Giuseppe Vitali: Real and Complex Analysis and Differential Geometry

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Abstract Giuseppe Vitali's mathematical output has been analysed from various points of view: his contributions to real analysis, celebrated for their importance in the development of the discipline, were accompanied by a more correct evaluation of his works of complex analysis and differential geometry, which required greater historical investigation since the language and themes of those research works underwent various successive developments, whereas the works on real analysis maintained a collocation within the classical exposition of that theory.

This article explores the figure of Vitali and his mathematical research through the aforementioned contributions and, in particular, the edition of memoirs and correspondence promoted by the *Unione Matematica Italiana*, which initiated and encouraged the analysis of his scientific biography.

Vitali's most significant output took place in the first 8 years of the twentieth century when Lebesgue's measure and integration were revolutionising the principles of the theory of functions of real variables. This period saw the emergence of some of his most important general and profound results in that field: the theorem on discontinuity points of Riemann integrable functions (1903), the theorem of the quasi-continuity of measurable functions (1905), the first example of a non-measurable set for Lebesgue measure (1905), the characterisation of absolutely continuous functions as antidervatives of Lebesgue integrable functions (1905), the covering theorem (1908). In the complex field, Vitali managed to establish fundamental topological properties for the functional spaces of holomorphic functions, among which the theorem of compacity of a family of holomorphic functions (1903–1904).

Vitali's academic career was interrupted by his employment in the secondary school and by his political and trade union commitments in the National Federation of Teachers of Mathematics (*Federazione Nazionale Insegnanti di Matematica*,

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FNISM), which brought about a reduction, and eventually a pause, in his publications.

Vitali took up his research work again with renewed vigour during the national competition for university chairs and then during his academic activity firstly at the University of Modena, then Padua and finally Bologna. In this second period, besides significant improvements to his research of the first years, his mathematical output focussed on the field of differential geometry, a discipline which in Italy was long renowned for its studies, and particularly on some leading sectors like connection spaces, absolute calculus and parallelism, projective differential geometry, and geometry of the Hilbertian space.

Vitali's connection with Bologna was at first related to his origins and formation. Born in Ravenna, Vitali spent the first 2 years of university at Bologna where he was taught by Federigo Enriques and Cesare Arzelà. Enriques commissioned him with his first publication, an article for the volume *Questioni riguardanti la geometria elementare* on the postulate of continuity. Vitali then received a scholarship for the *Scuola Normale Superiore* and later completed his university studies at Pisa under the guidance of Ulisse Dini and Luigi Bianchi.

From the end of 1902 until the end of 1904, when he was teaching in a secondary school in Voghera, Vitali returned to mix with the Bologna circles as he, at times, resided there. During his last years, after spending some time at the Universities of Modena and Padua, Vitali returned to teach at Bologna University, a role he carried out with energy and generosity but unfortunately not long enough to be able to found a school.

1 Formation and Career

Giuseppe Vitali was born in Ravenna on 24th August 1875, first of five children. His father worked in the Railway Company and his mother was a housewife. Before going to the university, he attended the Dante Alighieri Secondary School in Ravenna where he distinguished himself in mathematics, and his teacher, Giuseppe Nonni, strongly advised his father to allow him to continue his studies in this discipline.[1]

[1] A useful source for Vitali's biography is the Vitali Archive, donated by his daughter, Luisa, to the *Unione Matematica Italiana* in 1984. It included original documents of interest for his biography, and numerous letters sent to Vitali from various mathematicians, among which 100 letters judged to be the most significant, were chosen and inserted into the edition of his works on real and complex analysis: [1], which also includes a complete list of Vitali's publications. The correspondence, in particular, was edited by Maria Teresa Borgato and Luigi Pepe. Unedited letters and documents have contributed to the new biography of Giuseppe Vitali by Luigi Pepe, inserted into the volume: [1]: pp. 1–24. Specific studies on Vitali's contributions to real and complex analysis and differential geometry, in: [2, 4, 5]. On Vitali's treatises and production related to mathematics teaching see also: [6]. On Vitali's life and work see also Vitali's obituary [7]. For a general overview on the development of measure theory and integration, see: [8, 9].

So it was that Giuseppe attended the first 2 years of the degree course in Bologna University where he was taught by Federigo Enriques, who had just arrived from Rome to teach descriptive and projective geometry, and by Cesare Arzelà, who had the chair in infinitesimal analysis. Both Enriques and Arzelà recognised his ability and did their best to help him gain a scholarship. In the academic year 1897–98 he was accepted in the Scuola Normale of Pisa where, in July 1899, he graduated summa cum laude under the guidance of Luigi Bianchi. He wrote his thesis on analytic functions on Riemann surfaces. The teachers who had most influence over him were undoubtedly Ulisse Dini, who taught infinitesimal calculus, and Luigi Bianchi analytical geometry, other teachers were Eugenio Bertini for higher geometry, Gian Antonio Maggi for rational mechanics, and Cesare Finzi for algebra.

After graduating, Vitali became Dini's assistant for 2 years, during which time he took a teaching diploma with a thesis, under the guidance of Bianchi, on: "Le equazioni di Appell del 2° ordine e le loro equazioni integrali", (Appell's 2nd order equations and their integral equations) which was later published in a first work in 1902, and later in a more in-depth work in 1903.

In 1902, Vitali left university since he had accepted a position as a mathematics teacher in a secondary school (better paid than the post of assistant at university), first in Sassari and then in Voghera (until 1904) and finally in the "C. Colombo" Secondary School of Genoa. At that time it was not unusual for a person to begin a university career with a period of secondary school teaching (other examples are Luigi Cremona and Cesare Arzelà), but it was a great pity that it took the university up till 1922 to concede full research activity to such talent like that of Vitali.

Vitali, however, kept in contact with his mentors, and Arzelà in particular remained a constant point of reference for him, through exchange of letters and also meetings above all during the holiday periods (Vitali returned to Bologna during his teaching years in Voghera, and Arzelà, in his turn, spent long periods in Santo Stefano Magra in Liguria). He always maintained a friendship with Enriques, who had followed his transfer from the University of Bologna to the Scuola Normale Superiore of Pisa, but his lifelong friend was Guido Fubini, who had been a fellow student in Pisa.

