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On the HELP Inequality for Hill Operators on Trees

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Abstract. The validity of a generalised HELP inequality for a Schrödinger operator with periodic potential on a rooted homogeneous tree is related to the quasi-stability or quasi-instability of the associated differential equation. A numerical approach to the determination of the optimal constant in the HELP inequality is presented. Moreover, we give an example to illustrate that the generalised Weyl–Titchmarsh m function for the tree operator fails to capture all of its spectral properties.

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1. Introduction

The classical HELP inequality for a Sturm–Liouville operator

$$\tau f = \frac{1}{w}(-(pf')' + qf)$$

with locally integrable real-valued coefficients $w, \frac{1}{p}, q$; $w, p > 0$ on an interval (a, b) is

$$\left(\int_a^b (p\,|f'|^2 + q\,|f|^2)\right)^2 \leq K \int_a^b |f|^2\, w \int_a^b |\tau f|^2\, w$$

[6]. Typically one considers a situation where a is a regular, b a singular end-point in the limit-point case; the crucial point about the inequality is that, when valid, it holds for all functions f for which the derivatives exist in a weak sense and the right-hand side is finite, irrespective of the boundary values at a.

HELP is a generalisation of an inequality of Hardy and Littlewood [8] which covers the case $w = p = 1$, $q = 0$ and is valid with the optimal constant $K = 4$. The validity of the more general HELP inequality with some constant K can be shown to be equivalent to a certain property of the Weyl–Titchmarsh m function for the Sturm–Liouville operator in a cone-like neighbourhood of the imaginary axis [7]. The recent work [3] has shown that a similar criterion using a generalised m function can be obtained for a HELP inequality on trees of infinite length; in fact this is a particular instance of the abstract HELP inequality established by [1] and [10].

In the present study we focus on a situation which is analogous to Hill's equation (the periodic Sturm–Liouville equation) on a half-line; we consider a tree composed of infinitely many identical intervals, each carrying the same potential q, and such that at each end-point with a single exception (the tree root) a fixed number of intervals are joined together. The spectrum of the graph Laplacian on such trees has been studied by [11]; it consists of bands of purely absolutely continuous spectrum with an additional eigenvalue in each gap and thus is analogous to that of a Hill operator on a half-line, except that the eigenvalues of the tree Laplacian have infinite multiplicity. [11] then went on to consider the effect of adding a decaying potential which is symmetric in the sense of only depending on the distance from the tree root. The generalised Hill operator on a perfectly homogeneous, rootless tree has been analysed in detail by [5] under the assumption that the potential, equal on each tree edge, is an even function on the interval.

Our paper is organised as follows. In Section 2 we show, based on its strong limit-point property, that the maximal tree-Hill operator has deficiency indices $(1, 1)$ and that, as a consequence, the Weyl solutions for non-real spectral parameter are symmetric on the tree branches and their evolution is governed by a period-transfer matrix. This leads to an analogue to Floquet theory which we pursue in Section 3. In Section 4 we state the HELP inequality and study the associated generalised m function to relate the validity of the inequality to the quasi-stability or quasi-instability, in the sense of the generalised Floquet theory, of the tree-Hill equation at spectral parameter 0. Section 5 reports on a numerical approach to calculate the optimal HELP constant and shows results in the example of a shifted piecewise linear (sawtooth-type) potential in dependence on the constant offset. Finally, an appendix illustrates the observation that, due to the discrepancy between the finite deficiency indices of the operator and the a priori unboundedness of the multiplicity of its eigenvalues, the generalised m function does not carry complete spectral information, unlike the classical Sturm–Liouville case.

2. Hill operators on trees

Consider a regular tree Γ with constant branching number $b \in \mathbb{N}+1$ and fixed edge length l. Thus, b copies of the interval $[0, l]$ (1st generation edges) are attached to the right-hand end-point of the interval $[0, l]$ (the 0th generation),

b copies (2nd generation edges) are attached to each of the b right-hand endpoints of 1st generation edges, and so on ad infinitum.

The maximal domain for the tree Laplacian on Γ in the Hilbert space $L^2(\Gamma)$ is

$$\mathcal{D} := \{f : \Gamma \to \mathbb{C} \mid f, f' \text{ a.c. on edges,} \\ f \text{ continuous, Kirchhoff conditions, } f, f'' \in L^2(\Gamma)\}$$

(Kirchhoff conditions meaning, as usual, that at each junction the outgoing derivatives of f add up to 0). As shown in [3] Thm 4.1, the tree Laplacian on Γ, generally defined as $-f''$ for functions f satisfying the regularity, continuity and Kirchhoff conditions of $\mathcal{D}$, has the *Strong Limit-Point Property* that for all $f, g \in \mathcal{D}$,

$$\lim_{r\to\infty} \sum_{|x|=r} f(x)\overline{g}'(x) = 0.$$

(here $|\cdot|$ denotes the metric distance from the tree root, 0). In fact, this property holds for general trees of infinite length, i.e., those for which any forward path can be extended indefinitely.

In the following, we consider the *tree-Hill operator* on Γ, i.e., the operator

$$-f'' + qf \qquad (f \in \mathcal{D}),$$

where $q : \Gamma \to \mathbb{R}$ is an l-periodic, bounded function of the distance from the root only: in other words, each edge carries a (directed) copy of the same potential function. Thus the tree-Hill operator is a generalisation of the classical Hill, or one-dimensional Schrödinger, operator on a half-line.

Let

$$\Phi(\cdot, \lambda) = \begin{pmatrix} \theta & \phi \\ \theta' & \phi' \end{pmatrix} (\cdot, \lambda)$$

be the canonical fundamental system of the one-dimensional Schrödinger equation

$$-u'' + qu = \lambda u$$

on the interval $[0, l]$, $\lambda \in \mathbb{C}$; then $\Phi(x, \cdot)$ is an entire function for each $x \in [0, l]$.

For non-real spectral parameter, the strong limit-point property of the tree Laplacian yields the following characterisation of square-integrable solutions of the eigenvalue equation for the tree Laplacian.

