Chapter 2

Demimartingales

2.1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and suppose that a set of random variables $X_1, \ldots, X_n$ defined on the probability space $(\Omega, \mathcal{F}, P)$ have mean zero and are associated. Let $S_0 = 0$ and $S_j = X_1 + \ldots + X_j$, $j = 1, \ldots, n$. Then it follows that, for any componentwise nondecreasing function g,

$$E((S_{j+1} - S_j)g(S_1, \ldots, S_j)) \geq 0, \quad j = 1, \ldots, n \tag{2.1.1}$$

provided the expectation exists.

Throughout this chapter, we assume that all the relevant expectations exist unless otherwise specified. Recall that a sequence of random variables $\{S_n, n \geq 1\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is a martingale with respect to the natural sequence $\mathcal{F}_n = \sigma\{S_1, \ldots, S_n\}$ of σ-algebras if $E(S_{n+1}|S_1, \ldots, S_n) = S_n$ a.s. for $n \geq 1$. Here $\sigma\{S_1, \ldots, S_n\}$ denotes the σ-algebra generated by the random sequence $S_1, \ldots, S_n$. An alternate way of defining the martingale property of the sequence $\{S_n, n \geq 1\}$ is that

$$E((S_{n+1} - S_n)g(S_1, \ldots, S_n)) = 0, \quad n \geq 1$$

for all measurable functions $g(x_1, \ldots, x_n)$ assuming that the expectations exist.

Newman and Wright (1982) introduced the notion of demimartingales.

Definition. A sequence of random variables $\{S_n, n \geq 1\}$ in $L^1(\Omega, \mathcal{F}, P)$ is called a *demimartingale* if, for every componentwise nondecreasing function g,

$$E((S_{j+1} - S_j)g(S_1, \ldots, S_j)) \geq 0, \quad j \geq 1 \tag{2.1.2}$$

Remarks. If the sequence $\{X_n, n \geq 1\}$ is an L^1, mean zero sequence of associated random variables and $S_j = X_1 + \ldots + X_j$ with $S_0 = 0$, then the sequence $\{S_n, n \geq 1\}$ is a demimartingale.

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If the function g is required to be nonnegative (resp., non-positive) and nondecreasing in (2.1.2), then the sequence will be called a *demisubmartingale* (resp., *demisupermartingale*).

A martingale $\{S_n, \mathcal{F}_n,\ n \geq 1\}$ with the natural choice of σ-algebras $\{\mathcal{F}_n, n \geq 1\}$, $\mathcal{F}_n = \sigma\{S_1, \ldots, S_n\}$ is a demimartingale. This can be seen by noting that

$$\begin{aligned} E((S_{j+1} - S_j)g(S_1, \ldots, S_j)) &= E[E((S_{j+1} - S_j)g(S_1, \ldots, S_j)|\mathcal{F}_j)] \\ &= E[g(S_1, \ldots, S_j)E((S_{j+1} - S_j)|\mathcal{F}_j)] \\ &= 0 \end{aligned} \tag{2.1.3}$$

by the martingale property of the process $\{S_n, \mathcal{F}_n,\ n \geq 1\}$. Similarly it can be seen that every submartingale $\{S_n, \mathcal{F}_n,\ n \geq 1\}$ with the natural choice of σ-algebras $\{\mathcal{F}_n,\ n \geq 1\}$, $\mathcal{F}_n = \sigma(S_1, \ldots, S_n)$ is a demisubmartingale. However a demisubmartingale need not be a submartingale. This can be seen by the following example (cf. Hadjikyriakou (2010)).

Example 2.1.1. Let the random variables $\{X_1, X_2\}$ be such that

$$P(X_1 = -1, X_2 = -2) = p \ \text{ and } \ P(X_1 = 1, X_2 = 2) = 1 - p$$

where $0 \leq p \leq \frac{1}{2}$. Then the finite sequence $\{X_1, X_2\}$ is a demisubmartingale since for every nonnegative nondecreasing function $g(.)$,

$$\begin{aligned} E[(X_2 - X_1)g(X_1)] &= -pg(-1) + (1-p)g(1) \\ &\geq -pg(-1) + pg(1) \ \ (\text{since } \ p \leq \frac{1}{2}) \\ &= p(g(1) - g(-1)) \geq 0. \end{aligned} \tag{2.1.4}$$

However the sequence $\{X_1, X_2\}$ is not a submartingale since

$$E(X_2|X_1 = -1) = \sum_{x_2=-2,2} x_2 P(X_2 = x_2|X_1 = -1) = -2 < -1.$$

As remarked earlier, the sequence of partial sums of mean zero associated random variables is a demimartingale. However the converse need not hold. In other words, there exist demimartingales such that the demimartingale differences are not associated. This can be seen again by the following example (cf. Hadjikyriakou (2010)).

Example 2.1.2. Let X_1 and X_2 be random variables such that

$$P(X_1 = 5, X_2 = 7) = \frac{3}{8}, \quad P(X_1 = -3, X_2 = 7) = \frac{1}{8}$$

and

$$P(X_1 = -3, X_2 = 7) = \frac{4}{8}.$$

Let g be a nondecreasing function. Then the finite sequence $\{X_1, X_2\}$ is a demimartingale since

$$E[(X_2 - X_1)g(X_1)] = \frac{6}{8}[g(5) - g(-3)] \geq 0.$$

Let f be a nondecreasing function such that

$$f(x) = 0 \text{ for } x < 0, f(2) = 2, f(5) = 5 \text{ and } f(10) = 20.$$

It can be checked that

$$\mathrm{Cov}(f(X_1), f(X_2 - X_1)) = -\frac{75}{32} < 0$$

and hence X_1 and $X_2 - X_1$ are not associated.

Christofides (2004) constructed another example of a demimartingale. We now discuss his example.

Example 2.1.3. Let $X_1, \ldots, X_n$ be associated random variables, let $h(x_1, \ldots, x_m)$ be a "kernel" mapping R^m to R where $1 \leq m \leq n$. Without loss of generality, we assume that h is a symmetric function. Define the U-statistic

$$U_n = \binom{n}{m}^{-1} \sum_{1 \leq i_1 < \ldots < i_m \leq n} h(X_{i_1}, \ldots, X_{i_m}) \tag{2.1.5}$$

where $\sum_{1 \leq i_1 < \ldots < i_m \leq n} h(X_{i_1}, \ldots, X_{i_m})$ denotes the summation over the $\binom{n}{m}$ combinations of m distinct elements $\{i_1, \ldots, i_m\}$ from $\{1, \ldots, n\}$. Suppose that the function h is componentwise nondecreasing and $E(h) = 0$. Then the sequence $\{S_n = \binom{n}{m} U_n, \, n \geq m\}$ is a demimartingale. This can be checked in the following way.

Note that

$$\begin{aligned} S_{n+1} - S_n &= \sum_{1 \leq i_1 < \ldots < i_m \leq n+1} h(X_{i_1}, \ldots, X_{i_m}) - \sum_{1 \leq i_1 < \ldots < i_m \leq n} h(X_{i_1}, \ldots, X_{i_m}) \\ &= \sum_{1 \leq i_1 < \ldots < i_{m-1} \leq n} h(X_{i_1}, \ldots, X_{i_{m-1}}, X_{n+1}) \end{aligned} \tag{2.1.6}$$

Then, for any componentwise nondecreasing function g, and for $n \geq m$,

$$\begin{aligned} &E[(S_{n+1} - S_n)g(S_m, \ldots, S_n)] \\ &= E[\sum_{1 \leq i_1 < \ldots < i_{m-1} \leq n} h(X_{i_1}, \ldots, X_{i_{m-1}}, X_{n+1}) g(S_m, \ldots, S_n)] \\ &= \sum_{1 \leq i_1 < \ldots < i_{m-1} \leq n} E[h(X_{i_1}, \ldots, X_{i_{m-1}}, X_{n+1}) g(S_m, \ldots, S_n)] \end{aligned} \tag{2.1.7}$$

$$
\begin{aligned}
&= \sum_{1\le i_1<\ldots<i_{m-1}\le n} E[h(X_{i_1},\ldots,X_{i_{m-1}},X_{n+1})v(X_1,\ldots,X_n)]\\
&\ge 0
\end{aligned}
$$

where, for $n \geq m$,

$$
\begin{aligned}
v(x_1,\ldots,x_n) = g\Big(h(x_1,\ldots,x_m), &\sum_{1\le i_1<\ldots<i_m\le m+1} h(x_{i_1},\ldots,x_{i_m}),\ldots,\\
&\sum_{1\le i_1<\ldots<i_m\le n} h(x_{i_1},\ldots,x_{i_m})\Big).
\end{aligned}
$$

Note that the function $v(x_1,\ldots,x_n)$ is componentwise nondecreasing. The last inequality in equation (2.1.7) follows from the fact that the sequence $\{X_i,\, i\geq 1\}$ is an associated random sequence and the function $v(x_1,\ldots,x_n)$ is componentwise nondecreasing. Hence the sequence $\{S_n,\, n\geq m\}$ is a demimartingale.

Example 2.1.4 (Hadjikyriakou (2010)). Suppose the random sequence $\{X_n,\, n\geq 1\}$ is a sequence of associated identically distributed random variables such that the probability density function of X_1 is $f(x,\theta)$ with respect to a σ-finite measure μ and the support of the density function $f(x,\theta)$ does not depend on θ. Suppose the function

$$
h(x) = \frac{f(x,\theta_1)}{f(x,\theta_0)}
$$

is nondecreasing in x for any fixed θ_0 and θ_1. Define

$$
L_n = \prod_{k=1}^{n} \frac{f(X_k,\theta_1)}{f(X_k,\theta_0)}.
$$

Then, under the hypothesis $H_0 : \theta = \theta_0$, the sequence $\{L_n,\, n\geq 1\}$ is a demisubmartingale. This can be checked in the following manner. Observe that

$$
L_{n+1} - L_n = [\frac{f(X_{n+1},\theta_1)}{f(X_{n+1},\theta_0)} - 1]L_n
$$

and, for any componentwise nonnegative nondecreasing function g,

$$
\begin{aligned}
E[(L_{n+1}-L_n)g(L_1,\ldots,L_n)] &= E([\frac{f(X_{n+1},\theta_1)}{f(X_{n+1},\theta_0)} - 1]L_n g(L_1,\ldots,L_n)) \qquad (2.1.8)\\
&\geq E([\frac{f(X_{n+1},\theta_1)}{f(X_{n+1},\theta_0)} - 1])E(L_n g(L_1,\ldots,L_n))\\
&= 0.
\end{aligned}
$$

Note that the inequality in the above relations is a consequence of the fact that the random sequence $\{X_n,\, n\geq 1\}$ is associated and the last equality follows from the observation that

$$
E([\frac{f(X_{n+1},\theta_1)}{f(X_{n+1},\theta_0)}] = 1
$$

under $H_0; \theta = \theta_0$. Observe that L_n is the likelihood ratio when the random sequence $\{X_n, n \geq 1\}$ is an independent sequence of random variables.

The following result, due to Christofides (2000), shows that if the random sequence $\{S_n, n \geq 1\}$ is a demisubmartingale or a demimartingale, then the random sequence $\{g(S_n), n \geq 1\}$ is a demisubmartingale if g is a nondecreasing convex function.

Theorem 2.1.1. *Let the random sequence $\{S_n, n \geq 1\}$ be a demisubmartingale (or a demimartingale) and g be a nondecreasing convex function. Then the random sequence $\{g(S_n), n \geq 1\}$ is a demisubmartingale.*

Proof. Define

$$h(x) = \lim_{x \to y-0} \frac{g(x) - g(y)}{x - y}.$$

From the convexity of the function g, it follows that the function h is a nonnegative nondecreasing function. Furthermore

$$g(y) \geq g(x) + (y - x)h(x).$$

Suppose that f is a nonnegative componentwise nondecreasing function. Then

$$\begin{aligned} &E[(g(S_{n+1}) - g(S_n))f(g(S_1), \ldots, g(S_n))] \\ &\geq E[(S_{n+1} - S_n)h(S_n)f(g(S_1), \ldots, g(S_n))] \\ &= E[(S_{n+1} - S_n)f^*(S_1, \ldots, S_n)] \end{aligned} \tag{2.1.9}$$

where $f^*(x_1, \ldots, x_n) = h(x_n)f(g(x_1), \ldots, g(x_n))$. Note that f^* is a componentwise nondecreasing and nonnegative function. Since the sequence $\{S_n, n \geq 1\}$ is a demimartingale, it follows that the last term is nonnegative and hence the sequence $\{g(S_n), n \geq 1\}$ is a demimartingale. □

As an application of the above theorem, we have the following result.

Theorem 2.1.2. *If the sequence $\{S_n, n \geq 1\}$ is a demimartingale, then the sequence $\{S_n^+, n \geq 1\}$ is a demisubmartingale and the sequence $\{S_n^-, n \geq 1\}$ is also a demisubmartingale.*

Proof. Since the function $g(x) \equiv x^+ = \max(0, x)$ is nondecreasing and convex, it follows that the sequence $\{S_n^+, n \geq 1\}$ is a demisubmartingale from the previous theorem. Let $Y_n = -S_n, n \geq 1$. It is easy to see that the sequence $\{Y_n, n \geq 1\}$ is also a demimartingale and $Y_n^+ = S_n^-$ where $x^- = \max(0, -x)$. Hence, as an application of the first part of the theorem, it follows that the sequence $\{S_n^-, n \geq 1\}$ is a demisubmartingale. □

Suppose the sequence $\{S_n, n \geq 1\}$ is a demimartingale. The following result due to Hu et al. (2010) gives sufficient conditions for a stopped demisubmartingale to be a demisubmartingale.

Theorem 2.1.3. *Let the sequence* $\{S_n,\ n \geq 1\}$ *be a demisubmartingale,* $S_0 = 0$, *and* τ *be a positive integer-valued random variable. Furthermore suppose that the indicator function* $I_{[\tau \leq j]} = h_j(S_1, \ldots, S_j)$ *is a componentwise nonincreasing function of* $S_1, \ldots, S_j$ *for* $j \geq 1$. *Then the random sequence* $\{S_j^* = S_{\min(\tau,j)},\ j \geq 1\}$ *is a demisubmartingale.*

Proof. Note that

$$S_j^* = S_{\min(\tau,j)} = \sum_{i=1}^{j} (S_i - S_{i-1}) I_{[\tau \geq i]}.$$

We have to show that

$$E[(S_{j+1}^* - S_j^*) f(S_1^*, \ldots, S_j^*)] \geq 0, \quad j \geq 1$$

for any f which is componentwise nondecreasing and nonnegative. Since

$$g_j(S_1, \ldots, S_j) \equiv 1 - I_{[\tau \leq j]} = 1 - h_j(S_1, \ldots, S_j)$$

is a componentwise nondecreasing and and nonnegative function, we get that

$$u_j(S_1, \ldots, S_j) \equiv g_j(S_1, \ldots, S_j) f(S_1, \ldots, S_j)$$

is a componentwise nondecreasing nonnegative function. By the demisubmartingale property, we get that

$$\begin{aligned} E[(S_{j+1}^* - S_j^*) f(S_1^*, \ldots, S_j^*)] &= E[(S_{j+1} - S_j) I_{[\tau \geq j+1]} f(S_1^*, \ldots, S_j^*)] \\ &= E[(S_{j+1} - S_j) I_{[\tau \geq j+1]} f(S_1, \ldots, S_j)] \\ &= E[(S_{j+1} - S_j) u_j(S_1, \ldots, S_j)] \geq 0 \qquad (2.1.10) \end{aligned}$$

for $j \geq 1$. Hence the sequence $\{S_j^*, j \geq 1\}$ is a demisubmartingale. □

We now obtain some consequences of this theorem.

