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Minimal and Maximal Invariant Spaces of Holomorphic Functions on Bounded Symmetric Domains

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Dedicated to the memory of Israel Gohberg

Abstract. Let D be a Cartan domain in $\mathbb{C}^d$ and let $G = Aut(D)$ be the group of all biholomorphic automorphisms of G. Consider the projective representation of G on spaces of holomorphic functions on D

$$(U_\nu(g)f)(z) := \{J(g^{-1})(z)\}^{\nu/p} f(g^{-1}(z)), \quad g \in G, \quad z \in D,$$

where p is the genus of D and ν is in the Wallach set $W(D)$.

We identify the minimal and the maximal $U_\nu(G)$-invariant Banach spaces of holomorphic functions on D in a very explicit way: The minimal space $\mathfrak{M}_\nu$ is a Besov-1 space, and the maximal space $\mathcal{M}_\nu$ is a weighted H^∞-space. Moreover, with respect to the pairing under the (unique) $U_\nu(G)$-invariant inner product we have $\mathfrak{M}_\nu^* = \mathcal{M}_\nu$.

In the second part of the paper we consider invariant Banach spaces of vector-valued holomorphic functions and obtain analogous descriptions of the unique maximal and minimal space, in particular for the important special case of "constant" partitions which arises naturally in connection with non-tube type domains.

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1. Bounded symmetric domains and Jordan triples

Let D be a *Cartan domain* in $\mathbb{C}^d$, i.e., an irreducible bounded symmetric domain in its Harish-Chandra realization. Then $Z = \mathbb{C}^d$ is a hermitian Jordan triple. The main example is the *matrix ball*

$$D = D(I_{r,n}) = \{z \in M_{r,n}(\mathbb{C}),\ I_r - zz^* > 0\}, \quad 1 \le r \le n.$$

with triple product

$$\{x,y,z\} = \frac{1}{2}\,(xy^*z + zy^*x).$$

In this paper we only sketch the necessary background on Cartan domains and hermitian Jordan triples, for more details cf. [U2], [L2], [FK2]. Let $G = \mathrm{Aut}\,(D)$ be the group of holomorphic automorphisms, and let

$$K = \{g \in G;\ g(0) = 0\}$$

be the maximal compact subgroup. Using Cartan's linearity theorem, one proves that K consists of linear maps. Then $D \equiv G/K$ via the evaluation map $g \mapsto g(0)$. The *symmetries* of D have the form $s_0(z) = -z$ and, more generally, $s_z = g\, s_0 g^{-1}$, where $g \in G$ satisfies $g(0) = z$. For each $a \in D$ there exists a unique *midpoint symmetry* ϕ_a fixing the geodesic midpoint between 0 and a, and satisfying $\phi_a(0) = a$.

Example 1.1. For $D = D(I_{r,n})$ we have

$$G = SU\,(r,n) = \left\{ g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL\,(\mathbb{C}, r+n);\ gJg^* = J \right\}$$

where $J = \begin{pmatrix} I_r & 0 \\ 0 & -I_n \end{pmatrix}$. The action is given by Möbius transformations

$$g \cdot z = (\alpha z + \beta)(\gamma z + \delta)^{-1}$$

and the midpoint symmetry is

$$\phi_a(z) = (I_r - aa^*)^{-1/2}(a - z)(I - a^*z)^{-1}(I - a^*a)^{1/2}.$$

In the 1-dimensional case, this reduces to

$$\phi_a(z) = \frac{a - z}{1 - a^*z}.$$

The group

$$K \equiv S(U(r) \times U(n))$$

acts via $k(z) = uzv$, where $u \in U(r),\ \ v \in U(n)$ and $\det\,(u)\,\det\,(v) = 1$.

In general, the domain D is characterized by the *dimension* d, the *genus* p, and the *rank* r. Moreover we have *characteristic multiplicities* a, b defined via

$$d = r(r-1)\,\frac{a}{2} + r + rb,$$
$$p = (r-1)\,a + 2 + b.$$

In the matrix case $D = D\,(I_{r,n})$ for $1 \le r \le n$, we have $d = r \cdot n$, $p = r + n$, $a = 2$, $b = n - r$.

For any hermitian Jordan triple Z and $u, v \in Z$, the *Bergman operator* $B(u,v)$ acting on Z is defined by

$$B(u,v)\,z = z - 2\{u\,v^*z\} + Q_u\,Q_v\,z$$

where $Q_v\, z = \{v\, z^* v\}$. It is known that det $B(z,w) = h(z,w)^p$, where $h(z,w)$ is a K-invariant sesqui-holomorphic polynomial determined by

$$h(z,z) = \prod_{j=1}^{r} (1 - s_j^2(z)),$$

where $s_j(z)$ are the *singular values* of z. For matrices, we have $h(z,w) = \det(I - zw^*)$.

If $z, w \in Z$ and $B(z,w)$ is invertible, we define the *quasi-inverse* [L1], [L2]

$$z^w := B(z,w)^{-1}(z - Q_z\, w).$$

One can show [L2, p. 25, JP35] that $B(z,w)^{-1} = B(z^w, -w)$. The "transvection" $g_a \in G$ [L2, Proposition 9.8], defined by

$$g_a(z) = a + B(a,a)^{1/2}\, z^{-a} = \phi_a(-z)$$

for all $a, z \in D$, satisfies $g_a^{-1} = g_{-a}$ and

$$g_a'(z) = B(a,a)^{1/2}\, B(z,-a)^{-1} = B(a,a)^{1/2}\ B(z^{-a}, a).$$

2. Hilbert spaces of holomorphic functions

Let $dm(z)$ be the Lebesgue measure. The unique (up to a constant multiple) G-invariant measure on D has the form

$$h(z,z)^{-p}\, dm(z).$$

Given a parameter $\nu > p-1$ we define a probability measure

$$d\mu_\nu(z) = c_\nu \cdot h(z,z)^{\nu-p} dm(z)$$

on D, which has the quasi-invariance property

$$d\mu_\nu(g(z)) = |J(g,z)^{\frac{2\nu}{p}}|\, d\mu_\nu(z), \quad \forall\, g \in G. \tag{2.1}$$

Here $J(g,z) = \det g'(z)$ is the Jacobian of g at z. (2.1) follows from

$$B(g(z), g(w)) = g'(z)\, B(z,w)\, g'(w)^* \qquad \forall\, g \in G,\ \forall\, z, w \in D \tag{2.2}$$

which yields the quasi-invariance

$$h(g(z), g(w)) = J(g,z)^{\frac{1}{p}}\, h(z,w)\ \overline{J(g,w)}^{\frac{1}{p}}, \quad \forall\, g \in G \tag{2.3}$$

of h.

Proposition 2.1. *Each $g \in D$ has a unique "polar decomposition" $g = g_a \cdot k$ with $a = g(0),\ k \in K$.*

Proof. Define $a = g(0)$ and consider $k = g_a^{-1} \circ g$. Then $k \in G$ and $k(0) = 0$. Therefore $k \in K$ and $g = g_a \circ k$. □

Using Proposition 2.1, we define a cocycle $J_\nu : G \times D \to \mathbb{C}$ by putting

$$J_\nu\,(g_a\,k, z) := h(a,a)^{\nu/2}\;h(kz,-a)^{-\nu}, \tag{2.4}$$

using the sesqui-holomorphic branch of $h(z,w)^{-\nu}$ on $D \times D$ normalized by $h(0,0)^{-\nu} = 1$. Then

$$|J_\nu\,(g,z)| = |J\,(g,z)|^{\nu/p}.$$

The Jacobian of g_a has the form

$$J(g_a^{-1}, z) = h(a,a)^{p/2}\,h(z,a)^{-p}.$$

Since $g_a^{-1} = g_{-a}$, (2.4) implies

$$J_\nu(g_a^{-1}, z) = h(a,a)^{\nu/2}\,h(z,a)^{-\nu}.$$

Now consider the so-called *Wallach set*

$$W(D) := \{\nu;\ \ (z,w) \mapsto h(z,w)^{-\nu}\ \ \text{positive definite}\} \tag{2.5}$$

and, for $\nu \in W(D)$, define the reproducing kernel Hilbert space

$$\mathcal{H}_\nu = \overline{\text{span}}\,\{h(\cdot,w)^{-\nu};\ w \in D\}$$

with inner product determined by

$$\langle h(\cdot,w)^{-\nu},\ h(\cdot,z)^{-\nu}\rangle_\nu = h(z,w)^{-\nu}$$

for the reproducing kernel of $\mathcal{H}_\nu$. The corresponding group action

$$(U_\nu(g)f)(z) := J_\nu(g^{-1}, z)\,f((g^{-1}(z)) \tag{2.6}$$

on $\mathcal{H}_\nu$ acts *projectively:* $U_\nu(g_1 \circ g_2) = c(g_1,g_2)\,U_\nu(g_1)\,U_\nu(g_2)$ for a unimodular cocycle. Then $U_\nu(g) : \mathcal{H}_\nu \to \mathcal{H}_\nu$ acts isometrically, $\forall\ g \in G$, because (2.3) implies

$$J_\nu(g,z)\,h(g(z),g(w))^{-\nu}\ \overline{J_\nu(g,w)} = h(z,w)^{-\nu}.$$

One can show that $\mathcal{H}_\nu$ is *irreducible* for the action U_ν of G.

The primary examples are the *weighted Bergman space* $\mathcal{H}_\nu = L^2_a(D, \mu_\nu)$ for $\nu > p-1$, and the *Hardy space* $\mathcal{H}_{\frac{d}{r}} = H^2(S,\sigma)$, for $\nu = \frac{d}{r}$. Here S is the *Shilov boundary* of D and σ is the unique K-invariant probability measure on S.

For a deeper analysis of $\mathcal{H}_\nu$, we need the fine structure of the polynomial algebra $\mathcal{P}$ of Z. For $1 \le j \le r$ there exist *Jordan theoretic minors* $N_j(z)$ generalizing the principal $j \times j$-minors for matrices. In particular, $N_r = N$ is the Jordan determinant. The *conical polynomials*, for any signature

$$\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$$

satisfying $m_1 \ge m_2 \ge \cdots \ge m_r \ge 0$, are given by

$$N_{\mathbf{m}}(z) = \prod_{j=1}^{r} N_j(z)^{m_j - m_{j+1}},\ \ z \in Z,$$

where $m_{r+1} := 0$. For diagonal matrices (including the rectangular case), we have

$$N_{\mathbf{m}} \left[\begin{array}{cccc|c} t_1 & & & 0 & \\ & t_2 & & & \\ & & \ddots & & 0 \\ 0 & & & t_r & \end{array} \right] = \prod_{j=1}^{r} t_j^{m_j} = t^{\mathbf{m}}.$$

Denote by $\mathcal{P}_{\mathbf{m}}$ the span of $\{N_{\mathbf{m}} \circ k;\ k \in K\}$. It is well known [S], [U1], [FK1] that the $\{\mathcal{P}_{\mathbf{m}}\}_{\mathbf{m} \geq 0}$ are K-irreducible and K-inequivalent, and there is a direct sum decomposition

$$\mathcal{P} = \sum_{\mathbf{m} \geq 0}^{\oplus} \mathcal{P}_{\mathbf{m}}. \tag{2.7}$$

It follows that the $\{\mathcal{P}_{\mathbf{m}}\}_{m \geq 0}$ are pairwise orthogonal in any K-invariant inner product on $\mathcal{P}$. Consider the *Fischer inner product*

$$\langle f, F \rangle_{\mathcal{F}} = \frac{1}{\pi^d} \int_{\mathbb{C}^d} f(z)\ \overline{F(z)}\ e^{-|z|^2} dm(z) = f\Big(\frac{\partial}{\partial z}\Big)(F^*)(0) \tag{2.8}$$

on $\mathcal{P}$, where $F^*(z) := \overline{F(\overline{z})}$. Define $K^{\mathbf{m}}(z, w)$ as the reproducing kernel for $\mathcal{P}_{\mathbf{m}}$ in the Fischer inner product. Then

$$e^{\langle z, w \rangle} = \sum_{\mathbf{m} \geq 0} K^{\mathbf{m}}(z, w). \tag{2.9}$$

For $\nu \in \mathbb{C}$ and $z, w \in D$ there is a binomial expansion

$$h(z, w)^{-\nu} = \sum_{\mathbf{m} \geq 0} (\nu)_{\mathbf{m}}\ K^{\mathbf{m}}(z, w), \tag{2.10}$$

where

$$(\nu)_{\mathbf{m}} = \prod_{j=1}^{r} \prod_{\ell=0}^{m_j - 1} \Big(\nu + \ell - (j-1)\,\frac{a}{2}\Big) = \prod_{j=1}^{r} \Big(\nu - (j-1)\,\frac{a}{2}\Big)_{m_j}$$

is the multi-variable "Pochhammer symbol". As a consequence, one obtains a determination of the Wallach set

$$W(D) = \{\nu \in \mathbb{C};\ (\nu)_{\mathbf{m}} \geq 0\ \ \forall\ \mathbf{m}\} = \Big\{\frac{\ell a}{2}\Big\}_{\ell=0}^{r-1} \cup \Big((r-1)\,\frac{a}{2},\ \infty\Big)$$

as a union of a *discrete* and a *continuous* part [RV], [W], [LA], [FK1].

