The Asymptotic Shape of a Boundary Layer of Symmetric Willmore Surfaces of Revolution

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Dedicated to the memory of Prof. Wolfgang Walter, who uncovered so many deep insights into Analysis

Abstract We consider the Willmore boundary value problem for surfaces of revolution over the interval $[-1, 1]$ where, as Dirichlet boundary conditions, any symmetric set of position α and angle $\arctan\beta$ may be prescribed. Energy minimising solutions $u_{\alpha,\beta}$ have been previously constructed and for fixed $\beta \in \mathbb{R}$, the limit $\lim_{\alpha\searrow 0} u_{\alpha,\beta}(x) = \sqrt{1-x^2}$ has been proved locally uniformly in $(-1, 1)$, irrespective of the boundary angle. Subject of the present note is to study the asymptotic behaviour for fixed $\beta \in \mathbb{R}$ and $\alpha \searrow 0$ in a boundary layer of width $k\alpha$, $k > 0$ fixed, close to ± 1. After rescaling $x \mapsto \frac{1}{\alpha} u_{\alpha,\beta}(\alpha(x-1)+1)$ one has convergence to a suitably chosen cosh on $[1-k, 1]$.

Keywords Dirichlet boundary conditions · Willmore surfaces of revolution · Asymptotic shape · Boundary layer

Mathematics Subject Classification 49Q10 · 53C42 · 35J65 · 34L30

1 Introduction

Recently, the Willmore functional has attracted a lot of attention. For a smooth surface $\Gamma \subset \mathbb{R}^3$ we define it by

$$\mathcal{W}(\Gamma) := \int_\Gamma \left(H^2 - K\right) dS = \frac{1}{4}\int_\Gamma (\kappa_1 - \kappa_2)^2\, dS,$$

where κ_1, κ_2 denote the principal curvatures, $H = (\kappa_1 + \kappa_2)/2$ the mean curvature and K the Gaussian curvature of Γ. Apart from being of geometric interest [18, 19], the functional $\mathcal{W}$ and its variants are models for the elastic energy of thin shells [13] or biological membranes [8, 14]. Furthermore, they are used in image processing for

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problems of surface restoration and image inpainting [4]. In these applications one is usually concerned with minima, or more generally with critical points of the Willmore functional. It is well-known since Thomsen's work [18] that the corresponding surface Γ has to satisfy the Willmore equation

$$\Delta_\Gamma H + 2H\big(H^2 - K\big) = 0 \quad \text{on } \Gamma, \tag{1}$$

where Δ_Γ denotes the Laplace–Beltrami operator on Γ with respect to the induced metric. A solution of (1) is called a Willmore surface. Moreover, from the geometric point of view an essential property of the Willmore functional is its conformal invariance. This means that $\mathcal{W}(\Gamma) = \mathcal{W}(\Phi \circ \Gamma)$ for any Möbius transform Φ of $\mathbb{R}^3$ being regular on Γ. For an easily accessible derivation of these facts one may also see [7].

A particular difficulty in the analytical investigation of (1) arises from the fact that Δ_Γ depends on the unknown surface so that the equation is highly nonlinear. Moreover, it is of fourth order where many of the established techniques do not apply. Existence of closed Willmore surfaces of prescribed genus has been proved by Simon [17] and by Bauer and Kuwert [1]. Recently, Rivière [15] has developed a different approach which seems to open opportunities to address many further questions. For more detailed information and further references we refer to [6].

If one is interested in surfaces with boundaries, then appropriate boundary conditions have to be added to (1). Since this equation is of fourth order one requires two sets of conditions; a discussion of possible choices can be found in [13]. Of particular interest is the Dirichlet problem where at its boundary, the position and the direction of the unknown Willmore surface are prescribed. Existence results for the Dirichlet problem, which are not subject to unnatural smallness conditions, can be found e.g. in [5, 6, 16]. The result by Schätzle [16] is put into a very general context and so does not provide very detailed information about the topological and geometrical shape of the solutions. In [5, 6] the authors proceed just the other way round: They confine themselves to symmetric surfaces of revolution but at the same time they obtain rather precise information on the geometric shape of their energy minimising solutions.

