Chapter 7

Simultaneous Lipschitz Extensions

The chapter is designed for the study of metric spaces admitting simultaneous Lipschitz extensions for each of their subspaces. The theory of such spaces developed by the authors of the book will be presented here while some of its basic ingredients (spaces of pointwise homogeneous type and of metric balls, embeddings in spaceforms etc.) have been considered in Volume I. This allows us to simplify and shorten the exposition. The theory includes methods of constructing the corresponding linear extension operators. For the time being, three different methods have been developed independently and at the same time. The first of them, due to Lang and Schlichenmaier, was presented in Chapter 6. Because of the generality of their approach working both in nonlinear and linear settings, the corresponding estimates of extension constants are far from being optimal. Two other methods are much better in this respect, giving estimates close to optimal. One of them, due to Lee and Naor, exploits a probabilistic argument based on the stochastic "padding decompositions" that first appeared in Computer Science, see, e.g., Volume I, Section 4.1. Another one, due to the authors of the present book, is constructive and covers a wider class of metric spaces.

Now we briefly describe the content of the chapter.

Section 7.1 studies the basic characteristics of the spaces in question. The most important of them, the weak finiteness property, reduces the problem of constructing linear extension operators to that for finite metric spaces. This result is then exploited in a few situations here and in the forthcoming chapter.

Section 7.2 presents a construction of a linear extension operator for Lipschitz functions over metric spaces of pointwise homogeneous type. So, these spaces have the simultaneous Lipschitz extension property, the class of all spaces possessing this property is denoted by $\mathcal{SLE}$. As a consequence we prove that combinatorial metric trees, doubling metric spaces and direct sums of arbitrary finite combi-

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nations of these spaces belong to $\mathcal{SLE}$ and we obtain effective estimates of the corresponding extension constants.

In Section 7.3, we prove that, under mild restrictions, locally doubling metric spaces containing uniform lattices belong to $\mathcal{SLE}$.

In Section 7.4, we study metric spaces having the universal simultaneous Lipschitz extension property (shortly, this class is denoted by $\mathcal{ULE}$). This means that the isometric copy of a subspace of such a space $\mathcal{M}$ sitting in another metric space $\widetilde{\mathcal{M}}$ admits a linear continuous Lipschitz extension to $\widetilde{\mathcal{M}}$ with the extension constant bounded by $c = c(\mathcal{M})$. In particular, Gromov hyperbolic spaces of bounded geometry, combinatorial metric trees, doubling metric spaces and direct sums of their arbitrary finite combinations belong to $\mathcal{ULE}$. For single trees and doubling spaces this result was firstly established by Lee and Naor using the aforementioned probabilistic approach. Though nonconstructive, the method of these authors is of considerable interest. We briefly discuss its basic features at the final part of the section.

7.1 Characterization of simultaneous Lipschitz extension spaces

7.1.1 Basic notions

Let $(\mathcal{M}, d)$ be a metric space and X be a Banach space with norm $||\cdot||$. Recall that the space $\mathrm{Lip}(\mathcal{M}, X)$ of Banach-valued Lipschitz functions $f : \mathcal{M} \to X$ is defined by the seminorm

$$L(f; X) := \sup_{m \neq m'} \frac{||f(m) - f(m')||}{d(m, m')}.$$

In the case $X = \mathbb{R}$, we denote the corresponding space by $\mathrm{Lip}(\mathcal{M})$.

Further, we recall, see Volume I, Section 1.12,

Definition 7.1. A subset $S \subset \mathcal{M}$ admits a simultaneous Lipschitz extension with respect to X if there exists a linear bounded operator $T : \mathrm{Lip}(S, X) \to \mathrm{Lip}(\mathcal{M}, X)$ such that

$$Tf|_S = f, \quad f \in \mathrm{Lip}(S, X).$$

The set of all such operators is denoted by $\mathrm{Ext}(S, \mathcal{M}, X)$ and the optimal extension constant is given by

$$\lambda(S, \mathcal{M}, X) := \inf\{||T|| \ ; \ T \in \mathrm{Ext}(S, \mathcal{M}, X)\}.$$

(We assume that $\lambda(S, \mathcal{M}, X) = \infty$ if $\mathrm{Ext}(S, \mathcal{M}, X) = \emptyset$.) In the case $X = \mathbb{R}$, we denote the corresponding set and the constant by $\mathrm{Ext}(S, \mathcal{M})$ and $\lambda(S, \mathcal{M})$.

Definition 7.2. A metric space $(\mathcal{M}, d)$ has the simultaneous Lipschitz extension property (abbreviated $\mathcal{SLE}$), if $\mathrm{Ext}(S, \mathcal{M}) \neq \emptyset$ for each subspaces $S \in \mathcal{M}$.

Finally, we define the *global linear Lipschitz extension constant*

$$\lambda(\mathcal{M}, X) := \sup\{\lambda(S, \mathcal{M}, X) \; ; \; S \subset \mathcal{M}\}.$$

In the case $X = \mathbb{R}$, we denote this constant by $\lambda(\mathcal{M})$.

Our first result connects $\lambda(\mathcal{M}, X)$ and $\lambda(\mathcal{M})$ for certain Banach spaces X. Recall that a Banach space X is said to be *constrained in its bidual* if X is the range of a norm one linear projection when canonically embedded in its bidual X^{**}. It is well known (see Dixmier [Di-1948]) that an L_1-space or a dual Banach space X (i.e., $X = Y^*$ for a Banach space Y) meets this condition.

Proposition 7.3. *If X is constrained in its bidual, then for every subspace S of a metric space $(\mathcal{M}, d)$,*

$$\lambda(S, \mathcal{M}, X) = \lambda(S, \mathcal{M}).$$

In particular, it is true that

$$\lambda(\mathcal{M}, X) = \lambda(\mathcal{M}).$$

Proof. First we show that the constant in the left-hand side is bounded by that on the right. Since the latter remains unchanged after replacing $\mathrm{Lip}(\mathcal{M})$ by its subspace $\mathrm{Lip}_0(\mathcal{M})$ (containing functions vanishing at a fixed point of $\mathcal{M}$, say, m^*), we will work with the latter space and its canonical predual $\mathcal{F}(\mathcal{M})$ introduced in subsection 4.6.3 of Volume I. Moreover, S is now a *pointed subset* of $(\mathcal{M}, m^*, d)$.

Let $\lambda(S, \mathcal{M}) < \infty$ and $\epsilon > 0$ be fixed. Take an operator $T \in \mathrm{Ext}(S, \mathcal{M})$ with

$$||T|| \leq \lambda(S, \mathcal{M}) + \epsilon.$$

The conjugate operator T^* acts from $\mathrm{Lip}_0(\mathcal{M})^* = \mathcal{F}(M)^{**}$ into $\mathrm{Lip}_0(S)^* = \mathcal{F}(S)^{**}$. It is a matter of definition to check that T^* is a *linear continuous projection* of $\mathcal{F}(M)^{**}$ onto $\mathcal{F}(S)^{**}$. Since $\mathcal{F}(\mathcal{M})$ is isometrically embedded into $\mathcal{F}(\mathcal{M})^{**}$ and $\mathcal{M}$ is isometrically embedded into $\mathcal{F}(\mathcal{M})$, see Volume I, subsection 4.6.3, we may and will regard $\mathcal{M}$ as a (metric) subspace of $\mathcal{F}(\mathcal{M})^{**}$. Then $T^*|_{\mathcal{M}}$ is well-defined and is a Lipschitz map from $\mathcal{M}$ into $\mathcal{F}(\mathcal{M})^{**}$ such that

$$L(T^*|_{\mathcal{M}} \; ; \; \mathcal{F}(S)^{**}) \leq ||T||.$$

Using these facts we construct a linear extension operator $\widehat{T} : \mathrm{Lip}_0(S, X) \to \mathrm{Lip}_0(\mathcal{M}, X)$ satisfying $||\widehat{T}|| \leq \lambda(\mathcal{M}, S) + \epsilon$. This, clearly, implies the required inequality

$$\lambda(S, \mathcal{M}, X) \leq \lambda(S, \mathcal{M}).$$

To introduce $\widehat{T}$ we fix $f \in \mathrm{Lip}_0(S, X)$. According to Theorem 4.91 of Volume I, there exists a linear operator $T_f : \mathcal{F}(S) \to X$ such that

(i) $T_f|_S = f$;

(ii) T_f linearly depends on f;

(iii) $||T_f|| \le L(f; X)$.

We then have for $T_f^{**} : \mathcal{F}(S)^{**} \to X^{**}$,

$$T_f^{**}|_S = T_f|_S = f.$$

Since X is constrained in its bidual, there exists a linear projection $P : X^{**} \to X$ of norm 1. Composing all the operators introduced we define the function

$$\widehat{T}(f) := P \circ T_f^{**} \circ (T^*|_{\mathcal{M}})$$

acting from $\mathcal{M}$ into X. By linearity of T_f in f, the operator $f \mapsto \widehat{T}(f)$ is linear. Moreover, $\widehat{T}(f)|_S = f$ and $L(\widehat{T}(f); X) \le ||T|| L(f; X)$ by definition. Hence $\widehat{T} \in \mathrm{Ext}(S, \mathcal{M}, X)$ and its norm is bounded by $||T|| \le \lambda(S, \mathcal{M}) + \epsilon$, as required.

Conversely, assume that $\lambda(S, \mathcal{M}, X) < \infty$. Let $V \subset X$ be a one-dimensional subspace of X and $v \in V$ be such that $||v|| = 1$. According to the Hahn-Banach theorem there is a linear continuous functional $F_v : X \to \mathbb{R}$ of norm 1 such that $F_v(v) = 1$. We embed $\mathbb{R}$ isometrically into X by the formula $E(t) := t \cdot v, \quad t \in \mathbb{R}$. The linear operator $\widetilde{E}f := E \circ f$, maps $\mathrm{Lip}_0(S)$ isometrically into $\mathrm{Lip}_0(S, X)$. Let $T \in \mathrm{Ext}(S, \mathcal{M}, X)$ be such that $||T|| \le \lambda(S, \mathcal{M}, X) + \epsilon$. Then the operator $\widehat{T} := F_v \circ T \circ \widetilde{E}$ maps $\mathrm{Lip}_0(S)$ into $\mathrm{Lip}_0(\mathcal{M})$ and satisfies $||\widehat{T}|| \le \lambda(S, \mathcal{M}, X) + \epsilon$, and $\widehat{T} \in \mathrm{Ext}(S, \mathcal{M})$. This implies that

$$\lambda(S, \mathcal{M}) \le \lambda(S, \mathcal{M}, X).$$

Finally, if either $\lambda(S, \mathcal{M})$ or $\lambda(S, \mathcal{M}, X)$ is ∞ the arguments presented show that the remaining quantity must be ∞ as well.

This and the above two inequalities complete the proof of the proposition. $\square$

Problem. *Is it true that $\lambda(\mathcal{M}, X) = \lambda(\mathcal{M})$ for any Banach space X?*

One of the main questions is whether the $\mathcal{SLE}$ property of $\mathcal{M}$ is equivalent to the finiteness of $\lambda(\mathcal{M})$. In general the answer is unknown. In this part, we give the positive answer to this question for some important classes of metric spaces. To formulate the result let us recall that a metric space $\mathcal{M}$ is *proper* (or boundedly compact), if every closed ball in $\mathcal{M}$ is compact. We also require the following

Definition 7.4. A metric space $\mathcal{M}$ has the weak transition property (WTP), if for some $C \ge 1$ and every finite set F and open ball B in $\mathcal{M}$ there is a C-isometry $\sigma : \mathcal{M} \to \mathcal{M}$ such that

$$B \cap \sigma(F) = \emptyset.$$

Theorem 7.5. *Assume that $\mathcal{M}$ is either proper or has the WTP. Then $\mathcal{M} \in \mathcal{SLE}$ if and only if $\lambda(\mathcal{M}) < \infty$.*

One of the main tools of the proof is the *finiteness property* of the characteristic λ established in subsection 7.1.2, asserting, in particular, that

$$\lambda(\mathcal{M}) = \sup_F \lambda(F)$$

where F runs through all finite subspaces of $\mathcal{M}$ (with the induced metric).

Proof. We begin with the case of $\mathcal{M}$ possessing the WTP. Assume that $\mathcal{M} \in \mathcal{SLE}$ but $\lambda(\mathcal{M}) = \infty$. From here and the finiteness property for $\lambda(\mathcal{M})$ proved in subsection 7.1.2 it follows that there is a sequence of finite sets F_j with $\lambda(F_j) \geq j$, $j \in \mathbb{N}$. This, in turn, leads to the inequalities

$$\inf\{||E|| \ ; \ E \in \operatorname{Ext}(F_j, \mathcal{M})\} \geq j \ , \ j \in \mathbb{N}. \tag{7.1}$$

Using the WTP of $\mathcal{M}$ we may choose an appropriate sequence of C-isometries σ_j such that for $G_j := \sigma_j(F_j)$ the following holds:

$$\operatorname{dist}(G_j, \cup_{i \neq j} G_i) \geq C \operatorname{diam} F_j \ , \ j \in \mathbb{N}. \tag{7.2}$$

For every $j \in \mathbb{N}$, fix a point $m_j^* \in G_j$. From (7.2) we derive that the operator N_j given for every $f \in \operatorname{Lip}(G_j)$ by

$$(N_j f)(m) := \begin{cases} f(m), & m \in G_j, \\ f(m_j^*), & m \in \cup_{i \neq j} G_i \end{cases} \tag{7.3}$$

belongs to $\operatorname{Ext}(G_j, G_\infty)$ where $G_\infty := \cup_{i \in \mathbb{N}} G_i$, and, moreover,

$$||N_j|| = 1. \tag{7.4}$$

In fact, if $f \in \operatorname{Lip}(G_j)$ and $m' \in G_j$, $m'' \in G_\infty \setminus G_j = \cup_{i \neq j} G_i$, then

$$\begin{aligned} |(N_j f)(m') - (N_j f)(m'')| &= |f(m') - f(m_j^*)| \\ &\leq ||f||_{\operatorname{Lip}(G_j)} \operatorname{diam} G_j \leq (C \operatorname{diam} F_j) ||f||_{\operatorname{Lip}(G_j)}. \end{aligned}$$

Together with (7.2) this leads to

$$|(N_j f)(m') - (N_j f)(m'')| \leq ||f||_{\operatorname{Lip}(G_j)} d(m', m'').$$

Since this holds trivially for all other choices of m', m'', the equality (7.4) has been established.

Since $\mathcal{M} \in \mathcal{SLE}$, there is an operator $E \in \operatorname{Ext}(G_\infty, \mathcal{M})$ with $||E|| \leq A$ for some $A > 0$. By (7.4) the operator $E_j := EN_j \in \operatorname{Ext}(G_j, \mathcal{M})$ and $||E_j|| \leq A$. Then the operator $\widetilde{E}_j$ given for $m \in \mathcal{M}$, $f \in \operatorname{Lip}(F_j)$ by the formula

$$(\widetilde{E}_j f)(m) := (E_j(f \circ \sigma_j^{-1}))(\sigma_j(m))$$

belongs to $\operatorname{Ext}(F_j, \mathcal{M})$ and its norm is bounded by $C^2 A$. Comparing with (7.1), we get for each j,

$$C^2 A \geq j \, ,$$

a contradiction.

Now, let $\mathcal{M}$ be proper. In order to prove that $\lambda(\mathcal{M}) < \infty$ we need

Lemma 7.6. *For every $m \in \mathcal{M}$ there is an open ball B_m centered at $\mathcal{M}$ such that $\lambda(B_m) < \infty$.*

Proof. Assume that this assertion does not hold for some m. Then there is a sequence of balls $B_i := B_{r_i}(m)$, $i \in \mathbb{N}$, centered at m of radii r_i such that $\lim_{i\to\infty} r_i = 0$ and $\lim_{i\to\infty} \lambda(B_i) = \infty$. According to the finiteness property for $\lambda(B_i)$ this implies the existence of finite subsets $F_i \subset B_i$, $i \in \mathbb{N}$, such that

$$\inf\{||E|| \; ; \; E \in \mathrm{Ext}(F_i, B_i)\} \to \infty \quad \text{as } i \to \infty. \tag{7.5}$$

We may assume that m, the common center of all B_j, belongs to every F_j. Otherwise we simply replace F_j by $G_j := F_j \cup \{m\}$ and show that (7.5) is true with G_i substituting for F_i. In fact, let L_i be the operator given for every $f \in \mathrm{Lip}(F_i)$ by

$$(L_i f)(m') := \begin{cases} f(m_i), & \text{if } m' = m, \\ f(m'), & \text{if } m' \in F_i, \end{cases}$$

where m_i is the closest to $\mathcal{M}$ point from F_i. Then $L_i \in \mathrm{Ext}(F_i, G_i)$ and $||L_i|| \le 2$, since

$$\begin{aligned} |(L_i f)(m) - (L_i f)(m')| &= |f(m_i) - f(m')| \\ &\le ||f||_{\mathrm{Lip}(F_i)} d(m_i, m') \le 2||f||_{\mathrm{Lip}(F_i)} d(m, m'). \end{aligned}$$

If now (7.5) does not hold for $\{G_i\}$ substituted for $\{F_i\}$, then there is a sequence $E_i \in \mathrm{Ext}(G_i, B_i)$ such that $\sup_i ||E_i|| < \infty$. But then the same will be true for the norms of $\widetilde{E}_i := E_i L_i \in \mathrm{Ext}(F_i, B_i)$, $i \in \mathbb{N}$, in contradiction with (7.5).

The proof will be now finished by the following argument. Choose a subsequence $F_{i_k} \subset B_{i_k} := B_{r_{i_k}}(m)$, $k \in \mathbb{N}$, such that

$$r_{i_{k+1}} < \min\{r_{i_k}, \; \mathrm{dist}(F_{i_{k+1}} \setminus \{m\}, \cup_{s<k+1} F_{i_s} \setminus \{m\})\}.$$

Without loss of generality we assume that the sequence $\{F_i\}$ satisfies this condition, i.e.,

$$r_{i+1} < \min\{r_i, \; \mathrm{dist}(F_{i+1} \setminus \{m\}, \cup_{s<i+1} F_s \setminus \{m\})\}. \tag{7.6}$$

Set $F_\infty := \bigcup_{s\in\mathbb{N}} F_s$ and show that

$$\mathrm{Ext}(F_\infty, \mathcal{M}) = \emptyset \tag{7.7}$$

which gives the required contradiction to the $\mathcal{SLE}$ of $\mathcal{M}$.

To prove this we choose the center m as a marked point of $\mathcal{M}$. Then all F_i are subspaces of $(\mathcal{M}, m)$ and $f(m) = 0$ if $f \in \mathrm{Lip}_0(F_i)$.

Define now the operator N_i by

$$(N_i f)(m') := \begin{cases} f(m'), & \text{if } m' \in F_i, \\ 0, & \text{if } m' \in F_\infty \setminus F_i. \end{cases}$$

Then for $f \in \mathrm{Lip}_0(F_i)$ and $m' \in F_i \setminus \{m\}$ and $m'' \in F_\infty \setminus F_i$ we have

$$|(N_i f)(m') - (N_i f)(m'')| = |f(m') - f(m)| \le ||f||_{\mathrm{Lip}_0(F_i)} d(m', m).$$

Moreover, $m'' \in B_j$ for some $j \neq i$. Assume, first, that $j > i$. Then by (7.6)

$$\begin{aligned} d(m', m) &\le d(m', m'') + d(m'', m) \le d(m', m'') + r_j \\ &\le d(m', m'') + \mathrm{dist}(F_j \setminus \{m\}, F_i \setminus \{m\}) \le 2d(m', m''). \end{aligned}$$

If now $j < i$, then by (7.6) we have

$$d(m', m) \le r_i < \mathrm{dist}(F_i \setminus \{m\}, F_j \setminus \{m\}) \le d(m', m'').$$

Combining these we prove that $N_i \in \mathrm{Ext}(F_i, F_\infty)$ and $||N_i|| \le 2$.

If now (7.7) is not true, then there is an operator $E \in \mathrm{Ext}(F_\infty, \mathcal{M})$, and so every operator $\widetilde{E}_i := EN_i$ belongs to $\mathrm{Ext}(F_i, \mathcal{M})$ and $||\widetilde{E}_i|| \le 2||E||$, $i \in \mathbb{N}$, a contradiction to (7.5).

The proof is complete. □

Remark 7.7. In this proof the properness of $\mathcal{M}$ is not used.

To proceed we need

Lemma 7.8. *Let $\mathcal{U}$ be a finite open cover of a compact set $C \subset \mathcal{M}$. Then there is a partition of unity $\{\rho_U\}_{U \in \mathcal{U}}$ on C subordinate to $\mathcal{U}$ such that every ρ_U is Lipschitz with a constant depending only on the cover.*

Proof. Define $d_U : \mathcal{M} \to \mathbb{R}$ by

$$d_U(m) := \mathrm{dist}(m, \mathcal{M} \setminus U), \quad m \in \mathcal{M}.$$

This function is supported on U and has the Lipschitz constant 1; moreover, $\sum_{U \in \mathcal{U}} d_U > 0$ on C. Putting now

$$\rho_U(m) := \frac{d_U(m)}{\sum_{U \in \mathcal{U}} d_U(m)}, \quad m \in C \cap U, \quad U \in \mathcal{U},$$

we get the required partition. □

The next result implies finiteness of $\lambda(\mathcal{M})$ for compact $\mathcal{M}$.

Lemma 7.9. *For every compact subspace $C \subset \mathcal{M}$ the constant $\lambda(C)$ is finite.*

Proof. We have to show that for every $S \subset C$ there is an operator $E_S \in \mathrm{Ext}(S, C)$ such that

$$\sup_S ||E_S|| < \infty. \tag{7.8}$$

We may assume that S and C are subspaces of $(\mathcal{M}, m^*)$ so that $f(m^*) = 0$ for f belonging to $\mathrm{Lip}_0(S)$ or $\mathrm{Lip}_0(C)$. By compactness of C and Lemma 7.6 there is a finite cover $\{U_i\}_{1\le i\le n}$ of C by open balls such that for some constant $A > 0$ depending only on C we have $\lambda(U_i) < A$, $1 \le i \le n$. By the definition of λ this implies the existence of $E_i \in \mathrm{Ext}(S \cap U_i, U_i)$ with

$$||E_i|| \le A, \quad 1 \le i \le n. \tag{7.9}$$

For $f \in \mathrm{Lip}_0(S)$ one sets

$$f_i := \begin{cases} f|_{S\cap U_i}, & \text{if } S \cap U_i \ne \emptyset, \\ 0, & \text{if } S \cap U_i = \emptyset \end{cases} \tag{7.10}$$

and introduces a function f_{ij} given on $U_i \cap U_j$ by

$$f_{ij} := \begin{cases} E_i f_i - E_j f_j, & \text{if } U_i \cap U_j \ne \emptyset, \\ 0, & \text{if } U_i \cap U_j = \emptyset, \end{cases} \tag{7.11}$$

here $E_i f_i := 0$, if $f_i = 0$. Then (7.9) implies that

$$||f_{ij}||_{\mathrm{Lip}(U_i\cap U_j)} \le 2A. \tag{7.12}$$

Moreover, we get

$$f_{ij} = 0 \quad \text{on} \quad S \cap U_i \cap U_j. \tag{7.13}$$

Finally, introduce a function g_i given on $C \cap U_i$ by

$$g_i(m) := \sum_{1\le j\le n} \rho_j(m) f_{ij}(m) \tag{7.14}$$

where $\rho_j := \rho_{U_j}$, $1 \le j \le n$, is the partition of unity of Lemma 7.8.

A straightforward computation leads to the equalities:

$$g_i - g_j = f_{ij} \quad \text{on} \quad U_i \cap U_j \cap C \quad \text{and} \quad g_i|_{S\cap U_i} = 0. \tag{7.15}$$

Introduce now an operator E_S given for $f \in \mathrm{Lip}_0(S)$ and $m \in U_i \cap C$ by

$$(E_S f)(m) := (E_i f_i - g_i)(m). \tag{7.16}$$

We will show that E_S is an extension operator. In fact, if $m \in S$, then $m \in S \cap U_i$ for some $1 \le i \le n$ and by (7.15) and (7.10) we get

$$(E_S f)(m) = (E_i f_i)(m) = f(m).$$

We now show that $E_S \in \mathrm{Ext}(S, C)$ and $||E_S||$ is bounded by a constant depending only on C. To this end we denote by $\delta = \delta(C) > 0$ the *Lebesgue number* of the cover $\mathcal{U}$, see, e.g., Lemma C.6 of Volume I. So every subset of

$\cup_{i=1}^n U_i$ of diameter at most δ lies in one of the U_i. Using this we first establish the corresponding Lipschitz estimate for $m', m'' \in (\cup_{i=1}^n U_i) \cap C$ with

$$d(m', m'') \le \delta. \tag{7.17}$$

In this case both $m', m'' \in U_{i_0}$ for some i_0. Further, (7.14)-(7.16) imply that for $m \in U_{i_0} \cap C$,

$$(E_S f)(m) = \sum_{U_i \cap U_{i_0} \neq \emptyset} (\rho_i E_i f_i)(m). \tag{7.18}$$

In this sum, each ρ_i is Lipschitz with the constant L depending only on the subspace C; moreover, $0 \le \rho_i \le 1$. In turn, $E_i f_i$ is Lipschitz on $U_i \cap U_{i_0}$ with constant A, see (7.9) (recall that $E_i f_i = 0$ if $S \cap U_i = \emptyset$).

