Preface

The title refers to extension and trace problems for Lipschitz functions on metric spaces and smooth functions on subsets of $\mathbb{R}^n$, linked by a unified geometric analysis approach. While methods of Geometric Analysis are clearly relevant to the study of functions on metric spaces, the smooth extension and trace problems are of pure analytic origin and seem to require other tools for their study. An approach allowing us to involve Geometric Analysis in this study will be briefly considered below; to begin we discuss some features of the problems in question.

Seemingly very specific, mostly intended for applications, this topic has been, from its very beginning, a powerful source of ideas, concepts and methods that essentially influenced and sometimes even unified considerable parts of Analysis. These include Complex and Differential Analysis, PDE and Image Processing (to name but a few) but the most spectacular example is Algebraic Topology. Here the problem of existence (or nonexistence) of continuous extensions that arose about one hundred years ago (Lebesgue, Brower) has eventually turned into a main object of study (see, in particular, the Princeton Colloquium Lectures by N. Steenrod).

This amazing research power of the topic may apparently be explained by the nature of human knowledge that, citing philosopher W. V. Quine "… is a man-made fabric which impinges on experience only along the edges".

In the case of metric spaces (whose study forms a considerable part of the book), continuous maps are naturally replaced by their metric equivalents, Lipschitz and rough Lipschitz maps and the like. Moreover, together with the analysis of continuity, compactness and so forth, one should use concepts and tools of (interpreted broadly) Geometric Analysis such as geometric measures and probabilities, metric graphs, metric analogs of dimension and related methods and results.

In accordance with a diversity of geometric and analytic objects joined by the concept of metric space, the fundamental extension problem of topology is now divided into several components. The most natural analog asks about the existence of a Lipschitz extension for a map from a subset of one metric space to another. For real-valued functions a positive answer is given by a remarkably simple nonlinear operator (McShane, 1934) while sufficient conditions for the general problem were found only a few years ago (U. Lang-Schlichenmaier, 2005). The answer includes several concepts (Nagata dimension, Lipschitz n-connectivity etc.) belonging to a new developing area that may be naturally called Lipschitz Topology.

If the image of a Lipschitz map is a normed linear space, one can study a linear (*simultaneous*) Lipschitz extension of the map and the norm evaluation of the corresponding linear extension operator using geometric characteristics of the domain. Unlike the continuous case where a positive solution is due to Borsuk (1933) for separable metric spaces and Dugundji (1951) for the general case, the answer is now known to be negative even for real-valued Lipschitz functions (Pelczynski, 1960).

A wide class of metric spaces admitting simultaneous Lipschitz extensions was discovered simultaneously by three different approaches that exploit, respectively, tools of Lipschitz Topology (Lang-Schlichenmaier), Geometric Measure Theory (the authors) and Probabilistic Combinatorics (Lee-Naor). The first approach estimates the corresponding norms by unspecified constants while the two others give effective upper bounds.

A class of metric spaces with the required extension property may be essentially enlarged by using bi-Lipschitz embeddings into universal "nice" spaces such as the classical metric forms $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$. To successfully apply this approach one should really know much more about universal spaces for given classes of metric spaces, quasi-isometric invariants and the like. As model cases one may point to Urysohn's universal space (1928) containing isometric copies of all separable metric spaces, the Bonk-Schramm result (2000) on the universal property of $\mathbb{H}^n$ with respect to the class of Gromov hyperbolic spaces of bounded geometry and results of Lipschitz Topology (metric, Hausdorff and Nagata dimensions and so forth). Nevertheless, even the results proved so far have been applied to obtain extension theorems of considerable value.

The final part of the book is devoted to the extension and trace problems for spaces of multivariate differentiable and smooth functions[1] and related jet spaces. Given such a *smoothness space* X on $\mathbb{R}^n$ and a function f on a subset $S \subset \mathbb{R}^n$ the following problems will be studied:

Qualitative trace problem. Does there exist a function $F \in X$ whose trace $F|_S$ agrees with f?

Quantitative trace problem. Find an effective two-sided estimate for the trace norm of f, i.e., for $\inf\{\|F\|_X\, ;\, f = F|_S\}$.

Simultaneous extension problem. Does there exist a linear bounded extension operator from the trace space $X|_S$ into X?

These problems go back to the two classic papers that Whitney published in 1934 where the first solves all these problems for the jet space generated by functions from the space $C_b^k(\mathbb{R}^n)$ of all bounded k-times continuously differentiable functions with bounded higher derivatives. The second, less well known paper,

[1] i.e., with controlled moduli of continuity of given order of their higher derivatives

solves all of them for the space $C_b^k(\mathbb{R})$ of univariate functions. The indication of number I in the title of the second paper may be seen as Whitney's intention (or appeal?) to proceed with the multivariate case. However, considerable progress in this direction did not appear for more than fifty years.