More precisely, they look at surfaces of revolution, which are obtained by rotating a graph over the $x = x_1$-axis in $\mathbb{R}^3$ around the x_1-axis. These are described by a sufficiently smooth function

$$u : [-1, 1] \to (0, \infty),$$

which is moreover restricted to be even about $x = 0$, and are parametrised as follows:

$$(x, \varphi) \mapsto \big(x, u(x)\cos\varphi, u(x)\sin\varphi\big), \quad x \in [-1, 1],\ \varphi \in [0, 2\pi].$$

In the present chapter we consider the Willmore problem under Dirichlet boundary conditions, where the height $u(\pm 1) = \alpha > 0$ and the slope $u'(-1) = -u'(1) = \beta \in \mathbb{R}$ are symmetrically prescribed at the boundary. The focus will be on the asymptotic behaviour of energy minimising solutions as $\alpha \searrow 0$. However, in order to explain this one needs to recall first a bit of the underlying existence theory.

1.1 Some Basics

We consider the Willmore energy of the surface of revolution $\Gamma(u)$ generated by the graph of the smooth positive function $u : [-1, 1] \to (0, \infty)$

$$\begin{aligned}\mathcal{W}(u) &= \int_{\Gamma(u)} \left(H^2 - K\right) dS \\ &= \frac{\pi}{2} \int_{-1}^{1} \left(\frac{u''(x)}{(1+u'(x)^2)^{3/2}} - \frac{1}{u(x)\sqrt{1+u'(x)^2}} \right)^2 u(x)\sqrt{1+u'(x)^2}\, dx \\ &\quad + 2\pi \int_{-1}^{1} \frac{u''(x)}{(1+u'(x)^2)^{3/2}}\, dx.\end{aligned}$$

Definition 1 For $\alpha > 0$ and $\beta \in \mathbb{R}$ we introduce the function space

$$\begin{aligned}N_{\alpha,\beta} := \Big\{ u \in C^{1,1}\big([-1, 1], (0, \infty)\big) \,:\, &u \text{ positive, symmetric,} \\ &u(1) = \alpha \text{ and } u'(-1) = \beta \Big\}\end{aligned}$$

as well as

$$M_{\alpha,\beta} := \inf \big\{ \mathcal{W}(u) \,:\, u \in N_{\alpha,\beta} \big\}.$$

This notation here should not be mixed with that in [6]. We also need the following number

$$\alpha^* = \min \left\{ \frac{\cosh(b)}{b} : b > 0 \right\} = 1.5088795\ldots.$$

For α below α^* there is no catenary satisfying this boundary condition, irrespective of the prescribed slope at the boundary. In this regime—for $\beta < 0$—the existence proof and also the qualitative properties of solutions are different.

We recall as a special case from [6] the following existence result: For each $\alpha \in (0, \alpha^*)$ and each $\beta \in \mathbb{R}$ we find $u_{\alpha,\beta} \in N_{\alpha,\beta}$ satisfying

$$\mathcal{W}(u_{\alpha,\beta}) = M_{\alpha,\beta}.$$

The corresponding surface of revolution $\Gamma(u_{\alpha,\beta}) \subset \mathbb{R}^3$ is smooth and solves the Dirichlet problem for the Willmore equation

$$\begin{cases} \Delta_\Gamma H + 2H\left(H^2 - K\right) = 0 & \text{in } (-1, 1), \\ u_{\alpha,\beta}(-1) = u_{\alpha,\beta}(+1) = \alpha, & u'_{\alpha,\beta}(-1) = -u'_{\alpha,\beta}(+1) = \beta. \end{cases}$$