If now $m \in U_i$ and $S \cap U_i \neq \emptyset$, then for arbitrary $m_i \in S \cap U_i$,

$$\begin{aligned}|(E_i f_i)(m)| &\le |(E_i f_i)(m) - (E_i f_i)(m_i)| + |(E_i f_i)(m_i)| \\ &\le A||f||_{\mathrm{Lip}_0(S)} d(m, m_i) + |f(m_i) - f(m^*)| \\ &\le A||f||_{\mathrm{Lip}_0(S)} (d(m, m_i) + d(m_i, m^*)).\end{aligned}$$

This implies for all $m \in C \cap U_i$ the inequality

$$|(E_i f_i)(m)| \le 2A \ \operatorname{diam} C \ ||f||_{\mathrm{Lip}_0(S)}. \tag{7.19}$$

Together with (7.18) this leads to the estimate

$$|(E_S f)(m') - (E_S f)(m'')| \le An(2L \ \operatorname{diam} C + 1)||f||_{\mathrm{Lip}_0(S)} d(m', m''), \tag{7.20}$$

provided that $m', m'' \in U_{i_0} \cap C$. To prove a similar estimate for $d(m', m'') > \delta$, $m', m'' \in C$, we note that the left-hand side in (7.20) is bounded by

$$2 \sup_{m \in C} |(E_S f)(m)| \le 4A \ \operatorname{diam} C \ ||f||_{\mathrm{Lip}_0(S)},$$

see (7.19). In turn, the right-hand side of the last inequality is less than or equal to $4\delta^{-1} A \cdot \operatorname{diam} C \cdot ||f||_{\mathrm{Lip}_0(S)} d(m', m'')$. Together this implies that E belongs to $\mathrm{Ext}(S, C)$ and that its norm is bounded by a constant depending only on C. $\square$

Lemma 7.10. *Assume that for a family of finite spaces* $\{F_i \subset (\mathcal{M}, m^*);\ i \in \mathbb{N}\}$,

$$\sup_i \lambda(F_i, \mathcal{M}) = \infty\ . \tag{7.21}$$

Then for every closed ball $\overline{B}$ *centered at* m^*,

$$\sup_i \lambda(F_i \setminus \overline{B}, \mathcal{M}) = \infty\ . \tag{7.22}$$

Proof. Arguing as in the proof of Lemma 7.6, we can assume that $m^* \in F_i$ for all $i \in \mathbb{N}$. By the same reason we may and will assume that all F_i contain a fixed point $m' \in \mathcal{M} \setminus \overline{B}$. If now (7.22) is not true, then for some $A_1 > 0$ there are operators E_i^1, $i \in \mathbb{N}$, such that

$$E_i^1 \in \mathrm{Ext}(F_i \setminus \overline{B}, \mathcal{M}) \quad \text{and} \quad ||E_i^1|| \le A_1; \tag{7.23}$$

as above we set $E_i^1 := 0$, if $F_i \setminus \overline{B} = \emptyset$.
Let $2B$ be the open ball centered at m^* and of twice the radius of B. Introduce an open cover of $\mathcal{M}$ by

$$U_1 := \mathcal{M} \setminus \overline{B} \quad \text{and} \quad U_2 := 2B, \tag{7.24}$$

and let $\{\rho_1, \rho_2\}$ be the corresponding Lipschitz partition of unity (cf. Lemma 7.8) given for $m \in \mathcal{M}$ by

$$\rho_j(m) := \frac{d_{U_j}(m)}{d_{U_1}(m) + d_{U_2}(m)}, \quad j = 1, 2.$$

By this definition

$$|\rho_j(m_1) - \rho_j(m_2)| \le \frac{3d(m_1, m_2)}{\max_{k=1,2}\{d_{U_1}(m_k) + d_{U_2}(m_k)\}}, \quad m_1, m_2 \in \mathcal{M}. \tag{7.25}$$

Set now $H_i := F_i \cap 2B$, $i \in \mathbb{N}$. Since these are subsets of the compact set $2\overline{B}$, Lemma 7.9 gives

$$\sup_i \lambda(H_i, 2B) \le \lambda(2\overline{B}) < \infty.$$

This, in turn, implies the existence of operators E_i^2, $i \in \mathbb{N}$, such that

$$E_i^2 \in \mathrm{Ext}(H_i, 2B) \quad \text{and} \quad ||E_i^2|| \le A_2 \tag{7.26}$$

with A_2 independent of i.

We now follow the proof of Lemma 7.9 in which the set S and compact subspace $C \supset S$ are replaced by F_i and the (noncompact) space $\mathcal{M}$, respectively, and the cover by that in (7.24). Since

$$H_i = F_i \cap U_2 \quad \text{and} \quad F_i \setminus \overline{B} = F_i \cap U_1,$$

we can use in our derivation the operators E_i^j, $j = 1, 2$, instead of those in (7.9). By (7.23) and (7.26) inequalities similar to (7.9) hold for these operators. Then we set $f_j := f|_{F_i \cap U_j}$, and define for $f \in \mathrm{Lip}_0(F_i)$ functions $f_{12} := -f_{21}$ on $U_1 \cap U_2$ and g_1 on U_1 and g_2 on U_2 by

$$f_{12} := E_i^1 f_1 - E_i^2 f_2, \quad g_1 := \rho_2 f_{12}, \quad g_2 := \rho_1 f_{21}.$$

Finally, we introduce the required operator E_i on $\mathrm{Lip}_0(F_i)$ given for $F_i \cap U_j$, $j = 1, 2$, by
$$(E_i f)(m) := (E_i^j f_j)(m) - g_j(m).$$
As in Lemma 7.9, E_i is an operator extending functions from F_i to all of $\mathcal{M}$. To estimate the Lipschitz constant of $E_i f$ we use the McShane theorem to extend each function $E_i^j f_j$ outside U_j so that its extension $\widetilde{F}_j$ satisfies

$$||\widetilde{F}_j||_{\mathrm{Lip}(\mathcal{M})} = ||E_i^j f_j||_{\mathrm{Lip}(U_j)}.$$

Now, the definition of E_i implies that

$$E_i f := \rho_1 \widetilde{F}_1 + \rho_2 \widetilde{F}_2.$$

Then as in the proof of (7.19) we obtain for arbitrary $m \in \mathcal{M}$,

$$|\widetilde{F}_j(m)| \leq \begin{cases} A_1 ||f||_{\mathrm{Lip}_0(F_i)} (d(m, m') + d(m', m^*)), & \text{if } j = 1, \\ A_2 ||f||_{\mathrm{Lip}_0(F_i)} d(m, m^*), & \text{if } j = 2. \end{cases}$$

This implies for all $m \in \mathcal{M}$ the inequality

$$|\widetilde{F}_j(m)| \leq \; A(d(m, m^*) + d(m', m^*)) ||f||_{\mathrm{Lip}_0(F_i)} \tag{7.27}$$

with $A := 2\max(A_1, A_2)$.
Together with (7.25) this leads to the estimate

$$\begin{aligned} &|(E_i f)(m_1) - (E_i f)(m_2)| \\ &\leq \left(\frac{3(d(m_1, m^*) + d(m', m^*))}{\max_{k=1,2}\{d_{U_1}(m_k) + d_{U_2}(m_k)\}} + 1 \right) 2A \, ||f||_{\mathrm{Lip}_0(F_i)} d(m_1, m_2). \end{aligned} \tag{7.28}$$

Since $\max_{k=1,2}\{d_{U_1}(m_k) + d_{U_2}(m_k)\} \geq R$, the radius of B, and

$$\lim_{d(m_1, m^*) \to \infty} \frac{d(m_1, m^*) + d(m', m^*)}{d_{U_1}(m_1) + d_{U_2}(m_1)} = 1,$$

(7.28) implies that $E_i \in \mathrm{Ext}(F_i, \mathcal{M})$ and its norm is bounded by a constant independent of i. By definition this yields

$$\sup_i \lambda(F_i, \mathcal{M}) < \infty$$

in a contradiction with (7.21). □

Now we will complete the proof of Theorem 7.5. Recall that it has been already proved for compact $\mathcal{M}$, see Lemma 7.9. So it remains to consider the case of proper $\mathcal{M}$ with
$$\operatorname{diam} \mathcal{M} = \infty. \tag{7.29}$$

In this case we will show that

$$\sup_F \lambda(F, \mathcal{M}) < \infty, \tag{7.30}$$

where F runs through all finite subsets $F \subset (\mathcal{M}, m^*)$. Since $\sup_F \lambda(F)$ is bounded by the supremum in (7.30), which, in turn, equals $\lambda(\mathcal{M})$, see Corollary 7.13, this proves the result.

Suppose on the contrary that (7.30) does not hold. Then there is a sequence of finite subsets $F_i \subset (\mathcal{M}, m^*)$, $i \in \mathbb{N}$, that satisfies the assumption of Lemma 7.10, see (7.21). We use this to construct a sequence G_i, $i \in \mathbb{N}$, such that

$$\lambda(G_i, M) \geq i - 1 \quad \text{and} \quad \operatorname{dist}(G_{i+1}, G_i) \geq \operatorname{dist}(m^*, G_i) \quad \text{for all} \quad i \in \mathbb{N}. \tag{7.31}$$

If this will be done we set $G_\infty := \cup_{i \in \mathbb{N}} G_i$, and use with minimal changes the argument of Lemma 7.6 to show that

$$\operatorname{Ext}(G_\infty, \mathcal{M}) = \emptyset.$$

Since this contradicts the $\mathcal{SLE}$ property of $\mathcal{M}$, the result will be proved.

To construct the required $\{G_i\}$, set $G_1 := F_1$ and assume that the first j terms of this sequence have already been defined. Choose the closed ball $\overline{B}$ such that

$$\operatorname{dist}(G_j, \mathcal{M} \setminus \overline{B}) \geq \operatorname{dist}(m^*, G_j)$$

(it exists because of (7.29)). Then apply Lemma 7.10 to find $F_{i(j)}$ such that

$$\lambda(F_{i(j)} \setminus \overline{B}, \mathcal{M}) \geq j - 1.$$

Setting $G_{j+1} := F_{i(j)} \setminus \overline{B}$ we obtain the next term satisfying condition (7.31).

The proof is complete. $\square$

Problem. *Does there exist a metric space $\mathcal{M} \in \mathcal{SLE}$ but with $\lambda(\mathcal{M}) = \infty$?*

In conclusion of this subsection, we point out two other problems whose solutions may lead to a better understanding of the nature of the extension constant λ regarding it as a function from the category of metric spaces into $\mathbb{R}_+ \cup \{\infty\}$.

Problem. *Suppose $\lambda(\mathcal{M}_j) < \infty$, $j = 1, 2$. Are $\lambda(\mathcal{M}_1 \times \mathcal{M}_2)$ and $\lambda(\mathcal{M}_1 \times_S \mathcal{M}_2)$ finite?*

Here S is a common subspace of $\mathcal{M}_1$ and $\mathcal{M}_2$ and $\mathcal{M}_1 \times_S \mathcal{M}_2$ is the metric space obtaining by gluing $\mathcal{M}_1$ and $\mathcal{M}_2$ along S, see Volume I, Example 3.54 (d) for its definition.

7.1.2 Finiteness property

In order to formulate this property, we require

Definition 7.11.

(a) A sequence of metric spaces $\{\mathcal{M}_i\}_{i\in\mathbb{N}}$ is said to be finitely convergent to a metric space $\mathcal{M}$ if for every finite subspace $F \subset \mathcal{M}$ there exists a sequence of finite subspaces $\{F_i \subset \mathcal{M}_i\}_{i\in\mathbb{N}_0}$ where $\mathbb{N}_0 \subset \mathbb{N}$ such that $\lim_{\mathbb{N}_0} F_i = F$ (convergence in the Gromov-Hausdorff metric).

(b) This sequence δ-converges to $(\mathcal{M}, d)$ if every $\mathcal{M}_i$ is bi-Lipschitz homeomorphic to $\mathcal{M}$ with distortion D_i and $\lim\limits_{i\to\infty} D_i = 1$.

Due to the definition of the Gromov-Hausdorff convergence, the basic condition of Definition 7.11 (a) is equivalent to existence of bi-Lipschitz homeomorphisms of F_i onto F with distortion D_i, $i \in \mathbb{N}_0$, such that $\lim\limits_{i\to\infty} D_i = 1$, see subsection 3.1.8 and Proposition 3.55 of Volume I.

In particular, δ-convergence of $\{\mathcal{M}_i\}$ to $\mathcal{M}$ implies that this sequence finitely converges to $\mathcal{M}$.

Theorem 7.12. *Suppose that a sequence of metric spaces $\{(\mathcal{M}_i, d_i)\}_{i\in\mathbb{N}}$ finitely converges to a metric space $(\mathcal{M}, d)$. Then:*

(a) *It is true that*

$$\lambda(\mathcal{M}) \leq \varlimsup_{n\to\infty} \lambda(\mathcal{M}_i).$$

(b) *If, in addition, $\{(\mathcal{M}_i, d_i)\}_{i\in\mathbb{N}}$ δ-converges to a metric subspace $S \subset \mathcal{M}$, then*

$$\lambda(\mathcal{M}) = \sup_{F\subset S} \lambda(F)$$

where F runs over all finite subspaces of S.

Choosing $S = \mathcal{M}$ we immediately obtain from part (b)

Corollary 7.13.

$$\lambda(\mathcal{M}) = \sup_{F} \lambda(F) \tag{7.32}$$

where F runs over all finite subspaces of $\mathcal{M}$.

Together with Theorem 7.12 this implies

Corollary 7.14. *Let $S \subset \mathcal{M}$ satisfy the assumptions of Theorem 7.12. Then*

$$\lambda(\mathcal{M}) = \lambda(S).$$

In the forthcoming proofs, the duality results of subsection 4.6.3 of Volume I will play a considerable role. For convenience of the reader we recall them now.

Let $(\mathcal{M}, m^*)$ be a pointed metric space and S be its subspace, i.e., S is a subspace of $\mathcal{M}$ containing m^*. By $\mathrm{Lip}_0(S)$ we denote a subspace of $\mathrm{Lip}(S)$ determined by the condition $f(m^*) = 0$.

Further, the *Lipschitz-free space* $\mathcal{F}(S)$ is the closed linear span in the dual space $\mathrm{Lip}_0(\mathcal{M})^*$ of the set of *point evaluations* (δ-functionals) $\{\delta_S(m)\}_{m\in S}$ where

$$\delta_S(m)(f) := f(m).$$

Equivalently, see Volume I, formula (4.131),

$$\mathcal{F}(S) := \overline{\mathrm{span}\ \delta_S(S)} \quad (\text{closure in } \ \ell_\infty(B_S))$$

where B_S stands for the closed unit ball of $\mathrm{Lip}_0(S)$.

Then the required dual results of subsection 4.6.3 of Volume I are summarized in

Proposition 7.15. (a) *The dual space $\mathcal{F}(S)^*$ is linearly isometric to* $\mathrm{Lip}_0(\mathcal{M})$.

(b) *A map $J_S : S \to \mathcal{F}(S)$ given by*

$$J_S(m) := \delta_S(m) \tag{7.33}$$

is a (nonlinear) isometric embedding.

(c) *$\mathcal{F}(S)$ is the minimal closed linear subspace of $\ell_\infty(B_S)$ containing $J_S(S)$.*

Proof of Theorem 7.12. We begin with the case of $S = \mathcal{M}$, i.e., with Corollary 7.13.

Proposition 7.16. *Let S be a given finite subspace of $(\mathcal{M}, m^*)$. Assume that for every finite subspace $G \supset S$ there exists an extension operator $E_G \in \mathrm{Ext}(S, G)$ and*

$$A := \sup_G ||E_G|| < \infty. \tag{7.34}$$

Then there exists an operator $E \in \mathrm{Ext}(S, \mathcal{M})$ such that

$$||E|| \le A. \tag{7.35}$$

Proof. Using the map J_G, the canonical linear embedding $\kappa_G : \mathcal{F}(G) \to \mathcal{F}(G)^{**}$ and the conjugate operator E_G^* we then define a vector-valued function $\phi_G : G \to \mathcal{F}(S)^{**}$ by

$$\phi_G := E_G^* \kappa_G J_G. \tag{7.36}$$

The function is well-defined, since the conjugate to the operator $E_G \in \mathrm{Ext}(S, G)$ acts from $\mathrm{Lip}_0(G)^* = \mathcal{F}(G)^{**}$ into $\mathrm{Lip}_0(S)^* = \mathcal{F}(S)^{**}$.

Lemma 7.17. (a) $\phi_G \in \mathrm{Lip}(G, \mathcal{F}(S)^{**})$ *and its norm satisfies*

$$||\phi_G|| \le A. \tag{7.37}$$

(b) *For $m \in S$,*

$$\phi_G(m) = \kappa_G J_G(m). \tag{7.38}$$

In particular, $\phi_G(m^*) = 0$.

Proof. (a) Let $m \in G$ and $h \in \mathrm{Lip}_0(S)(= \mathcal{F}(S)^*)$. By (7.36)

$$\langle \phi_G(m),\ h\rangle = \langle \kappa_G J_G(m),\ E_G h\rangle = \langle E_G h,\ J_G(m)\rangle = (E_G h)(m).$$

This immediately implies that for $m', m'' \in G$,

$$|\langle \phi_G(m') - \phi_G(m''),\ h\rangle| \le ||E_G|| d(m', m'') ||h||_{\mathrm{Lip}_0(F)},$$

and (7.37) follows.
(b) Since $(E_G h)(m) = h(m)$, $m \in S$, and $h(m^*) = 0$, the previous identity implies (7.38). □

Our next aim is to find a limit point of the family $\{\phi_G\}$. To this end we extend every ϕ_G by zero to $\mathcal{M}$ and denote this extension by $\widehat{\phi}_G$. Due to Lemma 7.17 for every $m \in \mathcal{M}$,

$$||\widehat{\psi}_G(m)||_{\mathcal{F}(S)^{**}} \le Ad(m, m^*). \tag{7.39}$$

Further, by $\overline{B}_m$ denote the closed ball $\overline{B}_{Ad(m,m^*)}(0) \subset \mathcal{F}(S)^{**}$ and set

$$\Phi := \{\psi : \mathcal{M} \to \mathcal{F}(S)^{**}\, ;\, \psi(m) \in \overline{B}_m \text{ for all } m \in \mathcal{M}\}.$$

Using the map $\psi \mapsto (\psi(m))_{m \in \mathcal{M}}$ we then identify Φ with the direct product $\prod_{m \in \mathcal{M}} \overline{B}_m$ equipped with the product topology. Since $\dim \mathcal{F}(S)^{**} < \infty$, every $\overline{B}_m$ is compact. Hence, their direct product (and Φ) is also compact.

Now the collection of sets $G \supset S$ is partially ordered and therefore $\{\widehat{\phi}_G\}$ is a *net* containing by (7.39) in the compact set Φ. By compactness the net contains a subnet $\{\phi_{G_\alpha}\}_{\alpha \in A}$ such that for some $\phi \in \Phi$,

$$\lim_\alpha \widehat{\phi}_{G_\alpha} = \phi,$$

see, e.g., [Kel-1957, Ch. 5 Thm. 5]. By the definition of the product topology one also has for every $m \in \mathcal{M}$,

$$\lim_\alpha \widehat{\phi}_{G_\alpha}(m) = \phi(m) \quad (\text{convergence in } \mathcal{F}(S)^{**}). \tag{7.40}$$

We now show that ϕ is Lipschitz. Let $m', m'' \in \mathcal{M}$ be given, and $\widetilde{N}$ be a subnet of N containing those of $\widehat{\phi}_{G_\alpha}$ for which $m', m'' \subset G_\alpha$. Then by (7.36) and (7.40) one has for $h \in \mathrm{Lip}_0(S)$,

$$\begin{aligned}&\langle \phi(m') - \phi(m''),\ h\rangle = \lim_{\widetilde{N}} \langle \phi_{G_\alpha}(m') - \phi_{G_\alpha}(m''),\ h\rangle \\ &= \lim_{\widetilde{N}} \langle E_{G_\alpha} h,\ J_{G_\alpha}(m') - J_{G_\alpha}(m'')\rangle = \lim_{\widetilde{N}} [(E_{G_\alpha} h)(m') - (E_{G_\alpha} h)(m'')].\end{aligned}$$

Together with (7.34) this leads to the inequality

$$|\langle \phi(m') - \phi(m''),\ h\rangle| \le A||h||_{\mathrm{Lip}_0(S)} d(m', m''),$$

implying that,

$$||\phi||_{\mathrm{Lip}(\mathcal{M}, \mathcal{F}(S)^{**})} \le A. \tag{7.41}$$

Using (7.38) we also conclude that for $m \in S$,

$$\phi(m) = \kappa_S J_S(m), \quad \text{and} \quad \phi(m^*) = 0. \tag{7.42}$$

Finally, using the function ϕ we define the required extension operator $E : \mathrm{Lip}_0(S) \to \mathrm{Lip}_0(\mathcal{M})$ as follows. Let $\widetilde{\kappa}_S : \mathcal{F}(S)^* \to \mathcal{F}(S)^{***}$ be the canonical embedding (an isometry in this case). For $h \in \mathrm{Lip}_0(S) = \mathcal{F}(S)^*$ we set

$$(Eh)(m) := \langle \widetilde{\kappa}_S h,\ \phi(m)\rangle, \quad m \in \mathcal{M}. \tag{7.43}$$

Then by (7.41)

$$|(Eh)(m') - (Eh)(m'')| = |\langle \phi(m') - \phi(m''),\ h\rangle| \le A||h||_{\mathrm{Lip}_0(S)} d(m', m''),$$

and (7.35) follows.

Now by (7.42) we have for $m \in S$,

$$(Eh)(m) = \langle \phi(m),\ h\rangle = \langle \kappa_S J_S(m),\ h\rangle = h(m);$$

in particular, $(Eh)(m^*) = 0$.

The proof of the proposition is complete. □

We are now ready to prove Theorem 7.12(b) for the case of $S = \mathcal{M}$.

Since the inequality

$$A := \sup_F \lambda(F) \le \lambda(\mathcal{M}) \tag{7.44}$$

with F running through all finite subspaces of $(\mathcal{M}, m^*)$ is trivial, it remains to establish the converse. In other words, we must prove that for every $S \subset (\mathcal{M}, m^*)$ and $\epsilon > 0$ there exists an operator $E \in \mathrm{Ext}(S, \mathcal{M})$ with $||E|| \le A + \epsilon$.

In order to find the E we, as in Lemma 7.17, associate to every finite subspace $F \subset S$ a function $\phi_F : \mathcal{M} \to \mathcal{F}(S)^{**}$ and then find a limit point of the family $\{\phi_F\}$. The latter will be used to define the required extension operator E by a procedure similar to that in (7.43).

To define ϕ_F we use several linear operators first of which denoted by T_F is given as follows.

By the definition of the constant A, for each pair $F \subset G$ of finite subspaces of (S, m^*) and $\epsilon > 0$ there exists an extension operator $E_G \in \mathrm{Ext}(F, G)$ such that $||E_G|| \le A + \epsilon$. Applying then Proposition 7.16 to $\{E_G\}$ we find the required extension operator $T_F \in \mathrm{Ext}(F, \mathcal{M})$ such that

$$||T_F|| \le A + \epsilon. \tag{7.45}$$

Further, we need the following

Lemma 7.18. *For every subspace G of (S, m^*) there exists a linear isometric embedding*

$$I_G : \mathcal{F}(G) \to \kappa_S(\mathcal{F}(S)).$$

Proof. To define I_G we use the restriction operator $R_G : f \mapsto f|_G$. Due to McShane's Theorem 1.27 of Volume I, R_G regarding as a linear operator from $\mathrm{Lip}_0(S)$ into $\mathrm{Lip}_0(G)$ satisfies

$$||R_G|| = 1. \tag{7.46}$$

Then we set

$$I_G := R_G^* \kappa_G. \tag{7.47}$$

Since R_G^* maps $\mathrm{Lip}_0(G)^* = \mathcal{F}(G)^{**}$ into $\mathcal{F}(S)^{**}$ and the canonical embedding κ_G sends $\mathcal{F}(G)$ into $\mathcal{F}(G)^{**}$, the operator I_G is acting from $\mathcal{F}(G)$ into $\mathcal{F}(S)^{**}$.

Let us show that I_G ranges in $\kappa_S(\mathcal{F}(S))$. To this end we use the (nonlinear) isometric embedding $J_S : S \to \mathcal{F}(S)$, see (7.33). Let $m \in G$ and $h \in \mathrm{Lip}_0(S)$. By the definitions of I_G and J_S,

$$\langle (I_G J_S)(m)\, , h(m)\rangle = \langle ((\kappa_G J_S)(m)\, , h|_G\rangle = \langle h\, , J_S(m)\rangle = h(m).$$

This, clearly, means that

$$I_G(J_S(G)) \subset \kappa_S(J_S(S)).$$

Since $J_S(G) = J_G(G)$ and $\mathcal{F}(G)$ is the minimal closed subspace containing $J_G(G)$ (the similar is true for $\mathcal{F}(S)$ and $J_S(S)$), this embedding implies the required result:

$$I_G(\mathcal{F}(G)) \subset \kappa_S(\mathcal{F}(S)).$$

Finally, by (7.46) and (7.47),

$$\|I_G\| = \|R_G^*\| = \|R_G\| = 1. \qquad \square$$

Using the operators introduced and the canonical embeddings $\kappa_{\mathcal{M}}$ and κ_F we then define linear operators

$$P_F := T_F^* \kappa_{\mathcal{M}} \quad \text{and} \quad Q_F := I_F(\kappa_F)^{-1} P_F.$$

One notes that for a finite F the canonical embedding is invertible, since $\dim \mathcal{F}(F) < \infty$. Moreover, P_F acts from $\mathcal{F}(\mathcal{M})$ into $\mathcal{F}(F)^{**}$ and therefore Q_F maps $\mathcal{F}(\mathcal{M})$ into $\mathcal{F}(S)^{**}$. We also have, see (7.45),

$$\|Q_F\| \le \|P_F\| \le \|T_F\| \le A + \varepsilon. \tag{7.48}$$

Now, we define the desired vector function $\phi_F : \mathcal{M} \to \mathcal{F}(S)^{**}$ by setting

$$\phi_F := Q_F J_{\mathcal{M}}; \tag{7.49}$$

recall that the (nonlinear) isometric embedding $J_{\mathcal{M}} : \mathcal{M} \to \mathcal{F}(\mathcal{M})$ was introduced by (7.33).