The *multivariable hypergeometric functions* are defined as

$${}_pF_q \left(\begin{array}{c} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{array} \right) (z, w) = \sum_{\mathbf{m} \geq 0} \frac{\prod_1^p (\alpha_j)_{\mathbf{m}}}{\prod_1^q (\beta_j)_{\mathbf{m}}}\ K^{\mathbf{m}}(z, w).$$

For example, we have ${}_0F_0\,(z, w) = \exp \langle z, w \rangle$ by (2.9), and (2.10) yields

$${}_1F_0\,(\nu)(z, w) = h(z, w)^{-\nu}.$$

Let $\alpha_0, \alpha_1, \ldots, \alpha_q;\ \beta_1, \ldots, \beta_q > (r-1)\,\frac{a}{2}$. Put

$$\gamma = \sum_0^q \alpha_j - \sum_1^q \beta_j.$$

By [FK1], the hypergeometric functions have the following asymptotic behaviour, uniformly for $z \in D$:

$$\gamma > (r-1)\,\frac{a}{2} \Longrightarrow {}_{q+1}F_q \binom{\alpha}{\beta}(z,z) \approx h(z,z)^{-\gamma} \tag{2.11}$$

$$\gamma < -(r-1)\,\frac{a}{2} \Longrightarrow {}_{q+1}F_q \binom{\alpha}{\beta}(z,z) \approx 1. \tag{2.12}$$

Remark 2.1. For the unit ball ($r = 1$) and $\gamma = 0$, we have

$${}_{q+1}F_q \binom{\alpha}{\beta}(z,z) \approx \log\Big(\frac{1}{1-|z|}\Big).$$

For the exact asymptotics if z is scalar, see [Y]. For $r = 2$, exact asymptotics are given in [EZ].

In the following we consider Banach spaces of holomorphic functions on D which are "invariant" under the group action (2.6), with the aim to characterize the (unique) maximal and minimal invariant Banach spaces and describe them via explicit formulas. In later sections this study is extended to the case of vector-valued holomorphic functions associated with the holomorphic discrete series of G. In this context our main result concerns symmetric domains which are "not of tube type".

In this paper we only consider parameters ν belonging to the Wallach set (2.5). In a separate paper [AU4] we consider the so-called "pole set" arising from analytic continuation, and show that our results concerning the maximal and minimal invariant space can be generalized to this situation via suitable intertwining operators.

3. Invariant Banach spaces of holomorphic functions

In this section we assume that $\nu \in W(D)$ is a Wallach parameter and consider the weighted group action U_ν defined in (2.6). For the unweighted action ($\nu = 0$) and the unit disk, the results of this section have been obtained in [AF], [AFP].

Definition 3.1. Let X be a non-trivial Banach space of holomorphic functions on D. We say that X is $U_\nu(G)$-invariant if

(i) $f \in X,\ \ g \in G \Longrightarrow U_\nu(g)\,f \in X$ and $\|U_\nu(g)\,f\|_X = \|f\|_X$.

(ii) For any finite (complex) Borel measure μ on K, the linear operator (convolution by μ)

$$(T_\mu f)(z) = \int_K f(kz)\,d\mu(k)$$

maps X continuously into itself.

(iii) For every $z \in D$, the evaluation functional $f \mapsto \delta_z(f) := f(z)$ is bounded on X (it suffices to require the continuity of δ_0).

Note that condition (ii) holds if K acts on X strongly continuously via $\pi(k)f = f \circ k^{-1}$.

Proposition 3.1. *X contains the constant function 1 and, normalizing $\|1\|_X = 1$, we have for $f \in X$*

$$|f(0)| \leq \|f\|_X / \|1\|_X = \|f\|_X.$$

Proof. Since D is circular, we have by (ii) and (iii) for all $z \in D$

$$f(0)1 = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta} z)\, d\theta. \qquad \square$$

Corollary 3.1. *For $f \in X$ and $a \in D$, we have*

$$|f(a)| \leq h(a,a)^{-\nu/2} \|f\|_X.$$

Hence convergence in X implies uniform convergence on compact subsets of D.

Proof. Use the formula

$$|(U_\nu(g_a^{-1})\, f)(0)| = h(a,a)^{\nu/2} |f(a)| \leq \|U_\nu(g_a^{-1})\, f\|_X = \|f\|_X. \qquad \square$$

Corollary 3.2. *If $f = \sum\limits_{\mathbf{m} \geq 0} f_{\mathbf{m}} \in X$, then $f_{\mathbf{m}} \in X$ for all $\mathbf{m}$, and the projections $f \mapsto f_{\mathbf{m}}$ are continuous.*

Proof. In terms of the character $\chi_{\mathbf{m}}$ of K on $\mathcal{P}_{\mathbf{m}}$, we have

$$f_{\mathbf{m}}(z) = \int_K f(k^{-1} z)\, \chi_{\mathbf{m}}(k)\, dk. \qquad \square$$

Corollary 3.3. *If $\nu > (r-1)\frac{a}{2}$, then $\mathcal{P}$ is dense in X in the topology of uniform convergence on compact subsets. If $\nu = \frac{\ell a}{2}$, $0 \leq \ell \leq r-1$, the same holds for*

$$\mathcal{P}_\ell = \sum_{\substack{m_{\ell+1}=0 \\ \mathbf{m} \geq 0}}^{\oplus} \mathcal{P}_{\mathbf{m}}. \tag{3.1}$$

Proof. From $1 \in X$ (Proposition 3.1) it follows by (i) that

$$U_\nu(g_a)\, 1 = \mathrm{const}\, h(-,a)^{-\nu} \in X \quad \text{for all } a \in D.$$

Applying Corollary 3.2, we obtain $(\nu)_{\mathbf{m}}\, K^{\mathbf{m}}(-,a) \in X$, hence either $(\nu)_{\mathbf{m}} = 0$ or else $\mathcal{P}_{\mathbf{m}} = \mathrm{span}\,\{K^{\mathbf{m}}(-,a) : a \in D\} \subset X$. $\square$

Our main goal is to characterize the *maximal* and *minimal* invariant spaces.

Definition 3.2. Let $\mathcal{M}_\nu = \{f \in \mathcal{H}(D);\ \|f\|_{\mathcal{M}_\nu} < \infty\}$, where

$$\|f\|_{\mathcal{M}_\nu} = \sup_{z \in D} h(z,z)^{\nu/2}\, |f(z)| = \sup_{g \in G} |(U_\nu(g)\, f)(0)|.$$

It is easy to see that $\mathcal{M}_\nu$ satisfies (i), (ii) and (iii) of Definition 3.1. Hence using the second expression for the norm, it follows that $\mathcal{M}_\nu$ is $U_\nu(G)$-invariant. We remark that taking another base point $a \in D$ instead of 0 yields the same space with a norm proportional to $\|\cdot\|_{\mathcal{M}_\nu}$.

Proposition 3.2. *If X is $U_\nu(G)$-invariant, then $X \subseteq \mathcal{M}_\nu$ and $\|f\|_{\mathcal{M}_\nu} \leq \|f\|_X$, $\forall\, f \in X$.*

Proof. In view of Proposition 3.1, we have $|(U_\nu(g)\, f)(0)| \leq \|f\|_X, \;\; \forall\, f \in X$. $\square$

Corollary 3.4. *$\mathcal{M}_\nu$ is the unique maximal $U_\nu(G)$-invariant space, and it is a weighted H^∞-space, with weight $h(z,z)^{\nu/2}$.*

We remark that there exist spaces of holomorphic functions on D satisfying (i), (ii) of Definition 3.1, but not (iii). For instance, let f be any holomorphic function on D (possibly not in $\mathcal{M}_\nu$). Define $\mathfrak{M}_{\nu,d}(f)$ to be the space of all functions of the form

$$F(z) = \sum_{j=1}^{\infty} c_j \big(U_\nu(g_j)\, f\big)(z),$$

where $g_j \in G$ and $\sum\limits_{j=1}^{\infty} |c_j| < \infty$. For $F \in \mathfrak{M}_{\nu,d}(f)$ we define

$$\|F\|_{\mathfrak{M}_{\nu,d}(f)} = \inf \sum_{j=1}^{\infty} |c_j|,$$

where the infimum is taken over all admissible representations of F. Then it is easy to check that $\mathfrak{M}_{\nu,d}(f)$ is the smallest Banach space of holomorphic functions on D which contains f and satisfies (i) and (ii) of Definition 3.1.

Proposition 3.3. *The Banach space $\mathfrak{M}_{\nu,d}(f)$ satisfies condition* (iii) *if and only if $f \in \mathcal{M}_\nu$. More generally, let X be a Banach space of holomorphic functions on D satisfying* (i) *and* (ii). *Then X satisfies* (iii) *if and only if $X \subset \mathcal{M}_\nu$ continuously.*

Proof. If (iii) holds, then $\mathfrak{M}_{\nu,d}(f)$ (resp., X) is a $U_\nu(G)$-invariant Banach space and Proposition 3.2 implies $f \in \mathcal{M}_\nu$ (resp., $X \subset \mathcal{M}_\nu$ continuously). Conversely, if $f \in \mathcal{M}_\nu$, then

$$\sup_{g \in G} |U_\nu(g)\, f(0)| < \infty$$

and hence δ_0 is continuous on $\mathfrak{M}_{\nu,d}(f)$. Similarly, $X \subset \mathcal{M}_\nu$ continuously implies for all $f \in X$

$$|f(0)| \leq \|f\|_{\mathcal{M}_\nu} \leq c\, \|f\|_X.$$

Hence δ_0 is continuous on X. By (i), the continuity of $\delta_z, \; z \in D$, follows. $\square$

Definition 3.3. Let $\mathfrak{M}_\nu$ consist of all $f \in \mathcal{H}(D)$ such that

$$f(z) = \int_D d\mu(a)\, h(a,a)^{\nu/2}\, h(z,a)^{-\nu} \tag{3.2}$$

for some finite (complex) Borel measure μ on D. Define the norm

$$\|f\|_{\mathfrak{M}_\nu} = \inf\{\|\mu\|;\ \mu \text{ satisfies } (3.2)\}.$$

Proposition 3.4. *We have $f \in \mathfrak{M}_\nu$ if and only if*

$$f(z) = \int\limits_G d\mu(g)\,(U_\nu(g)\,1)(z), \quad \forall\ z \in D \tag{3.3}$$

for some finite Borel measure μ on G. Moreover

$$\|f\|_{\mathfrak{M}} = \inf\{\|\mu\|;\ \mu \quad \textit{satisfies } (3.3)\}.$$

Hence $\mathfrak{M}_\nu$ is $U_\nu(G)$-invariant.

The straightforward proof is omitted. Also, the condition

$$\|1\|_{\mathfrak{M}_\nu} = 1 \tag{3.4}$$

is satisfied. Indeed, if

$$1 = \int_D d\mu\,(a)\ h(a,a)^{\nu/2}\,h(z,a)^{-\nu}, \quad \forall\ z \in D$$

then for $z = 0$ we have

$$1 = \int_D d\mu\,(a)\ h(a,a)^{\nu/2} \leq \int_D d|\mu|\,(a)\ h(a,a)^{\nu/2} \leq \int_D d|\mu|\,(a) = \|\mu\|$$

and therefore

$$1 \leq \|1\|_{\mathfrak{M}_\nu} = \inf\{\|\mu\| : \mu \text{ representing measure}\}.$$

On the other hand, for $\mu = \delta_0$ we have

$$\int_D d\,\delta_0(a)\ h(a,a)^{\nu/2}\,h(z,a)^{-\nu} = 1$$

so $\|1\|_{\mathfrak{M}_\nu} \leq \|\delta_0\| = 1$. Hence (3.4) holds.

Proposition 3.5. *There is a canonical duality $\mathfrak{M}_\nu^* \equiv \mathcal{M}_\nu$ with respect to the pairing $\langle f, F\rangle_\nu$ of $\mathcal{H}_\nu$.*

Proof. Let $F \in \mathcal{M}_\nu$ and $f \in \mathfrak{M}_\nu$, with representation (3.3). Since

$$(U_\nu(g)\,F)(0) = \langle U_\nu(g)\,F, 1\rangle_\nu = \langle F, U_\nu(g^{-1})\,1\rangle_\nu,$$

it follows that

$$\begin{aligned}\langle f, F\rangle_\nu &= \int\limits_G d\mu(g)\ \langle U_\nu(g)\,1, F\rangle_\nu \\ &= \int\limits_G d\mu(g)\ \langle 1,\, U_\nu(g^{-1})\,F\rangle_\nu = \int\limits_G d\mu(g)\ \overline{U_\nu(g^{-1})\,F(0)}.\end{aligned}$$

Hence

$$|\langle f, F\rangle_\nu| \leq \int\limits_G d|\mu|\,(g)\ |(U_\nu(g^{-1})F)(0)| \leq \|\mu\|\|F\|_{\mathcal{M}_\nu}.$$

This holds for every representing measure μ for f, hence

$$|\langle f, F\rangle_\nu| \leq \|f\|_{\mathfrak{M}_\nu}\| \, \|F\|_{\mathcal{M}_\nu}. \tag{3.5}$$

Thus

$$\sup_{\|f\|_{\mathfrak{M}_\nu \leq 1}} |\langle f, F\rangle_\nu| \leq \|F\|_{\mathcal{M}_\nu}.$$

The converse inequality follows from

$$\|F\|_{\mathcal{M}_\nu} = \sup_{g\in G} |\langle F, U_\nu(g)\,1\rangle_\nu| \leq \sup_{\|f\|_{\mathfrak{M}_\nu} \leq 1} |\langle f, F\rangle_\nu|.$$

This means that the operator $V : \mathcal{M}_\nu \to \mathfrak{M}_\nu^*$ defined by

$$(VF)(f) = \langle f, F\rangle_\nu$$

is an isometry. We claim that V is surjective. Indeed, let $\Phi \in \mathfrak{M}_\nu^*$ and define $F(z) = \overline{\Phi(h(\cdot, z)^{-\nu})}$. Then F is holomorphic and

$$h(z, z)^{\nu/2}|F(z)| = |\Phi(h(z, z)^{\nu/2}\, h(\cdot, z)^{-\nu})| = |\Phi(U_\nu(g_z^{-1})\,1)| \leq \|\Phi\|_{\mathfrak{M}_\nu^*}.$$

So $F \in \mathcal{M}_\nu$ and $\|F\|_{\mathcal{M}_\nu} \leq \|\Phi\|_{\mathfrak{M}_\nu^*}$. Also, if $f \in \mathfrak{M}_\nu$ is represented as in (3.2), then

$$\begin{aligned}\Phi(f) &= \int_D d\mu(a)\, h(a, a)^{\nu/2}\, \Phi(h(\cdot, a)^{-\nu}) = \int_D d\mu(a)\, h(a, a)^{\nu/2}\, \overline{F(a)}\\ &= \int_D d\mu(a)\, h(a, a)^{\nu/2}\, \langle h(\cdot, a)^{-\nu}, F\rangle_\nu = \langle f, F\rangle_\nu.\end{aligned}$$

It follows that $V(F) = \Phi$, and so V is a surjective isometry. □

Definition 3.4. Let $\mathfrak{M}_{\nu,d}$ be the space of all $f \in \mathfrak{M}_\nu$ which are represented with respect to a discrete measure, i.e.,

$$f(z) = \sum_{j=1}^{\infty} c_j\, (U_\nu(g_j)\,1)(z) \tag{3.6}$$

with $g_j \in G$ and $c_j \in \mathbb{C}$ such that $\sum_j |c_j| < \infty$, with the norm

$$\|f\|_{\mathfrak{M}_{\nu,d}} = \inf \sum_{j=1}^{\infty} |c_j|$$

over all representations (3.6).