In [5, 6] the authors took advantage of looking at the Willmore energy of surfaces of revolution from a different point of view. It was observed by Bryant, Griffiths, and Pinkall (see [2, 3, 9]) and intensively exploited among others by Langer and Singer [11, 12] that the Willmore energy and the elastic energy of the profile curve considered in the hyperbolic half plane coincide up to a factor. The half-plane $\mathbb{R}^2_+ := \{(x,y)\in\mathbb{R}^2 : y>0\}$ is equipped with the hyperbolic metric $ds_h^2 := \frac{1}{y^2}\,(dx^2+dy^2)$. As explained in detail in [6, Sect. 2.2] one finds for the hyperbolic curvature of the curve $x\mapsto (x,u(x))$

$$\begin{aligned}\kappa_h(x) &= -\frac{u(x)^2}{u'(x)}\frac{d}{dx}\left(\frac{1}{u(x)\sqrt{1+u'(x)^2}}\right) = \frac{u(x)u''(x)}{(1+u'(x)^2)^{3/2}} + \frac{1}{\sqrt{1+u'(x)^2}}\\ &= \pm u(x)\big(\kappa_1(x)-\kappa_2(x)\big).\end{aligned}$$

The hyperbolic Willmore energy is defined in the following natural way and one observes that $\kappa_h^2 = 4u^2(H^2-K)$ and obtains so the following simple relation with the original energy:

$$\mathcal{W}_h(u) := \int_\gamma \kappa_h^2\, ds_h := \int_{-1}^{1} \kappa_h^2 \frac{\sqrt{1+u'^2}}{u}\, dx = \frac{2}{\pi}\,\mathcal{W}(u). \tag{2}$$

1.2 The Asymptotic Result

The previous work [6] also contains some asymptotic considerations. It should be mentioned that the numerically calculated pictures displayed there give the clear idea that for $\alpha\searrow 0$, the central part of any Willmore minimiser $u_{\alpha,\beta}$ looks pretty much like a sphere while close to the boundary the graph resembles a catenary. Combinations of these prototype functions were not only employed as initial data for the numerical flow method but were also used as comparison functions for precise estimates of the optimal Willmore energy, see [6, Sect. 5.1, Theorem 5.4]. Moreover, in [6, Theorem 5.8] it was proved for fixed $\beta\in\mathbb{R}$ and $\alpha\searrow 0$ that $u_{\alpha,\beta}(x)\to\sqrt{1-x^2}$ in $C^m([-1+\delta,1-\delta])$ for any $m\in\mathbb{N}_0$ and $\delta>0$. A related result under so called natural boundary conditions was proved by Jachalski [10].

It remains to study the asymptotic behaviour in boundary layers close to $x=\pm 1$. To this end it will be crucial to have the following comparison function which generates a minimal surface of revolution:

$$v_{\alpha,\beta}(x) := \frac{\alpha}{\sqrt{1+\beta^2}}\cosh\left(\frac{\sqrt{1+\beta^2}}{\alpha}(1-x)+\operatorname{arsinh}(\beta)\right).$$

We prove the following result:

Theorem 1 *Fix some $\beta\in\mathbb{R}$ and $k>0$. For $\alpha>0$ small enough let $u_{\alpha,\beta}\in N_{\alpha,\beta}$ minimise the Willmore energy in this class, i.e. $\mathcal{W}(u_{\alpha,\beta}) = M_{\alpha,\beta}$. Then we have uniform smooth convergence*

$$\lim_{\alpha\searrow 0}\frac{1}{\alpha}u_{\alpha,\beta}\big(\alpha(x-1)+1\big) = v_{1,\beta}(x).$$

on $[1-k,1]$.