Theorem 2.1. *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $\psi \in \mathcal{D}$ a solution of*

$$-f'' + qf = \lambda f \tag{1}$$

on Γ. Then

a) *ψ can be taken to be non-trivial, and any other solution of* (1) *in $\mathcal{D}$ is linearly dependent on ψ;*
b) *there is a function $\tilde{\psi} : [0, \infty) \to \mathbb{C}$ such that $\psi(x) = \tilde{\psi}(|x|)$ $(x \in \Gamma)$;*

c)

$$\begin{pmatrix} \tilde{\psi}(nl+) \\ \tilde{\psi}'(nl+) \end{pmatrix} = A(\lambda) \begin{pmatrix} \tilde{\psi}((n-1)l+) \\ \tilde{\psi}'((n-1)l+) \end{pmatrix}$$

for all $n \in \mathbb{N}$, *with the transfer matrix*

$$A(\lambda) = \begin{pmatrix} \theta(l,\lambda) & \phi(l,\lambda) \\ \frac{1}{b}\theta'(l,\lambda) & \frac{1}{b}\phi'(l,\lambda) \end{pmatrix}.$$

Remark 2.2. Part a) shows that the minimal tree-Hill operator, defined on the subspace of $\mathcal{D}$ of functions of compact support in the interior of Γ, has deficiency indices $(1,1)$. Self-adjoint realisations are obtained by restricting the maximal operator, defined on $\mathcal{D}$, by means of a boundary condition, e.g. of Dirichlet or Neumann type, at the tree root. Part b) shows that the generalised Weyl solutions, i.e., solutions in $\mathcal{D}$ for non-real spectral parameter, are all what we shall call *symmetric functions* in the following; these are functions which are symmetric under arbitrary permutations of the tree branches which leave the tree structure intact. We shall call $\mathcal{D}_{\mathrm{sym}} := \{f \in \mathcal{D} \mid f \text{ symmetric}\}$ the *symmetric subspace*.

Proof. a) Assume we have two linearly independent solutions $\psi, \xi \in \mathcal{D}$ of (1); let us assume for the moment that $\psi(0), \xi(0) \neq 0$ (the following argument will also show that this must always be the case). Then $\alpha := -\dfrac{\psi(0)}{\xi(0)} \in \mathbb{C}$ and $f := \psi + \alpha\xi \in \mathcal{D}$ is also a solution of (1), with $f(0) = 0$. Integrating by parts, we find

$$\int_{\Gamma_r} |f'|^2 = -\int_{\Gamma_r} f''\,\overline{f} + \sum_{|x|=r} f'(x)\,\overline{f(x)} - f'(0)\,\overline{f(0)},$$

where $\Gamma_r := \{x \in \Gamma \mid |x| \leq r\}$; so passing to the limit and using the strong limit-point property, the boundary value at the root and the eigenvalue equation,

$$\int_{\Gamma} |f'|^2 = \int_{\Gamma} (\lambda - q)\,|f|^2.$$

Taking the imaginary parts on either side gives $0 = (\operatorname{Im}\lambda) \int_{\Gamma} |f|^2$. In particular, if $\lambda \notin \mathbb{R}$, then ψ and ξ are linearly dependent. Clearly, if either $\psi(0) = 0$ or $\xi(0) = 0$, the same argument gives $\psi = 0$ or $\xi = 0$, respectively.

The existence of a non-trivial, symmetric solution of (1) can be inferred from the fact that, on the symmetric subspace, the tree-Hill operator is equivalent to a Sturm–Liouville operator with a singular right-hand end-point in the limit-point case; see [3] Section 3 for details.

b) By part a), the function arising from ψ by any rearrangement of tree branches which leaves the distance to the root invariant must be a solution of (1) linearly dependent on ψ; as it coincides with ψ on the first edge, the two must be identical. Hence ψ itself is a symmetric function.

c) In view of the symmetry shown in b), the continuity and Kirchhoff condition at the junctions joining the $(n-1)$-st to the n-th generation edges imply

$$\begin{pmatrix} \tilde{\psi}(nl+) \\ \tilde{\psi}'(nl+) \end{pmatrix} = \begin{pmatrix} \tilde{\psi}(nl-) \\ \frac{1}{b}\tilde{\psi}'(nl-) \end{pmatrix}$$

for all $n \in \mathbb{N}$. Also

$$\begin{pmatrix} \tilde{\psi}(nl-) \\ \tilde{\psi}'(nl-) \end{pmatrix} = \Phi(l,\lambda) \begin{pmatrix} \tilde{\psi}((n-1)l+) \\ \tilde{\psi}'((n-1)l+) \end{pmatrix}$$

by solving the differential equation along the edges, so we get the transfer relation with $A(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{b} \end{pmatrix} \Phi(l,\lambda)$. □

3. Quasi-Floquet theory

The transfer matrix $A(\lambda)$ plays a similar role to the standard monodromy matrix in the Floquet theory of Hill's equation. There is, however, the essential difference that it has determinant $\frac{1}{b}$ instead of 1. From Theorem 2.1 c) it is clear that if $\psi \in \mathcal{D}$ is a non-trivial solution of (1) for $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then $\begin{pmatrix} \psi(0) \\ \psi'(0) \end{pmatrix}$ will be an eigenvector of $A(\lambda)$. The corresponding eigenvalue will have modulus $< \frac{1}{\sqrt{b}}$ for the following reason.

Generally, if for any $\lambda \in \mathbb{C}$, μ is an eigenvalue of $A(\lambda)$ and $u : \Gamma \to \mathbb{C}$ is the corresponding unique *symmetric* solution of (1) whose start phase vector is an eigenvector,

$$A(\lambda) \begin{pmatrix} u \\ u' \end{pmatrix} (0) = \mu \begin{pmatrix} u \\ u' \end{pmatrix} (0),$$

then u will be equal on all tree edges, except for a factor μ^n on the n-th generation edges (of which there are b^n copies). Hence the square integral of u is

$$\int_\Gamma |u|^2 = \sum_{n=0}^{\infty} b^n |\mu|^{2n} \int_0^l |u|^2 = \left(\sum_{n=0}^{\infty} (\sqrt{b}|\mu|)^{2n} \right) \int_0^l |u|^2,$$

which is finite if and only if $|\mu| < \frac{1}{\sqrt{b}}$.

