Chapter 2

Global Existence and Exponential Stability for a Real Viscous Heat-conducting Flow with Shear Viscosity

2.1 Introduction

In this chapter we shall study the global existence and exponential stability of weak solutions for a real viscous compressible heat-conducting flow between two horizontal plates. The system describing this type of flow is derived from the following general $3D$ Navier-Stokes equations:

$$\rho_t + \mathrm{div}(\rho\vec{u}) = 0, \tag{2.1.1}$$

$$(\rho\vec{u})_t + \mathrm{div}\,(\rho\vec{u}\otimes\vec{u}) + \nabla p = \mathrm{div}\,(\lambda'(\mathrm{div}\,\vec{u}))\,\vec{I} + \mu(\nabla\vec{u} + (\nabla\vec{u})^T), \tag{2.1.2}$$

$$\zeta_t + \mathrm{div}\,(\vec{u}(\zeta + p)) = \mathrm{div}\,\big(\lambda'(\mathrm{div}\,\vec{u})\vec{u} + \mu\vec{u}(\nabla\vec{u} + (\nabla\vec{u})^T) + \kappa\nabla\theta\big) \tag{2.1.3}$$

where $x \in \mathbb{R}^3$ is the spatial variable and $t > 0$ is the time, $\rho \in \mathbb{R}, \vec{u} \in \mathbb{R}^3$ and $\theta \in \mathbb{R}^+$ denote the density, velocity and temperature, respectively, the total energy is $\zeta = \rho\left(e + \frac{1}{2}|\vec{u}|^2\right)$, with e the internal energy and $\frac{1}{2}|\vec{u}|^2$ the kinetic energy, the equations of state $p = p(\rho, \theta)$, $e = e(\rho, \theta)$ relate this pressure p and the internal energy e with the density and temperature of the flow, $(\nabla\vec{u})^T$ is the transpose of the matrix $\nabla\vec{u}$, $\lambda' = \lambda'(\rho, \theta)$ and $\mu = \mu(\rho, \theta)$ are the viscosity coefficients of the flow and $\kappa = \kappa(\rho, \theta)$ is the heat conductivity. The fluid in question is a Newtonian fluid, i.e., the stress tensor $-p\vec{I} + \lambda'(\mathrm{div}\,\vec{v})\vec{I} + \mu(\nabla\vec{u} + (\nabla\vec{u})^T)$ is a linear function of the deformation tensor $\frac{1}{2}(\nabla\vec{u} + (\nabla\vec{u})^T)$. The viscosity and heat conduction terms describe the dissipative processes in viscous flows.

Consider a $3D$ flow (2.1.1)–(2.1.3) with spatial variable $\vec{x} = (x, x_2, x_3)$, which is moving in the x direction and uniform in the transverse direction (x_2, x_3),

$$\rho = \rho(x,t), \quad \theta = \theta(x,t), \quad \vec{u} = (u, \vec{w})(x,t), \quad \vec{w} = (u_2, u_3), \tag{2.1.4}$$

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where $u \in \mathbb{R}$ is the longitudinal velocity and $\vec{w} \in \mathbb{R}^2$ is the transverse velocity. With this structure (2.1.4), equations (2.1.1)–(2.1.3) can be written as the following in one space dimension with $\lambda = \lambda' + 2\mu > 0$:

$$\rho_t + (\rho u)_x = 0, \tag{2.1.5}$$

$$(\rho u)_t + (\rho u^2 + p)_x = (\lambda u_x)_x, \tag{2.1.6}$$

$$(\rho\vec{w})_t + (\rho u\vec{w})_x = (\mu\vec{w}_x)_x, \tag{2.1.7}$$

$$\zeta_t + (u(\zeta + p))_x = (\lambda u u_x + \mu\vec{w}\cdot\vec{w}_x + \kappa\theta_x)_x\,, \tag{2.1.8}$$

where as in (2.1.1)–(2.1.3), $x \in \mathbb{R}$ is the spatial variable and $t > 0$ is the time variable, $\rho \in \mathbb{R}^+$, $u \in \mathbb{R}$, $\vec{w} \in \mathbb{R}^2$, $\theta \in \mathbb{R}$, p denote the density, longitudinal velocity, transverse velocity, temperature, pressure respectively and the total energy of the plane viscous flow is

$$\zeta = \rho\left(e + \frac{1}{2}(u^2 + |\vec{w}|^2)\right), \tag{2.1.9}$$

with internal energy e. The pressure p and internal energy e are expressed by the density and the temperature of the flow according to the state equations

$$p = p(\rho,\theta), \qquad e = e(\rho,\theta) \tag{2.1.10}$$

where $\lambda = \lambda(\rho,\theta)$ and $\mu = \mu(\rho,\theta)$ are the viscosity coefficients of the flow and $\kappa = \kappa(\rho,\theta)$ is the heat conductivity; $\mu = \mu(\rho,\theta)$ is particularly called the shear viscosity.