This means that in this sense

$$u_{\alpha,\beta}(x) \approx v_{\alpha,\beta}(x) \quad \text{for } x \in [1-k\alpha, 1]$$

for $\alpha \searrow 0$ while a careful analysis of the proof in [6] indicates that for any $\varepsilon > 0$ one may expect that

$$u_{\alpha,\beta}(x) \approx \sqrt{1-x^2} \quad \text{for } |x| \in \left[0, 1-\alpha^{1-\varepsilon}\right].$$

2 Rescaled Convergence to a Suitable cosh for $\alpha \searrow 0$

In this section, we choose any $\beta \in \mathbb{R}$, keep it fixed and study the singular limit $\alpha \searrow 0$, where the "holes" $\{\pm 1\} \times B_\alpha(0)$ in the cylindrical surfaces of revolution disappear.

2.1 Known Properties of Minimisers

We first recall for α small from [6, Sect. 5] the following properties of any minimiser $u_{\alpha,\beta} \in N_{\alpha,\beta}$ of $\mathcal{W}$, i.e. $\mathcal{W}(u_{\alpha,\beta}) = M_{\alpha,\beta}$.

Lemma 1 *We assume that* $\alpha < \min\{\alpha^*, 1/|\beta|\}$. *Let* $u \in N_{\alpha,\beta}$ *be such that* $\mathcal{W}(u) = M_{\alpha,\beta}$. *Then,* $u \in C^\infty([-1,1],(0,\infty))$ *and* u *has the following additional properties*:

1. *If* $\beta \geq 0$, *then* $u' < 0$ *in* $(0,1)$ *and*

$$\alpha \leq u(x) \leq \sqrt{1+\alpha^2-x^2} \quad \text{in } [-1,1], \qquad x + u(x)u'(x) > 0 \quad \text{in } (0,1).$$

2. *If* $\beta < 0$, *then* u *has at most one critical point in* $(0,1)$, *i.e. there exists* $x_0 \in [0,1)$ *such that* $u' > 0$ *in* $(x_0, 1]$, $u'(x_0) = 0$ *and* $u' < 0$ *in* $(0, x_0)$. *Moreover,*

$$x + u(x)u'(x) > 0 \quad \text{in } (0,1], \qquad u'(x) \leq \gamma := \max\left\{-\beta, \alpha^*\right\} \quad \text{in } [x_0,1]$$

$$\text{and} \quad u(x) \geq \min\left\{\frac{\alpha}{2\sqrt{1+\beta^2}}, \frac{\gamma}{2(e^C-1)}\right\} \quad \text{in } [-1,1],$$

with $C = 6\gamma\sqrt{1+\gamma^2} > 0$. *Moreover,*

$$\lim_{\alpha \searrow 0} x_0 = \lim_{\alpha \searrow 0} x_0(\alpha) = 1.$$

Lemma 2 *Keep some* $\beta \in \mathbb{R}$ *fixed. For* $\alpha > 0$ *small enough let* $\delta_\alpha > 0$ *be such that* $-u'_{\alpha,\beta}(1-\delta_\alpha)$ *is maximal. Then we know that*

$$\lim_{\alpha \searrow 0} \delta_\alpha = 0, \tag{3}$$

$$\lim_{\alpha \searrow 0} \left(-u'_{\alpha,\beta}(1-\delta_\alpha) \right) = \infty, \tag{4}$$

$$\lim_{\alpha \searrow 0} M_{\alpha,\beta} = \lim_{\alpha \searrow 0} \mathcal{W}(u_{\alpha,\beta}) = 4\pi - 4\pi \frac{\beta}{\sqrt{1+\beta^2}}, \tag{5}$$

$$\lim_{\alpha \searrow 0} \int_0^{1-\delta_\alpha} \kappa_h[u_{\alpha,\beta}]^2 \, ds_h[u_{\alpha,\beta}] = 0. \tag{6}$$

Proof Statements (3) and (4) follow from [6, Lemma 5.3, Theorem 5.8]. For (6), see the proof of [6, Corollary 5.5]. According to [6, Theorem 5.4] and (2) we finally have for $\alpha \searrow 0$

$$8 - \frac{8\beta}{\sqrt{1+\beta^2}} + o(1) = \mathcal{W}_h(u_{\alpha,\beta}) = \frac{2}{\pi} \mathcal{W}(u_{\alpha,\beta});$$

statement (5) follows. □

2.2 Further Comparison Results

In order to guarantee compactness in our limit process we need some further uniform bounds.