Theorem 2.1.4. *Let the sequence* $\{S_n,\ n \geq 1\}$ *be a demisubmartingale. and* τ *be a positive integer-valued random variable. Furthermore suppose that the indicator function* $I_{[\tau \leq j]} = h_j(S_1, \ldots, S_j)$ *is a componentwise nonincreasing function of* $S_1, \ldots, S_j$ *for* $j \geq 1$. *Then, for any* $1 \leq n \leq m$,

$$E(S_{\min(\tau,m)}) \geq E(S_{\min(\tau,n)}) \geq E(S_1), \qquad (2.1.11)$$

Suppose the sequence $\{S_n,\ n \geq 1\}$ *is a demimartingale and the indicator function* $I_{[\tau \leq j]} = h_j(S_1, \ldots, S_j)$ *is a componentwise nondecreasing function of* $S_1, \ldots, S_j$ *for* $j \geq 1$. *Then, for any* $1 \leq n \leq m$,

$$E(S_{\min(\tau,m)}) \leq E(S_{\min(\tau,n)}) \leq E(S_1). \qquad (2.1.12)$$

Proof. Suppose that the sequence $\{S_n, n \geq 1\}$ is a demisubmartingale and the indicator function $I_{[\tau \leq j]} = h_j(S_1, \ldots, S_j)$ is a componentwise nonincreasing function of $S_1, \ldots, S_j$ for $j \geq 1$. Then the sequence $\{S_n^*, n \geq 1\}$ is a demisubmartingale by Theorem 2.1.3. The inequalities stated in equation (2.1.11) follow from the demisubmartingale property by choosing $f \equiv 1$ in equation (2.1.10).

Suppose the sequence $\{S_n, n \geq 1\}$ is a demimartingale and that the indicator function $I_{[\tau \leq j]} = h_j(S_1, \ldots, S_j)$ is a componentwise nondecreasing function of $S_1, \ldots, S_j$ for $j \geq 1$. Since the sequence $\{S_n, n \geq 1\}$ is a demimartingale, we note that

$$\begin{aligned} -E(S_{j+1}^* - S_j^*) &= -E[(S_{j+1} - S_j)I_{[\tau \geq j+1]}] \\ &= E[(S_{j+1} - S_j)(h_j(S_1, \ldots, S_j) - 1)] \\ &\geq 0 \end{aligned} \tag{2.1.13}$$

for $j \geq 1$ from the demimartingale property. This in turn proves (2.1.12). □

2.2 Characteristic Function Inequalities

For any two complex-valued functions $f(x_1, \ldots, x_m)$ and $f_1(x_1, \ldots, x_m)$ defined on R^m, define $f \ll f_1$ if $f_1 - Re(e^{i\alpha} f)$ is componentwise nondecreasing for all real α. Note that

$$f_1 = \frac{(f_1 - Re(f)) + (f_1 - Re(-f))}{2}$$

and hence f_1 is real-valued and nondecreasing. Furthermore $f \ll f_1$ for real f if and only if $f_1 + f$ and $f_1 - f$ are both nondecreasing. We write $f \ll_A f_1$ if $f \ll f_1$ and both the functions f and f_1 depend only on x_j, $j \in A$. The following results are due to Newman (1984).

Lemma 2.2.1. *Suppose h is real and $h \ll h_1$ and ϕ is a complex-valued function on R such that*

$$|\phi(t) - \phi(s)| \leq |t - s|, \quad -\infty < t, s < \infty.$$

Then $\phi(h) \ll h_1$. In particular, this property holds for the function $\phi(h) = e^{ih}$.

Proof. For any function $g(x_1, \ldots, x_m)$, defined on R^m, let Δg denote the increment in the function whenever one or more of the components x_j, $1 \leq j \leq m$ is increased. We have to prove that

$$\Delta[h_1 - Re(e^{i\alpha}\phi(h))] \geq 0 \tag{2.2.1}$$

for any real α. Note that

$$|\Delta Re(e^{i\alpha}\phi(h))| \leq |\Delta(e^{i\alpha}\phi(h))| = |\Delta\phi(h)| \leq |\Delta h|$$

from the property of the function ϕ and

$$|\Delta h| \leq \Delta h_1$$

since $h \ll h_1$. □

Lemma 2.2.2. *Suppose $f \ll_A f_1$ and $g \ll_B g_1$. Define*

$$\langle f, g\rangle = \mathrm{Cov}(f(X_1, X_2, \ldots), g(X_1, X_2, \ldots))$$

where the X_j's are either associated or negatively associated. In the negatively associated case, assume in addition that the sets A and B are disjoint. Then

$$|\langle f, g\rangle| \le \langle f_1, g_1\rangle \text{ if } f \text{ and/or } g \text{ real}$$

and

$$|\langle f, g\rangle| \le 2\langle f_1, g_1\rangle \text{ otherwise.}$$

Proof. Suppose that f is real. Since $|\langle f, g\rangle| = \sup_{\alpha \in R} Re(e^{i\alpha}\langle f, g\rangle)$, it is sufficient to show that $Re(e^{i\alpha}\langle f, g\rangle) \le \langle f_1, g_1\rangle$ for every $\alpha \in R$. This can be seen from the observation that $h \equiv Re(e^{i\alpha} g) \ll g_1$ and $f \ll f_1$ and the identities

$$\langle f_1, g_1\rangle - \langle f, h\rangle = \frac{1}{2}[\langle f_1 + f, g_1 - h\rangle + \langle f_1 - f, g_1 + h\rangle] \ge 0 \tag{2.2.2}$$

in the associated case and

$$-\langle f_1, g_1\rangle - \langle f, h\rangle = \frac{1}{2}[\langle f_1 + f, g_1 + h\rangle + \langle f_1 - f, g_1 - h\rangle] \ge 0 \tag{2.2.3}$$

in the negatively associated case. If g is real, the arguments are the same. If neither f nor g is real, then

$$|\langle f, g\rangle| = |\langle Re\ f, h\rangle + i\langle Im\ f, g\rangle| \le |\langle Re\ f, g\rangle| + |\langle Im\ f, g\rangle| \tag{2.2.4}$$

and the required inequality follows from the inequality for the real f case discussed above. □

As a consequence of Lemma 2.2.1 and arguments similar to those given in the proof of Lemma 2.2.2, Newman (1984) proved the following theorem.

Theorem 2.2.3. *Let $\{S_n, n \ge 1\}$ be a demimartingale, $S_0 = 0$, and f and f_1 be complex-valued functions on R^j such that $f \ll f_1$. Then*

$$|E((S_{j+1} - S_j) f(S_1, \ldots, S_j))| \le E((S_{j+1} - S_j) f_1(S_1, \ldots, S_j)). \tag{2.2.5}$$

In particular, this is the case for the function $f(x_1, \ldots, x_j) = \exp\{i \sum_{k=1}^{j} r_k x_k\}$ and $f_1(x_1, \ldots, x_j) = \sum_{k=1}^{j} |r_k| x_k$.

The next result due to Newman (1984) gives sufficient conditions for a demimartingale to be a martingale with respect to the natural sequence of sub-σ-algebras.

Theorem 2.2.4. *Let $S_0 = 0$, and the sequence $\{S_n, n \ge 1\}$ be an L^2-demimartingale. Let $\mathcal{F}_n$ be the σ-algebra generated by the sequence $\{S_1, \ldots, S_n\}$. If the random sequence $\{S_n, n \ge 1\}$ has uncorrelated increments, that is, if*

$$\mathrm{Cov}((S_{j+1} - S_j), (S_{k+1} - S_k)) = 0, \quad 0 \le k \le j,$$

then the sequence $\{S_n, \mathcal{F}_n, n \ge 1\}$ is a martingale.

Proof. For any $0 \leq k \leq j$,

$$\begin{aligned}\mathrm{Cov}(S_{j+1} - S_j, S_k) &= E[(S_{j+1} - S_j)S_k] - E[(S_{j+1} - S_j)]E[S_k] \\ &= E[(S_{j+1} - S_j)S_k] \qquad (2.2.6)\end{aligned}$$

since $E(S_j)] = E[S_{j+1}]$ by the demimartingale property of the sequence $\{S_n, n \geq 1\}$. From the uncorrelated increment hypothesis, we get that

$$E[(S_{j+1} - S_j)S_k] = 0, \quad 1 \leq k \leq j.$$

Applying Theorem 2.2.3, we get that

$$E[(S_{j+1} - S_j)\exp(i\sum_{k=1}^{j} r_k S_k)] = 0$$

for $r_1, \ldots, r_k \in R$. This in turn proves that

$$E[S_{j+1} - S_j | \mathcal{F}_j] = 0 \text{ a.s } j \geq 1.$$

Hence the random sequence $\{S_n, \mathcal{F}_n, n \geq 1\}$ is a martingale. □

Remarks. Suppose $S_0 \equiv 0$ and $S_n = X_1 + \ldots + X_n$ where $\{X_n, n \geq 1\}$ is a Gaussian process. Then the sequence $\{S_n, n \geq 1\}$ is a *martingale* if and only if $\mathrm{Cov}(X_k, X_j) = 0$ for all $k > j$ and $E[X_j] = 0$ for all $j \geq 1$. It is the partial sum sequence of *associated random variables* if $\mathrm{Cov}(X_k, X_j) \geq 0$ for all $k > j$ and the sequence $\{S_n, n \geq 1\}$ is a *demimartingale* if and only if $\sum_{j=1}^{n} \mathrm{Cov}(X_k, X_j) \geq 0$ for all $k > n$ and $E[X_j] = 0$ for all $j \geq 1$.

2.3 Doob Type Maximal Inequality

We will now discuss a Doob type maximal inequality for demisubmartingales due to Newman and Wright (1982). Let

$$S_n^* = \max(S_1, \ldots, S_n). \qquad (2.3.1)$$

Define the rank orders $R_{n,j}$ by

$$R_{n,j} = \begin{cases} j\text{-th largest of } (S_1, \ldots, S_n), & \text{if } j \leq n, \\ \min(S_1, \ldots, S_n) = R_{n,n}, & \text{if } j > n. \end{cases}$$

The following theorem is due to Newman and Wright (1982).

Theorem 2.3.1. *Suppose $S_1, S_2, \ldots$ is a demimartingale (resp., demisubmartingale) and m is a nondecreasing (resp., nonnegative and nondecreasing) function on $(-\infty, \infty)$ with $m(0) = 0$; then, for any n and j,*

$$E(\int_0^{R_{n,j}} u\, dm(u)) \leq E(S_n m(R_{n,j})); \qquad (2.3.2)$$

and, for any $\lambda > 0$,

$$\lambda\, P(R_{n,j} \geq \lambda) \leq \int_{[R_{n,j}\geq\lambda]} S_n dP. \tag{2.3.3}$$

Proof. For fixed n and j, let $Y_k = R_{k,j}$ and $Y_0 = 0$. Then

$$S_n m(Y_n) = \sum_{k=0}^{n-1} S_{k+1}(m(Y_{k+1}) - m(Y_k)) + \sum_{k=1}^{n-1}(S_{k+1} - S_k)m(Y_k). \tag{2.3.4}$$

From the definition of $R_{n,j}$, it follows that

$$\text{for } k < j, \text{ either } Y_k = Y_{k+1} \text{ or } S_{k+1} = Y_{k+1};$$

and

$$\text{for } k \geq j, \text{ either } Y_k = Y_{k+1} \text{ or } S_{k+1} \geq Y_{k+1}.$$

Hence, for any k,

$$S_{k+1}(m(Y_{k+1} - m(Y_k)) \geq Y_{k+1}(m(Y_{k+1} - m(Y_k)) \geq \int_{Y_k}^{Y_{k+1}} u\, dm(u). \tag{2.3.5}$$

Combining the inequalities in equations (2.3.4) and (2.3.5), we get that

$$S_n m(Y_n) \geq \int_0^{Y_n} u\, dm(u) + \sum_{k=1}^{n-1}(S_{k+1} - S_k)m(Y_k). \tag{2.3.6}$$

Note that

$$E[(S_{k+1} - S_k)m(Y_k)] \geq 0, \quad 1 \leq k \leq n-1 \tag{2.3.7}$$

by the definition of demimartingale (resp., demisubmartingale) since the function $m(Y_k)$ is a nondecreasing (resp., nonnegative and nondecreasing) function of $S_1, \ldots, S_k$. Taking the expectations on both sides of the inequality (2.3.6) and applying the inequality (2.3.7), we obtain the inequality (2.3.2) stated in the theorem by observing that $Y_n = R_{n,j}$. . The inequality (2.3.3) is an easy consequence of (2.3.2) by choosing the function $m(u)$ to be the indicator function $I_{[u\geq\lambda]}$. □

We obtain some corollaries to the theorem stated above.

Theorem 2.3.2. *If the sequence* $\{S_n,\, n \geq 1\}$ *is an* L^2*-demimartingale, then*

$$E((R_{n,j} - S_n)^2) \leq E(S_n^2) \tag{2.3.8}$$

and if the sequence $\{S_n,\, n \geq 1\}$ *is an* L^2*-demisubmartingale, then*

$$E((R_{n,j}^+ - S_n)^2) \leq E(S_n^2). \tag{2.3.9}$$

Proof. In the demimartingale case, the corollary is proved by choosing $m(u) = u$ to obtain the inequality

$$E(R_{n,j}^2/2) \leq E(S_n R_{n,j})$$

which is equivalent to the inequality stated in (2.3.8). In the demisubmartingale case, we prove (2.3.9) by choosing $m(u) = uI_{[u \geq 0]}$. □

The following result gives a maximal inequality for the sequence of partial sums of mean zero associated random variables. This can be derived as a corollary to the theorem proved earlier. We omit the proof of this result. For details, see Newman and Wright (1982).