Consider the initial boundary value problem (2.1.5)–(2.1.8) in a bounded spatial domain $\Omega = (0,1)$ with the following initial condition and boundary conditions:

$$(\rho, u, \vec{w}, \theta)|_{t=0} = (\rho_0, u_0, \vec{w}_0, \theta_0)(x), \quad x \in \Omega, \tag{2.1.11}$$

$$(u,\vec{w})\big|_{\partial\Omega} = 0,\ \theta_x\big|_{\partial\Omega} = 0,\ \ \textit{or}\ \ \theta\big|_{\partial\Omega} = T_0 = \text{const}\,. > 0, \tag{2.1.12}$$

where the initial data are compatible with each other.

We assume that $\theta_0(x) > 0$, $\rho_0(x) > 0$, for any $x \in (0,1)$,

$$\int_0^1 \rho_0(x)dx = 1. \tag{2.1.13}$$

Now we introduce the Lagrangian variable,

$$y = y(x,t) = \int_0^x \rho(\xi,t)d\xi. \tag{2.1.14}$$

Then we have from (2.1.5), (2.1.13) and (2.1.14),

$$0 \le y \le 1,\ \int_0^1 \rho(x,t)dx = \int_0^1 \rho_0(x)dx = 1. \tag{2.1.15}$$

Therefore we can translate the problem (2.1.1)–(2.1.3) in Euler coordinates into the following initial boundary value problem in Lagrangian coordinates (y,t), $y \in \Omega = (0,1)$, a moving of coordinates along the particle path,

$$v_t - u_y = 0, \tag{2.1.16}$$

$$u_t + p_y = \left(\lambda \frac{u_y}{v}\right)_y, \tag{2.1.17}$$

$$\vec{w}_t = \left(\mu \frac{\vec{w}_y}{v}\right)_y, \tag{2.1.18}$$

$$E_t + (up)_y = \left(\frac{\lambda u u_y + \mu \vec{w}\vec{w}_y + \kappa\theta_y}{v}\right)_y, \tag{2.1.19}$$

with the initial boundary conditions

$$(v,u,\vec{w},\theta)\big|_{t=0} = (v_0,u_0,\vec{w}_0,\theta_0)(y), \quad y \in \Omega, \tag{2.1.20}$$

$$(u,\vec{w}) = 0, \quad \theta_y = 0, \quad \text{on } \partial\Omega \times [0,+\infty), \tag{2.1.21}$$

or

$$(u,\vec{w}) = 0, \quad \theta_y = T_0 > 0, \quad \text{on } \partial\Omega \times [0,+\infty), \tag{2.1.22}$$

where $v = \dfrac{1}{\rho}$ is specific volume, $e = e(v,\theta), p = p(v,\theta)$, and

$$E = e + \frac{1}{2}(u^2 + |\vec{w}|^2). \tag{2.1.23}$$

The second law of thermodynamics states the relation between p and e,

$$e_v(v,\theta) + p(v,\theta) = \theta p_\theta(v,\theta). \tag{2.1.24}$$

Now we assume that e, p and κ are C^3 functions on $0 < u < +\infty$ and $0 \le \theta < +\infty$. Let q and r be two positive constants (exponents of growth) satisfying one of the following relations:

$$0 \le r \le 1/3, \quad 1/3 < q, \tag{2.1.25}$$

$$1/3 < r < 4/7, \quad (2r+1)/5 < q, \tag{2.1.26}$$

$$4/7 \le r \le 1, \quad (5r+1)/9 < q, \tag{2.1.27}$$

$$1 < r \le 13/3, \quad (9r+1)/15 < q, \tag{2.1.28}$$

$$13/3 < r, \quad (11r+3)/19 < q. \tag{2.1.29}$$

Moreover, we further assume that there are positive constants ν, p_0, p_1, k_0 and for any $\underline{v} > 0$, there are positive constants $N(\underline{v})$, $p_2(\underline{v})$, $p_3(\underline{v})$ and $k_1(\underline{v})$ such that for any $v \ge \underline{v}$ and $\theta \ge 0$ the following conditions hold:

$$0 \le e(v,0), \quad \nu(1+\theta^r) \le e_\theta(v,\theta) \le N(\underline{v})(1+\theta^r), \tag{2.1.30}$$