We study first the simpler case $\beta \geq 0$.

Lemma 3 *Fix some* $\beta \geq 0$. *For* $0 < \alpha < \min\{\alpha^*, 1/|\beta|\}$ *we have for any Willmore minimiser* $u_{\alpha,\beta} \in N_{\alpha,\beta}$ *that*

$$u_{\alpha,\beta}(x) < v_{\alpha,\beta}(x) \quad \textit{for } x \in [0,1).$$

Proof Since both $u_{\alpha,\beta}$ and $v_{\alpha,\beta}$ are strictly decreasing on $[0,1]$ they may be considered as graphs over the angular variable. This means that for each $x \in [0,1]$ we find uniquely determined $\varphi, \psi \in [0, \pi/2]$ and $r_1(\varphi)$, $r_2(\psi)$ such that

$$\left(x, u_{\alpha,\beta}(x)\right) = r_1(\varphi)(\cos\varphi, \sin\varphi), \qquad \left(x, v_{\alpha,\beta}(x)\right) = r_2(\psi)(\cos\psi, \sin\psi).$$

Considering the curves $\varphi \mapsto r_j(\varphi)(\cos\varphi, \sin\varphi)$ let $T_j(\varphi) = (t_j^1(\varphi), t_j^2(\varphi))$ denote the corresponding unit tangent vectors with $t_j^2(\varphi) \leq 0$.

Let us assume by contradiction that $u_{\alpha,\beta} > v_{\alpha,\beta}$ on some subinterval of $[0,1]$. The case where the graphs touch tangentially in some point is simpler and can be treated similarly. Then we find $0 < \varphi_1 < \varphi_2$ such that

$$0 > \frac{t_2^2(\varphi_1)}{t_2^1(\varphi_1)} > \frac{t_1^2(\varphi_1)}{t_1^1(\varphi_1)} \quad \text{and} \quad 0 > \frac{t_1^2(\varphi_2)}{t_1^1(\varphi_2)} > \frac{t_2^2(\varphi_2)}{t_2^1(\varphi_2)}.$$

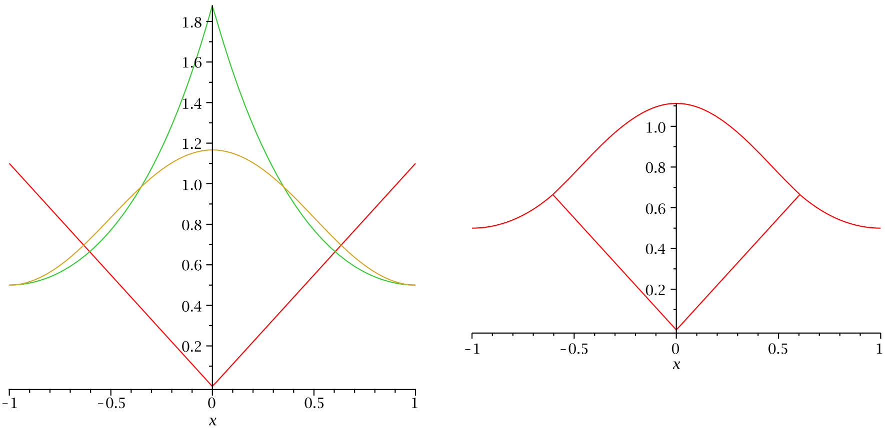

Fig. 1 *Left*: Assume that the minimiser is somewhere above the cosh. *Right*: Rescale minimiser inside the wedge and fit it into the cosh. This results in a decrease of Willmore energy

By the intermediate value theorem there exists a $\varphi_0 \in (\varphi_1, \varphi_2)$ satisfying

$$\frac{t_2^2(\varphi_0)}{t_2^1(\varphi_0)} = \frac{t_1^2(\varphi_0)}{t_1^1(\varphi_0)}.$$

Hence $T_1(\varphi_0) = T_2(\varphi_0)$, the tangents on the ray with angle φ_0 from the x-axis coincide.