Clearly, $\mathfrak{M}_{\nu,d}$ is a closed subspace of $\mathfrak{M}_\nu$ and $\|f\|_{\mathfrak{M}_\nu} \leq \|f\|_{\mathfrak{M}_{\nu,d}}$ for all $f \in \mathfrak{M}_{\nu,d}$.

Proposition 3.6. *The dual space of $\mathfrak{M}_{\nu,d}$ is identified isometrically with $\mathcal{M}_\nu$, with respect to the pairing $\langle f, F\rangle_\nu,\ f \in \mathfrak{M}_{\nu,d},\ F \in \mathcal{M}_\nu$. In particular, $\mathfrak{M}_{\nu,d} = \mathfrak{M}_\nu$ with equal norms.*

Proof. The fact that $\mathfrak{M}^*_{\nu,d} = \mathcal{M}_\nu$ isometrically is proved as in the proof of Proposition 3.5. This also yields that $\|f\|_{\mathfrak{M}_\nu} = \|f\|_{\mathfrak{M}_{\nu,d}}$ for all $f \in \mathfrak{M}_{\nu,d}$. To prove that $\mathfrak{M}_\nu = \mathfrak{M}_{\nu,d}$ it suffices (by the Hahn-Banach theorem) to prove that if $\Phi \in \mathfrak{M}^*_\nu$ vanishes on $\mathfrak{M}_{\nu,d}$ then it is zero. But this follows from the identification of $\mathfrak{M}^*_\nu$ with $\mathcal{M}_\nu$. $\square$

Proposition 3.7. *If $X \neq 0$ is $U_\nu(G)$-invariant, then $\mathfrak{M}_\nu \subseteq X$ and $\|f\|_X \leq \|f\|_{\mathfrak{M}_\nu}$, $\forall\ f \in \mathfrak{M}_\nu$. Hence $\mathfrak{M}_\nu$ is the unique minimal $U_\nu(G)$-invariant Banach space.*

Proof. Since $1 \in X$ and $\|1\|_X = 1$ we have $\|U_\nu(g)\,1\|_X = 1$ for all $g \in G$. Let $f \in \mathfrak{M}_\nu = \mathfrak{M}_{\nu,d}$, and let $f = \sum\limits_{j=1}^{\infty} c_j\, U_\nu(g_j)\, 1$ be an admissible representation. Then the series converges absolutely

$$\sum_{j=1}^{\infty} \|c_j\, U_\nu(g_j)\, 1\|_X = \sum_{j=1}^{\infty} |c_j| < \infty,$$

and the completeness of $\mathfrak{M}_\nu$ guarantees that the convergence is also in the norm of X. Therefore $f \in X$ and

$$\|f\|_X \leq \sum_j \|c_j\, U_\nu(g_j)\, 1\|_X = \sum_{j=1}^{\infty} |c_j|.$$

This holds for all discrete representations of f, hence $\|f\|_X \leq \|f\|_{\mathfrak{M}_\nu}$. $\square$

We remark that there exist functions $f \in \mathcal{M}_\nu$ for which the group action $g \mapsto U_\nu(g)\, f$ is not continuous in the norm of $\mathcal{M}_\nu$. This leads to the following

Definition 3.5. Let $\mathcal{M}^{(0)}_\nu = \{f \in \mathcal{M}_\nu;\ g \mapsto U_\nu(g) f$ is continuous in the $\mathcal{M}_\nu$ norm$\}$.

Proposition 3.8.

(i) *$\mathcal{M}^{(0)}_\nu$ is the maximal $U_\nu(G)$-invariant space X for which $g \mapsto U_\nu(g)\, f$ is continuous in norm for all $f \in X$;*

(ii) *$\mathcal{M}^{(0)*}_\nu = \mathfrak{M}_\nu$ with respect to $\langle \cdot, \cdot \rangle_\nu$;*

(iii) *The canonical embedding of $\mathcal{M}^{(0)}_\nu$ in $\mathcal{M}^{(0)**}_\nu = \mathcal{M}_\nu$ is the inclusion map.*

These statements will not be proved here, since they are not needed for our main problem: to identify $\mathfrak{M}_\nu$ via concrete integral formulas (not as a quotient space of the finite Borel measures on D or G).

Definition 3.6. The shift operator S^γ_α on $\mathcal{P}$ ("differentiation of order $\gamma - \alpha$") is defined by

$$S^\gamma_\alpha\Big(\sum_{\mathbf{m} \geq 0} f_{\mathbf{m}}\Big) = \sum_{\mathbf{m} \geq 0} \frac{(\gamma)_{\mathbf{m}}}{(\alpha)_{\mathbf{m}}}\, f_{\mathbf{m}}.$$

In view of the Faraut-Korányi-formula (2.10), we have

$$(S_\alpha^\gamma f)(z) = \langle f,\, h(\cdot, z)^{-\gamma}\rangle_\alpha,$$

and the reproducing kernel identity yields

$$S_\alpha^\gamma \,(h(\cdot, z)^{-\alpha}) = h(\cdot, z)^{-\gamma}.$$

It follows that

$$S_\alpha^\gamma \,(\mathcal{H}_\alpha) = \mathcal{H}_\gamma.$$

Remark 3.1. If $\alpha > (r-1)\,\frac{a}{2}$, then S_α^γ is defined on all of $\mathcal{P}$. If $\alpha = \frac{\ell a}{2},\; 0 \le \ell \le r-1$, then S_α^γ is defined only on $\mathcal{P}_\ell$ (cf. (3.1)).

Our first main result is

Theorem 3.1. *Let $\nu \in W(D),\;\; \nu > (r-1)\,a$. Choose $\beta \in \mathbb{R}$ such that $\beta + \frac{\nu}{2} > p-1$. Then there is a continuous embedding*

$$S_\nu^{\nu+\beta}(\mathfrak{M}_\nu) \subseteq L_a^1(D,\;\mu_{\beta+\frac{\nu}{2}}).$$

Here L_a^1 denotes the subspace of holomorphic functions in L^1.

Proof. It is enough to consider the "atoms": $f = h(a,a)^{\nu/2} h(\cdot, a)^{-\nu}$ for $a \in D$. We have

$$(S_\nu^{\nu+\beta} f)(z) = h(a,a)^{\frac{\nu}{2}}\; \langle h(\cdot, a)^{-\nu},\, h(\cdot, z)^{-(\nu+\beta)}\rangle_\nu = h(a,a)^{\frac{\nu}{2}}\, h(z,a)^{-(\nu+\beta)}.$$

Using the asymptotic behaviour of ${}_2F_1$, following from the assumption $\frac{\nu}{2} > (r-1)\,\frac{a}{2}$, we obtain

$$\begin{aligned}\|S_\nu^{\nu+\beta} f\|_{L^1(\mu_{\beta+\frac{\nu}{2}})} &= c_{\beta+\nu/2}\; h(a,a)^{\frac{\nu}{2}} \int_D \left| h(z,a)^{-\frac{\nu+\beta}{2}}\right|^2 h(z,z)^{\beta+\nu/2-p}\, dz \\ &= h(a,a)^{\frac{\nu}{2}}\, {}_2F_1\begin{pmatrix} \frac{\nu+\beta}{2} \;\; \frac{\nu+\beta}{2} \\ \beta + \frac{\nu}{2}\end{pmatrix}(a,a) \approx h(a,a)^{\frac{\nu}{2}}\, h(a,a)^{-\frac{\nu}{2}} = 1. \qquad \square\end{aligned}$$

Theorem 3.1 has the following converse

Theorem 3.2. *Let $\nu \in W(D)$ be arbitrary. Choose $\beta \in \mathbb{R}$ such that $\beta + \frac{\nu}{2} > p-1$. Let f be analytic on D such that $S_\nu^{\nu+\beta} f \in L_a^1(D, \mu_{\beta+\frac{\nu}{2}})$. Then $f \in \mathfrak{M}_\nu$ and*

$$\|f\|_{\mathfrak{M}_\nu} \le \frac{c_{\nu+\beta}}{c_{\beta+\nu/2}}\; \|S_\nu^{\nu+\beta} f\|_{L^1(\mu_{\beta+\frac{\nu}{2}})}.$$

Proof. Consider the finite Borel measure

$$d\mu(a) = (S_\nu^{\nu+\beta} f)(a)\, h(a,a)^{\beta+\nu/2-p}\, da.$$

Using the self-adjointness of $S_\nu^{\nu+\beta}$ with respect to $\mu_{\nu+\beta}$ and the reproducing property, we obtain

$$\begin{aligned}\int_D d\mu(a)\, h(a,a)^{\nu/2}\, h(z,a)^{-\nu} &= \int_D da\, h(a,a)^{\nu+\beta-p}\, (S_\nu^{\nu+\beta} f)(a)\, \overline{h(a,z)^{-\nu}} \\ &= \int_D da\, h(a,a)^{\nu+\beta-p}\, f(a)\, \overline{S_\nu^{\nu+\beta}\,(h(\cdot,z)^{-\nu})(a)} \\ &= \int_D da\, h(a,a)^{\nu+\beta-p}\, f(a)\, \overline{h(a,z)^{-(\nu+\beta)}} = \frac{1}{c_{\nu+\beta}}\, f(z).\end{aligned}$$

Hence $f \in \mathfrak{M}_\nu$ and $\|f\|_{\mathfrak{M}_\nu} \leq c_{\nu+\beta}\, \|\mu\| = \frac{c_{\nu+\beta}}{c_{\beta+\nu/2}}\, \|S_\nu^{\nu+\beta} f\|_{L^1(\mu_{\beta+\frac{\nu}{2}})}$. □

Corollary 3.5. *If $\frac{\nu}{2} > p-1$ we can choose $\beta = 0$. Hence*

$$\mathfrak{M}_\nu = L_a^1(D, \mu_{\frac{\nu}{2}}).$$

Corollary 3.6. *For each $f \in \mathfrak{M}_\nu$, the map $G \ni g \mapsto U_\nu(g)\, f \in \mathfrak{M}_\nu$ is continuous in the norm of $\mathfrak{M}_\nu$.*

Proof. This follows by realizing $\mathfrak{M}_\nu$ as $S_{\nu+\beta}^{\nu}(L_a^1(D,\, \mu_{\beta+\frac{\nu}{2}}))$ with $\beta+\frac{\nu}{2} > p-1$. □

Corollary 3.7. *Let $\nu > (r-1)\, a$ and choose $\beta \in \mathbb{R}$ such that $\beta + \frac{\nu}{2} > p-1$. Then*

$$f \in \mathfrak{M}_\nu \iff S_\nu^{\nu+\beta}\, f \in L_a^1\,(D,\, \mu_{\beta+\frac{\nu}{2}}). \tag{3.7}$$

Specializing to rank $r = 1$, we obtain

Corollary 3.8. *Let D be the open unit ball of $\mathbb{C}^d$. Let f be a holomorphic function on D and choose β such that $\beta + \frac{\nu}{2} > d$. Then (3.7) holds.*

4. Invariant Banach spaces of vector-valued holomorphic functions

We now turn to *vector-valued* holomorphic function spaces related to the *holomorphic discrete series.* In this section we describe the unique maximal space, and obtain a sufficient condition for membership in the unique minimal space.