$$p_0\theta^{r+1} < vp(v,\theta) \le p_1(1+\theta^{r+1}), \tag{2.1.31}$$

$$
\begin{aligned}
&- p_2(\underline{v})[l+(1-l)\theta+\theta^{r+1}] \le p_v(v,\theta)\\
&\quad \le -p_3(\underline{v})[l+(1-l)\theta+\theta^{r+1}], l=0 \ \ or\ 1, && (2.1.32)\\
&|p_\theta(v,\theta)| \le p_3(\underline{v})(1+\theta^r), && (2.1.33)\\
&k_0(1+\theta^q) \le k(v,\theta) \le k_1(\underline{v})(1+\theta^q), && (2.1.34)\\
&|k_v(v,\theta)| + |k_{vv}(v,\theta)| \le k_1(\underline{v})(1+\theta^q). && (2.1.35)
\end{aligned}
$$

For the viscosity $\lambda(v,\theta)$, we assume that

$$\lambda(v,\theta) = \lambda_0 > 0 \tag{2.1.36}$$

is a constant.

The notation in the chapter is standard. We put $\|\cdot\| = \|\cdot\|_{L^2}$ and denote by $C^k(I,B)$, $k \in \mathbb{N}_0 \equiv \{0\} \cup \mathbb{N}$, the space of k-times continuously differentiable functions from $I \subseteq \mathbb{R}$ into a Banach space B, and likewise by $L^{\bar{p}}(I,B)$, $1 \le \bar{p} \le +\infty$, the corresponding Lebesgue spaces. Subscripts t, x and y denote the (partial) derivatives with respect to t, x and y, respectively. We use C_i $(i = 1,2,3,4)$ to denote the universal positive constants depending only on the H^i norm of initial data, $\min_{y\in[0,1]} v_0(y)$, $\min_{y\in[0,1]} \vec{w}_0(y)$ and $\min_{y\in[0,1]} \theta_0(y)$. Without danger of confusion, we will use the same symbol to denote the state functions as well as their values along the thermodynamic process, e.g., $p(u,\theta)$ and $p(u(x,t), \theta(x,t))$ and $p(x,t)$.

Define the following three spaces:

$$
\begin{aligned}
H_+^1 = \Big\{&(v,u,\vec{w},\theta) \in (H^1[0,1])^5 : v(x) > 0, \theta(x) > 0, \forall x \in [0,1],\\
&u(0) = u(1) = 0, \vec{w}(0) = \vec{0},\ \theta(0) = \theta(1) = T_0 \ \text{ for } \ (2.1.22)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
H_+^i = \Big\{&(v,u,\vec{w},\theta) \in (H^i[0,1])^5 : v(x) > 0, \theta(x) > 0, \forall x \in [0,1],\\
&u(0) = u(1) = 0, \vec{w}(0) = \vec{0},\ \theta(0) = \theta(1) = T_0 \ \text{ for } \ (2.1.22),\\
&\theta'(0) = \theta'(1) = 0 \ \text{ for } \ (2.1.21)\Big\},\ i = 2,4.
\end{aligned}
$$

Our results read as follows, which are selected from [60, 64].

Theorem 2.1.1. *Assume that e, p and κ are C^2 functions on $0 < v < +\infty$ and $0 \le \theta < +\infty$, and assumptions* (2.1.23)–(2.1.36) *hold. Then for any $(v_0, u_0, \vec{w}_0, \theta_0) \in H_+^1$, there exists a unique global solution $(v(t), u(t), \vec{w}(t), \theta(t)) \in H_+^1$ to problem* (2.1.16)–(2.1.21) *or* (2.1.16)–(2.1.20), (2.1.22) *verifying that,*

$$0 < C_1^{-1} \le \theta(y,t) \le C_1, \quad 0 < C_1^{-1} \le u(y,t) \le C_1, \quad \forall (y,t) \in [0,1] \times [0,+\infty) \tag{2.1.37}$$

and for any $t > 0$,

$$\begin{aligned}
&\|v(t)-\bar{v}\|_{H^1}^2+\|u(t)\|_{H^1}^2+\|\vec{w}(t)\|_{H^1}^2+\|\theta(t)-\bar{\theta}\|_{H^1}^2\\
&\quad+\int_0^t\Big(\|v-\bar{v}\|_{H^1}^2+\|u\|_{H^2}^2+\|\vec{w}\|_{H^2}^2+\|\theta-\bar{\theta}\|_{H^2}^2+\|u_t\|^2\\
&\quad\quad+\|\vec{w}_t\|^2+\|\theta_t\|^2\Big)(\tau)\,\mathrm{d}\tau\le C_1. \qquad (2.1.38)
\end{aligned}$$