Arguing now as in Lemma 7.17 and using (7.48) we then obtain the estimate

$$||\phi_F||_{\mathrm{Lip}(\mathcal{M},\mathcal{F}(S)^{**})} \le A + \epsilon. \tag{7.50}$$

Further, by (7.49) and (7.47) we have for $m \in F$ and $h \in \mathrm{Lip}_0(\mathcal{M})$,

$$\langle \phi_F \, , h\rangle(m) := \langle R_F^* \kappa_F (\kappa_F)^{-1} P_F J_{\mathcal{M}} \, , h\rangle(m) = \langle h|_F \, , P_F J_{\mathcal{M}}\rangle(m) = \langle T_F h \, , J_{\mathcal{M}}\rangle(m).$$

Since $T_F h$ is an extension of h from F and $J_{\mathcal{M}}(m)$ is the δ-functional at m, the last term equals $h(m)$.

Hence for $m \in F$,

$$\phi_F(m) = \kappa_{\mathcal{M}} J_{\mathcal{M}}(m); \tag{7.51}$$

in particular, $\phi_F(m^*) = 0$.

From here and (7.50) we derive that the set $\{\phi_F(m)\}$ with F running through the set of all finite subspaces of S is a subset of the closed ball $\overline{B}_m \subset \mathcal{F}(S)^{**}$ centered at 0 and of radius $(A+\epsilon)d(m, m^*)$. In the weak* topology $\overline{B}_m$ is compact. From this point the proof repeats word for word that of Proposition 7.16. Namely, we consider the set Φ of functions $\psi : \mathcal{M} \to \mathcal{F}(S)^{**}$ satisfying

$$||\psi(m)||_{\mathcal{F}(S)^{**}} \le (A + \epsilon)d(m, m^*), \quad m \in \mathcal{M}.$$

Equip $\overline{B}_m$ with the weak* topology and introduce the set $Y := \prod_{m\in\mathcal{M}} B_m$ equipped with the product topology. Then Y is compact and, so, Φ is also compact in the topology induced by the bijection $\Phi \ni \psi \mapsto (\psi(m))_{m\in\mathcal{M}} \in Y$. Therefore there exists a subnet N of the net $\{\phi_F \ ; \ (F, m^*) \subset (S, m^*) \ , \ \mathrm{card}\, F < \infty\}$ such that

$$\lim_N \phi_F = \phi$$

for some $\phi \in \Phi$. By the definition of the product topology for every $m \in \mathcal{M}$,

$$\lim_N \phi_F(m) = \phi(m) \quad \text{(convergence in the weak}^* \text{ topology of } \mathcal{F}(S)^{**}).$$

Arguing as in the proof of Proposition 7.16, see (7.41), we derive from (7.50) that

$$||\phi||_{\mathrm{Lip}(\mathcal{M},\mathcal{F}(S)^{**})} \le A + \epsilon. \tag{7.52}$$

Moreover, for the subnet N' of the net $N = \{\phi_F\}$ determined by the condition $F \ni m$ and for every $m \in S$ we obtain from (7.51),

$$\phi(m) = \lim_{N'} \phi_F(m) = \kappa_{\mathcal{M}} J_{\mathcal{M}}(m). \tag{7.53}$$

Finally, we use the canonical embedding $\widetilde{\kappa}_S : \mathcal{F}(S)^* = \mathrm{Lip}_0(S) \to \mathcal{F}(S)^{***}$ to introduce the required extension operator $E \in \mathrm{Ext}(S, \mathcal{M})$ setting for $m \in \mathcal{M}$, $h \in \mathrm{Lip}_0(S)$,

$$(Eh)(m) := \langle \widetilde{\kappa}_S h, \phi(m)\rangle.$$

Since $\phi(m) \in \mathcal{F}(S)^{**}$, this is well defined. Then for $m', m'' \in \mathcal{M}$ we get from (7.52)

$$|(Eh)(m') - (Eh)(m'')| \le ||h||_{\mathrm{Lip}_0(S)} ||\phi(m') - \phi(m'')||_{\mathcal{F}(S)^{**}} \\ \le (A+\epsilon)||h||_{\mathrm{Lip}_0(S)} d(m', m'').$$

Moreover, by (7.53) we have for $m \in S$,

$$(Eh)(m) = \langle \widetilde{\kappa}_S h,\ \kappa_{\mathcal{M}} J_{\mathcal{M}}(m)\rangle = \langle h,\ J_{\mathcal{M}}(m)\rangle = h(m).$$

Hence, $E \in \mathrm{Ext}(S, \mathcal{M})$ and $||E|| \le A + \epsilon$. This implies the converse to (7.44), inequality

$$\lambda(\mathcal{M}) \le \sup_F \lambda(F).$$

The proof of Theorem 7.12 (b) for $S = \mathcal{M}$ is complete.

Now we prove Theorem 7.12 in the general case.

(a) Assume that a sequence of metric spaces $\{(\mathcal{M}_i, d_i)\}_{i\in\mathbb{N}}$ finitely converges to $(\mathcal{M}, d)$. Suppose first that $\lambda(\mathcal{M}) < \infty$. Then the first part of Theorem 7.12 (b) (i.e., Corollary 7.13) implies:

Given $\epsilon > 0$ there is a finite subspace $F \subset \mathcal{M}$ such that

$$\lambda(\mathcal{M}) - \epsilon \le \lambda(F) \le \lambda(\mathcal{M}). \tag{7.54}$$

Definition 7.11 and that of the Gromov-Hausdorff convergence imply that for some subsequence $\mathbb{N}_0 \subset \mathbb{N}$ there exist finite subsets $F_i \subset \mathcal{M}_i$ and bi-Lipschitz homeomorphisms of F_i onto F with distortions D_i, $i \in \mathbb{N}_0$, such that $\lim\limits_{i\to\infty} D_i = 1$. Then for every $i \in \mathbb{N}_0$ we clearly have

$$\lambda(F) \le \widetilde{D}_i^2 \lambda(F_i) \le \widetilde{D}_i^2 \lambda(\mathcal{M}_i).$$

Passing to the limit as $i \to \infty$ we get

$$\lambda(F) \le \varlimsup_{i\to\infty} \lambda(\mathcal{M}_i).$$

This and (7.54) then yield

$$\lambda(\mathcal{M}) \le \varlimsup_{i\to\infty} \lambda(\mathcal{M}_i).$$

Assume now that $\lambda(\mathcal{M}) = \infty$. Then the first part of Theorem 7.12 (b) implies for this case:

Given $l > 0$ there is a finite subspace $F \subset \mathcal{M}$ such that

$$l \le \lambda(F). \tag{7.55}$$

Using the above argument with (7.55) instead of (7.54) we obtain for arbitrary l that

$$l \le \varlimsup_{i\to\infty} \lambda(\mathcal{M}_i).$$

This completes the proof of Theorem 7.12 (a).

(b) Assume, in addition, that $\{(\mathcal{M}_i, d_i)\}_{i\in\mathbb{N}}$ δ-converges to a subspace $S \subset \mathcal{M}$. Then Definition 7.11 (b) and the definition of $\lambda(S)$ imply that

$$\lambda(S) = \lim_{i\to\infty} \lambda(\mathcal{M}_i).$$

This together with Theorem 7.12 (a) and Corollary 7.13 give

$$\lambda(\mathcal{M}) \leq \lambda(S) = \sup_F \lambda(F)$$

where F runs over all finite subspaces of S.

The proof of Theorem 7.12 has been completed. □

For the next corollary of Theorem 7.12 we require the notion of a *dilation*, a bi-Lipschitz homeomorphism of $\mathcal{M}$ of distortion 1.

Corollary 7.19. *Assume that S is a subspace of $\mathcal{M}$ such that for some dilation $\phi : \mathcal{M} \to \mathcal{M}$ we have*

(a) $S \subset \phi(S)$*;*

(b) $\cup_{j=0}^{\infty}\phi^j(S)$ *is dense in* $\mathcal{M}$*.*

Then

$$\lambda(\mathcal{M}) = \sup_{F\subset S} \lambda(F)$$

where F runs over all finite subspaces of S.

Proof. From condition (b) it follows that the sequence $\{(\phi^j(S), d)\}_{j\in\mathbb{N}}$ finitely converges to $(\mathcal{M}, d)$. Since ϕ is a dilation of $\mathcal{M}$, every ϕ^j, $j > 1$, is a dilation of $\mathcal{M}$ as well. Thus the sequence $\{(\phi^j(S), d)\}_{j\in\mathbb{N}}$ δ-converges to (S, d). Now the required result follows from Theorem 7.12 (b). □

For ϕ being the identity map this and Corollaries 7.13 and 7.14 imply that $\lambda(S) = \lambda(\mathcal{M})$ for a dense subset $S \subset \mathcal{M}$.

Finally, as a consequence of Theorem 7.12 we obtain a unique for now sharp result (preserving Lipschitz constants) on simultaneous Lipschitz extensions.

Corollary 7.20. *Let $(\mathcal{M}, d)$ be an ultrametric space. Then*

$$\lambda(\mathcal{M}) = 1.$$

Proof. According to Theorem 7.12 it suffices to prove that $\lambda(\mathcal{M}) = 1$ for any finite ultrametric space $(\mathcal{M}, d)$. So let $S \subset \mathcal{M}$ be a proper subset of such a space. Take a point $m' \in \mathcal{M} \setminus S$ and denote by $r(m')$ a point from S such that

$$d(m', r(m')) = d(m', S).$$

Now for each function $f \in \mathrm{Lip}(S)$ we define its extension $\widetilde{f} \in \mathrm{Lip}(S \cup \{m'\})$ by the formula

$$\widetilde{f}(m) := \begin{cases} f(m), & \text{if } m \in S, \\ f(r(m')), & \text{if } m = m'. \end{cases}$$

Then the strong triangle inequality for d implies

$$\begin{aligned} |\widetilde{f}(m') - \widetilde{f}(m)| &= |f(r(m')) - f(m)| \le L(f)d(r(m'), m) \\ &\le L(f) \max\{d(m', r(m')), d(m', m)\} = L(f)d(m', m). \end{aligned}$$

Therefore $L(\widetilde{f}) = L(f)$.

Applying the same extension procedure to $S \cup \{m'\}$ and points outside this set we construct by induction an extension $\widehat{f} \in \mathrm{Lip}(\mathcal{M})$ of f with $L(\widehat{f}) = L(f)$. This gives a linear extension operator $E \in \mathrm{Ext}(S, \mathcal{M})$ of norm 1. Then Theorem 7.12 implies the required result. □

Remark 7.21. (a) For the relative extension constant $\lambda(S, \mathcal{M})$ we obtain by repeating literally the proof of Corollary 7.13 that

$$\lambda(S, \mathcal{M}) = \sup_F \lambda(F, \mathcal{M}) \tag{7.56}$$

where F runs over all finite subspaces of S.
(b) Using the compactness argument of the proof of Proposition 7.16 one can also show that for every subspace $S \subset \mathcal{M}$ there exists an extension operator $E_{min} \in \mathrm{Ext}(S, \mathcal{M})$ such that

$$||E_{min}|| = \lambda(S, \mathcal{M}). \tag{7.57}$$

(c) The same argument allows to establish the following fact.

The set function $S \mapsto \lambda(S)$ defined on closed subspaces of $\mathcal{M}$ is lower semicontinuous in the Hausdorff metric.

7.2 Main extension result

We prove that the direct p-sum of spaces of pointwise homogeneous type has the simultaneous Lipschitz extension property and estimate the corresponding Lipschitz constants. For convenience of the reader we recall several basic notions introduced in subsection 3.2.5 of Volume I which will be used in the proof.

A triple $(\mathcal{M}, d, \mathcal{F})$ where $\mathcal{F} = \{\mu_m\}_{m \in \mathcal{M}}$ is a collection of Borel measures on $(\mathcal{M}, d)$ is said to be a *space of pointwise homogeneous type* (*$\mathcal{PHT}$ space*) if the following holds:

(a) $\mathcal{F}$ is *uniformly doubling*, i.e., its *doubling constant* $D(\mathcal{F})$ is finite.
Here $D(\mathcal{F}) := D(\mathcal{F}; 2)$ and the *dilation function* $D(\mathcal{F}; \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is given

for $l > 1$ by

$$D(\mathcal{F}; l) := \sup\left\{\frac{\mu_m(B_{lR}(m))}{\mu_m(B_R(m))}\,;\ R > 0 \text{ and } m \in \mathcal{M}\right\}, \tag{7.58}$$

see Volume I, Definitions 3.87 and 3.89.

(b) $\mathcal{F}$ is *consistent with the metric* d, i.e., for each $t > 0$ and all $R > 0$ and pairs $m_1, m_2 \in \mathcal{M}$ satisfying

$$d(m_1, m_2) \le tR$$

there exists a constant $C > 0$ such that the inequality

$$|\mu_{m_1} - \mu_{m_2}|(B_R(m)) \le \frac{C\mu_m(B_R(m))}{R}\, d(m_1, m_2) \tag{7.59}$$

holds for m equals m_1 or m_2.

The optimal C in this inequality denoted by $C(\mathcal{F}; t)$ is said to be the *consistency function* of $\mathcal{F}$ while the number $C(\mathcal{F}; 1)$ is denoted by $C(\mathcal{F})$ and is called the *consistency constant* of $\mathcal{F}$.

Finally, $\mathcal{F}$ is called K-*uniform* ($K \ge 1$) if for all $m_1, m_2 \in \mathcal{M}$ and $R > 0$

$$\mu_{m_1}(B_R(m_1)) \le K\mu_{m_2}(B_R(m_2)). \tag{7.60}$$

The theorem formulated below concerns the direct p-sum of the family $(\mathcal{M}_i, d_i, \mathcal{F}_i)$, $1 \le i \le N$, i.e., a triple $(\mathcal{M}, d_p, \mathcal{F})$ where

$$\mathcal{M} := \prod_{i=1}^{N} \mathcal{M}_i\,, \quad d_p := \left\{\sum_{i=1}^{N} d_i^p\right\}^{1/p}, \quad \mathcal{F} := \{\mu_{\widetilde{m}}\}_{\widetilde{m}\in\mathcal{M}}$$

$$\text{where for } \widetilde{m} = (m_1, \dots, m_N) \in \mathcal{M}\,, \quad \mu_{\widetilde{m}} := \bigotimes_{i=1}^{N} \mu_{m_i}.$$

Theorem 7.22. *Let $(\mathcal{M}_i, d_i, \mathcal{F}_i)$ be $\mathcal{PHT}$ spaces such that $\mathcal{F}_i$ is K_i-uniform, $1 \le i \le N$. Then its direct p-sum $(\mathcal{M}, d_p, \mathcal{F})$ ($1 \le p \le \infty$) has the simultaneous Lipschitz extension property and the extension constant satisfies*

$$\lambda(\mathcal{M}, X) \le c_0(\log_2 D + KC + 1), \tag{7.61}$$

where X is an arbitrary Banach space,

$$K := \prod_{i=1}^{N} K_i, \quad D := \prod_{i=1}^{N} D(\mathcal{F}_i), \quad C := \left\{\sum_{i=1}^{N} C(\mathcal{F}_i)^q\right\}^{1/q},$$

c_0 is a numerical constant and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 7.23. (a) For the special case of a single $\mathcal{PHT}$ space the assumption of K-uniformity can be excluded. It can be also eliminated for $N > 1$ but the estimate for the extension constant becomes in this case essentially worse.

(b) The linear operator $E \in \mathrm{Ext}(S, \mathcal{M}, X)$ constructed in the theorem extends a Lipschitz function $f : S \to X$ to that on $\mathcal{M}$ assigning its values in the closed convex hull of $f(S)$. Therefore the theorem holds for a target space being a closed convex subset of a Banach space.

(c) If some of the spaces $\mathcal{M}_i$ are of homogeneous type (with doubling measures μ_i), then for these $\mathcal{M}_i$ the family $\mathcal{F}_i = \{\mu_i\}$ and $D(\mathcal{F}_i) = D(\mu_i)$, $C(\mathcal{F}_i) = 0$ and K_i may be taken to be equal to 1.

Since the proof is long and rather involved, we briefly discuss its structure and basic steps.

We begin with the construction of the desired linear extension operator for a $\mathcal{PHT}$ space $(\mathcal{M}, d, \mathcal{F})$ and its subspace S. Since $\mathrm{Lip}_0(\mathcal{M})$ is 1-complemented in $\mathrm{Lip}(\mathcal{M})$, it suffices to find E to be acting from $\mathrm{Lip}_0(S)$ into $\mathrm{Lip}_0(\mathcal{M})$.

Hereafter m^* is a fixed point of $\mathcal{M}$ and every subspace $S \subset (\mathcal{M}, m^*)$ contains m^* whereas $\mathrm{Lip}_0(\mathcal{M})$ and $\mathrm{Lip}_0(S)$ are determined by the condition $f(m^*) = 0$. Further, without loss of generality we assume that S is a proper closed subspace of $\mathcal{M}$.

To introduce the operator E we exploit the Dugundji extension construction from the proof of Borsuk-Dugundji Theorem 1.8 of Volume I to construct a linear extension operator $f \mapsto \widehat{f}$ sending $\mathrm{Lip}_0(S, X)$ into the space of locally bounded continuous functions $C_b^{loc}(\mathcal{M}, X)$. Then we use the family $\mathcal{F} = \{\mu_m\}_{m \in \mathcal{M}}$ to "smooth" $\widehat{f}$ outside of S, exploiting for this goal an average operator given for $g \in C_b^{loc}(\mathcal{M}, X)$, $m \in \mathcal{M}$ and $R > 0$ by

$$I(g; m, R) := \frac{1}{\mu_m(B_R(m))} \int_{B_R(m)} g \, d\mu_m.$$

The required extension operator is then given for $f \in \mathrm{Lip}_0(S, X)$ by

$$(Ef)(m) := I(\widehat{f}; m; d(m))$$

where we set hereafter

$$d(m) := d(m, S) \, (= \inf_{m' \in S} d(m, m')). \tag{7.62}$$

In the next, considerably harder, part of the proof we will estimate the norm of $E : \mathrm{Lip}_0(S, X) \to \mathrm{Lip}_0(\mathcal{M}, X)$ via the basic characteristics of the family $\mathcal{F}$. In the derivation we will use the dilation function $t \mapsto D(\mathcal{F}; t)$ instead of the dilation constant $D(\mathcal{F})$ to minimize the final estimate over $t > 1$. For the dilation function equivalent to $t \mapsto t^\lambda$ for some $\lambda > 0$ an argument presented below will give the estimate of the norm:

$$||E|| \le c_0(\log_2 D(\mathcal{F}) + C(\mathcal{F}) + 1);$$

the uniformity constant K may be omitted in this case.

However, in general, this approach gives a result far beyond from the required. To improve the situation we will apply this method to a $\mathcal{PHT}$ space, say $(\widehat{\mathcal{M}}, \widehat{d}, \widehat{\mathcal{F}})$, which extends the initial space $(\mathcal{M}, d, \mathcal{F})$ in the sense that $\widehat{\mathcal{M}}$ contains an isometric copy of $\mathcal{M}$. Therefore the corresponding extension constants satisfy

$$\lambda(\mathcal{M}, X) \le \lambda(\widehat{\mathcal{M}}, X). \tag{7.63}$$

As soon as a suitable bound of the right-hand side via the basic characteristics of $\widehat{\mathcal{F}}$ would be established, it remained to bound them by those of $\mathcal{F}$. For a technical reason the latter derivation will be done before estimating the norm of the operator E.

These results lead to the required estimate of the simultaneous extension constant $\lambda(\mathcal{M}, X)$.

At the final step we apply the results obtained to a $\mathcal{PHT}$ space being the direct p-sum of $\mathcal{PHT}$ spaces $(\mathcal{M}_i, d_i, \mathcal{F}_i)$, $1 \le i \le N$. Combining this with the results of Proposition 3.92 of Volume I estimating the basic characteristics of the direct p-sum by those of $(\mathcal{M}_i, d_i, \mathcal{F}_i)$ we will complete the proof.

Proof of Theorem 7.22.

A. *Extension operator*

Let $(\mathcal{M}, d, m^*)$ be a pointed metric space and S be its subspace. We should construct a linear extension operator from $\mathrm{Lip}_0(S, X)$ into $\mathrm{Lip}_0(\mathcal{M}, X)$ where X is a Banach space. To this end we first recall the Dugundji extension method, see the proof of Borsuk-Dugundji Theorem 1.8 of Volume I.

Let g be a continuous function on a closed subspace S of a metric space $(\mathcal{M}, d)$ ranged into a Banach space X. Let $\{B_m\}_{m \in S^c}$ be a cover of the open set $S^c := \mathcal{M} \setminus S$ by the open balls

$$B_m := B_{r_m}(m), \quad \text{where} \quad r_m := \frac{1}{3} d(m, S). \tag{7.64}$$

Since any metric space is paracompact, there exists a continuous partition of unity $\{p_\alpha\}_{\alpha \in A}$ subordinate to the cover $\{B_m\}$ whose supports $U_\alpha := \{m \in S^c \,;\, p_\alpha(m) > 0\}$ form a *locally finite* cover of S^c, see Volume I, Proposition 3.17.

For every $\alpha \in A$, we now pick points

$$m_1(\alpha) \in S \quad \text{and} \quad m_2(\alpha) \in U_\alpha = \operatorname{supp} p_\alpha \tag{7.65}$$

such that

$$d(m_1(\alpha), m_2(\alpha)) < 2d(m_2(\alpha), S). \tag{7.66}$$

The Dugundji extension operator $g \mapsto \widehat{g}$ is then given for $m \in \mathcal{M}$ by

$$\widehat{g}(m) := \begin{cases} g(m), & \text{if } m \in S, \\ \sum_{\alpha \in A} g(m_1(\alpha)) p_\alpha(m), & \text{if } m \in S^c. \end{cases} \tag{7.67}$$

Repeating word-for-word the arguments of the proof of Theorem 1.8 of Volume I one obtains that $\widehat{g}$ is continuous on $\widehat{M}$.

Lemma 7.24. *Let $f \in \mathrm{Lip}_0(S, X)$. Then the extended function $\widehat{f} : \mathcal{M} \to X$ satisfies for all $m, m' \in \mathcal{M}$ the inequality*

$$||\widehat{f}(m) - \widehat{f}(m')||_X \le 7L(f)\{d(m, m') + d(m, S) + d(m', S)\}. \tag{7.68}$$

Proof. In the case $m, m' \in S$, inequality (7.68) (even with constant 1) is trivial, since $\widehat{f} = f$ on S and $d(m, S) = d(m', S) = 0$.

Let now $m \in S$ and $m' \in S^c$. We denote by V_m an open ball in the Banach space X given by the inequality

$$||\widehat{f}(m) - x||_X < (5d(m, m') + 2d(m', S))L(f), \quad x \in X. \tag{7.69}$$

Inequality (7.68) in this case clearly would follow from the inclusion

$$\widehat{f}(m') \in V_m. \tag{7.70}$$

Since $\widehat{f}(m')$ is a convex combination of the points $f(m_1(\alpha))$, $\alpha \in A_0$, where the finite set A_0 is given by

$$A_0 := \{\alpha \in A \ ; \ m' \in \operatorname{supp} p_\alpha\},$$

see (7.67), inclusion (7.70) follows from the condition

$$f(m_1(\alpha)) \in V_m, \quad \alpha \in A_0.$$

This, in turn, is a direct consequence of the inequality

$$d(m_1(\alpha), m) < 5d(m, m') + 2d(m', S) \tag{7.71}$$

and the fact that $f \in \mathrm{Lip}_0(S, X)$.

To prove (7.71) we choose for $\alpha \in A_0$ a point $m(\alpha) \in S^c$ so that

$$B_{m(\alpha)} \supset \operatorname{supp} p_\alpha \ \ (\ni m_2(\alpha)).$$

Then $m' \in B_{m(\alpha)}$, $m \in S$, and this and (7.64) imply that

$$d(m(\alpha), S) \le d(m(\alpha), m) \le d(m(\alpha), m') + d(m', m) \le \frac{1}{3} d(m(\alpha), S) + d(m, m').$$

Hence,

$$d(m(\alpha), S) \le d(m(\alpha), m) \le \frac{3}{2} d(m, m'). \tag{7.72}$$

Further, $m_2(\alpha) \in B_{m(\alpha)}$ and therefore

$$d(m_2(\alpha), m) \le d(m_2(\alpha), m(\alpha)) + d(m(\alpha), m) \le \frac{1}{3} d(m(\alpha), S) + d(m(\alpha), m).$$

Combining this with the previous inequality we obtain

$$d(m_2(\alpha), m) \le 3d(m, m').$$

Finally, this, (7.66) and (7.72) together with the inequality

$$d(m_1(\alpha), m) \le d(m_1(\alpha), m_2(\alpha)) + d(m_2(\alpha), m)$$

give the required inequality (7.71).

It remains to consider the case of $m, m' \in S^c$. For the sake of definiteness, let

$$d(m', S) \le d(m, S). \tag{7.73}$$

Given $\epsilon > 0$ we pick a point $m'' \in S$ satisfying

$$d(m', m'') \le d(m', S) + \epsilon$$

and write

$$||\widehat{f}(m) - \widehat{f}(m')||_X \le ||\widehat{f}(m) - \widehat{f}(m'')||_X + ||\widehat{f}(m'') - \widehat{f}(m')||_X.$$

Since $m'' \in S$, we can apply the estimate obtained in the previous part of the proof to bound the right-hand side by

$$L(f)\{2(d(m, S) + d(m', S)) + 5(d(m, m'') + d(m', m''))\}.$$

Moreover, by the choice of m'',

$$d(m, m'') \le d(m, m') + d(m', m'') \le d(m, m') + d(m', S) + \epsilon.$$

Therefore, the sum in the curly brackets is bounded by

$$2(d(m, S) + d(m', S)) + 5d(m, m') + 10d(m', S) + 10\epsilon.$$

This and (7.73), in turn, give the required inequality (7.68).