In these 20 years mild attempts were made to procure him a position, including a post as temporary lecturer (there were offers by Dini of a free course in 1907, an offer from Mario Pieri of a temporary position in Parma, a post teaching infinitesimal analysis in Genoa for the academic year 1917–1918) but when it came to making the decision a preference was shown for younger scholars mostly within the university in consolidated areas of research.

Vitali's biography [10] provides an excellent description of the academic situation and the widespread opinion which the university establishment held of Vitali's research and the new real analysis in general; the fruits of the review of the foundations of analysis initiated by Dini and his school were reaped above all in the field of integral equations thanks to Vito Volterra. It was along these lines that Leonida Tonelli found great success and his results gave ample space for application. In contrast, Vitali's problems appeared very limited to the discipline, above all those on the "groups of points", and only his works on integrability and the convergence of series, which fell within traditional interest, aroused general attention. On the other hand, also Lebesgue's research was initially received with some diffidence, so much so that his famous thesis was published in Italy not in France. Only when one considers that Italy and France were the countries in which this type of research was given most space, in an international setting of relative indifference, do we realise the initial difficulties in earning justified recognition. Lebesgue, in fact, reminds us that the new discipline had not yet received either definition or application: "Jusqu'à ces dernières temps, la plus part des travaux sur les fontions réelles, ceux concernant les séries trigonométriques exceptés, se réduisaient à des remarques, parfois très élégantes, mais sans lien, ne formant nul corps de doctrine et n'ayant servi pratiquement à rien... On en était encore à la phase d'observation: on explorait l'amas désordonné des fonctions pour y découvrir des catégories intéressantes, mais, comme on ne savait pas expérimenter sur les fonctions, c'est-à-dire calculer avec elles, s'en servir, on manquait totalement de critères pour juger qu'une catégorie était intéressante." In spite of this, a change of mind was swift in occurring abroad: Lebesgue's integral became part of courses on calculus as early as 1910 and Vitali was frequently cited in works by Montel and De La Vallée Poussin (1912, 1916), whereas in Italy Vitali continued to be excluded from academic research...

Let me quote a well-known passage from Giovanni Sansone' repertoire on famous mathematicians graduated from the Scuola Normale Superiore in the years 1860–1929. Sansone related that Vito Volterra, perhaps in 1922, had occasion to meet Lebesgue in Paris. Lebesgue asked Volterra for news about Vitali and when he heard that Vitali was teaching mathematics in secondary school in Genoa, surprised, he said: "I am pleased that Italy has the possibility to maintain mathematicians like Vitali in secondary schools" [11].

Finally in 1922, Vitali was one of the successful applicants of a national call for a university chair and on that occasion he had begun to publish once more: six works between the end of 1921 and 1922. Vitali came second, preceded by Gustavo Sannia (the Examining Board consisted of Guido Fubini, Tullio Levi-Civita, Salvatore Pincherle, Leonida Tonelli, and Gabriele Torelli. Only Fubini and Levi-Civita favoured Vitali for first place). Since Sannia did not take up the position, Vitali was called to the University of Modena where he remained for 3 years, then in the December of 1925, he transferred to the University of Padua, where he took up the chair in mathematical analysis which had previously been occupied by Gregorio Ricci Curbastro. His teaching, research and organisational activities over these 5 years in Padua were intense, in spite of the partial paralysis which afflicted him down the right side of his body in 1925. He explored new areas of research with various young students, founded the *Seminario matematico* of the University of Padua and was elected as Director: the aim of this initiative was to promote interest in mathematics in Padua by organising activities such as general conferences, elementary preparatory lessons for the teaching of mathematics, and seminars regarding ongoing research. The journal connected to this, the *Rendiconti*, founded in 1930, still continues publishing as the journal of the University of Padua. In 1928, Vitali was asked to return to Pisa to take up the chair previously occupied by Luigi Bianchi, but he refused. In order to remain close to his mother, he accepted the offer of transfer to Bologna in 1930, for the chair of infinitesimal calculus, which Tonelli was leaving in order to go to Pisa. Even in this brief time spent in Bologna he devoted himself with enthusiasm to his academic commitments until his sudden death under the portico of the city as he was going home from the university with Ettore Bortolotti after his lessons and speaking on recent mathematical papers.

2 First Research Works in Complex Analysis

Both in Bologna and Pisa, not surprisingly, Vitali's mathematical research was initially guided by his teachers, who influenced him in different ways. To Enriques, Vitali owes the focus on the foundations and didactics of mathematics. Enriques commissioned Vitali to write a chapter on the postulate of continuity for the volume: *Questioni riguardanti la geometria elementare*, of formative and cultural interest to teachers of mathematics, which was also his first published work [12].

Moreover, Vitali may have considered research themes in algebraic geometry: Enriques was developing his theory of algebraic curves by means of projective models which was to allow him a complete classification, and there are hints, in Enriques' correspondence with Vitali, of a few ideas or, rather, research plans, on algebraic varieties with elliptic section curves of higher dimensions, which were never developed.

The first two memoirs that Vitali published in 1900, in the same volume of the journal of the *Circolo Matematico* of Palermo, take their origin from his degree thesis and so the line of research carried out by Bianchi. The first one contains an extension of Mittag-Leffler's theorem to holomorphic functions with given singularities on Riemann surfaces [13], the second one, a property of holomorfic functions of one variable for which the limit of the n-derivatives at a regular point, exists as n goes to infinity [14].

Other important memoirs of 1902 and 1903 with the same title are linked to the teaching diploma thesis, thus still under the influence of Bianchi and on complex analysis [15, 16]: *On linear homogeneous differential equations with algebraic coefficients* where Abelian integrals and Abelian functions are applied to a class of fuchsian differential equations introduced by Appel [17] and called by Vitali "Appell equations" (with singular points in the Fuchs class). After a classical part derived from the Appell-Goursat treatise, Vitali concentrates on second order equations, and even if his results do not give a conclusion to the topic, they are still nowadays considered to be of interest and value, as the questions treated at that time, and then abandoned, by Appell and Vitali have had a successive revival.[2]

[2]Problems closely linked to those of Vitali were treated by André Weil: [18], and more recently in [137] much space is also given to differential equations of the 2nd order in the book: [138]. A description of this research work by Vitali, using the sheaf-theoretical notations, and its collocation in the research of his day, in [5]: 376–377.

