f|_{\Lambda^{s,\omega}(\mathbb{R}^n)} \tag{2.17}$$

while the corresponding homogeneous space $C^k\dot{\Lambda}^{s,\omega}(\mathbb{R}^n)$ is defined by the semi-norm

$$|f|_{C^k\Lambda^{s,\omega}(\mathbb{R}^n)} := \max_{|\alpha|=k} |D^\alpha f|_{\Lambda^{s,\omega}(\mathbb{R}^n)}. \tag{2.18}$$

Similarly we define other combinations; e.g., $J^k\Lambda^{s,\omega}(\mathbb{R}^n)$ is the space of k-jets $\vec{f} = \{f_\alpha\}_{|\alpha|\le k}$ from $J_b^k(\mathbb{R}^n)$, equipped with the norm

$$\|\vec{f}\|_{J^k\Lambda^{s,\omega}(\mathbb{R}^n)} := \max_{|\alpha|\le k} \sup_{\mathbb{R}^n} |f_\alpha| + \sup_{|\alpha|=k} |f_\alpha|_{\Lambda^{s,\omega}(\mathbb{R}^n)}. \tag{2.19}$$

The properties of the k-modulus of continuity presented in Theorem 2.7 immediately imply the following relations between some of these spaces.

Theorem 2.10. *Let $\omega \in \Omega_k$ and $0 < s < k$ be an integer. Then the following is true:*

(a) *If the function $\widetilde{\omega} : (0,+\infty) \to \mathbb{R}_+$ given by*

$$\widetilde{\omega}(t) := \int_0^t \frac{\omega(u)}{u^{s+1}}\, du$$

is finite, then

$$\Lambda^{k,\omega}(\mathbb{R}^n) \subset C^s\Lambda^{k-s,\widetilde{\omega}}(\mathbb{R}^n) \tag{2.20}$$

and the embedding constant[1] *is bounded by some $C = C(k, n)$.*[2]

(b) *If the function $\widehat{\omega} : (0,+\infty) \to \mathbb{R}_+$ given by*

$$\widehat{\omega}(t) := t^s \int_t^\infty \frac{\omega(u)}{u^{s+1}}\, du$$

is finite, then

$$\Lambda^{k,\omega}(\mathbb{R}^n) \subset \Lambda^{s,\widehat{\omega}}(\mathbb{R}^n) \tag{2.21}$$

and the embedding constant is bounded by $C = C(k, n)$.

In particular, let $\omega(t) := t^\sigma,\ 0 < \sigma \le k$, and s be the largest integer less than σ. Then the previous result yields the following equality:

$$\Lambda^{k,\omega}(\mathbb{R}^n) = C^s\Lambda^{2,\overline{\omega}}(\mathbb{R}^n), \tag{2.22}$$

where $\overline{\omega}(t) := t^{\sigma-s}$ and the corresponding norms are equivalent.

[1] that is, the norm of the linear embedding operator.

[2] We write $C = C(k, \ell, \dots)$ etc. to indicate dependence *only* on the arguments in the brackets.

Note that if σ is not integer or $\sigma = k$, one can replace the right-hand side by $C^s\Lambda^{1,\overline{\omega}}(\mathbb{R}^n)$. Otherwise $s = \sigma - 1$ and the space in the right-hand side is defined by the norm

$$\|f\|_{B^\sigma_\infty(\mathbb{R}^n)} := \|f\|_{\ell_\infty(\mathbb{R}^n)} + \max_{|\alpha|=\sigma-1} \sup_{t>0} \frac{\omega_2(t\,; D^\alpha f)}{t}\,. \tag{2.23}$$

For $s = 0$ (i.e., $\sigma = 1$) this space is sometimes called the Zygmund space after A. Zygmund who was the first to discover its role in Approximation Theory and Harmonic Analysis [Z-1945]. He also coined the term “smooth function”.

In the modern literature the space $\Lambda^{k,\omega}(\mathbb{R}^n)$ with $\omega(t) := t^\sigma,\ 0 < \sigma \le k$, is denoted by $B^\sigma_\infty(\mathbb{R}^n)$, since it is a member of the important family of the so-called Besov spaces. This explains the notation in (2.23); in the sequel we adopt this term and notation.

Remark 2.11. The embeddings of Theorem 2.10 also hold for the corresponding homogeneous spaces. E.g., using Theorem 2.7 (d) we have

$$\dot\Lambda^{k,\omega}(\mathbb{R}^n) \subset \dot\Lambda^{s,\widehat{\omega}}(\mathbb{R}^n)/\mathcal{P}_{k-1,n},$$

where $\mathcal{P}_{k,n} \subset \mathbb{R}[x_1,\dots,x_n]$ hereafter stands for the space of polynomials of degree k on $\mathbb{R}^n$.

Now we define similar families of spaces over a domain $G \subsetneq \mathbb{R}^n$. In this case, $f \in \ell^{\mathrm{loc}}_\infty(G)$ and the function $x \mapsto \Delta^k_h f(x)$ is defined on the set

$$G_{k,h} := \big\{x\,; x + jh \in G, \quad j = 0, 1, \dots, k\big\}.$$

We remark that the definition commonly used in the literature utilizes the set

$$G_{kh} := \big\{x\,; [x, x + kh] \subset G\big\}$$

which is smaller than the previous if G is not convex.

Note the distinction between these two sets. The former may be disconnected with infinitely many connected components (as, e.g., for a domain bounded by two spirals in the plane twisted infinitely many times around the origin) while the latter is connected.

Now the definition of k-modulus of continuity for a function $f \in \ell^{\mathrm{loc}}_\infty(G)$ is given by

$$\omega_k(t\,; f)_G := \sup_{\|h\|\le t} \big\|\Delta^k_h f\big\|_{\ell_\infty(G_{kh})}, \quad t > 0. \tag{2.24}$$

All of the properties of k-modulus of continuity formulated above for $\mathbb{R}^n$ remain to be true for *convex* domains G but most of them do not hold for general domains.

We leave to the reader to formulate the corresponding results and to define the spaces of Lipschitz functions $\dot\Lambda^{k,\omega}(G)$ and of smooth functions $C^k\dot\Lambda^{s,\omega}(G)$ and $\dot B^\lambda_\infty(G)$ and their nonhomogeneous counterparts.

Remark 2.12. (a) In accordance with the above introduced notation the spaces $\dot{C}^{k,\omega}(G)$, $\dot{J}^{k,\omega}(G)$ will be denoted by $C^k\dot{\Lambda}^{1,\omega}(G)$, $J^k\dot{\Lambda}^{1,\omega}(G)$.

(b) We emphasize the distinction between two Lipschitz spaces over an open set G, the space $\mathrm{Lip}^\omega(G)$ defined by the seminorm

$$|f|_{\mathrm{Lip}^\omega(G)} := \sup\left\{\frac{|f(x)-f(y)|}{\omega(\|x-y\|)}\,; x, y \in G\right\}$$

and the space $\dot{\Lambda}^{1,\omega}(G)$ where in both cases ω is a 1-majorant. Clearly,

$$\dot{\Lambda}^{1,\omega}(G) \supset \mathrm{Lip}^\omega(G),$$

but the second space may be essentially smaller for nonconvex G.

Let, e.g., G be the union of open balls (disks) $B_i := B_{r_i}(i,0) \subset \mathbb{R}^2$ where $r_i := \frac{1}{2} - \frac{1}{4i}$, $i \in \mathbb{N}$.

Define a function $f : G \to \mathbb{R}$ by letting f on B_i to be equal $(-1)^i$, $i \in \mathbb{N}$. Then, by (2.24),

$$\omega_1(t\,; f)_G = \sup_{\|h\|\le t} \left\|\Delta_h^1 f\right\|_{\ell^\infty(G_h)} = 0,$$

since every component of G_h is contained in some B_i.

On the other hand,

$$|f|_{\mathrm{Lip}^\omega(G)} \ge 2\sup_i \frac{1}{\omega((4i)^{-2})} = \infty.$$

2.1.4 Extension and trace problems for classical function spaces

Now let S be an arbitrary subset of $\mathbb{R}^n$, and F be one of the normed, seminormed or Fréchet spaces introduced above. In accordance with the definitions of Section 1.1, see (1.1)–(1.3) there, we define the trace space $F|_S$ and the corresponding trace norm or seminorm, etc. For example, for $F := \dot{C}_b^k(\mathbb{R}^n)$ the trace seminorm is given by

$$|f| := \inf\left\{\sum_{|\alpha|=k} \sup_{\mathbb{R}^n} |D^\alpha g|\ ;\ g|_S = f\right\}.$$

For these spaces we then pose the problems formulated in Section 1.2. In particular, the *Main Problem* for $\dot{C}_b^k(\mathbb{R}^n)$ asks for a complete characterization of the trace space $\dot{C}_b^k(\mathbb{R}^n)\big|_S$. This is naturally divided into two subproblems, the *Trace Problem* and the *Extension Problem*. Since all the spaces are linear, we may also ask about the existence of a linear extension operator (the *Simultaneous Extension Problem*).

Another possibility is to consider these problems only for special classes of subsets instead of the class of all closed ones. We will consider only one such case related to bounded domains in $\mathbb{R}^n$. For the space $C_b^k(\mathbb{R}^n)$ this, for instance, leads to the following modification of the Main Problem:

Restricted Main Problem. *Characterize bounded domains $G \subset \mathbb{R}^n$ such that*

$$C_b^k(\mathbb{R}^n)\big|_G = C_u^k(G). \tag{2.25}$$

Note that the restrictions of the higher derivatives of a function $f \in C_b^k(\mathbb{R}^n)$ to a bounded domain $G \subset \mathbb{R}^n$ are uniformly continuous on the closure $\overline{G}$. This explains the appearance of $C_u^k(G)$ in (2.25). In the same fashion, the Restricted Trace and Extension and Simultaneous Extension Problems are formulated for $C_b^k(\mathbb{R}^n)$ and bounded domains.

2.2 Whitney's extension theorem

We present a brief exposition of this classical result with emphasis on the Whitney extension method. The reader could restore the omitted details of the proof by consulting one of the books mentioned at the beginning of the chapter.

Our goal is to characterize the trace space $J^k(\mathbb{R}^n)\big|_S$ for an arbitrary subset $S \subset \mathbb{R}^n$ which in this settings may be assumed to be *closed*. The following remark motivates the appearance of *Taylor chains* in the formulation of Whitney's result.

Suppose that f is a function in $C^k(\mathbb{R}^n)$ and $T_y^k F$ is its Taylor polynomial, see (2.3). Then

$$f = T_y^k f + R_k$$

where R_k, the *remainder*, is a k-times continuously differentiable function in $x \in \mathbb{R}^n$ and continuous in y. Its mixed α-derivative in x with $|\alpha| \le k$,

$$D_x^\alpha R_k(x,y) = D^\alpha f(x) - \sum_{|\beta| \le k-|\alpha|} D^{\alpha+\beta} f(y) \cdot \frac{(x-y)^\beta}{\beta!}, \tag{2.26}$$

is clearly the remainder of order $k-|\alpha|$ for $D^\alpha F$ and by the Taylor–Peano theorem,

$$\big|D_x^\alpha R_k(x,y)\big| = o\big(\|x-y\|^{k-|\alpha|}\big) \quad \text{as} \quad y \to x.$$

The *reduced remainder*

$$r_\alpha(x,y) := D_x^\alpha R_k(x,y)/\|x-y\|^{k-|\alpha|}$$

is a continuous function on $(\mathbb{R}^n \times \mathbb{R}^n)\backslash\Delta$, where $\Delta := \{(x,y)\,;\, x=y\}$, and can be continuously extended by zero to $\mathbb{R}^n \times \mathbb{R}^n$.

Thus, we have the following

Taylor chain condition. *Assume that* $\vec{f} = \{f_\alpha\}_{|\alpha|\le k}$ *is a* k*-jet on the closed set* S *generated by a function* $f \in C^k(\mathbb{R}^n)$ [3]. *Then all its reduced remainders*

$$r_\alpha(\vec{f}\,; x, y) := \left| f_\alpha(x) - \sum_{|\beta|\le k-|\alpha|} f_{\alpha+\beta}(y)\frac{(x-y)^\beta}{\beta!} \right| \Big/ \|x-y\|^{k-|\alpha|} \quad (2.27)$$

are continuous on $(S\times S)\backslash\Delta$ *and can be continuously extended by zero to all of* $S\times S$.

In other words, if $\vec{f}$ belongs to the trace space $J^k(\mathbb{R}^n)\big|_S$, then $\vec{f}$ satisfies the Taylor chain condition. A theorem of Whitney states that this condition is also sufficient for $\vec{f}$ to be in $J^k(\mathbb{R}^n)\big|_S$.

Theorem 2.13 (Whitney). *A* k*-jet* $\vec{f} = \{f_\alpha\}_{|\alpha|\le k}$ *defined on a closed set* $S \subset \mathbb{R}^n$ *belongs to* $J^k(\mathbb{R}^n)\big|_S$ *if and only if* $\vec{f}$ *satisfies the Taylor chain condition.*

We single out the basic ingredients of Whitney's extension method and briefly explain how they work in the proof.

Whitney's covering lemma

Let $\mathcal{K} := \mathcal{K}(\mathbb{R}^n)$ stand for the collection of n-cubes in $\mathbb{R}^n$ homothetic to the cube

$$Q_0 := [-1, 1]^n. \quad (2.28)$$

We regard such a cube as the closed ball of ℓ^n_∞; so

$$Q_r(x) := \big\{y \in \mathbb{R}^n\ ;\ \|x-y\|_\infty := \max_{1\le i\le n} |y_i - x_i| \le r\big\}$$

is the ball of center x and radius r.

In the sequel, r_Q and c_Q stand for radius and center of $Q \in \mathcal{K}$. Further, by λQ with $\lambda > 0$ we denote the cube of center c_Q and radius λr_Q. Finally, we set

$$\mathcal{K}(S) := \{Q \in \mathcal{K}\ ;\ Q \subset S\};$$

this collection may be empty.

Lemma 2.14. *There is a cover of the open set* $S^c := \mathbb{R}^n\backslash S$ *by cubes of* $\mathcal{K}(S^c)$ *denoted hereafter by* $\mathcal{W}_S$ *such that*

(a) *interiors of distinct cubes do not intersect,*

(b) *for every* $Q \in \mathcal{W}_S$,

$$\frac{1}{5}\, r_Q \le \operatorname{dist}(Q, S) \le 5r_Q,$$

where the distance is measured in the ℓ_∞*-norm, i.e.,*

$$d(Q, S) := \inf\big\{\max_{1\le i\le n} |x_i - y_i|\ ;\ x \in Q,\ y \in S\big\}.$$

[3] i.e., $f_\alpha = D^\alpha F|_S$ for all $|\alpha| \le k$.

Let us explain how to find such a cover (cf. Lebesgue's decomposition in Figure 1.1 in Comments to Chapter 1). A cube $Q \in \mathcal{K}$ is said to be *dyadic*, if it has a form

$$Q = 2^{-j}(Q_0 + k)$$

for some $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$.

So, the dyadic cube has center $2^{-j}k$ and radius 2^{-j}.

Now we define $\mathcal{W}_S$ as follows. Cover S^c by dyadic cubes Q satisfying the condition

$$2Q \in \mathcal{K}(S^c) \quad \text{but} \quad 5Q \notin \mathcal{K}(S^c). \tag{2.29}$$

For every $x \in S^c$ take among these cubes the biggest containing x in its interior, say Q^x, and discard all smaller dyadic cubes contained in Q^x. Then take a point y from the open set $S^c \backslash Q^x$ and find for it the cube Q^y and so on. Since the interiors of two dyadic cubes either lie one in another or are mutually disjoint, this procedure yields a cover $\mathcal{W}_S$ of S^c satisfying condition (a) of the lemma. The second condition easily follows from (2.29).

The cover $\mathcal{W}_S$ has several additional properties which will be used in the sequel. We present them in the next statement where we use the notation

$$Q^* := \lambda Q \quad \text{with} \quad \lambda := \frac{9}{8}\,. \tag{2.30}$$

Because of the choice of λ and the estimate $\operatorname{dist}(Q, S) > \frac{1}{5}\,r_Q$, see Lemma 2.14, every Q^* is contained in the open set S^c, i.e.,

$$S^c = \cup\{Q^*\,;\, Q \in \mathcal{W}_S\}.$$

Corollary 2.15. *Let Q, K be cubes in $\mathcal{W}_S$. Then the following is true:*

(a) *$Q \cap K \neq \emptyset$ if and only if $Q^* \cap K^* \neq \emptyset$. Moreover, for some $c = c(n) > 0$ and every such pair Q^*, K^*, it is true that*

$$|Q^* \cap K^*| \geq c \min\{|Q^*|, |K^*|\}.$$

Here $|\cdot|$ stands for the Lebesgue measure in $\mathbb{R}^n$.

(b) *If $Q \cap K \neq \emptyset$, then*

$$\frac{1}{4}\,r_Q \leq r_K \leq 4r_Q.$$

(c) *The order (multiplicity) of cover $\mathcal{W}_S^* := \big\{Q^*\ ;\ Q \in \mathcal{W}_S\big\}$, i.e., the quantity*

$$\operatorname{ord} \mathcal{W}_S^* := \sup_{x \in S^c} \operatorname{card}\big\{Q^* \in \mathcal{W}_S^*\,;\, A^* \ni x\big\}$$

is bounded by a constant depending only on n.

Smooth partition of unity

Let φ be a C^∞ function supported by the cube $Q_0^* \,(= \frac{9}{8}\,[-1,1]^n)$ which equals 1 on Q_0. By scaling we define the function

$$\tilde{\varphi}_Q(x) := \varphi\left(\frac{x - c_Q}{r_Q}\right), \quad x \in \mathbb{R}^n, \quad Q \in \mathcal{K}.$$

Note that

$$\tilde{\varphi}_Q = 1 \ \text{ on } \ Q \ \text{ and } \ \operatorname{supp} \varphi_Q \subset Q^*.$$

Further, we define the C^∞ function φ_Q on S^c by

$$\varphi_Q := \tilde{\varphi}_Q \Big/ \sum_{Q \in \mathcal{W}_S} \tilde{\varphi}_Q$$

and extend it to $\mathbb{R}^n$ by $\varphi_Q := 0$ on S.

Since $\mathcal{W}_S$ is a locally finite cover of S^c, φ_Q is well defined. The collection $\{\varphi_Q \,;\, Q \in \mathcal{W}_S\}$ is the required *smooth partition of unity subordinate to the cover* $\mathcal{W}_S^*$.

From its definition and Corollary 2.15 we get

Lemma 2.16. (i) *For every $x \in S^c$,*

$$\sum_{Q \in \mathcal{W}_S} \varphi_Q(x) = 1;$$

(ii) *for each $\alpha \in \mathbb{Z}_+^n$,*

$$\sup_{\mathbb{R}^n} \left| D^\alpha \varphi_Q \right| \le C(\alpha, n) r_Q^{-|\alpha|}.$$

Linear extension operator

Let $\vec{f} = \{f_\alpha\}_{|\alpha| \le k}$ be a continuous vector function on a closed set $S \subset \mathbb{R}^n$. We introduce an analog of Taylor's polynomial for $\vec{f}$ by

$$T_x^k \vec{f}(y) := \sum_{|\alpha| \le k} f_\alpha(x) \, \frac{(y-x)^\alpha}{\alpha!} \, .$$

To define the required extension operator, we pick a point s_Q of S such that

$$d(s_Q, Q) = d(S, Q).$$

Now we define an extension operator E_k^S given for $\vec{f}$ by the formula

$$E_k^S \vec{f} := \begin{cases} \displaystyle\sum_{Q \in \mathcal{W}_S} \varphi_Q T_{s_Q}^k \vec{f} & \text{on } S^c, \\ f_0 & \text{on } S. \end{cases}$$

This is clearly a linear operator; moreover, $E_k^S \vec{f}$ is a C^∞ function on S^c. The basic two facts related to the extension operator are as follows.

Lemma 2.17. *There is a constant $\lambda = \lambda(n) \in (0, 1]$ such that for every cube Q with center at S,*

$$E_k^S \vec{f} = E_k^{S\cap Q} \vec{f} \quad \textit{on} \quad \lambda Q. \tag{2.31}$$

Further, let the reduced remainders r_α, $|\alpha| \le k$, see (2.27), satisfy the inequality

$$r_\alpha(\vec{f}; x, y) \le \omega(\|x - y\|), \quad x, y \in S, \tag{2.32}$$

for some continuous concave function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ which equals zero at zero (i.e., for $\omega \in \Omega_1$).

Lemma 2.18. *Under condition* (2.32) *the following is true:*

(a) *For every α with $|\alpha| \le k + 1$ and all points $x \in S$ and $y \in S^c$,*

$$\left|D_y^\alpha (E_k^S - T_x^k)\vec{f}(y)\right| \le C\omega(\|x - y\|) \tag{2.33}$$

for some $C = C(k, n)$;

(b) *for every cube Q with center at S there is a cube $\widetilde{Q} = \lambda Q$ with $0 < \lambda < 1$ such that*

$$\sum_{|\alpha|\le k} \sup_{Q\setminus S} \left|D^\alpha E_k^S \vec{f}\right| \le C \sum_{|\alpha|\le k} \sup_{\widetilde{Q}\cap S} |f_\alpha|; \tag{2.34}$$

here $\lambda = \lambda(n)$ and $C = C(S, k, n)$.

Outline of the proof for Whitney's Theorem

Let $\vec{f} = \{f_\alpha\}_{|\alpha|\le k}$ be a vector function on S satisfying the Taylor chain condition of Theorem 2.13. First, we must prove the existence and continuity of the derivatives for $E_k^S \vec{f}$ of order at most k on the whole of $\mathbb{R}^n$. It suffices to do this only for an open neighborhood of S. In turn, Lemma 2.17 allows us to work with $Q \cap S$ instead of S. Hence, in this part of the proof we can and will assume that S is compact.

Since every r_α is continuous on the compact set $S \times S$, $r_\alpha(x, y) \to 0$ as $\|x - y\| \to 0$, and this convergence is *uniform*. Therefore every r_α satisfies condition (2.32) with some function ω depending, clearly, on S and $\vec{f}$. Accordingly, inequality (2.33) holds for this ω. For $\alpha = 0$ this yields the continuity of $E_k^S \vec{f}$ on $\mathbb{R}^n$. Proceeding by induction on $|\alpha|$ we establish, with the help of (2.33), the existence and continuity of $D^\alpha E_k^S \vec{f}$ for all α with $|\alpha| \le k$. Hence, E_k^S is a linear extension operator acting from $J^k(\mathbb{R}^n)\big|_S$ to $C^k(\mathbb{R}^n)$. Continuity of the operator in the Fréchet topologies of these spaces follows from (2.34).

Whitney's theorem for other jet spaces

We present here variants of Theorem 2.13 for the spaces $\dot{J}_b^{k,\omega}(\mathbb{R}^n)$ and $J_b^{k,\omega}(\mathbb{R}^n)$. According to our notation, the former (with a dot!) consists of all vector functions $\vec{f} = \{f_\alpha\}_{|\alpha|\le k}$ that are continuous on $\mathbb{R}^n$ and such that the seminorm

$$|\vec{f}|_{k,\omega} := \sum_{|\alpha|=k} \sup_{x\neq y} \frac{|f_\alpha(x) - f_\alpha(y)|}{\omega(\|x-y\|)} < \infty.$$

In turn, $J_b^{k,\omega}(\mathbb{R}^n)$ is defined by the norm

$$\|\vec{f}\,\|_{k,\omega} := |\vec{f}\,|_{k,\omega} + \sum_{|\alpha|\le k} \sup_{\mathbb{R}^n} |f_\alpha|.$$

To justify the corresponding results, it is useful to notice that the Taylor chain condition for $f \in C_b^{k,\omega}(\mathbb{R}^n)$ is bounded as follows:

$$\left|D^\alpha(f - T_y^k f)(x)\right| \le C\|x-y\|^{k-|\alpha|}\omega\left(\|x-y\|\right)$$

where $C = C(k,n)$.

Hence the necessity condition for the reduced remainders $r_\alpha(\vec{f}\,)$, see (2.27), is

$$r_\alpha(\vec{f}\,;\ x,y) \le C\omega\left(\|x-y\|\right), \quad x,y \in S, \tag{2.35}$$

and this inequality coincides with (2.32). Therefore part (a) of Lemma 2.18 may be applied and this leads to the corresponding result for the homogeneous space $\dot{J}_b^{k,\omega}(\mathbb{R}^n)$. But we cannot estimate the uniform norms of the derivatives $D^\alpha E_k^S \vec{f}$ on $\mathbb{R}^n$ using part (b) of Lemma 2.18 (in fact, they may be unbounded). Therefore, we should modify the definition of the extension operator; namely, we set

$$\widehat{E}_k^S \vec{f} := \sum_{Q\in\widehat{\mathcal{W}}_S} \varphi_Q T_{s_Q}^k \vec{f} \quad \text{on} \quad S^c$$

and define $\widehat{E}_k^S \vec{f}$ to agree with f_0 on S. Here $\widehat{\mathcal{W}}_S$ is a part of the cover $\mathcal{W}_S$ containing only the cubes of radius at most 1. In other words, in the definition of $E_k^S \vec{f}$ we substitute $T_{s_Q}^k \vec{f}$ for zero, if $r_Q > 1$.

Estimate (2.33) remains to be true for this modification, but inequality (2.34) is replaced by the stronger inequality

$$\sum_{|\alpha|\le k} \sup_{S^c} \left|D^\alpha \widehat{E}_k^S \vec{f}\,\right| \le C(n,k) \sum_{|\alpha|\le k} \sup_S |f_\alpha|.$$

Using these facts one can derive the following result due to G. Glaser [Gl-1958].

Theorem 2.19. (a) *Let* $\vec{f} = \{f_\alpha\}_{|\alpha| \le k}$ *be a vector function defined on a closed set* $S \subset \mathbb{R}^n$. *Then* $\vec{f}$ *belongs to the trace space* $\dot{J}^{k,\omega}(\mathbb{R}^n)\big|_S$ *if and only if* $\vec{f}$ *satisfies* (2.32).

(b) $\vec{f}$ *belongs to the trace space* $J_b^{k,\omega}(\mathbb{R}^n)\big|_S$ *if and only if* $\vec{f}$ *is bounded on* S *and satisfies* (2.32).

In both cases, there exists a linear continuous extension operator.

Remark 2.20. For $k = 0$ and $\omega(t) := t,\ t > 0$, Theorem 2.19 yields the following:

For every closed $S \subset \mathbb{R}^n$, *there is a linear extension operator* $E : \mathrm{Lip}(\mathbb{R}^n)\big|_S \to \mathrm{Lip}(\mathbb{R}^n)$ *whose norm is bounded by a constant depending only on* n.

Since all Banach norms of $\mathbb{R}^n$ are equivalent, the same is true for $\mathrm{Lip}(X)$, where X is an n-dimensional Banach space.

It is worth noting that the extension operator $\widehat{E}_k^S$ is *universal* in the sense that it does not depend on the majorant ω. On the other hand, we have

$$\dot{J}_u^k(\mathbb{R}^n) = \bigcup_\omega \dot{J}_b^{k,\omega}(\mathbb{R}^n),$$

where ω runs over the set of 1-majorants. Let us recall that the space on the left-hand side consists of all vector functions $\vec{f} = \{f_\alpha\}_{|\alpha| \le k}$ generated by C^k functions whose higher derivatives are bounded and uniformly continuous on $\mathbb{R}^n$.

These facts lead to

Theorem 2.21. *A vector function* $\vec{f} = \{f_\alpha\}_{|\alpha| \le k}$ *defined and bounded on a closed set* $S \subset \mathbb{R}^n$ *belongs to the trace space* $\dot{J}_u^k(\mathbb{R}^n)\big|_S$ *if and only if the reduced remainders* $r_\alpha(\vec{f})$ *with* $|\alpha| = k$ *are uniformly continuous on* $S \times S$.

Moreover, $\widehat{E}_k^S$ *is a linear bounded extension operator from the trace space into* $\dot{C}_u^k(\mathbb{R}^n)$.

Whitney's theorem for C^∞ functions

Hestens [Hes-1941] modified the Whitney extension method to adapt it to C^∞ functions. In this case we deal with the space $J^\infty(\mathbb{R}^n)$ of vector functions $\vec{f} = \{f_\alpha\}_{|\alpha| < \infty}$ on $\mathbb{R}^n$ generated by C^∞ functions, i.e., for every such $\vec{f}$ there is a function $f \in C^\infty(\mathbb{R}^n)$ such that $f_\alpha = D^\alpha f$ for all α.

Given now $\vec{f} = \{f_\alpha\}_{|\alpha| < \infty}$ defined on a closed subset $S \subset \mathbb{R}^n$, we introduce as before the family $\{r_\alpha(\vec{f})\}_{|\alpha|,\infty}$ of reduced remainders. For this case the Taylor chain condition is:

For all $\alpha \in \mathbb{Z}_n^+$,

$$r_\alpha(\vec{f}\,;\ x, y) \to 0 \quad \text{as} \quad y \to x,\ y, x \in S. \tag{2.36}$$

Then the following is true.

Theorem 2.22. *A vector function $\vec{f} = \{f_\alpha\}_{|\alpha|<\infty}$ defined on a closed set $S \subset \mathbb{R}^n$ belongs to the trace space $J^\infty(\mathbb{R}^n)\big|_S$ if an only if $\vec{f}$ satisfies the Taylor chain condition* (2.36).

To prove sufficiency of this condition, Hestens used the following *nonlinear* extension operator.

Given $\vec{f} = \{f_\alpha\}_{|\alpha|<\infty}$ defined on S and a cube Q from Whitney's cover $\mathcal{W}_S$, a nondecreasing sequence of numbers $\{N_j\}_{j=1}^\infty$ is chosen so that

$$\Big|\sum_{|\alpha|=j} \frac{f_\alpha(x)}{\alpha!}\, y^\alpha\Big| < N_j$$

for every y with $\|y\|_\infty = 1$ and every x in $Q_j(s_Q) \cap S$. Recall that s_Q is a point of S nearest to Q.

Selecting constants $\delta_j > 0$ such that $\delta_j < \frac{1}{N_j}$ and $\delta_{j+1} < \delta_j,\ j = 1, 2, \dots$, we set

$$k_Q := \begin{cases} j, & \text{if } \delta_{j+1} \le d(Q,S) < \delta_j, \\ 0, & \text{if } d(Q,S) \ge \delta_j. \end{cases}$$

Finally, we define an extension operator E_∞^S on $\vec{f} = \{f_\alpha\}_{|\alpha|<\infty}$ by

$$E_\infty^S \vec{f} := \begin{cases} \displaystyle\sum_{Q\in\mathcal{W}_S} \varphi_Q\, T_{s_Q}^{k_Q} \vec{f} & \text{on } S^c, \\ f_0 & \text{on } S. \end{cases}$$

Using the estimates of Lemma 2.18 one then proves that $E_\infty^S \vec{f}$ belongs to $C^\infty(\mathbb{R}^n)$.

The Whitney–Hestens Theorem 2.22 is a far reaching generalization of the classical E. Borel Theorem [EBo-1895] concerning univariate functions and one-point sets $S = \{x_0\} \subset \mathbb{R}$. In this case, $J^\infty(\mathbb{R})\big|_S$ is the linear space of all sequences $\{c_i\}_{i\in\mathbb{Z}_+} \subset \mathbb{R}$, and the Taylor chain condition clearly holds. Therefore Theorem 2.22 states that, for every sequence $\{c_i\}_{i\in\mathbb{Z}_+} \subset \mathbb{R}$, there is a C^∞ function f on $\mathbb{R}$ such that

$$f^{(i)}(x_0) = c_i \quad \text{for every} \quad i \in \mathbb{Z}_+,$$

(Borel's theorem).

Remark 2.23. In fact, a Borel type result is true for every *countable* subset $S \subset \mathbb{R}^n$. Specifically, let $\{s_\alpha\,; \alpha \in \mathbb{Z}_+^n\}$ and $S := \{x_\alpha\,; \alpha \in \mathbb{Z}_+^N\}$ be arbitrary families of real numbers and points of $\mathbb{R}^n$, respectively. Then there is a function $f \in C^\infty(\mathbb{R}^n)$ such that for every $\alpha \in \mathbb{Z}_+^n$,

$$D^\alpha f(x_\alpha) = s_\alpha.$$

The result is a direct consequence of the classical Eidelheit theorem [Ei-1936] on the solvability of infinite linear systems on Fréchet spaces.

The Borel theorem gives a remarkably simple example of a closed subset $S \subset \mathbb{R}$ (a singleton $\{x_0\}$) for which there is no *linear* continuous extension operator from $J^\infty(\mathbb{R}^n)\big|_S$ to $J^\infty(\mathbb{R}^n)$. The result was due to Mitiagin [Mit-1961]; its proof presented now follows the argument from the paper [LyT-1969] by Yu. Lyubich and Tkachenko.

Theorem 2.24. *There is no linear continuous extension operator from $J^\infty(\mathbb{R}^n)\big|_{\{x_0\}}$ to $J^\infty(\mathbb{R}^n)$.*

Proof. Assume, on the contrary, that such an operator, say E, exists. Let $\{e_\alpha\}_{\alpha\in\mathbb{Z}_+^n}$ be the standard basis of the Fréchet space of all families $\{s_\alpha\}_{\alpha\in\mathbb{Z}_+^n}$, i.e., the coordinate $(e_\alpha)_\beta = 1$ if $\beta = \alpha$ and $(e_\alpha)_\beta = 0$ if $\alpha \neq \beta$. By the continuity of E,

$$E\big(\{s_\alpha\}\big) = \sum_\alpha s_\alpha E e_\alpha,$$

where the series converges in $C^\infty(\mathbb{R}^n)$ (identified with $J^\infty(\mathbb{R}^n)$) and therefore uniformly converges on $Q_0 := [-1,1]^n$. This implies that

$$\sup_\alpha |s_\alpha|\, \|Ee_\alpha\|_{C(Q_0)} < \infty.$$

Since this is true for an arbitrary family $\{s_\alpha\}$, there is only a *finite* number of α for which $Ee_\alpha \neq 0$. Hence, the image of E is a finite-dimensional subspace of $J^\infty(\mathbb{R}^n)$. On the other hand, the composition $R \circ E$, where $Rf := f\big|_{\{x_0\}}$ is the identity map, a contradiction. □

2.3 Divided differences, local approximation and differentiability

Turning to the trace and extension problems for C^k functions, we immediately notice their essential difference from those for k-jets. In the latter case, the following information is available:

(i) the traces to S of all the derivatives of the function up to order k;

(ii) a collection of inequalities relating the values of these traces for every *two-point* subset of S.

However, in the case of C^k functions, information is much more restricted: we know only the trace of a function to S. The deficiency in data should therefore be completed by a more expanded set of relations between the values of the trace. In Section 2.4, we prove the aforementioned Whitney theorem describing the complete set of these relations for univariate C^k functions. Every such relation includes at most $k+2$ distinct values and this number cannot be diminished. In Chapter 10, we present several results of this kind concerning different smoothness spaces in n variables; the number of values in all these cases is bounded by a constant depending only on smoothness order and n. Apparently, there exists a deep reason for this phenomenon called below the *Finiteness Property*, but for the time being we have only some heuristic argument for its explanation.

It is natural to begin the study with the simplest case of C^k functions defined on the whole of $\mathbb{R}^n$. The results presented below will provide several characteristics of C^k functions based on their behavior on subsets of cardinality $k+1$ and on that of their *local polynomial approximation*. The first group of results describes relations between k-differentiability of multivariate functions and behavior of its difference characteristics. For their formulations we need a few notions and facts concerning such characteristics.

Divided and mixed differences

Let $f : \mathbb{R} \to \mathbb{R}$ and let S be a finite subset of $\mathbb{R}$. The *divided difference of* f *on* S (denoted by $f[S]$) is described by induction on card S starting from $S := \{x_0\}$ as follows. Set

$$f\big[\{x_0\}\big] := f(x_0);$$

if $f[S]$ is now defined on all k-point subsets, then, for $S := \{x_0, \dots, x_k\}$, set

$$f[S] := \frac{f\big[\{x_0, \dots, x_{k-1}\}\big] - f\big[\{x_1, \dots, x_k\}\big]}{x_0 - x_k}\,.$$

Proposition 2.25. *Let* $S \subset \mathbb{R}$ *and* $\operatorname{card} S = k+1$. *Set* $\omega_S(t) := \prod_{s\in S}(t-s)$. *Then the following is true:*

(a)
$$f[S] = \sum_{s\in S} \frac{f(s)}{\omega_S'(s)}\,.$$

In particular, $f[S]$ *is a symmetric function in the arguments* $s \in S$.

(b) *Let* $L_S(f)$ *be the Lagrange interpolation polynomial for* f *with the nodes* $s \in S$. *Then*
$$L_S^{(k)}(f) = f[S].$$

(c) *If* f *belongs to* C^k *in some open interval* I *containing* S, *then*

$$f[S] = f^{(k)}(c)$$

for a point c *from* $(\operatorname{conv} S)^0$.

These classical results are well known, see, e.g., the book [deB-2001] by de Boor.

Now let the points of $S := \{x_0, \dots, x_k\}$ be equally spaced, i.e., $x_{j+1} - x_j = h$ for all $0 \le j < k$ and some $h \neq 0$. Then it is easy to see that

$$f[S] = \frac{k!}{h^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x_0 + jh). \tag{2.37}$$

Let us recall that for a multivariate function $f : \mathbb{R}^n \to \mathbb{R}$ and a vector $h \in \mathbb{R}^n$ the k-th-difference of f is given by the similar formula:

$$\Delta_h^k f(x) := \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh). \tag{2.38}$$

A variant of this notion, the k-th *difference of f in a direction* $e \in \mathbb{S}^{n-1}$, is given by

$$\Delta^k(te)f := \Delta_{te}^k f, \quad t \in \mathbb{R}.$$

Finally, we introduce *(mixed) α-difference* Δ_h^α by

$$\Delta_h^\alpha := \prod_{i=1}^{n} \Delta^{\alpha_i}(h_i e_i); \tag{2.39}$$

here $h = (h_1, \dots, h_n) \in \mathbb{R}^n$ and $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$.