The lemma has been proved. □

We are now ready to define the required extension operator E. It is given for $f \in \mathrm{Lip}_0(S, X)$ by

$$(Ef)(m) := \begin{cases} f(m), & \text{if } m \in S, \\ I(\widehat{f}; m, d(m)), & \text{if } m \in S^c. \end{cases} \tag{7.74}$$

Here $\widehat{f}$ is defined by (7.67), and for a continuous and locally bounded function (i.e., continuous and bounded on every bounded subset of $\mathcal{M}$) $g : \mathcal{M} \to X$ we set

$$I(g; m, R) := \frac{1}{\mu_m(B_R(m))} \int_{B_R(m)} g \, d\mu_m. \tag{7.75}$$

According to Lemma 7.24, the operator E is well defined, that is, the vector function $\widehat{f}$ is continuous and bounded on every bounded subset of $\mathcal{M}$.

The extension operator constructed will then be applied to the aforementioned extension of the $\mathcal{PHT}$ space $(\mathcal{M}, d, \mathcal{F})$ denoted now by $(\mathcal{M}_n, d_n, \mathcal{F}_n)$. Its underlying set is given by

$$\mathcal{M}_n := \mathcal{M} \times \mathbb{R}^n \tag{7.76}$$

and $\widetilde{m} = (m, x)$, $\widetilde{m}' = (m', x')$ etc. stand for its points.

Further, the metric d_n is given for $\widetilde{m} = (m, x)$, $\widetilde{m}' = (m', x')$ by

$$d_n(\widetilde{m}, \widetilde{m}') := d(m, m') + \sum_{i=1}^{n} |x_i - x_i'|. \tag{7.77}$$

The integer $n \geq 2$ will be chosen later to minimize the corresponding estimates.

We equip the space $(\mathcal{M}_n, d_n)$ with the family of measures $\mathcal{F}_n := \{\mu_{\widetilde{m}}\}_{\widetilde{m} \in \mathcal{M}_n}$ where for $\widetilde{m} = (m, x)$,

$$\mu_{\widetilde{m}} := \mu_m \otimes \lambda_n; \tag{7.78}$$

here λ_n is the Lebesgue measure on $\mathbb{R}^n$.

B. *Properties of the extended metric space*

It is easy to show that the $(\mathcal{M}_n, d_n, \mathcal{F}_n)$ is a $\mathcal{PHT}$ space but we need qualitative estimates of its basic parameters in terms of those for $(\mathcal{M}, d, \mathcal{F})$.

This goal will be achieved by several lemmas presented below. In the formulation of the first lemma $D_n(\cdot)$ denotes the dilation function for $(\mathcal{M}_n, d_n, \mathcal{F}_n)$ defined as in (7.58) with μ_m replaced by measure (7.78). We denote by $D := D(\mathcal{F}; 2)$ the doubling constant of $(\mathcal{M}, d, \mathcal{F})$.

Lemma 7.25. *Assume that n is related to the doubling constant D by*

$$n \geq \lfloor \log_2 D \rfloor + 5. \tag{7.79}$$

Then we have

$$D_n(1 + 1/n) \leq \frac{6}{5} e^4.$$

Proof. The result follows from Lemma 3.94 of Volume I where $(\mathcal{M}', d')$ is taken to be ℓ_1^n. □

Our next auxiliary result evaluates the constant $C_n(1/n) := C(\mathcal{F}_n; 1/n)$ in terms of $C(1/n) := C(\mathcal{F}; 1/n))$; recall that the consistency function $C(\mathcal{F}; \cdot)$ is defined in (7.59).

Lemma 7.26. $C_n(1/n) \le \left(1+\frac{4e}{3}\right) nC(1/n)$.

Proof. Using Fubini's theorem, we obtain

$$\mu_{\widetilde{m}}(B_R(\widetilde{m})) = \beta_n \int_0^R \mu_m(B_s(m))(R-s)^{n-1}ds \tag{7.80}$$

where β_n is the volume of the unit sphere in ℓ_1^n. Then for $\widetilde{m}_i = (m_i, x^i)$, $i = 1, 2$, we have

$$|\mu_{\widetilde{m}_1} - \mu_{\widetilde{m}_2}|(B_R(\widetilde{m}_i)) \le \beta_n \int_0^R |\mu_{m_1} - \mu_{m_2}|(B_s(m_i)) \cdot (R-s)^{n-1}ds.$$

Now let $d_n(\widetilde{m}_1, \widetilde{m}_2) \le \frac{R}{n}$. Divide the interval of integration into subintervals $[0, R/n]$ and $[R/n, R]$ and denote the corresponding integrals over these intervals by I_1 and I_2. Replacing $B_s(m_i)$ in I_1 by the larger ball $B_{s+R/n}(m_i)$ and applying (7.59) with $t = 1/n$ we obtain

$$I_1 \le C(1/n)\left(\beta_n \int_0^{R/n} \frac{\mu_{m_i}(B_{s+R/n}(m_i))}{s+R/n}(R-s)^{n-1}ds\right) d(m_1, m_2).$$

Replacing s by $t = s + R/n$ we bound the expression in the brackets by

$$\left(\beta_n \int_{R/n}^{2R/n} \mu_{m_i}(B_t(m_i))(R-t)^{n-1}dt\right) \max_{R/n \le t \le 2R/n} \frac{(R+R/n-t)^{n-1}}{t(R-t)^{n-1}}.$$

Since $[R/n, 2R/n] \subset [0, R]$ and the maximum is at most $\frac{n}{R}\left(1+\frac{1}{n-2}\right)^{n-1} < \frac{4e}{3}\frac{n}{R}$ for $n \ge 5$, this and (7.80) yield

$$I_1 \le \frac{4e}{3} C(1/n) n \frac{\mu_{\widetilde{m}_i}(B_R(\widetilde{m}_i))}{R} d(m_1, m_2).$$

For the second term we get from (7.59),

$$I_2 \le C(1/n)\left(\beta_n \int_{R/n}^{R} \frac{\mu_{m_i}(B_s(m_i))}{s}(R-s)^{n-1}ds\right) d(m_1, m_2)$$

and by (7.80) the term in the brackets is at most $\mu_{\widetilde{m}_i}(B_R(\widetilde{m}_i)) \cdot \frac{n}{R}$. Hence, we have

$$I_2 \le C(1/n) n \frac{\mu_{\widetilde{m}_i}(B_R(\widetilde{m}_i))}{R} d(m_1, m_2).$$

Further note that $d(m_1, m_2) \le d_n(\widetilde{m}_1, \widetilde{m}_2)$. Hence, we obtain finally the inequality

$$|\mu_{\widetilde{m}_1} - \mu_{\widetilde{m}_2}|(B_R(\widetilde{m}_i)) \le \left(1 + \frac{4e}{3}\right) nC(1/n) \frac{\mu_{\widetilde{m}_i}(B_R(\widetilde{m}_i))}{R} d_n(\widetilde{m}_1, \widetilde{m}_2)$$

whence $C_n(1/n) \le \left(1+\frac{4e}{3}\right) nC(1/n)$. □

Lemma 7.27. *Let* $A_n := \frac{6}{5}e^4 n$ *and* $n \geq \lfloor \log_2 D \rfloor + 6$*. Then for all* $R_2 \geq R_1 > 0$ *and* $\widetilde{m} \in \mathcal{M}_n$*,*

$$\mu_{\widetilde{m}}(B_{R_2}(\widetilde{m})) - \mu_{\widetilde{m}}(B_{R_1}(\widetilde{m})) \leq A_n \frac{\mu_{\widetilde{m}}(B_{R_2}(\widetilde{m}))}{R_2}(R_2 - R_1).$$

Proof. We write $\mathcal{M}_n = \mathcal{M}_{n-1} \times \mathbb{R}$ and $\mu_{\widetilde{m}} = \mu_{\widehat{m}} \otimes \lambda_1$ where $\widehat{m} \in \mathcal{M}_{n-1} := \mathcal{M} \oplus^{(1)} \mathbb{R}^{n-1}$. Then by Fubini's theorem we have for $0 < R_1 \leq R_2$,

$$\mu_{\widetilde{m}}(B_{R_2}(\widetilde{m})) - \mu_{\widetilde{m}}(B_{R_1}(\widetilde{m})) = 2\int_{R_1}^{R_2} \mu_{\widehat{m}}(B_s(\widehat{m}))ds \leq \frac{2R_2\mu_{\widehat{m}}(B_{R_2}(\widehat{m}))}{R_2}(R_2 - R_1).$$

We claim that for arbitrary $l > 1$ and $R > 0$,

$$R\mu_{\widehat{m}}(B_R(\widehat{m})) \leq \frac{lD_{n-1}(l)}{2(l-1)}\mu_{\widetilde{m}}(B_R(\widetilde{m})). \tag{7.81}$$

Together with the previous inequality this will yield

$$\mu_{\widetilde{m}}(B_{R_2}(\widetilde{m})) - \mu_{\widetilde{m}}(B_{R_1}(\widetilde{m})) \leq \frac{lD_{n-1}(l)}{l-1} \cdot \frac{\mu_{\widetilde{m}}(B_{R_2}(\widetilde{m}))}{R_2}(R_2 - R_1).$$

Finally we choose here $l = 1 + \frac{1}{n-1}$ and use Lemma 7.25. This will give the required inequality.

It remains to establish (7.81). By the definition of the dilation function D_{n-1} for $\mathcal{M}_{n-1}$ we have

$$\mu_{\widetilde{m}}(B_{lR}(\widetilde{m})) = 2l\int_0^R \mu_{\widehat{m}}(B_{ls}(\widehat{m}))ds \leq lD_{n-1}(l)\mu_{\widetilde{m}}(B_R(\widetilde{m})).$$

On the other hand, replacing $[0, R]$ by $[l^{-1}R, R]$ we also have

$$\mu_{\widetilde{m}}(B_{lR}(\widetilde{m})) \geq 2l\mu_{\widehat{m}}(B_R(\widehat{m}))(R - l^{-1}R) = 2(l-1)R\mu_{\widehat{m}}(B_R(\widehat{m})).$$

Combining the last two inequalities we get (7.81). $\square$

C. *Norm of the extension operator*

Let $\widehat{E}$ be the linear extension operator for the space $(\mathcal{M}_n, d_n, \mathcal{F}_n)$ constructed in part A. Hence, $\widehat{E} \in \mathrm{Ext}(S, \mathcal{M}_n, X)$ where S is an arbitrary closed proper subspace of $\mathcal{M}_n$.

Proposition 7.28. *For some numerical constant* $c_0 > 1$ *and*

$$n := \lfloor \log_2 D \rfloor + 6 \tag{7.82}$$

the following is true

$$\|\widehat{E}\| \leq c_0\left(n + C\left(\mathcal{F}_n; \frac{1}{n}\right)\right).$$

Proof. We need several auxiliary results. In their derivations we use the following notation:

$$K_n(l) := 42(A_n + \widehat{C}_n)(l+3)D_n(l) \tag{7.83}$$

where $\widehat{C}_n := C_n(\mathcal{F}_n; \frac{1}{n})$; recall that

$$A_n := \frac{6}{5}e^4 n \quad \text{and} \quad D_n(l) := D(\mathcal{F}_n; l).$$

Proposition 7.29. *The following inequality holds for* $l := 1 + \frac{1}{n}$,

$$||\widehat{E}|| \leq 56A_n + \max\left\{\frac{14(l+3)}{l-1}, K_n(l)\right\}. \tag{7.84}$$

Before our proof we derive from here Proposition 7.28. Since for $l = 1 + \frac{1}{n}$,

$$\frac{14(l+3)}{l-1} < 70n < 42A_n,$$

the maximum in (7.84) is attained at the second term. Moreover, by Lemma 7.25,

$$D\left(\mathcal{F}_n; 1 + \frac{1}{n}\right) \leq \frac{6}{5}e^4,$$

whence for some numerical constant $c_1 > 1$,

$$\|\widehat{E}\| \leq 56A_n + 42 \cdot \frac{6}{5}e^4\left(4 + \frac{1}{n}\right)[A_n + \widehat{C}_n] \leq c_1\left(n + C\left(\mathcal{F}_n; \frac{1}{n}\right)\right),$$

as required. □

Proof of Proposition 7.29. We assume without loss of generality that

$$||f||_{\mathrm{Lip}(S,X)} = 1 \tag{7.85}$$

and simplify the forthcoming computations by introducing the following notation:

$$\begin{aligned} &R_i := d_n(\widetilde{m}_i) := d_n(\widetilde{m}_i, S),\ \mu_i := \mu_{\widetilde{m}_i}, \\ &B_{ij} := B_{R_j}(\widetilde{m}_i),\ v_{ij} := \mu_i(B_{ij}),\quad 1 \leq i, j \leq 2. \end{aligned} \tag{7.86}$$

Assuming for definiteness that $0 < R_1 \leq R_2$ we have by the triangle inequality

$$0 \leq R_2 - R_1 \leq d_n(\widetilde{m}_1, \widetilde{m}_2). \tag{7.87}$$

Further, due to Lemma 7.27,

$$v_{i2} - v_{i1} \leq \frac{A_n v_{i2}}{R_2}(R_2 - R_1). \tag{7.88}$$

Next, using the consistency inequality for $\mathcal{F}_n$ with $t = \frac{1}{n}$, see (7.59), we also have

$$|\mu_1 - \mu_2|(B_{ij}) \le \frac{\widehat{C}_n v_{ij}}{R_j} d_n(\widetilde{m}_1, \widetilde{m}_2), \quad d_n(\widetilde{m}_1, \widetilde{m}_2) \le \frac{R}{n}. \tag{7.89}$$

Now, from inequality (7.68) applied to our setting and the triangle inequality we obtain

$$\max\{||\widetilde{f}(\widetilde{m})||_X \ ; \ \widetilde{m} \in B_{i2}\} \le 28R_2 + 7(i-1)d_n(\widetilde{m}_1, \widetilde{m}_2); \tag{7.90}$$

here $i = 1, 2$ and we set

$$\widetilde{f}(\widetilde{m}) := \widehat{f}(\widetilde{m}) - \widehat{f}(\widetilde{m}_1); \tag{7.91}$$

recall that $\widehat{f}$ is the Dugundji extension of f given by (7.67).

We now estimate $\|(\widehat{E}f)(\widetilde{m}_2) - (\widehat{E}f)(\widetilde{m}_1)\|_X$ for $\widetilde{m}_1 \in S$ and $\widetilde{m}_2 \notin S$. We begin with the evident inequality

$$||(\widehat{E}f)(\widetilde{m}_2) - (\widehat{E}f)(\widetilde{m}_1)||_X = \frac{1}{v_{22}} \left|\left| \int_{B_{22}} \widetilde{f}(\widetilde{m}) d\mu_2 \right|\right|_X \le \max_{B_{22}} ||\widetilde{f}||_X.$$

Applying (7.90) with $i = 2$ we then bound this maximum by $28R_2 + 7d_n(\widetilde{m}_1, \widetilde{m}_2)$. But $\widetilde{m}_1 \in S$ and therefore

$$R_2 = d_n(\widetilde{m}_2) \le d_n(\widetilde{m}_1, \widetilde{m}_2);$$

hence, in this case

$$||(\widehat{E}f)(\widetilde{m}_2) - (\widehat{E}f)(\widetilde{m}_1)||_X \le 35\|f\|_{\mathrm{Lip}(S,X)} d_n(\widetilde{m}_1, \widetilde{m}_2). \tag{7.92}$$

The remaining case $\widetilde{m}_1, \widetilde{m}_2 \notin S$ requires some additional auxiliary results. For their formulations we first write

$$(\widehat{E}f)(\widetilde{m}_1) - (\widehat{E}f)(\widetilde{m}_2) := D_1 + D_2, \tag{7.93}$$

where

$$\begin{aligned} D_1 &:= I(\widetilde{f}; \widetilde{m}_1, R_1) - I(\widetilde{f}; \widetilde{m}_1, R_2), \\ D_2 &:= I(\widetilde{f}; \widetilde{m}_1, R_2) - I(\widetilde{f}; \widetilde{m}_2, R_2); \end{aligned} \tag{7.94}$$

recall that the average operator I is given by (7.75).

Lemma 7.30. *We have*

$$||D_1||_X \le 56A_n d_n(\widetilde{m}_1, \widetilde{m}_2);$$

recall that $A_n := \frac{6}{5}e^4 n$.

Proof. Using the notations introduced, see (7.94), (7.91) and (7.86), we write

$$D_1 = \frac{1}{v_{11}} \int_{B_{11}} \widetilde{f} d\mu_1 - \frac{1}{v_{12}} \int_{B_{12}} \widetilde{f} d\mu_1 = \left(\frac{1}{v_{11}} - \frac{1}{v_{12}} \right) \int_{B_{11}} \widetilde{f} d\mu_1 - \frac{1}{v_{12}} \int_{B_{12} \setminus B_{11}} \widetilde{f} d\mu_1 .$$

This immediately implies that

$$||D_1||_X \leq 2 \cdot \frac{v_{12} - v_{11}}{v_{12}} \cdot \max_{B_{12}} ||\widetilde{f}||_X .$$

Applying now (7.88) and (7.87), and then (7.90) with $i = 1$ we get the desired estimate. □

To obtain a similar estimate for D_2 we will use the following two facts.

Lemma 7.31. *Assume that for a given $l \in [1, 1 + 1/n]$,*

$$d_n(\widetilde{m}_1, \widetilde{m}_2) \leq (l - 1) R_2 \tag{7.95}$$

and let for definiteness $v_{22} \leq v_{12}$. Then for symmetric difference $B_{12} \Delta B_{22}$ it is true that

$$\mu_2(B_{12} \Delta B_{22}) \leq 2(A_n + \widehat{C}_n) D_n(l) \frac{v_{12}}{R_2} d_n(\widetilde{m}_1, \widetilde{m}_2). \tag{7.96}$$

Proof. Set $R := R_2 + d_n(\widetilde{m}_1, \widetilde{m}_2)$. Then $B_{12} \cup B_{22} \subset B_R(\widetilde{m}_1) \cap B_R(\widetilde{m}_2)$ and therefore

$$\mu_2(B_{12} \Delta B_{22}) \leq (\mu_2(B_R(\widetilde{m}_1)) - \mu_2(B_{12})) + (\mu_2(B_R(\widetilde{m}_2)) - \mu_2(B_{22})). \tag{7.97}$$

The first term on the right-hand side is at most

$$|\mu_2 - \mu_1|(B_R(\widetilde{m}_1)) + |\mu_2 - \mu_1|(B_{R_2}(\widetilde{m}_1)) + (\mu_1(B_R(\widetilde{m}_1)) - \mu_1(B_{R_2}(\widetilde{m}_1)).$$

Estimating the first two terms by the consistency inequality, see (7.89), and the third by Lemma 7.27 we bound this sum by

$$\widehat{C}_n \left(\frac{\mu_1(B_R(\widetilde{m}_1))}{R} + \frac{\mu_1(B_{R_2}(\widetilde{m}_1))}{R_2} \right) d_n(\widetilde{m}_1, \widetilde{m}_2) + A_n \frac{\mu_1(B_R(\widetilde{m}_1))}{R} (R - R_2).$$

Moreover, $R_2 \leq R \leq l R_2$, see (7.95), and $R - R_2 := d_n(\widetilde{m}_1, \widetilde{m}_2)$. By the definition of the dilation function $D_n(l) := D(\mathcal{F}_n; l)$, see (7.58), we therefore have in notation (7.86),

$$\mu_2(B_R(\widetilde{m}_1)) - \mu_2(B_{12}) \leq [\widehat{C}_n(D_n(l) + 1) + A_n D_n(l)] \frac{v_{12}}{R_2} d_n(\widetilde{m}_1, \widetilde{m}_2).$$

Similarly, from Lemma 7.27 and the inequality $v_{22} \leq v_{12}$ we derive that

$$\begin{aligned}\mu_2(B_R(\widetilde{m}_2)) - \mu_2(B_{22}) &\leq A_n \frac{\mu_2(B_R(\widetilde{m}_2))}{R} (R - R_2) \\ &\leq A_n D_n(l) \frac{v_{22}}{R_2} d_n(\widetilde{m}_1, \widetilde{m}_2) \leq A_n D_n(l) \frac{v_{12}}{R_2} d_n(\widetilde{m}_1, \widetilde{m}_2).\end{aligned}$$

Combining the last two estimates with (7.97) we get the result. □

Lemma 7.32. *Under the assumptions of the previous lemma*

$$v_{12} - v_{22} \le 3(A_n + \widehat{C}_n)D_n(l)\frac{v_{12}}{R_2} d_n(\widetilde{m}_1, \widetilde{m}_2). \tag{7.98}$$

Proof. By (7.86) the left-hand side is bounded by

$$|\mu_1(B_{12}) - \mu_2(B_{12})| + \mu_2(B_{12}\Delta B_{22}).$$

Estimating these terms by (7.89) and (7.96) we get the required result. □

We now estimate D_2 from (7.94) beginning with

Lemma 7.33. *Under the assumptions of Lemma 7.31 we have*

$$||D_2||_X \le K_n(l) d_n(\widetilde{m}_1, \widetilde{m}_2);$$

recall that $K_n(l) := 42(A_n + \widehat{C}_n)D_n(l)(l+3)$.

Proof. By the definition of D_2 we have in the notation introduced

$$\begin{aligned} ||D_2||_X &:= \left|\left| \frac{1}{v_{12}} \int_{B_{12}} \widetilde{f} d\mu_1 - \frac{1}{v_{22}} \int_{B_{22}} \widetilde{f} d\mu_2 \right|\right|_X \\ &\le \frac{1}{v_{12}} \int_{B_{12}} ||\widetilde{f}||_X \, d|\mu_1 - \mu_2| + \frac{1}{v_{12}} \int_{B_{12}\Delta B_{22}} ||\widetilde{f}||_X \, d\mu_2 \\ &\quad + \left| \frac{1}{v_{12}} - \frac{1}{v_{22}} \right| \int_{B_{22}} ||\widetilde{f}||_X \, d\mu_2 := J_1 + J_2 + J_3. \end{aligned}$$

Now (7.89) and (7.90) with $i = 1$ imply that

$$J_1 \le \frac{1}{v_{12}} |\mu_1 - \mu_2|(B_{12}) \sup_{B_{12}} ||\widetilde{f}||_X \le \frac{\widehat{C}_n}{R_2} d_n(\widetilde{m}_1, \widetilde{m}_2) 28R_2 = 28\widehat{C}_n d_n(\widetilde{m}_1, \widetilde{m}_2).$$

In turn, by (7.96), (7.95) and (7.90) we have

$$\begin{aligned} J_2 &\le \frac{1}{v_{12}} \mu_2(B_{12}\Delta B_{22}) \sup_{B_{12}\Delta B_{22}} ||\widetilde{f}||_X \\ &\le \frac{2(A_n + \widehat{C}_n)D_n(l)}{R_2} d_n(\widetilde{m}_1, \widetilde{m}_2)(7d_n(\widetilde{m}_1, \widetilde{m}_2) + 28R_2) \\ &\le 14(A_n + \widehat{C}_n)D_n(l)(l+3)d_n(\widetilde{m}_1, \widetilde{m}_2). \end{aligned}$$

Finally, (7.98), (7.90) and (7.95) yield

$$J_3 \le 21(A_n + \widehat{C}_n)D_n(l)(l+3)d_n(\widetilde{m}_1, \widetilde{m}_2).$$

Combining these we get the required estimate. □

It remains to consider the case of points $\widetilde{m}_1, \widetilde{m}_2 \in \mathcal{M}_n$ satisfying the inequality

$$d_n(\widetilde{m}_1, \widetilde{m}_2) > (l-1)R_2$$

converse to (7.95). In this case, definition (7.94) of D_2 and (7.90) imply that

$$||D_2||_X \le 2 \sup_{B_{12}\cup B_{22}} ||\widetilde{f}||_X \le 2(28R_2 + 7d_n(\widetilde{m}_1, \widetilde{m}_2)) \le 14\left(\frac{4}{l-1}+1\right) d_n(\widetilde{m}_1, \widetilde{m}_2).$$

Combining this with the inequalities of Lemmas 7.30 and 7.33 and equality (7.92) we obtain the required estimate of the Lipschitz norm of the extension operator $\widehat{E}$:

$$||\widehat{E}|| \le 56A_n + \max\left(\frac{14(l+3)}{l-1}, K_n(l)\right) \tag{7.99}$$

where $K_n(l)$ is the constant in (7.83).

This proves Proposition 7.29 and, hence, Theorem 7.22 for $N = 1$. □

D. *Proof of Theorem* 7.22 *for* $N > 1$

At this stage we apply Proposition 7.29 to the space $(\mathcal{M}, d_p, \mathcal{F})$ being the direct p-product of $\mathcal{PHT}$ spaces $(\mathcal{M}_i, d_i, \mathcal{F}_i)$, $1 \le i \le N$. We change notation, denoting by $(\widehat{\mathcal{M}}, \widehat{d})$ the direct 1-sum $\mathcal{M} \oplus^{(1)} \ell_1^n$ and by $\widehat{\mathcal{F}}$ the tensor product $\mathcal{F} \otimes \{\lambda_n\}$ (previously these were denoted by $(\mathcal{M}_n, d_n)$ and $\mathcal{F}_n$). Then by Proposition 7.29,

$$\lambda(\widehat{\mathcal{M}}, X) \le c_0\left(n + C\left(\widehat{\mathcal{F}}; \frac{1}{n}\right)\right)$$

where $n := \lfloor \log_2 D(\mathcal{F}) \rfloor + 6$.