Vitali's later research took a different direction, under the influence of Cesare Arzelà, who, in 1899 and 1900, had published two extensive memoirs [19], in which he gave a systematic exposition of his results in the field of the theory of functions: on the sum of a series of continuous functions, on integrability of series, on developability in Fourier series, giving them the prominence they deserved [20]. Arzelà continued his research in the memoir "on the series of analytic functions" [21] in 1903 and, in the same year, over a short space of time, following Arzelà's research and using his techniques, Vitali published, under the same title, three memoirs that demonstrated four original theorems which in themselves were sufficient to guarantee him international renown and a place in the history of mathematical analysis [22–24].

The question, on which also Arzelà, Osgood and Montel worked, was to find the minimal sufficient conditions for a series of analytic functions in some range of the complex plane to converge to a holomorphic function. In the first memoir, using Arzelà's techniques, Vitali extended a theorem by Osgood and established that a sequence of holomorphic functions, uniformly bounded on a connected open set T and convergent at each point of a subset of T, with a cluster point in T, will converge uniformly, on compact subsets of T, to a holomorphic function.

The most famous result that Vitali obtained was the theorem of compacity, later taken up and generalised by Montel, in which, from any uniformly bounded sequence of holomorphic functions (on a connected open set), one can extract a convergent subsequence to a holomorphic function. Vitali, moreover, discovered another more important theorem (called "tautness theorem" by A. Vaz Ferreira),[3] even if his demonstration is incomplete, according to which a pointwise convergent sequence of holomorphic functions which omit two values is normally convergent.[4] In the last of the three memoirs, Vitali arrives at the conditions necessary and sufficient for the convergence of a series of holomorphic functions.

3 Research on Real Analysis

Still today the name of Giuseppe Vitali is mainly linked to his first research works on real analysis, whose basic topological concepts were not immediately successful due to the innovation in approach and language. The fact that his work fell within a line of research that had not yet attained complete international recognition may also account for the delay in his academic career.

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[3] For a detailed analysis of Vitali's results on this topic, Vitali's priority with respect to Montel, Arzelà and Osgood, and the attribution of the discovery of these theorems in following works (Carathéodory-Landau) and treatises see: [5]: 386–391.

[4] "Se $u_1, u_2, \ldots$ è una successione di funzioni analitiche finite e monodrome convergenti in ogni punto di un campo semplicemente connesso C, e se inoltre le funzioni suddette non assumono mai i valori $0, 1$ la successione converge verso una funzione analitica finita e monodroma in C."

His work was linked to the research into the theory of measure and integration which, in France, was being developed by Borel and Lebesgue. As for the Italian tradition, Ulisse Dini's work constituted the reference point, in particular the volume entitled *Fondamenti per la teorica delle funzioni di variabili reali* [25]. In it the basic concepts of the theory of functions of a real variable show organisation and precision, and provide a coherent and comprehensible structure to the entire theory which contains the preceding or contemporary research carried out by Weierstrass, Schwarz, Heine, Cantor, Du Boys-Reymond, Hankel, and Riemann as well as his own. Dini's *Fondamenti* were translated into German by J. Lüroth and A. Schepp, and then even 14 years after the Italian edition, as the editors stated, it was still considered to be "the only book of modern theory of a real variable" [26].

The volume includes the following topics: Dedekind's theory on real numbers, derived set and set of first and second category, least upper bound and greatest lower bound, the theory of the limits of sequences, continuous and various types of discontinuous functions (Weierstrass), uniform continuity (Cantor, Schwarz), pointwise discontinuous and totally discontinuous functions (Hankel), derivatives and derivability, intermediate value theorem (correctly demonstrated), types of discontinuity of the derivate function, series and series of functions, uniform convergence and its applications to continuity, differentiability, and integrability, Hankel's principle of condensation of singularities, continuous everywhere but nowhere derivable functions, functions of bounded variation, oscillation of a function, etc. Then in the section on finite integrals Dini introduced Riemann's integral and interesting criteria of integrability, correcting Hankel's condition, and based on the concept of negligible set. Continuing this line of research, one of his pupils, Giulio Ascoli published, in 1875, a work which gave a demonstration of a necessary and sufficient condition equivalent to Riemann integrability, based on the concept of oscillation intended as a point function. Vito Volterra, in his turn, in a work of 1881, presented an example of a nowhere dense not negligible set, to complete Dini's discussion, as well as new criteria of integrability through use of jump function.

The research carried out by Dini and his school formed the basis of successive development in France by Borel, Baire, Lebesgue and Fréchet. It is to be remembered that Baire was in Italy in 1898 when he worked with Volterra, and his thesis ("Sur les fonctions de variables réelles") was published in the *Annali di matematica pura ed applicata* of which Dini was the editor. Lebesgue's thesis, too, ("Integral, longueur, aire") was published in the *Annali di matematica* in 1902.

At the beginning of the twentieth century, after the premature death of Ascoli and Volterra who was particularly engaged in the field of integral equations, among Dini's pupils there remained Cesare Arzelà who could carry forth research in the field of real and complex analysis, and the *Fondamenti* by Dini as well as the long memoir *Sulle serie di funzioni* by Arzelà constituted the reference texts of the young Vitali.

Many of the most important results for which Vitali is still remembered, derived from the period between 1903 and 1908, after he had left his position as assistant in Pisa for the better paid job of secondary school teacher in Sassari and then in Voghera. Until his transfer to the Liceo Colombo in Genoa in 1904, Vitali continued to live in Bologna during the summer and Christmas holidays, as can be seen from his correspondence at that time.

It was during those years that the legacy left by Dini and Arzelà tied up with the new research that Borel and Lebesgue were carrying out on the theory of measure; in 1898 Borel had provided a vast class of measurable sets [27] and, in 1901, Lebesgue had come up with the definition of the integral which carries his name. In 1902 [28], in his famous doctoral thesis, Lebesgue had developed the properties of the measure and observed that the points of discontinuity of a Riemann integrable function form a set of measure zero. Vitali arrived on his own at the same concept of measure in 1903, and, in 1904,[5] demonstrated that the necessary and sufficient condition for a function to be Riemann-integrable was that its set of points of discontinuity is of measure zero, in a sense that coincided with Lebegue's, thus releasing the integrability from the behaviour of the function in these points. The same year Lebesgue also demonstrated the sufficiency of the condition.