Let us now consider a real spectral parameter. As uniqueness of solution of initial-value problems holds only on the symmetric subspace, the quasi-Floquet theory will not give a complete overview of the solutions of (1), but only of symmetric solutions; however, we shall be interested in the situation of real λ as limiting values of λ in the complex upper half-plane where all solutions in $\mathcal{D}$ are symmetric.

The two eigenvalues of $A(\lambda)$ satisfy $\mu_1\mu_2 = \frac{1}{b}$, $\mu_1 + \mu_2 = D(\lambda)$, where $D(\lambda) := \operatorname{trace} A(\lambda) = \theta(l,\lambda) + \frac{1}{b}\phi'(l,\lambda)$ is a *quasi-discriminant*; if $\lambda \in \mathbb{R}$, then $D(\lambda) \in \mathbb{R}$. Solving the characteristic equation, we find

$$\mu = \frac{D(\lambda) \pm \sqrt{D(\lambda)^2 - \frac{4}{b}}}{2}.$$

Thus the eigenvalues are real and of equal sign if $D(\lambda)^2 > 4/b$ (and then both eigenvalues have equal sign), complex conjugates if $D(\lambda)^2 < 4/b$, and in the limiting case they are $\pm\frac{1}{\sqrt{b}}$, with the same sign as $D(\lambda)$.

In the case of real eigenvalues, the one closer to 0 has modulus $|\mu| < \frac{1}{\sqrt{b}}$. Hence the corresponding symmetric (quasi-Floquet) solution u whose start phase vector is an eigenvector for this eigenvalue, will be square-integrable; in the exceptional case where it also satisfies the boundary condition imposed at the tree root, λ will be an eigenvalue of the corresponding self-adjoint tree-Hill operator with symmetric eigenfunction.

If the eigenvalues of $A(\lambda)$ are non-real complex conjugates, they will have modulus $|\mu| = \frac{1}{\sqrt{b}}$, so the corresponding quasi-Floquet solutions are not square-integrable.

Remark 3.1. We remark that the intervals on the real λ-axis where $D(\lambda)^2 > 4/b$ and where $D(\lambda)^2 < 4/b$ formally correspond to the instability and stability intervals of Hill's equation, respectively; the terminology refers to the fact that for the periodic Sturm–Liouville equation the trivial solution $u = 0$ is respectively unstable or stable in the two situations. By analogy, we shall refer to them as *quasi-instability* and *quasi-stability intervals*; but note that the trivial solution of the equation on the tree may still be asymptotically stable for λ inside a quasi-instability interval: this will happen when $\frac{1}{\sqrt{b}} < |\mu| < 1$ for the eigenvalue of larger modulus.

For the purpose of locating the quasi-stability and quasi-instability intervals, we observe that the transition points, i.e., the values of $\lambda \in \mathbb{R}$ where the quasi-discriminant is $D(\lambda) = \pm\frac{2}{\sqrt{b}}$ and $\mu = \pm\frac{1}{\sqrt{b}}$ correspondingly, are the eigenvalues of an associated quasi-(anti-)periodic boundary value problem on the interval $[0, l]$. To see this, assume μ is an eigenvalue of $A(\lambda)$ — for real λ — and v a corresponding eigenvector. Then the solution u of the differential equation

$$-u'' + qu = \lambda u$$

on $[0, l]$ with initial data $\begin{pmatrix} u \\ u' \end{pmatrix}(0) = v$ is given as $\begin{pmatrix} u \\ u' \end{pmatrix} = \Phi(\cdot, \lambda)\, v$, and hence

$$\begin{pmatrix} u \\ u' \end{pmatrix}(l) = \Phi(l, \lambda)\, v = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} A(\lambda)\, v = \mu \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} v = \mu \begin{pmatrix} u(0) \\ bu'(0) \end{pmatrix}.$$

At end-points of the quasi-stability intervals, $\mu = \pm\frac{1}{\sqrt{b}}$, so we find that, depending on the sign, u will be a solution of the quasi-periodic boundary value problem

$$u(l) = \frac{1}{\sqrt{b}}\, u(0), \qquad u'(l) = \sqrt{b}\, u'(0),$$

or of the quasi-anti-periodic boundary value problem

$$u(l) = -\frac{1}{\sqrt{b}}\, u(0) \qquad u'(l) = -\sqrt{b}\, u'(0).$$

4. Weyl–Titchmarsh m function and HELP inequality

For spectral parameter λ in the complex upper half-plane, we can construct a Weyl–Titchmarsh m function in analogy to the standard Sturm–Liouville theory. Indeed, by Theorem 2.1 there is exactly one non-trivial solution $\psi \in \mathcal{D}$ (up to multiplication with a constant) and it is symmetric under branch permutations of the tree. The solution ψ cannot have $\psi'(0) = 0$, as λ would then be a non-real eigenvalue of the self-adjoint realisation of our operator with Neumann boundary condition at the root; so by multiplication with a suitable constant we can assume that $\psi'(0) = 1$ and hence represent ψ on $[0, l]$ in terms of the canonical fundamental system,

$$\psi = \phi - m\theta.$$

The coefficient m gives the Weyl–Titchmarsh function; clearly $\psi(0) = -m$. In fact, m is the standard Weyl–Titchmarsh function for the Sturm–Liouville operator equivalent to the tree-Hill operator restricted to the symmetric subspace. In particular, m has the property that $\operatorname{Im} m(\lambda) > 0$ $(\operatorname{Im} \lambda > 0)$. As observed at the beginning of the previous section, the initial phase vector $\begin{pmatrix} \psi(0) \\ \psi'(0) \end{pmatrix} = \begin{pmatrix} -m \\ 1 \end{pmatrix}$ is an eigenvector of the transfer matrix $A(\lambda)$ with eigenvalue $\mu(\lambda)$ of modulus $|\mu(\lambda)| < \frac{1}{\sqrt{b}}$, which gives the following characterisation of the m as a function of the spectral parameter. (Note that $\mu(\lambda)$ denotes the eigenvalue of $A(\lambda)$ of smaller modulus.)