To illustrate the seemingly unsurmountable complexity of the multivariate problem one can compare the two simplest cases, namely, the spaces $C_b^1(\mathbb{R}^n)$ for the known case $n = 1$ and the recently studied case $n > 1$.

For the quantitative trace problem, the Whitney theorem characterizes the extendability of a univariate function $f : S \to \mathbb{R}$ to a function of $C_b^1(\mathbb{R})$ by behaviour of f on 3-point subsets of S and the number 3 of points is sharp (in the seemingly analogous case of Lipschitz functions, the number of required points is 2 for every n). However, the corresponding number of points for the multivariate case is of exponential growth in the dimension!

For the second trace problem Whitney's theorem gives an elegant explicit formula evaluating the trace norm via three point divided differences. In contrast, a result of this kind is unknown for the multivariate case.

Finally, a simultaneous extension for $n = 1$ is given by Whitney's (linear) extension construction developed in his first paper. Since this construction is, in a sense, universal, it also solves positively the simultaneous extension problem for the space $C_u^1(\mathbb{R})$ consisting of functions with uniformly continuous derivatives. However, this latter result does not hold for $n > 1$.

It is worth noting that Whitney's problem, as special as it may seen, is, in fact, one of the most challenging topics of a vast, intensively developing area, that studies problems with incomplete data. This, in particular, includes Differential Analysis on sets without differential structure (large finite subsets of $\mathbb{R}^n$, metric spaces etc.), inverse and incorrect problems of Mathematical Physics and such fields of Applied Mathematics as image restoration, mathematical tomography and computer graphics. The concepts and methods that have been and will be developed for the study of Whitney's problem will doubtless play a considerable role in the development of this area.

The road to applying geometric methods to the pure analytic issue in question is opened by Local Approximation Theory developed by the second named author in the 1960s. For a wide class of smoothness spaces including Sobolev and Besov spaces over $L_\infty(\mathbb{R}^n)$ and the associated jet spaces, the theory gives a complete description of their trace spaces to an arbitrary subset of $\mathbb{R}^n$. The sufficiency part of these criteria is equivalently reformulated as existence of Lipschitz selections for set-valued functions on specially constructed metric spaces or more involved spaces with values in the set $\mathcal{C}(\mathbb{R}^n)$ of nonempty convex subsets of $\mathbb{R}^n$.

A very attractive example of this kind is the next conjecture proved in several special cases.

Let $\varphi : \mathcal{M} \to \mathcal{C}(\mathbb{R}^n)$ be a set-valued map on a metric space $\mathcal{M}$. Assume that its trace to every 2^n point subset admits a 1-Lipschitz selection. Then φ itself admits a selection with Lipschitz constant depending only on n.

The first breakthrough in the realization of this program was due to Shvartsman who in his 1984 PhD thesis proved the aforementioned conjecture for the set of affine subspaces of $\mathbb{R}^n$ and derived from here a solution to the qualitative trace problem for Zygmund spaces[2]. He also constructed a very ingenious example showing that the cardinality of subsets involved in the solution is sharp (hence, the number 2^n in the Lipschitz selection conjecture is also sharp).

The next important result is a solution to the linear (simultaneous) extension problem for Zygmund spaces (Yu. Brudnyi and Shvartsman, 1985). The involved set-valued map now depends *linearly* on a functional parameter and the required Lipschitz selection should preserve this dependence.

Subsequent work in this direction was started only at the end of the previous century. New results obtained concern, in particular, spaces $C^{1,\omega}(\mathbb{R}^n)$ (the second of the only known sharp results) and the generalization of the Whitney-Glaeser theorem to the spaces of jets whose higher components satisfy the Zygmund condition. The research potential of the discussed method is by no means exhausted by these briefly discussed results. A challenging task justifying this claim is the proof of the trace-extension problem for the space of multivariate C^k functions whose higher derivatives satisfy the Zygmund condition.[3]

A new powerful method for the study of the classical Whitney problem (i.e., for spaces $C_b^k(\mathbb{R}^n)$ and $C^{k,\omega}(\mathbb{R}^n)$) was invented by Ch. Fefferman. In a series of papers (2003–2009) he solved for this case the qualitative trace and simultaneous extension problems and made important advancements (in partial collaboration with Klartag) in the solution of the quantitative trace problem. This breakthrough work is only surveyed in this book, since even a superficial explanation of its proofs requires over a hundred pages of an extremely complicated text. However, a comprehensive understanding of Fefferman's proofs and ideas beyond them is one of the concrete aims of the theory that we can believe will lead to new discoveries in the area.