In order to clarify the interconnection of these results we can examine Vitali's works in particular. In the first one, starting from one of Osgood's theorems, redemonstrated in an elementary form, Vitali provided his criterion for *Riemann integrabiliy*, that is that the *minimal extension* ("estensione minima") of the points of discontinuity must be equal to zero,[6] which he reformulated and clarified in the second note, in which he also observed that countable sets had minimal extension zero and that minimal extension coincides, for closed sets and only on these, with the *Inhalt* of Cantor. In the third note, Vitali, inspired by Borel [27], defined the measurable sets and studied the properties that make the *minimal extension* a measure for measurable linear sets of points: finite and countable additivity, that the family of measurable sets is closed under the operations of countable union, and intersection, comparing it with the measures of Jordan[7] and Borel (he demonstrated that both Jordan's and Borel's families of measurable sets were extended). While this memoir was being published, (it was presented on the 5th November 1803), Pincherle brought to the attention of Vitali Lebesgue's work of 1901, in which "mention is made to the concepts treated herein... for the construction of integrals of derived functions which are not integrable" ([31] p. 126).

It may be said that Vitali's *minimal extension* coincides with Lebesgue outer measure, but while he also introduced the inner measure by means of closed sets, and defined measurable sets as those for which inner and outer measure coincide, Vitali did not introduce an inner measure and defined measurable sets by means of a scalar function ("allacciamento" - *linking*) which links a set E of an interval (a, b)

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[5]Three of Vitali's memoirs refer to Riemann's integrability of a function in relation to the set of his points of discontinuity: [29–31].

[6]The *minimal extension* of a linear set of points in Vitali's theory is the least upper bound of the series of measures of (finite or countable) families of pairwise disjoint open intervals, which covers the set.

[7]One should say: "Peano-Jordan", see: [32]. See also Peano's letter of 29th February 1904 to Vitali, in which Peano compares the *minimal extension* with his outer measure.

to its complement E^*: $Z(E, E^*) = (\text{mis}_e E + \text{mis}_e E^*) - (b - a)$. Those sets for which the linking is zero are defined measurable.[8]

In 1905, Vitali published as many as seven articles, as well as two more up to the end of 1908, most of them originated from problems connected to the theory of measure and the integral of Lebesgue: the existence or otherwise of non measurable sets, the characterisation of measurable functions, the characterisation of the integrals of summable functions (Lebesgue integrable), the extension of the fundamental theorem of calculus to Lebesgue integrals, integration of series term by term, integrability on unlimited intervals, the extension to functions of two or more variables.

A brief note of 1905 [33] is famous as it contains the first example of non Lebesgue measurable set, usually cited in every critical presentation of measure theory. The counter-example is constructed using the axiom of choice in the form of the well-ordering theorem, which Vitali immediately accepted, whereas the mathematical milieu was divided over the matter. In Italy, for example, Tonelli wanted to work out a version of the calculus of variations independent from it. The fact that Vitali's counter-example was not inserted into a journal but was published separately is also significant. At the end of the note, Vitali concluded that, in any case, the possibility of measuring the sets of points of a straight line and that of well ordering the continuum could not coexist.[9] Vitali's statement is not, however, justified for the inverse part, or rather that, by denying the axiom of choice every set would be measurable. Several later research works refer to this problem, among which those of F. Hausdorff and W. Sierpiński.[10]

Naturally, the problem of extending the fundamental theorem of calculus involved functions of bounded variation. Lebesgue had demonstrated that the indefinite integral of a summable function has as a derivative this function except for a zero measure set ([35] pp. 123–125). Vitali introduced the concept of absolute continuity, and therefore demonstrated that the absolute continuity is the necessary and sufficient condition so that a function is the indefinite integral of a summable function [36]. In the same memoir, Vitali also provided the first example of a continuous function of bounded variation, but not absolutely continuous.[11] This work was contested by Legesgue who claimed authorship of various theorems in two letters of 16th and 18th February 1807 to Vitali[12] and pointed out his works were not adequately quoted in Vitali's memoirs. On the other hand, he admitted

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[8] A description of some of Vitali's works on real analysis of the first period in [3].

[9] "la possibilità del problema della misura dei gruppi di punti di una retta e quella di bene ordinare il continuo non possono coesistere".

[10] In more recent times, to this question is related the famous result [34].

[11] It corresponds to the function called *Cantor function* or *Devil's staircase*. Vitali presented this counterexample, with some more details, also in: [37].

[12] [1], pp. 457–462. Lebesgue also claimed authorship of the demonstration that the set of points in which a continuous function has a finite derived number is measurable, that Beppo Levi had criticised and that Vitali had redemonstrated. See [38, 39].

he had not read many of Vitali's works before Vitali sent them to him because "ici à Poitiers nous n'avons aucun periodique italien ainsi je ne connaissais aucun de vos notes (sauf "Sui gruppi di punti..." que Borel m'avait communiqué)". In reality, Lebesgue had only indicated[13] the result which was completely demonstrated by Vitali in this work of 1905, and was then redemonstrated by Lebesgue in 1907 [40] and once again by Vitali in 1908 with a different method, which could also be extended to functions of more than one variable (multiple integrals) [41]. Here Vitali gave, for the first time, a definition of bounded variation for functions of two variables, obtained from the variations of a function in four vertices of a rectangle.[14] Vitali adopted the following formula:

$$F(X,Y) = \int \int_{\rho} f(x,y)\mathrm{d}x\,\mathrm{d}y - F(0,0) + F(X,0) + F(0,Y)$$

to define an indefinite integral, where f is a summable function, and ρ the rectangle with vertices $(0,0),(X,Y),(X,0),(0,Y)$. The incremental ratio of any function $f(x,y)$:

$$\frac{f(x+h,y+k)+f(x,y)-f(x+h,y)-f(x,y+k)}{hk}$$

is used to define derived numbers, and its numerator to define functions of bounded variation and absolutely continuous functions.

Vitali's priority and contribution was then recognized by Lebesgue in many points of his *Notice* [45], in particular, with reference to the fundamental theorem of calculus ("proposition... la plus feconde" according to Lebesgue) Lebesgue says: "Toute intégrale indéfinie est continue et à variation bornée, mais la réciproque n'est pas vraie. Pour qu'une fonction soit une intégrale indéfinie, il faut que, de plus, la somme des valeurs absolues de ses accroissements dans des intervalles extérieurs les uns aux autres et de mesure ε, tende vers zéro avec ε. On dit alors, avec M. Vitali, qui à été le premier à publier une démonstration de cet énoncé que j'avais formulé, que la fonction est *absolument continue*... C'est aussi M. Vitali qui a publié, le premier, des résultats sur la dérivation des intégrales indéfinies des plusieurs variables".

