Preface

The volume contains an exposition of the recent development of the two main themes of the book, Lipschitz extension problems for maps between metric spaces and smooth extension-trace problems for functions on closed subsets of $\mathbb{R}^n$. The basic facts used for their study are presented in Volume I while the current volume consists only of detailed motivations and formulations within the corresponding proofs. The reader may find in the introduction of each chapter below a detailed description of the material of that chapter. Here we restrict ourselves to a discussion of the main features of the two parts forming the volume.

The first of the aforementioned topics is presented in Part 3 consisting of Chapters 6-8. Chapter 6 is mainly devoted to the Lang-Schlichenmaier theory relating the existence of the corresponding Lipschitz extensions to two important concepts of Lipschitz topology, *Nagata dimension* and *Lipschitz connectedness*, see Section 4.2 of Volume I. This culminates with an explicit construction of Lipschitz extension operators acting between large classes of metric spaces. If the target space in question is Banach, the corresponding operator becomes linear, i.e., it provides the simultaneous Lipschitz extension for all subsets of the domain. However, the extension constants in the latter case are either unspecified or have only coarse estimates of how they depend on the basic parameters.

In Chapter 7, we present two other methods for the Lipschitz simultaneous extension problem which give the corresponding results with extension constants close to optimal. One of them, due to Lee and Naor, exploits a probabilistic argument that first appeared in a computer science context, see, e.g., Section 4.1 of Volume I. A second, due to the authors of this book, is based on geometric analysis methods; this method is constructive and covers an essentially wider class of spaces.

Finally, Chapter 8 studies different relations between linear and nonlinear Lipschitz extensions and the corresponding extension constants. In particular, we study the influence of snowflake metric transforms on the existence of Lipschitz extensions, present an explicit formula relating the linear Lipschitz extension constants with the corresponding nonlinear ones and construct examples of metric spaces with Lipschitz extension constants that are finite for the nonlinear case but infinite for the linear case.

Finally, Part 4 of Volume II consisting of Chapters 9 and 10 is devoted to the smooth extension problems for functions on closed subsets of $\mathbb{R}^n$.

The first section of Chapter 9 presents the Yu. Brudnyi–Shvartsman characterization of the trace spaces to closed subsets for Lipschitz spaces of higher order and for the associated jet spaces. The proofs are strongly based on concepts and methods of Local Approximation Theory, see Section 2.3 and Appendices F, G of Volume I for some basic facts of the theory. This approach used throughout the chapter gives solutions to similar problems for Morrey-Campanato and higher order Lipschitz spaces and for a wide class of closed subsets of $\mathbb{R}^n$. This class, in particular, includes Ahlfors s-regular sets with $s > n-1$, some self-similar fractals with separation conditions, see Section 4.2 of Volume I, and the closure of *uniform* (or $\varepsilon - \delta$) domains, see Section 2.4 of Volume I.

Chapter 10 discusses two extension-trace problems going back to the classical 1934 Whitney papers. The first one asks how to distinguish traces to a closed set of functions from the space $C^k(\mathbb{R}^n)$ or the likes from those of other continuous functions on this set (*Whitney's trace problem*). The solution may be essentially simplified if one reduces the problem to the case of subsets containing only a fixed number of points depending only on parameters of the space in question (e.g., subsets of at most $3 \cdot 2^{n-1}$ points for $C^{1,1}(\mathbb{R}^n)$). This reduction is the direct consequence of the general Yu. Brudnyi-Shvartsman *finiteness principle* proved up to now only in some special cases, see subsections 10.3.2, 10.4.1 and 10.5.1 of this chapter.

The second Whitney problem asks about the existence of a linear continuous extension operator for the trace of a smoothness space into the space itself. The problem is solved constructively in several cases, see subsections 10.2.4, 10.2.5, 10.4.2 and 10.5.2 of this chapter.

All the above mentioned results are carried out in accordance with the geometric analysis approach of this book combining Lipschitz selection theorems of Sections 5.4 and 5.5 of Volume I with the local approximation extension-trace criteria of Sections 9.3 and 9.4 of this volume.

A completely new approach to the Whitney problems for the spaces $C^\ell(\mathbb{R}^n)$ and $C^{\ell,\omega}(\mathbb{R}^n)$ was proposed by Ch. Fefferman in a series of papers starting in 2003 and continuing to the present. We present a detailed account of his breakthrough results with a rather sketchy description of his methods in Sections 10.3 and 10.4. A deeper insight into Fefferman's methods is, from our point of view, the most actual problem in this area. Clarifying these very complicated and lengthy proofs and the ideas behind them surely will lead to important progress in the area.

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