For any fixed partition $\mathbf{m} = (m_1, \ldots, m_r)$ consider the $\mathbf{m}$-th Peter-Weyl component $\mathcal{P}_\mathbf{m}$ (cf. (2.7)) and parameters $\nu \in \mathbb{R}$ such that the integral

$$c_{\nu,\mathbf{m}}^{-1} = \int_D da\, h(a,a)^{\nu-p}\, \frac{K^\mathbf{m}\,(B(a,a)\, e, e)}{K^\mathbf{m}\,(e,e)} \tag{4.1}$$

is finite. Here $e \in Z$ is a maximal tripotent. It is well known that

$$K^\mathbf{m}\,(e,e) = \frac{d_\mathbf{m}}{(d/r)_\mathbf{m}}$$

where $d_\mathbf{m} = \dim\ \mathcal{P}_\mathbf{m}$. For example, in the rank 1 case (unit ball) we have

$$K^m\,(z,w) = \frac{(z|w)^m}{m!}$$

and $(d)_{\mathbf{m}} = d(d+1)\cdots(d+m-1) = \frac{(d+m-1)!}{(d-1)!}$. On the other hand, the space $\mathcal{P}_m$ of homogeneous polynomials on $Z = \mathbb{C}^d$ has dimension $\binom{m+d-1}{m}$, the number of solutions of $k_1 + \cdots + k_d = m$ in integers $k_i \geq 0$. Thus, for $e = (1, 0, \ldots, 0)$ we obtain

$$\frac{d_m}{(d)_m} = \frac{(m+d-1)!}{m!(d-1)!}\,\frac{1}{(d)_m} = \frac{1}{m!} = K^m\,(e,e).$$

Since K acts irreducibly on $\mathcal{P}_{\mathbf{m}}$ it follows that

$$\langle p|q\rangle_{\mathcal{F}} = c_{\nu,\mathbf{m}} \int_D da\ h(a,a)^{\nu-p}\,\langle p \circ B(a,a)^{1/2}\,\big|\,q \circ B(a,a)^{1/2}\rangle_{\mathcal{F}}$$

for all $p, q \in \mathcal{P}_{\mathbf{m}}$. Here $\langle p|q\rangle_{\mathcal{F}}$ is the Fischer-Fock norm (2.8). Equivalently,

$$p(\zeta) = c_{\nu,\mathbf{m}} \int_D da\, h(a,a)^{\nu-p} \cdot p(B(a,a)\zeta) \tag{4.2}$$

for all $p \in \mathcal{P}_{\mathbf{m}}$ and $\zeta \in Z$. Let $\mathcal{H}_{\nu,\mathbf{m}}$ denote the Hilbert space of all holomorphic functions

$$\Phi : D \to \mathcal{P}_{\mathbf{m}}, \quad z \mapsto \Phi_z(\zeta) = \Phi(z,\zeta)$$

such that

$$\|\Phi\|^2_{\nu,\mathbf{m}} = c_{\nu,\mathbf{m}} \int_D dz\, h(z,z)^{\nu-p} \|\Phi_z \circ B(z,z)^{1/2}\|^2_{\mathcal{F}} < +\infty.$$

Here we write

$$\Phi_z(\zeta) = \Phi(z,\zeta)$$

for $z \in D,\ \zeta \in Z$, noting that $\Phi(z,-)$ is a polynomial of type $\mathbf{m}$ in the ζ-variable. In this notation,

$$\Phi_z \circ B(z,z)^{1/2}(\zeta) = \Phi(z,\ B(z,z)^{1/2}\zeta).$$

Moreover the scalar parameter ν is chosen large enough so that $c_{\nu,\mathbf{m}} > 0$, and so $\mathcal{H}_{\nu,\mathbf{m}}$ contains all the "constant" functions

$$(1 \otimes p)(z,\zeta) = p(\zeta)$$

for $p \in \mathcal{P}_{\mathbf{m}}$. It is easily shown that

$$(U_{\nu,\mathbf{m}}(g^{-1})\Phi)(z,\zeta) = J_\nu(g,z)\,\Phi(g(z), g'(z)\zeta),$$

with $g \in G,\ \ \Phi \in \mathcal{H}_{\nu,\mathbf{m}},\ \ z \in D$ and $\zeta \in Z$, defines a unitary (projective) representation of G on $\mathcal{H}_{\nu,\mathbf{m}}$ belonging to the so-called *holomorphic discrete series* of G [AU3].

Proposition 4.1. *For $\Phi \in \mathcal{H}_{\nu,\mathbf{m}}$ we have the reproducing property*

$$\Phi_z(\zeta) = c_{\nu,\mathbf{m}} \int_D da\, h(a,a)^{\nu-p}\, h(z,a)^{-\nu} \cdot \Phi_a\left(B(a,a)\, B(z,a)^{-1}\,\zeta\right). \tag{4.3}$$

Proof. The reproducing formula, for a suitable constant, is proved in [AU3]. Applying the formula to $z = 0$, we obtain

$$\Phi_0(\zeta) = c_{\nu,\mathbf{m}} \int_D da\, h(a,a)^{\nu-p}\, \Phi_a(B(a,a)\,\zeta) \tag{4.4}$$

which reduces to (4.2) for $\Phi = 1 \otimes p$, and thus specifies the constant. $\square$

Definition 4.1. Let $X \subset \mathcal{O}(D, \mathcal{P}_{\mathbf{m}})$ be a non-trivial Banach space of $\mathcal{P}_{\mathbf{m}}$-valued holomorphic functions on D. We say that X is $U_{\nu,\mathbf{m}}(G)$-invariant if

(i) $\Phi \in X,\ \ g \in G \Longrightarrow U_{\nu,\mathbf{m}}(g)\,\Phi \in X$ and $\|U_{\nu,\mathbf{m}}(g)\,\Phi\|_X = \|\Phi\|_X$.

(ii) For any finite (complex) Borel measure μ on K, the linear operator (convolution by μ)

$$(T_\mu \Phi)(z,\zeta) = \int_K d\mu(k)\, \Phi(kz, k\zeta)$$

maps X continuously into itself.

(iii) For every $z \in D$, the evaluation map $\Phi \mapsto (\delta_z \otimes I)\,\Phi \in \mathcal{P}_{\mathbf{m}}$, defined by $(\delta_z \otimes I)\,\Phi(\zeta) := \Phi(z,\zeta)$, is bounded on X.

As before, condition (ii) is satisfied if the unweighted representation of K on X is strongly continuous.

Proposition 4.2. *Let $X \neq (0)$ be an invariant Banach space in $\mathcal{O}(D, \mathcal{P}_{\mathbf{m}})$. Then*

(i) $1 \otimes \mathcal{P}_{\mathbf{m}} \subset X$,

and there exists a constant c_X such that for all $\Phi \in X$

(ii) $\|\Phi_0\|_{\mathcal{F}} \le c_X\, \|\Phi\|_X$.

Proof. Put $m := m_1 + \cdots + m_r$, and consider the finite Borel measure $e^{i\,m\,t}\,dt/2\pi$. Since the polynomials in $\mathcal{P}_{\mathbf{m}}$ have total degree m, we have

$$\begin{aligned}\int_0^{2\pi} \frac{dt}{2\pi}\, e^{i\,m\,t}\, \Phi(e^{-it}z,\ e^{-it}\,\zeta) &= \int_0^{2\pi} \frac{dt}{2\pi}\, e^{i\,m\,t}\, e^{-i\,m\,t}\, \Phi(e^{-it}z, \zeta) \\ &= \int_0^{2\pi} \frac{dt}{2\pi}\, \Phi(e^{-it}z, \zeta) = \Phi(0,\zeta) = \Phi_0(\zeta).\end{aligned}$$

Since the action $U_{\nu,\mathbf{m}}$ on X is isometric and $dt/2\pi$ is a probability measure, it follows that

$$1 \otimes \Phi_0 = \int_0^{2\pi} \frac{dt}{2\pi}\, e^{i\,m\,t}\, U_{\nu,\mathbf{m}}(e^{it})\, \Phi \tag{4.5}$$

belongs to X, and $\|1 \otimes \Phi_0\|_X \le \|\Phi\|_X$. Choosing $\Phi \neq 0$, there exists $z \in D$ such that $\Phi_z(\zeta) = \Phi(z,\zeta) \not\equiv 0$. Applying a suitable $U_{\nu,\mathbf{m}}(g)$-transformation, we may assume $z = 0$, i.e., $\Phi_0(\zeta) = \Phi(0,\zeta) \not\equiv 0$. Since K acts irreducibly on $\mathcal{P}_{\mathbf{m}}$, it follows from (4.5) that $1 \otimes p \in X$ for all $p \in \mathcal{P}_{\mathbf{m}}$, i.e., $1 \otimes \mathcal{P}_{\mathbf{m}} \subset X$, and there exists $c_X > 0$ such that $\|p\|_{\mathcal{F}} \le c_X\, \|1 \otimes p\|_X$. $\square$

Definition 4.2. Let $\mathcal{M}_{\nu,\mathbf{m}} \subset \mathcal{O}(D, \mathcal{P}_{\mathbf{m}})$ be the Banach space of all holomorphic functions $\Phi : D \to \mathcal{P}_{\mathbf{m}}$ such that $\|\Phi\|_{\mathcal{M}_{\nu,\mathbf{m}}} < +\infty$, where

$$\|\Phi\|_{\mathcal{M}_{\nu,\mathbf{m}}} = \sup_{z\in D} \ h(z,z)^{\nu/2} \|\Phi_z \circ B(z,z)^{1/2}\|_{\mathcal{F}} = \sup_{g\in G} \ \|(U_{\nu,\mathbf{m}}(g)\,\Phi)_0\|_{\mathcal{F}}.$$

The requirements (ii) and (iii) in Definition 4.1 are easily checked, and hence, with the second expression for the norm, it follows that $\mathcal{M}_{\nu,\mathbf{m}}$ is $U_{\nu,\mathbf{m}}(G)$-invariant. Changing the K-invariant inner product on $\mathcal{P}_{\mathbf{m}}$, or taking another "base point" $a \in D$ instead of 0, changes the norm only by a proportionality constant.

Theorem 4.1. *Let $X \subset \mathcal{O}(D, \mathcal{P}_{\mathbf{m}})$ be a $U_{\nu,\mathbf{m}}$-invariant Banach space. Then $X \subset \mathcal{M}_{\nu,\mathbf{m}}$ continuously, i.e., $\mathcal{M}_{\nu,\mathbf{m}}$ is the unique maximal invariant space.*

Proof. Let $\Phi \in X$. Then Proposition 4.2 implies

$$\|(U_{\nu,\mathbf{m}}(g)\,\Phi)_0\|_{\mathcal{F}} \le c_X \cdot \|U_{\nu,\mathbf{m}}(g)\,\Phi\|_X = c_X\,\|\Phi\|_X$$

and hence

$$\sup_{g\in G} \|(U_{\nu,\mathbf{m}}(g)\,\Phi)_0\|_{\mathcal{F}} \le c_X \cdot \|\Phi\|_X.$$

The assertion follows. □

For $p \in \mathcal{P}_{\mathbf{m}}$ and $g \in G$, define

$$p^g := U_{\nu,\mathbf{m}}(g)\,(1 \otimes p) \in \mathcal{O}(D, \mathcal{P}_{\mathbf{m}}).$$

For $g = g_a$, we put $p^a := p^{g_a}$ and obtain

$$(p^a_z)(\zeta) = h(a,a)^{\nu/2} \ h(z,a)^{-\nu} \ p\,(B(a,a)^{1/2} \ B(z,a)^{-1}\,\zeta). \tag{4.6}$$

More generally,

$$p^{g_a k}_z(\zeta) = h(a,a)^{\nu/2}\, h(z,a)^{-\nu} \ p\,(k^{-1}\, B(a,a)^{1/2}\, B(z,a)^{-1}\,\zeta).$$

Lemma 4.1. *For large enough parameters α, β, γ we have the change of variables formula*

$$\begin{aligned}&\int dw\, h(w,w)^{\alpha-p} \ h(g_a(x),w)^{-\beta} \ h(w,g_a(y))^{-\gamma} \ f(g_a^{-1}(w))\\ &\quad = h(a,a)^{\alpha-\beta-\gamma} \ h(x,a)^{\beta} \ h(a,y)^{\gamma}\\ &\qquad \cdot \int_D dw\, h(w,w)^{\alpha-p} \ h(x,w)^{-\beta} \ h(w,y)^{-\gamma} \ h(w,a)^{\gamma-\alpha} \ h(a,w)^{\beta-\alpha} \ f(w)\end{aligned}$$

for all $a, x, y \in D$ and all $f \in L^1(D, \mu_\alpha)$.