Theorem 2.3.3. *Suppose the sequence $\{X_n,\ n \geq 1\}$ is a mean zero associated sequence of random variables and $S_n = X_1 + \ldots + X_n$ with $S_0 = 0$. Then the sequence $\{S_n,\ n \geq 1\}$ is a demimartingale and*

$$E(R_{n,j}^2) \leq E(S_n^2) = \sigma_n^2, \tag{2.3.10}$$

and, for $\lambda_1 < \lambda_2$,

$$(1 - \sigma_n^2/(\lambda_2 - \lambda_1)^2) P(S_n^* \geq \lambda_2) \leq P(S_n \geq \lambda_1), \tag{2.3.11}$$

so that for $\alpha_1 < \alpha_2$ with $\alpha_2 - \alpha_1 > 1$,

$$P(max(|S_1|, \ldots, |S_n|) \geq \alpha_2 \sigma_n) \leq \frac{(\alpha_2 - \alpha_1)^2}{(\alpha_2 - \alpha_1)^2 - 1} P(|S_n| \geq \alpha_1 \sigma_n). \tag{2.3.12}$$

Note that Theorem 2.3.3 holds for partial sums of mean zero associated random variables which form a demimartingale. However the following maximal inequality is valid for any demimartingale.

Theorem 2.3.4. *Suppose the sequence $\{S_n,\ n \geq 1\}$ is an L^2-demisubmartingale. Let $E(S_n^2) = \sigma_n^2$. Then, for $0 \leq \lambda_1 < \lambda_2$,*

$$P(S_n^* \geq \lambda_2) \leq (\sigma_n/(\lambda_2 - \lambda_1))(P(S_n \geq \lambda_1))^{1/2}. \tag{2.3.13}$$

If $S_1, S_2, \ldots$ is an L^2 demimartingale, then for $0 \leq \alpha_1 < \alpha_2$,

$$P(\max(|S_1|, \ldots, |S_n|) \geq \alpha_2 \sigma_n) \leq \sqrt{2}(\alpha_2 - \alpha_1)^{-1}(P(|S_n| \geq \alpha_1 \sigma_n))^{1/2}. \tag{2.3.14}$$

Proof. Let $\lambda = \lambda_2$ and $j = 1$ in the inequality proved in (2.3.3). Then, it follows that

$$\begin{aligned} \lambda_2\ P(S_n^* \geq \lambda_2) &\leq \int_{[S_n^* \geq \lambda_2]} S_n\, dP \\ &\leq \int_{[S_n \geq \lambda_1]} S_n\, dP + \int_{[S_n^* \geq \lambda_2, S_n < \lambda_1]} S_n\, dP \end{aligned}$$

$$\leq \int_{[S_n \geq \lambda_1]} S_n \, dP + \lambda_1 P(S_n^* \geq \lambda_2) \tag{2.3.15}$$

which implies that

$$P(S_n^* \geq \lambda_2) \leq (\lambda_2 - \lambda_1)^{-1} E(S_n I_{[S_n \geq \lambda_1]}). \tag{2.3.16}$$

Applying the Cauchy-Schwartz inequality to the term on the right-hand side of the above inequality, we get the inequality (2.3.13). In order to derive inequality (2.3.14), we take $\lambda_i = \alpha_i \sigma_n$ and add to (2.3.13) the corresponding inequality with all the S_i's replaced by $-S_i$'s which also forms a demimartingale. □

Chow (1960) proved a maximal inequality for submartingales. Christofides (2000) obtained an analogue of this inequality for demisubmartingales. Prakasa Rao (2002, 2007) and Wang (2004) derived other maximal inequalities for demisubmartingales. We will discuss these results later in this chapter.

2.4 An Upcrossing Inequality

The following theorem, due to Newman and Wright (1982), extends Doob's upcrossing inequality for submartingales to demisubmartingales. Given a set of random variables $S_1, S_2, \ldots, S_n$ and $a < b$, we define a sequence of stopping times $J_0 = 0, J_1, J_2, \ldots$ as follows (for $k = 1, 2, \ldots$):

$$J_{2k-1} = \begin{cases} n+1 & \text{if } \{j : J_{2k-2} < j \leq n \text{ and } S_j \leq a\} \text{ is empty,} \\ \min\{j : J_{2k-2} < j \leq n \text{ and } S_j \leq a\}, & \text{otherwise,} \end{cases}$$

and

$$J_{2k-1} = \begin{cases} n+1 & \text{if } \{j : J_{2k-2} < j \leq n \text{ and } S_j \geq b\} \text{ is empty,} \\ \min\{j : J_{2k-2} < j \leq n \text{ and } S_j \geq b\}, & \text{otherwise.} \end{cases}$$

The number of complete upcrossings of the interval $[a, b]$ by $S_1, \ldots, S_n$ is denoted by $U_{a,b}$ where

$$U_{a,b} = \max\{k : J_{2k} < n+1\}. \tag{2.4.1}$$

Theorem 2.4.1. *If the finite sequence $S_1, S_2, \ldots, S_n$ is a demisubmartingale, then for $a < b$,*

$$E(U_{a,b}) \leq \frac{E((S_n - a)^+) - E((S_1 - a)^+)}{b - a}. \tag{2.4.2}$$

The following theorem gives sufficient conditions for the almost sure convergence of a demimartingale. It is a consequence of the upcrossing inequality stated in Theorem 2.4.1 as in the case of martingales.

Theorem 2.4.2. *If the sequence $\{S_n, n \geq 1\}$ is a demimartingale and*

$$\limsup_{n \to \infty} E|S_n| < \infty,$$

then the S_n converge a.s. to a random variable X such that $E|X| < \infty$.

We now prove the upcrossing inequality.

Proof of Theorem 2.4.1. For $1 \le j \le n-1$, define

$$\epsilon_j = \begin{cases} 1 & \text{if for some } k = 1, 2, \dots, \ J_{2k-2} \le j < J_{2k-1} \\ 0 & \text{if for some } k = 1, 2, \dots, \ J_{2k-1} \le j < J_{2k} \end{cases} \tag{2.4.3}$$

so that $1 - \epsilon_j$ is the indicator function of the event that the time interval $[j, j+1)$ is a part of an upcrossing possibly incomplete; equivalently

$$\epsilon_j = \begin{cases} 1 & \text{if either } S_i > a \text{ for } i = 1, \dots, j \text{ or} \\ & \text{for some } i = 1, \dots, j, \ S_i \ge b \text{ and } S_k > a \text{ for } k = i+1, \dots, j \\ = 0 & \text{otherwise.} \end{cases} \tag{2.4.4}$$

Let Λ be the event that the sequence $S_1, \dots, S_n$ ends with an incomplete upcrossing, that is, $\tilde{J} \equiv J_{2U_{a,b}+1} < n$. Note that

$$(S_n - a)^+ - (S_1 - a)^+ = \sum_{j=1}^{n-1} [(S_{j+1} - a)^+ - (S_j - a)^+] = M_u + M_d \tag{2.4.5}$$

where

$$M_d = \sum_{j=1}^{n-1} \epsilon_j [(S_{j+1} - a)^+ - (S_j - a)^+] \ge \sum_{j=1}^{n-1} \epsilon_j (S_{j+1} - S_j). \tag{2.4.6}$$

This inequality follows from the observation that

$$(S_{j+1} - a)^+ \ge S_{j+1} - a$$

and

$$\epsilon_j (S_j - a)^+ = \epsilon_j (S_j - a)$$

since $\epsilon_j = 1$ implies that $S_j > a$ from the definition of ϵ_j. Observe that

$$\begin{aligned} M_u &= \sum_{j=1}^{n-1} (1 - \epsilon_j)[(S_{j+1} - a)^+ - (S_j - a)^+] \qquad (2.4.7) \\ &= \sum_{k=1}^{U_{a,b}} \sum_{j=J_{2k-1}}^{J_{2k}-1} [(S_{j+1} - a)^+ - (S_j - a)^+] + \sum_{j=\tilde{J}}^{n-1} [(S_{j+1} - a)^+ - (S_j - a)^+] \\ &= \sum_{k=1}^{U_{a,b}} [(S_{J_{2k}} - a)^+ - (S_{J_{2k-1}} - a)^+] + [(S_n - a)^+ - (S_{\tilde{J}} - a)^+] I_\Lambda \\ &= \sum_{k=1}^{U_{a,b}} (S_{J_{2k}} - a)^+ + (S_n - a)^+] I_\Lambda \\ &\ge (b - a) U_{a,b}. \end{aligned}$$

Combining equations (2.4.5), (2.4.6) and (2.4.7) and taking expectations, we get that

$$E[(S_n - a)^+ - (S_1 - a)^+] \geq (b-a)E[U_{a,b}] + \sum_{j=1}^{n-1} E[\epsilon_j(S_{j+1} - S_j)]. \tag{2.4.8}$$

Note that ϵ_j is a nonnegative nondecreasing function of $S_1, \ldots, S_n$ from the definition of ϵ_j. Since $\{S_j, j = 1, \ldots, n\}$ is a demisubmartingale, it follows that

$$E[\epsilon_j(S_{j+1} - S_j)] \geq 0, \quad j = 1, \ldots, n-1.$$

Hence

$$E[(S_n - a)^+ - (S_1 - a)^+] \geq (b-a)E[U_{a,b}] \tag{2.4.9}$$

which implies the upcrossing inequality stated in the theorem. $\square$

2.5 Chow Type Maximal Inequality

We now derive some more maximal inequalities for demimartingales which can be used to derive strong laws of large numbers for demimartingales. The following result, due to Christofides (2000), is an analogue of the maximal inequality for submartingales proved by Chow (1960).

Theorem 2.5.1. *Let the sequence $\{S_n,\ n \geq 1\}$ be a demisubmartingale with $S_0 = 0$. Let $\{c_n,\ n \geq 0\}$ be a nonincreasing sequence of positive numbers, Then, for any $\epsilon > 0$,*

$$\epsilon\ \ P[\max_{1 \leq k \leq n} c_k S_k \geq \epsilon] \leq \sum_{j=1}^{n} c_j\ E[S_j^+ - S_{j-1}^+] \tag{2.5.1}$$

where $x^+ = \max\{0, x\}$.

Proof. Let $\epsilon > 0$. Let $A = [\max_{1 \leq k \leq n} c_k S_k \geq \epsilon]$ and

$$A_j = [\max_{1 \leq i < j} c_i S_i < \epsilon,\ c_j\ S_j \geq \epsilon], \quad j = 1, \ldots, n.$$

Let E^c denote the complement of a set E. Let $I_{[E]}$ denote the indicator function of a set E. Note that the events A_j, $j = 1, \ldots, n$ are disjoint and $A = \cup_{j=1}^n A_j$. Hence

$$\begin{aligned} \epsilon\ P(A) &= \epsilon \sum_{j=1}^{n} P(A_j) \qquad (2.5.2) \\ &= \sum_{j=1}^{n} E[\epsilon I_{[A_j]}] \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{n} E[c_j\ S_j I_{[A_j]}] \\
&\le \sum_{j=1}^{n} E[c_j\ S_j^+ I_{[A_j]}] \\
&= c_1 E[S_1^+] - c_1 E[S_1^+ I_{[A_1^c]}] + \sum_{j=2}^{n} E[c_j\ S_j^+ I_{[A_j]}] \\
&\le c_1 E[S_1^+] - c_2 E[S_1^+ I_{[A_1^c]}] + c_2 E[S_2^+ I_{[A_2]}] + \sum_{j=3}^{n} E[c_j\ S_j^+ I_{[A_j]}] \\
&= c_1 E[S_1^+] + c_2 E[(S_2^+ - S_1^+) I_{[A_1^c]}] - c_2 E[S_2^+ I_{[A_1^c \cap A_2^c]}] + \sum_{j=3}^{n} E[c_j\ S_j^+ I_{[A_j]}].
\end{aligned}
$$

The last equality follows from the fact that $I_{[A_2]} = I_{[A_1^c]} - I_{[A_1^c \cap A_2^c]}$ which in turn holds since $A_2 \subset A_1^c$. The expression on the right-hand side of the last equality can be written in the form

$$
c_1 E[S_1^+] + c_2 E[S_2^+ - S_1^+] - c_2 E[(S_2^+ - S_1^+) I_{[A_1]}] - c_2 E[S_2^+ I_{[A_1^c \cap A_2^c]}] + \sum_{j=3}^{n} E[c_j\ S_j^+ I_{[A_j]}].
$$

Let $h(y) = \lim_{x \to y-o} \frac{x^+ - y^+}{x-y}$. Then $h(.)$ is a nonnegative nondecreasing function by the convexity of the function $x^+ = \max\{0, x\}$ and we have

$$
S_2^+ - S_1^+ \ge (S_2 - S_1) h(S_1).
$$

Therefore

$$
E[(S_2^+ - S_1^+) I_{[A_1]}] \ge E[(S_2 - S_1) h(S_1) I_{[A_1]}].
$$

Since $h(S_1) I_{[A_1]}$ is a nonnegative nondecreasing function of S_1, it follows that

$$
E[(S_2 - S_1) h(S_1) I_{[A_1]}] \ge 0
$$

by the demisubmartingale property of the sequence $\{S_n, n \ge 1\}$ which in turn shows that

$$
E[(S_2^+ - S_1^+) I_{[A_1]}] \ge 0.
$$

Hence the expression on the right-hand side of the last inequality in (2.5.2) is bounded above by J where

$$
J = c_1 E[S_1^+] + c_2 E[S_2^+ - S_1^+] - c_2 E[S_2^+ I_{[A_1^c \cap A_2^c]}] + \sum_{j=3}^{n} E[c_j\ S_j^+ I_{[A_j]}].
$$

Since the sequence c_k is a nondecreasing sequence,

$$
J \le c_1 E[S_1^+] + c_2 E[S_2^+ - S_1^+] - c_3 E[S_2^+ I_{[A_1^c \cap A_2^c]}] + \sum_{j=3}^{n} E[c_j\ S_j^+ I_{[A_j]}] \qquad (2.5.3)
$$

$$= c_1 E[S_1^+] + c_2 E[S_2^+ - S_1^+] + c_3 E[(S_3^+ - S_2^+) I_{[A_1^c \cap A_2^c]}]$$
$$- c_3 E[S_3^+ I_{[A_1^c \cap A_2^c \cap A_3^c]}] + \sum_{j=4}^{n} E[c_j \; S_j^+ I_{[A_j]}]$$

and the last equality follows from the fact that $I_{[A_3]} = I_{[A_1^c \cap A_2^c]} - I_{[A_1^c \cap A_2^c \cap A_3^c]}$ which in turn holds since $A_3 \subset A_1^c \cap A_2^c$. The expression on the right-hand side of the last equality can be written in the form

$$\sum_{j=1}^{3} c_j E[S_j^+ - S_{j-1}^+] - c_3 E[(S_3^+ - S_2^+) I_{[A_1 \cup A_2]}] - c_3 E[S_3^+ I_{[A_1^c \cap A_2^c \cap A_3^c]}]$$
$$+ \sum_{j=4}^{n} E[c_j \; S_j^+ I_{[A_j]}]. \tag{2.5.4}$$

Applying the convexity of the function x^+ again, we get that

$$E[(S_3^+ - S_2^+) I_{[A_1 \cup A_2]}] \geq E[(S_3 - S_2) h(S_2) I_{[A_1 \cup A_2]}].$$

Since the function $h(S_2) I_{[A_1 \cup A_2]}$ is a nonnegative componentwise nondecreasing function of (S_1, S_2), it follows that

$$E[(S_3 - S_2) h(S_2) I_{[A_1 \cup A_2]}] \geq 0$$

by the demisubmartingale property of the sequence $\{S_n, \, n \geq 1\}$. Hence the quantity defined by (2.5.4) is bounded above by

$$\sum_{j=1}^{3} c_j E[S_j^+ - S_{j-1}^+] - c_3 E[S_3^+ I_{[A_1^c \cap A_2^c \cap A_3^c]}] + \sum_{j=4}^{n} E[c_j \; S_j^+ I_{[A_j]}]. \tag{2.5.5}$$

Proceeding in this way, we get that

$$\epsilon P(A) \leq \sum_{j=1}^{n} c_j E[S_j^+ - S_{j-1}^+] - c_n E[S_n^+ I_{[A^c]}] \tag{2.5.6}$$
$$\leq \sum_{j=1}^{n} c_j E[S_j^+ - S_{j-1}^+]$$

since $c_n > 0$. □

A corollary to the Chow type maximal inequality is the following Doob type maximal inequality which was derived earlier by other methods.