Moreover, there exist constants $C_1 > 0$ and $\gamma_1 = \gamma_1(C_1) > 0$ such that and for any $t > 0$,

$$\begin{aligned}
&e^{\gamma t}\left(\|v(t)-\bar{v}\|_{H^1}^2+\|u(t)\|_{H^1}^2+\|\vec{w}(t)\|_{H^1}^2+\|\theta(t)-\bar{\theta}\|_{H^1}^2\right)\\
&\quad+\int_0^t e^{\gamma\tau}\Big(\|v-\bar{v}\|_{H^1}^2+\|u\|_{H^2}^2+\|\vec{w}\|_{H^2}^2+\|\theta-\bar{\theta}\|_{H^2}^2\\
&\quad\quad+\|u_t\|^2+\|\vec{w}_t\|^2+\|\theta_t\|^2\Big)(\tau)\,d\tau\le C_1 \qquad (2.1.39)
\end{aligned}$$

where $\bar{v} = \int_0^1 v_0(x)\,dx$, $\bar{\theta} = T_0$ *for* (2.1.22), $e(\bar{v},\bar{\theta}) = \int_0^1 (e(v_0,\theta_0)+\frac{v_0^2}{2})(x)\,dx$ *for* (2.1.21).

Theorem 2.1.2. *Assume that e, p and κ are C^3 functions on $0 < v < +\infty$, $0 \le \theta < +\infty$, and assumptions* (2.1.23)–(2.1.36) *hold. Then for any $(v_0, u_0, \vec{w}_0, \theta_0) \in H_+^2$, there exists a unique global solution $(v(t), u(t), \vec{w}(t), \theta(t)) \in H_+^2$ to problem* (2.1.16)–(2.1.21) *or* (2.1.16)–(2.1.20), (2.1.22) *verifying that for any $t > 0$,*

$$\begin{aligned}
&\|v(t)-\bar{v}\|_{H^2}^2+\|u(t)\|_{H^2}^2+\|\vec{w}(t)\|_{H^2}^2+\|\theta(t)-\bar{\theta}\|_{H^2}^2+\|u_t(t)\|^2\\
&\quad+\|\vec{w}_t(t)\|^2+\|\theta_t(t)\|^2+\int_0^t\Big(\|v-\bar{v}\|_{H^2}^2+\|u\|_{H^3}^2+\|\vec{w}\|_{H^3}^2\\
&\quad\quad+\|\theta-\bar{\theta}\|_{H^3}^2+\|u_{ty}\|^2+\|\vec{w}_{ty}\|^2+\|\theta_{ty}\|^2\Big)(\tau)\,d\tau\le C_2, \qquad (2.1.40)
\end{aligned}$$

and there exist constants $C_2 > 0$ and $\gamma_2 = \gamma_2(C_2) > 0$ such that for any fixed $\gamma \in (0, \gamma_2]$, the following estimates hold for any $t > 0$:

$$\begin{aligned}
&e^{\gamma t}\Big(\|v(t)-\bar{v}\|_{H^2}^2+\|u(t)\|_{H^2}^2+\|\vec{w}(t)\|_{H^2}^2+\|\theta(t)-\bar{\theta}\|_{H^2}^2+\|u_t(t)\|^2\\
&\quad+\|\vec{w}_t(t)\|^2+\|\theta_t(t)\|^2\Big)+\int_0^t e^{\gamma t}\Big(\|v-\bar{v}\|_{H^2}^2+\|u\|_{H^3}^2+\|\vec{w}\|_{H^3}^2\\
&\quad\quad+\|\theta-\bar{\theta}\|_{H^3}^2+\|u_{ty}\|^2+\|\vec{w}_{ty}\|^2+\|\theta_{ty}\|^2\Big)(\tau)d\tau\le C_2. \qquad (2.1.41)
\end{aligned}$$

Theorem 2.1.3. *Assume that e, p are C^5 functions on $0 < v < +\infty$ and $0 \le \theta < +\infty$, and assumptions* (2.1.23)–(2.1.36) *hold. Then for any $(v_0, u_0, \vec{w}_0, \theta_0) \in H_+^4$, there exists a unique global solution $(v(t), u(t), \vec{w}(t), \theta(t)) \in H_+^4$ to problem*