Our first result goes back to Brouwer [Bro-1908] who proved it for $n = 1$.

Theorem 2.26. *A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to $C^k(\mathbb{R}^n)$ if and only if for every $e \in \mathbb{S}^{n-1}$ and every cube Q the function $t^{-k}\Delta^k(te)f$ converges uniformly on Q as t tends to 0 through an arbitrary sequence $\{t_j\}$.*

Proof. (Necessity) If $f \in C^k(\mathbb{R}^n)$, then by Proposition 2.25 (c) and (2.37) we have

$$t^{-k}\Delta^k(te)f(x) = D_e^k f(c), \tag{2.40}$$

here c is a point in the interval $(x, x + kte)$ and the directional derivative D_e^k is defined by the iteration of D_e where

$$D_e f(x) := \lim_{t \to 0} t^{-1}\big(f(x + te) - f(x)\big).$$

Therefore $D_e = \sum_{i=1}^n e_i D_i$ and

$$t^{-k}\Delta^k(te)f(x) = \sum_{|\alpha|=k} e^\alpha D^\alpha f(c).$$

Since $f \in C^k(\mathbb{R}^n)$, the right-hand side is continuous in c (and therefore uniformly continuous) on every closed cube of $\mathbb{R}^n$. Hence, the left-hand side of (2.40) tends to $D_e^k f(x)$ uniformly in x as $t \to 0$.

(Sufficiency) We begin with

Proposition 2.27. *The assertion of Theorem* 2.26 *is true for* $n = 1$.

Proof. By the assumption of Theorem 2.26 for every closed interval $I := [a, b] \subset \mathbb{R}$ there is a function $\varphi_I \in C(I)$ such that

$$\max_I \left| t_j^{-k} \Delta_{t_j}^k f - \varphi_I \right| \to 0 \quad \text{as} \quad j \to \infty.$$

Let ψ be a C^∞ function compactly supported in $I^0 = (a, b)$. Then for a sufficiently small t,

$$\int_I (\Delta_t^k f)\psi dx = \int_I f\left(\Delta_{-t}^k \psi\right) dx.$$

Dividing by t^{-k} and passing to the limit as $t = t_j \to 0$, one then obtains

$$\int_I \varphi_I \cdot \psi dx = (-1)^k \int_I f \psi^{(k)} dx.$$

Finally, integrating by parts k-times we get

$$\int_I \left(f - \int_a^x \frac{(x-t)^{k-1}}{(k-1)!} \varphi_I(t) dt \right) \psi^{(k)} dx = 0.$$

Since this is true for an arbitrary ψ, the classical Dubois–Reymond lemma implies that the term in brackets is a polynomial of degree $k - 1$. Hence,

$$f(x) = \sum_{j=0}^{k-1} a_j x^j + \int_a^x \frac{(x-t)^{k-1}}{(k-1)!} \varphi_I(t) dt \in C^k(\mathbb{R}). \tag{2.41}$$

$\square$

Applying Proposition 2.27 to the restriction of $f \in C(\mathbb{R}^n)$ to a straight line in a direction $e \in \mathbb{S}^n$, we immediately get

Corollary 2.28. *Under the condition of Theorem* 2.26 f *has* k*-th continuous directional derivatives for any* $e \in \mathbb{S}^n$.

We derive from here the existence and continuity of all mixed derivatives for f up to order k. For this aim we need an algebraic identity concerning the difference operators, see Appendix E to this chapter for the proof.

In this identity, τ_v, $v \in \mathbb{R}^n$, denotes the shift operator given by $\tau_v f := f(\cdot + v)$, where f is a function on $\mathbb{R}^n$. We also set $\Delta(v) := \tau_v - 1$ for the difference in the direction v.

Let V be a collection of N vectors in $\mathbb{R}^n$ and $\varphi : V \to \left\{1, \frac{1}{2}, \dots, \frac{1}{N}\right\}$ be a bijection. Then

$$\prod_{v \in V} \Delta(v) = \sum_{\omega \subset V} (-1)^{\operatorname{card} \omega} \tau_{w_\omega} \Delta^N(v_\omega), \tag{2.42}$$

where the sum is taken over all nonempty subsets of V and

$$v_\omega := \sum_{v\in\omega} \varphi(v)v, \quad w_\omega := \sum_{v\notin\omega} v.$$

By standard convention, $w_V = 0$.

Let V_α, $|\alpha| = k$, be determined by

$$\prod_{v\in V_\alpha} \Delta(v) = \Delta_h^\alpha \left(:= \prod_{j=1}^n \Delta^{\alpha_j}(h_j e_j) \right).$$

Then (2.42) yields

$$\Delta_h^\alpha f(x) = \sum_{\omega\subset V_\alpha} (-1)^{\operatorname{card}\omega}\, \Delta_{v_\omega}^k f(x + w_\omega). \tag{2.43}$$

If here $h_j = t$ for $1 \le j \le n$ and some $t > 0$, then

$$v_\omega = t\hat{v}_\omega \quad \text{and} \quad w_\omega = t\widehat{w}_\omega,$$

where the vectors with the hat are independent of t. Therefore the condition of Theorem 2.26 and (2.43) imply that $|h|^{-k}\Delta_h^\alpha f$ converges uniformly on compact subsets to a continuous on $\mathbb{R}^n$ function as the vector $h := t\sum\limits_{j=1}^n e_j$ tends to zero. Hence we establish

Corollary 2.29. *Under the condition of Theorem* 2.26, $\|h\|^{-k}\Delta_h^\alpha f$ *tends uniformly on every cube* Q *to a continuous function* φ_Q *as* h *tends to zero along the ray* $\left\{ t\sum\limits_{i=1}^n e_i \,;\, t > 0 \right\}$.

Now we apply the argument of Proposition 2.27 to each variable x_i to get the following

Proposition 2.30. *If the assertion of Corollary* 2.29 *holds for a continuous function* f *and a cube* $Q := \prod\limits_{j=1}^n [a_j, b_j]$, *then for every* $x \in Q$,

$$f(x) = p_Q^\alpha(x) + \left(\prod_{j=1}^n I_j^{\alpha_j} \right)(\varphi_Q \;;\; x),$$

where the operator I_j *denotes integration with respect to the* j*-th variable from* a_j *to* x_j *and* p_Q^α *is a continuous function on* Q *given by*

$$p_Q^\alpha(x) = \sum_{j=1}^n \sum_{s=0}^{\alpha_j - 1} \varphi_{sj}(x) x_j^s \tag{2.44}$$

with φ_{sj} *being a continuous function independent of* x_j.

Now we can complete the proof of Theorem 2.26. By Corollary 2.28 f has first continuous derivatives. Proceeding by induction, assume that f has continuous mixed derivatives of order s with $1 \le s < k$ and prove the same for $s+1$. Let $|\alpha| = s+1$; then

$$\alpha = \beta + e_i \quad \text{for some} \quad |\beta| = s \;\text{ and }\; 1 \le i \le n. \tag{2.45}$$

Applying Proposition 2.30 for this α to have, for a cube Q,

$$f = p_Q^\alpha + \hat{f}_Q,$$

where p_Q^α is given by (2.44). Hence, $D^\gamma \hat{f}_Q$ exists and is continuous on Q for every multi-index γ with $\gamma_i \le \alpha_i,\ 1 \le i \le n$. Moreover, by the induction hypothesis, $D^\beta f$ exists and is continuous; hence, the same is true for $D^\beta p_Q^\alpha$. Now write

$$\Delta(te_i)D^\beta f = \Delta(te_i)D^\beta p_Q^\alpha + \Delta(te_i)D^\beta \hat{f}_Q.$$

Since $D^\beta p_Q^\alpha = s!\varphi_{si}$, i.e., is independent of x_i, the first term in the right-hand side is zero. Dividing then by $t \to 0$, we conclude that $\lim\limits_{t\to 0} t^{-1}\Delta(te_i)D^\beta f(x)$ exists and is equal on Q to the continuous function $D^\alpha \hat{f}_Q$, see (2.45) and (2.44).

Hence, $D^\alpha f$ exists and is continuous for every α with $|\alpha| = s+1$. □

The theorem proved leads to a similar difference characteristic of the space $\dot{C}^{k,\omega}(\mathbb{R}^n)$; see (2.7) for its definition.

Theorem 2.31. *A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to $\dot{C}^{k,\omega}(\mathbb{R}^n)$ if and only if for some constant $C > 0$ and all sufficiently small $h, g \in \mathbb{R}^N$, the inequality*

$$\|h\|^{-k}\left|\Delta_g \Delta_h^k f(x)\right| \le C\omega(\|g\|) \tag{2.46}$$

holds for all $x \in \mathbb{R}^n$. Moreover, the equivalence[4]

$$\inf C \approx |f|_{C^{k,\omega}(\mathbb{R}^n)} \tag{2.47}$$

holds (with constants independent of f).

Proof. By (2.46), the ratio $|h|^{-k}\Delta_h^k f$ converges uniformly as $h \to 0$ (recall that $\omega(0+) = 0$). The application of Theorem 2.26 then implies that $f \in C^k(\mathbb{R}^n)$. Passing to the limit in (2.46) we therefore obtain

$$\left|\Delta_g D_e^k f(x)\right| \le C\omega(\|g\|),$$

where $e \in \mathbb{S}^{n-1}$ and $x \in \mathbb{R}^n$. This and identity (2.43) yield

$$\left|\Delta_g D^\alpha f(x)\right| \le aC\omega(\|g\|), \quad |\alpha| = k,$$

[4] Hereafter we write $F \approx G$, if these functions satisfy a two-sided inequality $C_1 F \le G \le C_2 F$ with constants $C_1, C_2 > 0$ independent of the arguments of F, G.

for some constant $a = a(n,k)$, whence $|f|_{C^{k,\omega}(\mathbb{R}^n)} \le b\,C$ with $b = b(k,n)$.

In the opposite direction, it follows from Proposition 2.25 (c) and (2.38) that

$$\big|\Delta_g \Delta_h^k f(x)\big| \le \|h\|^k \,\big|\Delta_g D_e^k f(y)\big|,$$

provided that $f \in \dot{C}^{k,\omega}(\mathbb{R}^n)$; here $y \in (x, x+kh)$ and $e := \frac{h}{\|h\|}$.

Estimating the right-hand side as in Theorem 2.26 we then have

$$\|h\|^k \sum_{|\alpha|=k} \big|e^\alpha \Delta_g D^\alpha f(g)\big| \le C(n)|f|_{C^{k,\omega}(\mathbb{R}^n)}\|h\|^k \omega(\|g\|).$$

This completes the proof. □

For the case of $\omega(t) = t^\lambda,\ t \in \mathbb{R}_+$, with $0 < \lambda \le 1$ the criterion of Theorem 2.31 can be further simplified. In the consequent formulation, we use the notation

$$\dot{C}^{k,\omega} := \dot{C}^{k,\lambda} \quad \text{for} \quad \omega(t) := t^\lambda,\ t \in \mathbb{R}_+. \tag{2.48}$$

Theorem 2.32. *A continuous function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to the space $\dot{C}^{k,\lambda}(\mathbb{R}^n)$ if and only if*

$$\mathcal{M}(f) := \sup_h \|h\|^{-k-\lambda}\big\|\Delta_h^{k+1} f\big\|_{C(\mathbb{R}^n)} < \infty.$$

Moreover, $\mathcal{M}(f) \approx |f|_{C^{k,\lambda}(\mathbb{R}^n)}$.

The result is a consequence of Theorem 2.10 (a). The case of $\lambda = 1$ is true under the weaker assumption:

$$m(f) := \varliminf_{h\to 0} \|h\|^{-k-1}\big\|\Delta_h^{k+1} f\big\|_{C(\mathbb{R}^n)} < \infty. \tag{2.49}$$

We outline the proof of this result. Necessity may be proved in the same fashion as in Theorem 2.26 and leads to the inequality

$$\|h\|^{-k-1}\big\|\Delta_h^{k+1} f\big\|_{C(\mathbb{R}^n)} \le a(n,k)\|h\|^{-1} \sum_{|\alpha|=k} \big\|\Delta_h D^\alpha f\big\|_{C(\mathbb{R}^n)}.$$

This implies that

$$m(f) \le a(n,k)|f|_{C^{k,1}(\mathbb{R}^n)}.$$

The converse may be derived from the corresponding univariate case using the argument of Theorem 2.26 based on identity (2.42). Therefore it remains only to prove the following one-dimensional result:

If $f \in C(\mathbb{R})$ and $m(f) < \infty$, then $f \in C^{k,1}(\mathbb{R})$ and

$$|f|_{C^{k,1}(\mathbb{R})} \le a m(f)$$

with some constant a independent of f. As in the proof of Proposition 2.27, we pick a C^∞ function ψ compactly supported on $I = (a, b) \subset \mathbb{R}$. Then for a sufficiently small $t > 0$, we have

$$\int_I \left(t^{-k-1}\Delta_t^{k+1} f\right)\psi dx = \int_I f\left(t^{-k-1}\Delta_{-t}^{k+1}\psi\right)dx. \tag{2.50}$$

If $t \to 0$, the right-hand side becomes $(-1)^{k+1}\int_I f\psi^{(k+1)}dx$. On the other hand, for an appropriate sequence $t_j \to 0$, the sequence of functions $f_j := t_j^{-k-1}\Delta_{t_j}^{k+1} f$, $j = 1, 2, \dots$, is bounded in the L_∞-norm by (2.49). Since $L_\infty(I)$ is conjugate to $L_1(I)$, there exists a subsequence of $\{f_j\}$ converging in the weak* topology of $L_\infty(I)$ to some function φ_I, see, e.g., Dunford and Schwartz [DS-1958]. Passing to the limit in (2.50) as $t_j \to \infty$, we get

$$\int_I \varphi_I \psi dx = (-1)^{k+1}\int_I f\psi^{(k+1)}dx. \tag{2.51}$$

Moreover, by semicontinuity of the norm with respect to the weak* convergence,

$$\|\varphi_I\|_{L_\infty(I)} \le \varliminf_{j\to\infty} t_j^{-k-1}\left\|\Delta_t^{k+1}\right\|_{C(\mathbb{R})} =: m(f).$$

Integrating (2.51) by parts and using the Dubois–Reymond lemma we then have

$$f(x) = \sum_{j=0}^{k} a_j x^j + \int_a^x \frac{(x-t)^k}{k!}\varphi_I(t)dt.$$

This immediately implies that

$$\left|f^{(k)}(x+h) - f^{(k)}(x)\right| \le \int_x^{x+h} |\varphi_I(t)|\,dt \le |h| m(f).$$

Hence, $f \in \dot{C}^{k,1}(\mathbb{R})$ and its seminorm in this space is bounded by $m(f)$.

Remark 2.33. (a) This proof also yields the following useful fact:

$$\dot{C}^{k,1}(\mathbb{R}^n) = \dot{W}_\infty^{k+1}(\mathbb{R}^n) \tag{2.52}$$

with equivalence of the seminorms.

(b) The analog of Theorem 2.32 with a similar proof based on duality, holds for the homogeneous Sobolev space $\dot{W}_p^{k+1}(\mathbb{R}^n)$ with $1 < p < \infty$.

Let us recall that the (homogeneous) Sobolev space $\dot{W}_p^{k+1}(\mathbb{R}^n), 1 \le p \le \infty$, is defined by the seminorm

$$|f|_{W_p^{k+1}(\mathbb{R}^n)} := \sum_{|\alpha|=k+1} \|D^\alpha f\|_{L_p(\mathbb{R}^n)}.$$

Here $D^\alpha f$ are the distributional derivatives of $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ which are assumed to belong to $L_p(\mathbb{R}^n)$.

Local polynomial approximation and derivatives

The Whitney extension theorem relates the differentiable characteristics of a function to its local approximation by Taylor polynomials. Now we present other results of this kind which concern local Taylor approximation and also best approximation. The approximation of the latter type plays an essential role in the proofs of the extension theorems for Lipschitz functions of higher order, see Section 2.5 and Chapters 9 and 10. These proofs are fairly complicated and therefore it would be natural to consider this approach in the relatively simple situation of Theorem 2.38 below.

We begin with approximation by Taylor polynomials.

Theorem 2.34. *Let $\{T_y\}_{y\in\mathbb{R}^n}$ be a family of polynomials in $x \in \mathbb{R}^n$ of degree k, and f be a continuous function on $\mathbb{R}^n$. Assume that*

$$\big|f(x) - T_y(x)\big|/\|x-y\|^k \to 0 \quad \textit{as} \quad y \to x \tag{2.53}$$

uniformly on every closed cube. Then f belongs to $C^k(\mathbb{R}^n)$.

Proof. Write

$$T_y(x) = \sum_{|\alpha|\le k} f_\alpha(y)\,\frac{(x-y)^\alpha}{\alpha!}$$

and consider the k-jet

$$\vec{f} := \{f_\alpha\}_{|\alpha|\le k}\,.$$

By (2.53), $f_0 = f$ and we must show that $f_0 \in C^k(\mathbb{R}^n)$. We derive this from Whitney's Theorem 2.13 by checking that $\vec{f}$ satisfies the Taylor chain condition on every closed cube Q, see (2.27). Then Whitney's extension theorem implies that $f_0|_Q$ is the trace of a $C^k(\mathbb{R}^n)$ function and the result will follow.

To establish the required property of $\vec{f}$ we need

Lemma 2.35 (Markov's type inequality). *Let P be a polynomial on $\mathbb{R}^n$ of degree k. Then for every closed cube C of side length r and $|\alpha| \le k$, the inequality*

$$\max_C |D^\alpha P| \le \mu r^{-|\alpha|} \max_C |P| \tag{2.54}$$

holds with a constant μ depending only on k and n.

Proof. Since the space $\mathcal{P}_{k,n}$ of polynomials on $\mathbb{R}^n$ of degree k is affine-invariant, one can use scaling to replace C in (2.54) by the unit cube $Q_0 := [0,1]^n$. Then we should prove that

$$\max_{Q_0} |D^\alpha P| \le \mu(k,n) \max_{Q_0} |P|. \tag{2.55}$$

But every linear operator acting in a finite-dimensional Banach space is bounded. Applying this to the space $\mathcal{P}_{k,n}\big|_{Q_0}$ equipped with the uniform norm and to the operator D^α, we immediately get (2.55). □

We now introduce the polynomial in z,

$$P(z) := T_x(z) - T_y(z) \quad \text{with} \quad x, y \in Q$$

and apply Markov's inequality to P and the cube $C := \overline{Q}_{|x-y|}(x)$. Then we get

$$|D_z^\alpha P|\,\Big|_{z=x} \le \mu(k,n)\|x-y\|^{-|\alpha|} \max_C |P|.$$

The left-hand side equals

$$R_\alpha(\vec{f}\,;\ x, y) := f_\alpha(x) - \sum_{|\beta|\le k-|\alpha|} f_{\alpha+\beta}(y)\,\frac{(x-y)^\beta}{\beta!}\,.$$

On the other hand, by the definition of the cube C and (2.53),

$$\max_C |P| \le \max_C \big|f(z) - T_x(z)\big| + \max_C \big|f(z) - T_y(z)\big| \le \varepsilon_Q(|x-y|)|x-y|^k,$$

where $\varepsilon_Q(t) \to 0$ as $t \to 0^+$. Combining this with the previous two relations we obtain

$$\left|f_\alpha(x) - \sum_{|\beta|\le k-|\alpha|} f_{\alpha+\beta}(y)\,\frac{(x-y)^\beta}{\beta!}\right| \le \mu(k,n)\varepsilon_Q(\|x-y\|)\|x-y\|^{k-|\alpha|}$$

for all $x, y \in Q$ and $|\alpha| \le k$.

Hence, $\vec{f}$ satisfies the Taylor chain condition on Q. □

Another characterization of C^k functions is based on the notion of *local polynomial (best) approximation.* In its definition, $\mathcal{B}(\mathbb{R}^n)$ stands for the family of *bounded* subsets in $\mathbb{R}^n$.

Definition 2.36. The local polynomial (best) approximation of order k is a function $E_k : \ell_\infty^{\mathrm{loc}}(\mathbb{R}^n) \times \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}_+$ given by

$$E_k(S\,; f) := \inf\big\{\|f - P\|_{\ell_\infty(S)}\,;\, P \in \mathcal{P}_{k-1,n}\big\}. \tag{2.56}$$

Notice that the order k differs by 1 from the corresponding degree of the approximating polynomials.

We will discuss the basic properties of this set-function later; for now we only need the next profound fact whose univariate case was due to Whitney [Wh-1957] and the multivariate one was proved by Yu. Brudnyi [Br-1970a]. For the formulation of the latter we define the *k-oscillation* of $f : \mathbb{R}^n \to \mathbb{R}$ on a set $S \subset \mathbb{R}^n$ by

$$\omega_k(S\,; f) := \sup\big\{|\Delta_h^k f(x)|\ ;\ x + jh \in S, \quad j = 0, 1, \ldots, k\big\}. \tag{2.57}$$

The set $\bigcap_{j=0}^{k}(S+jh)$ may be empty for every $h \neq 0$; in this case the k-oscillation is assumed to be infinity. However, for S of positive Lebesgue measure this intersection is nonempty for a set of h of positive measure, a consequence of the general Ruziewicz theorem [Ruz-1925].

Theorem 2.37. *Let f be a locally bounded function on a convex subset C of $\mathbb{R}^n$ (which may be unbounded). Then there is a constant $w_k(C)$ such that*

$$E_k(C\,;f) \le w_k(C)\omega_k(C\,;f). \tag{2.58}$$

Moreover, the number

$$w_{k,n} := \sup_C w_k(C)$$

is finite.

We discuss this result in Appendix F to this chapter, while for now we apply it to the following characterization of the space $\dot{C}^{k,\lambda}(\mathbb{R}^n),\ 0 < \lambda \le 1$.

Theorem 2.38. *A locally bounded function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to $\dot{C}^{k,\lambda}(\mathbb{R}^n)$ if and only if*

$$M_{k,\lambda}(f) = \sup_Q \frac{E_{k+1}(Q\,;f)}{|Q|^{\frac{k+\lambda}{n}}}$$

is finite; here the supremum is taken over all closed cubes of $\mathbb{R}^n$. Moreover, the equivalence

$$M_{k,\lambda}(f) \approx |f|_{C^{k,\lambda}(\mathbb{R}^n)}$$

holds with constants depending only on k and n.

Proof. (Necessity) Let $f \in \dot{C}^{k,\lambda}(\mathbb{R}^n)$. By virtue of (2.38) and Proposition 2.25 (c),

$$\omega_{k+1}(Q\,;f) \le \sup\bigl\{|\Delta_h D_e^k f(x)| \cdot \|h\|^k \ ;\ e \in \mathbb{S}^n, \quad x, x+h \in Q\bigr\}.$$

As in the proof of necessity in Theorem 2.26, we then bound this upper estimate by

$$\begin{aligned} &\sum_{|\alpha|=k} \sup\bigl\{|\Delta_h D^\alpha f(x)| \cdot \|h\|^k \ ;\ x, x+h \in Q\bigr\} \\ &\le |f|_{C^{k,\lambda}(\mathbb{R}^n)} \sup\bigl\{\|h\|^{k+\lambda} \ ;\ x, x+h \in Q\bigr\} \\ &\le \bigl(\sqrt{n}|Q|^{\frac{1}{n}}\bigr)^{k+\lambda} |f|_{C^{k,\lambda}(\mathbb{R}^n)}. \end{aligned}$$

Applying now (2.58), we get the required inequality

$$E_{k+1}(Q\,;f) \le c(k,n)|Q|^{\frac{k+\lambda}{n}}\ |f|_{C^{k,\lambda}(\mathbb{R}^n)}.$$

(Sufficiency) Suppose that for every cube Q,

$$E_{k+1}(Q\,;f) \le M|Q|^{\frac{k+\lambda}{n}}. \tag{2.59}$$

Since $\Delta_h^{k+1} P = 0$ for $P \in \mathcal{P}_{k,n}$ and, moreover,

$$|\Delta_h^{k+1} f(x)| \le 2^{k+1} \max_{0 \le j \le k} |f(x + jh)|,$$

(see (2.38)), we therefore have

$$|\Delta_h^{k+1} f(x)| = \left|\Delta_h^{k+1} (f - P)(x)\right| \le 2^{k+1} \|f - P\|_{C(Q)},$$

where P is a polynomial from $\mathcal{P}_{k,n}$ and Q denotes the cube of minimal volume containing x and $x + kh$. Taking here the infimum over all polynomials P and using (2.59) we then obtain

$$|\Delta_h^{k+1} f(x)| \le cM \|h\|^{k+\lambda}$$

for all $x, h \in \mathbb{R}^n$ and some constant c depending only on k and n.

Hence, the assumption of Theorem 2.32 holds for f and therefore $f \in \dot{C}^{k,\lambda}(\mathbb{R}^n)$ and $|f|_{C^{k,\lambda}(\mathbb{R}^n)} \le cM$ with some constant $c = c(k, n)$. This and the first part of the proof also give the required equivalence

$$M_{k,\lambda}(f) \approx |f|_{C^{k,\lambda}(\mathbb{R}^n)}.$$

The proof is complete. $\square$

Remark 2.39. Using Theorem 2.37 and the inequality

$$2^{-k} \sup\left\{\left|\Delta_h^k f(x)\right| \; ; \; x, x + kh \in Q\right\} \le E_k(f \; ; \; Q),$$

(see the argument of the proof of (2.59)), we obtain in the very same way the equivalence

$$|f|_{\Lambda^{k,\omega}(\mathbb{R}^n)} \approx \sup_Q \frac{E_k(Q\,; f)}{\omega(|Q|^{\frac{1}{n}})}\,, \tag{2.60}$$

where the constants of equivalence are independent of f.

A characterization of C^k functions via local polynomial approximation has been proved only for the univariate case (S. Bernstein [Ber-1940]). The result is hardly known and its proof is only outlined in Bernstein's note. For this reason we present a complete proof of this remarkable theorem. We refer the reader to the books [Tim-1963] by Timan and [DeVL-1993] by DeVore and Lorentz for the results of classical Approximation Theory which will be used (but not proved) in the proof.

Theorem 2.40. *A continuous function $f : [-1, 1] \to \mathbb{R}$ is k-times continuously differentiable in $[-1, 1]$ if and only if the limit*

$$\lambda(f\,; x) := \lim_{Q \to x} \frac{E_k(Q\,; f)}{|Q|^k} \tag{2.61}$$

exists and converges uniformly in $[-1,1]$ [5]. *Moreover, in this case, for every* x

$$\lambda(f\,;\,x) = \frac{|f^{(k)}(x)|}{2^{2k-1}\cdot k!}\,. \tag{2.62}$$

Proof. (Necessity) Given $f \in C^k[-1,1]$ and $\{x\},\ Q \subset [-1,1]$, we write for $y \in Q$,

$$f(y) = T_x^{k-1}(y) + \frac{f^{(k)}(x)}{k!}\,(y-x)^k + R_k,$$

where the remainder R_k of the Taylor formula (in Cauchy's form) is bounded by $\frac{|y-x|^k}{k!}\,\omega_1\big(|y-x|\,;f^{(k)}\big)$. Since $f^{(k)}$ is uniformly continuous in $[-1,1]$,

$$\lim_{Q\to x} |R_k|\cdot|x-y|^{-k} = 0$$

uniformly for $y \in Q$.

Hence, for given $\varepsilon > 0$ and all $|x| \le 1$, there is $\delta > 0$ such that

$$\left|E_k(Q\,;f) - E_k\Big(Q\,;T_x^{k-1} + \frac{f^{(k)}(x)}{k!}\,(\cdot - x)^k\Big)\right| < \varepsilon|Q|^k,$$

provided that $|Q| + d(x,Q) < \delta$.

By the definition of E_k, the second term equals

$$\frac{|f^{(k)}(x)|}{k!}\,\min_{\{a_i\}}\max_{y\in Q}\left|x^k - \sum_{i=0}^{k-1} a_i y^i\right| = \frac{|f^{(k)}(x)|}{k!}\cdot 2\Big(\frac{|Q|}{4}\Big)^k,$$

where we use here the classical Chebyshev theorem on the polynomial of least deviation. Combining all these facts we get

$$\left||Q|^{-k}E_k(Q\,;f) - \frac{|f^{(k)}(x)|}{2^{2k-1}\cdot k!}\right| < \varepsilon$$

for all x satisfying $|Q| + d(x,Q) < \delta$.

This part of the result is proved.

(Sufficiency) We need some properties of best polynomial approximation which are collected in

Proposition 2.41. *Let* $f \in C(Q)$ *and differ from a polynomial of degree* $k-1$, $k \ge 1$. *Then the following facts are true.*

(a) *There exists a unique polynomial (of best approximation) of degree* $k-1$ *denoted by* $P_Q(f)$ *such that*

$$E_k(Q\,;f) = \|f - P_Q(f)\|_{C(Q)}.$$

---

[5] $Q \to x$ means that the endpoints of the interval Q tend to x and (2.61) converges uniformly in Q.

(b) *There is a collection of (equioscillation) points* $x_0 < x_1 < \cdots < x_k$ *in* Q *such that*

$$\bigl|f(x_i) - P_Q(f)(x_i)\bigr| = \max_Q |f - P_Q(f)| \bigl(= E_k(Q\,;f)\bigr), \quad i = 0, \dots, k,$$

and, moreover, for $\varepsilon_i := \operatorname{sgn}\bigl(f - P_Q(f)(x_i)\bigr)$,

$$\varepsilon_i \varepsilon_{i+1} < 0, \quad i = 0, 1, \dots, k-1.$$

Conversely, if for a polynomial P *of degree* $k-1$ *the difference* $f - P$ *has* $k+1$ *points of equioscillation in* Q*, then* $P = P_Q(f)$.

(c) *If* $f_\kappa \in C(Q)$ *continuously depends on a parameter* κ *ranging in a Hausdorff topological space, then* $E_k(Q\,; f_\kappa)$ *and* $P_Q(f_\kappa)$ *are continuous in* κ.

Assertion (c) and (2.61) imply that

Lemma 2.42. *The function* λ *is continuous.*

Proof. Let $T_Q : \mathbb{R} \to \mathbb{R}$ be an affine transform mapping $[-1, 1]$ onto Q. Then for $f_Q := f \circ T_Q$ we have, by the definition of E_k,

$$E_k(Q\,; f) = E_k\bigl([-1, 1]\,; f_Q\bigr).$$

Hence, the function $Q \mapsto |Q|^{-k} E_k(Q\,; f)$ continuously depends on Q and therefore the uniform limit in (2.61) is continuous. $\square$

The function λ is therefore bounded in $[-1, 1]$ and the condition of Theorem 2.38 holds for $n = 1$ and the exponent $(k-1) + 1$. Hence, f *belongs to the space* $C^{k-1,1}[-1, 1]$.

Since the Lipschitz function $f^{(k-1)}$ is absolutely continuous, $f^{(k)}$ exists almost everywhere, and for all $x \in Q$,

$$f(x) = T_a(x) + \frac{1}{(k-1)!} \int_a^x (x-t)^{k-1} f^{(k)}(t)dt, \tag{2.63}$$

where T_a is the (Taylor) polynomial of degree $k-1$ at a fixed point a in Q.

Set $S_f := \{x \in Q\,; f^{(k)}(x) \text{ exists}\}$. By the Taylor formula with the remainder in Peano form we have, for $x \in S_f$ and $y \in Q$,

$$f(y) = T_x^{k-1}(f\,;y) + \frac{f^{(k)}(x)}{k!}(y-x)^k + \varepsilon_k \cdot (y-x)^k$$

where $\varepsilon_k \to 0$ as $y \to x$.

The argument used for the proof of necessity then implies that

$$\lambda(x) := \lim_{Q \to x} \frac{E_k(Q\,; f)}{|Q|^k} = \frac{|f^{(k)}(x)|}{k! 2^{2k-1}}\,.$$

Hence, (2.63) may be written as

$$f(x) = T_a(x) + k \cdot 2^{2k-1} \int_a^x (x-t)^{k-1} \lambda(t)\sigma(t)dt, \tag{2.64}$$

where we set

$$\sigma := \operatorname{sgn} f^{(k)}.$$

Now let

$$G := \{x \in Q\,; \lambda(x) > 0\}.$$

This set is open, since λ is continuous. We will show below that σ admits a continuous extension from $G \cap S_f$ to G. In other words, we will prove that $f^{(k)}$ preserves its sign in any connected component (interval) of the open set G.

Let us assume for the time being that $\bar{\sigma}$ be this extension. Replacing σ by $\bar{\sigma}$ in (2.64) and differentiating k times the Riemann integral so obtained, we conclude that $f^{(k)}$ exists and is continuous in $[-1, 1]$, as required.

In turn, we derive the above stated extension result from the following

Claim. Given $\varepsilon > 0$ and a closed interval I from the open set

$$G_\varepsilon := \{x \in Q\,; \lambda(x) > \varepsilon\},$$

the function σ preserves its sign in $I \cap S(f)$.

We prove this by contradiction. Assume, on the contrary, that there are points z_1, z_2 in $I \cap S(f)$ such that

$$\sigma(z_1)\sigma(z_2) := \operatorname{sgn} f^{(k)}(z_1) \cdot \operatorname{sgn} f^{(k)}(z_2) < 0. \tag{2.65}$$

To proceed, we reformulate this condition using equioscillation points, see Proposition 2.41 (b). To this end, we fix a closed interval $J \subset Q$, denote by $x_0 < x_1 < \cdots < x_k$ the set of equioscillation points for $f - P_J(f)$ in J and set

$$\varepsilon_J(f) := \operatorname{sgn}\big(f(x_k) - P_J(f)(x_k)\big).$$

Lemma 2.43. *Let $f \in C^{k-1,1}(J)$ and $f^{(k)}(t_0) > 0$ for some $t_0 \in J \cap S_f$. Then there is $\delta > 0$ such that for every interval $\Delta \subset J$ of length $\le \delta$ which contains t_0, we have*

$$\varepsilon_\Delta(f) > 0.$$

Proof. Assume, on the contrary, that there is a sequence of intervals $\{\Delta_n\}_{n\ge 1}$ such that

$$t_0 \in \Delta_n \subset J,\ \varepsilon_{\Delta_n}(f) < 0 \ \text{ for all } \ n \ \text{ and } \ |\Delta_n| \to 0 \ \text{ as } n \to \infty. \tag{2.66}$$

Let Δ be one of these intervals and $x_0 < x_1 < \cdots < x_k$ be the equioscillation points in Δ for $f - P_\Delta(f)$. Due to (2.66) and Proposition 2.41, there is a point $y_{k-1} \in (x_{k-1}, x_k)$ such that

$$\big(f - P_\Delta(f)\big)'(y_{k-1}) < 0.$$

Similarly, between x_{k-2} and x_{k-1} there is a point y_{k-2} such that

$$\big(f - P_\Delta(f)\big)'(y_{k-2}) > 0,$$

and so forth.

Applying this argument to $(f - P_\Delta)'$, then to $(f - P_\Delta)''$ and so on, we finally find two points $w_0 < w_1$ in Δ such that

$$\operatorname{sgn}(f - P_\Delta)^{(k-1)}(w_i) = (-1)^i, \quad i = 0, 1.$$

Since $P_\Delta^{(k-1)}$ is a constant, this implies that

$$\frac{f^{(k-1)}(w_1) - f^{(k-1)}(w_0)}{w_1 - w_0} < 0.$$

Sending $|\Delta|$ to zero through the sequence $\{\Delta_n\}$ we obtain the inequality $f^{(k)}(t_0) \le 0$ which contradicts the assumption of the lemma. $\square$

From the lemma, we now immediately obtain

Corollary 2.44. *Assume that condition* (2.65) *holds for* $z_1, z_2 \in I \cap S(f)$. *Then there is* $\delta > 0$ *such that for arbitrary subintervals* $\Delta_i \ni z_i$, $i = 1, 2$, *in* I *of equal length less than* δ *we have*

$$\varepsilon_{\Delta_1}(f)\varepsilon_{\Delta_2}(f) < 0. \tag{2.67}$$

Now we proceed as follows. Let $q_\Delta(f) := \gamma_\Delta x^k + \cdots$ be a polynomial of degree k closest to $f|_\Delta$ in $C(\Delta)$. It will be shown that $\varepsilon_\Delta(f) = \operatorname{sgn}\gamma_\Delta$. Therefore, shifting Δ_1 toward Δ_2 and using (2.67) and the continuous dependence of γ_Δ on Δ, see Lemma 2.42, we find an intermediate segment Δ such that $\gamma_\Delta = 0$. But q_Δ has $k+2$ equioscillation points in Δ, say, $x_0 < x_1 < \cdots < x_{k+1}$, and the segment $\Delta' := [x_0, x_k] \subset \Delta$ contains $k+1$ of them. By Proposition 2.41 this implies

$$E_k(\Delta\,; f) = E_k(\Delta'\,; f). \tag{2.68}$$

Further, $\Delta \subset G_\varepsilon$, i.e., $\lambda(x) > \varepsilon$ on Δ, and $f^{(k-1)}$ satisfies the Lipschitz condition with the constant $\gamma(k)\max_Q \lambda$. This allows us to estimate the distance between adjacent equioscillation points and, hence, the size of Δ' by the inequality

$$\frac{|\Delta'|}{|\Delta|} \le q(\varepsilon, k, f) < 1. \tag{2.69}$$

Assuming this to be true for the moment we may derive from here the desired contradiction as follows. Let $\{\Delta_i^n\}_{n\in\mathbb{N}}$ be sequences of intervals in I such that $|\Delta_i^n| \to 0$ as $n \to \infty$ and $z_i \in \Delta_i^n$, $i = 1, 2$. Find for sufficiently large n, the aforementioned subintervals $\Delta_n' \subset \Delta_n$ related to the pairs Δ_1^n, Δ_2^n. Then we get for all n greater than some n_0,

$$E_k(\Delta_n' \,; f) = E_k(\Delta_n \,; f) \quad \text{and} \quad \frac{|\Delta_n'|}{|\Delta_n|} \le q < 1.$$

By compactness we may assume that Δ_n tends to some point x as $n \to \infty$. For this x we then have

$$\lambda(x) = \lim_{\Delta_n \to x} \frac{E_k(f; \Delta_n)}{|\Delta_n|^k} \le q^k \lim_{\Delta_n' \to x} \frac{E_k(f; \Delta_n')}{|\Delta_n'|^k} < \lambda(x),$$

a contradiction.