Theorem 2.5.2. *Suppose the sequence $\{S_n, \, n \geq 1\}$ is a demisubmartingale. Then, for any $\epsilon > 0$,*

$$\epsilon \; P[\max_{1 \leq k \leq n} S_k \geq \epsilon] \leq \int_{[\max_{1 \leq k \leq n} S_k \geq \epsilon]} S_n dP.$$

Applications of Chow Type Maximal Inequality

As an application of the Chow type maximal inequality, we can obtain the following results.

Theorem 2.5.3. *Let the sequence* $\{S_n, n \geq 1\}$ *be a demimartingale with* $S_0 = 0$. *Suppose that* $\{c_k, k \geq 1\}$ *is a positive nonincreasing sequence of numbers such that* $\lim_{k\to 0} c_k = 0$. *Suppose there exists* $\nu \geq 1$ *such that* $E[|S_k|^\nu] < \infty$ *for every* $k \geq 1$. *Assume that*

$$\sum_{k=1}^{\infty} c_k^\nu E(|S_k|^\nu - |S_{k-1}|^\nu) < \infty. \tag{2.5.7}$$

Then

$$c_n S_n \xrightarrow{a.s} 0 \ \ as \ \ n \to \infty.$$

Proof. Let $\epsilon > 0$. Note that

$$\begin{aligned} P[\sup_{k\geq n} c_k |S_k| \geq \epsilon] &= P[\sup_{k\geq n} c_k^\nu |S_k|^\nu \geq \epsilon^\nu] \\ &\leq P[\sup_{k\geq n} c_k^\nu (S_k^+)^\nu \geq \epsilon^\nu/2] + P[\sup_{k\geq n} c_k^\nu (S_k^-)^\nu \geq \epsilon^\nu/2]. \end{aligned}$$

Since the function x^ν, $x \geq 0$ is a nondecreasing convex function for any $\nu \geq 1$, it follows that the sequences $\{(S_k^+)^\nu, k \geq 1\}$ and $\{(S_k^-)^\nu, k \geq 1\}$ are both demisubmartingales. Applying the Chow type maximal inequality proved earlier, we get that

$$\begin{aligned} &P[\sup_{k\geq n} c_k^\nu (S_k^+)^\nu \geq \epsilon^\nu/2] + P[\sup_{k\geq n} c_k^\nu (S_k^-)^\nu \geq \epsilon^\nu/2] \\ &\leq 2\epsilon^{-\nu}(c_n^\nu E[(S_n^+)^\nu] + \sum_{k=n+1}^{\infty} c_k^\nu \; E[(S_k^+)^\nu - (S_{k-1}^+)^\nu] \\ &\quad + c_n^\nu E[(S_n^-)^\nu] + \sum_{k=n+1}^{\infty} c_k^\nu \; E[(S_k^-)^\nu - (S_{k-1}^-)^\nu]) \\ &= 2\epsilon^{-\nu}(c_n^\nu E[|S_n|^\nu] + \sum_{k=n+1}^{\infty} c_k^\nu \; E[|S_k|^\nu - |S_{k-1}|^\nu]) \end{aligned} \tag{2.5.8}$$

from the fact that $|S_n|^\nu = (S_n^+)^\nu + (S_n^-)^\nu$. The Kronecker lemma and the condition (2.5.7) imply that $\lim_{n\to\infty} c_n^\nu E|S_n|^\nu = 0$. From the inequality derived above, we get that

$$\lim_{n\to\infty} P[\sup_{k\geq n} c_k |S_k| \geq \epsilon] = 0, \tag{2.5.9}$$

equivalently

$$c_n S_n \xrightarrow{a.s} 0 \ \text{ as } \ n \to \infty. \tag{2.5.10}$$

□

Strong Law of Large Numbers for Associated Sequences

Consider a sequence of mean zero associated random variables. The following Kolmogorov type of strong law of large numbers can be proved for such a sequence as a consequence of Theorem 2.5.3.

Theorem 2.5.4. *Let the sequence $\{X_n,\ n \geq 1\}$ be a sequence of L^2-mean zero associated random variables. Let $S_n = X_1 + \ldots + X_n$ and $S_0 = 0$. Suppose that*

$$\sum_{n=1}^{\infty} n^{-2} \operatorname{Cov}(X_n, S_n) < \infty.$$

Then

$$\frac{S_n}{n} \xrightarrow{a.s} 0 \quad as \quad n \to \infty. \tag{2.5.11}$$

Proof. Since the sequence $\{X_n,\ n \geq 1\}$ is mean zero associated, it follows that $\{S_n,\ n \geq 1\}$ is a demimartingale. Applying Theorem 2.5.3 with $\nu = 2$ and $c_n = \frac{1}{n}$, it follows that

$$\frac{S_n}{n} \xrightarrow{a.s} 0 \text{ as } n \to \infty$$

provided

$$\sum_{n=1}^{\infty} n^{-2} E(S_n^2 - S_{n-1}^2) < \infty.$$

Observe that

$$E(S_n^2 - S_{n-1}^2) = 2E(X_n S_{n-1}) + E(X_n^2)$$

and

$$\begin{aligned} 2\sum_{n=1}^{\infty} n^{-2} E(X_n S_{n-1}) + \sum_{n=1}^{\infty} n^{-2} E(X_n^2) &\leq 2\sum_{n=1}^{\infty} n^{-2} E(X_n S_n) \qquad (2.5.12) \\ &= 2\sum_{n=1}^{\infty} n^{-2} \operatorname{Cov}(X_n, S_n) < \infty. \qquad \square \end{aligned}$$

This strong law of large numbers for associated sequences was also proved by Birkel (1989). Wang (2004) generalized Theorem 2.5.1 to obtain a maximal inequality for nonnegative convex functions of a demimartingale.

Theorem 2.5.5. *Let $S_0 = 0$ and the sequence $\{S_n,\ n \geq 1\}$ be a demimartingale. Let $g(.)$ be a nonnegative convex function on R with $g(0) = 0$. Suppose that the sequence $\{c_i,\ 1 \leq i \leq n\}$ is a non-increasing sequence of positive numbers. Let $S_n^* = \max_{1 \leq i \leq n} c_i g(S_i)$. Then, for any $\lambda > 0$,*

$$\lambda P(S_n^* \geq \lambda) \leq \sum_{i=1}^{n} c_i E[g(S_i) - g(S_{i-1})] - c_n E[g(S_n) I_{[S_n^* < \lambda]}] \tag{2.5.13}$$

$$\leq \sum_{i=1}^{n} c_i E[g(S_i) - g(S_{i-1})].$$

We now give a sketch of proof of this theorem following the ideas in Hadjikyriakou (2010) and Wang (2004).

Proof. Define the functions

$$u(x) = g(x)I_{[x\geq 0]} \text{ and } v(x) = g(x)I_{[x<0]}.$$

Note that the function $u(x)$ is a nonnegative nondecreasing convex function and the function $v(x)$ is a nonnegative nonincreasing convex function. It is obvious that

$$g(x) = u(x) + v(x) = \max\{u(x), v(x)\}.$$

Furthermore

$$\begin{aligned} P[\max_{1\leq i\leq n} c_i g(S_i) \geq \lambda) &= P[\max_{1\leq i\leq n} c_i \max\{u(S_i), v(S_i)\} \geq \lambda] \\ &\leq P[\max_{1\leq i\leq n} c_i u(S_i) \geq \lambda] + P[\max_{1\leq i\leq n} c_i v(S_i) \geq \lambda]. \end{aligned} \tag{2.5.14}$$

Applying Theorem 2.5.1, we get that

$$\lambda P[\max_{1\leq i\leq n} c_i u(S_i) \geq \lambda] \leq \sum_{i=1}^{n} c_i E[u(S_i) - u(S_{i-1})]. \tag{2.5.15}$$

Let

$$A_i = \{c_k v(S_k) < \lambda,\ 1 \leq k < i,\ c_i v(S_i) \geq \lambda\}, \quad i = 1, \ldots, n.$$

Following the method in the proof of Theorem 2.5.1, we get that

$$\begin{aligned} \lambda P[\max_{1\leq i\leq n} c_i v(S_i) \geq \lambda] \leq \sum_{i=3}^{n} c_i E[v(s_i)I_{A_i}] &+ c_1 E[v(S_1)] + c_2 E[v(S_2) - v(S_1)] \\ &- c_2 E[v(S_2)I_{A_1^c \cap A_2^c}] - c_2 E[(v(S_2) - v(S_1))I_{A_1}]. \end{aligned} \tag{2.5.16}$$

Let $h(.)$ be the left derivative of the function $v(.)$ Then $h(.)$ is a non-positive nondecreasing function and

$$v(x) - v(y) \geq (x - y)h(y).$$

Hence

$$v(S_2) - v(S_1) \geq (S_2 - S_1)h(S_1)$$

and

$$E[(v(S_2 - v(S_1))I_{A_1}] \geq E[(S_2 - S_1)h(S_1)I_{A_1}].$$

Since I_{A_1} is a nonincreasing function of S_1 and $h(.)$ is a non-positive nondecreasing function, it follows that $h(S_1)I_{A_1}$ is a nondecreasing function of S_1. By the demimartingale property, we get that

$$E[(S_2 - S_1)h(S_1)I_{A_1}] \geq 0.$$

Applying arguments similar to those in the proof of Theorem 2.5.1, we get that

$$\lambda P[\max_{1\leq i\leq n} c_i v(S_i) \geq \lambda] \leq \sum_{i=1}^{n} c_i E[v(S_i) - v(S_{i-1})]. \tag{2.5.17}$$

Combining the inequalities (2.5.14), (2.5.15) and (2.5.17), we obtain the inequality (2.5.13). $\square$

Suppose the sequence $\{S_n,\ n \geq 1\}$ is a nonnegative demimartingale. As a corollary to this theorem, it can be proved that

$$E(\max_{1\leq i\leq n} S_i) \leq \frac{e}{e-1}[1 + E(S_n \log^+ S_n)].$$

For a proof of this inequality , see Corollary 2.1 in Wang (2004).

2.6 Whittle Type Maximal Inequality

We now discuss a Whittle type maximal inequality for demisubmartingales due to Prakasa Rao (2002). This result generalizes the Kolmogorov inequality and the Hajek-Renyi inequality for independent random variables (Whittle (1969)) and is an extension of the results in Christofides (2000) for demisubmartingales.

Let the sequence $\{S_n,\ n \geq 1\}$ be a demisubmartingale. Suppose $\phi(.)$ is a nondecreasing convex function. Then the sequence $\{\phi(S_n),\ n \geq 1\}$ is a demisubmartingale by Theorem 2.1.1 (cf. Christofides (2000)).

Theorem 2.6.1. *Let $S_0 = 0$ and suppose the sequence of random variables $\{S_n, n \geq 1\}$ is a demisubmartingale. Let $\phi(.)$ be nonnegative nondecreasing convex function such that $\phi(0) = 0$. Let $\psi(u)$ be a positive nondecreasing function for $u > 0$. Let A_n be the event that $\phi(S_k) \leq \psi(u_k)$, $1 \leq k \leq n$, where $0 = u_0 < u_1 \leq \ldots \leq u_n$. Then*

$$P(A_n) \geq 1 - \sum_{k=1}^{n} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}. \tag{2.6.1}$$

If, in addition, there exist nonnegative real numbers $\Delta_k, 1 \leq k \leq n$ such that

$$0 \leq E[(\phi(S_k) - \phi(S_{k-1}))f(\phi(S_1), \ldots, \phi(S_{k-1}))] \leq \Delta_k E[f(\phi(S_1), \ldots, \phi(S_{k-1}))]$$

for $1 \leq k \leq n$ for all componentwise nonnegative nondecreasing functions f such that the expectation is defined and

$$\psi(u_k) \geq \psi(u_{k-1}) + \Delta_k, \quad 1 \leq k \leq n,$$

then

$$P(A_n) \geq \prod_{k=1}^{n}(1 - \frac{\Delta_k}{\psi(u_k)}). \tag{2.6.2}$$

Proof. Since the sequence $\{S_n, n \geq 1\}$ is a demisubmartingale by hypothesis and the function $\phi(.)$ is a nondecreasing convex function, it follows that the sequence $\{\phi(S_n), n \geq 1\}$ forms a demisubmartingale by Theorem 2.1.1. Hence

$$E\{(\phi(S_{n+1}) - \phi(S_n))f(\phi(S_1), \ldots, \phi(S_n))\} \geq 0, \quad n \geq 1 \tag{2.6.3}$$

for every nonnegative componentwise nondecreasing function f such that the expectation is defined. Let χ_j be the indicator function of the event $[\phi(S_j) \leq \psi(u_j)]$ for $1 \leq j \leq n$. Note that

$$\chi_n \geq (1 - \frac{\phi(S_n)}{\psi(u_n)})$$

and hence

$$\begin{aligned} P(A_n) = E(\prod_{i=1}^{n}\chi_i) &= E(\{\prod_{i=1}^{n-1}\chi_i\}\chi_n) \\ &\geq E(\{\prod_{i=1}^{n-1}\chi_i\}(1 - \frac{\phi(S_n)}{\psi(u_n)})). \end{aligned}$$

Note that

$$\begin{aligned} &E[\{\prod_{i=1}^{n-1}\chi_i\}\{(1 - \frac{\phi(S_n)}{\psi(u_n)}) - (1 - \frac{\phi(S_{n-1})}{\psi(u_n)})\} + \frac{\phi(S_n) - \phi(S_{n-1})}{\psi(u_n)}] \\ &\qquad = E[(1 - \prod_{i=1}^{n-1}\chi_i)(\frac{\phi(S_n) - \phi(S_{n-1})}{\psi(u_n)})] \geq 0 \end{aligned}$$

since the function $1 - \prod_{i=1}^{n-1}\chi_i$ is a nonnegative componentwise nondecreasing function of $\phi(S_i)$, $1 \leq i \leq n-1$. Hence

$$\begin{aligned} P(A_n) &\geq E(\{\prod_{i=1}^{n-1}\chi_i\}(1 - \frac{\phi(S_{n-1})}{\psi(u_n)})) - \frac{E\{\phi(S_n)\} - E\{\phi(S_{n-1})\}}{\psi(u_n)} \\ &\geq E(\{\prod_{i=1}^{n-2}\chi_i\}(1 - \frac{\phi(S_{n-1})}{\psi(u_{n-1})})) - \frac{E\{\phi(S_n)\} - E\{\phi(S_{n-1})\}}{\psi(u_n)}. \end{aligned}$$

The last inequality follows from the observation that the sequence $\psi(u_n), n \geq 1$ is positive and nondecreasing.