(2.1.16)–(2.1.21) *or* (2.1.16)–(2.1.20), (2.1.22) *verifying that for any* $t > 0$,

$$\begin{aligned}
&\|v(t)-\bar v\|^2_{H^4} + \|v(t)-\bar v\|^2_{W^{3,\infty}} + \|v_t(t)\|^2_{H^3} + \|v_{tt}(t)\|^2_{H^1} + \|u(t)\|^2_{H^4} \\
&\quad + \|u(t)\|^2_{W^{3,\infty}} + \|u_t(t)\|^2_{H^2} + \|u_{tt}(t)\|^2 + \|\theta(t)-\bar\theta\|^2_{H^4} \\
&\quad + \|\theta(t)-\bar\theta\|^2_{W^{3,\infty}} + \|\theta_t(t)\|^2_{H^2} + \|\theta_{tt}(t)\|^2 + \|\vec w(t)\|^2_{H^4} \\
&\quad + \|\vec w(t)\|^2_{W^{3,\infty}} + \|\vec w_t(t)\|^2_{H^2} + \|\vec w_{tt}(t)\|^2 \le C_4, \qquad (2.1.42)\\
&\int_0^t \Big(\|v-\bar v\|^2_{H^4} + \|v-\bar v\|^2_{W^{3,\infty}} + \|v_t\|^2_{H^4} + \|v_{tt}\|^2_{H^2} + \|v_{ttt}\|^2 \\
&\quad + \|u\|^2_{H^5} + \|u\|^2_{W^{4,\infty}} + \|u_t\|^2_{H^3} + \|u_{tt}\|^2_{H^1} + \|\theta-\bar\theta\|^2_{H^5} \\
&\quad + \|\theta-\bar\theta\|^2_{W^{4,\infty}} + \|\theta_t\|^2_{H^3} + \|\theta_{tt}\|^2_{H^1} + \|\vec w\|^2_{H^5} + \|\vec w\|^2_{W^{4,\infty}} \\
&\quad + \|\vec w_t\|^2_{H^3} + \|\vec w_{tt}\|^2_{H^1}\Big)(\tau)\, d\tau \le C_4, \qquad (2.1.43)
\end{aligned}$$

and there exist constants $C_4 > 0$ *and* $\gamma_4 = \gamma_4(C_4) > 0$ *such that for any fixed* $\gamma \in (0, \gamma_4]$, *the following estimates hold for any* $t > 0$:

$$\begin{aligned}
&\|v(t)-\bar v\|^2_{H^4} + \|v(t)-\bar v\|^2_{W^{3,\infty}} + \|v_t(t)\|^2_{H^3} + \|v_{tt}(t)\|^2_{H^1} + \|u(t)\|^2_{H^4} \\
&\quad + \|u(t)\|^2_{W^{3,\infty}} + \|u_t(t)\|^2_{H^2} + \|u_{tt}(t)\|^2 + \|\theta(t)-\bar\theta\|^2_{H^4} \\
&\quad + \|\theta(t)-\bar\theta\|^2_{W^{3,\infty}} + \|\theta_t(t)\|^2_{H^2} + \|\theta_{tt}(t)\|^2 + \|\vec w(t)\|^2_{H^4} \\
&\quad + \|\vec w(t)\|^2_{W^{3,\infty}} + \|\vec w_t(t)\|^2_{H^2} + \|\vec w_{tt}(t)\|^2 \le C_4 e^{-\gamma t}, \qquad (2.1.44)\\
&\int_0^t e^{\gamma\tau}\Big(\|v-\bar v\|^2_{H^4} + \|v-\bar v\|^2_{W^{3,\infty}} + \|v_t\|^2_{H^4} + \|v_{tt}\|^2_{H^2} + \|v_{ttt}\|^2 \\
&\quad + \|u\|^2_{H^5} + \|u\|^2_{W^{4,\infty}} + \|u_t\|^2_{H^3} + \|u_{tt}\|^2_{H^1} + \|\theta-\bar\theta\|^2_{H^5} \\
&\quad + \|\theta-\bar\theta\|^2_{W^{4,\infty}} + \|\theta_t\|^2_{H^3} + \|\theta_{tt}\|^2_{H^1} + \|\vec w\|^2_{H^5} + \|\vec w\|^2_{W^{4,\infty}} \\
&\quad + \|\vec w_t\|^2_{H^3} + \|\vec w_{tt}\|^2_{H^1}\Big)(\tau)\, d\tau \le C_4. \qquad (2.1.45)
\end{aligned}$$

2.2 Proof of Theorem 2.1.1

In this section, we shall complete the proof of Theorem 2.1.1 and take that the assumptions in Theorem 2.1.1 to be valid. We begin with the following lemma.