Lemma 4.1. *For $\lambda \in \mathbb{C} \setminus \mathbb{R}$,*

$$m(\lambda) = \frac{\phi(l, \lambda)}{\theta(l, \lambda) - \mu(\lambda)} = \frac{\phi'(l, \lambda) - b\mu(\lambda)}{\theta'(l, \lambda)}.$$

In particular, although m is the generalised Weyl–Titchmarsh function for the tree-Hill operator, it can be calculated from the solutions on the single interval $[0, l]$ only.

We remark that the relationship of this m function to the spectral properties of the tree-Laplace operator turns out to be more tenuous than in the case of Sturm–Liouville operators, where knowing the m function is equivalent to knowing the operator's spectral function.

From the point of view of Sturm–Liouville theory, it may already appear paradoxical that the minimal tree Laplacian has finite deficiency indices, as shown in Theorem 2.1, while it is well known [11] that the self-adjoint realisation of the Laplacian on Γ with Dirichlet boundary condition has eigenvalues of infinite multiplicity. Here it is important to observe that the statement of Theorem 2.1 a) only holds for *non-real* λ, but will be false for real λ in general, as is obvious from the proof. In particular, eigenfunctions for real λ need not be symmetric functions under branch permutations — each of the eigenvalues given in [11] has one symmetric eigenfunction, whereas the infinitely many additional eigenfunctions are not symmetric.

This observation may indicate some severe limitations to the scope of Titchmarsh-Weyl m function theory when applied to trees (or, more generally, graphs). The definition of the m function as characteristic coefficient in the representation of the distinguished square-integrable solution for non-real λ only captures the behaviour of the operator on the symmetric subspace and, in particular, will only flag a simple real eigenvalue where the tree Laplacian has in fact an eigenvalue of infinite multiplicity. The underlying reason for this effect is the lack of uniqueness of the solution of initial-value problems on the tree. Indeed, there are non-trivial solutions of (1) which vanish identically near the root.

This peculiarity may seem inconsequential in the present case of a fully regular tree with constant branching number and edge lengths, where the spectrum of the Laplacian just consists of infinitely many copies of the spectrum on the symmetric subspace; but for more general trees one may expect to lose spectral information in the m function. We illustrate this effect with a simple example in the Appendix below. A similar spectral incompleteness of the generalised m function has been observed for operators of a different type by [2].

The properties of the function m play a decisive role for the validity of a HELP inequality for the tree-Hill operator, as shown in [3].

Theorem 4.2. a) *The following statements are equivalent.*

(i) *(HELP Inequality) There is a constant $K > 0$ such that*

$$\left(\int_\Gamma (|f'|^2 + q|f|^2)\right)^2 \le K \int_\Gamma |f|^2 \int_\Gamma |-f'' + qf|^2 \qquad (f \in \mathcal{D}).$$

(ii) *There exist $\vartheta_+, \vartheta_- \in [0, \pi/2)$ such that*

$$\operatorname{Im}(-\lambda^2\, m(\lambda)) \ge 0 \qquad (\lambda \in \mathbb{C} \setminus \{0\}, \arg\lambda \in [\vartheta_+, \pi - \vartheta_-]). \tag{2}$$

The optimal constant for the HELP inequality is given by $K = (\cos\hat{\vartheta})^{-2}$, where $\hat{\vartheta} := \min\{\vartheta \in [0, \pi/2) \mid (2)$ holds for all $\lambda \in \mathbb{C} \setminus \{0\}$, $\arg\lambda \in [\vartheta, \pi - \vartheta]\}$.

b) *(2) is satisfied if and only if it holds locally both at 0 and at ∞.*

If $m(\lambda) \sim c\lambda^\alpha$ ($\lambda \to \infty$ in an open sector containing the imaginary axis) with $c \ne 0$, $\alpha \in [-1, 1] \setminus \{0\}$, then (2) holds at infinity.

If the non-tangential limit $m_0 := \lim_{\lambda \downarrow 0} m(\lambda)$ exists, then (2) is satisfied at 0 if $m_0 \in \mathbb{C} \setminus \mathbb{R}$ and (2) is not satisfied at 0 if $m_0 \in \mathbb{R} \setminus \{0\}$.

Moreover, if either m is analytic at 0 and $m(0) = 0$ or if $\frac{1}{m}$ is analytic at 0 and $\frac{1}{m}(0) = 0$, then (2) is satisfied at 0.

For the proof of these statements, we refer to the cited references, with the exception of the last sentence in the theorem, for which we give a proof now.

If m is analytic at 0 and $m(0) = 0$, then $m(\lambda) = a\lambda + o(|\lambda|)$ with $a > 0$ (since $\operatorname{Im} m(\lambda) > 0$ in the complex upper half-plane). Hence we find for all $\lambda = re^{i\vartheta}$ with sufficiently small $r > 0$ and $\vartheta \in (\frac{\pi}{3}, \frac{2\pi}{3})$ that

$$\operatorname{Im}(-\lambda^2 m(\lambda)) = -ar^3 \operatorname{Im} e^{3i\vartheta} + o(r^3) \ge 0.$$

Similarly, if $\frac{1}{m}$ is analytic at 0 and $\frac{1}{m}(0) = 0$, then $\frac{1}{m}(\lambda) = a\lambda + o(\lambda)$ with $a < 0$, and therefore for all $\lambda = re^{i\vartheta}$ with sufficiently small $r > 0$ and $\vartheta \in (0, \pi)$

$$\begin{aligned}\operatorname{Im}(-\lambda^2 m(\lambda)) &= |m(\lambda)|^2 \operatorname{Im}(-\lambda^2(a\overline{\lambda} + o(|\lambda|)) \\ &= |m(\lambda)|^2(-ar^3 \operatorname{Im} e^{i\vartheta} + o(r^3)) \geq 0.\end{aligned}$$

Part b) of the preceding theorem shows that, in order to decide whether a HELP inequality holds, we only need to study the limiting behaviour of the m function at ∞ and at 0. In the following we shall prove that for the tree-Hill operator, the condition at ∞ is always satisfied; the behaviour of m at 0 depends on whether we have quasi-stability or quasi-instability at $\lambda = 0$.