Applying this inequality repeatedly, we get that

$$P(A_n) \geq 1 - \sum_{k=1}^{n}\frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)} \tag{2.6.4}$$

completing the proof of the first part of the theorem. Note that

$$E\{\prod_{i=1}^{n-1}\chi_i(1-\frac{\phi(S_n)}{\psi(u_n)})-(1-\frac{\Delta_n}{\psi(u_n)})(1-\frac{\phi(S_{n-1})}{\psi(u_{n-1})})\prod_{i=1}^{n-1}\chi_i\}$$
$$\geq E\{\frac{\phi(S_{n-1})}{\psi(u_n)\psi(u_{n-1})}[\psi(u_n)-\psi(u_{n-1})-\Delta_n]\prod_{i=1}^{n-1}\chi_i\}$$

and the last term is nonnegative by hypothesis. Hence

$$P(A_n)\geq(1-\frac{\Delta_n}{\psi(u_n)})E(\{\prod_{i=1}^{n-2}\chi_i\}(1-\frac{\phi(S_{n-1})}{\psi(u_{n-1})})). \tag{2.6.5}$$

Applying this inequality repeatedly, we obtain that

$$P(A_n)\geq\prod_{k=1}^{n}(1-\frac{\Delta_k}{\psi(u_k)}). \tag{2.6.6}$$

□

Applications

As applications of the Whittle type maximal inequality derived above, the following results can be obtained.

Suppose the sequence $\{S_n,\ n\geq 1\}$ is a demisubmartingale. Then the sequences $\{(S_n^+)^p,\ n\geq 1\}$ and $\{(S_n^-)^p,\ n\geq 1\}$ are demisubmartingales for $p\geq 1$ by Theorem 2.1.1. Furthermore $|S_n|^p=(S_n^+)^p+(S_n^-)^p$ for all $p\geq 1$.

(1) Let $\psi(u)=u^p$, $p\geq 1$ in Theorem 2.6.1. Applying Theorem 2.6.1, we get that

$$P(S_j^+\leq u_j, 1\leq j\leq n)\geq 1-\sum_{j=1}^{n}\frac{E(S_j^+)^p-E(S_{j-1}^+)^p}{u_j^p} \tag{2.6.7}$$

and

$$P(S_j^-\leq u_j, 1\leq j\leq n)\geq 1-\sum_{j=1}^{n}\frac{E(S_j^-)^p-E(S_{j-1}^-)^p}{u_j^p}. \tag{2.6.8}$$

Hence , for every $\varepsilon>0$,

$$\begin{aligned}P(\sup_{1\leq j\leq n}\frac{|S_j|}{u_j}\geq\varepsilon)&=P(\sup_{1\leq j\leq n}\frac{|S_j|^p}{u_j^p}\geq\varepsilon^p)\\&=P(\sup_{1\leq j\leq n}\frac{(S_j^+)^p+(S_j^-)^p}{u_j^p}\geq\varepsilon^p)\\&=P(\sup_{1\leq j\leq n}\frac{(S_j^+)^p}{u_j^p}\geq\frac{1}{2}\varepsilon^p)+P(\sup_{1\leq j\leq n}\frac{(S_j^-)^p}{u_j^p}\geq\frac{1}{2}\varepsilon^p)\end{aligned}$$

$$
\begin{aligned}
&\leq 2\varepsilon^{-p} \sum_{j=1}^{n} \frac{E(S_j^+)^p - E(S_{j-1}^+)^p}{u_j^p} \\
&\quad + 2\varepsilon^{-p} \sum_{j=1}^{n} \frac{E(S_j^-)^p - E(S_{j-1}^-)^p}{u_j^p} \\
&= 2\varepsilon^{-p} \sum_{j=1}^{n} \frac{E|S_j|^p - E|S_{j-1}|^p}{u_j^p}.
\end{aligned}
$$

In particular, for $p = 2$, we have

$$
P(\sup_{1\leq j\leq n} \frac{|S_j|}{u_j} \geq \varepsilon) \leq 2\varepsilon^{-2} \sum_{j=1}^{n} \frac{ES_j^2 - ES_{j-1}^2}{u_j^2}. \tag{2.6.9}
$$

which is the Hajek-Renyi type inequality for associated sequences derived in the Corollary 2.3 of Christofides (2000).

Suppose $p = 1$. Let $\phi(x) = \max(0, x)$. Then the function $\phi(x)$ is a nonnegative nondecreasing convex function and it is clear that $S_n \leq S_n^+ = \phi(S_n)$ for every $n \geq 1$. Let $\psi(u) = u$ in Theorem 2.6.1. Then

$$
P(\sup_{1\leq j\leq n} \frac{S_j}{u_j} \geq \varepsilon) \leq P(\sup_{1\leq j\leq n} \frac{S_j^+}{u_j} \geq \varepsilon) \leq \varepsilon^{-1} \sum_{j=1}^{n} \frac{ES_j^+ - ES_{j-1}^+}{u_j}
$$

by Theorem 2.6.1 which is the Chow type maximal inequality derived in Theorem 2.5.1 (cf. Christofides (2000)).

(2) Let $p = 2$ again in the above discussion. If

$$
E(S_j^2 - S_{j-1}^2) \leq u_j^2 - u_{j-1}^2,
$$

for $1 \leq j \leq n$, then

$$
P(A_n) \geq \prod_{j=1}^{n} (1 - \frac{E(S_j^2) - E(S_{j-1}^2)}{u_j^2})
$$

which is an analogue of the Dufresnoy type maximal inequality for martingales (cf. Dufresnoy (1967)).

(3) Let the sequence $\{S_n, n \geq 1\}$ be a demisubmartingale and the function $\phi(.)$ be a nonnegative nondecreasing convex function such that $\phi(S_0) = 0$. Let $\psi(u)$ be a positive nondecreasing function for $u > 0$. Then, for any nondecreasing sequence u_n, $n \geq 1$ with $u_0 = 0$,

$$
P(\sup_{1\leq j\leq n} \frac{\phi(S_j)}{\psi(u_j)} \geq \varepsilon) \leq \varepsilon^{-1} \sum_{k=1}^{n} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}. \tag{2.6.10}
$$

In particular, for any fixed $n \geq 1$,

$$P(\sup_{k \geq n} \frac{\phi(S_k)}{\psi(u_k)} \geq \varepsilon) \leq \varepsilon^{-1}[E(\frac{\phi(S_n)}{\psi(u_n)}) + \sum_{k=n+1}^{\infty} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)}]. \quad (2.6.11)$$

As a consequence of this inequality, we get the following strong law of large numbers for demisubmartingales (cf. Prakasa Rao (2002)).

Theorem 2.6.2. *Let $S_0 = 0$ and let the sequence $\{S_n,\, n \geq 1\}$ be a demisubmartingale. Let $\phi(.)$ be a nonnegative nondecreasing convex function such that $\phi(0) = 0$. Let $\psi(u)$ be a positive nondecreasing function for $u > 0$ such that $\psi(u) \to \infty$ as $u \to \infty$. Further suppose that*

$$\sum_{k=1}^{\infty} \frac{E[\phi(S_k)] - E[\phi(S_{k-1})]}{\psi(u_k)} < \infty$$

for a nondecreasing sequence $u_n \to \infty$ as $n \to \infty$. Then

$$\frac{\phi(S_n)}{\psi(u_n)} \stackrel{a.s}{\to} 0 \;\; as \;\; n \to \infty.$$

2.7 More on Maximal Inequalities

Suppose the sequence $\{S_n,\, n \geq 1\}$ is a demisubmartingale. Let $S_n^{\max} = \max_{1 \leq i \leq n} S_i$ and $S_n^{\min} = \min_{1 \leq i \leq n} S_i$. As special cases of Theorem 2.3.1, we get that

$$\lambda \;\; P[S_n^{\max} \geq \lambda] \leq \int_{[S_n^{\max} \geq \lambda]} S_n dP \quad (2.7.1)$$

and

$$\lambda \;\; P[S_n^{\min} \geq \lambda] \leq \int_{[S_n^{\min} \geq \lambda]} S_n dP \quad (2.7.2)$$

for any $\lambda > 0$.

The inequality (2.7.1) can also be obtained directly without using Theorem 2.3.1 by the standard methods used to prove the Kolomogorov inequality. We now prove a variant of the inequality (2.7.2).

Suppose the sequence $\{S_n,\, n \geq 1\}$ is a demisubmartingale. Let $\lambda > 0$. Let

$$A = [\min_{1 \leq k \leq n} S_k < \lambda], A_1 = [S_1 < \lambda]$$

and

$$A_k = [S_k < \lambda, S_j \geq \lambda, 1 \leq j \leq k-1], \quad k > 1.$$

Observe that

$$A = \bigcup_{k=1}^{n} A_k$$

and $A_k \in \mathcal{F}_k = \sigma\{S_1, \ldots, S_k\}$. Furthermore the sets A_k, $1 \le k \le n$ are disjoint and

$$A_k \subset (\bigcup_{i=1}^{k-1} A_i)^c$$

where A^c denotes the complement of the set A in Ω. Note that

$$E(S_1) = \int_{A_1} S_1 dP + \int_{A_1^c} S_1 dP \le \lambda \int_{A_1} dP + \int_{A_1^c} S_2 dP.$$

The last inequality follows by observing that

$$\int_{A_1^c} S_1 dP - \int_{A_1^c} S_2 dP = \int_{A_1^c} (S_1 - S_2) dP = E((S_1 - S_2) I_{[A_1^c]}).$$

Since the indicator function of the set $A_1^c = [S_1 \ge \lambda]$ is a nonnegative nondecreasing function of S_1 and the fact that $\{S_k,\ 1 \le k \le n\}$ is a demisubmartingale, it follows that

$$E((S_2 - S_1) I_{[A_1^c]}) \ge 0.$$

Therefore

$$E((S_1 - S_2) I_{[A_1^c]}) \le 0$$

which implies that

$$\int_{A_1^c} S_1 dP \le \int_{A_1^c} S_2 dP.$$

This proves the inequality

$$E(S_1) \le \lambda \int_{A_1} dP + \int_{A_1^c} S_2 dP = \lambda P(A_1) + \int_{A_1^c} S_2 dP.$$

Observe that $A_2 \subset A_1^c$. Hence

$$\begin{aligned} \int_{A_1^c} S_2 dP &= \int_{A_2} S_2 dP + \int_{A_2^c \cap A_1^c} S_2 dP \\ &\le \int_{A_2} S_2 dP + \int_{A_2^c \cap A_1^c} S_3 dP \\ &\le \lambda \ \ P(A_2) + \int_{A_2^c \cap A_1^c} S_3 dP. \end{aligned}$$

The second inequality in the above chain follows from the observation that the indicator function of the set $A_2^c \cap A_1^c = I_{[S_1 \ge \lambda, S_2 \ge \lambda]}$ is a nonnegative nondecreasing function of S_1, S_2 and the fact that the sequence $\{S_k,\ 1 \le k \le n\}$ is a demisubmartingale. By repeated application of these arguments, we get that

$$\begin{aligned} E(S_1) &\le \lambda \sum_{i=1}^{n} P(A_i) + \int_{\cap_{i=1}^{n} A_i^c} S_n dP \\ &= \lambda\ \ P(A) + \int_{\Omega} S_n dP - \int_A S_n dP. \end{aligned}$$

Hence

$$\lambda\ \ P(A) \ge \int_A S_n dP - \int_{\Omega} (S_n - S_1) dP$$

and we have the following result.

Theorem 2.7.1 (Wood (1984)). *Suppose that the sequence $\{S_n,\ n \ge 1\}$ is a demisubmartingale. Let*

$$A = [\min_{1 \le k \le n} S_k < \lambda]$$

for any $\lambda > 0$. Then

$$\lambda\ \ P(A) \ge \int_A S_n dP - \int_{\Omega} (S_n - S_1) dP. \tag{2.7.3}$$

In particular, if the sequence $\{S_n,\ n \ge 1\}$ is a demimartingale, then it is easy to check that $E(S_n) = E(S_1)$ for all $n \ge 1$ and hence we have the following result as a corollary to Theorem 2.7.1.

Theorem 2.7.2. *Suppose that the sequence $\{S_n,\ n \ge 1\}$ is a demimartingale. Let $A = [\min_{1 \le k \le n} S_k < \lambda]$ for any $\lambda > 0$. Then*

$$\lambda\ \ P(A) \ge \int_A S_n dP. \tag{2.7.4}$$

We now prove some inequalities for $E(S_n^{\max})$ and $E(S_n^{\min})$ for nonnegative demisubmartingales $\{S_n,\ n \ge 1\}$. The following results are from Prakasa Rao (2007).