Linked to the fundamental theorem of the calculus is also another memoir [46] where it was shown that every summable function is "of null integral" (term coined by Dini), that is f is such that $\int_a^x f(t)\,dt = 0$ for every x in (a,b), if and only if $f = 0$ almost everywhere. Lebesgue had demonstrated the theorem for bounded functions, whereas Vitali's demonstration (of the non trivial part, the necessary one) extended to the unbounded functions by using a covering theorem of Lindelöf [47]. In another memoir Vitali dealt with the extension of integrability to unbounded

[13] [35] see footnote p. 129.

[14] Vitali's priority was also noted by E.W. Hobson [42]. Another definition was given by Lebesgue in 1910 in: [43] p. 364. See: [44].

intervals [48] in which he provided a characterisation of integrable functions in terms of their behaviour at infinity.

In his thesis of 1899, René-Louis Baire had provided the well-known classification of functions [49, 50], but Borel and Lebesgue had managed to construct functions that had escaped this classification. The problem of classifying Baire's functions arose.[15] In 1905 Vitali and Lebesgue demonstrated that all and only the functions of Baire are Borel measurable [52–54]. The same year, however, Vitali also proved that every Borel measurable function can be decomposed in the sum of a Baire function of first or second class, and a function equal to zero almost everywhere [55]. In the same memoir the so-called Luzin's theorem was also demonstrated [56], by which if a function f is finite and measurable on an interval (a, b) of length l, for every ε there exists a closed set in which f is continuous and whose measure is greater than $l - \varepsilon$. The theorem had been indicated by Borel and Lebesgue,[16]; it inspired various analysts in their research into new definitions of integral, in particular we may remember Leonida Tonelli, who posed the functions he called "quasi-continuous" at the basis of his definition of integral [59].

An important problem in relation to the theory of functions of real variables concerns term by term integration of series, which Vitali dealt with in two memoirs [60, 61]. In the first one, he generalised some results of Lebesgue, Borel and Arzelà managing to characterise term by term integrable series under suitable conditions. In the second more important memoir of 1907, Vitali considered the integration extended not only over an interval, but also to any measureable set and, to this purpose, introduced the concept of equi-absolute continuity of a sequence of functions and complete integrability of series.[17] He, therefore, provides the characterisation of term by term integrable series on the basis of the equi-absolute continuity of the integrals of its partial sums. Complete integrability of series implies term by term integrability of series in the ordinary sense but not vice versa unless the partial sums are positive; Vitali, also derived theorems demonstrated by Beppo Levi in 1906 [62] on positive term series, and, therefore, also the necessary and sufficient

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[15] See, for example, Note II of Lebesgue on the functions of class 1: "Démonstration d'un théorème de M. Baire", in: [51]:149–155 and Note III of Borel "Sur l'existence des fonctions de classe quelconque" [51]: 156–158 "On peut se demander si la classification de M. Baire n'est pas purement idéale, c'est-à-dire *s'il existe* effectivement des fonctions dans les diverses classes définies par M. Baire. Il est claire, en effet, que si l'on prouvait, par exemple, que toutes les fonctions sont de classe $0, 1, 2$ ou 3, la plus grande partie de la classification de M. Baire serait sans intérêt. Nous allons voir qu'il n'en rien:. . . Le raisonnement précédent ne permet pas d'exclure l'hypothèse où un théorème tel que le suivant serait exact: *toute fonction effectivement définie est nécessairement de classe* $0, 1, 2$ *ou* 3. Nous allons, au contraire, montrer qu'il est possible de définir effectivement une fonction dont la classe dépasse un nombre donné d'avance".

[16] [57, 58], [35] footnote p. 125.

[17] The equi-absolute continuity corresponds to the uniformity, in relaion to the family of functions, of the condition of absolute continuity of functions. Complete integrability of series extends, to measurabe sets, the ordinary concept of integrability of series (or rather when on every measurable subset Γ of G (measurable) the series of integrals and the integral of the series exist and can be exchanged: $\sum_{n=1}^{\infty} \int_{\Gamma} u_n(x)\,\mathrm{d}x = \int_{\Gamma} \sum_{n=1}^{\infty} u_n(x)\,\mathrm{d}x$).

condition for integrability of series over an interval by generalising the result of 1905. Vitali's characterisation turned out to be one of the basic results of the measure theory, extended by Hahn (1922), Nikodym (1931), Saks (1933), Dieudonné (1951) and Grothendieck (1953), and today a great deal of reformulations are inserted into general measure theory.

The results that Vitali produced in the first period are crowned by the so-called Vitali covering theorem, that Arnaud Denjoy evaluated 50 years later as one of the most important theorems in measure theory of Euclidean spaces.[18]

The covering theorem was demonstrated by Vitali as an intermediate result in a memoir written at the end of 1907[19] and was enunciated firstly for the points of the real straight line, and later extended to the plane and then, by analogy but without enunciating explicitly, to higher dimensions:

"Se Σ è un gruppo di segmenti, il cui nucleo abbia misura finita m_1, esiste un gruppo finito o numerabile di segmenti di Σ a due a due distinti, le cui lunghezze hanno una somma non minore di m_1."

This theorem is preceded by another one, known as the Vitali covering lemma:

"Se Σ è un gruppo di segmenti [quadrati] il cui corpo abbia misura finita μ e se ε è un numero maggiore di zero, esiste un numero finito di segmenti [quadrati] di Σ a due a due distinti, le cui lunghezze hanno una somma maggiore di $\mu/3-\varepsilon$ [$\mu/9-\varepsilon$]." As early as the note of 1904 "Sulla integrabilità delle funzioni", Vitali had provided a generalised version of the covering theorem known as the Heine – Pincherle – Borel theorem,[20] but the reach of this new covering theorem is much vaster; the aim is to cover, up to a measure zero set, a given set E by a disjoint sub-collection extracted from a *Vitali covering* for E: a *Vitali covering* ζ for E is a collection of sets such that, for every $x \in E$ and $\delta > 0$, there is a set U in the collection ζ such that $x \in U$ and the diameter of U is non-zero and less than δ. In the original version of Vitali the collection ζ was composed of intervals, squares, cubes, and so on.