Thus, it remains to prove (2.68) and (2.69). For this purpose we need the next

Lemma 2.45. *Suppose that the function f_λ is continuous on a closed interval I and depends continuously on $\lambda \in [\lambda_0, \lambda_1]$. Assume that the sign $\varepsilon_I(f_\lambda)$ satisfies*

$$\varepsilon_I(f_{\lambda_0})\varepsilon_I(f_{\lambda_1}) < 0. \tag{2.70}$$

Then there is a $\lambda \in (\lambda_0, \lambda_1)$ such that $f_\lambda - P_I(f_\lambda)$ has one more equioscillation point[6].

Proof. We denote the polynomial of degree k closest to f_λ in $C(I)$ by $q_I(f_\lambda)$. We will show that for some $\lambda \in (\lambda_0, \lambda_1)$ this equals $P_I(f_\lambda)$. If this will be proved, then $f - P_I(f_\lambda)$ has the same number of equioscillation points as $f - q_I(f_\lambda)$, i.e., $k+2$.

Assume, on the contrary, that for all $\lambda \in (\lambda_0, \lambda_1)$,

$$P_I(f_\lambda) \ne q_I(f_\lambda).$$

Then the difference

$$q_I(f_\lambda) - P_I(f_\lambda) = \gamma(\lambda)x^k + \cdots \tag{2.71}$$

has at every equioscillation point of $f_\lambda - P_I(f_\lambda)$ a sign that equals that of $f_\lambda - P_I(f_\lambda)$. Actually, at such a point x, we get

$$\big|f_\lambda(x) - q_I(f_\lambda)(x)\big| \le E_{k+1}(J\,; f_\lambda) < E_k(J\,; f_\lambda) = \big|f_\lambda(x) - P_I(f_\lambda)(x)\big|,$$

and the result follows from the equality

$$q_I(f_\lambda) - P_I(f_\lambda) = \big(f_\lambda - P_I(f_\lambda)\big) - \big(f_\lambda - q_I(f_\lambda)\big).$$

Thus, the polynomial in (2.71) changes sign at the $(k+1)$ equioscillation points. Let $x_I(f_\lambda)$ denote the largest of these points. Then the polynomial in (2.71) has k zeros less than $x_I(f_\lambda)$ and therefore

$$\operatorname{sgn}\big(q_I(f_\lambda) - P_I(f_\lambda)\big)(x_I(f_\lambda)) = \operatorname{sgn}\gamma(\lambda).$$

[6] i.e., $k+2$ points, since $\deg P_I = k-1$, see Proposition 2.41.

Since, as we have proved,

$$\varepsilon_I(f_\lambda) := \operatorname{sgn}\big(f_\lambda - P_I(f_\lambda)\big)(x_I(f_\lambda)) = \operatorname{sgn}\big(q_I(f_\lambda) - P_I(f_\lambda)\big)(x_I(f_\lambda)),$$

we get

$$\varepsilon_I(f_\lambda) = \operatorname{sgn}\gamma(\lambda).$$

By Proposition 2.41 (c), the function γ is continuous on $[\lambda_0, \lambda_1]$ while (2.70) shows that γ changes its sign at the endpoints. Hence there is $\lambda \in (\lambda_0, \lambda_1)$ such that $\gamma(\lambda) = 0$ and $P_I(f_\lambda) = q_I(f_\lambda)$ for this λ. □

Now we show how to find the intervals $\Delta' \subset \Delta \subset \operatorname{conv}(\Delta_1, \Delta_2)$ in (2.68). Set $\Delta_i := [a_i, b_i]$, $i = 1, 2$, and $\ell := b_1 - a_1$ ($= b_2 - a_2$). Assuming that $a_1 < a_2$, we define the function $f_\lambda \in C[0, \ell]$, where $\lambda \in [a_1, a_2]$, by

$$f_\lambda(x) := f(x + \lambda), \quad x \in [0, \ell].$$

Applying the previous lemma to the function f_λ and (2.70) we find $\lambda \in [a_1, a_2]$ such that $f_\lambda - P_{[0,\ell]}(f_\lambda)$ has $k+2$ equioscillation points in $[0, \ell]$. Translating by ℓ we then obtain the same for $f - P_\Delta(f)$ where $\Delta := [\lambda, \ell + \lambda] \subset \operatorname{conv}(\Delta_1 \cup \Delta_2)$. Let $x_0 < x_1 < \cdots < x_{k+1}$ be the equioscillation points of this difference in Δ. According to Proposition 2.41 (b), the polynomial P_Δ is of best approximation to f in $C[x_0, x_k]$. Setting

$$\Delta' := [x_0, x_k] \subset \Delta,$$

we then get

$$E_k(\Delta' ; f) = E_k(\Delta ; f),$$

and it remains to show that for this Δ',

$$\frac{|\Delta'|}{|\Delta|} \le q = q(k, \varepsilon, f) < 1; \tag{2.72}$$

this will prove (2.70) and the theorem.

In turn, (2.72) is a consequence of

Lemma 2.46. *Let the function $f \in C^{k-1,1}(I)$, where I is a closed interval in $\mathbb{R}$, satisfy the conditions*

$$\big|f^{(k-1)}(x) - f^{(k-1)}(y)\big| \le M_0|x - y|, \quad x, y \in I, \tag{2.73}$$

$$E_k(I ; f) \ge M_1|I|^k. \tag{2.74}$$

Then for every pair of adjacent equioscillation points x_i, x_{i+1} for $f - P_I(f)$, it is true that

$$|x_{i+1} - x_i| \ge c\frac{M_1}{M_0}\,|I|, \tag{2.75}$$

where $c > 0$ depends only on k and f.

We first derive from here (2.72). Let $\Delta_1, \Delta_2 \subset I$ be as in Corollary 2.44, where the closed interval $I \subset G_\varepsilon := \{x \in Q\,;\, \lambda(x) > \varepsilon\}$. By the uniform convergence in (2.61), we get for some $\delta_1 > 0$ and every subinterval $\widetilde{\Delta} \subset I$ of length $< \delta_1$,

$$\frac{E_k(\widetilde{\Delta}\,; f)}{|\widetilde{\Delta}|^k} \geq \frac{1}{2} \inf_{x \in \widetilde{\Delta}} \lambda(x) > \frac{1}{2}\,\varepsilon.$$

Choosing an interval $\Delta \subset \operatorname{conv}(\Delta_1 \cup \Delta_2)$ such that $|\Delta| < \delta_1$, we then get

$$E_k(\Delta\,; f) \geq \frac{1}{2}\,\varepsilon|\Delta|^k.$$

Applying now Lemma 2.46 with $M_0 := |f|_{C^{k-1,1}(\Delta)}$, $M_1 := \frac{1}{2}\,\varepsilon$, and $I := \Delta$ we obtain

$$|\Delta'| := [x_0, x_k] \leq |\Delta| - |x_{k+1} - x_k| \leq |\Delta| - \frac{\varepsilon c(k)}{2|f|_{C^{k-1,1}(\Delta)}}\,|\Delta|.$$

This gives the required inequality

$$\frac{|\Delta'|}{|\Delta|} \leq q = q(k, \varepsilon, f) < 1.$$

So, it remains to establish Lemma 2.46.

Proof. Let first $k = 1$. Then $P_I(f)$ is a constant and there are two adjacent equioscillation points x_0, x_1 in I. By the definition of these points, (2.74) and (2.73) we have

$$2M_1|I| \leq 2E_1(I\,; f) = \big|(f - P_I(f))(x_0) - (f - P_I(f))(x_1)\big| \leq M_0|f|_{C^{0,1}(I)}|x_0 - x_1|$$

and (2.75) for $k = 1$ is established with $c := \frac{2}{|f|_{C^{0,1}(I)}}$.

Now let $k \geq 2$ and $x_0 < x_1 < \cdots < x_k$ be the equioscillation points of $f - P_I$ in I. As before,

$$\begin{aligned} 2M_1|I|^k \leq 2E_k(I\,; f) &= \big|(f - P_I(f))(x_i) - (f - P_I(f))(x_{i+1})\big| \\ &\leq |x_i - x_{i+1}| \max_I |(f - P_I(f))'|. \end{aligned} \tag{2.76}$$

To estimate this maximum we use the Taylor formula and (2.73) to write

$$\max_I |f - T(f)| \leq \frac{M_0}{(k-1)!}\,|I|^k,$$

where $T(f)$ is the Taylor polynomial for f of degree $k-1$ at the left endpoint of I.

The same argument also gets

$$\max_I \big|(f - T(f))'\big| \le \frac{M_0}{(k-2)!}\,|I|^{k-1}. \tag{2.77}$$

Using the inequality

$$\max_I \big|\big(P_I(f) - T(f)\big)\big| \le E_k(I\,;f) + \max_I |f - T(f)| \le 2\max_I |f - T(f)|$$

we derive from here that

$$\max_I \big|P_I(f) - T(f)\big| \le \frac{2M_0}{(k-1)!}\,|I|^k.$$

This and the Markov inequality, see Lemma 2.35 or Appendix G, give

$$\max_I \big|\big(P_I(f) - T(f)\big)'\big| \le \frac{4M_0(k-1)^2}{(k-1)!}\,|I|^{k-1}.$$

Together with (2.77) this leads to the estimate

$$\max_I |(f - P_I)'| \le c(k)M_0|I|^{k-1}.$$

Combining this last inequality with (2.76), we prove the lemma. □

As it was explained above this gives the proof of Theorem 2.40. □

2.4 Trace and extension problems for univariate C^k functions

2.4.1 Whitney's theorem

The Whitney solution to the problem is based on a detailed study of the combinatorial properties of closed subsets of the real line and a subsequent use of such analytic tools as Lagrange interpolation and divided differences. The Lagrange interpolation polynomial which coincides with f on a $(k+1)$-point set X is denoted by $L(f\,;X)$; its degree equals k and the k-th derivative satisfies

$$L^{(k)}(f\ ;\ X) = k!f[X]. \tag{2.78}$$

Now let S be a closed subset of $\mathbb{R}$, and let $S^{\langle k\rangle}$ denote the class of all *k-point subsets* of S. The main role in the solution will play a class of functions on S satisfying the condition

$$\lim_{X\to x}\big\{|f[X]|\cdot \operatorname{diam} X\,;\, X \in S^{\langle k\rangle}\big\} = 0 \tag{2.79}$$

at every point $x \in S$; here $X \to x$ means that $d\big(\{x\}, X\big) \to 0$.

The linear space of continuous functions $f : S \to \mathbb{R}$ satisfying (2.79) is denoted by $D^k(S)$.

Theorem 2.47 (Whitney [Wh-1934b]). (a) *It is true that*

$$C^k(\mathbb{R})\big|_S = D^{k+2}(S). \tag{2.80}$$

(b) *There exists a linear extension operator from the trace space into $C^k(\mathbb{R})$.*

Proof. Assume that $f = F|_S$ where $F \in C^k(\mathbb{R})$. If $X = \{x_0, \dots, x_{k+1}\} \subset S$ and $x_i < x_{i+1}$, then for some c_1, c_2 from the interval $[x_0, x_{k+1}]$,

$$f[X] = F[X] = k!\,\frac{F^{(k)}(c_1) - F^{(k)}(c_2)}{x_0 - x_{k+1}}.$$

Since $F^{(k)}$ is continuous, the condition

$$\lim_{X \to x} \left\{ |f[X]| \cdot \operatorname{diam} X \ ; \ X \in S^{\langle k+2 \rangle} \right\} = 0$$

holds at every $x \in X$. Hence,

$$D^{k+2}(S) \supset C^k(\mathbb{R})\big|_S. \qquad \square$$

The proof of the converse embedding is essentially more complicated. We begin with a result whose proof will be postponed until the end of the section.

Lemma 2.48 (Combinatorial). *There exists a map $\phi : S \to S^{\langle 1 \rangle} \cup S^{\langle k+1 \rangle}$ such that*

(i) $x \in \phi(x)$;

(ii) *if x is a limit point, then*

$$\phi(x) = \{x\};$$

(iii) *if x is an isolated point, then*

$$\operatorname{card} \phi(x) = k + 1;$$

(iv) *if either of x_1, x_2 is a limit point, then*

$$\operatorname{diam} \phi(x_1) + \operatorname{diam} \phi(x_2) \le C|x_1 - x_2|$$

for some constant $C = C(k)$.

This is also true for isolated points if $\phi(x_1) \ne \phi(x_2)$.

Now let P be a univariate polynomial of degree k with k mutually distinct real roots situated in an interval I of length ℓ.

Lemma 2.49. *For every $j \le k$,*

$$\max_I |P^{(j)}| \le \frac{|P^{(k)}|}{j!(k-j)!}\,\ell^{k-j}.$$

Proof. Let $Z = \{z_1, \dots, z_k\} \subset I$ be the set of zeros. Then

$$P(x) = \frac{a}{k!} \prod_{z \in Z} (x - z),$$

where the constant $a := P^{(k)}$. Therefore,

$$P^{(j)}(x) = \frac{a}{k!} \sum_{Y \in Z^{\langle k-j \rangle}} \prod_{y \in Y} (x - y),$$

whence

$$\max_I |P^{(j)}| \leq \frac{|P^{(k)}|}{k!} \binom{k}{k-j} \ell^{k-j}. \qquad \square$$

Now let $X_1, X_2 \in S^{\langle k+1 \rangle}$ and $X_1 \neq X_2$. Suppose that I is an interval of length ℓ in $\mathbb{R}$ such that

$$X := X_1 \cup X_2 \subset I.$$

Then for the Lagrange polynomials $L(f\ ;\ X_i)$ the following holds.

Lemma 2.50. *For every $j \leq k$ and $x \in I$,*

$$\begin{aligned} &\big|\big(L(X_1, f) - L(X_2, f)\big)^{(j)}(x)\big| \\ &\leq c(k)\ell^{k-j} \cdot \max\big\{|f[Y]| \cdot \operatorname{diam} Y\ ;\ Y \in X^{\langle k+2 \rangle}\big\}. \end{aligned} \tag{2.81}$$

Proof. Connect X_1 with X_2 by a chain of $(k+1)$-point subsets $Y_i \in X^{\langle k+1 \rangle}$, $i \leq m := k+2$, such that $Y_1 = X_1$, $Y_m = X_2$ and

$$\operatorname{card}\big(Y_i \cap Y_{i+1}\big) = k \quad \text{for every} \quad i \leq m-1. \tag{2.82}$$

Setting $P_i := L(Y_i, f)$ we may bound the left-hand side of (2.81) by the sum $\sum\limits_{i<m} \max\limits_I \big|(P_i - P_{i+1})^{(j)}\big|$ and then bound the i-th term by Lemma 2.49 with $P := P_i - P_{i+1}$, a polynomial of degree k whose set of roots $Y_i \cap Y_{i+1}$ contains k points of the interval I.

Hence, the sum is bounded by

$$\frac{\ell^{k-j}}{j!(k-j)!} \sum_{i=1}^{m} \big|(P_i - P_{i+1})^{(k)}\big| = \binom{k}{j} \ell^{k-j} \big|f[Y_i] - f[Y_{i+1}]\big|,$$

see (2.78). Finally, by the definition of the divided difference and (2.82), we get

$$\begin{aligned} \big|f[Y_i] - f[Y_{i+1}]\big| &\leq \big|f[Y_i \cup Y_{i+1}]\big| \operatorname{diam}(Y_1 \cup Y_2) \\ &\leq \max\big\{|f[Y]| \operatorname{diam} Y\ ; Y \in X^{\langle k+2 \rangle}\big\}. \end{aligned}$$

Putting all these estimates together we establish (2.81). $\square$

Now we relate the space $D^{k+2}(S)$ to the trace space $J^k(\mathbb{R})\big|_S$. In this setting, $J^k(\mathbb{R})$ consists of vector functions $\{f_j\}_{j=0}^k$ generated by functions $f \in C^k(\mathbb{R})$ such that $f_j = f^{(j)}$.

Lemma 2.51. *There exists a linear operator $W : D^{k+2}(S) \to J^k(\mathbb{R})\big|_S$ such that*

$$(Wf)_0 = f.$$

Proof. By Theorem 2.13 the space $J^k(\mathbb{R})\big|_S$ consists of vector functions $\vec{f} = \{f_j\}_{j=0}^k$ continuous on S and satisfying the following condition:

For every $j \le k$ and $x, y \in S$,

$$r_j(\vec{f}; x, y) \to 0 \quad \text{as} \quad y \to x. \tag{2.83}$$

Let us recall that (in this case) the reduced remainders are defined by

$$r_j(\vec{f}\,;\, x, y) := \left| f_j(x) - \sum_{i=0}^{k-j} f_{i+j}(y)\, \frac{(x-y)^i}{i!} \right| \Big/ |x-y|^{k-j}.$$

Hence, given $f \in D^{k+2}(S)$, we must find a vector function $\vec{f} = \{f_j\}_{j=0}^k$ satisfying (2.83) that depends linearly on f. We define the function f_j as the limit of the j-th derivatives for the corresponding interpolation polynomials.

We begin with the value of f_j at a limit point $x \in S$. In this case, the set

$$T_\varepsilon(x) := \big\{X \in S^{\langle k+2\rangle}\ ;\ \operatorname{diam}(\{x\} \cup X) < \varepsilon\big\}$$

contains for sufficiently small ε, say, $\varepsilon < \varepsilon_0$, infinitely many points. This allows us to define f_j by

$$f_j(x) := \lim_{Y \to x} \big\{L^{(j)}(Y, f)(x)\,;\, Y \in S^{\langle k+1\rangle}\big\}; \tag{2.84}$$

the limit has the same meaning as that in (2.79).

To prove that the limit exists, we apply Lemma 2.50 to subsets $X_1, X_2 \in T_\varepsilon(x) \cap S^{\langle k+1\rangle}$ where $\varepsilon < \frac{\varepsilon_0}{2}$. This yields

$$\big|\big(L(X_1, f) - L(X_2, f)\big)^{(j)}(x)\big| \le c(k) \max\big\{|f[Y]| \operatorname{diam} Y\,;\, Y \in T_{2\varepsilon}(x) \cap S^{\langle k+2\rangle}\big\}.$$

Since $f \in D^{k+2}(S)$, the right-hand side tends to zero as $Y \to x$, see (2.79). Hence, the limit in (2.84) exists for all $j \le k$. Moreover, for such x,

$$f_0(x) = f(x),$$

since $L(X, f)(x) = f(x)$ as $x \in X$.

Further, we define $\vec{f}(x)$ at an *isolated point* $x \in X$ using the set $\phi(x)$ of Combinatorial Lemma 2.48. In this case, $\phi(x)$ consists of precisely $k+1$ points and $L(\phi(x), f)$ is a polynomial of degree k. We introduce now $f_j(x)$ by

$$f_j(x) := \frac{d^j}{dt^j}\, L(\phi(x), f)(t)\bigg|_{t=x} := L^{(j)}(\phi(x), f)(x) \tag{2.85}$$

noting that $x \in \phi(x)$ and therefore $f_0(x) = f(x)$ at isolated points as well.

Thus, $\vec{f}$ is now determined at all points of S. The operator $W : f \mapsto \vec{f}$ is clearly linear and satisfies

$$(Wf)_0 = f.$$

Let us show that $\vec{f} \in J^k(\mathbb{R})\big|_S$, that is, the reduced remainders r_j satisfy condition (2.83). To this end we choose, for a given $x \in S$, a sequence of sets $\{X_i(x)\}$ from $S^{\langle k+1\rangle}$ such that

$$\operatorname{diam} X_i(x) \to \operatorname{diam} \phi(x) \quad \text{as} \quad i \to \infty, \tag{2.86}$$

and, moreover, for $0 \le j \le k$,

$$f_j(x) = \lim_{i\to\infty} L^{(j)}(X_i(x), f)(x). \tag{2.87}$$

The existence of such a sequence is obvious for a limit point x; for an isolated x we simply set

$$X_i(x) := \phi(x), \quad i = 1, 2, \dots. \tag{2.88}$$

Since $L(\phi(x), f)$ is a polynomial of degree k, the remainder $r_j(\vec{f}\,; x, y)$ may be written in the form

$$r_j(\vec{f}\,; x, y) = \lim_{i\to\infty} \big|(P_{i,x} - P_{i,y})^{(j)}(x)\big| \Big/ |x-y|^{k-j}, \tag{2.89}$$

where we set for $x \in S$ and $z \in \mathbb{R}$,

$$P^{(j)}_{i,x}(z) := \frac{d^j}{dz^j}\, L\big(X_i(x), f\big)(z),$$

see (2.85) and (2.87).

It remains to estimate the j-th derivative in (2.89).

First, let x be a limit point. By Lemma 2.48,

$$\operatorname{diam} \phi(x) + \operatorname{diam} \phi(y) \le c(k)|x-y|.$$

This and (2.86) yield the estimate for the upper limit of lengths ℓ_i of intervals $J_i(x, y) := \operatorname{conv}\big(X_i(x) \cup X_i(y)\big)$:

$$\overline{\lim_{i\to\infty}}\, \ell_i \le \lim_{i\to\infty} \big(\operatorname{diam} X_i(x) + \operatorname{diam} X_i(y) + |x-y|\big) \le c(k)|x-y|.$$

Using this and Lemma 2.50 we bound the limit in (2.89) by

$$\begin{aligned} c(k)|x-y|^{k+j}\, \overline{\lim_{i\to\infty}} &\Big[\ell_i^{k-j} \max\big\{|f[X]| \operatorname{diam} X\,;\, X \in \big(X_i(x) \cup X_i(y)\big)^{\langle k+2\rangle}\big\}\Big] \\ &\le c(k)\, \overline{\lim_{i\to\infty}} \max\big\{|f[X]| \operatorname{diam} X\,;\, X \in \big(X_i(x) \cup X_i(y)\big)^{\langle k+2\rangle}\big\}. \end{aligned}$$

Since $f \in D^{k+2}(S)$, see (2.79), and $\ell_i = |J_i(x,y)| \to 0$ uniformly in i as $y \to x$, the right-hand side tends to zero as $y \to x$ and therefore

$$r_j(f\,;\,x,y) \to 0 \quad \text{as} \quad y \to x.$$

Now, let $x \in S$ be an isolated point. Then $y = x$ if $y \in S$ is sufficiently close to x and therefore, for such y,

$$r_j(f\,;\,x,y) = r_j(f\,;\,x,x) = 0. \qquad \square$$

Lemma 2.51 implies the required inverse embedding

$$D^{k+2}(S) \subset C^k(\mathbb{R})\big|_S,$$

since by Theorem 2.13, $(Wf)_0 = F\big|_S$ for some $F \in C^k(\mathbb{R})$ while $(Wf)_0 = f$.

Thus, part (a) of Theorem 2.47 is proved.

Finally, the required linear extension operator $\mathcal{E}_k^X$ from $D^{k+2}(S)$ to $C^k(\mathbb{R})$ is defined by composing Whitney's extension operator E_k^S acting from $J^k(\mathbb{R})\big|_S$ to $J^k(\mathbb{R})$ with W and the projection $P\vec{f} := f_0$, $\vec{f} \in J^k(\mathbb{R})$. In other words,

$$\mathcal{E}_k^S := PE_k^S W. \tag{2.90}$$

This completes the proof of Theorem 2.47. $\square$

We apply now the Whitney extension method to obtain a version of Theorem 2.47 concerning the space $\dot{C}^{k,\omega}(\mathbb{R})$ defined by the seminorm (2.7); recall that the continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the conditions

$$\omega(t) \ \text{ and } \ \omega^*(t) := \frac{t}{\omega(t)} \ \text{ increase as } \ t \to +\infty \ \text{ and } \ \omega(0+) = 0. \tag{2.91}$$

To formulate the result due to Merrien [Mer-1966] we introduce a space of functions $f : S \to \mathbb{R}$ by the condition

$$m_{k,\omega}(f) := \sup\big\{\big|f[X]\big|\omega^*(\operatorname{diam} X)\ ;\ X \in S^{\langle k+2\rangle}\big\} < \infty. \tag{2.92}$$

We denote this space by $\dot{D}^{k+2}_\omega(S)$.

Theorem 2.52. (a) $\dot{D}^{k+2}_\omega(S) = \dot{C}^{k,\omega}(\mathbb{R})\big|_S$ *and* $m_{k,\omega}(f)$ *is equivalent to the trace norm of* f *with constants depending only on* k.

(b) *The restriction of the extension operator* $\mathcal{E}_k^S$ *to* $\dot{C}^{k,\omega}(\mathbb{R})\big|_S$ *is a linear continuous extension operator from this space into* $\dot{C}^{k,\omega}(\mathbb{R})$.

Proof. Let $f = F|_S$ for some $F \in \dot{C}^{k,\omega}(\mathbb{R})$. Then we have, for $X \in S^{\langle k+2\rangle}$,

$$\big|f[X]\big| = k!\,\frac{\big|F^{(k)}(c_1) - F^{(k)}(c_2)\big|}{\operatorname{diam} X}\,,$$

where $c_1, c_2 \in \operatorname{conv} X$.

This immediately yields

$$\big|f[X]\big|\omega^*(\operatorname{diam} X) \le k!|F|_{C^{k,\omega}(\mathbb{R})},$$

whence

$$m_{k,\omega}(f) \le k!|f|_{C^{k,\omega}(\mathbb{R})|_S} \tag{2.93}$$

by the definition of the seminorm involved.

To prove the converse result, we first state that

$$\big|f[X]\big| \operatorname{diam} X \le m_{k,\omega}(f)\omega(\operatorname{diam} X), \tag{2.94}$$

see (2.92). But $\omega(0+) = 0$ and therefore finiteness of $m_{k,\omega}(f)$ implies that f belongs to $D^{k+2}(S)$. Hence, the operator $\mathcal{E}_k^S$ is defined on the space $\dot{D}_\omega^{k+2}(S)$ and it remains to estimate the $C^{k,\omega}$-seminorm of $\mathcal{E}_k^S f$ for $f \in \dot{D}_\omega^{k+2}(S)$. To this end we first apply the inequality established for the remainder $r_j(\vec{f}\,;\, x, y)$ with $x, y \in S$, see (2.89), and then use (2.94) to bound the right-hand side of this inequality by

$$\begin{aligned} C(k)m_{k,\omega}(f) \varlimsup_{i\to\infty} &\big\{\omega(\operatorname{diam} X)\ ;\ X \in J_i(x,y)^{\langle k+2\rangle}\big\} \\ &\le C(k)m_{k,\omega}(f)\omega\big(C_1(k)|x-y|\big); \end{aligned}$$

here $\vec{f}$ is the k-jet generated by f, i.e., $\vec{f} = Wf$, see Lemma 2.51.

Since ω^* is nondecreasing, $\omega(ct) \le \max(1,c)\omega(t)$ and therefore,

$$r_j(\vec{f}\ ;\ x, y) \le C(k)m_{k,\omega}(f)\omega(|x-y|)$$

provided that $x, y \in S$ and $0 \le j \le k$. This means that $\vec{f}$ satisfies the Taylor condition of Theorem 2.22, see (2.36), which then implies that the Whitney operator E_k^S extends $\vec{f}$ to the k-jet $E_k^S\vec{f}$ from $\dot{J}^{k,\omega}(\mathbb{R})$. Since, in addition, this operator is linear and continuous, the operator $\mathcal{E}_k^S := PE_k^SW$ (see (2.90)) extends functions from $\dot{D}_\omega^{k+2}(S)$ to $\dot{C}^{k,\omega}(\mathbb{R})$ and

$$|\mathcal{E}_k^S f|_{C^{k,\omega}(\mathbb{R})} \le C(k)m_{k,\omega}(f).$$

Together with (2.93) this completes the proof. $\square$

Remark 2.53. The same proof can be applied to the case of the Banach space $C_b^{k,\omega}(\mathbb{R})$ with norm

$$\|f\|_{C_b^{k,\omega}(\mathbb{R})} := \sup_{\mathbb{R}} |f| + |f|_{C^{k,\omega}(\mathbb{R})}$$

We leave to the reader to verify that up to the equivalence of the norms

$$C_b^{k,\omega}(\mathbb{R})\big|_S = D_\omega^{k+2}(S),$$

where the space of the right-hand side is defined by the norm $\sup_S |f| + m_{k,\omega}(f)$.

We now use *universality* of the extension operator $\mathcal{E}_k^S$ (meaning independence of ω) to obtain an extension result for the subspace $\dot{C}_u^k(\mathbb{R})$ of $C^k(\mathbb{R})$ consisting of functions whose higher derivatives are uniformly continuous on the real line.

Corollary 2.54. *The restriction of $\mathcal{E}_k^S$ to the space $\dot{C}_u^k(\mathbb{R})\big|_S$ is a linear continuous extension operator from this space into $\dot{C}_u^k(\mathbb{R})$.*

Proof. If $f \in \dot{C}_u^k(\mathbb{R})$, then the *modulus of continuity* of f defined for $f > 0$ by

$$\omega(t\,; f^{(k)}) := \sup_{|x-y|\le t} \left|f^{(k)}(x) - f^{(k)}(y)\right|$$

tends to zero as $t \to 0$.

Since modulus of continuity is a subadditive function, there exists a function $\widetilde{\omega} : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying conditions (2.91) and such that

$$\omega \le \widetilde{\omega} \le 2\omega;$$

for instance, one can choose $\widetilde{\omega}(t) := t \sup_{s\ge t} \frac{\omega(s)}{s}$. This implies the equality

$$\dot{C}_u^k(\mathbb{R}) = \bigcup_{\omega} \dot{C}^{k,\omega}(\mathbb{R}),$$

where the union is taken over all ω satisfying (2.91).

Then assertion (b) of Theorem 2.52 gives the required result. □

Proof of Combinatorial Lemma 2.48. We begin the definition of the required set function $\phi : S \to S^{\langle 1\rangle} \cup S^{\langle k+1\rangle}$ by setting

$$\phi(x) := \{x\}$$

for $x \in S$ being a limit point of S.

Now let x be an isolated point of S. Then we set

$$\phi(x) := \{x_0, x_1, \dots, x_k\},$$

where $x_0 := x$ and the remaining points are defined as follows.

Define x_1 by the condition

$$|x_0 - x_1| = d\big(\{x_0\}, S\backslash\{x_0\}\big), \quad x_1 \in S\backslash\{x_0\}$$

(recall that $d(S_1, S_2) := \inf\big\{|s_1 - s_2|\ ;\ s_i \in S_i,\quad i = 1, 2\big\}$). If there are two such points, choose x_1 to be the one to the right.

If now x_1 is a limit point of the closed set $S\backslash\{x_0\}$, we choose distinct points x_j from $S\backslash\{x_0, x_1\}$, $2 \le j \le k$, such that

$$(R) \qquad |x_j - x_1| = d\big(x_j, \{x_0, x_1, \dots, x_{j-1}\}\big) \le d\big(x_{j-1}\,, \{x_0, x_1, \dots, x_{j-2}\}\big).$$

Since x_1 is a limit point of $S\backslash\{x_0\}$, such a choice is possible. Otherwise $S\backslash\{x_0, x_1\}$ is closed and we define x_2 by the condition

$$d\big(x_2, \{x_0, x_1\}\big) = d\big(\{x_0, x_1\}, S\backslash\{x_0, x_1\}\big), \quad x_2 \in S\backslash\{x_0, x_1\};$$

as above, the point x_2 is to be the one to the right if the choice is not unique. Again, if x_2 is a limit point of $S\backslash\{x_0, x_1\}$, then we choose points $x_j \in S\backslash\{x_0, x_1, x_2\}$ by the above formulated rule (R) with x_2 replacing x_1.

Proceeding in this way, we define $\phi(x) := \{x_0, \dots, x_k\}$.

Claim. If $\phi(x) \neq \phi(y)$, then

$$|x_i - x_j| \leq \max\{i, j\}|x - y|, \quad 0 \leq i, j \leq k, \tag{2.95}$$

and the same holds for $|y_i - y_j|$.

To establish this, we first show that for $0 \leq i \leq k-1$,

$$\begin{aligned} d\big(x_{i+1}, \{x_0, \dots, x_i\}\big) &\leq |x_0 - y_0|, \\ d\big(y_{i+1}, \{y_0, \dots, y_i\}\big) &\leq |x_0 - y_0|. \end{aligned} \tag{2.96}$$

These inequalities will be proved by induction on i. The result is trivial for $i = 0$. Now let (2.96) hold for $i - 1 < k - 1$; we will prove it for i. By symmetry, it suffices to establish the first of these inequalities. Without loss of generality we may assume that the points $x_0, \dots, x_i$ are isolated; for otherwise the required inequality follows from the rule (R) and the induction hypothesis. It follows from the condition $\phi(x) \neq \phi(y)$ that there are two possibilities:

A. $\{y_0, \dots, y_i\}$ is a *proper* subset of $\{x_0, \dots, x_i\}$.

B. There is $j \leq i$ such that

$$y_j \notin \{x_0, \dots, x_i\}. \tag{2.97}$$

Case A is impossible, since all points $y_0, y_1, \dots, y_i$ are pairwise distinct.

In case B, let j be the minimal index satisfying (2.97). By the extremal property of x_{i+1}, we then have

$$d\big(x_{i+1}, \{x_0, \dots, x_i\}\big) \leq d\big(y_j, \{x_0, \dots, x_i\}\big).$$

If now $j = 0$, then the right-hand side of the inequality is clearly bounded by $|x_0 - y_0|$, as required. Otherwise, $\{y_0, \dots, y_{j-1}\}$ is a nonempty subset of $\{x_0, \dots, x_i\}$. Hence, in this case,

$$d\big(y_j, \{x_0, \dots, x_i\}\big) \leq d\big(y_j, \{y_0, \dots, y_{j-1}\}\big),$$

and it follows from the induction hypothesis that this is bounded by $|x_0 - y_0|$.

Using now inequalities (2.96) we prove (2.95) by induction on $\ell := \max\{i, j\}$.

For $\ell = 0$ this is trivial. For $\ell > 0$, we may assume that $i > j$. Let $i' < i$ be such that $|x_i - x_{i'}| = d\big(x_i, \{x_0, \dots, x_{i-1}\}\big) \le |x_0 - y_0|$. By the induction hypothesis, $|x_{i'} - x_j| \le \max\{i', j\}|x_0 - y_0| \le (\ell - 1)|x_0 - y_0|$. Hence,

$$|x_i - x_j| \le |x_i - x_{i'}| + |x_{i'} - x_j| \le \ell |x_0 - y_0|.$$

The proof of (2.95) is complete.

Now we are ready to prove the Combinatorial Lemma. By the definition of ϕ, the point x belongs to $\phi(x)$. Moreover, $\phi(x) = \{x\}$ at a limit point and $\operatorname{card} \phi(x) = k + 1$ at an isolated point.

We claim that if $x \ne y$ and one of these points is a limit point, then

$$\operatorname{diam} \phi(x) + \operatorname{diam} \phi(y) \le 2k|x - y|.$$

In fact, in this case $\phi(x) \ne \phi(y)$ and, by (2.95), the diameters of $\phi(x)$ and $\phi(y)$ are at most $k|x - y|$. The same argument can be applied to the case of isolated points x, y such that $\phi(x) \ne \phi(y)$ and so the lemma has been proved. □

2.4.2 Reformulation of Whitney's theorem

In an attempt to generalize the Whitney theorem to $C^k(\mathbb{R}^n)$, we immediately enter the following difficulty. Unlike the univariate case we cannot use for the definition of divided differences the linear order of the real line or the connection with Lagrange interpolation as in Proposition 2.25. In the multivariate case, it would be natural to define the divided difference $f[\Sigma]$ for a finite set $\Sigma \subset \mathbb{R}^n$ as a collection of derivatives $\big\{D^\alpha L_\Sigma(f)\,;|\alpha| = k\big\}$, where $L_\Sigma(f)$ is a polynomial of *minimal* degree interpolating the function f at points of Σ. Such a polynomial is not, in general, unique and one must select one of them. Unfortunately, such a choice cannot be done arbitrarily, since, given f being the trace of a C^k function to S, the chosen family of interpolation polynomials for f must satisfy quite strong restrictions. This leads to a very difficult *selection problem* whose geometric counterpart will be discussed in Section 5.3.

Therefore it seems to be reasonable to find an equivalent formulation of Whitney's theorem which eliminates divided differences (and related Lagrange interpolation) and then to extract from there the basic features of the multivariate extension problems. To achieve this purpose we need a new concept, the *Finiteness Property*, already mentioned in Section 2.3. For its motivation we first note that due to Theorem 2.52 for a $(k+2)$-point set $\Sigma \subset \mathbb{R}$ and a function $g : \Sigma \to \mathbb{R}$,

$$|g|_{C^{k,\omega}|_\Sigma} \approx \big|g[\Sigma]\omega^*(\operatorname{diam}\Sigma)\big|,$$

where $\omega^*(t) := t/\omega(t),\ t > 0$, and the constants of equivalence are independent of g and Σ.

Therefore Theorem 2.52 may be restated as follows.