Theorem 2.7.3. *Suppose that the sequence $\{S_n, n \ge 1\}$ is a positive demimartingale with $S_1 = 1$. Let $\gamma(x) = x - 1 - \log x$ for $x > 0$. Then*

$$\gamma(E(S_n^{\max})) \le E(S_n \log S_n) \tag{2.7.5}$$

and

$$\gamma(E(S_n^{\min})) \le E(S_n \log S_n). \tag{2.7.6}$$

Proof. Note that the function $\gamma(x)$ is a convex function with minimum $\gamma(1) = 0$. Observe that $S_n^{\max} \ge S_1 = 1$ and hence

$$\begin{aligned}
E(S_n^{\max}) - 1 &= \int_0^\infty P[S_n^{\max} \geq \lambda] d\lambda - 1 \\
&= \int_0^1 P[S_n^{\max} \geq \lambda] d\lambda + \int_1^\infty P[S_n^{\max} \geq \lambda] d\lambda - 1 \\
&= \int_1^\infty P[S_n^{\max} \geq \lambda] d\lambda \ \ (\text{since } \ S_1 = 1) \\
&\leq \int_1^\infty \{\frac{1}{\lambda} \int_{[S_n^{\max} \geq \lambda]} S_n dP\} d\lambda \ \ (\text{by } (2.7.2)) \\
&= E(\int_1^\infty \frac{S_n I_{[S_n^{\max} \geq \lambda]}}{\lambda} d\lambda) \\
&= E(S_n \int_1^{S_n^{\max}} \frac{1}{\lambda} d\lambda) \\
&= E(S_n \log(S_n^{\max})).
\end{aligned}$$

Using the fact that $\gamma(x) \geq 0$ for all $x > 0$, we get that

$$\begin{aligned}
E(S_n^{\max}) - 1 &\leq E[S_n(\log(S_n^{\max}) + \gamma(\frac{S_n^{\max}}{S_n E(S_n^{\max})}))] \\
&= E[S_n(\log(S_n^{\max}) + \frac{S_n^{\max}}{S_n E(S_n^{\max})} - 1 - \log(\frac{S_n^{\max}}{S_n E(S_n^{\max})}))] \\
&= 1 - E(S_n) + E(S_n \log S_n) + E(S_n) \log E(S_n^{\max}).
\end{aligned}$$

Rearranging the terms in the above inequality, we get that

$$\begin{aligned}
\gamma(E(S_n^{\max})) &= E(S_n^{\max}) - 1 - \log E(S_n^{\max}) \qquad (2.7.7) \\
&\leq 1 - E(S_n) + E(S_n \log S_n) + E(S_n) \log E(S_n^{\max}) - \log E(S_n^{\max}) \\
&= E(S_n \log S_n) + (E(S_n) - 1)(\log E(S_n^{(\max)}) - 1) \\
&= E(S_n \log S_n)
\end{aligned}$$

since $E(S_n) = E(S_1) = 1$ for all $n \geq 1$. This proves the inequality (2.7.5).

Observe that $0 \leq S_n^{\min} \leq S_1 = 1$ which implies that

$$\begin{aligned}
E(S_n^{\min}) &= \int_0^1 P[S_n^{\min} \geq \lambda] d\lambda \qquad (2.7.8) \\
&= 1 - \int_0^1 P[S_n^{\min} < \lambda] d\lambda \qquad (2.7.9) \\
&\leq 1 - \int_0^1 \{\frac{1}{\lambda} \int_{[S_n^{\min} < \lambda]} S_n dP\} d\lambda \ \ (\text{by Theorem } 2.7.2) \qquad (2.7.10) \\
&= 1 - E(\int_0^1 \frac{S_n I_{[S_n^{\min} < \lambda]}}{\lambda} d\lambda) \qquad (2.7.11)
\end{aligned}$$

$$= 1 - E(S_n \int_{S_n^{\min}}^{1} \frac{1}{\lambda} d\lambda) \tag{2.7.12}$$

$$= 1 + E(S_n \log(S_n^{\min})). \tag{2.7.13}$$

Applying arguments similar to those given above to prove the inequality (2.7.5), we get that

$$\gamma(E(S_n^{\min})) \leq E(S_n \log S_n) \tag{2.7.14}$$

which proves the inequality (2.7.6). □

The above inequalities for positive demimartingales are analogues of maximal inequalities for nonnegative martingales proved in Harremoës (2008).

2.8 Maximal ϕ-Inequalities for Nonnegative Demisubmartingales

Let $\mathcal{C}$ denote the class of *Orlicz functions* that is, unbounded, nondecreasing convex functions $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$. If the right derivative ϕ' is unbounded, then the function ϕ is called a *Young function* and we denote the subclass of such functions by $\mathcal{C}'$. Since

$$\phi(x) = \int_0^x \phi'(s) ds \leq x\phi'(x)$$

by convexity, it follows that

$$p_\phi = \inf_{x>0} \frac{x\phi'(x)}{\phi(x)}$$

and

$$p_\phi^* = \sup_{x>0} \frac{x\phi'(x)}{\phi(x)}$$

are in $[1, \infty]$. The function ϕ is called *moderate* if $p_\phi^* < \infty$, or equivalently, if for some $\lambda > 1$, there exists a finite constant c_λ such that

$$\phi(\lambda x) \leq c_\lambda \phi(x), \quad x \geq 0.$$

An example of such a function is $\phi(x) = x^\alpha$ for $\alpha \in [1, \infty)$. Example of a non-moderate Orlicz function is $\phi(x) = \exp(x^\alpha) - 1$ for $\alpha \geq 1$.

Let $\mathcal{C}^*$ denote the set of all differentiable $\phi \in \mathcal{C}$ whose derivative is concave or convex and $\mathcal{C}'$ denote the set of $\phi \in \mathcal{C}$ such that $\phi'(x)/x$ is integrable at 0, and thus, in particular $\phi'(0) = 0$. Let $\mathcal{C}_0^* = \mathcal{C}' \cap \mathcal{C}^*$.

Given $\phi \in \mathcal{C}$ and $a \geq 0$, define

$$\Phi_a(x) = \int_a^x \int_a^s \frac{\phi'(r)}{r} dr ds, \quad x > 0.$$

It can be seen that the function $\Phi_a I_{[a,\infty)} \in \mathcal{C}$ for any $a > 0$ where I_A denotes the indicator function of the set A. If $\phi \in \mathcal{C}'$, the same holds for $\Phi \equiv \Phi_0$. If $\phi \in \mathcal{C}_0^*$, then $\Phi \in \mathcal{C}_0^*$. Furthermore, if ϕ' is concave or convex, the same holds for

$$\Phi'(x) = \int_0^x \frac{\phi'(r)}{r} dr,$$

and hence $\phi \in \mathcal{C}_0^*$ implies that $\Phi \in \mathcal{C}_0^*$. It can be checked that ϕ and Φ are related through the differential equation

$$x\Phi'(x) - \Phi(x) = \phi(x), \quad x \geq 0$$

under the initial conditions $\phi(0) = \phi'(0) = \Phi(0) = \Phi'(0) = 0$. If $\phi(x) = x^p$ for some $p > 1$, then $\Phi(x) = x^p/(p-1)$. For instance, if $\phi(x) = x^2$, then $\Phi(x) = x^2$. If $\phi(x) = x$, then $\Phi(x) \equiv \infty$ but $\Phi_1(x) = x \log x - x + 1$. It is known that if $\phi \in \mathcal{C}'$ with $p_\phi > 1$, then the function ϕ satisfies the inequalities

$$\Phi(x) \leq \frac{1}{p_\phi - 1}\phi(x), \quad x \geq 0.$$

Furthermore, if ϕ is moderate, that is $p_\phi^* < \infty$, then

$$\Phi(x) \geq \frac{1}{p_\phi^* - 1}\phi(x), \quad x \geq 0.$$

The brief introduction for properties of Orlicz functions given here is based on Alsmeyer and Rosler (2006).

We now prove some maximal ϕ-inequalities for nonnegative demisubmartingales following the techniques in Alsmeyer and Rosler (2006).

Theorem 2.8.1. *Let the sequence $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale and let $\phi \in \mathcal{C}$. Then*

$$\begin{aligned} P(S_n^{\max} \geq t) &\leq \frac{\lambda}{(1-\lambda)t}\int_t^\infty P(S_n > \lambda s)ds \\ &= \frac{\lambda}{(1-\lambda)t} E(\frac{S_n}{\lambda} - t)^+ \end{aligned} \tag{2.8.1}$$

for all $n \geq 1$, $t > 0$ and $0 < \lambda < 1$. Furthermore,

$$E[\phi(S_n^{\max})] \leq \phi(b) + \frac{\lambda}{1-\lambda}\int_{[S_n > \lambda b]} (\Phi_a(\frac{S_n}{\lambda}) - \Phi_a(b) - \Phi_a'(b)(\frac{S_n}{\lambda} - b))dP \tag{2.8.2}$$

for all $n \geq 1$, $a > 0$, $b > 0$ and $0 < \lambda < 1$. If $\phi'(x)/x$ is integrable at 0, that is, $\phi \in \mathcal{C}'$, then the inequality (2.8.2) holds for $b = 0$.

Proof. Let $t > 0$ and $0 < \lambda < 1$. The inequality (2.7.1) implies that

$$\begin{aligned} P(S_n^{\max} \geq t) &\leq \frac{1}{t} \int_{[S_n^{\max} \geq t]} S_n dP \\ &= \frac{1}{t} \int_0^\infty P[S_n^{\max} \geq t, S_n > s] ds \\ &\leq \frac{1}{t} \int_0^{\lambda t} P[S_n^{\max} \geq t] ds + \frac{1}{t} \int_{\lambda t}^\infty P[S_n > s] ds \\ &\leq \lambda P[S_n^{\max} \geq t] ds + \frac{\lambda}{t} \int_t^\infty P[S_n > \lambda s] ds. \end{aligned} \tag{2.8.3}$$

Rearranging the last inequality, we get that

$$P(S_n^{\max} \geq t) \leq \frac{\lambda}{(1-\lambda)t} \int_t^\infty P(S_n > \lambda s) ds = \frac{\lambda}{(1-\lambda)t} E(\frac{S_n}{\lambda} - t)^+$$

for all $n \geq 1$, $t > 0$ and $0 < \lambda < 1$ proving the inequality (2.8.1). Let $b > 0$. Then

$$\begin{aligned} E[\phi(S_n^{\max})] &= \int_0^\infty \phi'(t) P(S_n^{\max} > t) dt \\ &= \int_0^b \phi'(t) P(S_n^{\max} > t) dt + \int_b^\infty \phi'(t) P(S_n^{\max} > t) dt \\ &\leq \phi(b) + \int_b^\infty \phi'(t) P(S_n^{\max} > t) dt \\ &\leq \phi(b) + \frac{\lambda}{1-\lambda} \int_b^\infty \frac{\phi'(t)}{t} [\int_t^\infty P(S_n > \lambda s) ds] dt \ \ \text{(by (2.7.1))} \\ &= \phi(b) + \frac{\lambda}{1-\lambda} \int_b^\infty (\int_b^s \frac{\phi'(t)}{t} dt) P(S_n > \lambda s) ds \\ &= \phi(b) + \frac{\lambda}{1-\lambda} \int_b^\infty (\Phi_a'(s) - \Phi_a'(b)) P(S_n > \lambda s) ds \\ &= \phi(b) + \frac{\lambda}{1-\lambda} \int_{[S_n > \lambda b]} (\Phi_a(\frac{S_n}{\lambda}) - \Phi_a(b) - \Phi_a'(b)(\frac{S_n}{\lambda} - b)) dP \end{aligned}$$

for all $n \geq 1$, $b > 0$, $t > 0$, $0 < \lambda < 1$ and $a > 0$. The value of a can be chosen to be 0 if $\phi'(x)/x$ is integrable at 0. □

As special cases of the above result, we obtain the following inequalities by choosing $b = a$ in (2.8.2). Observe that $\Phi_a(a) = \Phi_a'(a) = 0$.

Theorem 2.8.2. *Let the sequence $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale and let $\phi \in \mathcal{C}$. Then*

$$E[\phi(S_n^{\max})] \leq \phi(a) + \frac{\lambda}{1-\lambda} E[\Phi_a(\frac{S_n}{\lambda})] \tag{2.8.4}$$

for all $a \geq 0$, $0 < \lambda < 1$ *and* $n \geq 1$. *Let* $\lambda = \frac{1}{2}$ *in (2.8.4). Then*

$$E[\phi(S_n^{\max})] \leq \phi(a) + E[\Phi_a(2S_n)] \tag{2.8.5}$$

for all $a \geq 0$ *and* $n \geq 1$.

The following lemma is due to Alsmeyer and Rosler (2006).

Lemma 2.8.3. *Let* X *and* Y *be nonnegative random variables satisfying the inequality*

$$t \;\; P(Y \geq t) \leq E(XI_{[Y \geq t]})$$

for all $t \geq 0$. *Then*

$$E[\phi(Y)] \leq E[\phi(q_\phi X)] \tag{2.8.6}$$

for any Orlicz function ϕ *where* $q_\phi = \frac{p_\phi}{p_\phi - 1}$ *and* $p_\phi = \inf_{x>0} \frac{x\phi'(x)}{\phi(x)}$.

This lemma follows as an application of the Choquet decomposition

$$\phi(x) = \int_{[0,\infty)} (x-t)^+ \phi'(dt), x \geq 0.$$

In view of the inequality (2.8.2), we can apply the above lemma to the random variables $X = S_n$ and $Y = S_n^{\max}$ to obtain the following result.

Theorem 2.8.4. *Let the sequence* $\{S_n, n \geq 1\}$ *be a nonnegative demisubmartingale and let* $\phi \in \mathcal{C}$ *with* $p_\phi > 1$. *Then*

$$E[\phi(S_n^{\max})] \leq E[\phi(q_\phi S_n)] \tag{2.8.7}$$

for all $n \geq 1$.

Theorem 2.8.5. *Let the sequence* $\{S_n, n \geq 1\}$ *be a nonnegative demisubmartingale. Suppose that the function* $\phi \in \mathcal{C}$ *is moderate. Then*

$$E[\phi(S_n^{\max})] \leq E[\phi(q_\phi S_n)] \leq q_\Phi^{p_\phi^*} E[\phi(S_n)]. \tag{2.8.8}$$

The first part of the inequality (2.8.8) of Theorem 2.8.5 follows from Theorem 2.8.4. The last part of the inequality follows from the observation that if $\phi \in \mathcal{C}$ is moderate, that is,

$$p_\phi^* = \sup_{x>0} \frac{x\phi'(x)}{\phi(x)} < \infty,$$

then

$$\phi(\lambda x) \leq \lambda^{p_\phi^*} \phi(x)$$

for all $\lambda > 1$ and $x > 0$ (see equation (1.10) of Alsmeyer and Rosler (2006)).

