Preface

These are the notes from an Oberwolfach Seminar which we ran from 23–29 May 2010. There were 24 graduate student and postdoctoral participants. Each morning consisted of three lectures, one from each of the organisers. The afternoons consisted of problem sessions, apart from Wednesday which was reserved for the traditional hike to St. Roman. We have tried to be reasonably faithful to the lectures and problem sessions in these notes, and have added only a small amount of new material for clarification.

The seminar focused on recent developments in classification methods in commutative algebra, group representation theory and algebraic topology. These methods were initiated by Hopkins back in 1987 [35], with the classification of the thick subcategories of the derived category of bounded complexes of finitely generated projective modules over a commutative noetherian ring R, in terms of specialisation closed subsets of $\operatorname{Spec} R$. Neeman [44] (1992) clarified Hopkins' theorem and used analogous methods to classify the localising subcategories of the derived category of unbounded complexes of modules in terms of arbitrary subsets of $\operatorname{Spec} R$. In 1997, Benson, Carlson and Rickard [9] proved the thick subcategory theorem for modular representation theory of a finite p-group G over an algebraically closed field k of characteristic p. Namely, the thick subcategories of the stable category of finitely generated kG-modules are classified by the specialisation closed subsets of the homogeneous non-maximal prime ideals in $H^*(G,k)$, the cohomology ring. The corresponding theorem for the localising subcategories of the stable category of all kG-modules has only recently been achieved, in the paper [11] by the three organisers of the seminar.

	Thick subcategories of compact objects	Localising subcategories of all objects
$\mathsf{D}(R)$	Hopkins 1987	Neeman 1992
$\mathsf{StMod}(kG)$	Benson, Carlson and Rickard 1997	Benson, Iyengar and Krause 2008

In the process of achieving the classification of the localising subcategories of $\mathsf{StMod}(kG)$, a general machinery was established for such classification theorems in a triangulated category; see [10, 12]. It is also worth mentioning at this stage the work of Hovey, Palmieri and Strickland [36], who did a great deal to clarify the appropriate settings for these theorems.

The general setup involves a graded commutative noetherian ring R acting on a compactly generated triangulated category with small coproducts T. Write $\operatorname{Spec} R$ for the set of homogeneous prime ideals of R. For each $\mathfrak{p} \in \operatorname{Spec} R$ there is a *local cohomology functor* $\varGamma_{\mathfrak{p}} \colon T \to T$. The *support* of an object X is defined to be the subset of $\operatorname{Spec} R$ consisting of those $\mathfrak{p}$ such that $\varGamma_{\mathfrak{p}} X$ is non-zero.

The object of the game is to establish conditions under which this notion of support classifies the localising subcategories of T. This is given in terms of two conditions. The first is the *local-global principle* that says for each object X in T, the localising subcategory of T generated by X is the same as that generated by $\{\varGamma_{\mathfrak{p}} X\}$ as $\mathfrak{p}$ runs over the elements of $\operatorname{Spec} R$. The second is a minimality condition, which requires that each $\varGamma_{\mathfrak{p}}\mathsf{T}$ is either a minimal localising subcategory of T or it is zero. Under these two conditions, we say that T is *stratified* by the action of R, and then we obtain a classification theorem.

In the case of the derived category $\mathsf{D}(R)$, Neeman's classification made essential use of the existence of "field objects" – for a prime ideal $\mathfrak{p}$ of R, the field object is the complex consisting of the field of fractions of $R/\mathfrak{p}$, concentrated in a single degree. One of the principle obstructions to carrying out the classification in the finite group case is a lack of field objects; the obstruction theory of Benson, Krause and Schwede [15, 16] can be used to show that the required field objects usually do not exist. Circumventing this involves an elaborate series of changes of category, and machinery for transferring stratification along such changes of category. For a general finite group, the strategy is first to use Quillen stratification to reduce to elementary abelian p-groups, where there are still not enough field objects, but then to use a Koszul construction to reduce to an exterior algebra for which there are enough field objects. At this stage, a version of the Bernstein–Gelfand–Gelfand correspondence can be used to get to a graded polynomial ring, where the problem is solved. One consequence of this strategy is that we obtain classification theorems in a number of situations along the route.

In these notes we manage to give a complete proof in the case of characteristic two, where matters are considerably simplified by the fact that the group algebra of an elementary abelian 2-group is already an exterior algebra. We found it frustrating that in spite of having an entire week of lectures to explain the theory, we were not able to give a complete proof of the classification theorem for localising subcategories of $\mathsf{StMod}(kG)$, in odd characteristic. An overview of the classification in general characteristic is given in Section 3.3, while the proof in characteristic two may be obtained by combining Theorems 5.4 and 5.19 with results from Section 3.3.

A guide to these notes

In this volume, we have attempted to stick as closely as possible to the format of the Oberwolfach seminar. So the notes are divided into five chapters with four sections in each, corresponding to the five days with three lectures each morning and a problem session in the afternoon. The lecturing, and writing, styles of the three authors are different, and we have not tried to alter that for the purpose of these notes. In particular, there is a small amount of repetition. But we have tried to be consistent about important details such as notation, and grading everything cohomologically rather than homologically.

Prerequisites for this seminar consist of a solid background in algebra, including the basic theory of rings and modules, Artin–Wedderburn theory, Krull–Remak–Schmidt theorem; basic commutative algebra from the first chapters of the book of Atiyah and MacDonald; and basic homological algebra including derived functors, Ext and Tor. The appendix, describing the theory of support for modules over a commutative ring, is also necessary background material from commutative algebra that is not easy to find in the literature in the exact form in which we require it. The following books may be helpful.

[1] M. Atiyah and I. MacDonald, Commutative Algebra. Addison-Wesley, 1969.

[2] D. J. Benson, Representations and cohomology of finite groups I, II, Cambridge Studies in Advanced Mathematics 30, 31. Cambridge University Press, 2nd edition, 1998.

[3] W. Bruns and J. Herzog, Cohen–Macaulay rings, Cambridge Studies in Advanced Mathematics 39. Cambridge University Press, 2nd edition, 1998.

[4] R. Hartshorne, Local cohomology: A seminar given by A. Grothendieck (Harvard, 1961), Lecture Notes in Math. 41. Springer-Verlag, 1967.

[5] A. Neeman, Triangulated categories, Annals of Mathematics Studies 148. Princeton University Press, 2001.

[6] C. Weibel, Homological algebra, Cambridge Studies in Advanced Mathematics 38. Cambridge University Press, 1994.

About the exercises: These are from the problem sessions conducted during the seminar, though we have added a few more. Some are routine verifications/computations that have been omitted in the text, while others are quite substantial, and given with the implicit assumption (or hope) that, if necessary, readers would hunt for solutions in other sources.

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