The lemma and theorem of Vitali have been extended to other measures besides that of Lebesgue, and to more general spaces. Mention is to be made of the formulation which Costantin Carathéodory [66] gave a few years later, as well as Stefan Banach's extension, in a fundamental memoir in 1924 [67].

In the second chapter of the memoir, Vitali himself provided interesting applications of the covering theorem, on derived numbers of functions of bounded variation and on the integrals of summable functions, extending many of the results of his previous memoirs to functions of two variables, among which we may recall[21]:
The set of points in which a derived number of a continuous function is finite is measurable.

[18] A. Denjoy devoted several notes to Vitali's covering theorem and its generalizations: [63–65].

[19] [41]. Presented at the Academy meeting of 22nd December 1907.

[20] [30] p. 71: "Se si ha un'infinità numerabile di intervalli, tali che ogni punto di un gruppo lineare chiuso P sia dentro ad uno di essi, esiste un numero limitato di intervalli scelti tra gli intervalli dati e aventi la stessa proprietà."

[21] I have sometimes replaced the condition "with the exception of a measure zero set" with the modern terminology "almost everywhere".

The set of points in which a derived number of a continuous function of bounded variation is not finite is of measure zero.
A derived number of a function of bounded variation is summable and if the function is absolutely continuous the integral of the derived number coincides with the function with the exception of a set of measure zero.
A function of null integral is equal to zero with the exception of a set of measure zero.
Two functions with the same integral are equal almost everywhere.
Two derived numbers of the same absolutely continuous function are equal almost everywhere.
A summable function and a derived number of its integral are equal almost everywhere.

4 The Years of Secondary School Teaching

In the first years of his secondary school teaching Vitali managed to continue his research work which may be considered his best production thirteen: papers, seven of which in 1905 alone, the year he was transferred to Genoa and began a family. In Genoa, he taught at the "C. Colombo" Secondary School up to 1922. In 1906, he was promoted to a tenured position. Very soon after becoming a teacher he became involved in politics and the Trade Unions.

This was a period in which the Italian school system underwent many attempts at reform and experimentation which involved teachers in heated debates within the *Società Mathesis*, founded in 1895, and the *Federazione Nazionale Insegnanti Scuola Media* (FNISM) (National federation of Middle School Teachers), founded in 1902 by Giuseppe Kirner and Gaetano Salvemini.

In 1906, the Royal Commission for secondary schools issued a detailed questionnaire to the teachers in view of a reform of state education, which aimed to investigate: 1. the results and defects of the present system, 2. the type of system desired. 3. the value to be attributed to the final exams. Numerous documents and proposals were produced, above all, concerning the teaching of mathematics, and a Bill of Law was presented in 1908 which provided for a single middle school and the institution of a "liceo moderno".[22] The debate on the teaching of mathematics was still a European issue: the CIEM (Commission Internationale de l'Einseignement Mathématique) was also set up in 1908 with Felix Klein as its President.

Vitali was strongly committed to the FNISM and held the position of president of the association of Genoa and its province from 1908 to 1922. He supported positions

[22]On mathematics teaching in Italy, from Political Unification to Gentile Reform (1932) see [68]. The "liceo moderno", was a new type of secondary school, which did not substitute the classical one, with more foreign languages and a scientific vocation.

opposite to those of the idealists and Catholics led by Giovanni Gentile. Vitali was also elected town councillor of Genoa from 1910 to 1914.

From 1908 on Vitali's mathematical output was fairly scarce and up to 1921 he published only four short works: one on the mean value property of harmonic functions, already demonstrated in less general hypotheses by Eugenio Elia Levi, Leonida Tonelli and Vito Volterra [69], two on a generalization of Rolle's theorem to additive set functions, linked to contemporary works of Guido Fubini [70, 71], and a didactic note [72].

5 Between the Two Wars: Differential Geometry

The mathematical production of Vitali, which had considerably slowed down owing to his commitments in the secondary school, especially in concomitance with the first world war, was now taken up again with renewed energy on the occasion of examinations for chairs and successive academic activity.

Right from the very first works two trends in his scientific research may be distinguished since the works on real analysis are accompanied with those on absolute calculus and differential geometry.

The works on real analysis, about ten in all, did not achieve the important results which typified those of his youth, but were, nonetheless, carried out with the great insight, elegance and formal agility which constituted the hallmark of his works.

In the first memoir of 1921 [73], Vitali demonstrated how the condition of "closure" of an orthonormal system of square summable functions (in modern terms it corresponds to "completeness" namely, that an orthogonal function to the system is equal to zero almost everywhere) may be verified on the continuous functions alone, a result that turned out to be "precious for applications".[23] A brief note was linked to the research of Tonelli [59] in which the representation by means of absolutely continuous functions of a continuous rectifiable curve is made without resorting to the preventive rectification of the arc [74]. Some interest arose from the demonstration of the equivalence between the new definition of integral provided by Beppo Levi and that of Lebesgue [75]: Beppo Levi had already observed the coincidence for Lebesgue measurable functions [76] and Vitali concluded the matter by demonstrating that every limited function which is Beppo Levi integrable is Lebesgue measurable. Following Vitali's observations, Beppo Levi once more turned his attention to the matter.

The most interesting works in this group of Vitali's memoirs, however, concern the analysis of functions of bounded variation (now BV functions), and are those most closely linked to the research he carried out in his younger days. In the first one [37] Vitali showed that a continuous BV function may be divided into the sum of two functions: the absolutely continuous part, which is represented as

[23] [11] p. 41.

an integral, and the singular part (the "scarto") whose derivative vanishes almost everywhere, similarly to that which occurs for the functions of bounded variation that are the sum of a continuous function, and one that absorbs the discontinuities (the "saltus function" or "jump function"). A similar decomposition had already been obtained by De La Vallée Poussin in 1909,[24] and used by Fréchet in his Stieltjes integral [78], but the representation of the "scarto" given by Vitali as the sum of a countable infinity of elementary differences was considered "nouvelle et intéressante",[25] allowing classification of functions in accordance with the "scarto" (equal to zero, greater than zero, infinite) into: absolutely continuous, bounded variation, and infinite variation. The relation between his result and De La Vallée Poussin's theorem, is clarified by Vitali himself in a short note [79].