Theorem 2.8.6. *Let the sequence $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale. Suppose ϕ is a nonnegative nondecreasing function on $[0,\infty)$ such that $\phi^{1/\gamma}$ is also nondecreasing and convex for some $\gamma > 1$. Then*

$$E[\phi(S_n^{\max})] \leq (\frac{\gamma}{\gamma-1})^\gamma E[\phi(S_n)]. \tag{2.8.9}$$

Proof. The inequality

$$\lambda P(S_n^{\max} \geq \lambda) \leq \int_{[S_n^{\max}\geq\lambda]} S_n dP$$

given in (2.7.1) implies that

$$E[(S_n^{\max})^p] \leq (\frac{p}{p-1})^p E(S_n^p), \quad p > 1 \tag{2.8.10}$$

by an application of Hölder's inequality (cf. Chow and Teicher (1997), p. 255). Note that the sequence $\{[\phi(S_n)]^{1/\gamma}, n \geq 1\}$ is a nonnegative demisubmartingale by Theorem 2.1.1. Applying the inequality (2.8.10) for the sequence $\{[\phi(S_n)]^{1/\gamma}, n \geq 1\}$ and choosing $p = \gamma$ in that inequality, we get that

$$E[\phi(S_n^{\max})] \leq (\frac{\gamma}{\gamma-1})^\gamma E[\phi(S_n)]. \tag{2.8.11}$$

for all $\gamma > 1$. □

Examples of functions ϕ satisfying the conditions stated in Theorem 2.8.6 are $\phi(x) = x^p[\log(1+x)]^r$ for $p > 1$ and $r \geq 0$ and $\phi(x) = e^{rx}$ for $r > 0$. Applying the result in Theorem 2.8.6 for the function $\phi(x) = e^{rx}, r > 0$, we obtain the following inequality.

Theorem 2.8.7. *Let the sequence $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale. Then*

$$E[e^{rS_n^{\max}}] \leq eE[e^{rS_n}], \quad r > 0. \tag{2.8.12}$$

Proof. Applying the result stated in Theorem 2.8.6 to the function $\phi(x) = e^{rx}$, we get that

$$E[e^{rS_n^{\max}}] \leq (\frac{\gamma}{\gamma-1})^\gamma E[e^{rS_n}] \tag{2.8.13}$$

for any $\gamma > 1$. Let $\gamma \to \infty$. Then

$$(\frac{\gamma}{\gamma-1})^\gamma \downarrow e$$

and we get that

$$E[e^{rS_n^{\max}}] \leq eE[e^{rS_n}], r > 0. \tag{2.8.14}$$

□

The next result deals with maximal inequalities for functions $\phi \in \mathcal{C}$ which are k times differentiable with the k-th derivative $\phi^{(k)} \in \mathcal{C}$ for some $k \geq 1$.

Theorem 2.8.8. *Let the sequence $\{S_n, n \geq 1\}$ be a nonnegative demisubmartingale. Let $\phi \in \mathcal{C}$ which is differentiable k times with the k-th derivative $\phi^{(k)} \in \mathcal{C}$ for some $k \geq 1$. Then*

$$E[\phi(S_n^{\max})] \leq (\frac{k+1}{k})^{k+1} E[\phi(S_n)]. \tag{2.8.15}$$

Proof. The proof follows the arguments given in Alsmeyer and Rosler (2006) following the inequality (2.8.9). We present the proof here for completeness. Note that

$$\phi(x) = \int_{[0,\infty)} (x-t)^+ Q_\phi(dt)$$

where

$$Q_\phi(dt) = \phi'(0)\delta_0 + \phi'(dt)$$

and δ_0 is the Kronecker delta function. Hence, if $\phi' \in \mathcal{C}$, then

$$\begin{aligned} \phi(x) &= \int_0^x \phi'(y)dy \qquad (2.8.16)\\ &= \int_0^x \int_{[0,\infty)} (y-t)^+ Q_{\phi'}(dt)dy \\ &= \int_{[0,\infty)} \int_0^x (y-t)^+ dy Q_{\phi'}(dt) \\ &= \int_{[0,\infty)} \frac{((x-t)^+)^2}{2} Q_{\phi'}(dt). \end{aligned}$$

An inductive argument shows that

$$\phi(x) = \int_{[0,\infty)} \frac{((x-t)^+)^{k+1}}{(k+1)!} Q_{\phi^{(k)}}(dt) \tag{2.8.17}$$

for any $\phi \in \mathcal{C}$ such that $\phi^{(k)} \in \mathcal{C}$. Let

$$\phi_{k,t}(x) = \frac{((x-t)^+)^{k+1}}{(k+1)!}$$

for any $k \geq 1$ and $t \geq 0$. Note that the function $[\phi_{k,t}(x)]^{1/(k+1)}$ is nonnegative, convex and nondecreasing in x for any $k \geq 1$ and $t \geq 0$. Hence the process $\{[\phi_{k,t}(S_n)]^{1/(k+1)}, n \geq 1\}$ is a nonnegative demisubmartingale by Theorem 2.1.1. Following the arguments given to prove (2.8.10), we obtain that

$$E(([\phi_{k,t}(S_n^{\max})]^{1/(k+1)})^{k+1}) \leq (\frac{k+1}{k})^{k+1} E(([\phi_{k,t}(S_n)]^{1/(k+1)})^{k+1})$$

which implies that

$$E[\phi_{k,t}(S_n^{\max})] \leq (\frac{k+1}{k})^{k+1} E[\phi_{k,t}(S_n)]. \tag{2.8.18}$$

Hence

$$\begin{aligned} E[\phi(S_n^{\max}))] &= \int_{[0,\infty)} E[\phi_{k,t}(S_n^{\max})] Q_{\phi^{(k)}}(dt) \ \text{ (by (2.8.17))} \qquad (2.8.19) \\ &\leq (\frac{k+1}{k})^{k+1} \int_{[0,\infty)} E[\phi_{k,t}(S_n)] Q_{\phi^{(k)}}(dt) \ \text{ (by (2.8.18))} \\ &= (\frac{k+1}{k})^{k+1} E[\phi(S_n)] \end{aligned}$$

which proves the theorem. □

We now consider a special case of the maximal inequality derived in (2.8.2) of Theorem 2.8.1. Let $\phi(x) = x$. Then $\Phi_1(x) = x \log x - x + 1$ and $\Phi_1'(x) = \log x$. The inequality (2.8.2) reduces to

$$\begin{aligned} E[S_n^{\max}] &\leq b + \frac{\lambda}{1-\lambda} \int_{[S_n > \lambda b]} (\frac{S_n}{\lambda} \log \frac{S_n}{\lambda} - \frac{S_n}{\lambda} + b - (\log b) \frac{S_n}{\lambda}) dP \\ &= b + \frac{\lambda}{1-\lambda} \int_{[S_n > \lambda b]} (S_n \log S_n - S_n(\log \lambda + \log b + 1) + \lambda b) dP \end{aligned}$$

for all $b > 0$ and $0 < \lambda < 1$. Let $b > 1$ and $\lambda = \frac{1}{b}$. Then we obtain the inequality

$$E(S_n^{\max}) \leq b + \frac{b}{b-1} E(\int_1^{\max(S_n,1)} \log x \ \ dx), \quad b > 1, \ n \geq 1. \tag{2.8.20}$$

The value of b which minimizes the term on the right-hand side of equation (2.8.20) is

$$b^* = 1 + (E(\int_1^{\max(S_n,1)} \log x \ \ dx))^{1/2}$$

and hence

$$E(S_n^{\max}) \leq (1 + (E(\int_1^{\max(S_n,1)} \log x \ \ dx))^{1/2})^2. \tag{2.8.21}$$

Since

$$\int_1^x \log y \ \ dy = x \log^+ x - (x-1), \quad x \geq 1,$$

the inequality (2.8.20) can be written in the form

$$E(S_n^{\max}) \leq b + \frac{b}{b-1} (E(S_n \log^+ S_n) - E(S_n - 1)^+), \quad b > 1, \ n \geq 1. \tag{2.8.22}$$

Let $b = E(S_n - 1)^+$ in equation (2.8.22). Then we get the maximal inequality

$$E(S_n^{\max}) \leq \frac{1 + E(S_n - 1)^+}{E(S_n - 1)^+} E(S_n \log^+ S_n). \tag{2.8.23}$$

If we choose $b = e$ in equation (2.8.22), then we get the maximal inequality

$$E(S_n^{\max}) \leq e + \frac{e}{e-1}(E(S_n \log^+ S_n) - E(S_n - 1)^+), \quad b > 1, \ n \geq 1. \tag{2.8.24}$$

Results discussed in this section are due to Prakasa Rao (2007).

2.9 Maximal Inequalities for Functions of Demisubmartingales

We now derive some maximal inequalities due to Wang and Hu (2009) and Wang et al. (2010) for functions of demimartingales and demisubmartingales.

Theorem 2.9.1. *Let the sequence $\{S_n, n \geq 1\}$ be a demisubmartingale with $S_0 = 0$ and $g(.)$ be a nondecreasing convex function on R with $g(0) = 0$. Suppose that $E|g(S_i)| < \infty$, $i \geq 1$. Let the sequence $\{c_n, n \geq 1\}$ be a nonincreasing sequence of positive numbers. Then, for any $\epsilon > 0$,*

$$\epsilon\, P[\max_{1 \leq k \leq n]} c_k g(S_k) \geq \epsilon] \leq \sum_{k=1}^{n} c_k E[g^+(S_k) - g^+(S_{k-1})]. \tag{2.9.1}$$

Proof. This result follows from the fact that $\{g(S_n), n \geq 1\}$ is a demisubmartingale and applying Theorem 5.1 on Chow type maximal inequality for demisubmartingales (cf. Christofides (2000)). □

The following theorem is due to Wang (2004). We omit the proof.

Theorem 2.9.2. *Let the sequence $\{S_n, n \geq 1\}$ be a demimartingale, $S_0 = 0$, and $g(.)$ be a nonnegative convex function on R with $g(0) = 0$. Suppose that $E|g(S_i)| < \infty$, $i \geq 1$. Let $\{c_n, n \geq 1\}$ be a nonincreasing sequence of positive numbers. Then, for any $\epsilon > 0$,*

$$\begin{aligned}\epsilon\, P[\max_{1 \leq k \leq n]} c_k g(S_k) \geq \epsilon] &\leq \sum_{k=1}^{n} c_k E[(g(S_k) - g(S_{k-1}))I_{[\max_{1 \leq j \leq n} c_j g(S_j) \geq \epsilon]}] \\ &\leq \sum_{k=1}^{n} c_k E[g(S_k) - g(S_{k-1})]. \end{aligned} \tag{2.9.2}$$

As an application of the above theorem, we obtain the following result by choosing the function $g(x) = |x|$. For any $\epsilon > 0$,

$$\epsilon\, P[\max_{1 \leq k \leq n]} c_k |S_k| \geq \epsilon] \leq \sum_{k=1}^{n} c_k E[|S_k| - |S_{k-1}|]. \tag{2.9.3}$$

The following inequality gives a Doob type maximal inequality for functions of demisubmartingales.

Theorem 2.9.3. *Let the sequence* $\{S_n,\ n \geq 1\}$ *be a demisubmartingale,* $S_0 = 0$, *and* $g(.)$ *be a nondecreasing convex function on* R *with* $g(0) = 0$. *Suppose that* $E|g(S_i)| < \infty$, $i \geq 1$. *Then, for any* $\epsilon > 0$,

$$\epsilon\ P[\max_{1\leq k\leq n} g(S_k) \geq \epsilon] \leq \int_{[\max_{1\leq k\leq n} g(S_k)\geq \epsilon]} g(S_n)]\ dP. \tag{2.9.4}$$

Proof. This result follows from the fact that the sequence $\{g(S_n),\ n \geq 1\}$ is a demisubmartingale and applying Theorem 2.1 on Doob type maximal inequality for demisubmartingales. □

The following inequality gives a Doob type maximal inequality for nonnegative convex functions of demimartingales.

Theorem 2.9.4. *Let the sequence* $\{S_n,\ n \geq 1\}$ *be a demimartingale and* $g(.)$ *be a nonnegative convex function on* R *with* $g(0) = 0$. *Suppose that the random variables* $g(S_i)$, $i \geq 1$ *are integrable. Then, for any* $\epsilon > 0$,

$$\epsilon\ P[\max_{1\leq k\leq n} g(S_k) \geq \epsilon] \leq \int_{[\max_{1\leq k\leq n} g(S_k)\geq \epsilon]} g(S_n)\ dP. \tag{2.9.5}$$

For a proof, see Wang and Hu (2009). In particular, by choosing $g(x) = |x|^r$, $r \geq 1$, we obtain the following result.

Theorem 2.9.5. *Let the sequence* $\{S_n,\ n \geq 1\}$ *be a demimartingale,* $S_0 = 0$, *and suppose that* $E|S_n|^r < \infty$, $n \geq 1$ *for some* $r \geq 1$. *Then, for any* $\epsilon > 0$,

$$P[\max_{1\leq k\leq n} |S_k| \geq \epsilon] \leq \frac{1}{\epsilon^r}\int_{[\max_{1\leq k\leq n} |S_k|^r\geq \epsilon^r]} |S_n|^r\ dP \leq \frac{1}{\epsilon^r} E|S_n|^r. \tag{2.9.6}$$

Wang et al. (2010) derived some inequalities for expectations of maxima of functions of demisubmartingales. We now discuss some of these results.

Theorem 2.9.6. *Let the sequence* $\{S_n,\ n \geq 1\}$ *be a demimartingale and* $g(.)$ *be a nonnegative convex function on* R *with* $g(0) = 0$. *Suppose that* $E|g(S_k)|^p < \infty$, $k \geq 1$, *for some* $p > 1$. *Let* $\{c_n,\ n \geq 1\}$ *be a nonincreasing sequence of positive numbers. Then*

$$E[\max_{1\leq k\leq n} c_k g(S_k)]^p \leq (\frac{p}{p-1})^p E[\sum_{k=1}^n c_k(g(S_k) - g(S_{k-1}))]^p. \tag{2.9.7}$$

If $p = 1$, *then*

$$\begin{aligned} &E[\max_{1\leq k\leq n} c_k g(S_k)] \\ &\leq \frac{e}{e-1}(1 + E[(\sum_{k=1}^n c_k(g(S_k) - g(S_{k-1}))\log^+(\sum_{k=1}^n c_k(g(S_k) - g(S_{k-1})))]). \end{aligned} \tag{2.9.8}$$

Proof. Let $p > 1$. Then, by Theorem 2.9.2 due to Wang (2004) and Hölder's inequality, we have

$$\begin{aligned} &E[\max_{1\le k\le n]} c_k g(S_k)]^p \qquad (2.9.9)\\ &= p\int_0^\infty x^{p-1} P[\max_{1\le k\le n} c_k g(S_k) \ge x]\ dx \\ &\le p\int_0^\infty x^{p-2} E[\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))I_{[\max_{1\le k\le n} c_k g(S_k)\ge x]}]\ dx \\ &= \frac{p}{p-1}E([\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))(\max_{1\le k\le n} c_k g(S_k))^{p-1}) \\ &\le \frac{p}{p-1}(E([\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))]^p)^{1/p}(E(\max_{1\le k\le n} c_k g(S_k))^p)^{1/q} \end{aligned}$$

where q is such that $1/p+1/q=1$. Rearranging the last inequality, we get that

$$(E[\max_{1\le k\le n]} c_k g(S_k)]^p)^{1/p} \le \frac{p}{p-1}(E[\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))]^p)^{1/p} \qquad (2.9.10)$$

which implies that

$$E[\max_{1\le k\le n]} c_k g(S_k)]^p \le (\frac{p}{p-1})^p E[\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))]^p. \qquad (2.9.11)$$

Let us now consider the case $p=1$. Then

$$\begin{aligned} &E[\max_{1\le k\le n]} c_k g(S_k)] \qquad (2.9.12)\\ &\le 1+\int_1^\infty P[\max_{1\le k\le n} c_k g(S_k)\ge x]\ dx \\ &\le 1+\int_1^\infty x^{-1}E[\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1}))I_{[\max_{1\le k\le n} c_k g(S_k)\ge x]}]\ dx \\ &\le 1+E[(\sum_{j=1}^n c_j(g(S_j)-g(S_{j-1})))\log^+(\max_{1\le k\le n} c_k g(S_k))]. \end{aligned}$$

Note that, for any $a\ge 0$ and $b>0$,

$$a\log^+ b \le a\log^+ a + be^{-1}.$$

Applying this inequality, we get that

$$E[\max_{1\le k\le n} c_k g(S_k)] \tag{2.9.13}$$
$$\le 1+E[(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))) \log^+(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))]$$
$$+e^{-1}E[\max_{1\le k\le n} c_k g(S_k)].$$

Hence

$$E[\max_{1\le k\le n} c_k g(S_k)] \tag{2.9.14}$$
$$\le \frac{e}{e-1}(1+E[(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1}))\log^+(\sum_{j=1}^{n} c_j(g(S_j)-g(S_{j-1})))]). \quad \square$$

As a corollary to the above result, we get the following bound on the expectations for maximum of a nonnegative convex function of a demimartingale. Choose $c_k = 1$, $k \ge 1$.