Further, we apply Proposition 3.92 of Volume I estimating $D(\mathcal{F})$ and $C(\widehat{\mathcal{F}}; t)$ through the corresponding characteristics of $\mathcal{F}_j$, $1 \le j \le N$. For the former we get from there

$$D(\mathcal{F}) \le \prod_{j=1}^{N} D(\mathcal{F}_j) =: D,$$

while for the latter we have

$$C\left(\widehat{\mathcal{F}}; \frac{1}{n}\right) \le \gamma_p\left(\frac{1}{n}\right) \prod_{j=1}^{N} K_j \left\{ C\left(\{\lambda_n\}; \frac{1}{n}\right)^q + \sum_{j=1}^{N} C\left(\mathcal{F}_j; \frac{1}{n}\right)^q \right\}^{1/q}.$$

Since $C(\{\lambda_n\}; t) = 0$ for all $t > 0$ and $C(\mathcal{F}_j; \frac{1}{n}) \le C(\mathcal{F}_j; 1) =: C(\mathcal{F}_j)$, we conclude from here that

$$C\left(\widehat{\mathcal{F}}; \frac{1}{n}\right) \le \gamma_p\left(\frac{1}{n}\right) KC$$

where we recall that

$$K := \prod_{j=1}^{N} K_j \quad \text{and} \quad C := \left\{ \sum_{j=1}^{N} C(\mathcal{F}_j)^q \right\}^{1/q}.$$

Finally, by the definition of γ_p,

$$\gamma_p\left(\frac{1}{n}\right) := \inf_{a>0}\left(\left[(1+a)^p - \frac{1}{n^p}\right]^{-\frac{1}{p}} D(\widehat{\mathcal{F}}; (1+a))\right)$$
$$\leq \left[\left(1+\frac{1}{n}\right)^p - \frac{1}{n^p}\right]^{-\frac{1}{p}} D\left(\widehat{\mathcal{F}}; 1+\frac{1}{n}\right).$$

Since the first factor is less than $[(1+\frac{1}{n^p}) - \frac{1}{n^p}]^{-\frac{1}{p}} = 1$ and the second is at most $\frac{6}{5}e^4$, by Lemma 7.25 we finally get for some $c > 1$,

$$\lambda(\mathcal{M}, X) \leq \lambda(\widehat{\mathcal{M}}, X) \leq c(\log_2 D + KC + 1).$$

Theorem 7.22 has been established. $\square$

It is worth noting that the inequality for $C\left(\widehat{\mathcal{F}}_n; \frac{1}{n}\right)$ given by Proposition 3.92 of Volume I gives a slightly sharper result than that of Theorem 7.22 with K replaced by $K^1 := \prod_{j=2}^{N} K_j$ for $N > 1$ and $K^1 := 1$ for $N = 1$. Since the latter will be used below, we present it as

Corollary 7.34. *Let $(\mathcal{M}, d, \mathcal{F})$ be a $\mathcal{PHT}$ space and X be Banach. Then for some constant $c_0 > 1$,*

$$\lambda(\mathcal{M}, X) \leq c_0(\log_2 D_2(\mathcal{F}) + C(\mathcal{F}) + 1).$$

The subsequent two corollaries are, in fact, variants of Theorem 7.22 obtained by applying the argument used in its proof.

Corollary 7.35. *Let $(\mathcal{M}, d, \mathcal{F})$ be a $\mathcal{PHT}$ space whose dilation function satisfies for some $\sigma \geq 0$,*

$$a := \sup_{l \geq 1} \left\{ \frac{D(\mathcal{F}; l)}{l^\sigma} \right\} < \infty.$$

Then for some $c > 1$,

$$\lambda(\mathcal{M}, X) \leq c\left(n + C\left(\mathcal{F}; \frac{1}{n}\right)\right)$$

where $n := \lfloor \log_2 a \rfloor + \sigma + 1$.

Proof. We apply Proposition 7.29 to space $(\widehat{\mathcal{M}}, \widehat{d}, \widehat{\mathcal{F}})$ where

$$(\widehat{\mathcal{M}}, \widehat{d}) := (\mathcal{M}, d) \oplus^{(1)} \ell_1^k \quad \textit{and} \quad \widehat{\mathcal{F}} := \mathcal{F} \otimes \{\lambda_k\} \quad \textit{and} \quad k := \lfloor \log_2 a \rfloor + \sigma + 6.$$

This gives

$$\lambda(\widehat{\mathcal{M}}, X) \leq c\left(k + C\left(\widehat{\mathcal{F}}; \frac{1}{k}\right)\right).$$

Further, by Proposition 3.92 of Volume I and Lemma 7.27,

$$\begin{aligned} C\left(\widehat{\mathcal{F}}; \frac{1}{k}\right) &\leq \gamma_1\left(\frac{1}{k}\right)\left(C\left(\{\lambda_k\}; \frac{1}{k}\right) + C\left(\mathcal{F}; \frac{1}{k}\right)\right) \\ &\leq \frac{6}{5}e^4 C\left(\mathcal{F}; \frac{1}{k}\right) \leq \frac{6}{5}e^4 C\left(\mathcal{F}; \frac{1}{n}\right). \end{aligned}$$

The result is proved. □

We now single out a special case of the above result with a better estimate of $\lambda(\mathcal{M}, X)$. Specifically, suppose that $\mathcal{F}$ is n-homogeneous, i.e., for all balls in $\mathcal{M}$,

$$\mu_m(B_R(m)) = \gamma R^\sigma, \quad \gamma, \sigma > 0. \tag{7.100}$$

Under this assumption the following holds.

Corollary 7.36. *It is true that*

$$\lambda(\mathcal{M}, X) \leq c\left(\sigma^* + C\left(\mathcal{F}; \frac{1}{\sigma^*}\right)\right)$$

where $\sigma^* := \max\{\sigma, 1\}$ *and* $c < 311$.

Proof. In this case, we do not need the extended space trick. In fact, we exploit it only for the estimates of Lemmas 7.25 and 7.27. But the required variant of the latter for the space $(\mathcal{M}, d, \mathcal{F})$ satisfying (7.100) may be obtained by the next straightforward evaluation with $R_2 > R_1 > 0$,

$$\begin{aligned} \mu_n(B_{R_2}(m)) - \mu_m(B_{R_1}(m)) &:= \gamma(R_2^\sigma - R_1^\sigma) \leq \gamma \max\{\sigma, 1\} R_2^{\sigma-1}(R_2 - R_1) \\ &:= \sigma^* \cdot \frac{\mu_m(B_{R_2}(m))}{R_2}(R_2 - R_1). \end{aligned}$$

Further, repeating in this case the arguments of Proposition 7.29 with σ^* in place of n and $l := 1 + \frac{1}{3\sigma^*}$ and noting that $A_n := \frac{6}{5}e^4 n$ is now replaced by σ^* we get

$$\lambda(\mathcal{M}, X) \leq 56\sigma^* + \max\left\{42\sigma^*\left(4 + \frac{1}{3\sigma^*}\right), K_n\left(1 + \frac{1}{3\sigma^*}\right)\right\}.$$

Since under this choice of n,

$$K_n\left(1 + \frac{1}{3\sigma^*}\right) := 42\left(\sigma^* + C\left(\mathcal{F}; \frac{1}{\sigma^*}\right)\right)\left(4 + \frac{1}{3\sigma^*}\right) \cdot D\left(\mathcal{F}; 1 + \frac{1}{3\sigma^*}\right),$$

the maximum is attained at the second term. Moreover, by (3.121),

$$D\left(\mathcal{F}; 1+\frac{1}{3\sigma^*}\right) := \left(1+\frac{1}{3\sigma^*}\right)^{\sigma} < \sqrt[3]{e}.$$

Combining these estimates we prove the result. □

Now we present several consequences of Theorem 7.22 and its corollaries beginning with the following elegant result firstly established by Lee and Naor [LN-2005] in a nonconstructive way, see Section 7.3 for a discussion of their method.

Corollary 7.37. *Let $(\mathcal{M}, d)$ be a nontrivial doubling metric space with doubling constant δ. Then for some numerical constant $c > 1$,*

$$\lambda(\mathcal{M}, X) \le c \log_2 \delta.$$

Proof. Without loss of generality we may assume that $\mathcal{M}$ is complete. Then by the Koniagin-Vol'berg Theorem 4.43 of Volume I the space $\mathcal{M}$ carries a doubling measure μ with dilation function satisfying

$$D(\mu; l) \le c(s) l^s$$

where $s > \log_2 \delta + 1$ and

$$\log_2 c(s) := (\log_2 \delta + 1)\log_2 24 + 5s \max\left\{\frac{1+s}{s - \log_2 \delta - 1}, \log_2 21\right\},$$

see Volume I, Section 4.3.

For $\mathcal{M}$ containing at least two points its doubling constant is $\delta \ge 2$. Therefore choosing $s := 2\log_2 \delta + 1$, we obtain $\log_2 c(s) < \widetilde{c} \log_2 \delta$ for a numerical constant $\widetilde{c} > 1$. Then $(\mathcal{M}, d, \{\mu\})$ is a space of homogeneous type satisfying the condition of Corollary 7.35 with $\sigma := 2\log_2 \delta + 1$, $a := \delta^{\widetilde{c}}$. Moreover, $C(\{\mu\}; t) = 0$ for all $t > 0$ and we obtain from this corollary that

$$\lambda(\mathcal{M}, X) \le c \log_2 \delta$$

where $c > 1$ is a numerical constant.

The result is proved. □

Now we estimate $\lambda(\mathcal{M}, X)$ for $\mathcal{M}$ being the direct p-sum ($1 \le p \le \infty$) of the Riemannian manifolds $\mathbb{H}^{n_i}_{\omega_i}$, $1 \le i \le N$. Let us recall that $\mathbb{H}^n_\omega$ is defined by the Riemannian metric

$$ds^2 := \omega(x_n)^2 \cdot \sum_{i=1}^{n} dx_i^2$$

on the underlying set $H^n_+ := \{x \in \mathbb{R}^n \,;\, x_n > 0\}$ and is regarded as a metric space (with the geodesic metric). Further, $\omega : (0, +\infty) \to \mathbb{R}_+$ is a continuous nonincreasing function satisfying the condition

$$\int_1^\infty \omega(s) ds = \infty. \qquad (7.101)$$

Corollary 7.38. *Let $(\mathcal{M}, d)$ be the direct p-sum of spaces $\mathbb{H}_{\omega_i}^{n_i}$, $1 \le i \le N$. Assume that every ω_i satisfies* (7.101)*. Then for some $c > 1$ and $\frac{1}{q} := 1 - \frac{1}{p}$,*

$$\lambda(\mathcal{M}, X) \le c \left(\sum_{i=1}^{N} n_i + \left[\sum_{i=1}^{N} n_i^{2q} \right]^{1/q} \right) \cdot \max_{1 \le i \le N} \sqrt{n_i}.$$

Proof. According to Theorem 4.58 of Volume I the space $\mathbb{H}_\omega^n$ is bi-Lipschitz homeomorphic to the metric space of $(n-1)$-cubes in $\mathbb{R}^{n-1}$ denoted by $\mathcal{B}_\omega^{n-1} := (\mathcal{B}(\ell_\infty^{n-1}), d_\omega)$, see subsection 4.41 of Volume I for its definition and properties. In subsection 4.4.2 of Volume I, the family of measures $\mathcal{F}_\omega := \{\mu_Q\}_{Q \in \mathcal{B}(\ell_\infty^{n-1})}$ is constructed, see formula (4.88) there, such that $(\mathcal{B}_\omega^{n-1}, \mathcal{F})$ turns into a 1-uniform $\mathcal{PHT}$ space and for every ball $\overline{B}_R(Q) \subset \mathcal{B}_\omega^n$ and $0 < t < 1$,

$$\mu_Q(\overline{B}_R(Q)) = 2^{n-1} R^n \ \text{ and } \ C(\mathcal{F}_\omega; t) \le \frac{3n}{2} \cdot \frac{(1+t)^n - 1}{t}. \tag{7.102}$$

Now let $\widehat{\mathcal{M}} := \oplus^{(p)} \mathcal{B}_{\omega_i}^{n_i - 1}$ and $\widehat{\mathcal{F}} := \otimes_{i=1}^N \mathcal{F}_{\omega_i}$. By (7.102) a measure $\widehat{\mu} \in \widehat{\mathcal{F}}$ satisfies for all $\widehat{m} \in \widehat{M}$ and $R > 0$,

$$\widehat{\mu}(\overline{B}_R(\widehat{m})) = 2^{n-N} R^n$$

where $n := \sum_{i=1}^N n_i$. Hence, we may apply Corollary 7.36 to get

$$\lambda(\widehat{M}, X) \le c \left(n + C \left(\widehat{\mathcal{F}}; \frac{1}{n} \right) \right).$$

Further, by Proposition 3.92 of Volume I we have for $0 < t \le 1$,

$$C(\widehat{\mathcal{F}}; t) \le \gamma_p(t) \left\{ \sum_{i=1}^{N} C(\mathcal{F}_{\omega_i}; t)^q \right\}^{1/q}.$$

To estimate each term of this sum we use inequality (7.102) and the inequality

$$\frac{(1+t)^k - 1}{t} \le k(1+t)^{k-1}$$

to have

$$C \left(\mathcal{F}_{\omega_i}; \frac{1}{n} \right) \le \frac{3}{2} n_i^2 \left(1 + \frac{1}{n} \right)^{n_i - 1} < \frac{3e}{2} n_i^2.$$

Moreover, by (7.102) and the definition of γ_p,

$$\begin{aligned} \gamma_p \left(\frac{1}{n} \right) &\le \left[\left(1 + \frac{1}{n} \right)^p - \frac{1}{n^p} \right]^{-\frac{1}{p}} D \left(\widehat{\mathcal{F}}; 1 + \frac{1}{n} \right) \\ &\le D \left(\widehat{\mathcal{F}}; 1 + \frac{1}{n} \right) = \left(1 + \frac{1}{n} \right)^n < e. \end{aligned}$$

Combining these inequalities we have

$$\lambda(\widehat{\mathcal{M}}, X) \leq \frac{3}{2}e^2 c \left(n + \left[\sum_{j=1}^{N} n_i^{2q} \right]^{1/q} \right). \tag{7.103}$$

To conclude the proof we use the argument of Theorem 4.58 of Volume I asserting that $\mathbb{H}^n_\omega$ is bi-Lipschitz homeomorphic to the space of Euclidean balls $(\mathcal{B}(\mathbb{R}^{n-1}), d_\omega)$ with distortion at most 3, see formula (4.98) there. Further, the identity map of $\mathbb{R}^{n-1}$ $(= \ell_2^{n-1})$ onto ℓ^n_∞ has distortion $\sqrt{n-1}$. By Theorem 4.53 (b) of Volume I $(\mathcal{B}(\mathbb{R}^{n-1}), d_\omega)$ is bi-Lipschitz homeomorphic to $\mathcal{B}^{n-1}_\omega := (\mathcal{B}(\ell^{n-1}_\infty), d_\omega)$ with distortion at most $\sqrt{n}$. Then distortion of the bijection of $\widehat{\mathcal{M}}$ onto $\mathcal{M} := \oplus_{i=1}^N \mathbb{H}^{n_i}_{\omega_i}$ is at most $3(\max_{1\leq i\leq N} \sqrt{n_i})$ (use the Hölder inequality). This implies

$$\lambda(\mathcal{M}, X) \leq 3 \left(\max_{1\leq i\leq N} \sqrt{n_i} \right) \lambda(\widehat{\mathcal{M}}, X).$$

Together with (7.103) this yields the required result. □

Now from the results established we derive estimates of $\lambda(\mathcal{M}, X)$ for $\mathcal{M}$ being one of the classical spaceforms.

Corollary 7.39. *It is true that*

$$\lambda(\mathbb{S}^n, X), \quad \lambda(\mathbb{R}^n, X) \leq c_1 n \quad \textit{and} \quad \lambda(\mathbb{H}^n, X) \leq c_2 n^{5/2}$$

where $c_1 < 311$ *and* $c_2 < 20640$.

Proof. Since the surface measure of $\mathbb{S}^n$ and the Lebesgue measure of $\mathbb{R}^n$ are n-homogeneous, we may apply Corollary 7.36 with $\sigma = n$ and $C(\mathcal{F}; \cdot) = 0$ to prove the first two inequalities.

The third inequality follows from Corollary 7.38 with $N = 1$ and $\omega(t) = \frac{1}{t}$, $t > 0$. Under this choice $\mathbb{H}^n_\omega$ coincides with the hyperbolic space $\mathbb{H}^n$ while an accurate computation of the constants involved in the proof gives the above estimate for c_2. □

Using another method, discussed at the end of the chapter, one can establish that $\lambda(\mathbb{R}^n, X) \leq cn^{1/2}$ for a numerical constant $c > 0$, [LN-2005].

Problem. *What is the sharp order of growth for* $\lambda(\mathcal{M}, X)$ *as* $\dim \mathcal{M} \to \infty$ *where* $\mathcal{M}$ *is one of the classical spaceforms?*

We guess that the sharp order is $n^{1/2}$ for all these spaces. The proof of this conjecture apparently requires an approach which is considerably more profound than those described in the present book.

Finally, we briefly discuss another construction of the extension operator (7.74) which allows us to obtain essentially better numerical constants in all results of this section for an X-valued function with X being a Banach space constrained in its dual. For example, the constant in Corollary 7.36 will be less than 24 (see [BB-2007b, Cor. 2.27]).

The new extension operator is a simple modification of that given by (7.74) where the Dugundji operator $f \mapsto \widehat{f}$ is replaced by the operator composing f and a metric projection onto the subspace $S \subset \mathcal{M}$ denoted by p_S. For card $S < \infty$ and a suitable definition of p_S we obtain an operator $E \in \mathrm{Ext}(S, \mathcal{M})$ which yields better estimates than that in (7.74). To transfer from finite subsets S to the general case we, for scalar functions, exploit the finiteness property of Theorem 7.12 and then, for X-valued functions with X being constrained in its dual, Proposition 7.3.

Let us explain why this construction cannot be used for infinite subspaces S. In the definition of a metric or $(1+\varepsilon)$-metric projection, $\varepsilon > 0$, we encounter the following measurable selection problem.

Let P_S^ε, $\varepsilon \geq 0$, be a set-valued function on $\mathcal{M}$ given by

$$P_S^\varepsilon(m) := \{m' \in S\,;\, d(m, m') \leq (1+\varepsilon) d(m, S)\}.$$

If there exists a Borel measurable selection $p_S \in P_S^\varepsilon$, the composition $f \circ p_S$ with a continuous locally bounded function $f : S \to \mathbb{R}$ is also Borel measurable and the average operator I may be applied to $f \circ p_S$.

Unfortunately, such a selection may not exist in general, see Theorem 3.71 on the corresponding P. Novikov's counterexample. A careless choice of a selection may lead to a Borel nonmeasurable metric projection even for a finite S (M. Wojcieakowski, personal communication). However, in this case we simply enumerate arbitrarily the points of S and then select in $P_S^0(m)$ the point with the minimal number to obtain a Borel measurable p_S.

We present only two results which can be achieved in this way, see [BB-2007b] for more facts of this kind.

Corollary 7.40. *Let $(\mathcal{M}, d)$ be the direct p-sum of spaces $\mathbb{H}^{n_i}_{\omega_i}$, $1 \leq i \leq N$, and X be the dual space. Then it is true that*

$$\lambda(\mathcal{M}, X) \leq c \left(\sum_{i=1}^N n_i + \left[\sum_{i=1}^N n_i^{2q} \right]^{1/q} \right) \cdot \max_{1 \leq i \leq N} \sqrt{n_i}$$

where $c < 800$.

Here the weights ω_i do not necessarily satisfy condition (7.101).

Proof. To prove the result we first approximate each ω_i uniformly on compact subsets of $H_+^{n_i}$ by weights $w_{i,k}$, $k \in \mathbb{N}$, satisfying condition (7.101). Then we apply Corollary 7.38 to the direct p-sums $(\mathcal{M}_k, d_k)$ of spaces $\mathbb{H}^{n_i}_{\omega_{i,k}}$, $1 \leq i \leq N$. In fact, to get the required bound for c, we apply this corollary to finite subsets of $(\mathcal{M}_k, d_k)$ for the extension operator described above. Finally, using the finiteness property of Theorem 7.12 we obtain the result. $\square$

The second result concerns the direct p-sum of a finite family of metric trees. We restrict our consideration to scalar functions.

Corollary 7.41. *Let $(\mathcal{T}_i, d_i)$ be a nontrivial metric tree, $1 \leq i \leq N$. Then for some numerical constant $\widetilde{c} > 1$,*

$$\lambda(\oplus^p\{(\mathcal{T}_i, d_i)\}_{1\leq i\leq N}) \leq \widetilde{c}N$$

where $1 \leq p \leq \infty$.

Proof. Let $\mathcal{V}_i$ be the vertex set of $\mathcal{T}_i$. By Theorem 5.4, $\mathcal{V}_i$ admits a bi-Lipschitz embedding into $\mathbb{H}^2$ with distortion at most 257. Therefore $\oplus^p\{(\mathcal{V}_i, d_i)\}_{1\leq i\leq N}$ admits such an embedding into the direct p-sum of the spaces $\mathbb{H}^2$ taken N times. Due to Corollary 7.38 the simultaneous extension constant of the latter p-sum is bounded by

$$\sqrt{2}c\left(2N + \left(\sum_{i=1}^{N} 2^{2q}\right)^{1/q}\right) = \sqrt{2}c(2N + 4N^{1/q}).$$

Next, each finite $S \subset \oplus_p\{(\mathcal{T}_i, d_i)\}_{1\leq i\leq N}$ is a subset of $\oplus^p\{(S_i, d_i)\}_{1\leq i\leq N}$ where S_i are natural projections of S onto $\mathcal{T}_i$. In turn, each S_i is a subset of the vertex set $\mathcal{V}_{S_i}$ of a finite graph $(\mathcal{T}_{S_i}, d_i)$ in $\mathcal{T}_i$ with the underlying set being the union of geodesics in $\mathcal{T}_i$ joining distinct points of S_i and with the vertex set $\mathcal{V}_{S_i}$ consisting of points of S_i and those of $\mathcal{V}_i$ belonging to these geodesics.

Now, we apply the previous estimate to the direct p-sum of vertex sets $\mathcal{V}_{S_i}$, $1 \leq i \leq N$, to bound the extension constant $\lambda\left(S, \oplus^p\{\mathcal{T}_i\}_{1\leq i\leq N}\right)$. After that we use the finiteness property of Theorem 7.12 for the family of these extension constants with S running through all finite subsets of $\oplus^p\{(\mathcal{T}_i, d_i)\}_{1\leq i\leq N}$. Then we get

$$\lambda(\oplus^p\{(\mathcal{T}_i, d_i)\}_{1\leq i\leq N}) \leq 257\sqrt{2}c(2N + 4N^{1/q}) \leq \widetilde{c}N. \tag{7.104}$$

$\square$

Remark 7.42. For $N = 1$ the result was proved by Matoušek [Mat-1990] by another method admitting the Banach-valued functions.

7.3 Locally doubling metric spaces with uniform lattices

Locally doubling metric spaces do not in general have the simultaneous extension property (briefly, they are not $\mathcal{LE}$ spaces); in particular we present in Chapter 8 an example of a space which even homeomorphic to $\mathbb{R}$ but is not $\mathcal{LE}$. We, however, show that under a mild restriction the space in question is $\mathcal{LE}$, if it contains a certain discrete subspace of this kind. To formulate the result we recall the corresponding notions (discussed in details in Chapter 3 of Volume I, see, e.g., Definitions 3.29 and 3.45 there).

A subset Γ of a metric space $\mathcal{M}$ is said to be an *R-lattice with parameter* $c_\Gamma \in (0, \frac{1}{2}]$ if the open balls $B_R(\gamma)$, $\gamma \in \Gamma$, cover $\mathcal{M}$ while the open balls $B_{c_\Gamma R}(\gamma)$, $\gamma \in \Gamma$, are mutually disjoint.

The existence of R-lattices for an arbitrary metric space easily follows from Zorn's lemma.

Now let $\mathcal{M}$ belong to the class of locally doubling metric spaces $\mathcal{D}(R, N)$, i.e., each of its open balls of radius $r \le R$ can be covered by at most N open balls of radius $r/2$.

Theorem 7.43. *Let $(\mathcal{M}, d)$ be a metric space from $\mathcal{D}(2R, N)$. Assume that the extension constant $\lambda(\Gamma)$ of an R-lattice $\Gamma \subset \mathcal{M}$ is finite and, moreover,*

$$\lambda_R := \sup\{\lambda(B_{2R}(m)) \ ; \ m \in \mathcal{M}\} < \infty. \tag{7.105}$$

Then $\lambda(\mathcal{M})$ is bounded by a constant depending only on $\lambda(\Gamma)$, λ_R, c_Γ, R and N.

Proof. Let Γ be an R-lattice and $\mathcal{B} := \{B_R(\gamma)\}_{\gamma\in\Gamma}$. By definition $\mathcal{B}$ and $2\mathcal{B} := \{B_{2R}(\gamma)\}_{\gamma\in\Gamma}$ cover $\mathcal{M}$.

Lemma 7.44. *The order of the cover $2\mathcal{B}$ is bounded by a constant μ depending only on c_Γ and N.*

Proof. Let m be covered by balls $B_{2R}(\gamma_i)$, $1 \le i \le k$. Then all these γ_i lie in the open ball $B_{2R}(m)$. Since $d(\gamma_i, \gamma_j) \ge c_\Gamma R$ for $i \ne j$, any cover of $B_{2R}(m)$ by open balls of radius $\frac{c_\Gamma R}{2}$ separates the set $\{\gamma_i\}_{1\le i\le k}$, i.e., distinct γ_i lie in distinct balls of the cover. Hence, such a cover has cardinality at least k. On the other hand, $\mathcal{M}$ belongs to $\mathcal{D}(2R, N)$ and therefore there exists a cover of $B_{2R}(m)$ by open balls of radius $\frac{c_\Gamma R}{2}$ whose cardinality is at most N^s where $s := \left\lfloor \log_2 \frac{1}{c_\Gamma} \right\rfloor + 2$ (apply s times the doubling condition). Hence,

$$\operatorname{ord}(2\mathcal{B}) \le \max k \le N^s. \qquad \square$$

Next we prove

Lemma 7.45. *There is a partition of unity $\{\rho_\gamma\}_{\gamma\in\Gamma}$ subordinate to $2\mathcal{B}$ such that*

$$K := \sup_\gamma ||\rho_\gamma||_{\mathrm{Lip}(\mathcal{M})} < \infty \tag{7.106}$$

where K depends only on c_Γ, N and R.