Proof. Since $dw\ h(w,w)^{-p}$ is G-invariant, it follows that

$$\begin{aligned}&\int_D dw \ h(w,w)^{\alpha-p} \ h(g_a(x),w)^{-\beta} \ h(w,g_a(y))^{-\gamma} \ f(g_a^{-1}(w))\\ &= \int_D dw \ h(w,w)^{-p} \ h(g_a(w), g_a(w))^{\alpha}\\ &\qquad \cdot h(g_a(x), g_a(w))^{-\beta} \ h(g_a(w), g_a(y))^{-\gamma} \ f(w).\end{aligned}$$

Now the assertion follows from

$$
\begin{aligned}
&h(g_a(w), g_a(w))^{\alpha}\ h(g_a(x), g_a(w))^{-\beta}\ h(g_a(w), g_a(y))^{-\gamma} \\
&\quad = \left[h(a,a)\ h(w,a)^{-1}\ h(w,w)\ h(a,w)^{-1}\right]^{\alpha} \\
&\qquad \cdot \left[h(a,a)\ h(x,a)^{-1}\ h(x,w)\ h(a,w)^{-1}\right]^{-\beta} \\
&\qquad \cdot \left[h(a,a)\ h(w,a)^{-1}\ h(w,y)\ h(a,y)^{-1}\right]^{-\gamma} \\
&\quad = h(a,a)^{\alpha-\beta-\gamma}\ h(x,a)^{\beta}\ h(a,y)^{\gamma}\ h(w,w)^{\alpha}\ h(x,w)^{-\beta} \\
&\qquad \cdot h(w,y)^{-\gamma}\ h(w,a)^{\gamma-\alpha}\ h(a,w)^{\beta-\alpha}. \qquad \square
\end{aligned}
$$

Generalizing Definition 3.6, we define the *shift operator* $S_\nu^{\nu+\beta}$ acting on $\mathcal{O}(D, \mathcal{P}_{\mathbf{m}})$ by

$$
(S_\nu^{\nu+\beta}\,\Phi)_z(\zeta) = c_{\nu,\mathbf{m}} \int_D dw\, h(w,w)^{\nu-p}\ h(z,w)^{-(\nu+\beta)}\ \Phi_w\left(B(w,w)\, B(z,w)^{-1}\,\zeta\right)
$$

for all $z \in D$ and $\zeta \in Z$. The normalization is chosen so that $\beta = 0$ yields the identity. It is easily shown that $S_\nu^{\nu+\beta}$ commutes with the (unweighted) action of K on $\mathcal{O}(D, \mathcal{P}_{\mathbf{m}})$.

Proposition 4.3. *Let $p \in \mathcal{P}_{\mathbf{m}}$ and $a, z \in D$. Then, using the notation* (4.6), *we have*

$$
(S_\nu^{\nu+\beta}\, p^a)_z = h(z,a)^{-\beta}\ p_z^a.
$$

Proof. Using a $\mathbb{T}$-rotation in the anti-holomorphic variable w yields

$$
\begin{aligned}
&\int_D dw\, h(w,w)^{\nu-p}\ h(g_a(z),w)^{-(\nu+\beta)}\ h(a,w)^{\beta}\ p(B(w,w)\ B(g_a(z),w)^{-1}\ g_a'(z)\,\zeta) \\
&\quad = \int_D dw\, h(w,w)^{\nu-p} \int_{\mathbb{T}} \frac{d\vartheta}{2\pi}\ h(g_a(z),\, \vartheta w)^{-(\nu+\beta)}\ h(a,\, \vartheta w)^{\beta} \\
&\qquad\qquad \cdot p(B(w,w)\ B(g_a(z),\, \vartheta w)^{-1}\ g_a'(z)\,\zeta) \\
&\quad = \int_D dw\, h(w,w)^{\nu-p}\ h(g_a(z),\, 0)^{-(\nu+\beta)}\ h(a,0)^{\beta} \\
&\qquad\qquad \cdot p(B(w,w)\ B(g_a(z),0)^{-1}\ g_a'(z)\,\zeta) \\
&\quad = \int_D dw\, h(w,w)^{\nu-p}\ p(B(w,w)\ g_a'(z)\,\zeta) = c_{\nu,\mathbf{m}}^{-1}\ p(g_a'(z)\,\zeta).
\end{aligned}
$$

Applying Lemma 4.1 to $x = g_a(z),\ y = 0$ we obtain

$$
\begin{aligned}
&(S_\nu^{\nu+\beta}\, p^a)_z(\zeta) \\
&\quad = c_{\nu,\mathbf{m}} \int_D dw\, h(w,w)^{\nu-p}\ h(z,w)\ p_w^a\left(B(w,w)^{-(\nu+\beta)}\ B(z,w)^{-1}\,\zeta\right) \\
&\quad = c_{\nu,\mathbf{m}} \int_D dw\, h(w,w)^{\nu-p}\ h(z,w)^{-(\nu+\beta)}\ h(a,a)^{\nu/2} \\
&\qquad\qquad \cdot h(w,a)^{-\nu}\ p(g_a'(w)\ B(w,w)\ B(z,w)^{-1}\,\zeta)
\end{aligned}
$$

$$= c_{\nu,\mathbf{m}}\ h(a,a)^{\nu/2} \int_D dw\, h(w,w)^{\nu-p}\ h(z,w)^{-(\nu+\beta)} \\ \cdot\, h(w,a)^{-\nu}\ p(B(g_a(w),\, g_a(w))\ B(g_a(z),\, g_a(w))^{-1}\, g_a'(z)\,\zeta).$$

The general transformation formula (2.2) specializes to

$$B(g_a(z),\, g_a(w)) = g_a'(z)\ B(z,w)\ g_a'(w)^* \\ = B(a,a)^{1/2}\ B(z,a)^{-1}\ B(z,w)\ B(a,w)^{-1}\ B(a,a)^{1/2}.$$

As a consequence,

$$B(g_a(w),\, g_a(w))\ B(g_a(z),\, g_a(w))^{-1}\ g_a'(z) = g_a'(w)\ B(w,w)\ B(z,w)^{-1}.$$

Hence

$$(S_\nu^{\nu+\beta}\, p^a)_z(\zeta) = c_{\nu,\mathbf{m}}\ h(a,a)^{\nu/2}\ h(a,a)^{-(\nu+\beta)}\ h(g_a(z),a)^{\nu+\beta} \\ \cdot \int_D dw\, h(w,w)^{\nu-p}\ h(g_a(z),w)^{-(\nu+\beta)}\ h(a,w)^{\beta}\ p(B(w,w)\ B(g_a(z),w)^{-1}\ g_a'(z)\,\zeta) \\ = h(a,a)^{\nu/2}\ h(z,a)^{-(\nu+\beta)}\ p(g_a'(z)\,\zeta) = h(z,a)^{-\beta}\ p_z^a(\zeta). \qquad \square$$

Proposition 4.4. *The operators* S_ν^γ *are symmetric with respect to* $\langle\cdot,\cdot\rangle_{\nu,\mathbf{m}}$*, namely*

$$\langle S_\nu^\gamma\, \Phi, \Psi\rangle_{\nu,\mathbf{m}} = \langle \Phi,\, S_\nu^\gamma\, \Psi\rangle_{\nu,\mathbf{m}} \tag{4.7}$$

for all $\Phi, \Psi \in \mathcal{H}_{\nu,\mathbf{m}}$ *for which* $S_\nu^\gamma\, \Phi, S_\nu^\gamma\, \Psi \in \mathcal{H}_{\nu,\mathbf{m}}$*.*

Proof. For convenience we denote

$$d\sigma_{\nu,\mathbf{m}}(z) = c_{\nu,\mathbf{m}}\, dz\ h(z,z)^{\nu-p}.$$

Then we have

$$\langle S_\nu^\gamma\, \Phi, \Psi\rangle_{\nu,\mathbf{m}} = \int_D d\sigma_{\nu,\mathbf{m}}(z)\ \Big\langle (S_\nu^\gamma\, \Phi)(z,\, B(z,z)^{1/2}\cdot),\ \Psi(z,\, B(z,z)^{1/2}\cdot)\Big\rangle_{\mathcal{F}} \\ = \int_D d\sigma_{\nu,\mathbf{m}}(z) \Big\langle \int_D d\sigma_{\nu,\mathbf{m}}(w)\ h(z,w)^{-\gamma}\, \Phi(w,\, B(w,w)\, B(z,w)^{-1}\, B(z,z)^{1/2}\cdot), \\ \Psi(z, B(z,z)^{1/2}\cdot)\Big\rangle_{\mathcal{F}} \\ = \int_D d\sigma_{\nu,\mathbf{m}}(w) \Big\langle \Phi(w,\, B(w,w)\, B(z,w)^{-1}\, B(z,z)^{1/2}\cdot), \\ \int_D d\sigma_{\nu,\mathbf{m}}(z)\, h(w,z)^{-\gamma}\, \Psi(z,\, B(z,z)^{1/2}\cdot)\Big\rangle_{\mathcal{F}}.$$

Using the fact that for all $p, q \in \mathcal{P}_\mathbf{m}$ and $T \in K^\mathbb{C}$

$$\langle p \circ T,\, q\rangle_{\mathcal{F}} = \langle p,\, q \circ T^*\rangle_{\mathcal{F}}$$

we obtain (with $T = B(w,w)^{1/2}\, B(z,w)^{-1}\, B(z,z)^{1/2}$) that the last integral is equal to

$$\begin{aligned}
&\int_D d\sigma_{\nu,\mathbf{m}}(w)\, \Big\langle \Phi(w,\, B(w,w)^{1/2}\cdot), \\
&\qquad\qquad \int_D d\sigma_{\nu,\mathbf{m}}(z)\; h(w,z)^{\gamma}\; \Psi(z,\, B(z,z)\, B(w,z)^{-1}\, B(w,w)^{1/2}\cdot)\Big\rangle_{\mathcal{F}} \\
&\quad = \int_D d\sigma_{\nu,\mathbf{m}}(w)\, \Big\langle \Phi(w,\, B(w,w)^{1/2}\cdot),\; (S_\nu^\gamma\, \Psi)(w,\, B(w,w)^{1/2}\cdot)\Big\rangle_{\mathcal{F}} \\
&\quad = \langle \Phi,\, S_\nu^\gamma\, \Psi\rangle_{\nu,\mathbf{m}}. \qquad \square
\end{aligned}$$

The same arguments yield the following result.

Proposition 4.5. *For $\nu, \gamma \in \mathbb{R}$ let $\Phi \in \mathcal{H}_{\nu,\mathbf{m}} \cap \mathcal{H}_{\gamma,\mathbf{m}}$ and $\Psi \in \mathcal{H}_{\nu,\mathbf{m}}$ with $S_\nu^\gamma\, \Psi \in \mathcal{H}_{\gamma,\mathbf{m}}$. Then*

$$\langle \Phi, \Psi\rangle_{\nu,\mathbf{m}} = \langle \Phi,\; S_\nu^\gamma\, \Psi\rangle_{\gamma,\mathbf{m}}.$$

Proof.

$$\begin{aligned}
\langle \Phi,\; S_\nu^\gamma\, \Psi\rangle_{\gamma,\mathbf{m}} &= \int_D d\sigma_{\gamma,\mathbf{m}}(z)\, \Big\langle \Phi(z,\, B(z,z)^{1/2}\cdot),\; (S_\nu^\gamma\, \Psi)(z,\, B(z,z)^{1/2}\cdot)\Big\rangle_{\mathcal{F}} \\
&= \int_D d\sigma_{\gamma,\mathbf{m}}(z)\, \Big\langle \Phi(z,\, B(z,z)^{1/2}\cdot), \\
&\qquad \int_D d\sigma_{\nu,\mathbf{m}}(w)\, h(w,z)^{-\gamma}\, \Psi(w,\, B(w,w)\, B(z,w)^{-1}\, B(z,z)^{1/2}\cdot)\Big\rangle_{\mathcal{F}} \\
&= \int_D d\sigma_{\nu,\mathbf{m}}(w)\, \Big\langle \int_D d\sigma_{\gamma,\mathbf{m}}(z)\, \Phi(z,\, B(z,z)\, B(w,z)^{-1}\, B(w,w)^{1/2}\cdot)\, h(z,w)^{-\gamma}, \\
&\qquad \Psi(w,\, B(w,w)^{1/2}\cdot)\Big\rangle_{\mathcal{F}} \\
&= \int_D d\sigma_{\nu,\mathbf{m}}(w)\, \Big\langle \Phi(w,\, B(w,w)^{1/2}\cdot),\; \Psi(w,\, B(w,w)^{1/2}\cdot)\Big\rangle_{\mathcal{F}} = \langle \Phi, \Psi\rangle_{\nu,\mathbf{m}}
\end{aligned}$$

where we have used the reproducing property. $\square$

Corollary 4.1. *Let $\Psi, \Phi \in \mathcal{H}_{\nu,\mathbf{m}} \cap \mathcal{H}_{\gamma,\mathbf{m}}$ satisfy $S_\nu^\gamma\, \Psi,\; S_\nu^\gamma\, \Phi \in \mathcal{H}_{\gamma,\mathbf{m}}$. Then*

$$\langle S_\nu^\gamma\, \Phi,\, \Psi\rangle_{\gamma,\mathbf{m}} = \langle \Phi, \Psi\rangle_{\nu,\mathbf{m}} = \langle \Phi,\, S_\nu^\gamma\, \Psi\rangle_{\gamma,\mathbf{m}}.$$

Proof. The second equality follows from Proposition 4.5. For the first,

$$\langle S_\nu^\gamma\, \Phi,\, \Psi\rangle_{\gamma,\mathbf{m}} = \overline{\langle \Psi,\; S_\nu^\gamma\, \Phi\rangle_{\gamma,\mathbf{m}}} = \overline{\langle \Psi, \Phi\rangle}_{\nu,\mathbf{m}} = \langle \Phi, \Psi\rangle_{\nu,\mathbf{m}}. \qquad \square$$

Proposition 4.6. *We have*

$$\Phi_z(\zeta) = c_{\nu+\beta,\mathbf{m}} \int_D da\, h(a,a)^{\nu+\beta-p}\; h(z,a)^{-\nu}\; (S_\nu^{\nu+\beta}\, \Phi)_a\, (B(a,a)\; B(z,a)^{-1}\, \zeta).$$