We may now construct a new even function $\hat{u}_{\alpha,\beta} \in N_{\alpha,\beta}$ which coincides with the catenary $v_{\alpha,\beta}$ on $[r_2(\varphi_0)\cos\varphi_0, 1]$ and with

$$x \mapsto \frac{r_2(\varphi_0)}{r_1(\varphi_0)} u_{\alpha,\beta}\left(\frac{r_1(\varphi_0)}{r_2(\varphi_0)} x\right)$$

on $[0, r_2(\varphi_0)\cos\varphi_0]$. We emphasise that the Willmore energy is scaling invariant. Since $u_{\alpha,\beta}$ is nowhere locally equal to a cosh, we would end up with $\mathcal{W}(\hat{u}_{\alpha,\beta}) < \mathcal{W}(u_{\alpha,\beta})$, a contradiction.

See Fig. 1. □

Lemma 4 *Fix some* $\beta \geq 0$, $k \in \mathbb{N}$. *For* $0 < \alpha < \min\{\alpha^*, 1/|\beta|, 1/k\}$ *we have for any Willmore minimiser* $u_{\alpha,\beta} \in N_{\alpha,\beta}$ *that*

$$|u'_{\alpha,\beta}(x)| \leq \sinh\bigl(k\sqrt{1+\beta^2} + \operatorname{arsinh}(\beta)\bigr) \quad \textit{for } x \in [1 - k\alpha, 1].$$

Proof We proceed similarly as in the proof of Lemma 3 and consider rays which intersect $[0, 1] \ni x \mapsto (x, u_{\alpha,\beta}(x))$ and $[0, 1] \ni x \mapsto (x, v_{\alpha,\beta}(x))$. Using the same argument as before—cf. Fig. 2—we see that on each ray, the slope of $u_{\alpha,\beta}$ is less negative than the slope of $v_{\alpha,\beta}$. Since $v_{\alpha,\beta}(x) \geq u_{\alpha,\beta}(x)$, we find that the rays, which

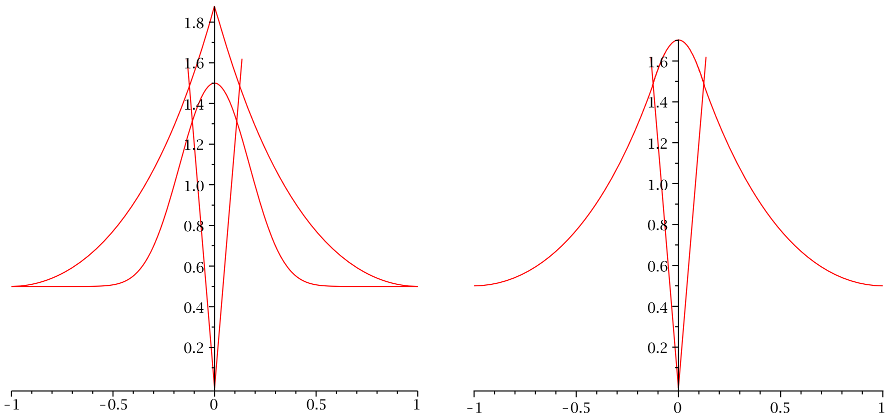

Fig. 2 *Left*: Assume that the minimiser is on some ray steeper than the cosh. *Right*: Rescale minimiser inside the wedge and fit it into the cosh. This results in a decrease of Willmore energy

cover $(x, u_{\alpha,\beta}(x))$ for $x \in [1-k\alpha, 1]$, cover $(x, v_{\alpha,\beta}(x))$ with x in a subinterval of $[1-k\alpha, 1]$:

$$\max_{x\in[1-k\alpha,1]} \left|u'_{\alpha,\beta}(x)\right| \le \max_{x\in[1-k\alpha,1]} \left|v'_{\alpha,\beta}(x)\right| \le \sinh\left(k\sqrt{1+\beta^2} + \operatorname{arsinh}(\beta)\right). \qquad \square$$

Combining the previous results with the statements from Lemma 1 we can also treat the case $\beta < 0$.