Proof. Set

$$B_\gamma := B_{2R}(\gamma) \quad \text{and} \quad B_\gamma^c := \mathcal{M} \setminus B_\gamma$$

and define

$$d_\gamma(m) := \operatorname{dist}(m, B_\gamma^c) \ , \quad m \in \mathcal{M}.$$

It is clear that

$$\operatorname{supp} d_\gamma \subset B_\gamma \quad \text{and} \quad ||d_\gamma||_{\mathrm{Lip}(\mathcal{M})} \le 1. \tag{7.107}$$

Let now $\phi : \mathbb{R}_+ \to [0,1]$ be continuous, equal to one on $[0, R]$, zero on $[2R, \infty)$ and be linear on $[R, 2R]$. Introduce the function

$$s := \sum_{\gamma} \phi \circ d_\gamma. \tag{7.108}$$

By Lemma 7.44 only at most μ terms here are nonzero at every point. Therefore

$$||s||_{\mathrm{Lip}(\mathcal{M})} \le 2\mu ||\phi||_{\mathrm{Lip}(\mathbb{R}_+)} \sup_{\gamma} ||d_\gamma||_{\mathrm{Lip}(\mathcal{M})}$$

and by (7.107) and the definition of ϕ we get

$$||s||_{\mathrm{Lip}(\mathcal{M})} \le 2\mu/R. \tag{7.109}$$

On the other hand, every $m \in \mathcal{M}$ is contained in some ball $B_R(\gamma)$ of the cover $\mathcal{B}$. For this γ,

$$(\phi \circ d_\gamma)(m) \ge \phi(R) = 1$$

and therefore

$$s \ge 1. \tag{7.110}$$

Introduce now the required partition by

$$\rho_\gamma := \frac{\phi \circ d_\gamma}{s}\ , \quad \gamma \in \Gamma.$$

Then $\{\rho_\gamma\}$ is clearly a partition of unity subordinate to $2\mathcal{B}$. Moreover, we have

$$|\rho_\gamma(m) - \rho_\gamma(m')| \le \frac{|\phi(d_\gamma(m)) - \phi(d_\gamma(m'))|}{s(m)} + \frac{\phi(d_\gamma(m'))}{s(m) \cdot s(m')} \cdot |s(m) - s(m')|$$

and application of (7.110), (7.109) and (7.107) leads to the desired inequality

$$||\rho_\gamma||_{\mathrm{Lip}(\mathcal{M})} \le \frac{2\mu + 1}{R}. \qquad \square$$

Lemma 7.46.

$$\mathrm{Ext}(\Gamma, \mathcal{M}) \ne \emptyset.$$

Proof. By assumption (7.105) of the theorem, for every $\gamma \in \Gamma$ there is a linear operator $E_\gamma \in \mathrm{Ext}(\Gamma \cap B_\gamma, B_\gamma)$ such that

$$||E_\gamma|| \le \lambda_R, \quad \gamma \in \Gamma. \tag{7.111}$$

Using this we introduce the required linear operator by

$$Ef := \sum_{\gamma \in \Gamma} (E_\gamma f_\gamma)\rho_\gamma, \quad f \in \mathrm{Lip}(\Gamma), \tag{7.112}$$

where $\{\rho_\gamma\}$ is the partition of unity from Lemma 7.45 and $f_\gamma := f|_{\Gamma\cap B_\gamma}$; here we assume that $E_\gamma f_\gamma$ is zero outside of B_γ. We have to show that

$$Ef|_\Gamma = f|_\Gamma \tag{7.113}$$

and estimate $||Ef||_{\mathrm{Lip}(\mathcal{M})}$.

Given $\hat\gamma \in \Gamma$ we can write

$$(Ef)(\hat\gamma) = \sum_{B_\gamma \ni \hat\gamma} (E_\gamma f_\gamma)(\hat\gamma)\rho_\gamma(\hat\gamma).$$

Since E_γ is an extension from $B_\gamma \cap \Gamma$, we get

$$(E_\gamma f_\gamma)(\hat\gamma) = f_\gamma(\hat\gamma) = f(\hat\gamma).$$

Moreover, $\sum\limits_{B_\gamma \ni \hat\gamma} \rho_\gamma(\hat\gamma) = 1$, and (7.113) is proved.

To estimate the Lipschitz constant of Ef, we use the the McShane Theorem 1.27 of Volume I to extend $E_\gamma f_\gamma$ outside of B_γ so that the (nonlinear) extension F_γ satisfies

$$||F_\gamma||_{\mathrm{Lip}(\mathcal{M})} = ||E_\gamma f_\gamma||_{\mathrm{Lip}(B_\gamma)}. \tag{7.114}$$

Since $\rho_\gamma F_\gamma = \rho_\gamma E_\gamma f_\gamma$, we have

$$Ef = \sum_\gamma F_\gamma \rho_\gamma. \tag{7.115}$$

Given $\hat\gamma \in \Gamma$ we then introduce a function $G_{\hat\gamma}$ by

$$G_{\hat\gamma} := \sum_\gamma (F_\gamma - F_{\hat\gamma})\rho_\gamma := \sum_\gamma F_{\gamma\hat\gamma}\rho_\gamma \tag{7.116}$$

and write for every $\hat\gamma$,

$$Ef = F_{\hat\gamma} + G_{\hat\gamma}. \tag{7.117}$$

It follows from (7.111) and (7.114) that

$$||F_{\hat\gamma}||_{\mathrm{Lip}(\mathcal{M})} \le \lambda_R ||f||_{\mathrm{Lip}(\Gamma)}. \tag{7.118}$$

We show now that for $m \in B_\gamma \cap B_{\hat\gamma}$,

$$|F_{\gamma\hat\gamma}(m)| \le 8R\lambda_R ||f||_{\mathrm{Lip}(\Gamma)}. \tag{7.119}$$

In fact, for these m,

$$\begin{aligned}|F_{\gamma\hat\gamma}(m)| = |(E_\gamma f_\gamma - E_{\hat\gamma} f_{\hat\gamma})(m)| &\le |f(\gamma) - f(\hat\gamma)| + |(E_\gamma f_\gamma)(m) - (E_\gamma f_\gamma)(\gamma)| \\ &+ |(E_{\hat\gamma} f_{\hat\gamma})(m) - (E_{\hat\gamma} f_{\hat\gamma})(\hat\gamma)|.\end{aligned}$$

Estimating the right-hand side by (7.111) we then get

$$|F_{\gamma\hat{\gamma}}(m)| \le \lambda_R ||f||_{\mathrm{Lip}(\Gamma)}(d(\gamma,\hat{\gamma}) + d(m,\gamma) + d(m,\hat{\gamma})) \le 8R\lambda_R ||f||_{\mathrm{Lip}(\Gamma)}.$$

Applying this to estimate $\Delta G_{\hat{\gamma}} := G_{\hat{\gamma}}(m) - G_{\hat{\gamma}}(m')$ for $m, m' \in B_{\hat{\gamma}}$ we get

$$|\Delta G_{\hat{\gamma}}| \le \sum_{B_\gamma \cap B_{\hat{\gamma}} \ni m} |\Delta\rho_\gamma| \cdot |F_{\gamma\hat{\gamma}}(m)| + \sum_{B_\gamma \cap B_{\hat{\gamma}} \ni m'} \rho_\gamma(m') \cdot |\Delta F_{\gamma\hat{\gamma}}|$$

(here the differences $\Delta\rho_\gamma$ and $\Delta F_{\gamma\hat{\gamma}}$ are defined similarly to $\Delta G_{\hat{\gamma}}$). The first sum is estimated by (7.119), (7.106) and Lemma 7.44, while the second one is at most $2\lambda_R ||f||_{\mathrm{Lip}(\Gamma)} d(m,m')$ by (7.114) and (7.111). This leads to the estimate

$$|\Delta G_{\hat{\gamma}}| \le (16RK\mu + 2) \cdot \lambda_R \cdot ||f||_{\mathrm{Lip}(\Gamma)} d(m,m'), \quad m, m' \in B_{\hat{\gamma}}.$$

Together with (7.118) this yields for these m, m':

$$|(Ef)(m) - (Ef)(m')| \le C ||f||_{\mathrm{Lip}(\Gamma)} d(m,m'). \tag{7.120}$$

Here and below C denotes a constant depending only on the basic parameters, that may change from line to line.

It remains to prove (7.120) for m, m' belonging to distinct balls B_γ. Let $m \in B_\gamma$ and m' be a point of some $B_R(\hat{\gamma})$ from the cover $\mathcal{B}$. Then $m \in B_\gamma \setminus B_{\hat{\gamma}}$ and therefore

$$d(m,m') \ge R. \tag{7.121}$$

Using now (7.117) we have

$$|(Ef)(m) - (Ef)(m')| \le |F_\gamma(m) - F_{\hat{\gamma}}(m')| + |G_\gamma(m) - G_{\hat{\gamma}}(m')| := I_1 + I_2.$$

By the definition of F_γ, we then get

$$I_1 \le |f(\gamma) - f(\hat{\gamma})| + |(E_\gamma f_\gamma)(m) - (E_\gamma f_\gamma)(\gamma)| + |(E_{\hat{\gamma}} f_{\hat{\gamma}})(m') - (E_{\hat{\gamma}} f_{\hat{\gamma}})(\hat{\gamma})|.$$

Together with (7.111) this leads to the estimate

$$\begin{aligned} I_1 &\le \lambda_R ||f||_{\mathrm{Lip}(\Gamma)}(d(\gamma,\hat{\gamma}) + d(m,\gamma) + d(m',\hat{\gamma})) \\ &\le 2\lambda_R ||f||_{\mathrm{Lip}(\Gamma)}(d(m,m') + d(m,\gamma) + d(m',\hat{\gamma})). \end{aligned}$$

Since $d(m,\gamma) + d(m',\hat{\gamma}) \le 4R \le 4d(m,m')$ by (7.121), we therefore have

$$I_1 \le C ||f||_{\mathrm{Lip}(\Gamma)} d(m,m'). \tag{7.122}$$

To estimate I_2, we note that for $m \in B_\gamma$,

$$G_\gamma(m) = \sum_{B_{\gamma'} \cap B_\gamma \ni m} (\rho_{\gamma'} F_{\gamma'\gamma})(m).$$

In combination with (7.119) and (7.121) this gives

$$|G_\gamma(m)| \le 8\lambda_R R||f||_{\mathrm{Lip}(\Gamma)} \le C||f||_{\mathrm{Lip}(\Gamma)} d(m, m').$$

The same argument estimates $G_{\hat\gamma}(m')$ for $m' \in B_{\hat\gamma}$. Hence

$$I_2 \le |G_\gamma(m)| + |G_{\hat\gamma}(m')| \le C||f||_{\mathrm{Lip}(\Gamma)} d(m, m').$$

Together with (7.122) and (7.120) this leads to the inequality

$$||Ef||_{\mathrm{Lip}(\mathcal{M})} \le C||f||_{\mathrm{Lip}(\Gamma)}.$$

Hence E is an operator from $\mathrm{Ext}(\Gamma, \mathcal{M})$. □

We are now in a position to complete the proof of Theorem 7.43. According to Theorem 7.12 we must show that

$$\sup_F \lambda(F) < \infty \tag{7.123}$$

where F runs through all finite subspaces of $\mathcal{M}$. To this end we consider the "Γ-envelope" of such F given by

$$\widehat{F} := \{\gamma \in \Gamma \ ; \ B_\gamma \cap F \ne \emptyset\}.$$

Then $\{B_\gamma \ ; \ \gamma \in \widehat{F}\} \subset 2\mathcal{B}$ is an open cover of F. By assumption (7.105) of the theorem for every $\gamma \in \widehat{F}$ there is an operator $E_\gamma \in \mathrm{Ext}(F \cap B_\gamma, B_\gamma)$ such that

$$||E_\gamma|| \le \lambda_R.$$

Introduce now a linear operator T given for $f \in \mathrm{Lip}(F)$, $\gamma \in \widehat{F}$ by

$$(Tf)(\gamma) := (E_\gamma f_\gamma)(\gamma) \quad where \quad f_\gamma := f|_{B_\gamma \cap F}.$$

We will show that

$$T : \mathrm{Lip}(F) \to \mathrm{Lip}(\widehat{F}) \quad \text{and} \quad ||T|| \le \lambda_R(2/c_\Gamma + 1). \tag{7.124}$$

Actually, let $\gamma_i \in \widehat{F}$ and $m_i \in B_{\gamma_i} \cap F$, $i = 1, 2$. Then $(E_{\gamma_i} f_{\gamma_i})(m_i) = f(m_i)$ and

$$\begin{aligned} |(Tf)(\gamma_1) - (Tf)(\gamma_2)| &\le \sum_{i=1,2} |(E_{\gamma_i} f_{\gamma_i})(\gamma_i) - (E_{\gamma_i} f_{\gamma_i})(m_i)| + |f(m_1) - f(m_2)| \\ &\le \lambda_R ||f||_{\mathrm{Lip}(F)} (d(\gamma_1, m_1) + d(\gamma_2, m_2) + d(m_1, m_2)). \end{aligned}$$

The sum in the brackets does not exceed $4R + d(m_1, m_2) \le 8R + d(\gamma_1, \gamma_2)$. Moreover, by the definition of an R-lattice, $d(\gamma_1, \gamma_2) \ge 4c_\Gamma R$. Combining these estimates, we have

$$|(Tf)(\gamma_1) - (Tf)(\gamma_2)| \le \lambda_R ||f||_{\mathrm{Lip}(F)} (2/c_\Gamma + 1) d(\gamma_1, \gamma_2).$$

This establishes (7.124).

Now assumption (7.105) of the theorem implies that there is an operator L from $\mathrm{Ext}(\widehat{F}, \Gamma)$ whose norm is bounded by λ_Γ. Composing T and L with the operator $E \in \mathrm{Ext}(\Gamma, \mathcal{M})$ of Lemma 7.46 we obtain the operator

$$\widetilde{E} := ELT : \mathrm{Lip}(F) \to \mathrm{Lip}(\mathcal{M}) \tag{7.125}$$

whose norm is bounded by a constant depending only on $\lambda(\Gamma)$, λ_R, c_Γ, R and N. This definition also implies that

$$(\widetilde{E}f)(\gamma) := (E_\gamma f_\gamma)(\gamma), \quad \gamma \in \widehat{F}. \tag{7.126}$$

The constructed operator $\widetilde{E}$ is not an extension from F, hence we need to modify it to obtain the required extension operator. To accomplish this we, first introduce an operator $\widehat{T}$ given for $f \in \mathrm{Lip}(F)$ by

$$(\widehat{T}f)(m) := \begin{cases} (\widetilde{E}f)(m), & \text{if } m \in \widehat{F}, \\ f(m), & \text{if } m \in F \setminus \widehat{F}. \end{cases} \tag{7.127}$$

Lemma 7.47. $\widehat{T} : \mathrm{Lip}(F) \to \mathrm{Lip}(F \cup \widehat{F})$ *and* $||\widehat{T}|| \leq C$.

Proof. It clearly suffices to estimate $I := |(\widehat{T}f)(m_1) - (\widehat{T}f)(m_2)|$ for $m_1 \in \widehat{F}$ and $m_2 \in F \setminus \widehat{F}$. First, let these points belong to a ball B_γ (hence $m_1 = \gamma$). Then (7.126) and the inclusion $E_\gamma \in \mathrm{Ext}(F \cap B_\gamma, B_\gamma)$ imply that

$$\begin{aligned} I &= |(E_\gamma f_\gamma)(\gamma) - f(m_2)| = |(E_\gamma f_\gamma)(\gamma) - (E_\gamma f_\gamma)(m_2)| \\ &\leq \lambda_R ||f||_{\mathrm{Lip}(F)} d(\gamma, m_2) := \lambda_R ||f||_{\mathrm{Lip}(F)} d(m_1, m_2). \end{aligned}$$

In the remaining case of $m_2 \in B_{\hat{\gamma}} \setminus B_\gamma$ for some $\hat{\gamma} \in \widehat{F}$,

$$d(m_1, m_2) = d(\gamma, m_2) \geq 2R. \tag{7.128}$$

Similarly to the previous estimate we obtain for this m_2,

$$\begin{aligned} I &\leq |(\widetilde{E}f)(\gamma) - (\widetilde{E}f)(\hat{\gamma})| + |(E_{\hat{\gamma}} f_{\hat{\gamma}})(\hat{\gamma}) - (E_{\hat{\gamma}} f_{\hat{\gamma}})(m_2)| \\ &\leq (||\widetilde{E}|| d(\gamma, \hat{\gamma}) + \lambda_R d(\hat{\gamma}, m_2)) ||f||_{\mathrm{Lip}(F)}. \end{aligned}$$

Moreover, (7.128) implies that

$$\begin{aligned} d(\gamma, \hat{\gamma}) + d(\hat{\gamma}, m_2) &\leq d(\gamma, m_2) + 2d(\hat{\gamma}, m_2) \leq d(\gamma, m_2) + 4R \\ &\leq 3d(\gamma, m_2) = 3d(m_1, m_2). \end{aligned}$$

Combining with the previous inequalities we estimate I and $\|\widehat{T}\|$, as required. $\square$

An operator that will be further used in our construction is defined for $f \in \mathrm{Lip}(F)$ by

$$(\widehat{S}f)(m) := (\widehat{T}f)(m) - (\widetilde{E}f)(m), \quad m \in F \cup \widehat{F}. \tag{7.129}$$

Lemma 7.48. $||\widehat{S}f||_{\ell_\infty(F\cup\widehat{F})} \le C||f||_{\mathrm{Lip}(F)}$ *and, moreover,*

$$\widehat{S} : \mathrm{Lip}(F) \to \mathrm{Lip}(F \cup \widehat{F}) \quad \text{and} \quad ||\widehat{S}|| \le C.$$

Proof. The second statement follows directly from definition (7.129). If, now, $m \in B_\gamma \cap (F \cup \widehat{F})$, then by the same definition

$$(\widehat{S}f)(m) = [(\widehat{T}f)(m) - (\widehat{T}f)(\gamma)] + [(\widetilde{E}f)(\gamma) - (\widetilde{E}f)(m)]$$

which implies that

$$|(\widehat{S}f)(m)| \le C||f||_{\mathrm{Lip}(F)}d(m,\gamma) \le CR||f||_{\mathrm{Lip}(F)}. \qquad \square$$

Finally we introduce an operator $\widehat{K}$ given for $g \in (\mathrm{Lip} \cap \ell_\infty)(F \cup \widehat{F})$ by

$$\widehat{K}g := \sum_\gamma (E_\gamma g_\gamma)\rho_\gamma; \tag{7.130}$$

here γ runs through the set $\{\gamma \in \Gamma\ ;\ (F \cup \widehat{F}) \cap B_\gamma \ne \emptyset\}$ and $\{\rho_\gamma\}$ is the partition of unity of Lemma 7.45, and $g_\gamma := g|_{(F\cup\widehat{F})\cap B_\gamma}$. Moreover, E_γ is an operator from $\mathrm{Ext}((F \cup \widehat{F}) \cap B_\gamma, B_\gamma)$ with

$$||E_\gamma|| \le \lambda_R \tag{7.131}$$

whose existence is provided by the assumption of Theorem 7.43.

Lemma 7.49.

$$||\widehat{K}g||_{\mathrm{Lip}(\mathcal{M})} \le C||g||_{(\mathrm{Lip}\cap\ell_\infty)(F\cup\widehat{F})}.$$

Proof. As in the proof of Lemma 7.46 it is convenient to extend every $E_\gamma g_\gamma$ outside B_γ so that the extension F_γ satisfies

$$||F_\gamma||_{\mathrm{Lip}(\mathcal{M})} = ||E_\gamma g_\gamma||_{\mathrm{Lip}(B_\gamma)}. \tag{7.132}$$

Then we clearly have

$$\widehat{K}g = \sum_\gamma F_\gamma \rho_\gamma. \tag{7.133}$$

Now, according to (7.130) we get for $m \in F \cup \widehat{F}$,

$$(\widehat{K}g)(m) = \sum \rho_\gamma(m)g(m) = g(m).$$

Hence, it remains to estimate the right-hand side of the inequality

$$\begin{aligned} I &:= |(\widehat{K}g)(m_1) - (\widehat{K}g)(m_2)| \\ &\le \sum_\gamma |\rho_\gamma(m_1) - \rho_\gamma(m_2)| \cdot |F_\gamma(m_1)| + \sum_\gamma \rho_\gamma(m_2)|F_\gamma(m_1) - F_\gamma(m_2)|. \end{aligned}$$

By (7.132) the second sum is at most $(\sup_\gamma ||E_\gamma||)||g||_{\mathrm{Lip}(F\cup\widehat{F})}d(m_1, m_2)$ and together with (7.131) this leads to the appropriate bound. In turn, the first sum is at most

$$2\mu \cdot K \cdot \max_\gamma |(E_\gamma g_\gamma)(m_1)| \cdot d(m_1, m_2),$$

where μ is the order of the cover $2\mathcal{B}$ and K is defined by (7.106). To estimate the maximum, one notes that $(E_\gamma g_\gamma)(\gamma) = g(\gamma)$ and therefore

$$\begin{aligned} &|(E_\gamma g_\gamma)(m_1)| \le |(E_\gamma g_\gamma)(m_1) - (E_\gamma g_\gamma)(\gamma)| + |g(\gamma)| \\ &\le \lambda_R d(m_1, \gamma))||g||_{\mathrm{Lip}(F\cup\widehat{F})} + ||g||_{\ell_\infty(F\cup\widehat{F})} \le (2R\lambda_R + 1)||g||_{(\mathrm{Lip}\cap\ell_\infty)(F\cup\widehat{F})}. \end{aligned}$$

Combining with the estimate of the second sum we prove the lemma. □

Now all is ready to define the required operator $\widehat{E}$ from $\mathrm{Ext}(F, \mathcal{M})$. Using the above introduced operators $\widetilde{E}$, $\widehat{K}$ and $\widehat{S}$ we set

$$\widehat{E}f := \widetilde{E}f + \widehat{K}(\widehat{S}f|_{F\cup\widehat{F}}).$$

Due to the properties of the operators involved we have for $m \in F$,

$$(\widehat{E}f)(m) = (\widetilde{E}f)(m) + (\widehat{S}f)(m) = (\widetilde{E}f)(m) + (\widehat{T}f)(m) - (\widetilde{E}f)(m) = f(m),$$

i.e., $\widehat{E}$ is an extension from F.

To obtain the required estimate of $||\widehat{E}f||_{\mathrm{Lip}(\mathcal{M})}$ it suffices by (7.125) to estimate $||\widehat{K}(\widehat{S}f|_{F\cup\widehat{F}})||_{\mathrm{Lip}(\mathcal{M})}$. The latter by Lemmas 7.49, 7.48, 7.47 and definition (7.125) is bounded by

$$C||\widehat{S}f||_{(\mathrm{Lip}\cap\ell_\infty)(F\cup\widehat{F})} \le C||f||_{\mathrm{Lip}(F)}.$$

Hence $\widehat{E} \in \mathrm{Ext}(F, \mathcal{M})$ and its norm is bounded, as required.

The proof of Theorem 7.43 is complete. □

Let us recall that a metric space $\mathcal{M}$ is of bounded geometry with parameters $n \in \mathbb{N}$, R, $D > 0$ (written $\mathcal{M} \in \mathcal{G}_n(R, D)$), if each of its open balls of radius R admits a bi-Lipschitz embedding into $\mathbb{R}^n$ with distortion D, see Volume I, subsection 3.3.2.

Note that if $B_R(m)$ is bi-Lipschitz homeomorphic to a subset S of $\mathbb{R}^n$ with distortion D, then

$$\frac{1}{D} \cdot \lambda(S) \le \lambda(B_R(m)) \le D \cdot \lambda(S),$$

and by the classical Whitney-Glaeser extension Theorem 2.19 of Volume I, $\lambda(S) \le \lambda(\mathbb{R}^n) < \infty$. Therefore the previous theorem leads to

Corollary 7.50. *Let $\mathcal{M} \in \mathcal{G}_n(2R, D)$. Then $\lambda(\mathcal{M})$ is finite if and only if for some R-lattice Γ in $\mathcal{M}$ we have*

$$\lambda(\Gamma) < \infty.$$

In the next corollary, we deal with finitely generated groups with word metrics, see the part of subsection 3.3.7 of Volume I devoted to these groups. Specifically, we consider groups of isometries acting *properly*, *freely* and *cocompactly* on a length metric space, see Volume I, Definition 3.132 for details. By the Efremovich-Švarc-Milnor Theorem 3.133 of Volume I such a group is finitely generated. If G is a group of this kind and A is its generating set, then d_A denotes the word metric of G. It is known that for distinct generating sets A and A',

$$c^{-1}d_A \le d_{A'} \le cd_A$$

for some constant $c > 1$.

Corollary 7.51. *Let $\mathcal{M}$ be a length space framed by a group G acting on $\mathcal{M}$ by isometries. Assume that*

(a) *$\mathcal{M}$ is a metric space of bounded geometry;*

(b) *G acts on $\mathcal{M}$ properly, freely and cocompactly.*

Then $\lambda(\mathcal{M})$ is finite if and only if G equipped with a word metric is an $\mathcal{LE}$ space.

Proof. Let A be a generating set of G and d_A be the associated word metric. If (G, d_A) is an $\mathcal{LE}$ space, then $(G, d_{A'})$ also is for any generating set A'. So we will work with a fixed A.

We first prove that the condition

$$\lambda(G, d_A) < \infty \tag{7.134}$$

is necessary for finiteness of $\lambda(\mathcal{M})$. Indeed, suppose that $\lambda(\mathcal{M})$ is finite. Then for a G-orbit $G(m) := \{g(m)\ ;\ g \in G\}$ we have

$$\lambda(G(m)) \le \lambda(\mathcal{M}) < \infty. \tag{7.135}$$

Theorem 3.133 of Volume I asserts that under the hypothesis (b) of the corollary there exists a constant $C \ge 1$ independent of m so that

$$C^{-1}d_A(g, h) \le d(g(m), h(m)) \le Cd_A(g, h) \tag{7.136}$$

for all $g, h \in G$. This, in particular, means that the metric subspace $G(m)$ is bi-Lipschitz homeomorphic to the metric space (G, d_A). Hence (7.135) implies the required inequality (7.134).