The second memoir [80] displays an even more incisive result, demonstrating a characteristic property of BV functions, thereby introducing an equivalent definition of total variation which may be extended to functions of more variables, different from the one proposed years before [41], with a view to establishing results for surfaces similar to those already obtained by Tonelli for the rectification of curves [81, 82]. It was demonstrated that if f is a continuous function of (a, b) on (c, d) and Γ_r is the set of points of (c, d) that f assumes at least r times, the condition necessary and sufficient in order that f is of bounded variation, is that the series of measures of Γ_r is convergent and in any case this series coincides with the total variation of f:

$$V = \sum_{r=1}^{\infty} \mu(\Gamma_r)$$

Moreover, if G_∞ is the subset of values in (c, d) that f assumes infinite times, and G_r represents the set of values assumed exactly r times, the necessary and sufficient condition so that f is of bounded variation is that G_∞ has zero measure and the series

$$\sum_{r=1}^{\infty} r\, \mu(G_r)$$

is convergent: in this case the sum of this series equals the total variation of f.

In his own studies, Stefan Banach had obtained similar results with different procedures [84]. This memoir is connected to other research works of the Polish School as Waclaw Sierpiński, Stefan Mazurkiewicz and the Russian, Nikolai Luzin, were, at the same time, studying the measurability and cardinality of the set of values that a continuous function assumes a number of times equal to an assigned cardinal.[26] It is to be remembered that, in 1924, from one of Banach's articles he had summarised for the *Bollettino UMI* [86], Vitali drew inspiration for an

[24] [77] I: 277, Vitali himself had recognised the priority of De La Vallée Poussin in a letter addressed to Fréchet (Vitali Archive, Bologna, 1-V-23).

[25] See two letters from Fréchet to Vitali of 30th March and 4th May 1923, in [1]: 483–485.

[26] [85]. See the letter of 3rd October 1942 from Sierpiński to Vitali in [1]: 490–491.

interesting note of measure theory [87]. In the same year, Banach had also published a simplified demonstration of Vitali's covering theorem [67].

Relationships with the Polish researchers were frequent and imbued with esteem; Vitali corresponded not only with Sierpiński, but also with Otton Nikodym, with whom he discussed questions of projective geometry (collineations of the complex projective plane, subsets of the complex projective plane which intersect each line in a certain number of points)[27] and whom he had occasion to meet at the International Congress of Bologna in 1829 (Nikodym held a lecture on the principles in local reasoning of classical analysis). Later, Vitali became a member (presented by Nikodym and his wife) of the *Société Polonaise de Mathématique* and published some of his memoirs in the *Annales de la Société Polonaise de Math.* as well as in *Fundamenta Mathematicae* [80, 88, 89]. At the *Congrès des Mathématiciens des Pays Slaves*, held in Warsaw in 1829, Vitali gave a speech on the definitions of measurable sets and summable functions which he had used to introduce Lebesgue integral in his treatise *Geometria nello spazio hilbertiano* [90] to include, from the beginning, non limited sets and functions [91].

Notwithsatanding this, Vitali's output on real analysis in the interim period between the two wars remains an interesting complement to the research works of his first years (a new demonstration of the Lebesgue-Vitali theorem is to be remembered [92]) whereas his work on differential geometry, a discipline with a strong tradition in Italy, provides much wider and more significant results as it became his main field of investigation.

Even if studies on differential geometry were well established in Italy by the middle of the nineteenth century, there is no doubt that this line of research was boosted by the works of Eugenio Beltrami in the years 1864–1884, after he had entered into contact with Betti and Riemann in Pisa. From then on differential geometry in Italy went through a period of a frenetic development which lasted until the beginning of the Second World War. Several basic techniques of the discipline were honed by Luigi Bianchi, Gregorio Ricci-Curbastro and Tullio Levi-Civita between 1880 and 1920, and later the applications achieved in the field of theoretical physics and, above all, in theory of relativity were so successful that a wide range of researchers were drawn to these studies, so much so that no mathematician of a certain level in Italy had not been involved in the study of differential geometry at some time in the years from 1920 to 1940. Thus alongside the specialists in the discipline, like Enrico Bompiani and Enea Bortolotti, questions of differential geometry were also being tackled by analysts, algebraic geometers, and mathematical physicists such as Pia Nalli, Guido Fubini, Giuseppe Vitali, Francesco Severi, Beniamino Segre, Enrico Fermi and many more. As pointed out in a previous paper, works of a certain importance alone, involving differential geometry produced in Italy from 1880 to 1940 numbered several hundreds.[28] These may be divided into various groups:

[27] See the letter of 11th May 1928 from Nikodym to Vitali, in [1]: 494–496.

[28] See [4] p. 49.

– Connection spaces and relative absolute calculus and parallelism, differential geometry of immersed varieties)
– Projective differential geometry
– Theory of surfaces, questions of applicability, transformations and deformations
– Geometric variational problems

Giuseppe Vitali worked mainly in the first two fields, publishing about thirty works from 1922 to 1932. Some of his most important contributions include the discovery of a covariant derivative associated with n covariant systems of first order [93, 94], later named Weitzenböck-Vitali [95–98] also used by Einstein in one of his works in 1928 [99]. The Weitzenböck–Vitali parallel transport was later characterised by Enea Bortolotti as the most general integrable Euclidean transport [100, 101]. Mention must also be made of the generalised absolute calculus, introduced by Vitali as a generalisation of the Ricci calculus [90, 102, 103] as well as the geometric applications made by Vitali himself [88, 89, 104–108] and his followers like Angelo Tonolo and Vitali's less known pupils (Aliprandi, Baldoni, Sacilotto, and Liceni), using the techniques of immersion of varieties in Hilbertian space (of an infinite number of dimensions), in the absence of modern concepts of tangent bundle, normal bundle and so on. Vitali's absolute calculus was used in the thirties by Enea Bortolotti [109–112], in order to arrange Bompiani's work in the domain of what was then called "geometrie riemanniane di specie superiore", and to study the immersed subvarieties in a Riemannian space (Gauss, Codazzi and Ricci equations).