The behaviour of m at ∞

We have the following observation, valid under general hypotheses.

Lemma 4.3. *Let $q : [0, l] \to \infty$ be integrable and $Z := \{z \in \mathbb{C} \setminus \{0\} \mid \arg z \notin (-\alpha, \alpha) \bmod 2\pi\}$, with any $\alpha > 0$. Then $m(\lambda) \sim \frac{i}{\sqrt{\lambda}}$ $(\lambda \to \infty, \lambda \in Z)$.*

Proof. We begin with obtaining the asymptotics of $\phi(l, \lambda)$ and $\theta(l, \lambda)$ by a variation of constants estimate (cf., e.g., [4]).

Let $\lambda \in Z$. Then the free Schrödinger equation on $[0, l]$, $-y'' = \lambda y$, has the canonical fundamental system

$$\Phi_0(x) = \begin{pmatrix} \cos(\sqrt{\lambda}x) & \sin(\sqrt{\lambda}x)/\sqrt{\lambda} \\ -\sqrt{\lambda}\sin(\sqrt{\lambda}x) & \cos(\sqrt{\lambda}x) \end{pmatrix} \qquad (x \in [0, l]),$$

using the convention $\operatorname{Im}\sqrt{\lambda} > 0$; so solving the initial-value problem $-u'' + q(x)u = \lambda u$ with given $u(0), u'(0)$ by variation of constants, we find

$$\begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} = \begin{pmatrix} u_0(x) \\ u_0'(x) \end{pmatrix} + \int_0^l \Phi_0(x)\Phi_0^{-1}(t) \begin{pmatrix} 0 & 0 \\ q(t) & 0 \end{pmatrix} \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix} dt \qquad (x \in [0, l])$$

and hence

$$u(x) - u_0(x) = \int_0^x q(t) \frac{\sin(\sqrt{\lambda}(x - t))}{\sqrt{\lambda}} u(t)\, dt \qquad (x \in [0, l]),$$

where u_0 is the solution of $-y'' = \lambda y$ with the same initial data as u.

Thus, writing

$$u(t) = (u(t) - u_0(t)) + \cos(\sqrt{\lambda}t)\, u(0) + \frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}\, u'(0),$$

we can estimate

$$
\begin{aligned}
&|e^{i\sqrt{\lambda}x}|\,|u(x)-u_0(x)| \\
&\qquad \le \frac{1}{|i\sqrt{\lambda}|}\int_0^x |q(t)|\,\frac{|e^{2i\sqrt{\lambda}(x-t)}-1|}{2}\Big(|e^{i\sqrt{\lambda}t}|\,|u(t)-u_0(t)| \\
&\qquad\qquad + \frac{|e^{2i\sqrt{\lambda}t}+1|}{2}|u(0)| + \frac{|e^{2i\sqrt{\lambda}t}-1|}{2|i\sqrt{\lambda}|}|u'(0)|\Big)\,dt \\
&\qquad \le \int_0^x \frac{|q(t)|}{|\sqrt{\lambda}|}\,|e^{i\sqrt{\lambda}t}|\,|u(t)-u_0(t)|\,dt + \int_0^x \frac{|q(t)|}{|\sqrt{\lambda}|}\,dt\,(|u(0)| + \frac{|u'(0)|}{|\sqrt{\lambda}|}).
\end{aligned}
$$

This is an integral inequality of Gronwall type,

$$f(x) \le c\int_0^x g + \int_0^x fg$$

with $g \ge 0$; it follows that

$$
\begin{aligned}
f(x)\,e^{-\int_0^x g} &\le \Big(c\int_0^x g + \int_0^x fg\Big)\,e^{-\int_0^x g} \\
&= \int_0^x \Big(c + (f(y) - c\int_0^y g - \int_0^y fg)\Big)\,g(y)\,e^{-\int_0^y g}\,dy \\
&\le c\int_0^x f(y)\,e^{-\int_0^y g}\,dy = c(1 - e^{-\int_0^x g})
\end{aligned}
$$

(for the second line, we rewrite the previous expression as the integral of its derivative). Thus we obtain

$$|u(l) - u_0(l)| \le (|u(0)| + \frac{|u'(0)|}{|\sqrt{\lambda}|})\,|e^{-i\sqrt{\lambda}l}|\,(e^{\frac{1}{\sqrt{\lambda}}\int_0^l |q|} - 1).$$