Theorem 2.55. *Let S be a closed subset of $\mathbb{R}$ containing at least $k+2$ points.*

(a) *A function* $f : S \to \mathbb{R}$ *belongs to the trace space* $\dot{C}^{k,\omega}\big|_S$ *if and only if its restrictions* $f|_\Sigma$ *to subsets* Σ *from* $S^{\langle k+2\rangle}$ *satisfy*

$$\sup\Big\{ \big|f|_\Sigma\big|_{C^{k,\omega}|_\Sigma} ;\, \Sigma \in S^{\langle k+2\rangle} \Big\} < \infty.$$

Moreover, the supremum is equivalent to $|f|_{C^{k,\omega}|_S}$.

(b) *There is a linear continuous extension operator from* $\dot{C}^{k,\omega}\big|_S$ *to* $\dot{C}^{k,\omega}(\mathbb{R})$ *independent of* ω.

In order to reformulate Whitney's Theorem 2.47, we note that the entity $\big|f[\Sigma]\,\mathrm{diam}\,\Sigma\big|$, where $\Sigma := \{x_0, \dots, x_{k+1}\} \subset \mathbb{R}$ may be rewritten up to the multiplier $k!$ as $\big|L_{\Sigma_0}(f) - L_{\Sigma_1}(f)\big|_{C^k(\mathbb{R})}$ for $\Sigma_i := \{x_i, \dots, x_{k+i}\}$, $i = 0, 1$, see Proposition 2.25 (b). Moreover, the Lagrange polynomial (of degree k) $L_{\Sigma_i}(f)$ is an extension of the trace $f|_{\Sigma_i}$ to a function of $\dot{C}^k(\mathbb{R})$.

These facts lead to the following form of Whitney's Theorem 2.47.

Theorem 2.56. *Let* S *be a closed subset of* $\mathbb{R}$ *containing at least* $k+2$ *points.*

(a) *A function* $f : S \to \mathbb{R}$ *belongs to the trace space* $\dot{C}^k\big|_S$ *if and only if every trace* $f|_\Sigma$ *to a* $(k+1)$*-point subset* $\Sigma \subset S$ *admits an extension to a function* F_Σ *of* $\dot{C}^k(\mathbb{R})$ *such that for every* $x \in S$,

$$|F_\Sigma - F_{\Sigma'}|_{C^k(\mathbb{R})} \to 0 \quad \text{as} \quad \Sigma, \Sigma' \to \{x\}.$$

(b) *There exists a linear continuous extension operator* $\mathcal{E}_k^S : \dot{C}^k\big|_S \to \dot{C}^k(\mathbb{R})$ *such that for every* 1*-majorant* ω,

$$\mathcal{E}_k^S\big(\dot{C}^{k,\omega}\big|_S\big) \subset \dot{C}^{k,\omega}(\mathbb{R}).$$

In the case of the space $C_b^k(\mathbb{R})$ the family $\{F_\Sigma\}_{\Sigma \in S^{\langle k+2\rangle}}$ may be taken to be uniformly bounded on $\mathbb{R}$. This fact and condition (a) allow us to apply the Arcelá-Ascoli theorem which leads to the following version of Whitney's result.

Theorem 2.57. *There exists a constant* $c(k) > 0$ *such that a function* $f : S \to \mathbb{R}$ *belongs to* $C_b^k(\mathbb{R})|_S$ *if and only if some family of* $C_b^k(\mathbb{R})$ *extensions* $\{F_\Sigma\}_{\Sigma \in S^{\langle k+2\rangle}}$ *is contained in a compact subset of a ball in* $C_b^k(\mathbb{R})$ *centered at* 0 *and of radius at most* $c(k)\sup_S |f|$.

2.4.3 Finiteness and linearity

Now we are ready to introduce these general concepts and formulate the basic conjectures. An extensive study of this area will be presented in Chapter 10 (Volume II).

Let X be a space of smooth functions on $\mathbb{R}^n$ equipped with a seminorm $|\cdot|$, e.g., $\dot{C}^{k,\omega}(\mathbb{R}^n)$ or $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$, or norm $f \mapsto \sup_{\mathbb{R}^n}|f| + |f|_X$. For a closed subset S

of $\mathbb{R}^n$ we, for brevity, denote by $X(S)$ the trace space $X|_S$, and by $|\cdot|_{X(S)}$ the corresponding trace seminorm, i.e.,

$$|f|_{X(S)} := \inf\{|g|_X \,;\, g|_S = f\}.$$

Given an integer $N > 1$ and a closed subset $S \subset \mathbb{R}^n$ containing at most N points, we now introduce a functional $\delta_N(\cdot\,;\,S\,;\,X)$ defined on functions $f : S \to \mathbb{R}$ by

$$\delta_N(f\,;\,S\,;\,X) := \sup\bigl\{|f|_{X(\Sigma)}\,;\,\Sigma \in S^{\langle N\rangle}\bigr\}. \tag{2.98}$$

Using this we introduce the basic

Definition 2.58. (a) The space X possesses the finiteness property on a closed subset $S \subset \mathbb{R}^n$ if for some integer N the following is true.

A function $f : S \to \mathbb{R}$ belongs to the trace space $X(S)$ if

$$\delta_N(f\,;\,S\,;\,X) < \infty.$$

Moreover, the equivalence

$$\delta_N(f\,;\,S\,;\,X) \approx |f|_{X(S)}, \tag{2.99}$$

holds with constants independent of f.

(b) X has the (uniform) finiteness property if (a) holds for every closed S with the constants in (2.99) independent of f and S.

The minimal N here is denoted by $\mathcal{F}_S(X)$ and is called the *local finiteness constant* of X.

In turn, the (uniform) *finiteness constant* of X is given by

$$\mathcal{F}(X) := \sup\bigl\{\mathcal{F}_S(X)\,;\,S \ \text{ is closed}\bigr\}. \tag{2.100}$$

In these terms, Theorem 2.55 (a) simply asserts that

$$\mathcal{F}\bigl(\dot{C}^{k,\omega}(\mathbb{R})\bigr) = k + 2. \tag{2.101}$$

To formulate in the same manner Theorems 2.56 and 2.57 we need a new concept.

Definition 2.59. The space X possesses the strong finiteness property if the following is true:

A function $f : S \to \mathbb{R}$ belongs to the trace space $X(S)$ if

(a) for the seminormed space X, some family $\{F_\Sigma\}_{\Sigma \in S^{\langle N\rangle}}$ satisfies

$$\bigl|F_\Sigma - F_{\Sigma'}\bigr|_X \to 0 \quad \text{if} \quad \Sigma, \Sigma' \to \{x\}$$

for every $x \in S$;

(b) for the normed space X, some family $\{F_\Sigma\}_{\Sigma\in S^{\langle N\rangle}}$ is contained in a compact subset of a ball in X centered at 0 of radius at most $c(n,k)\sup_S |f|$.

The minimal N of this definition is said to be the *strong finiteness constant* of X denoted by $\mathcal{F}_{\mathrm{str}}(X)$.

In these terms, Theorem 2.56 (a) simply states that

$$\mathcal{F}_{\mathrm{str}}\big(\dot{C}^k(\mathbb{R})\big) = k+1. \tag{2.102}$$

One more illustration of these properties yields the Shevchuk theorem [She-1992, Thm.11.1] generalizing Theorem 2.55 to the space $C^k\dot{\Lambda}^{s,\omega}(\mathbb{R})$.

Let us recall that this space (over $\mathbb{R}^n$) is defined by the seminorm

$$|f|_{C^k\dot{\Lambda}^{s,\omega}(\mathbb{R}^n)} := \sup_{|\alpha|=k}\sup_{t>0}\frac{\omega_s(t\,;D^\alpha f)}{\omega(t)},$$

where ω is an s-majorant, see Section 2.1 for details.

The aforementioned theorem asserts that $f : X \to \mathbb{R}$ belongs to the trace space $C^k\dot{\Lambda}^{s,\omega}(\mathbb{R})\big|_S$ if

$$\sup\big\{\big|f[\Sigma]\big|\cdot\Psi_{k,s}(\omega,\Sigma)\,;\,\Sigma\in S^{\langle k+s+1\rangle}\big\} \tag{2.103}$$

is finite. Here $\Psi_{k,s}(\omega,\Sigma)$ is a rather complicated function of ω and Σ for which an analytic expression can be given only in few cases; for instance,

$$\Psi_{k,1}(\omega,\Sigma)\approx\omega^*(\operatorname{diam}\Sigma) \tag{2.104}$$

so that (2.103) implies Theorem 2.55 (a).

Another interesting case concerns the Besov space $\dot{B}^\sigma_\infty(\mathbb{R})$, see (2.23). For noninteger σ we have $\dot{B}^\sigma_\infty(\mathbb{R}) = \dot{C}^{k,\omega}(\mathbb{R})$ with $\omega(t) := t^{\sigma-k}$, $0<\sigma-k<1$; hence the corresponding result is given by (2.104) with $\omega(t) := t^\sigma$, $k := \lfloor\sigma\rfloor$ and $s=1$. However, for integer σ we obtain a new result; in this case, $k=\sigma-1$, $s=2$ and $\omega(t) := t$, and for this k,s,ω,

$$\Psi_{k,s}(\omega,\Sigma)\approx\frac{\operatorname{diam}\Sigma}{\big|\log\big(\frac{\operatorname{diam}\Sigma_1}{\operatorname{diam}\Sigma_0}\big)\big|+1},$$

where $\Sigma := \{x_0<x_1<\cdots<x_{k+s}\}$ and $\Sigma_i := \{x_i,\dots,x_{k+i}\}$, $i=0,1$.

Hence, $f : S\to\mathbb{R}$ belongs to the trace space $\dot{B}^\sigma_\infty(\mathbb{R})\big|_S$, where σ is an integer, if (and only if)

$$\sup\left\{\big|f[\Sigma]\big|\frac{\operatorname{diam}\Sigma}{\big|\log\big(\frac{\operatorname{diam}\Sigma_1}{\operatorname{diam}\Sigma_0}\big)\big|+1}\,;\,\Sigma\in S^{\langle\sigma\rangle}\right\}$$

is finite.

For $\sigma=1$ this result was firstly proved by A. Jonsson [Jon-1980].

The final case concerns the space $\dot{\Lambda}^{2,\omega}(\mathbb{R})$ with arbitrary 2-majorant ω. In this setting,

$$\Psi_{0,2}(\omega,\Sigma) \approx (\operatorname{diam}\Sigma)\left(\int\limits_{\delta(\Sigma)}^{\operatorname{diam}\Sigma} \frac{\omega(t)}{t^2}\,dt\right)^{-1},$$

where $\delta(\Sigma) := \min\{x_{i+1} - x_i\,;\, i = 0,1\}$ provided that $\Sigma := \{x_0 < x_1 < x_2\}$.

Then (2.103) yields the trace description result established independently by Shvartsman [Shv-1982] and Dziadik and Shevchuk [DShe-1983].

We now clarify the Shevchuk theorem using the finiteness constants. The result is naturally divided into two parts the first of which simply asserts that

$$\mathcal{F}\big(C^k\dot{\Lambda}^{s,\omega}(\mathbb{R})\big) = k + s + 1. \tag{2.105}$$

The second, the computational part of the theorem, evaluates the trace norm of $f : \Sigma \to \mathbb{R}$ for subsets Σ of cardinality $k + s + 1$. The Shevchuk result yields for such subsets

$$|f|_{C^k\dot{\Lambda}^{s,\omega}(\mathbb{R})|_\Sigma} \approx \Psi_{k,s}(\omega,\Sigma).$$

2.4.4 Basic conjectures

The previous discussion forms a basis for several conjectures for multivariate functions.

We begin with

Finiteness Conjecture 2.60. (a) *The space $C^k\dot{\Lambda}^{s,\omega}(\mathbb{R}^n)$ has the finiteness property.*

(b) *The space $\dot{C}^k(\mathbb{R}^n)$ has the strong finiteness property.*

Linearity Conjecture 2.61. *Let $X(\mathbb{R}^n)$ be one of the spaces of the previous conjecture. Then for every closed subset $S \subset \mathbb{R}^n$ there is a linear bounded extension operator from the trace space $X|_S$ into $X(\mathbb{R}^n)$. In other words, the trace space admits a simultaneous extension.*

Until the early 1980s the only result was Whitney's Theorem 2.56, confirming the first conjecture for $n = 1$. Since then the situation has essentially improved due to the works of Yu. Brudnyi, Shvartsman, Ch. Fefferman, Bierstone and P. Milman. We discuss their results in the final part of the book; for now we only single out the next striking fact established by Shvartsman [Shv-1987]:

$$\mathcal{F}\big(\dot{\Lambda}^{2,\omega}(\mathbb{R}^n)\big) = 3 \cdot 2^{n-1}. \tag{2.106}$$

Remark 2.62. (a) Unlike the one-dimensional case the extension operator for $n > 1$ depends on the majorant ω. This follows from the following fact:

There is a subset $S \subset \mathbb{R}^2$ such that $\dot{C}^1_u(\mathbb{R}^2)\big|_S$ does not admit a simultaneous extension to $\dot{C}^1_u(\mathbb{R}^2)$.

This result due to Yu. Brudnyi and P. Shvartsman [BSh-1999] will be also discussed in the final part of the book.

(b) "Interpolating" the Shevchuk and Shvartsman results (2.105) and (2.106) one may guess that

$$\mathcal{F}\big(C^k\dot{\Lambda}^{s,\omega}(\mathbb{R}^n)\big) = (k+s+1)2^{\gamma(k,n)}$$

where $\gamma(k,n) := -1 + \binom{n+s-2}{s-1}$. Note that the binomial coefficient is the dimension of the space of $(s-1)$-homogeneous polynomial in $x_1, \dots, x_n$.

This quantitative version of Conjecture 2.60 (a) seems to be extremely difficult to prove (if true).

(c) The local extension constant $\mathcal{F}_S(X)$ for "massive" subsets $S \subset \mathbb{R}^n$ may have a strictly lesser rate of growth in n. For example, let S be n-regular, i.e., for some $c_0 > 0$ and $r_0 > 0$ and all cubes $Q_r(x)$ with $r \le r_0$ and $x \in S$,

$$\big|S \cap Q_r(x)\big| \ge c_0 r^n.$$

Then the extension theorem from the paper by Yu. Brudnyi [Br-1970b] and the Remez–Shnirelman finiteness principle, see Appendix B of Chapter 1, yield for these S and power k-majorants ω the inequality

$$\mathcal{F}_S\big(\dot{C}^{k,\omega}(\mathbb{R}^n)\big) \le 1 + \dim \mathcal{P}_{k,n} = 1 + \binom{n+k}{k}.$$

A general result of this kind concerning the space $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ and the so-called Markov subsets S will be presented in Section 9.2 (Volume II).

(d) In all these formulations we may replace seminorms of X by its normed counterpart defined by the norm $\|f\|_{C_b(\mathbb{R}^n)} + |f|_X$. The known results cited above remain to be true for the normed spaces as well.

2.5 Restricted main problem for some classes of domains in $\mathbb{R}^n$

2.5.1 Quasiconvex domains

We derive from Theorems 2.13 and 2.19 trace and extension results for C^k functions on open sets. We restrict our consideration to the space $\dot{C}^{k,\omega}(G)$, where hereafter $G \subset \mathbb{R}^n$ stands for a domain. Let us recall that this space is defined by the seminorm

$$|f|_{C^{k,\omega}(G)} := \sum_{|\alpha|=k} \sup_{[x,y]\subset G} \frac{\big|D^\alpha f(x) - D^\alpha f(y)\big|}{\omega(\|x-y\|)}, \tag{2.107}$$

where ω is a nondecreasing nonnegative function on $\mathbb{R}_+$ such that $\omega(t)/t$ is nonincreasing and $\omega(0+) = 0$.

In particular, ω is *subadditive*, i.e., for all $t_1, t_2 > 0$,

$$\omega(t_1 + t_2) \le \omega(t_1) + \omega(t_2); \tag{2.108}$$

indeed, $\frac{\omega(t_1+t_2)}{t_1+t_2} \le \frac{\omega(t_i)}{t_i}$, and multiplying this by t_i and summing on i, one gets (2.108).

In the case under consideration, the Restricted Main Problem is formulated as follows:

Problem. (a) *Under what geometric characteristics of a domain G is it true that*

$$\dot{C}^{k,\omega}(G) = \dot{C}^{k,\omega}(\mathbb{R}^n)\Big|_G ? \tag{2.109}$$

(b) *Under what conditions on G is the equality* (2.109) *a linear isomorphism, i.e., there exists a linear bounded extension operator from the trace space into $\dot{C}^{k,\omega}(\mathbb{R}^n)$?*

We present here only a partial solution to the problem, namely, we describe a subclass of domains which satisfies (2.109). Since the proof of the result will be based on Whitney's extension theorems, the answer to question (b) is automatically affirmative.

To introduce the subclass in question, we need several geometric notions.

Suppose that $\gamma : [0, 1] \to \mathbb{R}^n$ is a *curve* (continuous map). Recall that its *length* is defined by

$$\ell(\gamma) := \sup \sum_{i=0}^{n} \big\|\gamma(t_{i+1}) - \gamma(t_i)\big\|, \tag{2.110}$$

where the supremum is taken over all *polygonal lines* with associated segments $[\gamma(t_i), \gamma(t_{i+1})]$, $0 \le t_1 < \cdots < t_n \le 1$, and all n.

A curve γ is *rectifiable* if $\ell(\gamma) < \infty$.

Further, fixing ω one defines the *ω-length* of a *polygonal line* $\gamma : [0, 1] \to \mathbb{R}^n$ with the segments $[\gamma(t_i), \gamma(t_{i+1})]$, where $t_0 = 0 < t_1 < \cdots < t_n = 1$, by

$$\ell_\omega(\gamma) := \sum_{i=0}^{n-1} \omega\big(\big\|\gamma(t_{i+1}) - \gamma(t_i)\big\|\big). \tag{2.111}$$

Now let G be a domain in $\mathbb{R}^n$. One defines the *(geodesic) ω-distance* in G setting for $x, y \in G$,

$$d_\omega(x, y) := \inf \ell_\omega(\gamma), \tag{2.112}$$

where γ runs over all polygonal lines γ joining x, y in G (i.e., $\gamma : [0, 1] \to G$ and $\gamma(0) = x$, $\gamma(1) = y$).

Due to (2.108), $d_\omega(x,y) = \omega(\|x-y\|)$ if the segment $[x,y]$ lies in G and therefore d_ω is a metric on $\mathbb{R}^n$. It will be shown below, see the proof of Proposition 2.68, that d_ω is a metric in G as well.

Definition 2.63. A domain $G \subset \mathbb{R}^n$ is said to be (C,ω)-convex, if for every two points x, y in G there is a polygonal line γ joining them in G and such that

$$\ell_\omega(\gamma) \le C\omega(\|x-y\|).$$

The optimal C will be denoted by $C_\omega(G)$.

If $\omega(t) = t,\ t \in \mathbb{R}_+$, then such a domain is said to be C-*quasiconvex* (*quasiconvex* if the constant does not matter).

A 1-quasiconvex domain is clearly convex. Moreover, for C close to 1, C-quasiconvex domains inherit some basic features of convex ones. In particular, such a domain is contractible if $C < \frac{\pi}{2}$, and *simply connected*[7], if $C < \frac{\pi\sqrt{2}}{2}$, see Gromov [Gr-2000, pp. 11–12]. On the other hand, the geometry of (C,ω)-convex domains may be rather complicated even for C close to 1. As an elementary example we consider a planar domain $G_\lambda,\ \lambda \ge 1$, in the open unit disk $\mathbb{D}$ centered at (0,0) given by $G_\lambda := \{(x,y) \in \mathbb{D}\,;\, x \le 0 \ \text{ or } \ |y| > x^\lambda\}$:

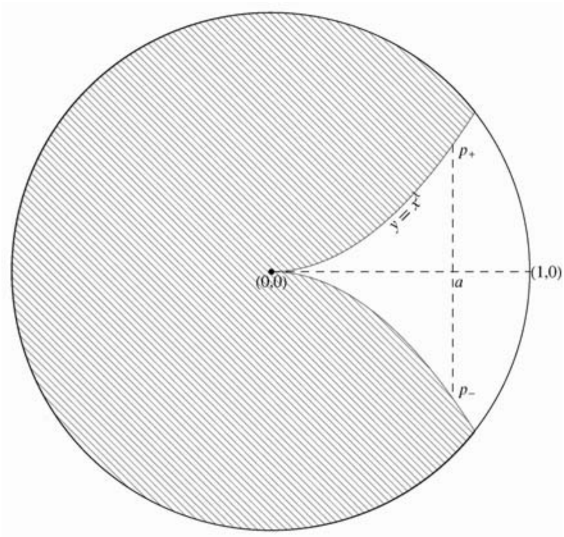

Figure 2.1: An example of a (C,ω)-convex domain.

This domain is (C,ω)-convex for $\omega(t) := t^{\frac{1}{\lambda}}$. In fact, it suffices to check the condition of Definition 2.63 only near the vertex $(0,0)$ of the cusp of G_λ.

[7] i.e., every closed curve into the domain is homotopic to a constant map.

In this case, fix a sufficiently small $a > 0$ and let γ_a be a polygonal line with the endpoints $p_\pm := (a, \pm a^\lambda)$ and with very small segments inscribed in the arc $|y| = x^\lambda,\ 0 \le x \le a$. Then

$$\ell_\omega(\gamma_a) \approx 2\int_0^a \left(1 + \left(\frac{d}{dt}\,\omega(t^\lambda)\right)^2\right)^{\frac{1}{2}} dt \approx a,$$

while

$$\omega\big(\|p_+ - p_-\|\big) = (2a^\lambda)^{\frac{1}{\lambda}} \approx a$$

and the required condition follows.

G_λ has a single inner cusp at 0; it is easy to construct a (C,ω)-domain with infinitely many inner and outer cusps.

Now we formulate and prove the basic result of this section following, in essence, Whitney's paper [Wh-1934d], where a simultaneous extension for $\dot{C}^k_u(G)$ in the case of a quasiconvex bounded domain is studied.

Theorem 2.64. *Assume that $G \subset \mathbb{R}^n$ is a (C,ω)-convex domain. Then for every k equality* (2.109) *holds up to equivalence of seminorms. Moreover, there exists a linear bounded extension operator from $\dot{C}^{k,\omega}(G)$ into $\dot{C}^{k,\omega}(\mathbb{R}^n)$.*

Proof. Since the restriction of $f \in \dot{C}^{k,\omega}(\mathbb{R}^n)$ to G obviously belongs to $\dot{C}^{k,\omega}(G)$, the embedding

$$\dot{C}^{k,\omega}(\mathbb{R}^n)\big|_G \subset \dot{C}^{k,\omega}(G) \tag{2.113}$$

holds with the embedding constant 1.

We prove the converse embedding using the Whitney extension operator E_k^G, see Section 2.2. To this end, given a function $f \in \dot{C}^{k,\omega}(G)$, we introduce a k-jet $\vec{f} := \{f_\alpha\}_{|\alpha| \le k}$ defined on G by

$$f_\alpha := D^\alpha f.$$

We claim that under the assumption on G, the continuous extension of k-jet to $\overline{G}$ satisfies the Whitney–Glaeser Theorem 2.19. This can be done if we prove for the reduced remainders

$$r_\alpha(\vec{f}\,;\ x, y) := \left| D^\alpha f(x) - \sum_{|\beta| \le k - |\alpha|} \frac{D^{\alpha+\beta} f(y)}{\beta!}\,(x-y)^\beta \right| \Big/ \|x - y\|^{k-|\alpha|}$$

the inequality

$$r_\alpha(\vec{f}\,;\ x, y) \le C|f|_{C^{k,\omega}(G)}\,\omega(\|x - y\|) \tag{2.114}$$

with C depending only on k, n and the constant $C_\omega(G)$.

To establish (2.114), set

$$R_\alpha(f\,;\, x, y) := D^\alpha(f - T_y^k)(x),$$

where $T_y^k f$ is the k-th Taylor polynomial of f at point y.

Lemma 2.65. *Let* $f \in \dot{C}^{k,\omega}(G)$ *and* $[x, y] \subset G$. *Then for every* α *and* $z \in \mathbb{R}^n$,

$$R_\alpha(f\,; x, y) \le c|f|_{C^{k,\omega}(G)}\big(\|z - x\|^{k-|\alpha|} + \|z - y\|^{k-|\alpha|}\big) \cdot \omega(\|x - y\|), \tag{2.115}$$

where $c = c(k, n)$.

Proof. Applying the Taylor formula with the remainder in the integral form we represent $R_\alpha(f\,;\, x, y)$ with $|\alpha| < k$ in the form

$$R_\alpha(f\,;\, x, y) = (k - |\alpha|) \sum_{|\beta|=k-|\alpha|} \frac{(x-y)^\beta}{\beta!} \int_0^1 (1-t)^{k-|\alpha|-1}$$
$$\times \big[D^{\alpha+\beta} f(y + t(x - y)) - D^{\alpha+\beta} f(y)\big]\, dt.$$

Since $y + t(x - y) \in [x, y] \subset G$, the expression in the square brackets does not exceed in absolute value $|f|_{C^{k,\omega}(G)} \cdot \omega(\|x - y\|)$. Hence, we have

$$\big|R_\alpha(f\,;\, x, y)\big| \le c(k, n)|f|_{C^{k,\omega}(G)}\, \|x - y\|^{k-|\alpha|}\, \omega(\|x - y\|).$$

This estimate is true also for $|\alpha| = k$, because in this case

$$\big|R_\alpha(f\,;\, x, y)\big| = \big|D^\alpha f(x) - D^\alpha f(y)\big|.$$

Apply now these estimates to the identity

$$D^\alpha\big[T^k_x f - T^k_y f\big](z) = \sum_{|\beta|\le k-|\alpha|} \frac{1}{\beta!}\, R_{\alpha+\beta}(f\,;\, x, y)(z - x)^\beta.$$

This gives for the absolute value of the right-hand side the upper bound

$$c(k, n)|f|_{C^{k,\omega}(G)}\omega(\|x - y\|) \sum_{|\beta|\le k-|\alpha|} \frac{1}{\beta!}\, \|x - y\|^{k-|\alpha+\beta|}\|z - x\|^{|\beta|}.$$

The sum is clearly bounded by $c(k)\big(\|x - z\|^{k-|\alpha|} + \|y - z\|^{k-|\alpha|}\big)$, and the result follows. $\square$

Going back to the proof of (2.114), choose a polygonal curve $\gamma : [0, 1] \to G$ connecting points $x, y \in G$ and such that

$$\ell_\omega(\gamma) \le 2C_\omega(G)\omega(\|x - y\|).$$

Let $\{y_i\}_{0\le i\le m} \subset \gamma([0, 1])$ be the vertices of γ, so that $y_0 = x$, $y_m = y$, $[y_i, y_{i+1}] \subset G$; then

$$\ell_\omega(\gamma) \ge \sum_{i=0}^{m-1} \omega\big(\|y_{i+1} - y_i\|\big). \tag{2.116}$$

Applying Lemma 2.65 we get

$$\begin{aligned}\big|R_\alpha(f\,;\,x,y)\big| &= \big|D^\alpha\big(T_x^k f - T_y^k f\big)(x)\big| \le \sum_{i=0}^{m-1}\big|D^\alpha\big(T_{y_{i+1}}^k f - T_{y_i}^k f\big)(x)\big| \\ &\le C(k,n)|f|_{C^{k,\omega}(G)}\sum_{i=0}^{m-1}\omega\big(\|y_{i+1}-y_i\|\big)\big(\|x-y_i\|^{k-|\alpha|}+\|x-y_{i+1}\|^{k-|\alpha|}\big).\end{aligned}$$

Setting $\mu := \sup_i\big(\|x-y_i\|^{k-|\alpha|}+\|x-y_{i+1}\|^{k-|\alpha|}\big)$ we therefore have

$$\big|R_\alpha(f\,;\,x,y)\big| \le c(k,n)\mu\,|f|_{C^{k,\omega}(G)}\ell_\omega(\gamma). \tag{2.117}$$

We now show that

$$\mu \le c(k)\big[C_\omega(G)\|x-y\|\big]^{k-|\alpha|}. \tag{2.118}$$

Together with (2.117) and (2.112) this would yield the desired inequality (2.114) with

$$\widetilde{c} = c(k,n)c(k)C_\omega(G)^k. \tag{2.119}$$

To establish (2.118) we use

Lemma 2.66. *Let $\gamma : [0,1] \to \mathbb{R}^n$ be a curve with endpoints $\gamma(0) := x$ and $\gamma(1) := y$. Then*

$$\ell(\gamma) \le \frac{\|x-y\|}{\omega(\|x-y\|)}\,\ell_\omega(\gamma).$$

Proof. Since the map $t \mapsto \ell\big(\gamma|_{[0,t]}\big)$ is uniformly continuous on $[0,1]$, one can choose $\{y_i\} \in \gamma([0,1])$ such that $\|y_{i+1}-y_i\| := \big\|\gamma(t_{i+1})-\gamma(t_i)\big\| \le \|x-y\|$ for all i. Since the function $t \mapsto \frac{t}{\omega(t)}$ is nondecreasing, we then have

$$\big\|\gamma(t_{i+1})-\gamma(t_i)\big\| \le \frac{\|x-y\|}{\omega(\|x-y\|)}\;\omega\big(\big\|\gamma(t_{i+1})-\gamma(t_i)\big\|\big).$$

Summing these inequalities over i and taking the corresponding supremum we then obtain

$$\ell(\gamma) \le \frac{\|x-y\|}{\omega(\|x-y\|)}\sum\omega\big(\big\|\gamma(t_{i+1})-\gamma(t_i)\big\|\big) \le \frac{\|x-y\|}{\omega(\|x-y\|)}\;\ell_\omega(\gamma).$$

The result is thus established. $\square$

To apply the result just proved to estimate μ, we write

$$\begin{aligned}\|x-y_i\|^{k-|\alpha|}+\|x-y_{i+1}\|^{k-|\alpha|} &\le c(k)\big(\|x-y_i\|+\|x-y_{i+1}\|\big)^{k-|\alpha|} \\ &\le 2^k c(k)\,\ell\,(\gamma)^{k-|\alpha|}.\end{aligned}$$

Estimating here $\ell(\gamma)$ by Lemma 2.66 and recalling the definition of μ we obtain the inequality

$$\mu \le 2^k c(k)\left(\frac{\|x-y\|}{\omega(\|x-y\|)}\ \ell_\omega(\gamma)\right)^{k-|\alpha|}$$
$$\le 2^k c(k)\left(\frac{\|x-y\|}{\omega(\|x-y\|)}\cdot C_\omega(G)\omega(\|x-y\|)\right)^{k-|\alpha|}.$$

This proves (2.118) and (2.114).

Now (2.114) with $|\alpha| = k$ implies that

$$\big|D^\alpha f(x) - D^\alpha f(y)\big| \le O(1)|f|_{C^{k,\omega}(G)}\,\omega(\|x-y\|).$$

Hence all derivatives $D^\alpha f$ with $|\alpha| = k$ are uniformly continuous on G. This and (2.114) with $|\alpha| = k-1$ then imply the uniform continuity of all $D^\alpha f$ with $|\alpha| = k-1$. Proceeding this way we prove that $D^\alpha f$ for all $|\alpha| \le k$ are uniformly continuous on G. Extending these derivatives by continuity to the closure $\overline{G}$ we obtain the k-jet satisfying condition (2.114) on the closed set.

It remains to apply the Whitney–Glaeser Theorem 2.19 to complete the proof. □

Remark 2.67. Estimates (2.114) and (2.119) yield the upper bound $\tilde{c}(k,n)C_\omega(G)^k$ for the extension operator of Theorem 2.64. It can be shown that the estimate is asymptotically sharp (up to a constant) as $C_\omega(G)$ tends to 1.

Now we return to the Restricted Main Problem, see (2.109). With this in mind, we reformulate Theorem 2.64 as follows.

Let $\mathcal{D}_{\text{ext}}(k,\omega)$ denote the class of domains in $\mathbb{R}^n$ satisfying (2.109). Then Theorem 2.64 asserts that ω-convexity of a domain is sufficient for being a member of $\mathcal{D}_{\text{ext}}(k,\omega)$.

This condition is, in general, not necessary. Nevertheless, this is the case for a few situations presented below.

Proposition 2.68. *A domain $G \subset \mathbb{R}^n$ belongs to the class $\mathcal{D}_{\text{ext}}(0,\omega)$ if and only if G is ω-convex.*

Proof. According to Theorem 2.64 only ω-convexity of G should be established. For this purpose we first show that the distance d_ω introduced by (2.112) is a metric in G. It is easy to verify the metric axioms for d_ω, except that of finiteness (i.e., $d_\omega < \infty$). To show the finiteness of d_ω, we fix $x \in G$ and denote by $G(x)$ the set of all $y \in G$ that can be connected with x by a polygonal curve $\gamma : [0,1] \to G$ with $\ell_\omega(\gamma) < \infty$. Finiteness of d_ω means that $G = G(x)$ for every $x \in G$. But $G(x)$ is clearly open and it can be easily checked that it is also closed in G. Since G is connected, the result follows.

Fix now a point $x \in G$ and define a function $f_0 : G \to \mathbb{R}$ by

$$f_0(y) := d_\omega(x,y), \quad y \in G. \tag{2.120}$$

Let $[y, y'] \subset G$. The triangle inequality and the definition of d_ω then yield

$$\big|f_0(y) - f_0(y')\big| \le d_\omega(y, y') = \omega(\|y - y'\|).$$

Hence $f_0 \in \dot{C}^{0,\omega}(G)$ and $|f|_{C^{0,\omega}(G)} \le 1$. Since (2.109) with $k = 0$ holds for the domain G, every $f \in \dot{C}^{0,\omega}(G)$ admits an extension $F \in \dot{C}^{0,\omega}(\mathbb{R}^n)$ such that

$$|F|_{C^{0,\omega}(\mathbb{R}^n)} \le C|f|_{C^{0,\omega}(G)}$$

with a constant C independent of f. Applying this to the function f_0 and denoting its extension by F_0, we obtain, for $x, y \in G$, the inequality

$$\begin{aligned} d_\omega(x, y) &= \big|f_0(x) - f_0(y)\big| = \big|F_0(x) - F_0(y)\big| \\ &\le |F_0|_{C^{0,\omega}(\mathbb{R}^n)}\omega(\|x - y\|) \le C\omega(\|x - y\|). \end{aligned}$$

That is to say, for every pair $x, y \in G$, there is a curve $\gamma : [0, 1] \to G$ connecting them and such that $\ell_\omega(\gamma) \le C\omega(\|x - y\|)$.

Hence, G is (C, ω)-convex. □

The next result, proved by Zobin [Zo-1999], shows that ω-convexity is a necessary condition for the space $\dot{C}^{k,1}(G)$, i.e., $\dot{C}^{k,\omega}(G)$ with $\omega(t) = t,\ t > 0$, and some class of domains $G \subset \mathbb{R}^2$ to satisfy (2.109). Let us recall that a ω-convex domain with this ω is called quasiconvex.

Theorem 2.69. (a) *Assume that G is a simply connected planar domain belonging to $\mathcal{D}_{\text{ext}}(k, \omega)$ with $\omega(t) := t,\ t > 0$, i.e.,*

$$\dot{C}^{k,1}(G) = \dot{C}^{k,1}(\mathbb{R}^2)\big|_G. \tag{2.121}$$

Then G is quasiconvex.

(b) *The same is true for finitely connected bounded planar domains of the class $\mathcal{D}_{\text{ext}}(k, \omega)$ with $\omega(t) := t,\ t > 0$.*

(c) *There is a bounded planar domain of this class which is not quasiconvex.*

Proof. We present the proof of assertion (a) and outline the proof of assertion (b) referring to Zobin's paper [Zo-1998] for a counterexample proving assertion (c).

(a) By the assumption, for every $f \in \dot{C}^{k,1}(G)$ there exists an extension $F \in \dot{C}^{k,1}(\mathbb{R}^2)$ such that

$$|F|_{C^{k,1}(\mathbb{R}^2)} \le A|f|_{C^{k,1}(G)} \tag{2.122}$$

with A independent of f. To derive from here the quasiconvexity of G, we construct, for every $x \in G$, a function $f_x \in \dot{C}^{k,1}(G)$ such that its seminorm satisfies

$$|f_x|_{C^{k,1}(G)} \le \sqrt{2}, \tag{2.123}$$

and, moreover, for every $y \in G$, the inequality

$$d_\omega(x, y) \le \sum_{|\alpha|=k} \left|D^\alpha f_x(x) - D^\alpha f_x(y)\right| \tag{2.124}$$

is true with $\omega(t) := t$.

We first show that this function can be used to complete the proof. Let $F_x \in \dot{C}^{k,1}(\mathbb{R}^2)$ be an extension of f_x satisfying (2.122). Then (2.122) and (2.124) yield the inequality

$$\begin{aligned} d_\omega(x, y) &\le \sum_{|\alpha|=k} \left|D^\alpha F_x(x) - D^\alpha F_x(y)\right| \le |F_x|_{C^{k,1}(\mathbb{R}^2)} \|x - y\| \\ &\le A|f_x|_{C^{k,1}(G)} \|x - y\| \le \sqrt{2}\, A\|x - y\|. \end{aligned}$$

By the definition of d_ω (with the linear ω) this estimate means that every pair $x, y \in G$ is connected within G by a curve of length at most $C(k)A\|x - y\|$, and the result follows.