Vitali also provided a direct contribution to projective differential geometry, whose main author was his old university friend Guido Fubini, using the generalised absolute calculus [113, 114]. Finally, Vitali devoted a few works to issues of variational geometry, still within the subvarieties of a Hilbertian space [115–117] Other geometric works were published by Vitali on ruled surfaces and geodesics before the introduction of his geometry in the Hilbertian space [118, 119].

The essential parts of Vitali's research works concerning differential geometry and absolute differential calculus was later re-elaborated and re-exposed in an organic way in a monography: *Geometria nello spazio hilbertiano*.[29] It is divided into five parts: Lebesgue Integral, Developments in series of orthogonal functions and first notions on the Hilbertian space, Complements of algebra, Absolute differential calculus, Differential geometry. In the fourth part, the most original, an extremely general definition of absolute systems is given. In the last part the applications were given, above all, to metric differential geometry, we may particularly remember the notion of principal systems of normals and the study of minimal varieties.

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[29][90] "Geometry in Hilbertian space is not only an expansion of the field of research, it is not only the passage from finite to countable of the number of dimensions of the ambient space. That would be too little. It is a method of geometric representation that, substituting the usual Cartesian, allows more simple formulas, more concise demonstrations and a clearer and vaster view of the problems" (from the Preface).

The *Geometria nello spazio hilbertiano* was conceived as a basic text for courses on higher analysis: in fact Vitali always had a great interest in the teaching of mathematics. For the courses on analysis held in the first 2 years of university he devoted much time to producing a text which was published in lithograph form in 1930.[30] Another treatise, a monography on functions of one real variable, destined to enter a collection on mathematical topics edited by the National Research Committee (CNR), remained unfinished and was published posthumously in 1935, completed, for the second volume, by Giovanni Sansone [121].

In his last years, Vitali held some important conferences and reports addressed not only to specialists, in which he expressed his strong desire to communicate the ideas that formed the basis of his method and how he envisaged mathematical research [122, 123]. Still within this context are to be found some notes published in the *Bollettino della'Unione Matematica Italiana* or in the *Periodico di Matematiche* [72, 74, 92, 124–128] as well as the article on "Limits, series, continued fractions, infinite products" inserted into the *Enciclopedia delle matematiche elementari e complementi* published from 1929 on and edited by L. Berzolari, G. Vivanti and D. Gigli [129].

Finally, we may remember that after having devoted his research to pure mathematics, in the end Vitali also turned his attention to problems of analytical mechanics, structure of matter, and astrophysics, trying to use refined techniques of measure theory and differential geometry [130–134].

I should underline the fact that it was Vitali's fate to be considered something of an outsider. Even his research on differential geometry during his life did not attain great recognition within university circles. No one doubted his worth, but these research works were not considered of great importance to the discipline. Beniamino Segre's study on the development of geometry in Italy starting from 1860 [135], gives evidence of this when, presenting differential geometry, he only cited Vitali as the author of the treatise *Geometria nello spazio hilbertiano*. Further confirmation to this regard comes from the report on the Competition for the Royal Award for Mathematics (which expired on 31st December 1931). The commission (composed of S. Pincherle, G. Castelnuovo, E. Pascal, F. Severi, and G. Fubini) unanimously decided to share the prize between A. Commessatti and L. Fantappié. As for Vitali, whose untimely death occurred during the competition, the Commission said that, were Vitali's entire research works to have been judged, he would have been the winner, but since their judgement was limited to the last 10 years, they had to admit that the introduction Vitali produced of absolute systems and their derivates,

[30] [120] Concerning Vitali's treatises and his production linked to the teaching of mathematics, see: [6].

although of great value, did not justify its great formal complication in relation to the importance of the results obtained.[31]

It was only after some time, in a long report on differential geometry, that Enea Bortolotti, following the applications and developments that he, himself, had produced, gave formal recognition to Vitali's work [136].

6 Conclusions

In hindsight, we may say that Vitali's absence from university constituted a significant loss to mathematics in Italy; according to contemporary evidence Vitali emerges as a generous person and enthusiastic teacher, with a real genius for identifying key problems and results in mathematical research which he skilfully expressed without excessive formalism in elegant, comprehensible language showing rigour and sobriety and clarity in the exposition of the ideas which formed the basis of his output. The depth of affection and sorrow expressed when news of his sudden death arrived is surprising.[32] Tullio Viola, one of his pupils in Bologna, who followed different lines of research, made reference to his grief during a seminar in Paris, as the words "Vitali est mort!" spread through the audience, and he gave tribute to his mentor [3]: "The Maestro was struck down,... under the portico of the city in which I had spent the best days of my student life..." In a moving obituary, which was also an accurate description of Vitali's research, Angelo Tonolo, who had been one of his colleagues in Padua, defined his personality [7]: gifted with discerning intuition, he not only had great skills in algorithm, but was also able to predict the truth of a proposition even without having logical proof, as well as identifying the general outline of the problem. Original and independent in his approach to a topic, he felt the need to elaborate even well-known results on his own. Devoted to his role as advisor for his students' research, his pupils repaid him with admiration and affection. Tonolo also pointed out, along with his mental attributes,

[31] [4] p. 54: "Certainly, if the entire output of a competitor were to be judged, there is no doubt that the prize should go to Vitali, whose work has given the highest honour to mathematics in Italy. The Commission, however, regrets that judgement must be limited exclusively to the works presented in this competition and is not able to take into consideration the research carried out before the last decade. Having these limitations posed on its jurisdiction, the Commission has to admit that the introduction Vitali produced of absolute systems and their derivatives, although of great value, did not justify its great formal complication in relation to the importance of the results obtained".

[32] Let me quote the letter sent by N. Luzin to the Seminario Matematico of Padua University on 18th March 1932 ([2] p. 201): "Messieurs et chers Collègues, C'est avec la plus vive douleur que j'ai appris la mort inattendue de notre cher et inoubliable confrère, le professeur Giuseppe Vitali. Permettez moi de vous exprimer ma condoléance profonde sur cette perte irréparable d'un grand savant dont la vie consacré tout entière sans mélange et sans partage aux recherches scientifiques et aux travaux de l'enseignement et dont les belles découvertes dans la Théorie des Fonctions font l'honneur et la gloire de notre Science".

his great sensitivity, kindness and gentleness, qualities which had prevented him from being assertive in obtaining the role that he deserved.

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