Proof. Let $z \in D$ and $p \in \mathcal{P}_{\mathbf{m}}$ be fixed. The reproducing formula (4.6) applied to $\nu + \beta$ yields

$$
\begin{aligned}
&c_{\nu+\beta,\mathbf{m}}^{-1} \cdot h(z,z)^{\nu/2} \, (\Phi_z \,|\, p \circ B(z,z)^{1/2})_{\mathcal{F}} \\
&\quad = \int_D da\, h(a,a)^{\nu+\beta-p} \; h(z,a)^{-(\nu+\beta)} \; h(z,z)^{\nu/2} \\
&\qquad\qquad \cdot (\Phi_a \circ B(a,a) \; B(z,a)^{-1} \,|\, p \circ B(z,z)^{1/2})_{\mathcal{F}} \\
&\quad = \int_D da\, h(a,a)^{\nu+\beta-p} \; h(z,a)^{-(\nu+\beta)} \; h(z,z)^{\nu/2} \\
&\qquad\qquad \cdot (\Phi_a \circ B(a,a) \; \,|\, p \circ B(z,z)^{1/2} \; B(a,z)^{-1})_{\mathcal{F}} \\
&\quad = \int_D da\, h(a,a)^{\nu+\beta-p} \\
&\qquad\qquad \cdot (\Phi_a \circ B(a,a) \,|\, h(a,z)^{-\beta} \cdot h(a,z)^{-\nu} \; h(z,z)^{\nu/2} \, p \circ B(z,z)^{1/2} \; B(a,z)^{-1})_{\mathcal{F}} \\
&\quad = \int_D da\, h(a,a)^{\nu+\beta-p} \, (\Phi_a \circ B(a,a) \,|\, h(a,z)^{-\beta} \cdot p_a^z)_{\mathcal{F}} \\
&\quad = \int_D da\, h(a,a)^{\nu+\beta-p} \, (\Phi_a \circ B(a,a) \,|\, (S_\nu^{\nu+\beta} \; p^z)_a)_{\mathcal{F}}.
\end{aligned}
$$

Using Proposition 4.4 for the parameter $\nu + \beta$, we obtain

$$
\begin{aligned}
&c_{\nu+\beta,\mathbf{m}}^{-1} \cdot h(z,z)^{\nu/2} \, (\Phi_z \,|\, p \circ B(z,z)^{1/2})_{\mathcal{F}} \\
&\quad = \int_D da\, h(a,a)^{\nu+\beta-p} \, \big((S_\nu^{\nu+\beta} \, \Phi)_a \circ B(a,a) \,|\, p_a^z \big)_{\mathcal{F}} \\
&\quad = \int_D da\, h(a,a)^{\nu+\beta-p} \\
&\qquad\qquad \cdot \Big((S_\nu^{\nu+\beta} \, \Phi)_a \circ B(a,a) \,|\, h(a,z)^{-\nu} \; h(z,z)^{\nu/2} \, p \circ B(z,z)^{1/2} \; B(a,z)^{-1} \Big)_{\mathcal{F}} \\
&\quad = h(z,z)^{\nu/2} \int_D da\, h(a,a)^{\nu+\beta-p} \; h(z,a)^{-\nu} \\
&\qquad\qquad \cdot \Big((S_\nu^{\nu+\beta} \, \Phi)_a \circ B(a,a) \,|\, p \circ B(z,z)^{1/2} \; B(a,z)^{-1} \Big)_{\mathcal{F}} \\
&\quad = h(z,z)^{\nu/2} \int_D da\, h(a,a)^{\nu+\beta-p} \; h(z,a)^{-\nu} \\
&\qquad\qquad \cdot \Big((S_\nu^{\nu+\beta} \, \Phi)_a \circ B(a,a) \; B(z,a)^{-1} \,|\, p \circ B(z,z)^{1/2} \Big)_{\mathcal{F}}.
\end{aligned}
$$

Since any polynomial in $\mathcal{P}_{\mathbf{m}}$ has the form $h(z,z)^{\nu/2} \; p \circ B(z,z)^{1/2}$, the assertion follows. □

Remark 4.1. Proposition 4.6 can be written as

$$S_{\nu+\beta}^{\nu} \; S_\nu^{\nu+\beta} \, \Phi = \Phi$$

for Φ in a dense subspace of $\mathcal{H}_{\nu,\mathbf{m}}$. Thus, formally,

$$S_\gamma^\nu\ S_\nu^\gamma = I$$

for all $\nu, \gamma \in \mathbb{R}$ large enough.

Up to now, the polynomial $p \in \mathcal{P}_\mathbf{m}$ was arbitrary. We now specialize to

$$A(\zeta) = K_e^\mathbf{m}(\zeta) = K^\mathbf{m}(\zeta, e)$$

where $e \in Z$ is a maximal tripotent. Then we have

$$\begin{aligned} A_z^{g_a k}(\zeta) &= h(a,a)^{\nu/2}\ h(z,a)^{-\nu}\, K^\mathbf{m}(k^{-1}\ B(a,a)^{1/2}\, B(z,a)^{-1}\,\zeta, e) \\ &= h(a,a)^{\nu/2}\, h(z,a)^{-\nu}\, K^\mathbf{m}(B(a,a)^{1/2}\, B(z,a)^{-1}\,\zeta, ke). \end{aligned} \tag{4.8}$$

Definition 4.3.

(i) Let $\mathfrak{M}_{\nu,\mathbf{m}}$ denote the Banach space of all holomorphic functions $\Phi : D \to \mathcal{P}_\mathbf{m}$ which have a representation

$$\Phi_z(\zeta) = \int_G d\mu(g)\ A_z^g(\zeta)$$

for some finite $\mathbb{C}$-valued Borel measure on G. The norm is defined as the infimum

$$\|\Phi\|_{\mathfrak{M}_{\nu,\mathbf{m}}} = \inf_\mu\ \|\mu\|$$

taken over all such representations.

(ii) Define a vector-valued L^1-space $\mathcal{L}_\gamma^1$ to consist of all $\Phi \in \mathcal{O}(D, \mathcal{P}_\mathbf{m})$ such that

$$\|\Phi\|_{\mathcal{L}_\gamma^1} := c_{\gamma,\mathbf{m}} \int_D dz\, h(z,z)^{\gamma-p}\, \|\Phi_z \circ B(z,z)^{1/2}\|_\mathcal{F} < \infty.$$

Here $\|\cdot\|_\mathcal{F}$ is the Fischer norm on $\mathcal{P}_\mathbf{m}$.

Our main theorem in this section is

Theorem 4.2. *Let $\Phi \in \mathcal{O}(D, \mathcal{P}_\mathbf{m})$ and suppose that $S_\nu^{\nu+\beta}\,\Phi \in \mathcal{L}_{\beta+\nu/2}^1$. Then $\Phi \in \mathfrak{M}_{\nu,\mathbf{m}}$ and*

$$\|\Phi\|_{\mathfrak{M}_{\nu,\mathbf{m}}} \leq (d/r)_\mathbf{m}^{1/2}\ d_\mathbf{m}^{1/2}\ \frac{c_{\nu+\beta,\mathbf{m}}}{c_{\beta+\nu/2}}\ \|S_\nu^{\nu+\beta}\,\Phi\|_{\mathcal{L}_{\beta+\nu/2}^1}.$$

Proof. Define a complex measure μ on G by

$$d\mu(g_a k) = dk\ da\, h(a,a)^{\beta+\nu/2-p}\,(S_\nu^{\nu+\beta}\,\Phi)_a(B(a,a)^{1/2}\, ke).$$

For each $k \in K$ the Cauchy-Schwarz inequality yields

$$\begin{aligned} \left|(S_\nu^{\nu+\beta}\,\Phi)_a\,(B(a,a)^{1/2}\, ke)\right| &= \left|((S_\nu^{\nu+\beta}\,\Phi)_a \circ B(a,a)^{1/2}\,|\,K_{ke}^\mathbf{m})_\mathcal{F}\right| \\ &\leq \|(S_\nu^{\nu+\beta}\,\Phi)_a \circ B(a,a)^{1/2}\|_\mathcal{F} \cdot K^\mathbf{m}(e,e)^{1/2} \\ &= \frac{d_\mathbf{m}^{1/2}}{(d/r)_\mathbf{m}^{1/2}}\ \|(S_\nu^{\nu+\beta}\,\Phi)_a \circ B(a,a)^{1/2}\|_\mathcal{F} \\ &= d_\mathbf{m}^{1/2}\ \|(S_\nu^{\nu+\beta}\,\Phi)_a \circ B(a,a)^{1/2}\|_{d/r}. \end{aligned}$$

Hence

$$\begin{aligned}\|\mu\| &= \int\limits_K dk \int_D da\, h(a,a)^{\beta+\nu/2-p} \left|(S_\nu^{\nu+\beta}\,\Phi)_a\,(B(a,a)^{1/2}\,ke)\right| \\ &\leq \frac{d_{\mathbf{m}}^{1/2}}{(d/r)_{\mathbf{m}}^{1/2}} \int_D da\, h(a,a)^{\beta+\nu/2-p} \left\|(S_\nu^{\nu+\beta}\,\Phi)_a \circ B(a,a)^{1/2}\right\|_{\mathcal{F}} \\ &= \frac{1}{c_{\beta+\nu/2}} \frac{d_{\mathbf{m}}^{1/2}}{(d/r)_{\mathbf{m}}^{1/2}} \left\|S_\nu^{\nu+\beta}\,\Phi\right\|_{\mathcal{L}^1_{\beta+\nu/2}}.\end{aligned}$$

Hence μ is a finite measure on G. Moreover, (4.8) implies

$$\begin{aligned}&\int\limits_G d\mu(g)\; A_z^g(\zeta) \\ &= \int_D da h(a,a)^{\beta+\nu/2-p}\; h(a,a)^{\nu/2}\; h(z,a)^{-\nu} \\ &\qquad \cdot \int\limits_K dk\; (S_\nu^{\nu+\beta}\Phi)_a(B(a,a)^{1/2}ke)\; K^{\mathbf{m}}(B(a,a)^{1/2}B(z,a)^{-1}\zeta, ke) \\ &= \int_D da h(a,a)^{\nu+\beta-p}\; h(z,a)^{-\nu} \int\limits_K dk\; (S_\nu^{\nu+\beta}\Phi)_a(B(a,a)^{1/2}ke) \\ &\qquad \cdot \overline{K^{\mathbf{m}}(ke, B(a,a)^{1/2}B(z,a)^{-1}\zeta)} \\ &= \int_D da h(a,a)^{\nu+\beta-p}\; h(z,a)^{-\nu} \big((S_\nu^{\nu+\beta}\Phi)_a \circ B(a,a)^{1/2} | K^{\mathbf{m}}_{B(a,a)^{1/2}B(z,a)^{-1}\zeta}\big)_{d/r} \\ &= (d/r)_{\mathbf{m}}^{-1} \int_D da h(a,a)^{\nu+\beta-p}\; h(z,a)^{-\nu} \\ &\qquad \cdot \big((S_\nu^{\nu+\beta}\Phi)_a \circ B(a,a)^{1/2} | K^{\mathbf{m}}_{B(a,a)^{1/2}B(z,a)^{-1}\zeta}\big)_{\mathcal{F}} \\ &= (d/r)_{\mathbf{m}}^{-1} \int_D da h(a,a)^{\nu+\beta-p}\; h(z,a)^{-\nu} (S_\nu^{\nu+\beta}\Phi)_a\big(B(a,a)^{1/2}B(a,a)^{1/2}B(z,a)^{-1}\zeta\big) \\ &= (d/r)_{\mathbf{m}}^{-1} \int_D da h(a,a)^{\nu+\beta-p}\; h(z,a)^{-\nu} (S_\nu^{\nu+\beta}\Phi)_a\; \big(B(a,a)\; B(z,a)^{-1}\zeta\big) \\ &= (d/r)_{\mathbf{m}}^{-1}\; c_{\nu+\beta,\mathbf{m}}^{-1}\; \Phi_z(\zeta)\end{aligned}$$

using Proposition 4.4. Thus Φ is represented by μ, up to a constant. □

5. Minimal spaces for non-tube type domains

In this section we obtain a "converse" of Theorem 4.2, and thus a complete characterization of the minimal space, for the special partitions $\mathbf{s} = (s, \ldots, s)$, where $s \in \mathbb{N}$. These "constant" partitions arise naturally in the study of highest quotients (Dirichlet spaces) for domains which are not of tube type (cf. [AU3]). The integration formulas developed here may be of independent interest.

We consider the *Peirce decomposition*

$$Z = Z_1 \oplus Z_{1/2} = \begin{pmatrix} Z_1 \\ Z_{1/2} \end{pmatrix} \tag{5.1}$$

of Z for a maximal tripotent e, and write $z \in Z$ as $z = z_1 + z_{1/2}$, with $z_1 \in Z_1$ and $z_{1/2} \in Z_{1/2}$.