Lemma 5 *Fix some $\beta < 0$, then there exists a constant $C = C(\beta) > 0$ such that for all $0 < \alpha < \min\{\alpha^*, 1/|\beta|\}$ we have for any Willmore minimiser $u_{\alpha,\beta} \in N_{\alpha,\beta}$ that*

$$\frac{\alpha}{C} \le u_{\alpha,\beta}(x) < \alpha \cosh\left(\frac{C}{\alpha}(1-x)\right) \quad \textit{for } x \in [0, 1).$$

Proof Let $x_0 \in [0, 1)$ be as mentioned in Lemma 1, i.e. $u_{\alpha,\beta}(x_0) = \min\{u_{\alpha,\beta}(x) : x \in [-1, 1]\} =: u_{\min,\alpha}$. We take further from this lemma that there exists a constant $C = C(\beta) > 0$ such that for all $0 < \alpha < \min\{\alpha^*, 1/|\beta|\}$

$$u_{\min,\alpha} \ge \frac{\alpha}{C}.$$

Then, according to Lemma 3 we have for $x \in [0, x_0)$:

$$\begin{aligned} u_{\alpha,\beta}(x) &< u_{\min,\alpha} \cosh\left(\frac{1}{u_{\min,\alpha}}(x_0 - x)\right) \\ &\le \alpha \cosh\left(\frac{1}{u_{\min,\alpha}}(1-x)\right) \le \alpha \cosh\left(\frac{C}{\alpha}(1-x)\right). \end{aligned}$$

Since $u_{\alpha,\beta}(x) < \alpha \le \alpha \cosh(\frac{C}{\alpha}(1-x))$ on $[x_0, 1)$, the proof is complete. $\square$

Lemma 6 *Fix some $\beta < 0$, $k \in \mathbb{N}$. There exists a bound $C = C(k, \beta)$ such that for all $\alpha > 0$ small enough we have for any Willmore minimiser $u_{\alpha,\beta} \in N_{\alpha,\beta}$ that*

$$\left|u'_{\alpha,\beta}(x)\right| \leq C \quad \textit{for } x \in [1 - k\alpha, 1].$$

Proof We proceed similarly as in the previous Lemma 5. Let x_0, $u_{\min,\alpha}$ be as there and let $C_1 = C_1(\beta) > 0$ be the constant used there. In particular we use that $u_{\min,\alpha} \geq \frac{\alpha}{C_1}$ From Lemma 4 we see that there exists a constant $C_2 = C_2(k, \beta) > 0$ such that for $\alpha > 0$ small enough

$$\left|u'_{\alpha,\beta}(x)\right| \leq C_2 \quad \text{on } [x_0 - kC_1 u_{\min,\alpha}, x_0] \supset [x_0 - k\alpha, x_0].$$

One should observe that $\lim_{\alpha \searrow 0} x_0(\alpha) = 1$. According to Lemma 1 we have that

$$0 \leq u'_{\alpha,\beta}(x) \leq \max\left\{-\beta, \alpha^*\right\} \quad \text{in } [x_0, 1].$$

Putting all together proves the claim. □

Corollary 1 *For any $k \in \mathbb{N}$ we have that for $\alpha > 0$ small enough*

$$\delta_\alpha > k\alpha.$$

2.3 Concentration of the Willmore Energy

According to Lemma 2, for $\alpha \searrow 0$, the hyperbolic Willmore energy concentrates close to ± 1. The following lemma shows the reverse result for $\int H^2 \, dS$.