To prove sufficiency of condition (7.134) for finiteness of $\lambda(M)$, we choose a point m_0 of the generating compact set K_0, see Volume I, Definition 3.132, and show that the G-orbit $G(m_0)$ is an R-lattice for some $R > 0$. Let $B_{R_0}(m_0)$ be a ball containing K_0. Then due to the definition of the generating compact we have for $\Gamma := G(m_0)$

$$\bigcup_{m\in\Gamma} B_{R_0}(m) = G(B_{R_0}(m_0)) \supset G(K_0) = \mathcal{M}.$$

Hence the family of balls $\{B_{R_0}(m)\ ;\ m \in \Gamma\}$ covers $\mathcal{M}$. Moreover, (7.136) implies that for $m := g(m_0)$, $m' := h(m_0)$ with $g \neq h$,

$$d(m, m') \geq C^{-1} d_A(g, h) \geq C^{-1},$$

i.e., the family $\{B_{cR_0}(m)\ ;\ m \in \Gamma\}$ with $c = c_\Gamma := \frac{1}{2CR_0}$ consists of pairwise disjoint balls. Hence Γ is an R-lattice, $R := R_0$, satisfying, by (7.134) and (7.136), the condition

$$\lambda(\Gamma) < \infty.$$

We now apply Theorem 7.43 with that R-lattice Γ to derive the finiteness of $\lambda(\mathcal{M})$. To this end we should establish the validity of the assumptions of the theorem with this R.

First we prove that $\mathcal{M}$ belongs to the class of locally doubling metric spaces $\mathcal{D}(2R, N)$ for some $N = N(R, \mathcal{M})$. In other words, we show that every ball $B_r(m)$ with $r \leq 2R$ can be covered by at most N balls of radius $r/2$. Indeed, by hypothesis (a) of the corollary, $\mathcal{M} \in \mathcal{G}_n(\widetilde{R}, \widetilde{D})$ for certain $\widetilde{R}, \widetilde{D}$ and n. This implies that $\mathcal{M} \in \mathcal{D}(\widetilde{R}/2, N)$ for some $N = N(\widetilde{D}, n)$ and shows that the required statement is true for $2R \leq \widetilde{R}/2$. Suppose now that

$$\widetilde{R}/2 \leq r \leq 2R. \tag{7.137}$$

Note that it suffices to consider balls with $m \in K_0$. In fact, $G(K_0) = \mathcal{M}$ and therefore $g_0(m) \in K_0$ for some isometry $g_0 \in G$. Hence $g_0(B_r(m)) = B_r(g_0(m))$ and we can work with $B_r(m)$ for $m \in K_0$. Let us fix a point $m_0 \in K_0$ and set $R' := 2R + \operatorname{diam} K_0$. Then

$$B_r(m) \subset B_{R'}(m_0), \quad m \in K_0, \tag{7.138}$$

and it remains to show that $B_{R'}(m_0)$ can be covered by a finite number, say N, of (open) balls of radius $r/2$ with N independent of r. We use the following

Lemma 7.52. *Suppose that G acts properly, freely and cocompactly on a length space $\mathcal{M}$ by isometries. Then every bounded closed set $S \subset \mathcal{M}$ is compact.*

Proof. For every $m \in S$ there is a finite number of isometries $g_{im} \in G$, $i = 1, \dots, k_m$, such that $g_{im}(m) \in K_0$. Here K_0 is a generating compact set with respect to the action of G. Let $H := \{g_{im}^{-1} \in G\ ;\ 1 \leq i \leq k_m,\ m \in S\}$. Then $S \subset H(K_0)$, and, by definition, $\operatorname{diam} H(K_0) < \infty$. For a fixed $m_0 \in K_0$ let us consider the orbit $H(m_0)$. We claim that $H(m_0)$ consists of a finite number of points. Otherwise there is a sequence of points $m_i = h_i(m_0) \in H(m_0)$ such that

$d_A(h_i, 1) \to \infty$ as $i \to \infty$. This and inequality (7.136) imply $d(m_i, m_0) \to \infty$ in $\mathcal{M}$ as $i \to \infty$ and this contradicts the condition diam $H(K_0) < \infty$. From the finiteness of $H(m_0)$ we also obtain that H is finite. Thus S is covered by a finite number of compact sets, and, since S is closed, it is compact. □

According to this lemma the closed ball $\overline{B}_{R'}(m_0)$ is compact. Thus $B_{R'}(m_0)$ can be covered by a finite number N of open balls of radius $\widetilde{R}/4$. This, (7.137) and (7.138) show that $B_r(m)$ can be covered by N open balls of radius $r/2$ as is required.

To establish the second condition of Theorem 7.43, i.e., the finiteness of

$$\lambda_R := \sup\{\lambda(B_{2R}(m))\ ;\ m \in \mathcal{M}\},$$

we first use the previous argument and (7.138) which immediately yield

$$\lambda_R \leq \lambda(B_{R'}(m_0)).$$

We now show that the right-hand side is bounded. Since $\mathcal{M} \in \mathcal{G}_n(\widetilde{R}, \widetilde{D})$, for every $m \in \mathcal{M}$,

$$\lambda(B_{\widetilde{R}}(m)) < \infty.$$

This, compactness of $\overline{B}_{R'}(m_0)$ and the argument used in the proof of Lemma 7.9 lead to the required inequality

$$\lambda(B_{R'}(m_0)) < \infty.$$

The proof of Corollary 7.51 is complete. □

Corollary 7.50 tells us that the desired extension property for a metric space may be reduced to that for some of its lattices regarded as a metric space. We assume that for the class of the, so-called, *uniform lattices* the answer is positive, i.e., each such a lattice belongs to $\mathcal{SLE}$.

Uniformity of a lattice Γ means that for some increasing function $\phi_\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ and constant $0 < c \leq 1$, the number of points of $\Gamma \cap B_R(m)$ for every $R > 0$ and $m \in \Gamma$ satisfies

$$c\phi_\Gamma(R) \leq |\Gamma \cap B_R(m)| \leq \phi_\Gamma(R). \tag{7.139}$$

According to Theorem 3.134 of Volume I uniform lattices bi-Lipschitz homeomorphic to finitely generated groups of polynomial growth (in this case $\phi_\Gamma(R) = aR^n$ for some $a, n \geq 0$) equipped with word metrics are of homogeneous type. Hence, they are $\mathcal{SLE}$ spaces, see Corollary 7.37. Also, it follows from the results of Lang and Schlichenmaier [LSchl-2005] and of the authors of the present book [BB-2007b] that the result holds for uniform lattices bi-Lipschitz homeomorphic to Gromov hyperbolic groups, see Volume I, Definition 3.137 (in this case the function ϕ is generally of exponential growth). However, one can construct a uniform lattice bi-Lipschitz homeomorphic to a countable group with infinite number of generators equipped with the word metric which does not belong to $\mathcal{SLE}$, see subsection 8.3.3.

These facts lead to the following

Conjecture 7.53. *A uniform lattice bi-Lipschitz homeomorphic to a finitely generated group equipped with the word metric belongs to $\mathcal{SLE}$.*

Returning to Corollary 7.51 we will single out the case of a group acting on a metric space for which we can give rather sharp estimates of the characteristics λ. To formulate the results we introduce the functional

$$\lambda_{\text{conv}}(\mathcal{M}) := \sup_{C} \lambda(C, \mathcal{M})$$

where C runs through all convex subsets of a normed linear space $\mathcal{M}$. As before we set

$$\lambda(S, \mathcal{M}) := \inf\{||E|| \ ; \ E \in \text{Ext}(S, \mathcal{M})\}.$$

Theorem 7.54. *There exists a numerical constant $c_0 > 1/4$ such that for all n and $1 \le p \le \infty$,*

$$c_0 \le n^{-\left|\frac{1}{p}-\frac{1}{2}\right|} \cdot \lambda_{\text{conv}}(\ell_p^n) \le 1.$$

Proof. We first prove that

$$\lambda_{\text{conv}}(\ell_2^n) = 1. \tag{7.140}$$

Since the lower bound 1 is clear, we have to prove that

$$\lambda_{\text{conv}}(\ell_2^n) \le 1. \tag{7.141}$$

Let $C \subset (\ell_2^n, 0)$ be a closed convex set containing 0, and $p_C(x)$ be the (unique) closest to x point from C. Then, see, e.g., [BL-2000, Sect. 3.2], the metric projection p_C is Lipschitz and

$$||p_C(x) - p_C(y)||_2 \le ||x - y||_2. \tag{7.142}$$

Using this we introduce a linear operator E given on $\text{Lip}_0(C)$ by

$$(Ef)(x) := (f \circ p_C)(x), \quad x \in \ell_2^n. \tag{7.143}$$

Since p_C is the identity on C and $p_C(0) = 0$ as $0 \in C$, this operator belongs to $\text{Ext}(C, \ell_2^n)$. Moreover, by (7.142),

$$||(Ef)(x) - (Ef)(y)||_2 \le ||f||_{\text{Lip}_0(C)} ||x - y||_2,$$

i.e., $||E|| \le 1$, and (7.141) is established.

Using now the inequalities

$$||x||_p \le ||x||_2 \le n^{\frac{1}{2}-\frac{1}{p}} \ ||x||_p \quad \text{for} \quad 2 \le p \le \infty, \tag{7.144}$$

and

$$n^{\frac{1}{2}-\frac{1}{p}} \ ||x||_p \le ||x||_2 \le ||x||_p \quad \text{for} \quad 1 \le p \le 2 \tag{7.145}$$

we derive from (7.141) the required upper bound:

$$\lambda_{\text{conv}}(\ell_p^n) \le n^{\left|\frac{1}{2}-\frac{1}{p}\right|}. \tag{7.146}$$

In order to prove the lower estimate we need the following result where Banach spaces are regarded as pointed metric spaces with $m^* = 0$.

Proposition 7.55. *Let Y be a linear subspace of a finite-dimensional Banach space X, and the operator E belong to* $\mathrm{Ext}(Y,X)$. *Then there is a linear projection P from X onto Y such that*

$$||P|| \le ||E||. \tag{7.147}$$

Proof. We use an argument similar to that of the proof of Theorem 1.49 of Volume I. First, we introduce an operator $S : \mathrm{Lip}_0(Y) \to \mathrm{Lip}_0(X)$ given at $z \in X$ by

$$(Sf)(z) := \int_X \left\{ \int_Y [(Ef)(x+y+z) - (Ef)(x+y)]dy \right\} dx. \tag{7.148}$$

Here $\int_A \dots da$ is a translation invariant mean on the space $\ell_\infty(A)$ of all bounded functions on the abelian group A. Since the function within [ ] is bounded for every fixed z (recall that $Ef \in \mathrm{Lip}(X)$), this operator is well defined. Moreover, as $\int_A da = 1$ we get

$$||Sf||_{\mathrm{Lip}_0(X)} \le ||E|| \cdot ||f||_{\mathrm{Lip}_0(Y)}. \tag{7.149}$$

By the translation invariance of dx we then derive from (7.148) that

$$(Sf)(z_1 + z_2) = (Sf)(z_1) + (Sf)(z_2), \quad z_1, z_2 \in X.$$

Together with (7.149) and the equality

$$||f||_{\mathrm{Lip}_0(X)} = ||f||_{X^*}, \quad f \in X^*, \tag{7.150}$$

this shows that Sf belongs to X^* and therefore S maps $\mathrm{Lip}_0(Y)$ linearly and continuously into X^*. Further, Y^* is a linear subset of $\mathrm{Lip}_0(Y)$ whose norm coincides with that induced from $\mathrm{Lip}_0(Y)$. Therefore the restriction

$$T := S|_{Y^*}$$

is a bounded linear operator from Y^* to X^*. As in the proof of Theorem 1.49 of Volume I we obtain that

$$(Tf)(z) = f(z), \quad z \in Y, \tag{7.151}$$

i.e., T is an extension from Y^*.

Consider now the conjugate to T operator T^* acting from $X^{**} = X$ to $Y^{**} = Y$. Since T is a linear extension operator from Y^*, its conjugate is a projection onto Y. Finally, (7.149) and (7.150) give for the norm of this projection the required estimate

$$||T^*|| = ||T|| \le ||E||.$$

The proposition is proved. □

Proposition 7.56. *Let X be either ℓ_1^n or ℓ_∞^n. Then there is a subspace $Y \subset X$ such that* $\dim Y = \lfloor n/2 \rfloor$ *and its projection constant* $\pi(Y,X) := \inf ||P||$ *where P runs through all linear projections from X onto Y and satisfies*

$$\pi(X,Y) \ge c_0\sqrt{n} \tag{7.152}$$

with c_0 independent of n.

Proof. The inequality follows from Theorem 1.2 of the paper [So-1941] by Sobczyk with the optimal c_0 greater than 1/4. □

We now complete the proof of Theorem 7.54. Applying Propositions 7.55 and 7.56 we get for an arbitrary $E \in \mathrm{Ext}(Y, \ell_1^n)$ the inequality

$$||E|| \geq c_0\sqrt{n} \tag{7.153}$$

with $c_0 > 0$ independent of n. A similar estimate is valid for $E \in \mathrm{Ext}(Y^\perp, \ell_\infty^n)$ as well. Hence for $p = 1, \infty$,

$$\lambda_{\mathrm{conv}}(\ell_p^n) \geq c_0\sqrt{n}. \tag{7.154}$$

Using this estimate for $p = 1$ and applying an inequality similar to (7.145) comparing $||x||_1$ and $||x||_p$, we get for $1 \leq p \leq 2$ the estimate

$$\lambda_{\mathrm{conv}}(\ell_p^n) \geq c_0 n^{\frac{1}{p}-\frac{1}{2}}.$$

Then using (7.154) for $p = \infty$ and an inequality similar to (7.144) comparing $||x||_p$ and $||x||_\infty$, we get for $2 \leq p \leq \infty$ the inequality

$$\lambda_{\mathrm{conv}}(\ell_p^n) \geq c_0 n^{\frac{1}{2}-\frac{1}{p}}.$$

The proof of the theorem is complete. □

As a corollary of Theorem 7.54 we get

Corollary 7.57. *For a Banach space X,*

$$c_0\sqrt{n} \leq \lambda(\ell_p^n, X) \leq cn, \quad p = 1, \infty,$$

for some numerical constants $c \leq 311$ and $c_0 > \frac{1}{4}$.

Proof. The right-hand inequality follows directly from Corollary 7.36. And, the left-hand inequality follows from Theorem 7.54 and the inequalities (see the proof of Proposition 7.3)

$$\lambda(\ell_p^n, X) \geq \lambda(\ell_p^n) \geq \lambda_{\mathrm{conv}}(\ell_p^n) \geq c_0\sqrt{n}.$$ □

Now we derive from here a result for the abelian group $\mathbb{Z}^n$ with the word metric d_A where A is the standard orthonormal basis of $\mathbb{R}^n$. The reader may easily check that d_A coincides with the restriction to $\mathbb{Z}^n$ of ℓ_1^n-metric. This fact and the result of Corollary 7.19 with $(\mathcal{M}, d) = (\mathbb{Z}^n, d_A)$ and the dilation $\phi : x \mapsto \frac{1}{2}x$ yield

$$\lambda(\mathbb{Z}^n, d_A) = \lambda(\ell_1^n).$$

This and Corollary 7.57 immediately imply the following result.

Corollary 7.58.

$$c_0\sqrt{n} \leq \lambda((\mathbb{Z}^n, d_A), X) \leq cn.$$

In a similar way we may consider other finitely generated groups introduced in subsection 3.3.7 of Volume I. We present several illustrating examples of groups having the simultaneous extension property referring to the paper [BB-2007b] for more results and missing proofs.

Example 7.59. (a) **Carnot groups.** Let G be an m-step Carnot group whose Lie algebra is stratified into the direct sum of nontrivial vector spaces V_i, $1 \leq i \leq m$. The number $Q := \sum_{i=1}^{m} i \dim V_i$ is recalled to be the *fractal* (or homogeneous) *dimension* of G see Volume I, formula (3.152).

Let d_C be the Carnot-Carathéodory metric on G. Then the Haar measure of G is Q-homogeneous, see Volume I, formula (3.151). Applying Corollary 7.36 we then have

$$\lambda((G, d_C), X) \leq cQ$$

for a dual space X with the constant $c < 24$, see [BB-2007b, Cor. 2.27].

For instance, $\mathbb{R}^n$ equipped with any Banach norm $\|\cdot\|$ is a Carnot group with $Q = n$ and d_C generated by this norm. The above inequality, in particular, gives the Johnson-Lindenstrauss-Schechtman theorem [JLS-1986] for $\lambda((\mathbb{R}^n, \|\cdot\|), X)$ with a better constant but for a dual X.

Another special case is the Heisenberg group H_n, see Volume I, Example 3.141 (b), in which case $Q = 2n + 2$ and therefore

$$\lambda(H_n, d_C) \leq 48(n + 1).$$

Finally, let us consider the discrete subgroup $H_n(\mathbb{Z})$ of H_n consisting of elements from the set $\mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}$, see Volume I, Example 3.135. It is readily seen that the map $\phi : (x, y, t) \mapsto \frac{1}{2}(x, y, t)$ is a dilation in the sense of Corollary 7.19. Since $\phi(H_n(\mathbb{Z})) \supset H_n(\mathbb{Z})$ and $\cup_{j=0}^{\infty} \phi^j(H_n(\mathbb{Z}))$ is dense in (H_n, d_C) the aforementioned corollary yields

$$\lambda(H_n(\mathbb{Z})) = \lambda(H_n) \leq 48(n + 1).$$

(b) **Groups of polynomial growth.** Let now G be a finitely generated group with a word metric associated with a generating set A. Assume that G is of polynomial growth, i.e., for all $R > 0$ the cardinality of a ball of radius R is bounded by CR^{ρ} for some $C > 0$ and $\rho \geq 0$. Then the counting measure of G is equivalent to a Q-homogeneous measure where the constants of equivalence depend on G and A, and Q is the degree of the maximal torsion free nilpotent subgroup of G, see Volume I, Theorem 3.134. Therefore due to Corollary 7.36 for some $c = c(G, A)$,

$$\lambda(\mathcal{M}, X) \leq c.$$

If, in addition, G is torsion free, the constants of equivalence in the previous statement depend only on the degree Q and the constant c does as well.

(c) **Hyperbolic groups.** Let G be a (finitely generated) hyperbolic group and d_A be some of its word metrics. Then (G, d_A) is a metric subspace in a δ-hyperbolic space with $\delta \geq 0$ (its Cayley graph), see Definition 3.137 and Theorem 3.138 of Volume I. If G is not virtually cyclic (i.e., its quotient by a finite subgroup is cyclic), the ambient δ-hyperbolic space is of bounded geometry. Due to Corollary 6.38 $\lambda((G, d_A), X) < \infty$ for such G.

7.4 Spaces with the universal linear Lipschitz extension property

The basic characteristic studied in this section is described by

Definition 7.60. A metric space $\mathcal{M}$ is said to be universal with respect to simultaneous Lipschitz extensions if, for an arbitrary metric space $\widetilde{\mathcal{M}}$ and every subspace S of $\widetilde{\mathcal{M}}$ isometric to a subspace of $\mathcal{M}$,

$$\lambda(S, \widetilde{\mathcal{M}}, X) \leq c$$

where c depends only on $\mathcal{M}$.

The optimal constant in this inequality will be denoted by $\lambda_u(\mathcal{M})$.

Remark 7.61. In fact, in all our results related to universality we will establish a much stronger property: if S is C-isometric ($C \geq 1$) to a subspace of $\mathcal{M}$, then

$$\lambda(S, \widetilde{\mathcal{M}}, X) \leq C^2 c$$

with c depending only on $\mathcal{M}$. This clearly implies the universality of $\mathcal{M}$.

We will show that the direct sum of a finite combination of Gromov hyperbolic spaces of bounded geometry and homogeneous metric spaces is universal.

It was shown in Section 6.4, see Corollary 6.41 and Remark 6.42, that all these results with unspecified estimates of the extension constants follow from the Lang-Schlichenmaier theory [LSchl-2005]. Universality of doubling metric spaces with an almost optimal extension constant was firstly proved by Lee and Naor, see [LN-2005, Thm. 1.6], in a nonconstructive way. Universality of Gromov hyperbolic spaces of bounded geometry was proved by the authors of this book [BB-2007c].

In this section, we prove all of the aforementioned universality results by the method of this paper. This method is constructive and allows us to obtain relatively good estimates of the extension constants. We first apply this approach to the proof of the result equivalent to the Lee-Naor theorem, cf. Corollary 7.37. Since the basic ideas had already presented in the proof of Theorem 7.22, we refer to that proof to shorten our derivation.

Theorem 7.62. *Let $\mathcal{M}_0$ be a metric subspace of an arbitrary metric space $\mathcal{M}$. Assume that $\mathcal{M}_0$ is of homogeneous type with respect to a doubling measure μ_0.*

Then for any Banach space X there exists a simultaneous extension operator E : $\mathrm{Lip}(\mathcal{M}_0, X) \to \mathrm{Lip}(\mathcal{M}, X)$ satisfying

$$||E|| \leq c(\log_2 D(\mu_0) + 1)$$

with some numerical constant $c \geq 1$.

Proof. We begin with the following remark reducing the required result to a special case. Let $\mathcal{M}$ and $\mathcal{M}_0$ be isometric to subspaces of a new metric space $\widehat{\mathcal{M}}$ and its subspace $\widehat{\mathcal{M}}_0$, respectively. Assume that there exists a simultaneous extension operator $\widehat{E} : \mathrm{Lip}(\widehat{\mathcal{M}}_0, X) \to \mathrm{Lip}(\widehat{\mathcal{M}}, X)$. Then, after the corresponding identification, the operator $\widehat{E}$ gives rise to a simultaneous extension operator $E : \mathrm{Lip}(\mathcal{M}_0, X) \to \mathrm{Lip}(\mathcal{M}, X)$ satisfying

$$||E|| \leq ||\widehat{E}||.$$

If, in addition, $||\widehat{E}||$ is bounded by $c(\log_2 D(\mu_0) + 1)$ with a numerical constant $c \geq 1$, then the desired result immediately follows.

We choose as the above pair $\widehat{\mathcal{M}}_0 \subset \widehat{\mathcal{M}}$ metric spaces denoted by $\mathcal{M}_{0N}$ and $\mathcal{M}_N$ defined as follows.

The underlying sets of these spaces are

$$\mathcal{M}_N := \mathcal{M} \times \mathbb{R}^N, \quad \mathcal{M}_{0N} := \mathcal{M}_0 \times \mathbb{R}^N; \tag{7.155}$$

a metric d_N on $\mathcal{M}_N$ is given by

$$d_N((m, x), (m', x')) := d(m, m') + \|x - x'\|_1 \tag{7.156}$$

where $m, m' \in \mathcal{M}$ and $x, x' \in \mathbb{R}^N$, and $\|x\|_1 := \sum_{i=1}^N |x_i|$ is the ℓ_1^N-metric of $x \in \mathbb{R}^N$. Further, d_{0N} denotes the metric on $\mathcal{M}_{0N}$ induced by d_N.

Finally, we define a Borel measure μ_{0N} on $\mathcal{M}_{0N}$ by

$$\mu_{0N} := \mu_0 \otimes \lambda_N. \tag{7.157}$$

We extend this measure to the σ-algebra consisting of subsets $S \subset \mathcal{M}_N$ such that $S \cap \mathcal{M}_{0N}$ is a Borel subset of $\mathcal{M}_{0N}$ and for these S,

$$\overline{\mu}_N(S) := \mu_{0N}(S \cap \mathcal{M}_{0N}).$$

It is important for the subsequent part of the proof that each open ball $B_R((m, x)) \subset \mathcal{M}_N$ belongs to this σ-algebra. In fact, its intersection with $\mathcal{M}_{0N}$ is a Borel subset of this space, since the function $(m', x') \mapsto d_N((m, x), (m', x'))$ is continuous on $\mathcal{M}_{N0}$. Hence,

$$\overline{\mu}_N(B_R((m, x))) = \mu_{0N}(B_R((m, x)) \cap \mathcal{M}_{0N}). \tag{7.158}$$

Hereafter we denote by $\widehat{m}$ the pair (m, x) with $m \in \mathcal{M}$ and $x \in \mathbb{R}^N$, and by $B_R^o(\widehat{m})$ the open ball in $\mathcal{M}_{0N}$ centered at $\widehat{m} \in \mathcal{M}_{0N}$ and of radius R. The open ball $B_R(\widehat{m})$ of $\mathcal{M}_N$ relates to that by

$$B_R^o(\widehat{m}) = B_R(\widehat{m}) \cap \mathcal{M}_{0N}$$

provided that $\widehat{m} \in \mathcal{M}_{0N}$.

Since the measure μ_{0N} is clearly doubling, its dilation function given for $l \geq 1$ by

$$D_{0N}(l) := \sup\left\{\frac{\mu_{0N}(B_{lR}^o(\widehat{m}))}{\mu_{0N}(B_R^o(\widehat{m}))}\ ;\ \widehat{m} \in \mathcal{M}_{0N} \text{ and } R > 0\right\}$$

is finite.

We also define a (modified) dilation function D_N for the extended measure $\overline{\mu}_N$. This is given for $l \geq 1$ by

$$D_N(l) := \sup\left\{\frac{\overline{\mu}_N(B_{lR}(\widehat{m}))}{\overline{\mu}_N(B_R(\widehat{m}))}\right\} \tag{7.159}$$

where the supremum is taken over all R satisfying

$$R > 4d(\widehat{m}, \mathcal{M}_{0N}) := 4\inf\{d_N(\widehat{m}, \widehat{m}')\ ;\ \widehat{m}' \in \mathcal{M}_{0N}\} \tag{7.160}$$

and then over all $\widehat{m} \in \mathcal{M}_N$.

Due to (7.158) and (7.160) the denominator in (7.159) is not zero and $D_N(l)$ is well defined.