Lemma 2.2.1. *Assume that* e, p *and* κ *are* C^2 *functions on* $0 < v < +\infty$ *and* $0 \le \theta < +\infty$, *and assumptions* (2.1.23)–(2.1.36) *hold. Then for any* $(v_0, u_0, \vec w_0, \theta_0) \in H^1_+$, *there exists a unique global solution* $(v(t), u(t), \vec w(t), \theta(t)) \in H^1_+$ *to problem* (2.1.16)–(2.1.21) *or* (2.1.16)–(2.1.20), (2.1.22) *verifying that,*

$$\begin{aligned}
&\|v(t)-\bar v\|^2_{H^1} + \|u(t)\|^2_{H^1} + \|\vec w(t)\|^2_{H^1} + \|\theta(t)-\bar\theta\|^2_{H^1} \qquad (2.2.1)\\
&\quad + \int_0^t \Big(\|v-\bar v\|^2_{H^1} + \|u\|^2_{H^2} + \|\vec w\|^2_{H^2} + \|\theta-\bar\theta\|^2_{H^2}\Big)(\tau)\,\mathrm{d}\tau \le C_1, \quad \forall t > 0
\end{aligned}$$

and for any $(y,t)\in[0,1]\times[0,+\infty)$,

$$0<C_1^{-1}\le\theta(y,t)\le C_1,\quad 0<C_1^{-1}\le v(y,t)\le C_1. \tag{2.2.2}$$

Proof. For the case $\vec{w}=0$, Qin [55] proved the result to problems (2.1.16), (2.1.17), (2.1.19)–(2.1.21) or (2.1.16), (2.1.17), (2.1.19)–(2.1.20), (2.1.22) under the assumptions (2.1.23)–(2.1.36). The proof is the same as that in Qin [50, 51, 52, 54] and Chapter 3 of [59] for the case of $\vec{w}=0$. The proof is complete. □

In what follows we shall prove the exponential stability in H_+^1. Set

$$\Psi(v,\theta)=e(v,\theta)-\theta\eta(\theta,v) \tag{2.2.3}$$

where $\eta(\theta,v)$ is defined by the relations

$$e_\theta=\theta\eta_\theta,\quad \eta_v=p_\theta. \tag{2.2.4}$$

Now we introduce the density of the gas, $\rho=1/v$, then $\eta=\eta(1/\rho,\theta)$ satisfies

$$\frac{\partial\eta}{\partial\rho}=\frac{-p_\theta}{\rho^2},\ \frac{\partial\eta}{\partial\theta}=\frac{e_\theta}{\theta}. \tag{2.2.5}$$

We consider the transform

$$A:(\rho,\theta)\in D_{\rho,\theta}=\{(\rho,\theta):\rho>0,\theta>0\}\to(v,\eta)\in AD_{\rho,\theta}. \tag{2.2.6}$$

Owing to the Jacobian, $|\partial(v,\eta)/\partial(\rho,\theta)|=-e_\theta/\rho^2\theta<0$ on $AD_{\rho,\theta}$. Thus the functions e,p can also be regarded as the smooth functions of (v,η). We denote them by

$$\begin{aligned}&e=e(v,\eta):\equiv e(v,\theta(v,\eta))=e(1/\rho,\theta),\\&p=p(v,\eta):\equiv p(v,\theta(v,\eta))=p(1/\rho,\theta).\end{aligned} \tag{2.2.7}$$

Then it follows from (2.2.3)–(2.2.7) that

$$\begin{aligned}&e_v=-p,\quad e_\eta=\theta,\\&p_v=-(\rho^2p_\rho+\theta p_\theta^2/e_\theta),\quad p_\eta=\theta p_\theta/e_\theta,\\&\theta_v=-\theta p_\theta/e_\theta,\quad \theta_\eta=\theta/e_\theta.\end{aligned} \tag{2.2.8}$$

We define the energy form

$$V(v,u,\vec{w},\eta)=\frac{u^2}{2}+\frac{|\vec{w}|^2}{2}+e(v,\eta)-e(\bar{v},\bar{\eta})-\frac{\partial e}{\partial v}(\bar{v},\bar{\eta})(v-\bar{v})-\frac{\partial e}{\partial\eta}(\bar{v},\bar{\eta})(\eta-\bar{\eta}), \tag{2.2.9}$$

where

$$\bar{v}=1/\bar{\rho},\quad \bar{\eta}=\eta(1/\bar{\rho},\bar{\theta}), \tag{2.2.10}$$

$$\bar{\theta}=T_0,\ \text{ for }\ (2.1.22),\quad e(\bar{v},\bar{\theta})=\int_0^1\Big(e(v_0,\theta_0)+\frac{v_0^2}{2}\Big)(x)\,dx\ \text{ for }\ (2.1.21).\quad \square$$