Lemma 7 *Fix some $\beta \in \mathbb{R}$, $k \in \mathbb{N}$. Let $u_{\alpha,\beta} \in N_{\alpha,\beta}$ be any Willmore minimiser and let $H_{\alpha,\beta}$ denote its mean curvature. Then*

$$\lim_{\alpha \searrow 0} \int_{1-k\alpha}^{1} H^2_{\alpha,\beta} u_{\alpha,\beta} \sqrt{1 + \left(u'_{\alpha,\beta}\right)^2} \, dx = 0.$$

Proof According to Lemma 2, we have for $\alpha \searrow 0$:

$$\begin{aligned} 4\pi + o(1) &= \mathcal{W}(u_{\alpha,\beta}) + 4\pi \frac{\beta}{\sqrt{1 + \beta^2}} \\ &= \mathcal{W}(u_{\alpha,\beta}) + 2\pi \int_{-1}^{1} K[u_{\alpha,\beta}] u_{\alpha,\beta} \sqrt{1 + \left(u'_{\alpha,\beta}\right)^2} \, dx \\ &= 4\pi \int_{1-\delta_\alpha}^{1} H^2_{\alpha,\beta} u_{\alpha,\beta} \sqrt{1 + \left(u'_{\alpha,\beta}\right)^2} \, dx \\ &\quad + 4\pi \int_{0}^{1-\delta_\alpha} H^2_{\alpha,\beta} u_{\alpha,\beta} \sqrt{1 + \left(u'_{\alpha,\beta}\right)^2} \, dx \end{aligned}$$

$$
\begin{aligned}
&= 4\pi \int_{1-\delta_\alpha}^{1} H_{\alpha,\beta}^2 u_{\alpha,\beta} \sqrt{1+\left(u'_{\alpha,\beta}\right)^2}\, dx \\
&\quad + \pi \int_0^{1-\delta_\alpha} \kappa_h[u_{\alpha,\beta}]^2\, ds_h[u_{\alpha,\beta}] + 4\pi \frac{|u'_{\alpha,\beta}(1-\delta_\alpha)|}{\sqrt{1+u'_{\alpha,\beta}(1-\delta_\alpha)^2}} \\
&= 4\pi \int_{1-\delta_\alpha}^{1} H_{\alpha,\beta}^2 u_{\alpha,\beta} \sqrt{1+\left(u'_{\alpha,\beta}\right)^2}\, dx + o(1) + 4\pi + o(1).
\end{aligned}
$$

This yields

$$
\int_{1-\delta_\alpha}^{1} H_{\alpha,\beta}^2 u_{\alpha,\beta} \sqrt{1+\left(u'_{\alpha,\beta}\right)^2}\, dx = o(1)
$$

which in view of Corollary 1 proves the claim. □

2.4 *Limit of the Rescaled Solutions, Proof of Theorem 1*

We introduce the rescaled solutions

$$
\hat{u}_{\alpha,\beta} := \frac{1}{\alpha} u_{\alpha,\beta}\big(\alpha(x-1)+1\big)
$$

and keep some $k \in \mathbb{N}$ fixed in what follows. Lemmas 3–6 show that $(\hat{u}_{\alpha,\beta})_{\alpha \searrow 0}$ is uniformly bounded in $C^1([1-k, 1])$ and uniformly bounded from below on $[1-k, 1]$ while Lemma 7 proves that its mean curvature converges to 0 in $L^2([1-k, 1])$. By standard arguments (cf. [6, Proof of Theorem 5.8]) we find a strong C^1- and weak H^2-limit $u : [1-k, 1] \to (0, \infty)$ satisfying

$$
u(1) = 1, \qquad u'(1) = -\beta, \qquad H[u](x) \equiv 0.
$$

By direct integration this gives

$$
u(x) = v_{1,\beta}(x) = \frac{1}{\sqrt{1+\beta^2}} \cosh\!\Big(\sqrt{1+\beta^2}(1-x) + \operatorname{arsinh}(\beta)\Big)
$$

and so, the proof of Theorem 1. As for convergence in higher order norms one may see the proof of [6, Theorem 5.8].

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