It remains to find the function $f_x \in \dot{C}^{k,1}(G)$. For this purpose, we denote by $\mathcal{L}(x, y)$ the set of piecewise linear curves in G connecting x and y and having segments parallel either to the x_1-axis (x_1-segments) or to the x_2-axis (x_2-segments). For such a curve, say β, one sets

$$\ell_i(\beta) := \text{sum of lengths of } x_i\text{-segments of } \beta.$$

Now define a pseudometric $d_G^i : G \times G \to \mathbb{R}_+$ by setting

$$d_G^i(x, y) := \inf\left\{\ell_i(\beta)\,;\, \beta \in \mathcal{L}(x, y)\right\}.$$

For the segment $[x, y] \subset G$ the Pythagoras theorem gives

$$\|x - y\| = \sqrt{d_G^1(x, y)^2 + d_G^2(x, y)^2}\,.$$

This implies the inequality

$$d_G \ge \frac{1}{\sqrt{2}}\,(d_G^1 + d_G^2), \tag{2.125}$$

where one sets $d_G := d_\omega$ for $\omega(t) := t$.

It is important that for simply connected planar domains inequality (2.125) can be conversed. This is based on the following geometric fact.

Lemma 2.70. *Let β_1, β_2 be piecewise linear curves from the class $\mathcal{L}(x, y)$. Then there exists a piecewise linear curve β from this class whose length satisfies the inequality*

$$\ell(\beta) \le \ell_1(\beta_1) + \ell_2(\beta_2). \tag{2.126}$$

We postpone the proof of the lemma to the final part of the derivation. For now, we only conclude from (2.126) that

$$\inf \ell(\beta) \le d_G^1(x,y) + d_G^2(x,y),$$

where the infimum is taken over all $\beta \in \mathcal{L}(x,y)$. Evidently, this infimum equals $d_G(x,y)$, see (2.112) with $\omega(t) = t$. Hence, the desired converse inequality

$$d_G \le d_G^1 + d_G^2 \tag{2.127}$$

is true.

Define now the required function $f_x : G \to \mathbb{R}$, where $x \in G$, by

$$f_x(y) := \frac{1}{(k-1)!} \int_\beta \sum_{i=1}^{2} d_G^i(z,x)(y_i - z_i)^{k-1}\, dz, \tag{2.128}$$

where β is an arbitrary curve from $\mathcal{L}(x,y)$.

This f_x is well defined, i.e., the integral does not depend on the choice of β. By the Green formula for such Lipschitz differential 1-forms, see, e.g., Federer [Fe-1969], and simply connectedness of G, this claim will follow from the identity

$$\frac{\partial}{\partial z_1}\, P_2(z) = \frac{\partial}{\partial z_2}\, P_1(z), \quad z \in G, \tag{2.129}$$

where one sets $P_i(z) := d_G^i(z,x)(y_i - z_i)^{k-1}$.

Further, by definition, the function $z \mapsto d_G^1(z,x)$ is constant on every interval parallel to the x_2-axis, and the similar assertion holds for d_G^2 with respect to x_1-axis. Hence, $\frac{\partial}{\partial z_1} P_2(x) = \frac{\partial}{\partial z_2} P_1(z) = 0$ and f_x is well defined.

The same argument works for the evaluation of mixed derivatives of f_x. To compute $D_i f_x \left(:= \frac{\partial}{\partial z_i} f_x\right)$, choose curves $\beta \in \mathcal{L}(x,y)$ and $\beta' \in \mathcal{L}(x, y+he_i)$, where $e_i = (\delta_j^i)_{j=1,2}$ and $h \ne 0$, and use them in (2.128) to obtain

$$\begin{aligned}
&\frac{1}{h}\left(f_x(y+he_i) - f_x(y)\right) \\
&= \frac{1}{(k-1)!}\left\{\int_\beta d_G^i(z,x) \cdot \frac{(y_i + h - z_i)^{k-1} - (y_i - z_i)^{k-1}}{h}\, dz_i \right.\\
&\quad \left. + \int_{[y,y+he_i]} \frac{d_G^i(z,x)(y_i - z_i)^{k-1}}{2}\, dz_i \right\}.
\end{aligned}$$

Then passing to the limit as $h \to 0$ we have

$$D_i f_x(y) = \frac{1}{(k-2)!} \int_\beta d_G^i(z,x)(y_i - z_i)^{k-2} dz_i \quad \text{for} \quad k \ge 2$$

and, moreover,

$$D_i f_x(y) = d_G^i(x,y) \quad \text{for} \quad k = 1.$$

Continuing this way we obtain for all $y \in G$ and $0 < m < k$,

$$D_i^m f_x(y) = \frac{1}{(k-1-m)!} \int_\beta d_G^i(z,x)(y_i - z_i)^{k-1-m}\, dy_i$$

and, for $m = k$,

$$D_i^k f_x(y) = d_G^i(x,y). \tag{2.130}$$

In addition to this, we also have

$$D^\alpha f_x = 0 \quad \text{for} \quad |\alpha| := \alpha_1 + \alpha_2 \le k \quad \text{and} \quad \alpha_1, \alpha_2 > 0. \tag{2.131}$$

This follows from the fact that the derivatives in (2.129) are zero. These calculations and (2.125) lead to the following estimate:

$$\begin{aligned} |f_x|_{C^{k,1}(G)} &:= \sum_{i=1}^{2} \sup_{[y,y']\subset G} \frac{\left|D_i^k f_x(y) - D_i^k f_x(y')\right|}{\|y-y'\|} \\ &\le \sum_{i=1}^{2} \sup_{[y,y']\subset G} \frac{d_G^i(y,y')}{\|y-y'\|} \le \sqrt{2}. \end{aligned}$$

Hence, f_x satisfies inequality (2.123). Moreover, inequality (2.127) and equalities (2.130) and (2.131) yield

$$d_G(x,y) \le d_G^1(x,y) + d_G^2(x,y) = \sum_{|\alpha|=k} \left|D^\alpha f_x(x) - D^\alpha f_x(y)\right|,$$

and inequality (2.124) follows as well.

It remains to prove Lemma 2.70. We derive it from an elementary geometric result whose proof is postponed to the end of this subsection. In its formulation we use the following notion. A non-self-intersecting polygonal curve in $\mathbb{R}^2$ with segments parallel to the coordinate axes is called a *bolt*. If β is a *closed bolt* (i.e., homeomorphic to a circle) then $\mathcal{U}(\beta)$ designates the open (simply connected) polygon with boundary β. Clearly, all angles of this polygon equal $\pm\frac{\pi}{2}$.

Geometric Lemma. *Let x, y be distinct points of a closed bolt β, and let β_1, β_2 be bolts connecting x and y and such that*

$$\beta = \beta_1 \cup \beta_2 \quad \textit{and} \quad \beta_1 \cap \beta_2 = \{x, y\}.$$

Then there is a bolt β_3 with endpoints x, y such that $\beta_3 \subset \overline{\mathcal{U}(\beta)}$ and its length satisfies the inequality

$$\ell(\beta_3) \le \ell_1(\beta_1) + \ell_2(\beta_2).$$

Let us recall that $\ell_i(\beta)$ is the sum of the x_i-segments lengths of β.

Now we apply the lemma to prove inequality (2.126). Let $\beta_1, \beta_2 \in \mathcal{L}(x, y)$; we should find $\beta_3 \in \mathcal{L}(x, y)$ whose length satisfies the inequality

$$\ell(\beta_3) \leq \ell_1(\beta_1) + \ell_2(\beta_2).$$

Obviously, the piecewise linear curves β_1 and β_2 intersect in a finite number of points, say, $x = x^0, x^1, \dots, x^k = y$. Then the parts β_1^i, β_2^i of β_1, β_2 from x^i to x^{i+1} form a closed bolt which is denoted by β^i. By the Geometric Lemma, there is a bolt β_3^i in $\overline{\mathcal{U}(\beta^i)}$ such that β_3^i connects x^i and x^{i+1} and such that

$$\ell(\beta_3^i) \leq \ell_1(\beta_1^i) + \ell_2(\beta_2^i).$$

Then $\beta_3 := \bigcup\limits_{i=0}^{k-1} \beta_3^i$ belongs to $\mathcal{L}(x, y)$; summing the above inequalities over i we establish inequality (2.126) for β_3.

Part (a) of Theorem 2.69 has been proved.

(b) Now let G be a bounded finitely connected domain in $\mathbb{R}^2$ such that

$$\dot{C}^{k,1}(G) = \dot{C}^{k,1}(\mathbb{R}^2)\big|_G, \quad k \geq 1.$$

We must prove that G is quasiconvex. To this end, we first note that for *bounded* domains quasiconvexity is equivalent to *local quasiconvexity*. The latter means that there is a constant $\delta > 0$ such that

$$\sup\left\{\frac{d_G(x, y)}{\|x - y\|}\,;\ x, y \in G,\ \|x - y\| < \delta\right\} < \infty.$$

This fact may be established by a standard compactness argument.

Secondly, we may choose a sufficiently small $\varepsilon > 0$ such that every open disk $B_\varepsilon(x)$ in $\mathbb{R}^2$ of radius ε intersects at most one connected component of the complement $G^c := \mathbb{R}\backslash G$ and such that the intersection of this disk with G is simply connected or empty. This can be done, since G^c contains a finite number of connected components.

Finally, for a nonempty simply connected set $G \cap B_\varepsilon(x)$ where $x \in G$, we may, as in part (a), find a function $F \in \dot{C}^{k,1}(G)$ such that for every $y \in B_\varepsilon(x)$,

$$\min\left\{d_G(x, y), \frac{\varepsilon}{2}\right\} \leq 2 \sum_{|\alpha|=k} \left|D^\alpha F(x) - D^\alpha F(y)\right|$$

and, moreover, for some constant $C = C(k)$,

$$|F|_{C^{k,1}(G)} \leq C(k).$$

As in the proof of part (a) we conclude from those inequalities that for every $y \in B_\varepsilon(x)$,

$$d_G(x, y) \leq C(\varepsilon, k)\|x - y\|.$$

Notice that if $B_\varepsilon(x)$ does not intersect G^c, then $d_G(x,y)$ simply equals $\|x-y\|$ and the above inequality trivially holds as well.

By definition, this means that G is locally (and therefore globally) quasiconvex, as required.

Proof of Geometric Lemma. (Igonin and Yanishevski [IYa-1998]) Let β be a closed bolt with fixed points $x, y \in \beta$. The parts of β joining x and y are denoted by β_1 and β_2. We must find a bolt β_3 joining x and y inside the closure of the domain $U(\beta)$ bounded by β, and such that its length satisfies

$$\ell(\beta) \le \ell_1(\beta_1) + \ell_2(\beta_2).$$

Recall that $\ell_i(\beta)$ is the sum of lengths of x_i-segments[8] in β.

This is trivial if the closed polygon $\overline{U(\beta)}$ is a rectangle (put, e.g., $\beta_3 := \beta_1$). In the sequel we assume that β has at least six segments. In particular, β has at least one *outer angle* vertex meaning that the angle of $\overline{U(\beta)}$ associated to this vertex equals $-\frac{\pi}{2}$ and we call a vertex of $\overline{U(\beta)}$ an *inner angle* one if the associated angle equals $\frac{\pi}{2}$.

Now let $\lambda_i = [v_i, v_{i+1}]$, $i = 1, 2, 3$, be the sequential segments of β, i.e., vertex v_{i+1} is common for λ_i and λ_{i+1}, $i = 1, 2$. We say that $(\lambda_1, \lambda_2, \lambda_3)$ forms a *marked triple* if the following holds:

(a) v_2 and v_3 are inner angle vertices of β.

(b) The orthogonal projection of v_1 onto the straight line containing λ_3 lies in λ_3.

The simplest example of this object is any triple of sides of a rectangle.

Claim. For every segment λ of β there exists a marked triple $(\lambda_1, \lambda_2, \lambda_3) \subset \beta$ such that $\lambda \ne \lambda_i$, $i = 1, 2, 3$.

We use induction on the number of segments. If $U(\beta)$ is a rectangle, the result is evident. Otherwise, this number is at least six and therefore β has an outer vertex. Let v be such a vertex with the maximal vertical coordinate. We extend the horizontal segment ending at v inside $\overline{U(\beta)}$ and denote by w the closest to v vertex of the intersection of this extension and β. The intersection denoted by $[w, w']$ is a horizontal segment or a point, where $w' = w$ in the latter case. Then the points v, w divide β into two bolts denoted by β_1 and β_2. Since v has the maximal x_2-coordinate, one of the polygons $P_i := \overline{U}\big(\beta_i \cup [v, w]\big)$, $i = 1, 2$, say, P_1, is a rectangle, see Figure 2.2.

First, let the chosen segment λ belong to β_2, and λ_i be a side of P_1 that does not contain $[v, w]$, $i \in \{1, 2, 3\}$. If $w' \ne w$, then $(\lambda_1, \lambda_2, \lambda_3)$ is the required marked triple. Otherwise, we extend the side ending at w (say, λ_3) down to the closest

[8] In the sequel, an x_i-segment is said to be *horizontal* if $i = 1$ and *vertical* if $i = 2$.

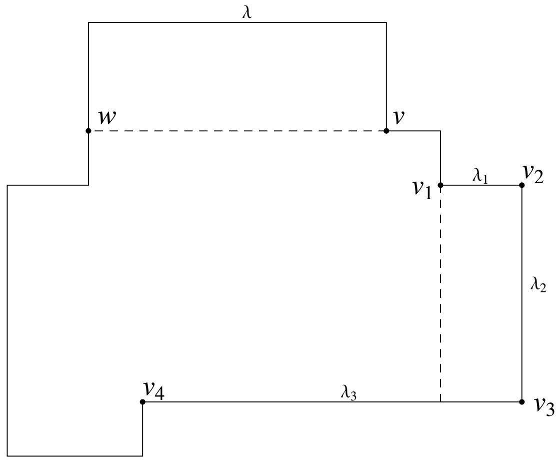

Figure 2.2: Marked triple $(\lambda_1, \lambda_2, \lambda_3)$.

to w vertex w' of β. The triple $\big(\lambda_1, \lambda_2, \lambda_3 \cup [w, w']\big)$ is then the required marked triple.

Now, let λ belong to β_1. The closed bolt $\widetilde{\beta}_2 := \beta_2 \cup [v, w]$ has clearly less segments than β. By the induction hypothesis there exists a marked triple $\big(\lambda_1, \lambda_2, \lambda_3\big)$ of the bolt $\widetilde{\beta}_2$ which does not contain its segment determined by $[v, w]$. If $w' \neq w$, then $\big(\lambda_1, \lambda_2, \lambda_3\big)$ is a marked triple of β which does not contain λ. Otherwise, we extend the segment containing w, say λ_3, up to the closest to w vertex w'' of β. Then the triple $\big(\lambda_1, \lambda_2, \lambda_3 \cup [w, w'']\big)$ is required and the result follows.

Now we complete the proof of the lemma using again induction on the number of segments. Let β have at least six segments and $\lambda_i = [v_i, v_{i+1}],\ i = 1, 2, 3,$ be a marked triple of β which does not contain the given point x. We assume, for definiteness, that $\ell(\lambda_1) \leq \ell(\lambda_3)$ and denote by v_1' the projection of v_1 onto λ_3. Then we replace $\bigcup\limits_{i=1}^{3} \lambda_i$ by $[v_1, v_1'] \cup [v_1', v_4]$ and denote by β' the closed bolt obtained in this way. Further, we consider two cases, see Figure 2.3 below.

Assume first that the point $y \in \beta'$, and denote by β_1' and β_2' the parts of β' determined by x and y. Clearly, β' has less segments than β. Therefore, by the induction hypothesis, there exists a bolt $\beta_3 \subset \overline{U(\beta')}$ joining x and y and such that

$$\ell(\beta_3) \leq \ell_1(\beta_1') + \ell_2(\beta_2').$$

Since $\ell_i(\beta_i') \leq \ell_i(\beta_i)$ and $\overline{U(\beta')} \subset \overline{U(\beta)}$, β_3 is the required curve.

Secondly, let $y \notin \beta'$ and therefore belongs to $\lambda_1 \cup \lambda_2 \cup [v_3, v_1']$. Without loss of generality we may assume that $v_1 \in \beta_1,\ v_1' \in \beta_2$ and λ_1 is horizontal. The points x and v_1' divide the bolt β' into parts denoted by β_1' and β_2'. We choose this notation so that in a neighborhood of x the polygonal curve β_i' coincides with $\beta_i,\ i = 1, 2$. It is easily seen that $\ell_i(\beta_i') \leq \ell_i(\beta_i),\ i = 1, 2$.

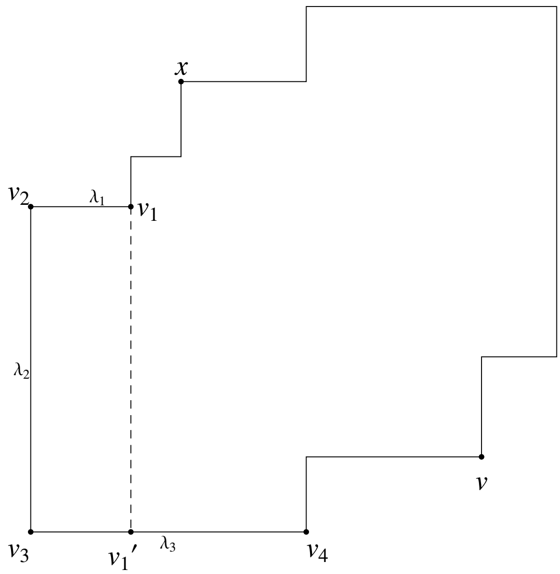

Figure 2.3: First case: $y = v$. Second case: $y = v_2$ or v_3.

We construct the required bolt β_3 by the juxtaposition of two bolts β_3' and β_3'' defined as follows. The former joins v_1 and x, contained in $\overline{U(\beta')}$ and satisfies the inequality

$$\ell(\beta_3') \leq \ell_1(\beta_1') + \ell_2(\beta_2').$$

This exists by the induction hypothesis.

Further, β_3'' is the smallest bolt joining v_1 and y inside the closed rectangle with vertices v_1, v_2, v_3, v_1'. In accordance with this definition

$$\ell(\beta_3'') \leq \ell_1(\beta_1) - \ell_1(\beta_1') + \ell_2(\beta_2) - \ell_2(\beta_2').$$

Then $\beta_2 := \beta_3' \cup \beta_3''$ satisfies the required inequality

$$\ell(\beta_3) \leq \ell_1(\beta_1) + \ell_2(\beta_2),$$

joins x and y and is contained in the polygon $\overline{U(\beta)}$, as required. $\square$

The proof of Theorem 2.69 is complete. $\square$

2.5.2 Lipschitz domains

For this subclass of the class of quasiconvex domains Theorem 2.64 may be strengthened in two respects. Firstly, the extension theorem can be proved for all spaces $\dot{\Lambda}^{k,\omega}(G)$ (without any restriction on ω) and the same may be done for $C^k\dot{\Lambda}^{s,\omega}(G)$. Secondly, the corresponding extension operator is "universal" in the sense that it extends simultaneously the family of all Lipschitz spaces $\dot{\Lambda}^{k,\omega}(G)$ with $1 \le k \le N$, where N is arbitrary (but the seminorm for $k = n$ tends to infinity as $N \to \infty$).

We discuss an approach leading to these results and formulate the main facts obtained. The corresponding proofs will only be outlined.

We introduce the required class of domains using the following more general concept, see, e.g., Stein [Ste-1970, Sec. VI.3.3].

Definition 2.71. Let $G \subset \mathbb{R}^n$ be an open set. We say that its boundary ∂G is *minimally smooth* if there exist numbers $\varepsilon,\ L > 0$, integers $N, M \ge 1$ and a sequence of open sets $G_j,\ j \in \mathbb{N}$, such that

(a) the ε-neighborhoods $(G_j)_\varepsilon,\ j \in \mathbb{N}$, cover the boundary ∂G;

(b) the order of the cover $\{G_j\}_{j\in\mathbb{N}}$ is at most M;

(c) for every j, there is a special Lipschitz domain D_j with the Lipschitz constant at most L such that
$$G_j \cap G = G_j \cap D_j.$$

A domain $D \subset \mathbb{R}^n$ is *special Lipschitz with the Lipschitz constant* $L(D)$ if D is a subgraph of a Lipschitz function f with $L(f) \le L(D)$ defined on a hypersubspace of $\mathbb{R}^n$. In other words, in a suitable coordinate system, f is defined on $\mathbb{R}^{n-1}$ and $D := \big\{(x, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}\,;\, x_n < f(x)\big\}$.

Definition 2.72. A domain $G \subset \mathbb{R}^n$ is said to be Lipschitz if G is bounded and has a minimally smooth boundary as in Definition 2.71.

This class of Lipschitz domains is denoted by $\mathcal{L}ip$.

The boundary of such a domain is *locally Lipschitz*, i.e., can be locally represented as the graph of a Lipschitz function defined on some open ball of $\mathbb{R}^{n-1}$. This follows from a simple fact noticed by Gagliardo [Ga-1958]. For its formulation we denote by $K := K(e, r, \varphi)$ the cone with axis in the direction $e \in \mathbb{S}^{n-1}$, apex at 0, height $r > 0$ and angle $0 < \varphi < \frac{\pi}{2}$; that is,
$$K := \big\{x \in \mathbb{R}^n\,;\, \|x\|\cos\varphi < x\cdot e < r\big\}.$$
Recall that $x \cdot y$ and $\|x\|$ stand for the standard scalar product and norm in $\mathbb{R}^n$, respectively.

Proposition 2.73. *A domain $G \subset \mathbb{R}^n$ belongs to the class $\mathcal{L}ip$ if and only if, for some integer N, there are open balls B_i centered at ∂G and cones K_i, $1 \le i \le N$, such that*

(a) $\{B_i\}$ *covers* ∂G;

(b) *for every* $1 \le i \le N$,

$$(B_i \cap \partial G) + K_i \subset G.$$

Proof. Because of boundedness of G we may take only a finite number of sets G_i and D_i in Definition 2.71 and then note that for every special domain D_i there is a cone K_i satisfying $D_i + K \subset D_i$. □

The locally Lipschitz structure of ∂G may be derived from here as follows. The surface of the shifted cone $x + K_i$ is clearly the graph of a Lipschitz function f_x with Lipschitz constant $c = c(K_i) > 0$ which is defined on a ball in $\mathbb{R}^{n+1}$. Extending f_x to $\mathbb{R}^{n-1}$ with the same Lipschitz constant and preserving the notation f_x for the extended function, we represent the part $B_i \cap \partial G$ of the boundary as the graph of the function $f := \sup\{f_x \,;\, x \in B_i \cap \partial G\}$. But f is Lipschitz as the supremum of Lipschitz functions with uniformly bounded Lipschitz constants.

From Proposition 2.73 we also easily derive that

$$\mathcal{L}ip \subset \mathcal{C} \tag{2.132}$$

where $\mathcal{C}$ stands for the class of quasiconcave domains (see Definition 2.63). However, there are bounded quasiconcave domains which are not Lipschitz. As an example we point out the domain G_λ following Definition 2.63.

Now we discuss the aforementioned extension results for $\dot{\Lambda}^{k,\omega}(G)$ with $G \in \mathcal{L}ip$ and an arbitrary k-majorant ω. To describe the main features of the extension method we begin with the case of a special Lipschitz domain $G \subset \mathbb{R}^n$. So let

$$G := \big\{(x, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}\,;\, x_n < \varphi(x)\big\}, \tag{2.133}$$

where $\varphi \in \mathrm{Lip}(\mathbb{R}^{n-1})$.

We use for such G the so-called "mirror reflection" method originated by L. Lichtenstein [Lich-1929] and developed to full extent by Hestens [Hes-1941]. This gives the following result (a special case of the Calderón–Stein theorem, see [Ste-1970, Sec. VI.3.2.1]).

Theorem 2.74. *Let G be given by (2.133) and $N \ge 1$ be a fixed integer. There is a linear extension operator $T_N : C_b(G) \to C_b(\mathbb{R}^n)$ such that for every $k \in \{0, 1, \dots, N\}$,*

$$T_N\big(C_b^k(G)\big) \subset C_b^k(\mathbb{R}^n)$$

and, moreover, for some $c = c(N, G) > 0$,

$$\max_{|\alpha|=k} \sup_{\mathbb{R}^n} |D^\alpha T_N f| \le c \max_{|\alpha|=k} \sup_{G} |D^\alpha f|. \tag{2.134}$$

For G an open half-space given by (2.133) with $\varphi = 0$, the Lichtenstein–Hestens operator T_N for $(x, x_n) \in \mathbb{R}^{n-1} \times (0, \infty)$ is given by

$$T_N f(x, x_n) := \int_0^\infty f(x - \lambda x_n)\psi(\lambda)d\lambda, \tag{2.135}$$

where $\psi : (0, \infty) \to \mathbb{R}$ is a compactly supported function with moments

$$\int_0^\infty \lambda^k \psi(\lambda) d\lambda = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } 1 \le k \le N. \end{cases}$$

For a detailed proof for special Lipschitz domains see the above cited section of Stein's book.

Remark 2.75. In his proof, Stein exploits a function ψ decreasing at infinity faster than any power λ^{-N}, $N \ge 1$, and having zero moments for all $k = 1, 2, \dots$. Using such a ψ he obtains a linear extension operator T_∞ such that

$$T_\infty\big(C_b^\infty(G)\big) \subset T_\infty\big(C_b^\infty(\mathbb{R}^n)\big).$$

Here $C_b^\infty(G) := \bigcap\limits_{k \ge 0} C_b^k(G)$.

The second basic ingredient of the extension method is the equivalence of the so-called K-functional of the pair $C_b(G), \dot{C}_b^s(G)$ and the s-modulus of continuity $\omega_s(\cdot\,; f)_G$. Let us recall the definition of the former concept introduced by J. Peetre [Peet-1963].

Let X_0, X_1 be a *Banach couple*, i.e., a pair of Banach spaces linearly and continuously embedded into a topological vector space. This embedding allows us to define the sums $x_0 + x_1$ with $x_i \in X_i$.

Definition 2.76. The K-functional of a Banach couple X_0, X_1 is a function on $(X_0 + X_1) \times (0, +\infty)$ given by

$$K\big(t\,;\, x\,;\, X_0, X_1\big) := \inf_{x = x_0 + x_1} \big\{\|x_0\|_{X_0} + t\|x_1\|_{X_1}\big\}. \tag{2.136}$$

We will also use a modification of this definition when one or both spaces X_i are only seminormed (but complete).

The function K is a Banach norm on $X_0 + X_1$ for every fixed $t > 0$, and is a nondecreasing concave function in t, see, e.g., Bergh and Löfström [BLo-1976, Sec. 3.3.1].

Now let $X_0 := C_b(G)$ and $X_1 := \dot{C}_b^s(G)$; in this case, we use the notation $K_s(t\,;\, f\,;\, G)$, i.e.,

$$K_s(t\,; f\,;\, G) := \inf_{f = f_0 + f_1} \Big\{\sup_G |f| + t \max_{|\alpha| = s} \sup_G |D^\alpha f|\Big\}.$$

The following result was proved in an equivalent form by J. L. Lions and Peetre [LP-1964] for $G := \mathbb{R}^n$. The proof presented below is based on the method of the paper [Br-1964] by Yu. Brudnyi.

Theorem 2.77. *Let $G \subset \mathbb{R}^n$ be a special Lipschitz domain and $f \in C_b(G)$. Then, for every $s \ge 1$,*

$$K_s(t^s\,; f\,; G) \approx \omega_s(t\,; f)_G, \quad t > 0, \tag{2.137}$$

with constants of equivalence independent of f and t.

Proof. If G is given by (2.133), then there exists an *infinite* cone $K_\infty := K(e_0, \infty, \psi)$ such that

$$G + K_\infty \subset G. \tag{2.138}$$

Now let e be a unit vector in K_∞ and $t > 0$. We define an operator $S_t(e)$ on functions $f \in C_b(G)$ by

$$S_t(e)f := \frac{1}{t^s} \int_0^t \dots \int_0^t \sum_{j=1}^{s} c(j) f(\cdot + j\tau e) d\tau_1 \dots d\tau_s, \tag{2.139}$$

where $c(j) := (-1)^{s-j}\binom{s}{j}$ and $\tau := \tau_1 + \cdots + \tau_s$. Due to (2.138) and the choice of e, this operator acts continuously from $C_b(G)$ into $C_b(G)$.

Transforming the multiple integral, we rewrite (2.139) as

$$S_t(e)f(x) = \int_0^1 \big(f(x) - \Delta^s_{t\tau e} f(x)\big)\vartheta(\tau) d\tau, \tag{2.140}$$

where ϑ is a certain bounded function.

Moreover, $S_t(e)f$ is clearly s-times continuously differentiable on G and its derivative D^s_e in direction e equals

$$D^s_e S_t(e) f = \left(\sum_{j=1}^{s} d(j) \Delta^s_{js^{-1}te} f\right) \Big/ t^s, \tag{2.141}$$

where $d(j) := (sj)^{-1} c(j)$.

Choosing now $N := \begin{pmatrix} h+s+1 \\ s \end{pmatrix}$ vectors e_j in the cone K_∞ we set

$$S_t := \prod_{1 \le j \le N} S_t(e_j).$$

Then $S_t : C_b(G) \to C_b(G)$ and

$$f - S_t f = \sum_{j=1}^{N} \left(\prod_{i \ge j+1} S_t(e_i)\right) \int_0^1 (\Delta^s_{t\tau e_j} f)\vartheta(\tau) d\tau,$$

where the product is assumed to be the identity operator if $j + 1 > N$.

This immediately leads to the inequality

$$\sup_G |f - S_t f| \le O(1)\omega_s(t\,; f)_G. \tag{2.142}$$

Now we use identity (2.43) for mixed derivatives D^α with $|\alpha| = s$ which gives

$$D^\alpha = \sum_{j=1}^{N} a_j D^s_{e_j},$$

where a_j are some constants. Using this and (2.141) we have

$$\begin{aligned}\left|D^\alpha(S_t f)(x)\right| &\le \sum_{j=1}^{N} |a_j| \left| \left(\prod_{i\neq j} S_t(e_i) \right) D^s_{e_j}\big(S_t(e_j)f\big)(x) \right| \\ &\le O(1)t^{-s} \sup_{1\le i\le s} \left|\Delta^s_{is^{-1}te} f(x)\right| \le O(1)t^{-s}\omega_s(t\,; f)_G .\end{aligned}$$

Combining this with (2.142) we obtain

$$K_s\big(t^s\,; f\,; G\big) \le \sup_G \big| f - S_t f\big| + t^s \max_{|\alpha|=s} \sup_G \big| D^\alpha S_t f\big| \le O(1)\omega_s(t\,; f)_G .$$

The converse inequality is easy. In fact, if $f = f_0 + f_1$ and $f_1 \in \dot{C}^s_b(G)$, then the known properties of ω_s imply

$$\omega_s(t\,; f)_G \le \omega_s(t\,;\, f_0)_G + \omega_s(t\,;\, f_1)_G \le 2^s \sup_G |f_0| + t^s \max_{|\alpha|=s} \sup_G |D^\alpha f_1|.$$

Taking here the infimum over all decompositions $f = f_0 + f_1$ we get the desired inequality

$$\omega_s(t\,; f)_G \le O(1)K_s(t^s\,; f\,; G). \tag{2.143}$$

□

Remark 2.78. (a) In fact, we have shown that the couple $\big(C_b(G), \dot{C}^s_b(G)\big)$ is K-*linearized* meaning that there is the family $\{S_t\}_{t>0}$ of *linear* operators realizing nearly optimal decompositions for $K_s(t^s\,;\, f\,;\, G)$:

$$f = (1 - S_t)f + S_t f.$$

In general, a Banach couple (X_0, X_1) is said to be K-*linearized* if there exist families of linear operators $\{S^i_t\}_{t>0}$ acting from X_0+X_1 into X_i, $i = 0, 1$, such that $S^0_t + S^1_t = Id_{X_0+X_1}$ for each $t > 0$, and for every $x \in X_0 + X_1$ and $t > 0$,

$$K(x\,;\, t\,;\, X_0, X_1) \approx \|S^0_t x\|_{X_0} + t\|S^1_t x\|_{X_1}$$

with constants independent of x and t.

(b) Inequality (2.143) is true for *any* open set G. The converse inequality does not hold even for domains with good extension properties, e.g., for bounded uniform domains, see Definition 2.81 below.

As a consequence of Theorems 2.74 and 2.77 we now obtain the desired extension result.

Theorem 2.79. *Let $G \subset \mathbb{R}^n$ be a special Lipschitz domain and $N \geq 1$ be an integer. Then the linear extension operator T_N of Theorem* 2.74 *maps every homogeneous Lipschitz space $\dot{\Lambda}^{s,\omega}(G)$ into $\dot{\Lambda}^{s,\omega}(\mathbb{R}^n)$, $1 \leq s \leq N$. Moreover, for some $c = c(N,G) > 0$ and every $f \in \dot{\Lambda}^{s,\omega}(G)$,*

$$|T_N f|_{\Lambda^{s,\omega}(\mathbb{R}^n)} \leq c|f|_{\Lambda^{s,\omega}(G)}.$$

Proof. Let $f \in C_b(G)$ and $f = f_0 + f_1$ be such a decomposition that

$$\sup_G |f_0| + t^s \max_{|\alpha|=s} \sup_G \big|D^\alpha f_1\big| \leq O(1)\omega_s(t\,;\, f)_G, \tag{2.144}$$

see Theorem 2.77. By the definition of ω_s, we have for the function $T_N f = T_N f_0 + T_N f_1$,

$$\omega_s(t\,; T_N f) \leq O(1)\Big\{\sup_{\mathbb{R}^n} |T_N f_0| + t^s \max_{|\alpha|=s} \sup_{\mathbb{R}^n}\big|D^\alpha T_N f_1\big|\Big\}.$$

Estimating the right-hand side first by (2.134) and then by (2.144) we get

$$\omega_s(t\,; T_N f) \leq O(1)\omega_s(t\,; f)_G.$$

Dividing this by an s-majorant ω and taking the supremum over $t > 0$ we obtain the required inequality

$$|T_N f|_{\Lambda^{s,\omega}(\mathbb{R}^n)} \leq O(1)|f|_{\Lambda^{s,\omega}(G)}. \qquad \square$$

Remark 2.80. In other words, this theorem asserts that for some $c = c(N,G)$ and every $f \in C_b(G)$, $t > 0$ and $1 \leq s \leq N$,

$$\omega_s(t\,; T_N f) \leq c\,\omega(t\,; f)_G.$$

It is highly probable that this result is true for $N = \infty$.

The situation with Lipschitz domains is more complicated. According to the Calderón–Stein theorem, the extension operator analogous to that of Theorem 2.74 exists in this case even for open sets with minimally smooth boundaries, see [Ste-1970, Sec. VI.3.1]. However, inequality (2.134) is replaced by

$$\max_{|\alpha|=s} \sup_{\mathbb{R}^n}\big|D^\alpha T_N f\big| \leq O(1)\Big\{\max_{|\alpha|=s} \sup_G |D^\alpha f| + \sup_G |f|\Big\}, \tag{2.145}$$

where $0 \leq s \leq N$. We might eliminate the second term here by assuming that T_N preserves polynomials of degree $N-1$. Applying, in this case, (2.145) to the function $f - p$, where p is a polynomial of degree $s-1$, and taking the infimum over all such p, we replace the second summand in (2.145) by

$$E_s(G\,; f) := \inf\Big\{\sup_G |f - p|\,;\, p \in \mathcal{P}_{s-1,n}\Big\}.$$

This, in turn, is bounded above, for a bounded Lipschitz domain G, by $O(1)(\operatorname{diam} G)^s \max_{|\alpha|=s} \sup_G |D^\alpha f|$, see, e.g., the paper [BrH-1970] by Bramble and Hilbert. Therefore, (2.145) can be rewritten as

$$\max_{|\alpha|=s} \sup_{\mathbb{R}^n} \big|D^\alpha T_N f\big| \leq O(1) \max_{|\alpha|=s} \sup_G |D^\alpha f|.$$

Further, Theorem 2.77 is also true for Lipschitz domains, see the papers [Br-1976] by Yu. Brudnyi and [JSch-1977] by H. Johnen and K. Scherer.

Thus, the version of Theorem 2.79 for bounded Lipschitz domains would be true, if the extension operator T_N preserved polynomials of degree $N-1$. Unfortunately, the Calderón–Stein operator does not possess this property. Their construction is modified as required in the paper [Br-1980] by Yu. Brudnyi where Theorem 2.79 is proved also for bounded Lipschitz domains.

2.6 Sobolev spaces: selected trace and extension results

2.6.1 P. Jones' theorem and related results

The local polynomial approximation methods partially discussed in Section 2.3 have a much wider range of applications. In particular, they may be applied to the study of trace and extension problems for spaces of weakly differentiable functions such as Sobolev or Besov spaces. Unfortunately, the corresponding results remained in manuscript form or were published in almost unaccessible journals. In Chapter 9, we present several applications of this approach to the extension and trace problems for Lipschitz functions of higher order, while in the Comments to that chapter we formulate several results contained in the aforementioned sources. In particular, we prove the extension theorem for the space $\dot{\Lambda}^{k,\omega}(G)$, where ω is a quasipower majorant and $G \subset \mathbb{R}^n$ is a so-called *uniform domain*. The class $\mathcal{U}$ of these domains plays a considerable role in Analysis; for now we only formulate the corresponding definition leaving a detailed discussion to Chapter 9.

Let $\gamma : [0,1] \to \mathbb{R}^n$ be a rectifiable curve with the endpoints x, y. A *(curvilinear) cone with axis* γ *and parameter* $c > 0$ is the open set

$$Kn(\gamma\,;\, c) := \bigcup \big\{ B_{c\rho(z)}(z)\,;\, z \in \gamma\big([0,1]\big) \big\}, \tag{2.146}$$

where

$$\rho(z) := \min\big\{ \|x - z\|,\, \|y - z\| \big\}. \tag{2.147}$$

Such a cone is also defined for the case of $y = \infty$ (i.e., for γ being unbounded and locally rectifiable).