In particular,

$$\theta_0(l,\lambda) = \cos(\sqrt{\lambda}l) \sim \frac{e^{-i\sqrt{\lambda}l}}{2} \qquad (\operatorname{Im}\sqrt{\lambda} \to \infty)$$

and

$$|\theta(l,\lambda) - \theta_0(l,\lambda)| \le |e^{-i\sqrt{\lambda}l}|\,(e^{\frac{1}{|\sqrt{\lambda}|}\int_0^l |q|} - 1) \sim |e^{-i\sqrt{\lambda}l}|\,\frac{1}{|\sqrt{\lambda}|}\int_0^l |q|,$$

so $\theta(l,\lambda) \sim \frac{e^{-i\sqrt{\lambda}l}}{2}$ $(\operatorname{Im}\sqrt{\lambda} \to \infty)$; similarly,

$$\phi_0(l,\lambda) = \frac{\sin(\sqrt{\lambda}l)}{\sqrt{\lambda}} \sim \frac{i\,e^{-i\sqrt{\lambda}l}}{2\sqrt{\lambda}} \qquad (\operatorname{Im}\sqrt{\lambda} \to \infty)$$

and

$$|\phi(l,\lambda) - \phi_0(l,\lambda)| \le \frac{|e^{-i\sqrt{\lambda}l}|}{|\sqrt{\lambda}|}\,(e^{\frac{1}{|\sqrt{\lambda}|}\int_0^l |q|} - 1) \sim |e^{-i\sqrt{\lambda}l}|\,\frac{1}{|\lambda|}\int_0^l |q|,$$

so $\phi(l,\lambda) \sim \frac{i\,e^{-i\sqrt{\lambda}l}}{2\sqrt{\lambda}}$ $(\operatorname{Im}\sqrt{\lambda} \to \infty)$.

The assertion now follows, bearing in mind that $|\mu(\lambda)| \leq \frac{1}{\sqrt{b}} = o(|e^{-i\sqrt{\lambda}l}|)$ and that $\operatorname{Im}\sqrt{\lambda} \to \infty$ whenever $\lambda \to \infty$ in Z. □

The behaviour of m at 0

The type of limiting value of m at 0 essentially depends on whether the tree-Hill equation with spectral parameter 0 has quasi-stability or quasi-instability.

Theorem 4.4. a) *If $\lambda = 0$ is a point of quasi-stability, then* $\lim_{\lambda \downarrow 0} m(\lambda) \in \mathbb{C} \setminus \mathbb{R}$. *Consequently, a HELP inequality holds.*

b) *If $\lambda = 0$ is a point of quasi-instability, then either*

- $\lim_{\lambda \downarrow 0} m(\lambda) \in \mathbb{R} \setminus \{0\}$ *and there is no valid HELP inequality, or*
- $m(0) = 0$, *0 is a Dirichlet eigenvalue, or* $\frac{1}{m}(0) = 0$, *0 is a Neumann eigenvalue, and in either case, a HELP inequality holds.*

Proof. First we observe that if $\phi(l,0) = 0$ or $\theta'(l,0) = 0$, then 0 is not a point of quasi-stability. Indeed, then either the top-right or the bottom-left entries of $A(0)$ vanish, so in view of the determinant of $A(0)$, $\phi'(l,0) = \frac{1}{\theta(l,0)}$, and the quasi-discriminant satisfies

$$D(0)^2 = \left(\theta(l,0) + \frac{1}{\theta(l,0)b}\right)^2 = \frac{1}{b}\left(\sqrt{b}\theta(l,0) + \frac{1}{\sqrt{b}\theta(l,0)}\right)^2 \geq \frac{4}{b}.$$

a) If 0 is a point of quasi-stability, then observing that all entries of A are entire functions of λ and hence approach a finite real limit as $\lambda \to 0$, while $\mu(\lambda) \in \mathbb{C} \setminus \mathbb{R}$ in the limit, the formulae given in Lemma 4.1 show that m has a non-real limit, and the assertion follows.

b) If 0 is a point of quasi-instability and $m(0) \neq 0 \neq \frac{1}{m}(0)$, then m will have a finite limit as the ratio of non-vanishing analytic functions given in Lemma 4.1 (note μ is analytic except at the quasi-(anti-)symmetric eigenvalues; cf. [9] p. 64); moreover, as all entries of A and μ are real in the limit, so will m be, and by Theorem 4.2 b) there will be no valid HELP inequality.

If $m(0) = 0$, then the formulae of Lemma 4.1 show that $\phi(l,0) = 0$ and $\phi'(l,0) = b\mu(0)$, so the transfer matrix must have the form

$$A(0) = \begin{pmatrix} \frac{1}{\mu(0)b} & 0 \\ \frac{\theta'(l,0)}{b} & \mu(0) \end{pmatrix}.$$

This matrix clearly has eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalue $\mu(0)$ of modulus $|\mu(0)| < \frac{1}{\sqrt{b}}$; in other words, the extension of ϕ as a symmetric function on the tree is a Dirichlet eigenfunction. As the two eigenvalues of $A(0)$ are distinct, $\theta(l,0) - \mu(0) \neq 0$, so m is analytic at 0. Theorem 4.2 b) shows that a HELP inequality holds.

The situation $\frac{1}{m}(0) = 0$ is analogous; in this case,

$$A(0) = \begin{pmatrix} \mu(0) & \phi(l,0) \\ 0 & \frac{1}{\mu(0)b} \end{pmatrix},$$

Now the extension of θ as a symmetric function on the tree is a Neumann eigenfunction, $\phi'(l,0) \neq b\mu(0)$, and $\frac{1}{m}$ is analytic at 0. Again by Theorem 4.2 b), we conclude that a HELP inequality holds. □

Remark 4.5. If 0 is a point of transition between quasi-stability and quasi-instability, i.e., a quasi-(anti-)periodic eigenvalue, then the analysis is very much like part b) of the preceding theorem, with the difference that in the exceptional cases $m(0) = 0$ or $\frac{1}{m}(0) = 0$, the two eigenvalues of $A(0)$ are equal and ϕ or θ, resp., will not be Dirichlet or Neumann eigenfunctions. It would seem that in this exceptional situation the local validity of (2) will need to be verified separately for the particular operator under study.

5. Calculating the optimal constant

Theorem 4.4 only provides a criterion for the existence or otherwise of a HELP inequality with a finite constant; however, the actual determination of that constant will require more detailed knowledge of the m function, beyond the bare asymptotics at 0 and at ∞. To illustrate the process, we studied the tree-Hill operator with edge length $l = 1$, branching number $b = 3$ and the potential $q_\tau(x) := x + \tau$ $(x \in [0,1])$ on each edge. The offset τ is introduced with a view to investigating the effect of closeness of $\lambda = 0$ to an end-point of the region of quasi-stability. More precisely, the first two quasi-stability intervals for potential q_0 are approximately $[0.68772, 7.15072]$ and $[14.0335, 33.5907]$; we therefore considered the cases $\tau = -3$, $\tau = -0.7$ and $\tau = -1$ in order to study situations where 0 is near the middle, close to the end, or at an intermediate position in the quasi-stability interval, respectively.