Lemma 2.2.2. *The unique global solution* $(v(t), u(t), \vec{w}(t), \theta(t)) \in H_+^1$ *to problem* (2.1.16)–(2.1.21) *or* (2.1.16)–(2.1.20), (2.1.22) *satisfies the following estimates:*

$$\begin{aligned}\frac{u^2}{2} + \frac{|\vec{w}|^2}{2} + C_1^{-1}\left[(v-\bar{v})^2 + (\eta-\bar{\eta})^2\right] &\le V(v,u,\vec{w},\eta) \\ &\le \frac{u^2}{2} + \frac{|\vec{w}|^2}{2} + C_1\left[(v-\bar{v})^2 + (\eta-\bar{\eta})^2\right].\end{aligned} \tag{2.2.11}$$

Proof. By the mean value problem theorem, there exists a point $(\tilde{v}, \tilde{\eta})$ between (v, η) and $(\bar{v}, \bar{\eta})$ such that

$$\begin{aligned}e(v,\eta) = e(\bar{v},\bar{\eta}) &+ \frac{\partial e}{\partial v}(\bar{v},\bar{\eta})(v-\bar{v}) + \frac{\partial e}{\partial \eta}(\bar{v},\bar{\eta})(\eta-\bar{\eta}) + \frac{1}{2}\left[\frac{\partial^2 e}{\partial v^2}(\tilde{v},\tilde{\eta})(v-\bar{v})^2\right. \\ &\left.+ 2\frac{\partial^2 e}{\partial v\partial\eta}(\tilde{v},\tilde{\eta})(v-\bar{v})(\eta-\bar{\eta}) + \frac{\partial^2 e}{\partial \eta^2}(\tilde{v},\tilde{\eta})(\eta-\bar{\eta})^2\right].\end{aligned} \tag{2.2.12}$$

By (2.2.9) and (2.2.12), we get

$$\begin{aligned}V(v,u,\vec{w},\eta) = \frac{u^2}{2} + \frac{|\vec{w}|^2}{2} &+ \frac{1}{2}\left[\frac{\partial^2 e}{\partial v^2}(\tilde{v},\tilde{\eta})(v-\bar{v})^2\right. \\ &\left.+ 2\frac{\partial^2 e}{\partial v\partial\eta}(\tilde{v},\tilde{\eta})(v-\bar{v})(\eta-\bar{\eta}) + \frac{\partial^2 e}{\partial \eta^2}(\tilde{v},\tilde{\eta})(\eta-\bar{\eta})^2\right]\end{aligned} \tag{2.2.13}$$

where

$$\tilde{v} = \lambda_0\bar{v} + (1-\lambda_0)v,\ \tilde{\eta} = \lambda_0\bar{\eta} + (1-\lambda_0)\eta, \quad 0 \le \lambda_0 \le 1.$$

By (2.2.2), we get

$$0 < C_1^{-1} \le \left|\tilde{v}(1/\tilde{\rho}, \tilde{\theta})\right| \le C_1, \quad 0 < C_1^{-1} \le \left|\tilde{\eta}(1/\tilde{\rho}, \tilde{\theta})\right| \le C_1,$$

which implies

$$\left|\frac{\partial^2 e}{\partial v^2}(\tilde{v},\tilde{\eta})\right| + \left|\frac{\partial^2 e}{\partial v\partial\eta}(\tilde{v},\tilde{\eta})\right| + \left|\frac{\partial^2 e}{\partial \eta^2}(\tilde{v},\tilde{\eta})\right| \le C_1. \tag{2.2.14}$$

Thus (2.2.14) and the Cauchy inequality give

$$V(u,v,\vec{w},\eta) \le \frac{u^2}{2} + \frac{|\vec{w}|^2}{2} + C_1\left[(v-\bar{v})^2 + (\eta-\bar{\eta})^2\right]. \tag{2.2.15}$$

On the other hand, we infer from (2.2.8) that

$$e_{vv} = -p_v = \rho^2 p_\rho + \theta p_\theta^2/e_\theta, \quad e_{v\eta} = -p_\eta = \theta_v = -\theta p_\theta/e_\theta, \quad e_{\eta\eta} = \theta_\eta = \theta/e_\theta,$$

which yields that the Hessian of $e(v, \eta)$ is positive definite for any $v > 0$ and $\theta > 0$. Thus we deduce from (2.2.12) that