Definition 2.81. A domain $G \subset \mathbb{R}^n$ is said to be (c_0, c_1)-uniform if for every pair $x, y \in G$ satisfying

$$\|x - y\| \leq c_0$$

there is a curve γ joining x and y in G such that the length of γ satisfies

$$\ell(\gamma) \leq c_1 \|x - y\| \tag{2.148}$$

and for the associated cone it is true that

$$Kn(\gamma\,;\, c_1) \subset G. \tag{2.149}$$

Due to (2.148) G is locally quasiconvex, but Example 2.79 (c) below shows that the class $\mathcal{U}$ is a proper subclass of the class of locally quasiconvex domains.

The aforementioned extension result for $\dot{\Lambda}^{k,\omega}(G)$, see Theorem 9.51 of Volume II, and the argument of Theorem 2.32, see also Remark 2.33, lead for $p = \infty$ to the isomorphism

$$W_p^k(G) = W_p^k(\mathbb{R}^n)\big|_G, \tag{2.150}$$

provided that G is a uniform domain.

Let us recall that the Sobolev space $W_p^k(G)$ is defined by the norm

$$\|f\|_{W_p^k(G)} := \max_{|\alpha| \leq k} \|D^\alpha f\|_{L_p(G)}. \tag{2.151}$$

Here the mixed weak derivative $D^\alpha f$ is an L_p-function such that for every C^∞ function φ supported on a compact subset of G,

$$\int_G f \cdot D^\alpha \varphi dx = (-1)^{|\alpha|} \int_G D^\alpha f \cdot \varphi dx.$$

The following fundamental fact was due to P. Jones [Jon-1981].

Theorem 2.82. *Let $G \subset \mathbb{R}^n$ be a (c_0, c_1)-uniform domain. Then there is a linear continuous extension operator from $W_p^k(G)$ to $W_p^k(\mathbb{R}^n)$.*

The P. Jones theorem leads to the isomorphism (2.150) for all $1 \leq p \leq \infty$.

There are examples due to Maz'ya [Maz-1981] showing that the uniformity of G is not necessary for validity of this result. However, for planar simply connected domains and $kp = 2$ the uniformity of G is also necessary. This result was due to V. Gol'dshtein and Vodop'yanov for $k = 1$ and $p = 2$, see [GV-1980] and [GV-1981]; a generalization including the cases $k = 2$ and $p = 1$ was then proved by Christ [Chr-1984].

Theorem 2.83. *Let $G \subset \mathbb{R}^2$ be a simply connected domain. Then for $kp = 2$ equality* (2.150) *holds up to the equivalence of the seminorms if and only if G is a uniform domain.*

Proof. In view of Theorem 2.82, it suffices to prove that if (2.150) holds for $n = 2$ and $kp = 2$, then G is uniform. To establish this we need the following characteristics of uniform domains, see Gehring [Ge-1982, Thm. 3.6].

Proposition 2.84. *Assume that $G \subset \mathbb{R}^2$ is uniform. Then there is a constant $\lambda \geq 1$ such that for every $x \in \mathbb{R}^2$ and $r \in (0, \infty)$ the following holds:*

(a) *every pair of points of $G \cap B_r(x)$ can be joined in $G \cap \overline{B}_{\lambda r}(x)$;*

(b) *every pair of points of $G \backslash B_r(x)$ can be joined in $G \backslash \overline{B}_{\frac{r}{\lambda}}(x)$.*

Here "joined" means joined by a rectifiable curve lying within the specified set.

Assume now that (2.151) holds with the specified k and p but G is not uniform. Then according to Proposition 2.84 there are closed disks denoted by $D_r(x^0)$ and $D_R(x^0)$ with an arbitrary large ratio $\frac{R}{r}$ of their radii satisfying one of the following conditions:

(i) $G \cap D_R(x^0)$ contains two different connected components G_0, G_1 for which

$$D_r(x^0) \cap G_j \neq \emptyset, \quad j = 0, 1.$$

(ii) $G \backslash D_r(x^0)$ contains two different connected components G_0, G_1 for which

$$\big(\mathbb{R}^2 \backslash D_R(x^0)\big) \cap G_j \neq \emptyset, \quad j = 0, 1.$$

To derive a contradiction we need

Proposition 2.85. *Suppose that a domain G satisfies either (i) or (ii) with a fixed $r < \frac{R}{4}$. Suppose also that for every $f \in \dot{W}_p^k(G)$, where $kp = 2$, there exists an extension $F \in \dot{W}_p^k(\mathbb{R}^2)$ of f satisfying*

$$|F|_{W_p^k(\mathbb{R}^2)} \leq A|f|_{W_p^k(G)}. \tag{2.152}$$

Then the inequality

$$C\Big(\log \frac{R}{r}\Big)^{\frac{1}{p}} \leq A \tag{2.153}$$

holds with a numerical constant $C > 0$.

Before proving this, we derive the theorem from (2.153). If (2.150) holds for the domain G with $kp = 2$, then (2.152) does as well. But since G is non-uniform, one of the two conditions (i), (ii) holds for an arbitrary large $\frac{R}{r}$. This contradicts (2.153).

It remains to prove Proposition 2.85. We begin with the following auxiliary result.

Lemma 2.86. *Let $f \in C^\infty(G)$ and let γ be a rectifiable curve of length ℓ joining fixed points x and y in the domain G. Then for the Taylor polynomial $T_y^{k-1} f$ it is true that*

$$\big|f(x) - T_y^{k-1} f(x)\big| \leq C(k, n)\ell^{k - \frac{1}{p}} \bigg(\sum_{|\alpha| = k} \int_\gamma |D^\alpha f|^p ds\bigg)^{\frac{1}{p}}; \tag{2.154}$$

here $1 \leq p \leq \infty$.

Proof. It suffices to prove the inequality for a polygonal curve $\gamma \in G$. We apply to each segment $[a_i, a_{i+1}]$ of γ the inequality

$$\big|\big(T_u^{k-1} f - T_v^{k-1} f\big)(z)\big| \le C(k,n)\big(\|u-z\| + \|z-v\|\big)^{k-1} \sum_{|\alpha|=k} \int_{[u,v]} |D^\alpha f| ds;$$

here $[u,v] \subset G$ and $z \in \mathbb{R}^n$. Its proof is a simple modification of the proof of Lemma 2.65 and can be left to the reader. Taking $u = a_i,\ v = a_{i+1}$ and summing the inequalities obtained, we get

$$\big|f(x) - T_y^{k-1} f(x)\big| \le C(k,n) \sum_i \big(\|x-a_i\| + \|x-a_{i+1}\|\big)^{k-1} \sum_{|\alpha|=k} \int_{[a_i,a_{i+1}]} |D^\alpha f| ds.$$

Since $\|x-a_i\| + \|x-a_{i+1}\| \le 2\ell$, the right-hand side does not exceed

$$2^{k-1} \cdot C(k,n) \ell^{k-1} \sum_{|\alpha|=k} \int_\gamma |D^\alpha f| ds.$$

Estimating the integral in the right-hand side by the Hölder inequality we obtain the desired result. $\square$

Now let G_0, G_1 be domains in $\mathbb{R}^2$ satisfying the following condition: For every $t \in (r,R)$ the circle $C_t := \partial D_t(0)$ satisfies

$$G_j \cap C_t \ne \emptyset, \quad j = 0,1. \tag{2.155}$$

Denote by $A_{r,R}$ the circular annulus $\big\{x \in \mathbb{R}^2\,;\, r < \|x\| < R\big\}$. In this setting, the following holds.

Lemma 2.87. *Let $f \in \dot W_p^k(\mathbb{R}^2)$ and $kp = 2$. Assume that*

$$f\big|_{G_j \cap A_{r,R}} = j, \quad j = 0,1. \tag{2.156}$$

Then there is a constant $C > 0$ such that

$$|f|_{W_p^k(\mathbb{R}^2)} \ge C\Big(\log \frac{R}{r}\Big)^{\frac{1}{p}}.$$

Proof. Since $p < \infty$, the set $C_0^\infty(\mathbb{R}^2)$ is dense in $\dot W_p^k(\mathbb{R}^2)$, see, e.g., Maz'ya [Maz-1985], we can assume that $f \in C_0^\infty(\mathbb{R}^2)$. By the Fubini theorem,

$$\int_{A_{r,R}} \Bigg(\sum_{|\alpha|=k} |D^\alpha f|^p\Bigg) dx = \int_r^R \Bigg(\sum_{|\alpha|=k} \int_{C_t} |D^\alpha f|^p ds\Bigg) dt.$$

According to (2.155), there exist points $x_j(t) \in G_j \cap C_t,\ j = 0, 1$. Since $f = 0$ in the neighborhood of $x_0(t)$, the Taylor polynomial $T^{k-1}_{x_0(t)} f$ equals 0. Moreover, $f(x_1(t)) = 1$, see (2.156). Hence, by (2.154),

$$1 = \left| f(x_1(t)) - T^{k-1}_{x_0(t)} f(x_1(t)) \right|$$
$$\leq C_1 \ell(C_t)^{k-\frac{1}{p}} \left(\sum_{|\alpha|=k} \int_{C_t} |D^\alpha f|^p \, ds \right)^{\frac{1}{p}} = C_1 (2\pi t)^{\frac{1}{p}} \left(\sum_{|\alpha|=k} \int_{C_t} |D^\alpha f|^p ds \right)^{\frac{1}{p}}.$$

Combining this inequality with the previous identity we get

$$|f|_{W^k_p(\mathbb{R}^2)} \geq \left(\int_r^R \frac{1}{C_1^p 2\pi t} \, dt \right)^{\frac{1}{p}} = C_2 \left(\log \frac{R}{r} \right)^{\frac{1}{p}},$$

and the result follows. $\square$

Now we are ready to prove Proposition 2.85. Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ be a test-function satisfying

$$\varphi\big|_{D_{\frac{1}{2}}(0)} = 1, \quad \operatorname{supp}(\varphi) \subset D_{\frac{3}{4}}(0).$$

Consider first the case of the connected component of $G \cap D_R(x^0)$ for which the corresponding G_0, G_1 satisfy condition (i) with $r < \frac{R}{4}$. Let us define a function $f : G \to \mathbb{R}$ by

$$f(x) := \begin{cases} \varphi\left(\dfrac{x - x^0}{R} \right) & \text{for } x \in G_1, \\ 0 & \text{for } x \in G \backslash G_1. \end{cases}$$

Since $\operatorname{supp} \varphi \left(\dfrac{\cdot - x^0}{R} \right) \subset D_{\frac{3}{4}R}(x^0)$, the function f is well defined and belongs to $C_0^\infty(G)$. By its definition and the equality $kp = 2$,

$$|f|_{W^k_p(G)} \leq \left| \varphi\left(\frac{\cdot - x^0}{R} \right) \right|_{W^k_p(\mathbb{R}^2)} = R^{\frac{2}{p} - k} |\varphi|_{W^k_p(\mathbb{R}^2)} = |\varphi|_{W^k_p(\mathbb{R}^2)}.$$

In the case of condition (ii), we define f as above but replace 0 by 1. Since $\varphi\left(\dfrac{x - x^0}{2r} \right) = 1$ for $x \in D_r(x^0)$, the function f is well defined and belongs to $C_0^\infty(G)$. In this case, we have a similar estimate:

$$|f|_{W^k_p(G)} = |f|_{W^k_p(G \backslash G_1)} = (2r)^{\frac{2}{p} - k} |\varphi|_{W^k_p(\mathbb{R}^2)} = |\varphi|_{W^k_p(\mathbb{R}^2)}.$$

Since $f \in \dot{W}^k_p(G)$, for its extension $F \in \dot{W}^k_p(\mathbb{R}^2)$ we get by (2.152)

$$|F|_{W^k_p(\mathbb{R}^2)} \leq A |\varphi|_{W^k_p(\mathbb{R}^2)}.$$

Applying now Lemma 2.87 to the function F and the annulus $x_0+A_{r,\frac{R}{2}}$ in case (i), and the annulus $x_0 + A_{2r,R}$ in case (ii), we get

$$A|\varphi|_{W_p^k(\mathbb{R}^2)} \geq |F|_{W_p^k(\mathbb{R}^2)} \geq C\left(\log \frac{R}{2r}\right)^{\frac{1}{p}}.$$

This is clearly equivalent to the required inequality (2.153).

The proof of Proposition 2.85 is complete. □

2.6.2 Peetre's nonexistence theorem

We first formulate a special case of the trace theorems for Sobolev spaces. Let $x = (x', x_n) \subset \mathbb{R}^n_+$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \geq 0$. Since C^∞ functions are dense in $W_p^k(\mathbb{R}^n_+)$ with $1 \leq p < \infty$, the trace operator $\mathrm{Tr} : f(x) \to f(x', 0)$ is defined on a dense subset of $W_p^k(\mathbb{R}^n_+)$ and can be continuously extended to the whole of this space.

The following classical result was due to Gagliardo [Ga-1957].

Theorem 2.88. Tr *is a linear continuous operator mapping* $W_1^1(\mathbb{R}^n_+)$ *onto* $L_1(\mathbb{R}^{n-1})$. *In other words,*

$$W_1^1(\mathbb{R}^n_+)\big|_{\mathbb{R}^{n-1}} = L_1(\mathbb{R}^{n-1}),$$

where the trace space is defined as the image of the trace operator.

Remark 2.89. Gagliardo's proof also gives the following result which will be used below.

If $f \in W_1^1(\mathbb{R}^n_+)$, then the function $x' \mapsto f(x', x_n)$ belongs to $L_1(\mathbb{R}^{n-1})$ for almost all x_n and the limit

$$f(x', 0) := \lim_{x_n \to 0} f(x', x_n) \quad (\text{in } L_1(\mathbb{R}^{n-1})) \tag{2.157}$$

exists and coincides with $\mathrm{Tr}\, f(x')$ almost everywhere on $\mathbb{R}^{n-1}$.

The extension operator used in the Gagliardo proof is nonlinear. The following result obtained by Peetre [Peet-1979] shows that the simultaneous extension problem is unsolvable in this case.

Theorem 2.90. *There is no linear continuous extension operator from* $L_1(\mathbb{R}^{n-1})$ *to* $W_1^1(\mathbb{R}^n_+)$ *for* $n \geq 2$.

Proof. Assume, on the contrary, that there is a linear continuous extension operator $E : L_1(\mathbb{R}^{n-1}) \to W_1^1(\mathbb{R}^n_+)$. As an element of $W_1^1(\mathbb{R}^n_+)$ the function Ef satisfies

$$\lim_{x_n \to \infty} Ef(\cdot, x_n) = 0 \text{ (convergence in } L_1(\mathbb{R}^{n-1})) \tag{2.158}$$

for every $f \in L_1(\mathbb{R}^{n-1})$, see, e.g., Maz'ya [Maz-1985]. This and (2.157) imply by integration by parts that

$$f = -\int_0^\infty (D_n Ef)(\cdot, x_n) dx_n, \tag{2.159}$$

where the integral is defined as the limit of $\int_\varepsilon^N \dots dx_n$ as $\varepsilon \to 0$ and $N \to \infty$ (convergence in $L_1(\mathbb{R}^{n-1})$).

We use this representation to find a sequence of "smoothing" operators $\{E_j\}_{j\in\mathbb{Z}}$ acting from $L_1(\mathbb{R}^{n-1})$ to $W_1^1(\mathbb{R}^{n-1})$ whose properties are given by

Lemma 2.91. (a) *For every $f \in L_1(\mathbb{R}^{n-1})$*

$$f = \sum_j E_j f \quad (\textit{convergence in } L_1(\mathbb{R}^{n-1})). \tag{2.160}$$

(b) *For some constant $C > 0$ and every f,*

$$\sum_j \|E_j f\|_{L_1(\mathbb{R}^{n-1})} \le C\|f\|_{L_1(\mathbb{R}^{n-1})}; \tag{2.161}$$

$$\sum_j 2^j |E_j f|_{W_1^1(\mathbb{R}^{n-1})} \le C\|f\|_{L_1(\mathbb{R}^{n-1})}. \tag{2.162}$$

Proof. We define the required sequence using test-functions $\chi_j : \mathbb{R}_+ \to \mathbb{R}$, $j \in \mathbb{Z}$, satisfying the conditions:

$$\operatorname{supp} \chi_j \subset S_j := \left[2^{j-2}, 2^{j+1}\right] \quad \text{and} \quad \sum_j \chi_j = 1; \tag{2.163}$$

$$\sup |\chi_j| \le C \quad \text{and} \quad \sup |\chi_j'| \le 2^{-j} C, \quad j \in \mathbb{Z}, \tag{2.164}$$

where $C > 0$ is a constant independent of j.

Then the required operator E_j is given for $f \in L_1(\mathbb{R}^{n-1})$ by

$$E_j f := -\int_{\mathbb{R}_+} \chi_j(x_n) D_n Ef(\cdot, x_n) dx_n. \tag{2.165}$$

The identity (2.160) follows from (2.159) and (2.163). The assertions in (b) are proved by the same argument; we derive only (2.162). Integrating by parts in (2.165) and applying (2.163) and (2.164), we obtain

$$|E_j f|_{W_1^1(\mathbb{R}^{n-1})} \le C \cdot 2^{-j} \int_{S_j} \sum_{i=1}^{n-1} \left\| D_i Ef(\cdot, x_n) \right\|_{L_1(R^{n-1})} \le C \cdot 2^{-j} |Ef|_{W_1^1(\mathbb{R}^{n-1}\times S_j)}.$$

Since the family $\{\mathbb{R}^{n-1} \times S_j\}_{j\in\mathbb{Z}}$ covers $\mathbb{R}^n_+$ with order (multiplicity) ≤ 4, we obtain (2.162) by summing over j. □

We further transform the sequence $\{E_j\}_{j\in\mathbb{Z}}$ into a sequence $\{\widehat{E}_j\}_{j\in\mathbb{Z}}$ of *translation invariant* linear operators on $L_1(\mathbb{R}^{n-1})$ satisfying (2.160)–(2.162). To accomplish this, we use the *invariant mean* $\mathcal{M}$ on the space $\mathcal{B}(\mathbb{R}^{n-1})$ of all functions bounded on $\mathbb{R}^{n-1}$ equipped with the uniform norm. Let us recall, see, e.g., [HR-1963], that $\mathcal{M}$ is a linear functional on $\mathcal{B}(\mathbb{R}^{n-1})$ satisfying

$$\big|\mathcal{M}[f]\big| \le \sup_{\mathbb{R}^{n-1}} |f|; \tag{2.166}$$

$$\mathcal{M}[1] = 1; \tag{2.167}$$

$$\mathcal{M}\big[f(\cdot + y)\big] = \mathcal{M}[f] \tag{2.168}$$

for all $y \in \mathbb{R}^{n-1}$.

Now let $C_0(\mathbb{R}^{n-1})$ be the space of continuous functions vanishing at infinity, i.e., for every $\varepsilon > 0$ and a function f from this space, there is a compact set $S_\varepsilon \subset \mathbb{R}^{n-1}$ such that $|f(x)| < \varepsilon$ for $x \notin S_\varepsilon$.

Given functions $\varphi \in C_0(\mathbb{R}^{n-1})$ and $f \in L_1(\mathbb{R}^{n-1})$ we define a function $\psi_j = \psi_j(f, \varphi) : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$\psi_j : h \longmapsto \int\limits_{\mathbb{R}^{n-1}} \tau_{-h}(E_j \tau_h f)\varphi dx, \tag{2.169}$$

where $\tau_h g := g(\cdot + h), \quad h \in \mathbb{R}^{n-1}$.

Since by (2.161)

$$\sup_h \big|\psi_j(h)\big| \le C\|f\|_{L_1(\mathbb{R}^{n-1})}\|\varphi\|_{L_\infty(\mathbb{R}^{n-1})}, \tag{2.170}$$

the function ψ_j belongs to $\mathcal{B}(\mathbb{R}^{n-1})$.

Applying the invariant mean $\mathcal{M}$ to ψ_j, we define the bilinear functional

$$(f, \varphi) \to \mathcal{M}\big[\psi_j(f, \varphi)\big].$$

Due to (2.166) and (2.170), this functional is continuous in φ for each $f \in L_1(\mathbb{R}^{n-1})$, and its norm is bounded by $C\|f\|_{L_1(\mathbb{R}^{n-1})}$.

By the Riesz representation theorem there is a bounded Borel measure $\mu := \mu_{f,j}$ on $\mathbb{R}^{n-1}$ such that

$$\mathcal{M}\big[\psi_j(f, \varphi)\big] = \int\limits_{\mathbb{R}^{n-1}} \varphi d\mu.$$

By inequality (2.170),

$$\|\mu\| \le C\|f\|_{L_1(\mathbb{R}^{n-1})},$$

where $\|\mu\|$ stands for the variation of μ over $\mathbb{R}^{n-1}$, i.e., $\|\mu\| := \sup\limits_{S\subset\mathbb{R}^{n-1}} |\mu(S)|$.

Thus, the correspondence $f \to \mu_{f,j}$ defines a linear bounded operator from $L_1(\mathbb{R}^{n-1})$ into the Banach space $M(\mathbb{R}^{n-1})$ of bounded Borel measures on $\mathbb{R}^{n-1}$. We denote this operator by $\widehat{E}_j$ and show that it satisfies conditions similar to those in Lemma 2.91.

Lemma 2.92. (a) *For every $f \in L_1(\mathbb{R}^{n-1})$ and the measure μ_f given by $d\mu_f := f dx$, it is true that*

$$\mu_f = \sum_j \widehat{E}_j f \text{ (convergence in } M(\mathbb{R}^{n-1})). \tag{2.171}$$

(b) *For some constant $C > 0$ and every $f \in L_1(\mathbb{R}^{n-1})$,*

$$\sum_j \big\| \widehat{E}_j f \big\|_{M(\mathbb{R}^{n-1})} \le C \|f\|_{L_1(\mathbb{R}^{n-1})}. \tag{2.172}$$

(c) *For every $h \in \mathbb{R}^{n-1}$,*

$$\tau_h \widehat{E}_j = \widehat{E}_j \tau_h,$$

i.e., $\widehat{E}_j$ is translation invariant.[9]

Proof. Assertions (a) and (b) immediately follow from the corresponding ones of Lemma 2.91 and the definition of $\widehat{E}_j$.

To prove (c), we use (2.169) to conclude that for $\varphi \in C_0(\mathbb{R}^{n-1})$ and $g, h \in \mathbb{R}^{n-1}$,

$$\begin{aligned}\langle \widehat{E}_j \tau_g f, \varphi\rangle &= \mathcal{M}\big[\langle \tau_{-h} E_j \tau_h \tau_g f, \varphi\rangle\big] \\ &= \mathcal{M}\big[\langle \tau_{-h-g} E_j \tau_{h+g} f, \tau_{-g}\varphi\rangle\big] = \mathcal{M}\big[\tau_g(\psi_j(f, \tau_{-g}\varphi))\big].\end{aligned}$$

Due to (2.168) and the translation invariance of $\mathcal{M}$, the last term of the above identity equals

$$\mathcal{M}\big[\psi_j(f, \tau_{-g}\varphi)\big] = \langle \widehat{E}_j f, \tau_{-g}\varphi\rangle := \langle \tau_g \widehat{E}_j, \varphi\rangle.$$

Combining these identities we get

$$\langle \widehat{E}_j \tau_g f, \varphi\rangle = \langle \tau_g \widehat{E}_j f, \varphi\rangle,$$

as required. □

By virtue of the equivariance of $\widehat{E}_j$ and the Riesz representation theorem, there is a measure $\mu_j \in M(\mathbb{R}^{n-1})$ such that for all $f \in L_1(\mathbb{R}^{n-1})$,

$$\widehat{E}_j f = \mu_j * f := \int_{\mathbb{R}^{n-1}} f(\cdot + y) d\mu_j(y) \quad \text{(equality in } \ M(\mathbb{R}^{n-1})).$$

Inserting this in (2.172) we get

$$\sum_j \|\mu_j * f\|_{M(\mathbb{R}^{n-1})} \le C\|f\|_{L_1(\mathbb{R}^{n-1})}.$$

[9] Recall that translation $\tau_h \mu$ of a measure μ is a linear functional defined by $\langle \tau_h \mu, \varphi\rangle := \int \tau_{-h}\varphi d\mu$, $\varphi \in C_0(\mathbb{R}^{n-1})$.

Now let $\{f_N\}_{N\in\mathbb{N}}$ be an approximate identity in $C_0(\mathbb{R}^{n-1})$, i.e., for every N and $\varepsilon > 0$,

$$\int\limits_{\mathbb{R}^{n-1}} f_N dx = 1,\ f_N \geq 0 \quad \text{and} \quad \max_{|x|>\varepsilon} f_N \to 0 \quad \text{as} \quad N \to \infty.$$

Then $\mu_j * f_N \to \mu_j$ in $M(\mathbb{R}^{n-1})$. This and the previous inequality imply that

$$\sum_j \|\mu_j\|_{M(\mathbb{R}^{n-1})} \leq C. \tag{2.173}$$

Lemma 2.93. *For each j the measure μ_j is absolutely continuous with respect to the Lebesgue $(n-1)$-measure. In other words, there is a function $g_j \in L_1(\mathbb{R}^{n-1})$ such that*

$$g_j * f = \mu_j * f \quad \textit{and} \quad \|\mu_j\|_{M(\mathbb{R}^{n-1})} = \|g_j\|_{L_1(\mathbb{R}^{n-1})}. \tag{2.174}$$

Proof. First, we show that the first-order distributional derivatives $D_i\mu_j$, $1 \leq i \leq n-1$, (tempered distributions over the Schvartz space $\mathcal{S}(\mathbb{R}^{n-1})$) are bounded Borel measures over $\mathbb{R}^{n-1}$.

Using the definition of $\widehat{E}_j$ and (2.163), we obtain, for $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})$,

$$\begin{aligned}\langle D_i\mu_j * f, \varphi\rangle &:= \langle D_i\widehat{E}_j f, \varphi\rangle := -\langle \widehat{E}_j f, D_i\varphi\rangle \\ &:= \mathcal{M}\big[\langle \tau_{-h}E_j\tau_h f, D_i\varphi\rangle\big] = \mathcal{M}\big[\langle \tau_{-h}D_i(E_j\tau_h f), \varphi\rangle\big].\end{aligned}$$

The last equality is proved by integrating by parts inside the square brackets using the fact that E_jT_hf belongs to $W_1^1(\mathbb{R}^{n-1})$. Since $D_iE_jT_hf$ belongs to $L_1(\mathbb{R}^{n-1})$ and $S(\mathbb{R}^{n-1})$ is dense in $C_0(\mathbb{R}^{n-1})$, the last term of the above equality can be extended to a linear continuous functional on $C_0(\mathbb{R}^{n-1})$. This means, see, e.g., Stein [Ste-1970, Ch. 2], that $D_i\mu_j * f$ is a bounded measure on $\mathbb{R}^{n-1}$ for every $f \in L_1(\mathbb{R}^{n-1})$; therefore $D_i\mu_j \in M(\mathbb{R}^{n-1})$, as required.

We conclude from here that μ_j is absolutely continuous as follows. Using a C^∞-approximate identity $\{f_N\}_{n\in\mathbb{N}}$ we define the differentiable function $F_N := f_N * \mu_j$, $N \in \mathbb{N}$. By the Gagliardo embedding theorem [Ga-1957]

$$\|F_N\|_{L_p(\mathbb{R}^{n-1})} \leq C|F_N|_{W_1^1(\mathbb{R}^{n-1})}, \quad N \in \mathbb{N},$$

where $\frac{1}{p} := 1 - \frac{1}{n-1}$ and $C = C(n) > 0$.

The right-hand side is bounded by $C \sum\limits_{i=1}^{n-1} \|D_i\mu_j\|_{M(\mathbb{R}^{n-1})}$. Since, by definition, $1 < p < \infty$ if $n > 2$ and $p = \infty$ if $n = 2$, the space $L_p(\mathbb{R}^{n-1})$ is reflexive for $n > 2$ and dual to $L_1(\mathbb{R}^{n-1})$ for $n = 2$. Hence in both cases there is a subsequence of $\{F_N\}$ (which, without loss of generality, may be identified with $\{F_N\}$) and a function, say g_j, such that for every $f \in L_q(\mathbb{R}^{n-1})$, where $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int F_N \cdot f dx \to \int g_j \cdot f dx \quad \text{as} \quad N \to \infty.$$

But $F_N \to \mu_j$ in $M(\mathbb{R}^{n-1})$ as $N \to \infty$. Hence $\int_{\mathbb{R}^{n-1}} f d\mu_j = \int_{\mathbb{R}^{n-1}} f g_j \, dx$ for every $f \in C_0(\mathbb{R}^{n-1})$ and therefore $d\mu_j = g_j dx$.

This proves (2.174). □

Now we derive the desired contradiction to the existence of the linear extension operator $E : L_1(\mathbb{R}^{n-1}) \to W_1^1(\mathbb{R}^n_+)$.

Combining (2.174) and (2.173) we have

$$\sum_j \|g_j\|_{L_1(\mathbb{R}^{n-1})} \leq C.$$

Then the sum $g := \sum_j g_j$ is an element of $L_1(\mathbb{R}^{n-1})$. On the other hand, $f = g * f$ for every $f \in L_1(\mathbb{R}^{n-1})$ by (2.171). Hence g is the δ-measure which clearly does not belong to $L_1(\mathbb{R}^{n-1})$.

The proof is complete. □

Comments

Spaces of C^k functions whose higher derivatives satisfy the Hölder or Zygmund conditions and the associated spaces of k-jets are customary objects of modern analysis. The general class of Lipschitz spaces of higher order introduced in Section 2.1 was first introduced in Quantitative Approximation Theory originated by S. Bernstein, D. Jackson, de la Vallée–Poussin and A. Zygmund, see, e.g., comments to Chapter 4 of the book [TB-2004] by Trigub and Belinsky.

The notion of k-modulus of continuity was due to Lebesgue ($k = 1$) and S. Bernstein [Ber-1912] ($k > 1$). Its basic properties for univariate continuous functions were studied by Marchaud [Mar-1927].

Another approach to Lipschitz spaces of higher order and the associated spaces of smooth functions $C^k\Lambda^{s,\omega}$ is based on a generalized modulus of continuity; various definitions of this notion were given by A. Calderón [Cal-1964], Yu. Brudnyi [Br-1965b] and H. Shapiro, see his book [Sha-1971] and references therein. For our goal the following definition is the most suitable.

Let μ be a bounded Borel measure on $\mathbb{R}^n$ *orthogonal to the space of polynomials* $\mathcal{P}_{k-1,n}$. Then the μ-modulus of continuity for a function $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ and $t > 0$ is given by

$$\omega_\mu(t\,;\, f\,;\, L_p) := \sup_{0<s\leq t} \left\| \int_{\mathbb{R}^n} f(\cdot + sy) d\mu(y) \right\|_p .$$

For compactly supported μ this definition can be extended to functions $f \in L_p(G)$ where G is a domain in $\mathbb{R}^n$. In particular, choosing $\mu := \sum_{j=0}^{k} c_{kj}\delta_{kj}$ where $c_{kj} := (-1)^{k-j}\binom{k}{j}$, we obtain k-modulus of continuity $\omega_k(\cdot\,;\, f\,;\, L_p)_G$.

Theorem. (a) *For a convex domain $G \subset \mathbb{R}^n$ and $f \in L_p(G)$, $1 \le p \le \infty$,*

$$\omega_\mu(\cdot\,;\, f\,;\, L_p)_G \le c(k,n)|\mu|\omega_k(\cdot\,;\, f\,;\, L_p)_G.$$

(b) *If the Fourier transform of μ does not vanish identically on any ray emanating from the origin, then for $t > 0$,*

$$\omega_k(t\,;\, f\,;\, L_p) \le c(k,n,\mu) \int_{\mathbb{R}_+} \min\left[\left(\frac{t}{u}\right)^k, 1\right] \frac{\omega_\mu(u\,;\, f\,;\, L_p)}{u}\, du.$$

If μ is compactly supported, the inequality holds for $f \in L_p(G)$ and $0 < t \le \operatorname{diam} G$.

Part (a) of the theorem was proved by Yu. Brudnyi [Br-1965a], see also [Br-1970a]. Part (b) for $d\mu = w dx$, where w is a spherically symmetric compactly supported C^∞ function, was, in essence, due to A. Calderón [Cal-1964]; in general, assertion (b) follows from the general Boman and Shapiro *comparison theorem* estimating ω_μ by an integral transform of another μ-modulus of continuity, see the book by H. Shapiro [Sha-1971] and the consequent Boman's paper [Bom-1977].

The Whitney–Hestens extension Theorem 2.22 for the jet space $J^\infty(\mathbb{R}^n)$ naturally poses the following problem.

We say that a closed subset $S \subset \mathbb{R}^n$ has the C^∞ *simultaneous extension property* if there is a linear continuous extension operator from the trace space $J^\infty(\mathbb{R}^n)\big|_S$ to $J^\infty(\mathbb{R}^n)$.

Let us recall that $J^\infty(\mathbb{R}^n)$ is a Frechet space consisting of ∞-jets associated to C^∞ functions; its topology is defined by the family of seminorms given on elements $\vec{f} := \{f_\alpha\}_{\alpha \in \mathbb{Z}_+^n}$ of $J^\infty(\mathbb{R}^n)$ by

$$\|\vec{f}\|_k := \max_{|\alpha| \le k} \sup_{\mathbb{R}^n} |f_\alpha|, \quad k \in \mathbb{Z}_+.$$

In turn, the Frechet topology of the trace space $J^\infty(\mathbb{R}^n)\big|_S$ is defined by the family of seminorms

$$\|\vec{f}\|_k^S := \inf_F \big\| \{D^\alpha F\}_{\alpha \in \mathbb{Z}_+^n} \big\|_k, \quad k \in \mathbb{Z}_+,$$

where F runs over all C^∞ functions on $\mathbb{R}^n$ satisfying $D^\alpha F\big|_S = f_\alpha$, $\alpha \in \mathbb{Z}_+^n$.

Due to the Whitney–Hestens Theorem 2.19 the topology of the trace space may be equivalently defined by the reduced Taylor remainders $r_\alpha(\vec{f})$, $\alpha \in \mathbb{Z}_+^n$.

Problem. *Characterize closed subsets of $\mathbb{R}^n$ possessing the C^∞ simultaneous extension property.*

By virtue of Theorem 2.24 a set S from the class defined by this property (briefly, $S \in \mathrm{SEP}_\infty$) has no isolated points. The next criterion was due to Tidten [Tid-1979] and Vogt [Vogt-1983] who generalized the one-dimensional Mitiagin result [Mit-1961]. For its formulation let us denote by s the space of rapidly decreasing sequences $x := (x_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ whose Frechet topology is defined by the sequence of seminorms $\|x\|_k$ given by

$$\|x\|_k := \sum_{j\in\mathbb{N}} |x_j| j^k, \quad k \in \mathbb{Z}_+.$$

Theorem. *Let S be a compact set in $\mathbb{R}^n$ such that $\overline{S^o} = S$. Then $S \in \mathrm{SEP}_\infty$ if and only if the trace space $J^\infty(\mathbb{R}^n)\big|_S$ is isomorphic to the space s.*

In general, the trace space is isomorphic to some factor space of the space s, see the papers [Tid-1979] by Tidten and [BS-1983] by Bierstone and G. Schwartz. The following example from the first paper demonstrates that $J^\infty(\mathbb{R}^n)\big|_S$ may be nonisomorphic to s.

The set $S := \left\{(x,y) \in \mathbb{R}^2 \ ; \ x \geq 0, \ |y| \leq e^{-\frac{1}{x}}\right\}$ does not belong to SEP_∞ (but $\overline{\mathbb{R}^2\backslash S}$ does!).

The condition for a compact subset $S \subset \mathbb{R}^n$ to be the closure of its interior is not necessary. For example, the classical Cantor set has empty interior but belongs to SEP_∞ (Tidten [Tid-1983]).

A considerable number of papers has been devoted to the study of other subclasses of the class SEP_∞: Nash subanalytic sets (Bierstone and P. Milman [BM-1991]), sets with real analytic boundaries having polynomial cusps (Pawlucki and Pleśniak [PP-1988]), sets for which a Markov type inequality is valid (Pleśniak [Pl-1990]).

There are some variants of the extension problem in question which consider spaces of subanalytic functions (Kurdyka and Pawlucki [KP-1997]) and spaces of ultradifferentiable functions $\mathcal{E}_\omega(\mathbb{R}^n)$ in the Beurling–Bjork sense, see, e.g., the paper by Meise and B. Taylor [MT-1989] and references therein. Note that $\mathcal{E}_\omega(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ for $\omega(t) := \log(1+t), \ t > 0$; other choices of the weight give the Denjoy–Carleman classes of quasianalytic functions and (nonquasianalytic) Gevrey classes.

Theorems 2.26 and 2.31 were proved by Brouwer for univariant continuous functions in his almost forgotten paper [Bro-1908]. The multivariate generalizations presented in Section 2.3 were proved in Yu. Brudnyi's papers [Br-1965b] and [Br-1967].

Identity (2.43) is due to Kemperman; the problem of finding such identities was formulated by A. Timan and solved in different ways by Yu. Brudnyi [Br-1965b] (see [Br-1970a] for the proof) and M. Timan [MTim-1969]. The discussion of these results is presented in Appendix E.

Theorem 2.32 for $C(\mathbb{R}_+)$-functions was due to Marchaud [Mar-1927]; the argument for the proof of the multivariate case is taken from [Br-1967].

Theorem 2.34 for univariate continuous functions was proved by Brouwer [Bro-1908] and rediscovered by Whitney [Wh-1934c].

Theorem 2.37 for bounded functions on intervals of the real line was due to Whitney [Wh-1957, Wh-1959]. For convex domains in $\mathbb{R}^n$ and integrable functions the result was proved by Yu. Brudnyi in [Br-1970a]; see Appendix F for the corresponding proofs and other references.