For the calculation, we use (2) directly on semicircular arcs in the complex upper half-plane, i.e., for $\lambda = Re^{i\vartheta}$, $\vartheta \in (0,\pi)$; $r > 0$. The Weyl–Titchmarsh function m is computed, using Lemma 4.1, with Mathematica software from numerical solutions of

$$-y'' + q_\tau y = \lambda y$$

on $[0,1]$. For fixed $r > 0$,

$$\frac{1}{|\lambda|^2}\,\mathrm{Im}(\lambda^2 m(\lambda)) = \mathrm{Im}(e^{2i\vartheta} m(re^{i\vartheta}))$$

then is a function taking a non-negative value at $\vartheta = 0$, a negative value at $\vartheta = \frac{\pi}{2}$ and a non-negative value at $\vartheta = \pi$ (see Fig. 1).

To determine the maximal symmetric interval of negativity around $\vartheta = \frac{\pi}{2}$ for this function, we computed its nearest zero $\vartheta_0(r)$ by the bisection method, taking $\vartheta_0(r) \in [0, \frac{\pi}{2})$ w.l.o.g. (otherwise subtract from π). The supremum $\hat{\vartheta}$ of $\vartheta_0(r)$, taken over all $r > 0$, will then determine the optimal constant $K = (\cos\hat{\vartheta})^{-2}$.

The numerical results show three different situations.

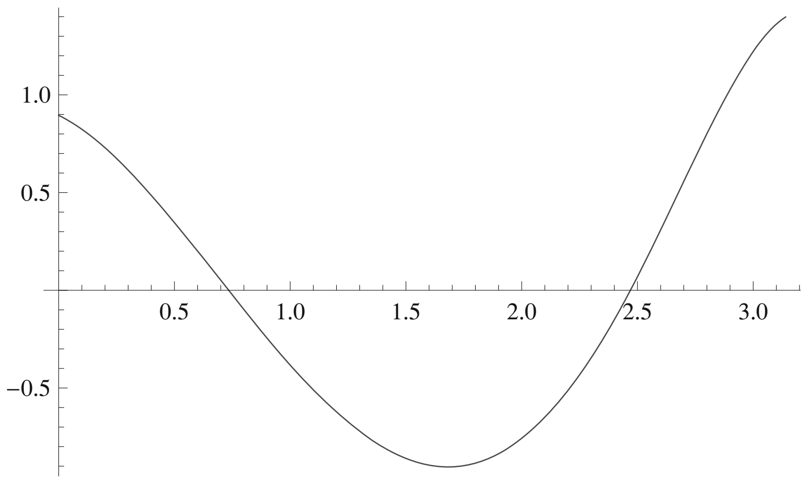

FIGURE 1. Fig 1. Plot of $\mathrm{Im}(e^{2i\vartheta}\, m(e^{i\vartheta}))$.

In the first case, we have $\tau = -3$, and the profile curve of the zero is given in Fig. 2 a). The curve approaches its supremum asymptotically as $r \to \infty$, and $\hat{\vartheta} = \frac{\pi}{3}$ gives the optimal constant $K = 4$. Since the asymptotic value at infinity will always be the same, as shown in Lemma 4.3, $K = 4$ is clearly the minimum value for an optimal HELP constant.

In the second case, where $\tau = -0.7$, the behaviour of the curve near ∞ is the same as before, but the supremum is now achieved near 0, giving a HELP constant of about 60, cf. Fig. 2 b). This behaviour seems to be typical of a situation where 0 is close to the end of an interval of quasi-stability; it is plausible that the HELP constant should tend to infinity as 0 approaches a region of quasi-instability, in the interior of which the inequality was shown to fail.

Finally, the intermediate choice $\tau = -1$ gives rise to a situation where the optimal constant is determined by a local maximum, cf. Fig. 2c), illustrating the point that knowledge of the asymptotics is not sufficient in general.

6. Appendix: m function and spectra

We give a simple example to show that the m function for the Schrödinger operator on a tree does not provide full information on its spectral properties. Consider the tree-Hill operator with potential $q = 0$, i.e., the tree Laplacian on our regular tree Γ, with a Dirichlet boundary condition at 0. As shown in [11], the spectrum of this operator consists of absolutely continuous bands coinciding with the closure of the quasi-stability intervals and of eigenvalues of infinite multiplicity, situated in quasi-instability intervals. The high multiplicity arises from the fact that any Dirichlet eigenfunction can be shifted down, multiplied with the b-th roots of unity, into the first-generation