Lemma 5.1. *For $u \in Z_1,\ v \in Z_{1/2}$ the Bergman operator $B(u, v)$ has a block-matrix decomposition*

$$B(u, v) = \begin{pmatrix} I_1 & -2u \,\Box\, v^* \\ 0 & I_{1/2} \end{pmatrix} \tag{5.2}$$

with respect to (5.1). *Here I_ν denotes the identity operator on Z_ν.*

Proof. For $z \in Z$, we have $\{u\, v^*\, z_1\} \in Z_{3/2} = (0)$ and $Q_v z_1 \in Z_0 = (0)$, since e is maximal. Moreover, $Q_v\, z_{1/2} \in Z_{1/2}$ and hence $Q_u\, Q_v\, z_{1/2} \in Z_{3/2} = (0)$. Thus

$$\begin{aligned} B(u, v)\, z &= z - 2\{u\, v^* z\} + Q_u\, Q_v\, z \\ &= z_1 + z_{1/2} - 2\{u\, v^*(z_1 + z_{1/2})\} + Q_u\, Q_v\, (z_1 + z_{1/2}) \\ &= z_1 + z_{1/2} - 2\{u\, v^* z_{1/2}\}, \end{aligned}$$

with $z_1 - 2\{u\, v^* z_{1/2}\} \in Z_1$. The assertion follows. □

Corollary 5.1. *For $u \in Z_1,\ v \in Z_{1/2}$, we have $\det_Z B(u, v) = 1$. In particular, $B(u, v)$ is invertible, with inverse given by*

$$B(u, v)^{-1} = B(u, -v) = \begin{pmatrix} I_1 & 2u \,\Box\, v^* \\ 0 & I_{1/2} \end{pmatrix}$$

Lemma 5.2. *If $B(z, w)$ is invertible and $Q_z\, w = Q_w\, z = 0$, then $z^w = z$.*

Proof. By assumption, we have

$$B(z,w)z = z - 2\{z w^* z\} + Q_z Q_w z = z - 2Q_z w + Q_z Q_w z = z = z - Q_z w = B(z,w)\, z^w.$$

Since $B(z, w)$ is invertible, we conclude that $z = z^w$. □

Proposition 5.1. *Suppose $v, u \in D$ and*

$$Q_u v = Q_v u = 0. \tag{5.3}$$

Then we have

$$\begin{aligned} &B\big(u + B(u, u)^{1/2}\, v,\ \ u + B(u, u)^{1/2}\, v\big) \\ &\quad = B(u, u)^{1/2}\, B(v, u)\, B(v, v)\, B(u, v)\, B(u, u)^{1/2}. \end{aligned} \tag{5.4}$$

Proof. Since $v^{-u} = v$ by Lemma 5.2, we have $g_u(v) = u + B(u,u)^{1/2}\, v^{-u} = u + B(u,u)^{1/2}\, v$ and $g'_u(v) = B(u,u)^{1/2}\ B(v^{-u},u) = B(u,u)^{1/2}\ B(v,u)$. Now apply (2.2). □

For any tripotent, the Peirce spaces are hermitian Jordan subtriples of Z, and Z_1 and Z_0 are always irreducible if Z is irreducible. One can show that in our case of a maximal tripotent (i.e., $Z_0 = (0)$) the Peirce $\frac{1}{2}$-space $Z_{1/2}$ is also irreducible. Let $D_1 = D \cap Z_1$ and $D_{1/2} = D \cap Z_{1/2}$ denote the respective open unit balls.

Corollary 5.2. *Let $u \in D_1$ and $v \in D_{1/2}$. Then* (5.3) *holds and, in addition, we have*

$$h(u + B(u,u)^{1/2}\, v,\ \ u + B(u,u)^{1/2}\, v) = h(u,u)\, h(v,v). \tag{5.5}$$

Proof. By Lemma 5.1 and Lemma 5.2, the assumption of Proposition 5.1 is satisfied, showing that (5.4) holds. Moreover, $h(u,v) = 1 = h(v,u)$ by Lemma 5.1. Therefore (5.5) follows from (5.4) by taking determinants. □

Proposition 5.2. *For $u \in Z_1$ and $v \in Z_{1/2}$, we have $u + B(u,u)^{1/2}\, v \in D$ if and only if $u \in D_1$ and $v \in D_{1/2}$.*

Proof. As a consequence of the spectral theorem for Jordan triples, we have $h(z,z) > 0$ for $z \in D$ and $h(z,z) = 0$ for all $z \in \partial D$. Hence D is a connected component of

$$M := \{z \in Z :\ h(z,z) > 0\}.$$

Define $\pi : D \to Z_{1/2}$ by

$$\pi(w) := B(w_1,w_1)^{-1/2}\, w_{1/2}$$

for all $w = w_1 + w_{1/2} \in D$ with $w_\nu \in Z_\nu$. Since Peirce projections are contractive, we have $\|w_1\| \le \|w\| < 1$. Therefore $w_1 \in D_1$ and $B(w_1,w_1)$ is invertible. By Corollary 5.2, we have

$$h(w_1,w_1)\, h(\pi(w),\ \pi(w)) = h(w,w) \neq 0.$$

It follows that $h(\pi(w),\ \pi(w)) \neq 0$ and therefore $\pi(w) \in Z_{1/2} \cap M$. Since π is continuous and D is connected, it follows that $\pi(D)$ belongs to the 0-connected component of $M \cap Z_{1/2}$, which coincides with $D_{1/2}$. This shows that $w = u + B(u,u)^{1/2}\, v \in D$ implies $u \in D_1$ and $v = \pi(w) \in D_{1/2}$.

Conversely, let $u \in D_1$. Define $F_u : Z_{1/2} \to Z$ by

$$F_u(v) := u + B(u,u)^{1/2}\, v.$$

Then Corollary 5.2 implies

$$h(F_u(v),\ F_u(v)) = h(u,u)\, h(v,v).$$

If $v \in D_{1/2}$, then $h(v,v) \neq 0$ and hence $F_u(v) \in M$. Since $F_u(0) = u \in D_1 \subset D$, $F_u(D_{1/2})$ belongs to the u-connected component of M, which coincides with D. Therefore $w = F_u(v) \in D$. □

According to Proposition 5.2 the map

$$F(u,v) := u + B(u,u)^{1/2}\, v$$

defines a real-analytic isomorphism from $D_1 \times D_{1/2}$ onto D, with inverse

$$F^{-1}(w_1 + w_{1/2}) = w_1 + B(w_1, w_1)^{-1/2}\, w_{1/2}.$$

Put $\beta(u) := B(u,u)^{1/2} \in \mathrm{End}(Z)$. Then F has the derivative

$$F'(u,v)(x,y) = x + \beta(u)\, y + (\beta'(u)\, x)\, v$$

for $x \in Z_1,\ y \in Z_{1/2}$. Since $\beta(u)$ preserves both Peirce spaces, the same is true for $\beta'(u)x \in \mathrm{End}(Z)$. Thus we have a block-matrix decomposition

$$F'(u,v) = \begin{pmatrix} I_1 & T \\ 0 & B(u,u)^{1/2} \end{pmatrix}$$

with respect to (5.1), where

$$Tx := (\beta'(u)\, x)\, v = \frac{\partial}{\partial t}\Big|_{t=0} \beta(u + tx)\, v.$$

It follows that

$$\det\nolimits_Z\ F'(u,v) = \det\nolimits_{Z_{1/2}} B(u,u)^{1/2} = h(u,u)^{b/2}.$$

Hence $F'(u,u)$ has the "real" determinant

$$\det\ F'(u,v) = \left|\det\nolimits_{Z_{1/2}}\ B(u,u)^{1/2}\right|^2 = h(u,u)^b. \tag{5.6}$$

Making the change of variables

$$w = u + B(u,u)^{1/2}\, v \quad (u \in D_1,\ v \in D_{1/2}) \tag{5.7}$$

(5.6) yields

$$dw = h(u,u)^b\, du\, dv. \tag{5.8}$$

Proposition 5.3. *Let $u \in Z_1,\ v \in Z_{1/2}$ and $a = a_1 + a_{1/2} \in Z$ with $a_\nu \in Z_\nu$. Suppose that $B(a_1,u)$ is invertible. Then*

$$h(u + B(u,u)^{1/2}\, v, a) = h(u,a_1) \cdot h(v,\ B(u,u)^{1/2}\, B(a_1,u)^{-1}\ a_{1/2}). \tag{5.9}$$

Proof. Polarizing the identity (5.5) yields

$$h(u + B(u,a_1)^{1/2}\, v_1,\ \ a_1 + B(a_1,u)^{1/2}\, v_2) = h(u,a_1)\, h(v_1,v_2) \tag{5.10}$$

whenever $v_1, v_2 \in Z_{1/2}$. Putting

$$v_1 = B(u,a_1)^{-1/2}\ B(u,u)^{1/2}\, v \quad \text{and} \quad v_2 = B(a_1,u)^{-1/2}\ a_{1/2},$$

the left-hand sides of (5.9) and (5.10) agree, whereas

$$\begin{aligned} h(v_1,v_2) &= h\big(B(u,a_1)^{-1/2}\, B(u,u)^{1/2}\, v,\ B(a_1,u)^{-1/2}\ a_{1/2}\big) \\ &= h\big(B(u,u)^{1/2}\, v,\ B(a_1,u)^{-1}\, a_{1/2}\big) \\ &= h\big(v,\ B(u,u)^{1/2}\, B(a_1,u)^{-1}\ a_{1/2}\big). \end{aligned}$$

□

Lemma 5.3. *Let $u \in D_1$ and $a = a_1 + a_{1/2} \in D$ with $a_\nu \in Z_\nu$. Then $B(a_1, u)$ is invertible and*

$$B(u,u)^{1/2}\ B(a_1,u)^{-1}\ a_{1/2} \in D_{1/2}.$$

Proof. Since $a_1 \in D_1$, it follows that $B(a_1, u)$ is invertible. Therefore the addition formula [L2, p.26] yields

$$a^u = (a_1 + a_{1/2})^u = a_1^u + B(a_1,u)^{-1}\ a_{1/2}^{(u^{a_1})} = a_1^u + B(a_1,u)^{-1}\ a_{1/2}$$

since $u^{a_1} \in Z_1$ and hence $a_{1/2}^{(u^{a_1})} = a_{1/2}$ by Lemma 5.2. It follows that

$$g_{-u}(a) = -u + B(u,u)^{1/2}\, a^u = -u + B(u,u)^{1/2}\ a_1^u + B(u,u)^{1/2}\ B(a_1,u)^{-1}\ a_{1/2}.$$

Since $a \in D$, we have $g_{-u}(a) \in D$.

Therefore the Peirce $\frac{1}{2}$-component $B(u,u)^{1/2}\ B(a_1,u)^{-1}\ a_{1/2} \in D_{1/2}$. □

Let $P_1 : Z \to Z_1$ denote the Peirce 1-projection.

Lemma 5.4. *For $u \in Z_1$ and $v \in Z_{1/2}$, we have $P_1\, B(v,u) = P_1$.*

Proof. Using Lemma 5.1 and $B(v,u) = B(u,v)^*$ we write

$$P_1\, B(v,u) = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_1 & 0 \\ -2v \,\Box\, u^* & I_{1/2} \end{pmatrix} = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} = P_1.$$

Here I_ν is the identity map on Z_ν. □

Lemma 5.5. *Let $\mathbf{s} = (s, \ldots, s)$ and $w = w_1 + w_{1/2} \in D$ with $w_\nu \in Z_\nu$. Then*

$$K_e^{\mathbf{s}}\,(B(w,w)\,e) = \frac{d_{\mathbf{s}}}{(d/r)_{\mathbf{s}}}\ h(w,w)^s\ h(w_1,w_1)^s. \tag{5.11}$$

Proof. Let N be the Jordan algebra determinant of Z_1, normalized by $N(e) = 1$. Then

$$K_e^{\mathbf{s}}\,(z) = K^{\mathbf{s}}(e,e)\, N(P_1 z)^s = \frac{d_{\mathbf{s}}}{(d/r)_{\mathbf{s}}}\ N(P_1 z)^s.$$

Writing $w = u + B(u,u)^{1/2}\, v$ with $u \in D_1$ and $v \in D_{1/2}$, Proposition 5.1 and Lemma 5.4 imply

$$\begin{aligned} P_1\, B(w,w)\, e &= P_1\, B(u,u)^{1/2}\, B(v,u)\ B(v,v)\, B(u,v)\, B(u,u)^{1/2}\, e \\ &= P_1\, B(u,u)^{1/2}\ P_1\, B(v,u)\, B(v,v)\, B(u,v)\ \ P_1\, B(u,u)^{1/2}\, e \\ &= P_1\, B(u,u)^{1/2}\ P_1\, B(v,v)\ P_1\, B(u,u)^{1/2}\, e = B(u,u)^{1/2}\ B(v,v)\ B(u,u)^{1/2}\, e. \end{aligned}$$

The invertible transformations $P_1\, B(u,u)^{1/2}\, P_1$ and $P_1\, B(v,v)\, P_1$ on Z_1 belong to the "structure group" $K_1^{\mathbb{C}}$ of Z_1, and N has the semi-invariance property

$$N(\gamma z) = N(\gamma e)\, N(z) = (\mathrm{Det}\,\gamma)^{r/d_1}\, N(z)$$

for all $\gamma \in K_1^{\mathbb{C}}$ and $z \in Z_1$.

It follows that

$$N(P_1\,B(w,w)\,e) = N((P_1\,B(u,u)^{1/2}\;P_1)\,(P_1B(v,v)\,P_1)\;(P_1B(u,u)^{1/2})\,e)$$
$$= N(B(u,u)^{1/2}\,e)^2\;N(B(v,v)\,e) = h(u,u)^2\;h(v,v).$$

Since $h(w,w) = h(u,u)\;h(v,v)$ by (5.5), the assertion follows. □

Let d_1, r_1, a_1, p_1 and $d_{1/2}, r_{1/2}, a_{1/2}, p_{1/2}$ denote the respective invariants for the (irreducible) Jordan triples Z_1 and $Z_{1/2}$.