Raikov [Rai-1939] was the first to characterize differential properties of functions by behavior of their local best approximation. His paper contains Theorem 2.38 for univariate continuous functions; the result is an easy consequence of the aforementioned Marchaud theorem. A new approach based on Whitney's inequality from [Wh-1957] and on a local Markov inequality was proposed by Yu. Brudnyi and Gopengauz [BGo-1961], see also [BGo-1963] for the proofs. In particular, the (generalized) Marchaud theorem is a consequence of their results. These ideas were then developed in a series of papers by Yu. Brudnyi's and his students' and collaborators' papers beginning with [Br-1965a], see the bibliography to this book. In these papers, the basic properties of the classical spaces of smooth multivariate functions were studied using the local approximation methods. Some results of this kind will be discussed and proved in Chapter 9.

The n-dimensional generalization of the Raikov theorem, Theorem 2.38, was due to Yu. Brudnyi [Br-1965a] (see [Br-1971] for the proof). A generalization to a wide class of subsets in $\mathbb{R}^n$ including Lipschitz domains and fractals was due to A. and Yu. Brudnyis [BB-2007a]; the result will be presented in subsection 9.2.3 (Volume II).

Theorem 2.38 raises the following extremal

Problem. *Find the sharp constant $\gamma = \gamma(k, \lambda, n, p)$ for the inequality*

$$M_{k,\lambda}(f) \leq \gamma |f|_{C^{k,\lambda}(G)}.$$

The only known result concerns univariate functions and $\lambda = 1$; in this case $\gamma = \frac{1}{2^{2k+1}(k+1)!}$.

The proof of Bernstein's Theorem 2.40 was outlined in his note [Ber-1940]; the derivation in the book follows his basic ideas.

In the multivariate case, the following claim might be true.

Conjecture. *A function $f : G \to \mathbb{R}$, where G is a convex domain of $\mathbb{R}^n$, belongs to the space $C^k_u(G)$ if and only if the limit*

$$\lim_{Q \to x} \frac{E_k(Q \cap G; f)}{|Q|^{\frac{k}{n}}}$$

exists at every $x \in G$ and convergence is uniform.

Here $C^k_u(G)$ stands for the space of k-times continuously differentiable on G functions whose higher derivatives are uniformly continuous.

The variant of Whitney's extension Theorem 2.47 for the space $\dot{C}^{k,\omega}(\mathbb{R})$, i.e., Theorem 2.52, is due to Merrien [Mer-1966]. The derivations of both theorems follow the Whitney argument [Wh-1934b] but the proof of combinatorial Lemma 2.48 is taken from the book [KM-1997] by Kriegl and Michor.

Theorem 2.64 is motivated by the Whitney extension theorem [Wh-1934d] for the space of k-times continuously differentiable functions on $G \subset \mathbb{R}^n$ with uniformly continuous higher derivatives; in this case G is quasiconvex.

The proof of Zobin's Theorem 2.69 (a) is based on his argument but was essentially simplified by Shvartsman in [BSh-2001a]; he, in particular, formulated the Geometrical Lemma which then was proved by Igonin and Yanishevski.

All the results of subsection 2.5.2 are also true for p-integrable functions with $1 \le p \le \infty$; the proofs, up to trivial changes, are the same. In connection with the Yu. Brudnyi result [Br-1980] mentioned in Remark 2.80, the following conjecture seems to be valid.

Conjecture. *Let $f \in L_p(G)$, where $G \subset \mathbb{R}^n$ is a special Lipschitz domain. There exists a linear continuous operator $T : L_p(G) \to L_p(\mathbb{R}^n)$ such that, for all $k \ge 0$ and some $c(k,n) > 0$,*

$$\omega_k(\cdot\,; Tf)_{L_p(\mathbb{R}^n)} \le c(k,n)\omega_k(\cdot\,; f)_{L_p(G)}.$$

The local approximation approach to Jones type extension theorems (see subsection 2.6.1) will be used in Chapter 9 for general spaces of smooth functions. The results of subsection 2.6.1 yield the following (very difficult)

Problem. *Find a geometric characteristic of domains $G \subset \mathbb{R}^n$ which admit a simultaneous extension from $W_p^k(G)$ into $W_p^k(\mathbb{R}^n)$.*

The proof of the nonexistence theorem of subsection 2.6.2 was outlined in Peetre's note [Peet-1979]; the proof presented here is due to Yu. Brudnyi in [BSh-2001a]. Another variant of this proof was then proposed by Pelchyński and Wojciechowski [PW-2002] who used this result to show that the Sobolev space $W_1^k(\mathbb{R}^n)$ for $n \ge 2$ is not isomorphic to any Banach lattice (unlike the case $n = 1$, see Borsuk [Bor-1933a]).

It is worth noting that a Peetre type nonexistence result holds only for the embedding in $L_p(\mathbb{R}^{n-1})$ with $p = 1$; a simultaneous extension does exist for $p > 1$. This, clearly, means that the set of linear continuous operators acting in L_1-spaces is very small. Actually, the L_1-space is a "border point" between L_p spaces with $p < 1$, where there is no nontrivial linear continuous operators, and L_p spaces with $p > 1$, where the set of these operators is as large as in L_2.

Another nonexistence phenomenon was discovered by Burenkov and Goldman [BuGo-1979]. In particular, they proved that for the embedding operator

$$B_p^{\sigma,1}(\mathbb{R}^n) \subset L_p(\mathbb{R}^m)$$

there is no inverse linear extension operator.

Here $1 \le p < \infty$, $1 \le m < n$, $\sigma = \frac{n-m}{p}$ and $\mathbb{R}^m$ is identified with an m-dimensional coordinate subspace of $\mathbb{R}^n$; for the definition of the Besov space standing in the left-hand side see, e.g., Triebel [Tri-1992].

Appendices

E. Difference identities

E.1. Kemperman's identity

We present the identity for mixed differences formulated in Section 2.3, see (2.43), and also a certain modification which will be more relevant for applications. The former was due to J. H. B. Kemperman and was first published in [JSch-1977]. Previous identities of this kind were found by Yu. Brudnyi [Br-1965b] and M. Timan [MTim-1969]. All of them may be written in a form which uses the following notation.

By $\tau(h)$, $h \in \mathbb{R}^n$, we denote the shift by h acting on functions $f : \mathbb{R}^n \to \mathbb{R}$ as $\tau(h)f := f(\cdot + h)$. Then the kth difference operator is written as

$$\Delta^k(h) := \Delta(h)^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \tau(jh),$$

where $\Delta(h) := \tau(h) - \tau(0)\,(:= \tau(h) - 1)$, and the α-difference operator, $\alpha \in \mathbb{Z}_+^n$, as

$$\Delta^\alpha(h) := \prod_{i=1}^{n} \Delta^{\alpha_i}(h_i e_i);$$

here $\{e_i\}_{1 \le i \le n}$ is the standard basis of $\mathbb{R}^n$.

We deviate here from the notation of Section 2.3, where $\Delta^k(h)$ is denoted by Δ_h^k, etc.

Now the desired identity looks as follows:

$$\Delta^\alpha(h) = \int\limits_{\mathbb{R} \times \mathrm{Mat}_n(\mathbb{R})} \Delta^k(th)\tau(Mh)d\lambda(t)d\mu(M), \tag{E.1}$$

where $|\alpha| = k$ and λ and μ are compactly supported Borel measures on $\mathbb{R}$ and on $\mathrm{Mat}_n(\mathbb{R})$, respectively; the latter stands for the space of real $n \times n$ matrices.

Since $\Delta^\alpha(h)$ and $\Delta^k(th)$ are finitely supported (as linear combinations of δ-measures), it will be natural to look for finitely supported measures λ and μ in this identity.

This is the case of the Kemperman and M. Timan identities; in the proof of Yu. Brudnyi, the measures are unspecified but the identity has some additional property which the previous ones do not possess. Specifically, in this identity, for every number $t \in \operatorname{supp} \lambda$ and every matrix $M \in \operatorname{supp} \mu$,

$$Mh + \operatorname{conv}(\operatorname{supp} \Delta^k(th)) \subset \operatorname{conv}(\operatorname{supp} \Delta_h^\alpha). \tag{E.2}$$

Let us note that $\mathrm{conv}(\mathrm{supp}\,\Delta^k(h))$ is the closed *h-interval* $[0, kh] := \mathrm{conv}\{0, kh\}$ while the right-hand side is the rectangular box $\prod_{i=1}^{h}[0, \alpha_i h_i]$; here we adopt the convention $[0, a] := [a, 0]$ if $a < 0$.

Now we formulate and prove a modification of Kemperman's identity which also satisfies (E.2).

Theorem E.1. *There are finitely supported measures λ and μ such that* (E.1) *and* (E.2) *hold.*

Proof. We begin with the proof of Kemperman's identity.

Proposition E.2. *Let V be a collection of k vectors in $\mathbb{R}^n$ and $\varphi : V \to \{1, \frac{1}{2}, \ldots, \frac{1}{k}\}$ be a bijection. Then*

$$\prod_{v \in V} \Delta(v) = \sum_{\omega \in V} (-1)^{\mathrm{card}\,\omega} \tau(w_\omega) \Delta^k(v_\omega), \tag{E.3}$$

where the sum is taken over all nonempty subsets ω of V and

$$v_\omega := \sum_{v \in \omega} \varphi(v) v, \quad w_\omega := \sum_{v \notin \omega} v.$$

Note that $w_V = 0$, since $\{v \notin V\} = \emptyset$.

Proof. We enumerate the vectors of V such that $v = v_i$ if $\varphi(v) = \frac{1}{i}$. Then we identify V with the set $\{1, \ldots, k\}$ and regard the ω's in (E.3) as nonempty subsets of the latter set. For arbitrary $0 \le i \le k$ we have

$$\begin{aligned}\prod_{j=1}^{k} \Delta\big((j-i)v_j\big) &= \prod_{j=1}^{k} \big[\tau\big((j-i)v_j\big) - 1\big] \\ &= \sum_{\omega} (-1)^{k-|\omega|}\, \tau\Big(\sum_{j \in \omega} j v_j\Big) \tau\Big(-i \sum_{j \in \omega} v_j\Big).\end{aligned}$$

Here ω runs over *all* subsets of $\{1, \ldots, k\}$; the term with $\omega = \emptyset$ equals $(-1)^k \tau(0)$.

The left-hand side is not zero only for $i = 0$. Therefore, multiplying both sides by $(-1)^{k-i}\binom{k}{i}$ and summing over i, we obtain

$$(-1)^k \prod_{j=1}^{k} \Delta(j v_j) = \sum_{\omega} (-1)^{k-|\omega|} \tau\Big(\sum_{j \in \omega} j v_j\Big) \Delta^k\Big(-\sum_{j \in \omega} v_j\Big),$$

where we may exclude $\omega = \emptyset$, since the corresponding term is zero.

Next we replace v_j by $-v_j$ and apply to the right-hand side the formula

$$\Delta(-j v_j) = -\tau(-j v_j)\Delta(j v_j). \tag{E.4}$$

This leads to the equality

$$\tau\Big(-\sum_{j=1}^{k} jv_j\Big)\prod_{j=1}^{k}\Delta(jv_j) = \sum_{\omega}(-1)^{k-|\omega|}\tau\Big(-\sum_{j\in\omega} jv_j\Big)\Delta^k\Big(\sum_{j\in\omega} v_j\Big).$$

Multiplying both sides by $\tau\Big(\sum\limits_{j=1}^{k} jv_j\Big)$ and then substituting v_j for jv_j one gets (E.3). □

Now we choose here the set V in such a way that $\prod_{v\in V}\Delta(v)$ becomes the mixed α-difference $\Delta^\alpha(h) := \prod\limits_{i=1}^{n}\Delta^{\alpha_i}(h_ie_i)$. Hence, k equals $|\alpha|$ and $V = \{v_1,\dots,v_k\}$, where $v_j := h_1e_1$ if $1 \le j \le \alpha_1$, $v_j := h_2e_2$ if $\alpha_1 < j \le \alpha_1+\alpha_2$, etc. To write the corresponding vectors v_ω and w_ω in (E.3), we define a surjection $\psi : \{1,\dots,k\} \to \{1,\dots,n\}$ by

$$\psi(j) := i \quad \text{if} \quad v_j = h_ie_i.$$

Then, in accordance with (E.3), we have

$$\begin{aligned} v_\omega &:= v_\omega(h) = \sum_{j\in\omega} j^{-1}h_{\psi(j)}e_{\psi(j)}, \\ w_\omega &:= w_\omega(h) = \sum_{j\notin\omega} h_{\psi(j)}e_{\psi(j)}. \end{aligned} \tag{E.5}$$

Here the ω is regarded as a subset of $\{1,2,\dots,k\}$. In particular, for $\omega := \{1,\dots,k\}$, $v_\omega(h) = \sum\limits_{i=1}^{n} c_i\alpha_ih_ie_i$ and $w_\omega(h) = 0$, where $c_i := \sum\limits_{\alpha_{i-1}<j\le\alpha_j} j^{-1}$.

Now condition (E.2) requires that for every nonempty ω the n-interval

$$I_\omega(h) := \operatorname{conv}\big\{w_\omega(h),\ w_\omega(h)+kv_\omega(h)\big\}$$

be contained in the rectangular box $\Pi_\alpha(h) := \prod\limits_{i=1}^{n}[0,\alpha_ih_i]$. That is to say, for every nonempty ω we must have

$$w_\omega(h) \quad \text{and} \quad w_\omega(h)+kv_\omega(h) \in \Pi_\alpha(h). \tag{E.6}$$

It is readily seen that this is not yet the case for $|\alpha| = k > 1$.

We now modify the Kemperman identity to get rid of this shortage. To this end we fix an integer $N > 1$ and apply the Multinomial Theorem to the identity

$$\Delta^k(Nh) := (\tau_h^N - 1)^k = (\tau_h - 1)^k\Big(\sum_{j=0}^{N-1}\tau_h^j\Big)^k.$$

This yields

$$\Delta^k(Nh) = \sum_{\beta} \tau(|\beta|h)\Delta_h^k,$$

where the vectors $(\beta_0, \ldots, \beta_{N-1})$ run over the lattice

$$L(k,N) := \mathbb{Z}_+^N \cap [0,(k-1)N]^N.$$

Substituting k for α_i and h for $\frac{1}{N} h_i e_i$ and multiplying the identities so obtained over $1 \le i \le n$ we then get

$$\Delta^\alpha(h) = \sum_{\beta} \tau\big(N^{-1}u_\beta(h)\big)\Delta^\alpha(N^{-1}h), \tag{E.7}$$

where $\beta := (\beta^1, \ldots, \beta^n)$, vector β^i runs over the lattice $L(\alpha_i, N)$ and

$$u_\beta(h) := \sum_{i=1}^{n} |\beta^i| h_i e_i.$$

Note that all points $N^{-1}u_\beta(h)$ satisfy

$$N^{-1}u_\beta(h) \in \Pi_\alpha(h). \tag{E.8}$$

Now we associate to each β a signature $\varepsilon_\beta \in \{-1, 1\}^n \in \mathbb{Z}^n$ as follows. We divide $\Pi_\alpha(h)$ into 2^n congruent subboxes and fix an arbitrary one, say Π. There exists a unique vertex of Π common to $\Pi_\alpha(h)$, and every vertex of $\Pi_\alpha(h)$ has the form $\frac{1}{2}\sum\limits_{i=1}^{n}(1+\varepsilon_i)\alpha_i h_i e_i$, where $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_n)$ is a signature. We denote the signature related to the common vertex by $\varepsilon(\Pi)$.

Because of (E.8) the point $N^{-1}u_\beta(h)$ belongs to one of such Π and we set

$$\varepsilon_\beta := -\varepsilon(\Pi). \tag{E.9}$$

If $N^{-1}u_\beta(h)$ lies on the boundary of Π and therefore belongs also to other subboxes of the subdivision, we choose in (E.9) one of them arbitrarily.

Now we modify each term in (E.7) in the following way. Fix a signature and set for $h \in \mathbb{R}^n$,

$$h_\varepsilon := \sum_{i=1}^{n} \varepsilon_i h_i e_i.$$

Raising identity (E.4) to the power α_i we get

$$\Delta^{\alpha_i}(h_i e_i) = (-1)^{\alpha_i}\tau(-\alpha_i h_i e_i)\Delta^\alpha(-h_i e_i).$$

Multiplying these equalities together for those i where $\varepsilon_i = -1$ we further obtain

$$\Delta^\alpha(h) = \pm\tau(z_\alpha(h_\varepsilon))\Delta^\alpha(h_\varepsilon), \tag{E.10}$$

where

$$z_\alpha(h_\varepsilon) := \sum_{i=1}^{n} \min(0, \varepsilon_i)\alpha_i h_i e_i.$$

Replacing finally h by $N^{-1}h$ and choosing $\varepsilon := \varepsilon_\beta$ we express the β-summand of (E.7) in the form

$$\tau\big(N^{-1}u_\beta(h) + N^{-1}z_\alpha(h_\varepsilon)\big)\Delta^\alpha(N^{-1}h_\varepsilon), \quad \text{where} \quad \varepsilon = \varepsilon_\beta.$$

Then we apply the Kemperman identity to $\Delta^\alpha(N^{-1}h_{\varepsilon_\beta})$ and insert the result into (E.7). This gives an identity of the type (E.1) and we should only explain why condition (E.2) holds for this case.

According to (E.5) and (E.6) this condition is equivalent to the belonging of the points $x_\omega := N^{-1}u_\beta(h)+N^{-1}\big(z_\alpha(h_{\varepsilon_\beta})+w_\omega(h_{\varepsilon_\beta})\big)$ and $y_\omega := x_\omega+N^{-1}v_\omega(h_{\varepsilon_\beta})$ to the parallelotope $\Pi_\alpha(h)$. Setting $N := 2k$ we establish this directly computing the coordinates of x_ω and y_ω.

The result has been proved. □

To formulate a consequence of identity (E.1) we introduce several notions.

Let $x, y \in \mathbb{R}^n$. The *pointwise multiplication* of these vectors is given by

$$x \cdot y := \sum_{i=1}^{n} x_i y_i e_i. \tag{E.11}$$

Further, the *ordered interval* determined by x and y is the rectangular box defined by

$$\Pi[x, y] := \big\{z \in \mathbb{R}^n \,;\, z_i \in [x_i, y_i], \quad i = 1, \dots, n\big\}. \tag{E.12}$$

Now let G be a domain in $\mathbb{R}^n$. We say that G is *orderly convex* if G along with every two points x, y contains the ordered interval $\Pi[x, y]$.

Now we introduce an analog of the k-th modulus of continuity, see (2.24), based on mixed differences.

Definition E.3. Let G be a domain in $\mathbb{R}^n$. Then the α-modulus of continuity, where $\alpha \in \mathbb{Z}^n_+$, is a function $\omega_\alpha : C(G) \times (0, +\infty)^n \to \mathbb{R}_+$ given by

$$\omega_\alpha(h\,; f)_G := \sup\big\{\|\Delta_t^\alpha f\|_{C(G_{\alpha\cdot t})} \,;\, t \in \Pi[0, h]\big\},$$

where $G_y := \big\{x \in G\,; \Pi[x, x + y] \subset G\big\}$.

Using the notions now introduced we derive from identity (E.1)

Corollary E.4. *Let $G \subset \mathbb{R}^n$ be orderly convex. Then for every $f \in C(G)$ and $t > 0$,*

$$\omega_k(t\,; f)_G \approx \sup_{|\alpha|=k} \omega_\alpha(te\,; f)_G, \tag{E.13}$$

where the constants of equivalence depend only on n and k.

Here $e := \sum\limits_{i=1}^{n} e_i = (1, \dots, 1)$.

Proof. Identity (E.1) and condition (E.2) immediately imply, for $|\alpha| = k$, $h := te$ and the orderly convex domain G the inequality

$$\omega_\alpha(te\,; f)_G \leq c(k,n)\omega_k(t\,; f)_G.$$

The converse follows from an identity expressing Δ_h^k via the corresponding mixed differences. In fact, by raising the identity

$$\Delta(h) = \sum_{i=1}^{n} \tau\Big(\sum_{j=1}^{i-1} h_j e_i\Big)\Delta(h_i e_i)$$

to the k-th power we get

$$\Delta^k(h) = \sum_{|\alpha|=k} \frac{k!}{\alpha!}\,\tau(h_\alpha)\Delta_h^\alpha, \tag{E.14}$$

where $h_\alpha := \sum\limits_{i=1}^{n-1} \alpha_i\Big(\sum\limits_{j=1}^{i-1} h_j e_j\Big)$. This and the orderly convexity of G immediately imply that

$$\omega_k(t\,; f)_G \leq n^k \sup_{|\alpha|=k} \omega_\alpha(te\,; f)_G.$$

□

This result can be easily extended to a wide class of (quasi-)norms and arbitrary bases in $\mathbb{R}^n$. We begin with

Definition E.5. Suppose that the linear space X of measurable (classes of) functions on $\mathbb{R}^n$ is equipped with the quasinorm [10] $\|\cdot\|_X$. This space is said to be a quasi-Banach translation-invariant lattice if the following conditions hold:

(a) (Translation-invariance) For every $h \in \mathbb{R}^n$,

$$\|\tau(h)f\|_X = \|f\|_X.$$

(b) (Monotonicity) If $|f| \leq |g|$ almost everywhere and $g \in X$, then $f \in X$ and

$$\|f\|_X \leq \|g\|_X.$$

(c) (Completeness) $(X, \|\cdot\|)$ is complete.

Property (b) allows us to define for every measurable subset $S \subset \mathbb{R}^n$ a quasi-norm $\|\cdot\|_{X(S)}$ given for measurable $f : S \to \mathbb{R}$ by

$$\|f\|_{X(S)} := \|\bar{f}\|_X, \tag{E.15}$$

[10] i.e., the triangle inequality holds in a weaker form: for a fixed $c > 1$ and all $f, g \in X$ $\|f+g\|_X \leq c\{\|f\|_X + \|g\|_X\}$.

where $\bar{f}$ is the extension of f to $\mathbb{R}^n$ by zero.

It can be easily checked that the linear space $X(S)$ defined by this quasinorm is complete and the quasinorm is monotone.

Now let G be a domain in $\mathbb{R}^n$ and $f \in X(G)$. Then similarly to the case of continuous functions we define the k-modulus of continuity for f by

$$\omega_k(t\,; f)_{X(G)} := \sup_{\|h\| \le t} \|\Delta^k(h) f\|_{X(G_{kh})}; \tag{E.16}$$

recall that $G_y := \{x \in G\,; [x, y] \subset G\}$.

Next, let $\mathcal{B} := \{b_1, \ldots, b_n\}$ be a basis of $\mathbb{R}^n$. Replacing the standard orthonormal basis $\{e_1, \ldots, e_n\}$ by $\mathcal{B}$ in all the above definitions we introduce orderly convex domains and mixed moduli of continuity *with respect to the basis* $\mathcal{B}$. For instance, the latter notion is given for $f \in X(G)$ and $h \in \mathbb{R}^n$ by

$$\omega_\alpha^{\mathcal{B}}(h\,; f)_{X(G)} := \sup\{\|\Delta_t^\alpha f\|_{X(G_{\alpha\cdot t})}\,; t \in \Pi_{\mathcal{B}}[0, h]\}, \tag{E.17}$$

where $x \cdot y$ and $\Pi_{\mathcal{B}}[x, y]$ are now defined by (E.11) and (E.12) with $\mathcal{B}$ substituted for $\{e_1, \ldots, e_n\}$.

The applications of identities (E.1) and (E.14) immediately lead to

Corollary E.6. *Let $G \subset \mathbb{R}^n$ be an orderly convex domain with respect to a basis $\mathcal{B}$ of $\mathbb{R}^n$. Then for every $f \in X(G)$ and $h \in \mathbb{R}^n$,*

$$\omega_k(t\,; f)_{X(G)} \approx \sup_{|\alpha|=k} \omega_\alpha^{\mathcal{B}}(te\,; f)_{X(G)}. \tag{E.18}$$

Example E.7. Let $G \subset \mathbb{R}^n$ be a special Lipschitz domain, i.e., the subgraph of a Lipschitz function, say $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$,

$$G := \{(x, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}\,; x_n < \varphi(x)\}.$$

Then there is an infinite (circular) cone K such that $\partial G + K \subset G$. Choose a basis $\mathcal{B} := \{b_1, \ldots, b_n\}$ such that all the b_i are in K. It can be then easily checked that G is orderly convex with respect to this $\mathcal{B}$, and we may apply (E.18) to this setting.

E.2. Marchaud's identity

This identity expresses the k-th difference via differences of bigger orders in the following fashion.

Proposition E.8. *Let $k, s \ge 1$ be integers and $h \in \mathbb{R}^n$. Then the following holds:*

$$\Delta^k(h) = 2^{-sk}\Delta^k(2^s h) + \sum_{j=0}^{s-1} 2^{-jk} T_k(2^j h)\Delta^{k+1}(2^j h), \tag{E.19}$$

where $T_k(h) := \sum_{i=0}^{k-1} c_k(i)\tau(ih)$ *and the numbers* $c_k(i) \geq 0$ *and satisfy*

$$\sum_{i=1}^{k-1} c_k(i) = \frac{k}{2}\,.$$

Proof. Using the identity preceding (E.7) for $N = 2$ we get

$$\Delta^k(2h) - 2^k\Delta^k(h) = \sum_{j=0}^{k}\binom{k}{j}\left[\tau_{jh}\Delta_h^k - \Delta_h^k\right] = \sum_{j=1}^{k}\binom{k}{j}\sum_{i=0}^{j-1}\tau(ih)\Delta^{k+1}(h).$$

Further, we set $c_k(i) := 2^{-k}\sum_{j=i+1}^{k}\binom{k}{j}$, $\quad 0 \leq i \leq k-1$, and define $T_k(h)$ using these numbers. Then $\sum_i c_k(i) = \frac{k}{2}$ and the previous identity yields

$$2^{-k}\Delta^k(2h) - \Delta^k(h) = T_k(h)\Delta^{k+1}(h).$$

Substituting here h for $2^j h$, dividing by 2^{-jk} and summing over $j = 0, 1, \ldots, s-1$ we then get

$$2^{-sk}\Delta^k(2^{sh}) - \Delta^k(h) = \sum_{j=0}^{s-1} 2^{-jk} T_k(2^j h)\Delta^{k+1}(2^j h),$$

as required. □

As a consequence we obtain the following Marchaud inequality, see [Mar-1927] for $n = 1$.

Theorem E.9. *Let* X *be a translation-invariant Banach lattice on* $\mathbb{R}^n$*, and* $G \subset \mathbb{R}^n$ *be a convex domain. Given integers* $0 \leq k < \ell$*, there is a constant* $c(\ell) > 0$ *such that for every* $f \in X(G)$ *and* $0 < t \leq \frac{1}{\ell}\operatorname{diam} G$

$$\omega_k(t\,; f)_{X(G)} \leq c(\ell)t^k\left\{\int_t^d \frac{\omega_\ell(u; f)_{X(G)}}{u^{k+1}}\,du + \frac{\|f\|_{X(G)}}{(\operatorname{diam} G)^k}\right\}.$$

Here $d := \frac{1}{\ell}\operatorname{diam} G$ *and the second term in the sum is zero if* $\operatorname{diam} G = \infty$.

Proof. It suffices to consider the case of $\ell = k+1$ and then iterate the inequality obtained. Applying identity (E.19) with this ℓ we get

$$\|\Delta^k(h)f\|_{X(G_{kh})} \leq \frac{k}{2}\sum_{j=0}^{s-1} 2^{-jk}\omega_{k+1}(\|h\|\,; f)_{X(G)} + 2^{-sk}\cdot 2^{k+1}\|f\|_{X(G)}. \quad \text{(E.20)}$$

Now let $\|h\| \le t$ and s satisfies the condition

$$\frac{\operatorname{diam} G}{2\ell} \le 2^{st} < \frac{\operatorname{diam} G}{\ell}\,;$$

in the case $\operatorname{diam} G = \infty$, we replace $\operatorname{diam} G$ by an integer N and let it tend to infinity in the final inequality. Applying inequality (E.20) and using the monotonicity of ω_{k+1} we obtain for this s,

$$\omega_k(t\,; f)_{X(G)} \le c(k)\left\{\int_t^d \frac{\omega_{k+1}(u; f)_{X(G)}}{u^{k+1}}\,du + \frac{\|f\|_{X(G)}}{(\operatorname{diam} G)^k}\right\}.$$

The result has been proved. □

Remark E.10. Now let X be a translation-invariant quasi-Banach lattice on $\mathbb{R}^n$. In this setting, inequality (E.20) should be multiplied by c^s, where $c > 1$ is the constant in the triangle inequality for X. This clearly destroys the proof. To avoid this obstacle one uses the Aoki–Rolewica theorem, see, e.g., [BLo-1976, Lemma 3.10.2], which states that for some $\rho = \rho(X) \in (0, 1)$,

$$\left\|\sum_{i=1}^{\infty} f_i\right\|_X \le 2\left\{\sum_{i=1}^{\infty} \|f_i\|_X^{\rho}\right\}^{\frac{1}{\rho}}. \tag{E.21}$$

Using this replacement of the triangle inequality we derive the Marchaud inequality for a quasi-Banach X in the form

$$\omega_k(t\,;\, f)_{X(G)} \le c(\ell, p)t^k\left\{\left(\int_t^d \left[\frac{\omega_\ell(u; f)_{X(G)}}{u^k}\right]^{\rho} \frac{du}{u}\right)^{\frac{1}{\rho}} + \frac{\|f\|_{X(G)}}{(\operatorname{diam} G)^k}\right\}.$$

Let, e.g., $X = L_p$ where $0 < p < 1$. Then $\rho(L_p) = p$ and (E.21) holds without the factor 2 because of concavity of the function $t \mapsto t^p,\ t > 0$.

F. Local polynomial approximation and moduli of continuity

We will prove Theorem 2.37 which was formulated and widely used in Section 2.3 and will be intensively exploited in Chapter 9, and then present several related results.

F.1. Degree of local polynomial approximation

We first prove Theorem 2.37 for continuous functions. Let us recall (in an equivalent form) its formulation.

Theorem F.1. *Let $V \subset \mathbb{R}^n$ be a closed convex body*[11]*. Let $f : V \to \mathbb{R}$ be continuous and satisfy the condition, for $x, x + kh \in V$,*

$$\left|\Delta_h^k f(x)\right| \le 1. \tag{F.1}$$

Then there is a constant $w = w(k, n)$ and a polynomial p of degree $k - 1$ such that

$$\sup_V |f - p| \le w(k, n). \tag{F.2}$$

Remark F.2. Local best approximation of order 1 may be evaluated by

$$E_1(f\,;\,C) = \frac{1}{2}\sup\bigl\{|f(x) - f(y)|\,;\,x, y \in C\bigr\}.$$

Hence, (F.2) holds with $w(1, n) = \frac{1}{2}$ and we may assume in the sequel that $k \ge 2$.

Proof. We begin with V being a cube. Using scaling we reduce the proof to the case

$$V = Q := [0, 1]^n.$$

Lemma F.3. *Let $Q_k := [0, 1 - \frac{1}{k}]^n$. For every $f \in C(Q)$ there exists a function $f_k \in C^{k-1,1}(Q_k)$ such that*

$$\begin{aligned} &\|f - f_k\|_{C(Q_k)} \le c(k, n)\omega_k(Q\,;\,f), \\ &\sup_{|\alpha|=k} \|D^\alpha f_k\|_{C(Q_k)} \le c(k, n)\omega_k(Q\,;\,f). \end{aligned} \tag{F.3}$$

Here $\omega_k(S\,;\,f)$ is the k-oscillation of f on $S \subset \mathbb{R}^n$, i.e., $\omega_k\left(\frac{\operatorname{diam} S}{k}\,;\,f\right)_S$, see (2.57).

Proof. We use the operator $S_t := \prod_{1 \le j \le N} S_t(e_j)$ introduced in the proof of Theorem 2.77 with all the unit vectors e_j contained in the cube Q. According to its definition, S_t is an integral operator whose kernel is supported on $\operatorname{conv}\{te_j\}_{1 \le j \le n}$. Choosing an appropriate $t = t(k, n)$ we obtain for every $x \in Q_k$,

$$x + \operatorname{conv}\{te_j\}_{1 \le j \le N} \subset Q.$$

Under this choice of t and e_j the function $f_k := S_t f$ is defined on a set containing Q_k. In turn, inequalities (F.3) follow from those proved in Theorem 2.77, see (2.142) and the subsequent inequality for $D^\alpha S_t f$. $\square$

Lemma F.4. *Theorem* F.1 *is true for $V = Q$.*

Proof. Define the desired polynomial $p \in \mathcal{P}_{k-1,n}$ as the Taylor polynomial of order $k - 1$ for f_k at 0 , i.e., $p := T_0^{k-1} f_k$. Then for $x \in Q_k$,

$$|(f - p)(x)| = \bigl|\bigl(f_k - T_0^{k-1} f_k\bigr)(x)\bigr| \le c(k, n)|f_k|_{C^k(Q_k)}$$

[11] i.e., the interior of V is nonempty.

by the Taylor formula.

Combining this with (F.3) we get

$$\|f-p\|_{C(Q_k)} \le c(k,n)\omega_k(Q\,;f). \tag{F.4}$$

Now let $x \in Q\backslash Q_k$. Then for an appropriate h,

$$x - jh \in Q_k, \quad j = 1, \ldots, k.$$

Moreover, $\Delta_h^k p = 0$ and therefore

$$(f-p)(x) = \Delta_h^k f(x) - \sum_{j=1}(-1)^{k-j}\binom{k}{j}(f-p)(x-jh).$$

Applying (F.4) to each term of the sum we then have

$$\big|(f-p)(x)\big| \le \omega_k(Q\,;f) + (2^k-1)c(k,n)\omega_k(Q\,;f) \le 2^k c(k,n).$$

Together with (F.4) this proves the result. □

Lemma F.5. *Theorem* F.1 *holds for Euclidean balls.*

Proof. Without loss of generality we may assume that f is continuous and satisfies (F.1) on the closed unit ball B centered at 0. Let Q be a cube of maximal volume contained in B. Then Q is of center 0 and side length $\frac{2}{\sqrt{n}}$. Restricting f to Q we use Lemma F.4 to find a polynomial $p \in \mathcal{P}_{k-1,n}$ with

$$\big|f(x) - p(x)\big| \le c(k,n) \quad \text{for} \quad x \in Q.$$

Fix any $x \in B$. By the one-dimensional variant of Lemma F.4, there is a polynomial $\varphi \in \mathcal{P}_{k-1,1}$ such that

$$\big|f(tx) - \varphi(t)\big| \le c(k,1)$$

for $|t| \le 1$. Hence for $|t| \le n^{-\frac{1}{2}}$ we have

$$\big|p(tx) - \varphi(t)\big| \le c(k,1) + c(k,n).$$

Now we apply the Remez type inequality, see Corollary G.2 of Appendix G, to extend this to all of $t \in [-1,1]$. This gives for these t,

$$\big|p(tx) - \varphi(t)\big| \le \left(\frac{4}{n^{-\frac{1}{2}}}\right)^{k-1}\big(c(k,1) + c(k,n)\big) =: \tilde{c}(k,n).$$

Combining this with the inequality for $f(tx)$ and choosing $t = 1$ we finally get

$$\big|f(x) - p(x)\big| \le c(k,1) + \tilde{c}(k,n) \quad \text{for any} \quad x \in B.$$

□

We now establish the result for *bounded* convex bodies. To this end, we need the classical F. John result [Jo-1948].

Lemma F.6. *Let V be a bounded closed convex body in $\mathbb{R}^n$. There is an ellipsoid E containing V and such that the homothety $\frac{1}{n}E$ of E with respect to its center is contained in V.*

Lemma F.7. *Theorem* F.1 *holds for bounded closed convex bodies.*

Proof. The assertion of Theorem F.1 may be reformulated as follows.