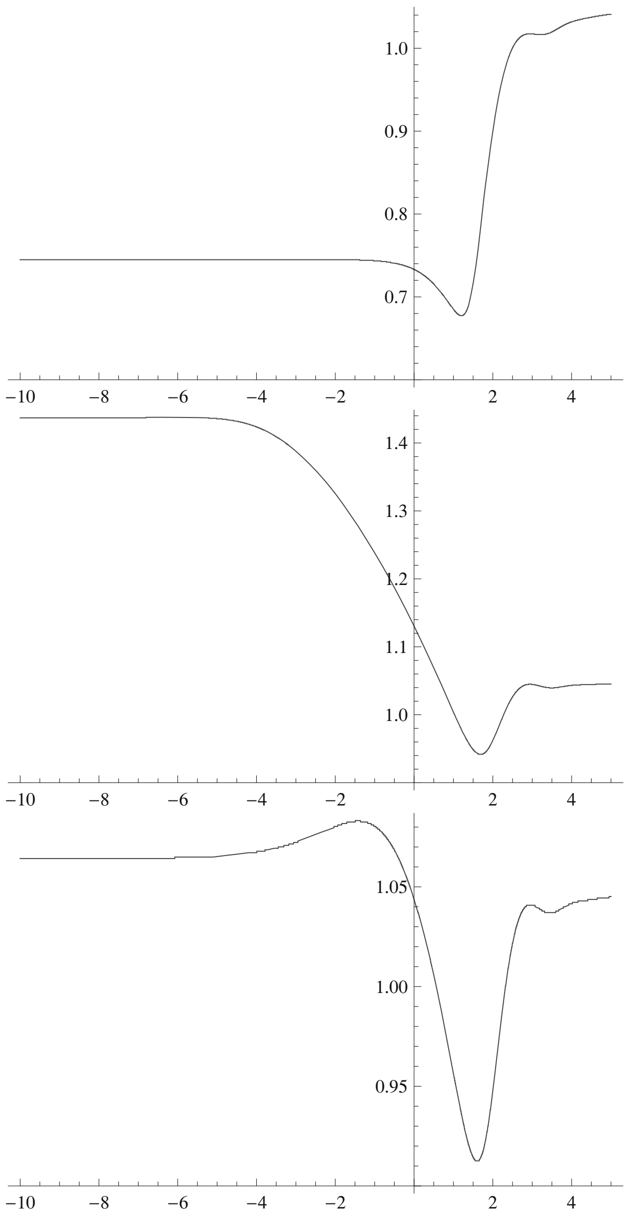

FIGURE 2. Fig. 2. Zero ϑ_0 as a function of $\log r$ for $\tau = -3$ (a), $\tau = -0.7$ (b), and $\tau = -1$ (c).

subtrees and extended by 0 on the 0-th generation edge, to obtain another Dirichlet solution in $\mathcal{D}$. Thus, starting from a symmetric eigenfunction, an unlimited number of linearly independent eigenfunctions can be created by repeated application of this mechanism.

As apparent from the considerations in the proof of Theorem 4.4 b) above, the Dirichlet eigenvalue will, by virtue of its symmetric eigenfunction, be accompanied with a zero of the m function. While the m function thus flags up all eigenvalues for the operator with full tree-Hill symmetry, although not their multiplicities, the disparity becomes more obvious when this symmetry of periodic type is broken.

Let us therefore consider a modified operator which differs from the above by the addition of a potential $\tilde{q}$ of compact support in $(0, l)$ on the 0th generation edge only. The essential spectrum of this perturbed operator will be the same as before, and the infinitely many non-symmetric Dirichlet eigenfunctions constructed above will still be Dirichlet eigenfunctions, as they vanish on the 0-th generation edge and hence know nothing about the potential. However, the Dirichlet eigenvalues on the symmetric subspace will feel the perturbation and hence change in general, as will the m function.

Regarding the latter, we can calculate it for the perturbed operator as follows. Let m be the original m function, calculated as previously for the tree-Hill operator from the canonical fundamental system Φ, which in our case is equal to the Φ_0 of the proof of Lemma 4.3 above. Now let

$$\tilde{\Phi} = \begin{pmatrix} \tilde{\theta} & \tilde{\phi} \\ \tilde{\theta}' & \tilde{\phi}' \end{pmatrix}$$

be the canonical fundamental system on $[0, l]$ of $-u'' + \tilde{q}\, u = \lambda u$.

For $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the distinguished solution $\psi \in \mathcal{D}$ will be the same, except on the 0-th generation edge, for the unperturbed and perturbed operators, so denoting the solution in the perturbed case with $\tilde{\psi}$, we find $\tilde{\psi}(l-) = \psi(l-)$, $\tilde{\psi}'(l-) = \psi'(l-)$ and therefore

$$\begin{aligned} \begin{pmatrix} \tilde{\psi} \\ \tilde{\psi}' \end{pmatrix}(0) &= \tilde{\Phi}^{-1}(l) \begin{pmatrix} \tilde{\psi} \\ \tilde{\psi}' \end{pmatrix}(l) = \tilde{\Phi}^{-1}(l) \Phi(l) \begin{pmatrix} \psi \\ \psi' \end{pmatrix}(0) \\ &= \begin{pmatrix} (\theta\, \tilde{\phi}' - \phi\, \tilde{\theta}')(l)\, \psi(0) + (\phi\, \tilde{\theta} - \theta\, \tilde{\phi})(l)\, \psi'(0) \\ (\theta'\, \tilde{\phi}' - \phi'\, \tilde{\theta}')(l)\, \psi(0) + (\phi'\, \tilde{\theta} - \theta'\, \tilde{\phi})(l)\, \psi'(0). \end{pmatrix}. \end{aligned}$$

For the perturbed m function, we thus obtain

$$\tilde{m} = -\frac{\tilde{\psi}(0)}{\tilde{\psi}'(0)} = \frac{(\theta\, \tilde{\phi}' - \phi\, \tilde{\theta}')(l)\, m - (\phi\, \tilde{\theta} - \theta\, \tilde{\phi})(l)}{(\phi'\, \tilde{\theta} - \theta'\, \tilde{\phi})(l) - (\theta'\, \tilde{\phi}' - \phi'\, \tilde{\theta}')(l)\, m}.$$

This function $\tilde{m}$ has a zero at the position of a simple Dirichlet eigenvalue for the perturbed operator with symmetric eigenfunction. However, at the position of the Dirichlet eigenvalue of infinite multiplicity, m has a zero and hence (cf. Lemma 4.1) $\phi(l) = 0$ and, by linear independence, $\theta(l) \neq 0$. Thus we obtain

$$\tilde{m} = \frac{\theta(l)\, \tilde{\phi}(l)}{\phi'(l)\, \tilde{\theta}(l) - \theta'(l)\, \tilde{\phi}(l)},$$

which will be zero only if $\tilde{\phi}(l) = 0$. This will not be the case for most perturbations $\tilde{q}$, and hence $\tilde{m}$ cannot be used to detect the eigenvalue of infinite multiplicity inherited from the unperturbed problem.

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