Now, we briefly discuss the contents of the book. More information can be found in the introduction to each chapter.

The book is divided into two volumes each consisting of two parts. Volume II is devoted to the study of the main themes, extension and trace problems for Lipschitz and smooth functions, respectively; see Preface to this volume for more information. Part 1 of Volume I (Chapters 1, 2) is introductory and gives background material and the important initial results that motivated and shaped the area. These include classical results of Lebesgue, Brouwer, Whitney, Bernstein and Valentine that have never appeared in book form.

Appendices to each chapter intend to familiarize the beginners with some facts and concepts used in the subsequent text. Those to Chapter 1 contain, in

[2]defined by the norm $\sup_{\mathbb{R}^n}|f| + \sup_{x\neq y}\dfrac{|f(x)-2f\left(\frac{x+y}{2}\right)+f(y)|}{\omega(\|x-y\|)}$

[3]Note that Taylor polynomials of order $k+1$ may nowhere exist for functions of this space. This explains the inapplicability to this case of the Fefferman approach that we discuss below.

particular, the basic material on covering dimension and its relation to continuous extensions and Helly's topological generalization of his classical convex body result. This includes some topics in Algebraic Topology used in the proofs there and in the subsequent parts of the book.

Appendices to Chapter 2 contain, in particular, two fundamental facts of Local Polynomial Approximation Theory: the multivariate Whitney and Remez type inequalities.

Finally, Part 2 of Volume I (Chapters 3–5) contains concepts and results of Metric Space Theory used in the subsequent parts of the book. To avoid fragmenting the text into a discontinuous string of theorems, we add some background and several classical results that turn this part into a specifically oriented course on Metric Space Theory. Most of the basic results there were proved not long ago and have never appeared in book form. They, along with several classical results, are accompanied by detailed proofs.

Comments to each chapter discuss generalizations and related results. The latter are presented partly in a historical context; we believe that the reader may learn something important from such a presentation.

Reader's Guide

The reader whose main interest is Lipschitz Analysis may begin with Chapter 1 (Volume I) and its appendices presenting topological background and motivations for Lipschitz extension results studied in Chapters 6–8 (Volume II). Geometric Analysis results used for this study are contained in Sections 3.1, 3.2, 4.2, 4.3, 4.6, 5.1 and 5.2 of Volume I.

The reader interested in Whitney's problems may begin with Chapter 2 of Volume I and its appendices presenting background material on Multivariate Differential Analysis, the aforementioned classical Whitney theorems and related conjectures and several basic facts of Local Approximation Theory. This material forms an introduction to the intensive study of Whitney's extension and trace problems presented in Chapters 9, 10 of Volume II. The Geometric Analysis results used in this study are contained in Sections 3.2, 4.4, 5.3 and 5.4 of Volume I.

The prerequisites for reading this book include the material covered by first year graduate study (in particular, Linear Algebra, Real Analysis, Functional Analysis and some General Topology). More specialized topics are carefully presented in survey or more extended form. We either give a complete proof or else a detailed outline of the proof for very recent results. Many of these proofs are based on rather complicated geometric constructions; their study may be essentially facilitated by using appropriate geometric presentations. We give some of them in the book and strongly recommend the readers to draw their own geometric sketches in every such proof.

The vast majority of the main results presented in the book appear for the first time in book form. This holds not only for recent results but also for some of the deep classical results mentioned above.

Acknowledgments. It is a pleasure to express our thanks to several colleagues who helped us in preparation of this book.

Len Bos' suggestions and remarks allowed us to essentially improve the exposition of Chapters 1–3 and 9 while Peter Zwengrowski helped us to improve the exposition of Chapter 5. Charles Fefferman prepared for us a survey of his extension results and provided us with preprints of his papers. Our special thanks go to Pavel Shvartsman whose PhD Thesis and collaboration with us are reflected in many deep results of this book.

We thank also Ms. Galya Khanin whose marvelous work transformed our rough handwritten text into an excellent manuscript.

http://www.springer.com/978-3-0348-0208-6

Methods of Geometric Analysis in Extension and Trace Problems
Volume 1
Brudnyi, A.; Technion R&D Foundation Ltd, P.Y.B.
2012, XXIII, 560 p. 11 illus., Hardcover
ISBN: 978-3-0348-0208-6
A product of Birkhäuser Basel