There is a constant $c > 0$ such that for any $f \in C(V)$,

$$E_k(f\,;V) \le c\,\omega_k(V\,;f).$$

Set $w(k,V) := \inf c$. It is easily seen that this constant is affine invariant. Using an appropriate affine transform we may then assume that the John ellipsoid for V is the closed unit ball B centered at 0. Then

$$\frac{1}{n}B \subset V \subset B.$$

Now let $f \in C(V)$. Restricting f to the ball $\frac{1}{n}B$ and applying Lemma F.5 we find a polynomial $p \in \mathcal{P}_{k-1,n}$ such that for $x \in \frac{1}{n}B$,

$$\big|f(x) - p(x)\big| \le w_k\Big(\frac{1}{n}B\,;f\Big) = w(B\,;f) =: \tilde{c}(k,n).$$

Repeating then the argument of the proof of Lemma F.5 we extend this inequality to all points of B as follows:

$$\big|f(x) - p(x)\big| \le c(k,1) + (4n)^k\big(c(k,1) + \tilde{c}(k,n)\big). \qquad \square$$

At the final stage V is *unbounded*. Let V_N denote the intersection of V with the closed ball of radius N centered at a fixed point of V. Suppose that $f \in C(V)$ and satisfies (F.1). Then there is a polynomial $p_N \in \mathcal{P}_{k-1,n}$ such that

$$\big|f(x) - p_N(x)\big| \le c(k,n) \quad \text{for} \quad x \in V_N. \tag{F.5}$$

The sequence $\{p_N\}_{N\ge1}$ is then uniformly bounded on the compact set V_1; therefore there is a subsequence which converges uniformly on V_1 to some polynomial p of degree $k-1$. Since $\mathcal{P}_{k-1,n}$ is finite-dimensional, this subsequence converges to p on any compact subset of $\mathbb{R}^n$. Restricting (F.5) to the set V_{N_0} with fixed $N_0 \le N$ and passing to the limit as $N \to \infty$, we get

$$\big|f(x) - p(x)\big| \le c(k,n) \quad \text{for} \quad x \in V_{N_0}.$$

Since N_0 is arbitrary, this proves the result for unbounded bodies. $\square$

A generalization of Theorem F.1 *is also true for bounded (maybe, non-measurable) functions and for measurable functions.*

Let $B(V)$ be the Banach space of bounded on V functions equipped with the uniform norm. As in the proof of Theorem F.1, the main point is the derivation of the result for $Q := [0,1]^n$. According to Whitney's theorem [Wh-1959] there is a constant w_k such that for every $f \in B([0,1])$ the Lagrange polynomial interpolating f at k equally distributed points of $[0,1]$ satisfies

$$\|f - L(f)\|_B \le w_k \sup\big\{|\Delta_h^k f(x)|\,;\, 0 \le x \le x + kh \le 1\big\}. \tag{F.6}$$

Let $f \in B(Q)$ and $L_i(f)$ be defined by applying the interpolation operator L to the function $x_i \mapsto f(x)$, $0 \le x_i \le 1$. Then $L_i(f)$ is a polynomial of degree $k-1$ in x_i with the coefficients being bounded functions on Q independent of x_i. By their definition, the operators L_i mutually commute. Therefore the function $\widehat{L}(f) := (L_1 \cdots L_n)(f)$ is a polynomial of degree $k-1$ in each variable. Moreover, (F.6) implies that

$$\|f - L_i(f)\|_{B(Q)} \le w_k \sup\big\{\big|\Delta_{he_i}^k f(x)\big|\,;\, [x, x + khe_i] \subset Q\big\}.$$

In particular, the norm of L_i is bounded by $(2^k + 1)w_k$. This implies that for $f \in B(Q)$,

$$\|f - \widehat{L}(f)\|_{B(Q)} \le \sum_{i=1}^n \Bigg(\prod_{j \ne i} \|L_j\|\Bigg) \|f - L_i f\|_{B(Q)} \le c(k,n)\omega_k(Q\,; f). \tag{F.7}$$

Now we derive from here the required inequality

$$E_k(Q\,; f) \le c(k,n)\omega_k(Q\,; f).$$

The left-hand side is a Banach norm on the factor space $B(Q)/\mathcal{P}_{k-1,n}$. From inequality (F.7) we conclude that the functional $f \mapsto \omega_k(Q\,; f)$ is also a norm on this factor space. In fact, it suffices to check that $\omega_k(Q\,; f) = 0$ implies that $f \in \mathcal{P}_{k-1,n}$. But due to (F.7) f is a polynomial. Moreover, $\Delta_h^k f = 0$ for every h and therefore all its derivatives of order k equal zero. Hence, f is a polynomial of degree $k-1$.

Further, the norm $\omega_k(Q\,; \cdot)$ is Banach, see Lemma 6 of the paper [Br-1970a] by Yu. Brudnyi. Finally, for every $p \in \mathcal{P}_{k-1,n}$,

$$\omega_k(Q\,; f) = \omega_k(Q\,; f - p) \le 2^k \|f - p\|_{B(Q)}$$

and therefore

$$\omega_k(Q\,; f) \le 2^k E_k(Q\,; f).$$

By the Banach open mapping theorem, see, e.g., [DS-1958], this implies the inverse inequality

$$E_k(Q\,; f) \le c\,\omega_k(Q\,; f)$$

with $c > 0$ independent of f. That is to say, the required result for cubes is true, and Theorem F.1 for bounded functions then follows.

The theorem for measurable functions may be reduced to the case of bounded functions. To this end it suffices to prove that a measurable function $f : [0,1] \to \mathbb{R}$ satisfying the inequality

$$|\Delta_h^k f(x)| \le 1 \tag{F.8}$$

for $x, x+kh \in [0,1]$, is bounded on $[0,1]$. This allows us to use for f Whitney's theorem (F.6) and to complete the proof as above.

Let $f : [0,1] \to \mathbb{R}$ be measurable, and let (F.8) hold. By measurability, for any $\varepsilon > 0$ there is $N_\varepsilon > 0$ such that the measure of the set

$$S_\varepsilon := \big\{x \in [0,1]\,;\, |f(x)| > N_\varepsilon\big\}$$

is bounded by ε.

We then fix an $x \in \big[0, 1 - \frac{1}{k}\big]$ and consider the set $\{h > 0\,;\, x + ih \in S_\varepsilon\}$ for $1 \le i \le k$. The measure of this set is bounded by $\big|\frac{1}{i}(S_\varepsilon - x)\big| = \frac{1}{i}|S_\varepsilon| < \frac{\varepsilon}{i}$; here λS is the λ-homothety of $S \subset [0,1]$ with respect to zero. Then the measure of the set of all $h > 0$ satisfying $x + ih \in S_\varepsilon$ for some $1 \le i \le k$ is at most $\Big(\sum\limits_{i=1}^{k} \frac{1}{i}\Big)\varepsilon$. Choose ε such that this number becomes less than $\frac{1}{k}$. Then there is $h \in \big(0, \frac{1}{k}\big)$ such that all points $x + ih$ belong to $[0,1]\backslash S_\varepsilon$.

Finally, we write for these x and h,

$$|f(x)| \le \big|\Delta_h^k f(x)\big| + \sum_{i=1}^{k} \binom{k}{i} |f(k+ih)| \le 1 + (2^k - 1)N_\varepsilon.$$

Hence, f is bounded on $\big[0, 1 - \frac{1}{k}\big]$.

Applying the same argument (with negative h) to $x \in \big[\frac{1}{k}, 1\big]$ we prove boundedness of f on $[0,1]$.

As a consequence, we now derive the precise form of Marchaud's inequality formulated in Theorem 2.7 (d).

Corollary F.8. *Let f be a locally bounded function on the closure of a convex domain $G \subset \mathbb{R}^n$. Assume that $\omega_k(f\,;G) < \infty$ for a fixed integer $k \ge 1$. Then there exists a polynomial p of degree $k-1$ and a constant $c = c(n,k) > 0$ such that for every integer $0 \le \ell < k$ the inequality*

$$\omega_\ell(t\,; f - p)_G \le c t^\ell \int_t^{2d} \frac{\omega_k(s\,; f)_G}{s^{\ell+1}}\, ds$$

holds for all $t \le d$; here $d := \frac{1}{\ell}\operatorname{diam} G$.

Proof. First let G be bounded and $p \in \mathcal{P}_{k-1,n}$ be such that

$$\sup_G |f - p| \le c\,\omega_k(G\,; f)\left(:= c\,\omega_k\Big(\frac{\operatorname{diam} G}{k}\,; f\Big)_G\right) \tag{F.9}$$

for some $c = c(k,n)$. Applying Theorem E.8 to $f - p$ we obtain, for $0 < t \le d$,

$$\omega_\ell(t\,; f-p)_G \le c_1(k,n)t^\ell\left(\int_t^d \frac{\omega_k(s; f)_G}{s^{\ell+1}}\,ds + \frac{\sup_G |f-p|}{(\operatorname{diam} G)^\ell}\right). \tag{F.10}$$

Since ω_k is equivalent to a k-majorant, see Theorem 2.7, we get for $t \le s \le 2d$,

$$\frac{\omega_k(s\,; f)_G}{s^{\ell+1}} \ge 2^{-k}\frac{\omega_k(G\,; f)}{(\operatorname{diam} G)^k}\, s^{k-\ell-1}.$$

Integrating this over $[t, 2d]$ and noting that for some $c(k) > 0$ and $t \le d$,

$$\int_t^{2d} s^{k-\ell-1}\,ds \ge c(k)(\operatorname{diam} G)^{k-\ell},$$

we finally obtain for this t,

$$\int_t^{2d} \frac{\omega_k(s\,; f)_G}{s^{\ell+1}}\,ds \ge 2^{-k}c(k)\frac{\omega_k(G\,; f)}{(\operatorname{diam} G)^\ell}\,,$$

i.e., the second term in the sum of (F.10) is absorbed by the first.

Now let G be unbounded and G_R denote the intersection of G with a Euclidean ball of radius R centered at a fixed point of G. Let $p \in \mathcal{P}_{k-1,n}$ be chosen as in (F.9). Then applying Marchaud's inequality to $f - p$ and G_R, we get

$$\omega_\ell(t\,; f-p)_{G_R} \le c(k,n)t^\ell\left(\int_t^\infty \frac{\omega_k(s; f)_G}{s^{\ell+1}}\,ds + \frac{\omega_k(G; f)}{(\operatorname{diam} G_R)^\ell}\right).$$

Sending R to infinity we prove the corollary for this case. □

Remark F.9.

(a) Let $S := \overline{G}$ be the closure of a bounded domain in $\mathbb{R}^n$. Assume that S is *star-like*, i.e., there is a point $x_0 \in S$ such that for every line L passing through x_0 the set $S \cap L$ is a nontrivial line segment and, moreover, G contains a ball B centered at x_0. Applying the argument of Lemma F.7 to the closed set $S := \overline{G}$ we get the inequality

$$E_k(S\,; f) \le c\,\omega_k(S\,; f) \tag{F.11}$$

with the constant c depending on n, k and on the star-like coefficient

$$\sigma(S) := \sup\left\{\frac{\operatorname{diam} S}{\operatorname{diam} B}\,;\, B \subset S\right\},$$

where B runs over all balls in S centered at x_0.

Because of the last dependence, inequality (F.10) may be incorrect for unbounded star-like domains.

(b) Another extension of the class of bounded convex bodies can be obtained by applying the extension theorems of subsection 2.6.3. This gives inequality (F.8) for (bounded) Lipschitz domains and (unbounded) special Lipschitz domains, see Definitions 2.71 and 2.72.

Example F.10.

(a) Unlike Lipschitz domains the boundary of star-like domains may be far from any kind of regular. The classical von Koch snowflake, see, e.g., Falconer [Fal-1999, p. XV and Example 9.5], bounds a star-like domain whose boundary is a "thick" curve (its Hausdorff dimension[12] is strictly greater than 1). In particular, any arc of this curve is of infinite length.

(b) The following example mentioned in the paper [BI-1985] by Binev and Ivanov demonstrates the role of cusps for the validity of Theorem F.1.

The set $S := \left\{x \in \mathbb{R}^2\,;\, \frac{1}{2}\,x_1^2 \leq x_2 \leq x_1^2, \quad 0 \leq x_1 \leq 1\right\}$ has a cusp at $(0,0)$:

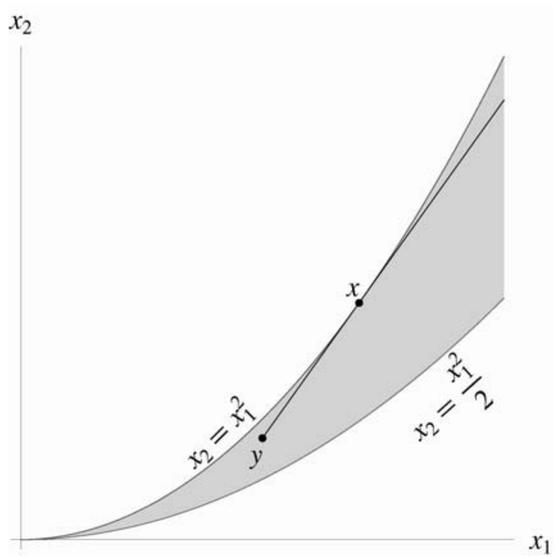

Figure 2.4: The role of cusps for the validity of Theorem F.1.

We define the function $f_\varepsilon : S \to \mathbb{R}$ by

$$f_\varepsilon(x) := \sin\Big(\varepsilon \log \frac{1}{x_1}\Big).$$

[12] see subsection 3.2.4 below for the definition of this concept.

Since f_ε assumes values ± 1 infinitely many times and $|f_\varepsilon| \le 1$, we have

$$E_k(S\,;f_\varepsilon) = 1.$$

On the other hand,

$$\omega_k(S\,;f_\varepsilon) \le 2^{k-1}\,\omega_1(S\,;f_\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{F.12}$$

In fact, by definition,

$$\omega_1(S\,;f_\varepsilon) = \sup\bigl|f_\varepsilon(x_1) - f_\varepsilon(y_1)\bigr|,$$

where the supremum is taken over all x, y such that the segment $[x, y] \subset S$. Note that this interval with a fixed x is of maximal length if y is a point of the lower parabola $x_2 = \frac{1}{2}\,x_1^2$ and $[x, y]$ is tangent to the upper parabola $x_2 = x_1^2$. A direct computation shows that in this case $y_1 \le c_0 x_1$ with some numerical constant $c_0 > 1$.

Finally, by the Mean Value Theorem,

$$\bigl|f_\varepsilon(x_1) - f_\varepsilon(y_1)\bigr| \le \varepsilon\left|\log\frac{x_1}{y_1}\right| \le \varepsilon \log c_0.$$

Hence, (F.12) holds and Theorem F.1 is not true for this setting.

However, the domain obtained by replacing the lower parabola by $x_2 = -\frac{1}{2}\,x_1^2$ is star-like. Hence, in this case the cusp does not prevent the validity of (F.8). $\square$

F.2. Whitney constants

Let S be a closed subset of $\mathbb{R}^n$. We define the *Whitney constant* $w_k(S)$ by

$$w_k(S) := \sup\bigl\{E_k(S\,;f)\,;\, f \in C(S) \quad \text{and} \quad \omega_k(S\,;f) \le 1\bigr\}. \tag{F.13}$$

We also define the *global Whitney constant* $w_k(n)$ by

$$w_k(n) := \sup\{w_k(S)\,;\, S \subset \mathbb{R}^n \quad \text{bounded and convex}\}. \tag{F.14}$$

In the spirit of Whitney's paper [Wh-1957] who considered the case of dimension one[13], let us consider also the constants $w_k^*(n)$ and $w_k^{**}(n)$ defined by (F.13) with $S := \mathbb{R}^n_+$ and $S := \mathbb{R}^n$. Using the argument of Beurling, see [Wh-1957], it is easy to prove the following estimates:

$$w_k^*(n) \le 2, \quad w_k^{**}(n) \le \min_{1\le j\le n} 1\Big/\binom{n}{j}.$$

[13] In this case $w_k(1) = w_k([0,1])$.

In contrast, the sharp upper bound for $w_k(n)$ depends on the dimension, and in fact, $\lim_{n\to\infty} w_k(n) = \infty$ if $k \geq 2$. We discuss this situation below following the paper [BKa-2000] by Yu. Brudnyi and Kalton, but first consider the one-dimensional Whitney constant $\omega_k := \omega_k(1)$.

It is easy to show that $\omega_2 = \frac{1}{2}$ (H. Burkill [Bu-1952]) but the value of w_3 is still unknown. In the above mentioned paper, Whitney proved that $\frac{8}{15} \leq w_3 \leq \frac{7}{10}$ and $\omega_k < \infty$ for all k. The latter result is essentially improved by Sendov [Sen-1987] whose method (after a slight modification) leads to the inequality $\omega_k \leq 3$. Further improvement was due to Giliwicz, Kryakin and Shevchuk [GKS-2002]; the result states that $w_k \leq 2 + e^{-2}$. It was conjectured by Sendov for all of k and proved for $k \leq 7$ by Kryakin and Zhelnov that $w_k \leq 1$, see the survey by Kryakin [Kry-2002] for more information.

Now we present several conjectures and results of the aforementioned paper by Yu. Brudnyi and Kalton.

There is a fairly precise estimate for $w_2(n)$, i.e.,

$$\frac{1}{2}\log_2\Big(\Big\lfloor\frac{n}{2}\Big\rfloor + 1\Big) \leq w_2(n) \leq \frac{1}{2}\lfloor\log_2 n\rfloor + \frac{5}{4}.$$

Curiously enough, $w_2(n)$ is almost attained not for the unit n-simplex S^n as it may be thought, but for $S^n \oplus S^n \subset \mathbb{R}^{2n}$. Meanwhile for S^n the precise asymptotic is given by

$$\lim_{n\to\infty} \frac{w_2(S^n)}{\log_2 n} = \frac{1}{4}\,.$$

In the sequel we will write $w_k(\ell_p^n)$ instead of $w_k(S)$ when S is the closed unit ball of ℓ_p^n. Then $w_2(\ell_1^n) \approx \log n$ while $w_2(\ell_p^n)$ with $1 < p \leq \infty$ is equivalent, up to a logarithmic factor, to $(p-1)^{-1}$ as $p \to 1$. This striking difference in asymptotic behavior is explained by Theorem 3.12 of [BKa-2000] which gives the upper bound of $w_2(B_X)$, where B_X is the unit ball of a finite-dimensional Banach space X, in terms of the p-type constant of X.

Now let $w_k^{(\mathrm{sym})}(n)$ be defined as in (F.14) but for centrally symmetric convex bodies. Then for some numerical constants $c_1, c_2 > 0$,

$$c_1\sqrt{n} \leq w_3^{(\mathrm{sym})}(n) \leq c_2\sqrt{n}\log(n+1).$$

As in the linear approximation case, this result can be improved for $w_3(\ell_p^n)$. For example, $w_3(\ell_2^n) \approx \log(n+1)$ and

$$c_1\log(n+1) \leq w_3(\ell_\infty^n) \leq c_2\big(\log(n+1)\big)^2.$$

There are also a few estimates for $k \geq 4$. In particular,

$$w_k^{(\mathrm{sym})}(X) \leq cn^{\frac{k}{2}-1}\log(n+1),$$

while

$$w_k(\ell_p^n) \leq cn^{\frac{(k-3)}{2}}\log(n+1)$$

for $2 \leq p \leq \infty$ and $w_k(\ell_1^n) \approx \log(n+1)$.

F.3. Conjectures

(a) If $k \geq 2$, then

$$w_k(n) \approx w_k^{(\mathrm{sym})}(n) \approx n^{\frac{k}{2}-1} \log(n+1)$$

as $n \to \infty$.

This is proved for $k = 2$ while the upper estimate for $w_k^{(\mathrm{sym})}(n)$ is established for $k \geq 2$. As for the lower bound, we only have $w_k(n) \geq w_k^{(\mathrm{sym})}(n) \geq c\sqrt{n}$ for $k \geq 3$.

(b) If $k \geq 3$ and $1 \leq p < \infty$, then

$$w_k(\ell_p^n) \approx \log(n+1)$$

as $n \to \infty$.

The result is established for $p = 1$ and all $k \geq 2$ and for $k = 3$ and $2 \leq p < \infty$, while the lower bound is established for all $k \geq 3$. It is quite possible, that it is way off the mark when $k \geq 4$.

(c) $w_2(\ell_\infty^n)$ is "small", say, $w_2(\ell_\infty^n) \leq 2$. The only known results are $w_2(\ell_\infty^1) = \frac{1}{2}$, and $w_2(\ell_\infty^2) = 1$, and $w_2(\ell_\infty^n) \leq 802$ for $n \geq 3$. If the conjecture held, then for every convex function f on an n-cube Q we would have the inequality

$$E_2(Q\,; f) \leq \omega_2(Q\,; f).$$

(d) If X is an infinite-dimensional Banach space, then $w_3(X) = \infty$.

G. Local inequalities for polynomials

We will prove several inequalities estimating the uniform or L_p-norm of a polynomial on a convex body via that on a subset of positive measure. The results have their origin in the classical Chebyshev inequality, see, e.g., [Tim-1963, pp. 67–68]. It asserts that a polynomial $p \in \mathcal{P}_{k,1}$ satisfying the inequality

$$\max_I |p| \leq M$$

on the interval $I := [a, b] \subset \mathbb{R}$ grows outside the interval as

$$|p(x)| \leq M t_k^I(x), \quad x \notin I.$$

Here t_k^I is the normalized k-th *Chebyshev polynomial*, i.e.,

$$t_k^I(x) := t_k\Big(\frac{2x - a - b}{b - a}\Big),$$

where $t_k(x) := \cos(k \arccos x)$.

If, in particular, $[a,b] \subset [0,1]$ then this implies the inequality

$$\max_{[0,1]} |p| \le t_k\Big(\frac{2-\ell}{\ell}\Big) \max_I |p|, \tag{G.1}$$

where $\ell := b - a$.

In a little known and hardly available paper [Rem-1936], Remez generalizes (G.1) by replacing $[a,b]$ by an arbitrary measurable subset $S \subset \mathbb{R}$ of measure ℓ. This result has been rediscovered several times, see, e.g., Dudley and Randall [DR-1962] and Yu. Brudnyi and Ganzburg [BrG-1973, Lemma 2] but the Remez proof presented below remains the most simple and elegant.

A multidimensional result of this kind is due to Yu. Brudnyi and Ganzburg [BrG-1973] presented now.

Theorem G.1. *Let $V \subset \mathbb{R}^n$ be a compact convex body and $S \subset V$ be a subset of relative Lebesgue measure*

$$\ell := \frac{|S|}{|V|} > 0.$$

Then for every polynomial $p \in \mathcal{P}_{k,n}$ the sharp inequality

$$\max_V |p| \le t_k\left(\frac{1+\sqrt[n]{1-\ell}}{1-\sqrt[n]{1-\ell}}\right) \sup_S |p| \tag{G.2}$$

holds.

Proof. We begin with the aforementioned Remez result (and proof).

Claim I. (G.2) is true for $V := [a,b] \subset \mathbb{R}$ and a subset $S \subset V$ of relative Lebesgue measure $\ell > 0$.

We may assume that $V = [0,1]$; then $S \subset [0,1]$ is of measure ℓ. Without loss of generality, we may also assume that S is closed and has no isolated points. Moreover, for S being an interval, the result follows from (G.1). Otherwise, $[0,1]\backslash S$ contains at least one open interval, say, (b,c), and therefore

$$S \subset [a,b] \cup [c,d],$$

where $a := \min S$ and $d := \max S$. Further, let $\xi \in [0,1]$ be an extreme point for a polynomial $p \in \mathcal{P}_{k,1}$, i.e.,

$$|p(\xi)| = \max_{[0,1]} |p|.$$

As the case $\xi \in S$ is trivial, we should consider only three possibilities:

(a) $\xi \in [0,a]$ or $\xi \in [d,1]$;

(b) $\xi \in (b,c)$.

First let $\xi \in [0, a]$ and t_k^I be the normalized Chebyshev polynomial for $I := [a, a + \ell]$, and let

$$a =: x_1 < x_2 < \cdots < x_{k+1} := a + \ell$$

be points where t_k^I assumes alternatively the values ± 1. Then we choose points $\hat{x}_1 < \hat{x}_2 < \cdots < \hat{x}_{k+1}$ in S as follows. Set $\hat{x}_1 := a$ and determine $\hat{x}_i$ for $2 \le i \le k+1$ by the condition

$$\big|S \cap [\hat{x}_1, \hat{x}_i]\big| = x_i - x_1.$$

These points are correctly defined, since $|S| = \ell = x_{k+1} - x_1$. Due to this choice,

$$a + \ell - x_j \ge |\xi - \hat{x}_j| \quad \text{and} \quad |\hat{x}_i - \hat{x}_j| \ge |x_i - x_j|.$$

Further, $\hat{x}_i \in S$ and therefore $|p(\hat{x}_i)| \le \sup\limits_S |p|$.

Using Lagrange interpolation we then have

$$\begin{aligned} |p(\xi)| &\le \sum_{i=1}^{k+1} \Bigg(\prod_{i \ne j} \frac{|\xi - \hat{x}_j|}{|\hat{x}_i - \hat{x}_j|} \Bigg) |p(\hat{x}_j)| \\ &\le \left(\sum_{i=1}^{k+1} \Bigg(\prod_{j \ne i} \frac{a + \ell - x_j}{|x_i - x_j|} \Bigg) \big|t_k^I(x_i)\big| \right) \sup_S |p|. \end{aligned}$$

The sum in the right-hand side equals

$$\left| \sum_{i=1}^{k+1} \Bigg(\prod_{j \ne i} \frac{a + \ell - x_j}{x_i - x_j} \Bigg) t_k^I(x_i) \right| = \big|t_k^I(a + \ell)\big| = t_k\Big(\frac{2 - \ell}{\ell}\Big),$$

and the result follows.

The case of $\xi \in [d, 1]$ is considered similarly.

Finally, let $\xi \in (b, c)$. Then at least one of the fractions

$$\ell_1 := \frac{\big|[0, \xi] \cap S\big|}{\xi}, \quad \ell_2 := \frac{\big|[\xi, 1] \cap S\big|}{1 - \xi}$$

is greater than or equal to

$$\ell = \frac{\big|[0, \xi] \cap S\big| + \big|[\xi, 1] \cap S\big|}{\xi + (1 - \xi)}.$$

If, e.g., $\ell_1 \ge \ell$, then we apply the just proved result to the interval $I_1 := [0, \xi]$ and the subset $S_1 := S \cap [0, \xi]$ to have

$$\max_{[0,1]} |p| = |p(\xi)| \le t_k\Big(\frac{2 - \ell_1}{\ell_1}\Big) \max_{S_1} |p| \le t_k\Big(\frac{2 - \ell}{\ell}\Big) \max_S |p|.$$

Claim I is proved.

To proceed we need the following geometric fact.

Let x_0 be an interior point of the body V and $0 < \ell \le 1$. Let R stand for a ray emanating from x_0. Consider the extreme problem

$$\gamma(\ell) := \sup_S \left(\operatorname*{ess\,inf}_R \frac{|V \cap R|_1}{|S \cap R|_1} \right),$$

where S runs over all subsets of V of relative measure $\frac{|S|}{|V|} \ge \ell$. Here $|\cdot|_1$ is the Lebesgue 1-measure.

Claim II. The following is true:

$$\gamma(\ell) = \frac{1}{1 - \sqrt[n]{1-\ell}}\,. \tag{G.3}$$

To prove this we use the spherical coordinate system $(r, \varphi) = (r, \varphi_1, \dots, \varphi_{n-1})$ with the origin at x_0. Let

$$r = H(\varphi) = H(\varphi_1, \dots, \varphi_{n-1})$$

be the equation of the surface ∂V.

Then we define the subset $\widetilde{S} \subset V$ by the inequalities

$$(\sqrt[n]{1-\ell})H(\varphi) \le r \le H(\varphi).$$

Comparing the n-volume of V and the Lebesgue n-measure of $\widetilde{S}$, we get

$$|\widetilde{S}| = \ell|V|.$$

Moreover, a similar computation gives

$$\frac{|V \cap R|_1}{|\widetilde{S} \cap R|_1} = \frac{1}{1 - \sqrt[n]{1-\ell}} \tag{G.4}$$

for every ray R from x_0.

It remains to show that for a subset $S \subset V$ satisfying $\frac{|S|}{|V|} \ge \ell$ the following inequality holds:

$$\sup_S \left(\operatorname*{ess\,inf}_R \frac{|V \cap R|_1}{|S \cap R|_1} \right) \le \frac{1}{1 - \sqrt[n]{1-\ell}}\,. \tag{G.5}$$

Suppose, on the contrary, that for some $S_0 \subset V$ with $\frac{|S_0|}{|V|} \ge \ell$, the converse inequality holds. This and (G.4) then imply that for almost all R,

$$|S_0 \cap R|_1 < |\widetilde{S} \cap R|_1. \tag{G.6}$$

Further, let $I_0 \subset V \cap \mathbb{R}$ be the closed interval of length $|S_0 \cap R|_1$ with right endpoint $|V \cap R|_1$. By monotonicity of the power function,

$$\int\limits_{S_0 \cap R} r^{n-1} dr \le \int\limits_{I_0} r^{n-1}\, dr. \tag{G.7}$$

Due to (G.6), I_0 is a proper subinterval of an interval $\widetilde{I}$ which has the same right endpoint $|V \cap R|_1$ and is of length $|\widetilde{S} \cap R|_1$. Substituting in (G.7) $\widetilde{I}$ for I_0, then multiplying by $\prod_{i=1}^{n-1} \cos^{i-1} \varphi_i$ and integrating over φ we have

$$|S_0| = \int_{S_0 \cap R} dv < \int_{\widetilde{S}} dv = |\widetilde{S}| = \ell |V|.$$

This contradicts the inequality $|S_0| \geq \ell|V|$ for the chosen subset S_0.

The result has been proved.

Now we complete the proof of (G.2). Let $S \subset V$ and $|S| \geq \ell|V|$ for some $0 < \ell \leq 1$. Let $p \in \mathcal{P}_{k,n}$ and $x_0 \in V$ be such that

$$\max_V |p| = |p(x_0)|.$$

We may assume that x_0 is an inner point of V; otherwise we use a point close to x_0 and then pass to the limit.

Apply Claim I to the univariate polynomial $p|_{R \cap V}$, where R is a ray from x_0, to obtain

$$\max_V |p| = |p(x_0)| \leq t_k\left(\frac{2|V \cap R|}{|S \cap R|} - 1\right) \max_{S \cap R} |p|.$$

Taking $\operatorname{ess\,inf}$ over R and sup over S, and using monotonicity of t_k on $[1, +\infty)$ and (G.3) we derive that

$$\max_V |p| \leq t_k\big(2\gamma(\ell) - 1\big) \max_S |p| = t_k\left(\frac{1 + \sqrt[n]{1-\ell}}{1 - \sqrt[n]{1-\ell}}\right) \max_S |p|,$$

as required.

Finally, let us show that inequality (G.1) is sharp for the class of compact convex bodies. To this end we define V to be a circular cone of height 1, say,

$$V := \left\{x \in \mathbb{R}^n \,;\, x_1^2 \leq \sum_{i=2}^n x_i^2, \quad 0 \leq x_1 \leq 1\right\}.$$

Let $\ell \in (0,1)$ and V_h be a subcone of V of height h determined by the condition $|V \backslash V_h| = \ell|V|$; then $h = \sqrt[n]{1-\ell}$. Set now $S := V \backslash V_h$ and let

$$p(x) := t_k\left(\frac{2x_1 - 1 - h}{1 - h}\right)$$

be the Chebyshev polynomial associated to interval $[h, 1]$. Then $|S| = \ell|V|$ and

$$\max_V |p| = t_k\left(\frac{1+h}{1-h}\right) = t_k\left(\frac{1 + \sqrt[n]{1-\ell}}{1 - \sqrt[n]{1-\ell}}\right) \max_S |p|,$$

i.e., (G.1) becomes equality.

The result has been proved. $\square$

In applications the following consequence of the theorem is of common use.

Corollary G.2. *Under the assumption of Theorem* G.1 *it is true that*

$$\max_V |p| \le \frac{1}{2}\left(\frac{4n}{\ell}\right)^k \max_S |p|. \tag{G.8}$$

Proof. The function $\ell \mapsto 1 - \sqrt[n]{1-\ell}$ is convex on $(0,1]$ and therefore

$$\frac{1+\sqrt[n]{1-\ell}}{1-\sqrt[n]{1-\ell}} \le \frac{2n}{\ell} - 1.$$

This, the definition of t_k and its monotonicity on $[1,\infty)$ imply the result. □

In fact, inequality (G.8) may be generalized to integral norms as follows.

Corollary G.3. *Let $0 < r \le q \le \infty$ and let S be a subset of V of relative measure $\ell \in (0,1]$. Then for every polynomial P of degree k the inequality*

$$\left\{\frac{1}{|V|}\int_{|V|} |P|^q dx\right\}^{\frac{1}{q}} \le (rk+1)^{\frac{1}{r}}\gamma(k,n)\ell^{-k}\left\{\frac{1}{|S|}\int_S |P|^r dx\right\}^{\frac{1}{r}} \tag{G.9}$$

holds for $\gamma(k,n) := \frac{1}{2}(4n)^k$.

Proof. It suffices to consider the case $q = \infty$. Due to the homogeneity of (G.9) we may assume that

$$\max_V |P| = 1.$$

Further, for $t \in (0,1]$, we define the sublevel set of P by

$$L_t := \{x \in V\,;\, |P(x)| \le t\}.$$

Applying to this subset inequality (G.8) we get

$$1 = \max_V |P| \le \gamma(k,n)\left(\frac{|V|}{|L_t|}\right)^k \cdot t$$

which implies

$$|L_t| \le |V|\big(\gamma(k,n)t\big)^{\frac{1}{k}}. \tag{G.10}$$

To proceed, we need the notion of *rearrangement*, see, e.g., [Zi-1989, sec. 1.8] for details.

Let (Σ,μ) be a measure space and $f : \Sigma \to \mathbb{R}$ be μ-measurable. The nonincreasing function $m(f) : (0,+\infty) \to \mathbb{R}_+ \cup \{+\infty\}$ is then given by

$$m(f\,;\,t) := \mu\{\sigma \in \Sigma\,;\, |f(\sigma)| > t\},$$

while the rearrangement $f^* : (0, \mu(\Sigma)] \to \mathbb{R}_+ \cup \{+\infty\}$ is defined by

$$f^*(s) := \inf\big\{t\,;\, m(f\,;\,t) \le s\big\}.$$

The functions $|f|$ and f^* are *equimeasurable*; therefore, for $0 < r < \infty$,

$$\int\limits_0^{\mu(\Sigma)} f^*(s)^r ds = \int_\Sigma |f|^r d\mu.$$

Using these definitions we relate $|L_t|$ to the rearrangement of the trace $P|_V$. Specifically,

$$|L_t| = |V| - m\big(P|_V\,;\,t\big)$$

and therefore the converse to the function $t \mapsto |L_t|$ is equal to the function $t \mapsto \big(P|_V\big)^*\big(|V| - t\big)$.

The latter is estimated by (G.10) to give

$$\big(P|_V\big)^*\big(|V| - t\big) \ge \frac{1}{\gamma(k,n)}\left(\frac{t}{|V|}\right)^k.$$

It remains to note that for $S \subset V$ and $0 \le t \le |S|$,

$$\big(P|_S\big)^*(t) \ge \big(P|_V\big)^*(t)$$

and therefore

$$\begin{aligned}\int\limits_0^{|S|}\left[\frac{1}{\gamma(k,n)}\left(\frac{t}{|V|}\right)^k\right]^r dt &\le \int\limits_0^{|S|}\Big[\big(P|_S\big)^*(|S| - t)\Big]^r dt \\ &= \int\limits_0^{|S|}\big(P|_S\big)^*(t)^r dt = \int_S |P|^r dx.\end{aligned}$$

Integrating and raising to the power $\frac{1}{r}$ we get the inequality

$$\frac{1}{(rk+1)^{\frac{1}{r}}}\,\frac{1}{\gamma(k,n)}\left(\frac{|S|}{|V|}\right)^k \le \left(\frac{1}{|S|}\int_S |P|^r dx\right)^{\frac{1}{r}}$$

which is equivalent to (G.8) with $q = \infty$. □

Remark G.4.

(a) Let $S = V$; then (G.9) yields the *inverse Hölder inequality* for polynomials. The constant obtained is, up to a numerical factor, optimal for $r \le 1$ and $q = \infty$ but may be essentially improved for other values of r and q, see the paper [CW-2001] by Carberry and Wright and the references therein.

(b) Fix a compact convex body V and denote by $\rho_k(\ell, V)$ the optimal constant in inequality (G.2). By $\rho_k^{\text{conv}}(\ell, V)$ we denote the similar constant, where S now runs over *convex* subsets of V of relative measure ℓ. It is conjectured by Yu. Brudnyi and Ganzburg in [BrG-1973] that

$$\rho_k^{\text{conv}}(\ell, V) = \rho_k(\ell, V). \tag{G.11}$$

If this were true, we would obtain the following asymptotic for $\rho_k(\ell, V)$ in the case of V being the Euclidean unit ball B^n:

$$\rho_k(\ell, B^n) = \frac{1}{2}\left(\frac{c_n}{\ell}\right)^k + o(\ell^{-k}),$$

where $c_1 := 8$ and

$$c_n := 4|B^n|\left(1 + \frac{1}{n}\right)^{\frac{n+1}{2}}\left(1 - \frac{1}{n}\right)^{\frac{n-1}{2}}, \quad n \geq 2,$$

see [BrG-1973, Corollary 1].

Finally, we present a Markov type inequality for derivatives of a polynomial. The classical Markov inequality asserts that

$$\max_{|t|\leq 1} \left|P'(t)\right| \leq k^2 \max_{|t|\leq 1} \left|P(t)\right|, \tag{G.12}$$

provided that P is a univariate polynomial of degree k.

Now let $Q \subset \mathbb{R}^n$ be a cube of side length $2r$ and consider polynomial $P \in \mathcal{P}_{k,n}$. Then (G.12) immediately implies that

$$\max_Q |\partial^\alpha P| \leq \left(\frac{k^2}{r}\right)^{|\alpha|} \max_Q |P|.$$

Combining this and inequality (G.9) we get, for this setting and S being a subset of Q of relative measure $\ell \in (0, 1)$, the following.

Corollary G.5. *For $0 < r \leq q \leq \infty$ the inequality*

$$\left\{\frac{1}{|Q|}\int_Q |D^\alpha P|^q dx\right\}^{\frac{1}{q}} \leq c\ell^{-k}\left(\frac{k^2}{r}\right)^{|\alpha|}\left\{\frac{1}{|S|}\int_S |P|^r dx\right\}^{\frac{1}{q}} \tag{G.13}$$

holds with $c := (rk+1)^{\frac{1}{r}}\gamma(k, n)$.

The first inequality of this kind was due to Di Giorgi and was published in the paper [Cam-1964] by Campanato. It states that there exists an unspecified constant $c(k, n, \ell) < \infty$ such that (G.13) holds with the factor $\frac{c(k,n,\ell)}{r^{|\alpha|}}$ in the right-hand side. The proof is based on a compactness argument.

http://www.springer.com/978-3-0348-0208-6

Methods of Geometric Analysis in Extension and Trace Problems
Volume 1
Brudnyi, A.; Technion R&D Foundation Ltd, P.Y.B.
2012, XXIII, 560 p. 11 illus., Hardcover
ISBN: 978-3-0348-0208-6
A product of Birkhäuser Basel