Theorem 2.9.7. *Let the sequence $\{S_n,\ n \ge 1\}$ be a demimartingale and $g(.)$ be a nonnegative convex function on R with $g(0) = 0$. Suppose that $E|g(S_k|^p < \infty$, $k \ge 1$, for some $p \ge 1$. If $p > 1$, then*

$$E[\max_{1\le k\le n} g(S_k)]^p \le (\frac{p}{p-1})^p E[g(S_n)]^p \tag{2.9.15}$$

and

$$E[\max_{1\le k\le n} g(S_k)] \le \frac{e}{e-1}[1+E(g(S_n)\log^+ g(S_n))]. \tag{2.9.16}$$

As an additional corollary, we obtain the following Doob type maximal inequality for demimartingales.

Theorem 2.9.8. *Let the sequence $\{S_n,\ n \ge 1\}$ be a demimartingale and $p \ge 1$. Suppose that $E|S_k|^p < \infty$, $k \ge 1$. Then, for every $n \ge 1$,*

$$E[\max_{1\le k\le n} |S_k|]^p \le (\frac{p}{p-1})^p E[|S_n|^p], \quad p > 1 \tag{2.9.17}$$

and

$$E[\max_{1\le k\le n} |S_k|] \le \frac{e}{e-1}[1+E(|S_n|\log^+ |S_n|)]. \tag{2.9.18}$$

Remarks. As corollaries to the above results, one can obtain a strong law of large numbers for functions of demimartingales and partial sums of mean zero associated random variables using conditions developed in Fazekas and Klesov (2001) and Hu and Hu (2006). For related results, see Hu et al. (2008). We will not discuss these results.

2.10 Central Limit Theorems

As was mentioned in Chapter 1, Newman (1980,1984) obtained the central limit theorem for partial sums of mean zero stationary associated random variables which form a demimartingale This result also holds for weakly associated random sequences as defined below. This was pointed out by Sethuraman (2000). However it is not known whether the central limit theorem holds for any demimartingale in general.

Let $\{\mathbf{v}(t) = (v_1(t), \ldots, v_m(t)),\ t \geq 0\}$ be an m-dimensional L^2-process with stationary increments. This process is said to have *weakly positive associated increments* if

$$E[\phi(\mathbf{v}(t+s) - \mathbf{v}(s))\psi(\mathbf{v}(s_1), \ldots, \mathbf{v}(s_n))] \geq E[\phi(\mathbf{v}(t))]E[\psi(\mathbf{v}(s_1), \ldots, \mathbf{v}(s_n)]$$

for all componentwise nondecreasing functions ϕ and ψ, for all $s, t \geq 0$ and for $0 \leq s_1 < \ldots, s_n = s$, $n \geq 1$.

Sethuraman (2000, 2006) proved the following invariance principle for processes which have weakly positive associated increments. This theorem is a consequence of the central limit theorem due to Newman (1980) and the maximal inequalities for demimartingales discussed earlier in this chapter (cf. Newman and Wright (1982)). Let $\mathbf{d}$ denote a column vector in R^m and $\mathbf{d}'$ the corresponding row vector.

Theorem 2.10.1. *Let $\{\mathbf{v}'(t) = (v_1(t), \ldots, v_m(t)),\ t \geq 0\}$ be an m-dimensional L^2-process in $C[0,\infty)$ with stationary and weakly positive associated increments such that $E[v_i(t)] = 0$ for $1 \leq i \leq m$ and $t \geq 0$. Further suppose that*

$$\lim_{t\to\infty} t^{-1}E[v_i(t)v_j(t)] = \sigma_{ij} < \infty. \tag{2.10.1}$$

Then

$$\alpha^{-1/2}\mathbf{v}'(\alpha t)\mathbf{d} \to W(\mathbf{d}'\Sigma\mathbf{d}t) \ \text{ as } \ \alpha \to \infty \tag{2.10.2}$$

weakly in the uniform topology where $\Sigma = ((\sigma_{ij})_{m\times m}$ is the covariance matrix, $\mathbf{d} \in R^m$ and W is the standard Brownian motion.

Newman (1984) conjectured the following result: Let $S_0 \equiv 0$ and the sequence $\{S_n, n \geq 1\}$ be an L^2- demimartingale whose difference sequence $\{X_n = S_n - S_{n-1}$, $n \geq 1\}$ is strictly stationary and ergodic with

$$0 < \sigma^2 = Var(X_1) + 2\sum_{j=2}^{\infty} \mathrm{Cov}(X_1, X_j) < \infty.$$

Then

$$n^{-1/2}S_n \xrightarrow{\mathcal{L}} \sigma Z \text{ as } \ n \to \infty$$

where Z is a standard normal random variable. It is not known whether the above conjecture is true. The problem remains open. We will come back to the discussion on Newman's conjecture in Chapter 6.

2.11 Dominated Demisubmartingales

Let $M_0 = N_0 = 0$ and the sequence $\{M_n, n \geq 0\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that

$$E[(M_{n+1} - M_n)f(M_0, \ldots, M_n)|\zeta_n] \geq 0$$

for any nonnegative componentwise nondecreasing function f given a filtration $\{\zeta_n, n \geq 0\}$ contained in $\mathcal{F}$. Then the sequence $\{M_n, n \geq 0\}$ is said to be a *strong demisubmartingale* with respect to the filtration $\{\zeta_n, n \geq 0\}$. It is obvious that a strong demisubmartingale is a demisubmartingale in the sense discussed earlier.

Definition. Let $M_0 = 0 = N_0$. Suppose $\{M_n, n \geq 0\}$ is a strong demisubmartingale with respect to the filtration generated by a demisubmartingale $\{N_n, n \geq 0\}$. The strong demisubmartingale $\{M_n, n \geq 0\}$ is said to be *weakly dominated* by the demisubmartingale $\{N_n, n \geq 0\}$ if for every nondecreasing convex function $\phi : R_+ \to R$, and for any nonnegative componentwise nondecreasing function $f : R^{2n} \to R$,

$$E[(\phi(|e_n|) - \phi(|d_n|)f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1})|N_0, \ldots, N_{n-1}] \geq 0 \text{ a.s.,} \tag{2.11.1}$$

for all $n \geq 1$ where $d_n = M_n - M_{n-1}$ and $e_n = N_n - N_{n-1}$. We write $M \ll N$ in such a case.

In analogy with the inequalities for dominated martingales developed in Osekowski (2007), we will now prove an inequality for domination between a strong demisubmartingale and a demisubmartingale.

Following Osekowski (2007), define the functions $u_{<2}(x, y)$ and $u_{>2}(x, y)$ as given below. Let

$$u_{<2}(x, y) = \begin{cases} 9|y|^2 - 9|x|^2 & \text{if } (x, y) \in D, \\ 2|y| - 1 + 8|y|^2 I_{[|y| \leq 1]} + (16|y| - 8)I_{[|y|>1]} & \text{if } (x, y) \in D^c \end{cases} \tag{2.11.2}$$

and

$$u_{>2}(x, y) = \begin{cases} 0 & \text{if } (x, y) \in E, \\ 9|y|^2 - (|x| - 1)^2 - 8(|x| - 1)^2 I_{[|x| \geq 1]} & \text{if } (x, y) \in E^c, \end{cases} \tag{2.11.3}$$

where

$$D = \{(x, y) \in R^2 : |y| + 3|x| \leq 1\} \text{ and } E = \{(x, y) \in R^2 : 3|y| + |x| \leq 1\}.$$

We now state a weak-type inequality between dominated demisubmartingales.

Theorem 2.11.1. *Suppose the sequence $\{M_n, n \geq 0\}$ is a strong demisubmartingale with respect to the filtration generated by the sequence $\{N_n, n \geq 0\}$ which is a demisubmartingale . Further suppose that $M \ll N$. Then, for any $\lambda > 0$,*

$$\lambda\, P(|M_n| \geq \lambda) \leq 6E|N_n|, \quad n \geq 0. \tag{2.11.4}$$

We will first state and sketch the proof of a lemma which will be used to prove Theorem 2.11.1. The method of proof is the same as that in Osekowski (2007).

Lemma 2.11.2. *Suppose the sequence $\{M_n, n \geq 0\}$ is a strong demisubmartingale with respect to the filtration generated by the sequence $\{N_n, n \geq 0\}$ which is a demisubmartingale. Further suppose that $M \ll N$. Then*

$$\begin{aligned} &E[u_{<2}(M_n, N_n) f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1})] \\ &\geq E[u_{<2}(M_{n-1}, N_{n-1}) f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1})] \end{aligned} \tag{2.11.5}$$

and

$$\begin{aligned} &E[u_{>2}(M_n, N_n) f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1})] \\ &\geq E[u_{>2}(M_{n-1}, N_{n-1}) f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1})] \end{aligned} \tag{2.11.6}$$

for any nonnegative componentwise nondecreasing function $f : R^{2n} \to R$, $n \geq 1$.

Proof. Define $u(x, y)$ where $u = u_{<2}$ or $u = u_{>2}$. From the arguments given in Osekowski (2007), it follows that there exist a nonnegative function $A(x, y)$ nondecreasing in x and a nonnegative function $B(x, y)$ nondecreasing in y and a convex nondecreasing function $\phi_{x,y}(.) : R_+ \to R$, such that, for any h and k,

$$u(x, y) + A(x, y)h + B(x, y)k + \phi_{x,y}(|k|) - \phi_{x,y}(|h|) \leq u(x + h, y + k). \tag{2.11.7}$$

Let $x = M_{n-1}$, $y = N_{n-1}$, $h = d_n$ and $k = e_n$. Then, it follows that

$$\begin{aligned} &u(M_{n-1}, N_{n-1}) + A(M_{n-1}, N_{n-1})d_n + B(M_{n-1}, N_{n-1})e_n \\ &\quad + \phi_{M_{n-1},N_{n-1}}(|e_n|) - \phi_{M_{n-1},N_{n-1}}(|d_n|) \\ &\leq u(M_{n-1} + d_n, N_{n-1} + e_n) = u(M_n, N_n). \end{aligned} \tag{2.11.8}$$

Note that,

$$E[A(M_{n-1}, N_{n-1})d_n f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1})|N_0, \ldots, N_{n-1}] \geq 0 \text{ a.s.}$$

from the fact that $\{M_n, n \geq 0\}$ is a strong demisubmartingale with respect to the filtration generated by the process $\{N_n, n \geq 0\}$ and that the function

$$A(x_{n-1}, y_{n-1}) f(x_0, \ldots, x_{n-1}; y_0, \ldots, y_{n-1})$$

is a nonnegative componentwise nondecreasing function in $x_0, \ldots, x_{n-1}$ for any fixed $y_0, \ldots, y_{n-1}$. Taking expectation on both sides of the above inequality, we get that

$$E[A(M_{n-1}, N_{n-1})d_n f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1})] \geq 0. \tag{2.11.9}$$

Similarly we get that

$$E[B(M_{n-1}, N_{n-1})d_n f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1})] \geq 0. \tag{2.11.10}$$

Since the sequence $\{M_n,\ n \geq 0\}$ is dominated by the sequence $\{N_n,\ n \geq 0\}$, it follows that

$$E[(\phi_{M_{n-1},N_{n-1}}(|e_n|) - \phi_{M_{n-1},N_{n-1}}(|d_n|)f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1})] \geq 0 \tag{2.11.11}$$

by taking the expectations on both sides of (2.11.1). Combining the relations (2.11.7) to (2.11.11), we get that

$$\begin{aligned} &E[u(M_n, N_n)f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1})] \\ &\geq E[u(M_{n-1}, N_{n-1})f(M_0, \ldots, M_{n-1}; N_0, \ldots, N_{n-1})]. \end{aligned} \tag{2.11.12}$$

$\square$

Remarks. Let $f \equiv 1$. Repeated application of the inequality obtained in Lemma 2.11.2 shows that

$$E[u(M_n, N_n)] \geq E[u(M_0, N_0)] = 0. \tag{2.11.13}$$

Proof of Theorem 2.11.1. Let

$$v(x, y) = 18\ \ |y| - I[|x| \geq \frac{1}{3}].$$

It can be checked that (cf. Osekowski (2007))

$$v(x, y) \geq u_{<2}(x, y). \tag{2.11.14}$$

Let $\lambda > 0$. It is easy to see that the strong demisubmartingale $\{\frac{M_n}{3\lambda},\ n \geq 0\}$ is weakly dominated by the demisubmartingale $\{\frac{N_n}{3\lambda},\ n \geq 0\}$. In view of the inequalities (2.11.7) and (2.11.8), we get that

$$6\ \ E|N_n| - \lambda\ \ P(|M_n| \geq \lambda) = \lambda E[v(\frac{M_n}{3\lambda}, \frac{N_n}{3\lambda})] \geq \lambda E[u_{<2}(\frac{M_n}{3\lambda}, \frac{N_n}{3\lambda})] \geq 0 \tag{2.11.15}$$

which proves the inequality

$$\lambda\ \ P(|M_n| \geq \lambda) \leq 6\ \ E|N_n|, n \geq 0. \tag{2.11.16}$$

$\square$

Results discussed in this section are from Prakasa Rao (2007).

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