Comparison with the dilation function for M_{0N} shows that $D_{0N}(l) \leq D_N(l)$. We will see that the converse is also true for l close to 1.

Lemma 7.63. *Assume that N and the doubling constant $D := D(\mu_0)$ are related by*

$$N \geq \lfloor 3\log_2 D\rfloor + 5. \tag{7.161}$$

Then the following is true:

$$D_N(1 + 1/N) \leq \frac{6}{5}e^4.$$

Proof. The proof repeats the proof of Lemma 3.94 of Volume I. □

Next, we estimate $\overline{\mu}_N$-measure of the spherical layer $B_{R_2}(\widehat{m}) - B_{R_1}(\widehat{m})$ with $R_2 \geq R_1$ by a kind of a surface measure. For its formulation we set

$$A_N := \frac{12}{5}e^4 N. \tag{7.162}$$

Lemma 7.64. *Assume that*

$$N \geq \lfloor 3 \log_2 D \rfloor + 6.$$

Then for all $\widehat{m} \in \mathcal{M}_N$ *and* $R_1, R_2 > 0$ *satisfying*

$$R_2 \geq \max\{R_1, 8d_N(\widehat{m}, \mathcal{M}_{0N})\}$$

the following is true:

$$\overline{\mu}_N(B_{R_2}(\widehat{m}) \setminus B_{R_1}(\widehat{m})) \leq A_N \frac{\overline{\mu}_N(B_{R_2}(\widehat{m}))}{R_2}(R_2 - R_1).$$

Proof. By definition $\mathcal{M}_N = \mathcal{M}_{N-1} \times \mathbb{R}$ and $\overline{\mu}_N = \overline{\mu}_{N-1} \otimes \lambda_1$. Then by the Fubini theorem we have for $R_1 \leq R_2$ and $\widehat{m} = (\widetilde{m}, t)$,

$$\begin{aligned}\overline{\mu}_N(B_{R_2}(\widehat{m})) - \overline{\mu}_N(B_{R_1}(\widehat{m})) &= 2\int_{R_1}^{R_2} \overline{\mu}_{N-1}(B_s(\widetilde{m}))ds \\ &\leq \frac{2R_2\overline{\mu}_{N-1}(B_{R_2}(\widetilde{m}))}{R_2}(R_2 - R_1).\end{aligned}$$

We claim that for arbitrary $l > 1$ and $R \geq 8d_N(\widehat{m}, \mathcal{M}_{0N}) := 8d_{N-1}(\widetilde{m}, \mathcal{M}_{0\,N-1})$

$$R\overline{\mu}_{N-1}(B_R(\widetilde{m})) \leq \frac{lD_{N-1}(l)}{l-1}\overline{\mu}_N(B_R(\widehat{m})). \tag{7.163}$$

Together with the previous inequality this would yield

$$\overline{\mu}_N(B_{R_2}(\widehat{m})) - \overline{\mu}_N(B_{R_1}(\widehat{m})) \leq \frac{2lD_{N-1}(l)}{l-1} \cdot \frac{\overline{\mu}_N(B_{R_2}(\widehat{m}))}{R_2}(R_2 - R_1).$$

Finally choose here $l = 1 + \frac{1}{N-1}$ and use Lemma 7.64 to obtain the required inequality.

Hence, it remains to establish (7.81). By the definition of $D_{N-1}(l)$ and the previous lemma we have for $l > 1$,

$$\begin{aligned}\overline{\mu}_N(B_{lR}(\widehat{m})) &= 2l\int_0^R \overline{\mu}_{N-1}(B_{ls}(\widetilde{m}))ds \leq 4l\int_{R/2}^R \overline{\mu}_{N-1}(B_{ls}(\widetilde{m}))ds \\ &\leq 4lD_{N-1}(l)\int_{R/2}^R \overline{\mu}_{N-1}(B_s(\widetilde{m}))ds \leq 2lD_{N-1}(l)\overline{\mu}_N(B_R(\widehat{m})).\end{aligned}$$

On the other hand, replacing $[0, R]$ by $[l^{-1}R, R]$ we also have

$$\overline{\mu}_N(B_{lR}(\widehat{m})) \geq 2l\overline{\mu}_{N-1}(B_R(\widetilde{m}))(R - l^{-1}R) = 2(l-1)R\overline{\mu}_{N-1}(B_R(\widetilde{m})).$$

Combining the last two inequalities we get (7.81). $\square$

Extension operator

We define the required simultaneous extension operator $E : \mathrm{Lip}(\mathcal{M}_{0N}, X) \to \mathrm{Lip}(\mathcal{M}_N, X)$ using the standard average operator Ave defined on continuous and locally bounded functions $g : \mathcal{M}_{0N} \to X$ by

$$Ave(g; \widehat{m}, R) := \frac{1}{\overline{\mu}_N(B_R(\widehat{m}))} \int_{B_R(\widehat{m})} g \, d\overline{\mu}_N.$$

Then we define the E on functions $f \in \mathrm{Lip}(\mathcal{M}_{0N}, X)$ by

$$(Ef)(\widehat{m}) := \begin{cases} f(\widehat{m}), & \text{if } \widehat{m} \in \mathcal{M}_{0N}, \\ Ave(f; m, R(\widehat{m})), & \text{if } \widehat{m} \notin \mathcal{M}_{0N}, \end{cases} \tag{7.164}$$

where we set $R(\widehat{m}) := 8d_N(\widehat{m}, \mathcal{M}_{0N})$. Since $\overline{\mu}_N(B_{R(\widehat{m})}(\widehat{m})) > 0$, this definition is correct.

To give the required estimate of $||E||$ we set

$$K_N(l) := A_N D_N(l)(4l + 1) \tag{7.165}$$

where the first of two factors are defined by (7.162) and (7.159).

Proposition 7.65. *The following inequality*

$$||E|| \le 20A_N + \max\left(\frac{4l+1}{2(l-1)}, K_N(l)\right)$$

is true provided that $l := 1 + 1/N$.

Proof. The proof repeats the steps of the proof of Proposition 7.29 where we use A_N instead of A_n and do not use any results related to $\widehat{C}_n$, see also [BB-2006] for details. □

Choosing here

$$N := \lfloor 3 \log_2 D \rfloor + 6$$

and using Lemma 7.63 and (7.162) to estimate $D_N(1 + 1/N)$ and A_N we get

$$||E|| \le C(\log_2 D + 2)$$

with some numerical constant C. This clearly gives the required result. □

Our next result, proved in [BB-2007b, Thm. 1.5], establishes the universality of the direct sum of Gromov hyperbolic spaces of bounded geometry, see subsection 3.3.4 of Volume I for the corresponding definitions.

Theorem 7.66. *Let* $\mathcal{M} := \oplus^p \{(\mathcal{M}_i, d_i)\}_{1 \le i \le N}$ *where every* $(\mathcal{M}_i, d_i)$ *is a (Gromov) hyperbolic metric space of bounded geometry. Then* $\mathcal{M}$ *is universal.*

Here $\oplus^{(p)}\{(\mathcal{M}_i, d_i)\}_{1\le i\le N}$ is the metric space with underlying set $\prod_{i=1}^N \mathcal{M}_i$ and metric $d_p := \left\{\sum_{i=1}^N d_i^p\right\}^{1/p}$.

Proof. Clearly it suffices to proof the result for $p=\infty$ only.

We need several auxiliary results the first of which is Theorem 5.14 of Volume I and the second result used is a variant of Corollary 7.39 which requires trivial changes in its proof.

Lemma 7.67. *Let $(\mathcal{M}, d)$ be the direct sum $\oplus^{(\infty)}\{(\mathcal{M}_i, d_i)\}_{0\le i\le N}$ where $(\mathcal{M}_0, d_0)$ is an n_0-dimensional Banach space and $\mathcal{M}_i = \mathbb{H}^{n_i}$, $1\le i\le N$. Then for some $c>1$ we have*

$$\lambda(S,\mathcal{M},X) \le c\left(n_0 + \sum_{i=0}^N [n_i^2 + n_i]\right)\cdot \max_{1\le i\le N}\sqrt{n_i}.$$

We are now ready to prove Theorem 7.66. So, let S be a subspace of an arbitrary metric space $(\widehat{\mathcal{M}}, \widehat{d})$ and let $\phi : S \to \oplus^{(\infty)}\{(\mathcal{M}_i, d_i)\}_{1\le i\le N}$ be a C-isometric embedding, i.e., for all $m, m' \in S$,

$$C^{-1}\widehat{d}(m,m') \le d_\infty(\phi(m),\phi(m')) \le C\widehat{d}(m,m').$$

We must find a linear extension operator $E : \mathrm{Lip}(S,X) \to \mathrm{Lip}(\widehat{\mathcal{M}}, X)$ whose norm is bounded by a constant depending only on the geometric characteristics of the spaces $\mathcal{M}_i$ and the embedding constant C.

For this aim we first use Theorem 5.14 of Volume I to find a C_1-isometric embedding ψ of $\oplus^{(\infty)}\{(\mathcal{M}_i, d_i)\}_{1\le i\le N}$ into the space $(\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{1\le i\le N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}$. Note that C_1 depends only on the characteristics of the spaces $\mathcal{M}_i$. Then $\psi\circ\phi$ is a CC_1-isometric embedding of S into $(\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{1\le i\le N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}$.

Set

$$\widehat{S} := \mathrm{Image}(\psi\circ\phi) \subset (\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{1\le i\le N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}$$

and define the linear operator E_1 on $\mathrm{Lip}(S,X)$ by the formula

$$E_1 f := f\circ\phi^{-1}\circ\psi^{-1}. \tag{7.166}$$

Then $E_1 : \mathrm{Lip}(S,X) \to \mathrm{Lip}(\widehat{S}, X)$ and

$$||E_1|| \le CC_1. \tag{7.167}$$

Further, we use Lemma 7.67 to find a linear bounded operator

$$E_2 : \mathrm{Lip}(\widehat{S}, X) \to \mathrm{Lip}((\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{1\le i\le N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}, X)$$

such that

$$E_2 g|_{\widehat{S}} = g \quad \textit{for} \quad g\in \mathrm{Lip}(\widehat{S}, X) \quad \textit{and} \quad ||E_2|| \le c(\overline{n}) \tag{7.168}$$

where

$$c(\overline{n}) := c(n_0, n_1, \ldots, n_N) \leq c\left(n_0 + \sum_{i=0}^{N}[n_i^2 + n_i]\right) \cdot \max_{1 \leq i \leq N} \sqrt{n_i}.$$

Finally, the coordinatewise application of the Lang-Pavlović-Schroeder Theorem 6.38 allows us to extend the map $\psi \circ \phi : S \to (\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{1 \leq i \leq N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}$ to a Lipschitz map $\Phi : \widehat{\mathcal{M}} \to (\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{1 \leq i \leq N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}$ such that

$$\Phi|_S = \psi \circ \phi \quad \text{and} \quad L(\Phi) \leq c_1(\overline{n})CC_1 \tag{7.169}$$

where

$$c_1(\overline{n}) \leq 2\sqrt{2}\left(\max_{1 \leq i \leq N} \sqrt{n_i}\right).$$

Next, define a linear operator E_3 on $\mathrm{Lip}((\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{1 \leq i \leq N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}, X)$ given by

$$E_3 h := h \circ \Phi.$$

Then $\mathrm{Lip}(\widehat{\mathcal{M}}, X)$ is the target space of E_3 and

$$||E_3|| \leq L(\Phi) \leq c_1(\overline{n})CC_1. \tag{7.170}$$

Moreover, by (7.169),

$$(E_3 h)|_S = h(\Phi|_S) = h \circ \psi \circ \phi. \tag{7.171}$$

Finally, define the desired linear extension operator E by

$$E = E_3 E_2 E_1.$$

According to (7.166), (7.168) and (7.171), E acts from $\mathrm{Lip}(S, X)$ into $\mathrm{Lip}(\widehat{\mathcal{M}}, X)$ and

$$Ef|_S = f.$$

In addition, (7.167), (7.168) and (7.170) imply that

$$||E|| \leq C^2 C_1^2 c(\overline{n}) c_1(\overline{n}).$$

Hence, the simultaneous extension constant $\lambda(S, \widehat{\mathcal{M}}, X)$ is bounded by a constant which depends only on the characteristics of the spaces $\mathcal{M}_i$ and the embedding constant C of the map ϕ. □

Theorem 7.68. *Let $\mathcal{M} := \oplus^{(p)}\{(\mathcal{M}_i, d_i)\}_{1 \leq i \leq N}$ where every $(\mathcal{M}_i, d_i)$ is either of homogeneous type or a Gromov hyperbolic space of bounded geometry. Then $\mathcal{M}$ is universal.*

Proof. Without loss of generality we assume that $p = \infty$ and $(\mathcal{M}_i, d_i)$ is homogeneous for $i = 1$ and Gromov hyperbolic of bounded geometry for $i \geq 2$.

Let S be a subspace of an arbitrary metric space $(\widetilde{\mathcal{M}}, \widetilde{d})$ and $\phi : S \to \mathcal{M}$ be a C-isometric embedding. Set $\mathcal{M}^1 := \oplus^{(\infty)}\{(\mathcal{M}_i, d_i)\}_{2 \leq i \leq N}$ so that $\mathcal{M} = \mathcal{M}_1 \oplus^{(\infty)} \mathcal{M}^1$. By Theorem 5.14 of Volume I we embed $\mathcal{M}^1$ bi-Lipschitz homeomorphically into $H := (\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{2 \leq i \leq N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}$. Further, using Lemma 4.92 of Volume I we embed $\mathcal{M}_1$ isometrically into the predual space $\mathcal{F}(\mathcal{M}_1)$ of the space $\mathrm{Lip}_0(\mathcal{M}_1)$. In turn, we embed $\mathcal{F}(\mathcal{M}_1)$ isometrically into the Banach space $\ell_\infty(B)$ where B is the unit ball of $\mathcal{F}(\mathcal{M}_1)$. This allows us to identify the set $\mathcal{M}$ with its image in $\ell_\infty(B) \oplus^{(\infty)} H$ and the map $\phi : S \to \mathcal{M}$ with a bi-Lipschitz embedding into this image, here $\phi = \phi_1 \oplus \phi_2$ where $\phi_1 : S \to \ell_\infty(B)$ and $\phi_2 : S \to H$.

Next, by the McShane extension Theorem 1.27 of Volume I, ϕ_1 admits a Lipschitz extension to all of $\widetilde{\mathcal{M}}$ preserving its Lipschitz constant while ϕ_2 can be extended to all of $\widetilde{\mathcal{M}}$ with Lipschitz constant bounded by $c(\sum_{i=2}^{N} n_i, n_0)L(\phi_2)$, by Theorem 6.38. Hence there is a Lipschitz map $\widetilde{\phi} : \widetilde{\mathcal{M}} \to \ell_\infty(B) \oplus H$ such that $\widetilde{\phi}|_S = \phi$ and $L(\widetilde{\phi})$ is bounded by a constant $c(\mathcal{M})C$.

Following the arguments of the proof of Theorem 7.66, we now determine continuous linear extension operators $E_1 : \mathrm{Lip}(\phi(S), X) \to \mathrm{Lip}(\mathcal{M}_1 \oplus^{(\infty)} H, X)$ and $E_2 : \mathrm{Lip}(\mathcal{M}_1 \oplus^{(\infty)} H, X) \to \mathrm{Lip}(\ell_\infty(B) \oplus^{(\infty)} H, X)$ with bounds of their norms depending only on the basic parameters of $\mathcal{M}$. Setting then

$$E(f)(x) := (E_2 E_1)(f \circ \phi^{-1})(\widetilde{\phi}(x)), \quad x \in \widetilde{\mathcal{M}}, \quad f \in \mathrm{Lip}(S, X),$$

we obtain a linear extension operator $\mathrm{Lip}(S, X) \to \mathrm{Lip}(\widetilde{\mathcal{M}}, X)$ whose norm is bounded by the basic parameters of $\mathcal{M}$ and C. This would complete the proof of the corollary.

The operator E_1 has already been defined by Theorem 7.22 with $\mathcal{M}_1$ being a metric space of homogeneous type.

To define E_2 we first use Theorem 7.62 to find a bounded linear extension operator $\widetilde{E} : \mathrm{Lip}(\mathcal{M}_1, X) \to \mathrm{Lip}(\ell_\infty(B), X)$ whose norm is controlled by some constant $D(\mathcal{M}_1)$. Moreover, since $\widetilde{E}$ is an average operator,

$$\widetilde{E} f \subset \overline{\mathrm{conv}\, f(\mathcal{M}_1)} \quad \text{(closure in } X).$$

Now for every $h \in H$ we define a linear operator $\pi_h : \mathrm{Lip}(\mathcal{M}_1 \oplus^{(\infty)} H, X) \to \mathrm{Lip}(\mathcal{M}_1, X)$ by setting $\pi_h f := f(\cdot, h)$, and then introduce the required operator E_2 given for $f \in \mathrm{Lip}(\mathcal{M}_1 \oplus^{(\infty)} H, X)$ by

$$(E_2 f)(m, h) := (\widetilde{E} \pi_h f)(m), \quad (m, h) \in \ell_\infty(B) \oplus^{(\infty)} H.$$

Using this definition we get

$$\begin{aligned} &||(E_2 f)(m_1, h_1) - (E_2 f)(m_2, h_2)||_X \\ &\leq ||\widetilde{E}(\pi_{h_1} f)(m_1) - \widetilde{E}(\pi_{h_1} f)(m_2)||_X + ||\widetilde{E}[(\pi_{h_1} - \pi_{h_2}) f](m_2)||_X. \end{aligned}$$

The first summand is at most $||\widetilde{E}||L(f)d_1(m_1, m_2)$ while the second one is bounded by

$$\begin{aligned} &\sup\{||x||_X \ ; \ x \in \operatorname{conv}[f(\cdot, h_1) - f(\cdot, h_2)]\} \\ &= \sup\{||\sum \alpha_i[f(m_i, h_1) - f(m_i, h_2)]||_X \ ; \ \alpha_i \geq 0, \ \sum \alpha_i = 1 \text{ and } \{m_i\} \subset \mathcal{M}_1\} \end{aligned}$$

which is clearly bounded by $L(f)d_H(h_1, h_2)$.

Together with the previous estimates this bounds the Lipschitz constant of $E_2 f$ in $\operatorname{Lip}(\ell_\infty(B) \oplus^{(\infty)} H, X)$ by that of f, as required. □

It seems to be highly plausible that Theorem 7.68 is true for more general hyperbolic metric spaces. In particular, one can ask the following question.

Problem. *Is it true that a complete simply connected length space $\mathcal{M}$ of non-positive curvature in the Alexandrov sense is universal?*

Finally we present three main results of the Lee and Naor paper [LN-2005] and briefly discuss the extension method of these authors. To their formulation we need to add the notion of a *graph minor*.

Let $G = (V, E)$ be a combinatorial graph. Its minor is a graph obtained from G by a finite sequence of the following two operations:

- Removal an edge.
- Contracting an edge to a (new) vertex which replaces the endpoints of the edge.

A finite graph G is said to be *n-incomplete*, if the complete graph K_n on n vertices differs from any minor of G.

Theorem 7.69. (a) *Let $\mathcal{M}$ be doubling with the doubling constant $\delta_{\mathcal{M}}$. Then its universal extension constant satisfies*

$$\lambda_u(\mathcal{M}) \leq c_1 \log_2 \delta_M.$$

(b) *Let $G = (V, E)$ be an n-incomplete finite graph equipped by a weight $w : E \to \mathbb{R}_+ \cup \{+\infty\}$. Then the universal extension constant of the associated metric space $(\mathcal{M}_G, d_w)$ satisfies*

$$\lambda_u((\mathcal{M}_G, d_w)) \leq c_2 n^2.$$

(c) *For the Euclidean space $\mathbb{R}^n$,*

$$\lambda_u(\mathbb{R}^n) \leq c_3\sqrt{n}.$$

Here $c_1, c_2, c_3 > 0$ are numerical constants.

The first assertion of the theorem is equivalent to that of Theorem 7.62, cf. the proof of Corollary 7.37.

At the present time constructive proofs of the remaining assertions of Theorem 7.69 are unknown.

Using the limiting procedure of the proof of Theorem 7.12 we easily derive from part (b) of the theorem the following result:

Let $G = (V, E)$ *be a (not necessarily finite) graph and* $w : E \to \mathbb{R}_+ \cup \{+\infty\}$ *be a weight. Assume that for some* $n \in \mathbb{N}$ *each finite subgraph of* G *is* n*-incomplete. Then*

$$\lambda_u^*((\mathcal{M}_G, d_w)) \le c_2 n^2.$$

Here λ_u^* *is defined similarly to* λ_u *but with target spaces being dual Banach spaces.*

This result implies an extension of the Matoušek theorem [Mat-1990] on the simultaneous Lipschitz extension property of metric trees to the class of planar graphs. Let us recall that a metric graph is *planar* if each of its finite subgraphs is homeomorphic to a subset of $\mathbb{R}^2$ (in particular, a metric tree is a planar graph).

To derive the above assertion we use the Kuratowski planarity theorem stating that every finite planar graph is 5-incomplete. This implies that any planar graph $G = (V, E)$ with a weight $w : E \to \mathbb{R}_+ \cup \{+\infty\}$ satisfies $\lambda_u^*((\mathcal{M}_G, d_w)) \le 25c_2$.

Now we describe the extension method of these authors.

Let S be a closed subspace of a metric space $(\mathcal{M}, d)$. As in other proofs the main point for constructing an extension operator is existence of a Lipschitz partition of unity subordinate to a suitable cover of $\mathcal{M} \setminus S$. In the Lee-Naor proof, these objects are obtained by averaging a family of partitions of unity $\{P_i^\omega\}_{i \in I(\omega)}$ depending on stochastic variable ω. In turn, this family is constructed using padding stochastic decompositions, see, e.g., Lemma 4.2 and Remark 4.4 of Section 4.1 of Volume I where such an object is described. Existence of these decompositions is a deep fact established by Gupta, Krauthgamer and Lee [GKL-2003] for doubling metric spaces, and followed from the result of Klein, Plotkin and Rao [KPR-1993] for n-incomplete graphs.

Now the construction of the required extension operator is as follows.

Using a padded stochastic decomposition one defines, for a metric space $(\mathcal{M}, d)$ and some measure space (Ω, μ), a function $\Psi : \Omega \times \mathcal{M} \to [0, +\infty]$ with the following properties:

(i) For every $m \in \mathcal{M} \setminus S$ the function $\omega \mapsto \Psi(\omega, m)$ is μ-measurable and

$$\int_\Omega \Psi(\omega, m) d\mu(\omega) = 1;$$

(ii) For all $\omega \in \Omega$

$$\Psi(\omega, \cdot)|_S = 0;$$

(iii) There exists a function $\gamma : \Omega \to S$ such that for some constant $K > 0$ and all $m, m' \in \mathcal{M}$,

$$\int_\Omega d(\gamma(\omega), m)|\Psi(\omega, m) - \Psi(\omega, m')|d\mu(\omega) \leq Kd(m, m').$$

Clearly, Ψ may be seen as the aforementioned stochastic partition of unity. Using Ψ we define the required extension operator $E : \mathrm{Lip}(S, X) \to \mathrm{Lip}(\mathcal{M}, X)$ by setting

$$Ef(m) := \int_\Omega f(\gamma(\omega))\Psi(\omega, m)d\mu(\omega)$$

for $m \in \mathcal{M} \setminus S$ and

$$Ef(m) := f(m)$$

for $m \in S$.

An easy computation exploiting (i)-(iii) gives for the Lipschitz constant of Ef the inequality

$$L(Ef; X) \leq 3K.$$

It is worth also noting that this promising method is applied in [LN-2005] to several other interesting settings. In particular, it established an estimate of $\lambda_u(\mathcal{M})$ for an n-point metric space $\mathcal{M}$ which improves the estimate $O(\log n)$ given in the paper [JLS-1986]. It was also proved that $\lambda_u(S)$ where S is a subset of a Riemann surface of genus g is majorated by $O(g + 1)$.

Comments

Unlike the continuous extension case where linear operators play a role right from the beginning (see Chapter 1 of Volume I) the small number of results on Lipschitz simultaneous extensions arise only as by-products in the papers devoted to the nonlinear case. It was noted by Pelczyński [Pe-1968] that "... our knowledge of existence of linear extension operators for uniformly continuous or Lipschitz functions is rather unsatisfactory." However, only several isolated results have been obtained since then, the most important of which are those of the seminal papers by Marcus and Pisier [MP-1984], Johnson and Lindenstrauss [JL-1984], and Johnson, Lindenstrauss and Schechtman [JLS-1986].

The methods and results presented in the last two chapters were developed independently and at about the same time by Lang and Schlichenmaier, Lee and Naor, and the authors of this book, and presented in the papers [LSchl-2005], [LN-2005] and [BB-2007b], [BB-2007c], respectively.

The simultaneous extension results of the first two papers are consequences of more general theories devoted, respectively, to a Lipschitz generalization of the Hurewicz theory of continuous extensions and to a probabilistic approach to Computer Science combinatorial algorithms.

The third approach due to the authors of this book stems from the work of Yu. Brudnyi and Shvartsman on Whitney's linear extension problem for functions of Zygmund class announced in [BSh-1985] and published in detail in [BSh-1999]. To construct the corresponding extension operators these authors introduced generalized hyperbolic spaces $\mathbb{H}^n_\omega$ and associated spaces of balls and established the simultaneous Lipschitz extension property for them. The corresponding operator is given there as the composition of the average operator and the metric $(1+\varepsilon)$-projection (difficulties of this construction are discussed in Section 7.2 and the way to overcome them in Section 7.4).

The account of most of the results of Chapter 7 follows, with some improvements, that of aforementioned papers of A. and Yu. Brudnyis.

Our attempt to present in detail the proofs of the Lee-Naor results requires the understanding of a great deal of material that for now is mostly available only in the proceedings of Computer Science conferences in the form of extended abstracts. In fact, a detailed description of the methods and results of this area would probably form a volume comparable in size with this book.

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