Theorem 5.1. *The integral defining $c_{\nu,\mathbf{s}}^{-1}$ is finite (i.e., $c_{\nu,\mathbf{s}} > 0$) if and only if $s+\nu > p-1$. In this case we have*

$$c_{\nu,\mathbf{s}} = \frac{\Gamma_\Omega(2s+\nu)\;\Gamma_{\Omega_{1/2}}\,(s+\nu-p+p_{1/2})}{\pi^d\;\Gamma_\Omega\,(2s+\nu-\frac{d_1}{r})\;\Gamma_{\Omega_{1/2}}\,(s+\nu-p+p_{1/2}-\frac{d_{1/2}}{r_{1/2}})}.$$

Proof. Combining (5.11), (5.5) and (5.7) we see that

$$c_{\nu,\mathbf{s}}^{-1} = \int_D dw\,h(w,w)^{s+\nu-p}\;h(w_1,w_1)^s = \int_{D_1} du\,h(u,u)^{2s+\nu+b-p}\int_{D_{1/2}} dv\,h(v,v)^{s+\nu-p}.$$

Since $p-b=p_1$ (the genus of Z_1), we have

$$\int_{D_1} du\,h(u,u)^{2s+\nu+b-p} = \pi^{d_1}\,\Gamma_\Omega\,(2s+\nu-\frac{d_1}{r})/\Gamma_\Omega(2s+\nu)$$

which is finite if and only if $2s+\nu > (r-1)\,a+1 = p_1-1$. Also,

$$\int_{D_{1/2}} dv\,h(v,v)^{s+\nu-p} = \pi^{d_{1/2}}\,\Gamma_{\Omega_{1/2}}\Big(s+\nu-p+p_{1/2}-\frac{d_{1/2}}{r_{1/2}}\Big)/\Gamma_{\Omega_{1/2}}(s+\nu-p+p_{1/2})$$

which is finite if and only if

$$s+\nu-p+p_{1/2}-\frac{d_{1/2}}{r_{1/2}} > (r_{1/2}-1)\,\frac{a_{1/2}}{2}.$$

Since $p_{1/2}-\frac{d_{1/2}}{r_{1/2}} = (r_{1/2}-1)\,\frac{a_{1/2}}{2}+1$, this is equivalent to $s+\nu > p-1$. □

Proposition 5.4. *Let $a\in D$ and $\zeta\in Z$. Then*

$$\frac{1}{c_{\beta+\nu/2}}\;\|S_\nu^{\nu+\beta}\,(K_\zeta^{\mathbf{m}})^a\|_{\mathcal{L}^1_{\beta+\nu/2}} = \int_D dz\,h(z,z)^{\beta+\nu/2-p}\,|h(z,a)^{-\beta}|\cdot K_\zeta^{\mathbf{m}}\,(B(z,z)\,\zeta)^{1/2}$$
$$= \int_D dz\,h(z,z)^{\beta+\nu/2-p}\cdot|h(z,a)^{-\beta}|\cdot\|K_\zeta^{\mathbf{m}}\circ B(z,z)^{1/2}\|_{\mathcal{F}} \tag{5.12}$$

Proof. Proposition 4.3 and (4.8) imply

$$(S_\nu^{\nu+\beta}\,(K_\zeta^{\mathbf{m}})^a)_z\circ B(z,z)^{1/2} = h(z,a)^{-\beta}\,(K_\zeta^{\mathbf{m}})^a_z\circ B(z,z)^{1/2}$$
$$= h(a,a)^{\nu/2}\;h(z,a)^{-(\beta+\nu)}\;K_\zeta^{\mathbf{m}}\circ B(a,a)^{1/2}\,B(z,a)^{-1}\,B(z,z)^{1/2}.$$

Since

$$\begin{aligned}&\langle K^{\mathbf{m}}_\zeta \circ B(a,a)^{1/2}\, B(z,a)^{-1}\, B(z,z)^{1/2}\rangle^2_{\mathcal{F}}\\ &\quad = K^{\mathbf{m}}_\zeta\,(g'_{-a}(z)\, B(z,z)\, g'_{-a}(z)^*\zeta) = K^{\mathbf{m}}_\zeta\,(B(g_a^{-1}(z),\, g_a^{-1}(z))\zeta),\end{aligned}$$

it follows that

$$\begin{aligned}&\|(S^{\nu+\beta}_\nu\,(K^{\mathbf{m}}_\zeta)^a)_z \circ B(z,z)^{1/2}\|_{\mathcal{F}}\\ &\quad = h(a,a)^{\nu/2}\,\left|h(z,a)^{-(\beta+\nu)}\right|\, K^{\mathbf{m}}_\zeta\,(B(g_a^{-1}\,(z),\, g_a^{-1}\,(z))\,\zeta)^{1/2}.\end{aligned}$$

Applying Lemma 4.1 to $x = y = 0$ yields

$$\begin{aligned}&\frac{1}{c_{\beta+\nu/2}}\ \|S^{\nu+\beta}_\nu\,(K^{\mathbf{m}}_\zeta)^a\|_{\mathcal{L}^1_{\beta+\nu/2}}\\ &= \int_D dz\, h(z,z)^{\beta+\nu/2-p}\, \|(S^{\nu+\beta}_\nu\,(K^{\mathbf{m}}_\zeta)^a)_z \circ B(z,z)^{1/2}\|_{\mathcal{F}}\\ &= h(a,a)^{\nu/2} \int_D dz\, h(z,z)^{\beta+\nu/2-p} \cdot \left|h(z,a)^{-(\beta+\nu)}\right|\\ &\qquad \cdot K^{\mathbf{m}}_\zeta\,(B(g_a^{-1}\,(z),\, g_a^{-1}\,(z))\,\zeta)^{1/2}\\ &= h(a,a)^{\nu/2}\ h(a,a)^{-\nu/2}\\ &\qquad \cdot \int_D dz\, h(z,z)^{\beta+\nu/2-p}\ h(z,a)^{-\beta/2}\ h(a,z)^{-\beta/2}\ K^{\mathbf{m}}_\zeta\,(B(z,z)\,\zeta)^{1/2}\\ &= \int_D dz\, h(z,z)^{\beta+\nu/2-p}\ \left|h(z,a)^{-\beta}\right| K^{\mathbf{m}}_\zeta\,(B(z,z)\,\zeta)^{1/2}\\ &= \int_D dz\, h(z,z)^{\beta+\nu/2-p}\ \left|h(z,a)^{-\beta}\right| \cdot \|K^{\mathbf{m}}_\zeta \circ B(z,z)^{1/2}\|_{\mathcal{F}}.\end{aligned}$$

□

Our main result in this section is

Theorem 5.2. *Let $s \in \mathbb{N}$ and ν satisfy*

$$s + \frac{\nu}{2} > \frac{a}{2}\,(r-1)$$

and

$$\frac{\nu+s}{2} > \frac{a_{1/2}}{2}\ (r_{1/2}-1) + p - p_{1/2}.$$

Let $\beta \in \mathbb{R}$ satisfy $\beta + \frac{\nu+s}{2} > p-1$. Then we have for $\Phi \in \mathcal{O}(D, \mathcal{P}_{\mathbf{s}})$

$$\Phi \in \mathfrak{M}_{\nu,\mathbf{s}} \Longleftrightarrow S^{\nu+\beta}_\nu\,\Phi \in \mathcal{L}^1_{\beta+\nu/2}.$$

Proof. Let $p_{1/2}$ be the genus of $Z_{1/2}$, and put

$$\alpha = \beta + \frac{\nu+s}{2} + p_{1/2} - p.$$

Then

$$\beta - \alpha = p - p_{1/2} - \frac{\nu+s}{2} < -\frac{a_{1/2}}{2}\ (r_{1/2}-1)$$

by assumption. This implies

$$C^{(1/2)} := \sup_{y \in D_{1/2}} {}_2F_1^{(1/2)} \begin{pmatrix} \beta/2 & \beta/2 \\ & \alpha \end{pmatrix} (y, y) < +\infty.$$

By Lemma 5.3, $B(u,u)^{1/2} B(a_1,u)^{-1}\ a_{1/2} \in D_{1/2}$ and hence

$${}_2F_1^{(1/2)} \begin{pmatrix} \beta/2 & \beta/2 \\ & \alpha \end{pmatrix} (B(u,u)^{1/2}\ B(a_1,u)^{-1}\ a_{1/2},\ B(u,u)^{1/2}\ B(a_1,u)^{-1}\ a_{1/2}) \\ \leq C^{(1/2)}$$

for all $u \in D_1$ and $a = a_1 + a_{1/2} \in D$. Now consider $\Phi = A^{g_a} = (K_e^{\mathbf{s}})^a$. Specializing Proposition 5.4 to the constant partition $\mathbf{s} = (s, \ldots, s)$ and making the change of variables $w = u + B(u,u)^{1/2}\, v$ as in (5.7), we obtain with Proposition 5.3 and Lemma 5.5.

$$\begin{aligned}
&\frac{1}{c_{\beta+\nu/2}}\ \frac{(d/r)_{\mathbf{s}}^{1/2}}{d_{\mathbf{s}}^{1/2}}\ \|S_\nu^{\nu+\beta}\, \Phi\|_{\mathcal{L}^1_{\beta+\nu/2}} \\
&\quad = \frac{(d/r)_{\mathbf{s}}^{1/2}}{d_{\mathbf{s}}^{1/2}} \int_D dw\, h(w,w)^{\beta+\nu/2-p} \left|h(w,a)^{-\beta}\right| K_e^{\mathbf{s}}\, (B(w,w)\, e)^{1/2} \\
&\quad = \int_D dw\, h(w,w)^{\beta+\nu/2-p} \left|h(w,a)^{-\beta}\right|\ h(w,w)^{s/2}\ h(w_1,w_1)^{s/2} \\
&\quad = \int_D dw\ h(w,w)^{\beta+\frac{\nu+s}{2}-p}\ h(w_1,w_1)^{s/2} \left|h(w,a)^{-\beta}\right| \\
&\quad = \int_{D_1} du\ h(u,u)^{\beta+\frac{\nu}{2}+s+b-p} \left|h(u,a_1)^{-\beta}\right| \\
&\qquad\qquad \cdot \int_{D_{1/2}} dv\ h(v,v)^{\beta+\frac{\nu+s}{2}-p} \left|h(v, B(u,u)^{1/2}\, B(a_1,u)^{-1}\, a_{1/2})^{-\beta}\right| \\
&\quad = \frac{1}{c_\alpha^{(1/2)}} \int_{D_1} du\, h(u,u)^{\beta+s+\nu/2-p_1} \left|h(u,a_1)^{-\beta}\right| {}_2F_1^{(1/2)} \begin{pmatrix} \beta/2 & \beta/2 \\ & \alpha \end{pmatrix} \\
&\qquad\qquad \cdot \left(B(u,u)^{1/2}\ B(a_1,u)^{-1}\ a_{1/2},\ B(u,u)^{1/2}\ B(a_1,u)^{-1}\ a_{1/2}\right) \\
&\quad \leq \frac{C^{(1/2)}}{c_\alpha^{(1/2)}} \int_{D_1} du\, h(u,u)^{\beta+s+\nu/2-p_1} \cdot \left|h(u,a_1)^{-\beta}\right| \\
&\quad = \frac{C^{(1/2)}}{c_{\beta+s+\nu/2}^{(1)}\, c_\alpha^{(1/2)}}\ {}_2F_1^{(1)} \begin{pmatrix} \beta/2 & \beta/2 \\ \beta+s+\nu/2 \end{pmatrix} (a_1,a_1) \\
&\quad \leq \frac{C^{(1/2)} \cdot C^{(1)}}{c_{\beta+s+\nu/2}^{(1)}\ c_\alpha^{(1/2)}},
\end{aligned}$$

where

$$C^{(1)} := \sup_{a_1 \in D_1}\ {}_2F_1^{(1)} \begin{pmatrix} \beta/2 & \beta/2 \\ \beta+s+\nu/2 \end{pmatrix} (a_1,a_1) < +\infty$$

since our assumption on the parameters implies

$$\frac{\beta}{2} + \frac{\beta}{2} - (\beta + \frac{\nu}{2} + s) = -s - \frac{\nu}{2} < -(r-1)\,\frac{a}{2}.$$

Every $\Phi \in \mathfrak{M}_{\nu,\mathbf{m}}$ has a representation

$$\Phi = \int_G d\mu(g)\; A^g$$

for a finite complex measure μ on G. Then

$$S_\nu^{\nu+\beta}\,\Phi = \int_G d\mu(g)\; S_\nu^{\nu+\beta}\,A^g$$

and the previous calculation shows

$$\begin{aligned} \|S_\nu^{\nu+\beta}\,\Phi\|_{\mathcal{L}^1_{\beta+\nu/2}} &\leq \|\mu\| \cdot \sup_{g\in G}\; \|S_\nu^{\nu+\beta}\,A^g\|_{\mathcal{L}^1_{\beta+\nu/2}} \\ &\leq \frac{d_{\mathbf{s}}^{1/2}}{(d/r)_{\mathbf{s}}^{1/2}}\;\; \frac{c_{\beta+\nu/2}}{c^{(1)}_{\beta+s+\nu/2}\; c^{(1/2)}_\alpha}\;\; C^{(1/2)}\; C^{(1)} \cdot \|\mu\|. \end{aligned}$$

It follows that $S_\nu^{\nu+\beta}\,\Phi \in \mathcal{L}^1_{\beta+\nu/2}$, as asserted. Thus we obtain the implication

$$\Phi \in \mathfrak{M}_{\nu,\mathbf{s}} \Longrightarrow S_\nu^{\nu+\beta}\,\Phi \in \mathcal{L}^1_{\beta+\nu/2}.$$

The converse implication follows from Theorem 4.2, applied to the partition $\mathbf{s} = (s, \ldots, s)$. □

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