$$\begin{aligned}V(v,u,\vec{w},\eta) &\ge \frac{u^2}{2} + \frac{|\vec{w}|^2}{2} + \lambda_{\min}(\tilde{v},\tilde{\eta})\left[(v-\bar{v})^2 + (\eta-\bar{\eta})^2\right] \\ &\ge \frac{u^2}{2} + \frac{|\vec{w}|^2}{2} + C_1^{-1}\left[(v-\bar{v})^2 + (\eta-\bar{\eta})^2\right],\end{aligned} \tag{2.2.16}$$

where $\lambda_{\min}(\tilde{v},\tilde{\eta})$ $(\geq C_1^{-1})$ is the smaller characteristic root of the Hessian of $e(\tilde{v},\tilde{\eta})$. Thus by the combination of (2.2.15) and (2.2.16), we deduce that

$$\begin{aligned}\frac{u^2}{2}+\frac{|\vec{w}|^2}{2}+C_1^{-1}\left[(v-\bar{v})^2+(\eta-\bar{\eta})^2\right] &\leq V(v,u,\vec{w},\eta)\\ &\leq \frac{u^2}{2}+\frac{|\vec{w}|^2}{2}+C_1\left[(v-\bar{v})^2+(\eta-\bar{\eta})^2\right].\end{aligned}$$

The proof is complete. □

Lemma 2.2.3. *There exist constants $C_1>0$ and $\gamma_1=\gamma_1(C_1)>0$ such that for any fixed $\gamma\in(0,\gamma_1]$, the global solution $(v(t),u(t),\vec{w}(t),\theta(t))\in H_+^1$ to problem* (2.1.16)–(2.1.21) *or* (2.1.16)–(2.1.20), (2.1.22) *satisfies the estimate*

$$\begin{aligned}&e^{\gamma t}\Big(\|u(t)\|^2+\|v(t)-\bar{v}\|_{H^1}^2+\|\vec{w}(t)\|^2+\|\theta(t)-\bar{\theta}\|^2\Big)\\ &\quad+\int_0^t e^{\gamma t}\Big(\|u_y\|^2+\|v_y\|^2+\|\vec{w}_y\|^2+\|\theta_y\|^2\Big)(\tau)d\tau\leq C_1,\quad\forall\ \ t>0.\end{aligned}\tag{2.2.17}$$

Proof. By (2.1.19) and (2.1.23), we get

$$\left[e+\frac{1}{2}(u^2+|\vec{w}|^2)\right]_t=(-up+\lambda\rho uu_y+\mu\vec{w}\cdot\vec{w}_y+\kappa\rho\theta_y)_y.$$

That is,

$$e_\eta\eta_t+e_vv_t+uu_t+\vec{w}\cdot\vec{w}_t=\left(-up+\lambda\rho uu_y+\mu\vec{w}\cdot\vec{w}_y+\kappa\rho\theta_y\right)_y.\tag{2.2.18}$$

Thus it follows from (2.1.17)–(2.1.18) that

$$e_\eta\eta_t+e_vv_t=-pu_y+\lambda\rho u_y^2+(\kappa\rho\theta_y)_y.\tag{2.2.19}$$

By (2.1.16) and (2.2.8), we get

$$\eta_t=\frac{\lambda\rho u_y^2}{\theta}+\left(\frac{\kappa\rho\theta_y}{\theta}\right)_y+\kappa\rho\left(\frac{\theta_y}{\theta}\right)^2.\tag{2.2.20}$$

Since $\bar{u}$, $\bar{\theta}$, $e(\bar{v},\bar{\eta})$ are constants, $\bar{u}_t=0,\bar{\theta}_t=0,e_t(\bar{v},\bar{\eta})=0$, by (2.2.8), (2.2.9) and (2.2.20), we get

$$\begin{aligned}V_t&=\left(\frac{u^2}{2}+\frac{|\vec{w}|^2}{2}+e\right)_t-\frac{\partial e}{\partial v}(\bar{v},\bar{\eta})v_t-\frac{\partial e}{\partial\eta}(\bar{v},\bar{\eta})\eta_t\\ &=\left(\frac{u^2}{2}+\frac{|\vec{w}|^2}{2}+e\right)_t-\bar{\theta}\eta_t+\bar{p}v_t,\end{aligned}\tag{2.2.21}$$

which, together with (2.2.20), gives

$$V_t + \frac{\bar{\theta}}{\theta}\left(\lambda\rho u_y^2 + \frac{\kappa\rho\theta_y^2}{\theta}\right) = \left[\lambda\rho u u_y + \mu\rho\vec{w}\cdot\vec{w}_y + (1-\frac{\bar{\theta}}{\theta})\kappa\rho\theta_y - (p-\bar{p})u\right]_y. \tag{2.2.22}$$

Differentiating ρ_θ with respect to y, we have

$$\frac{\partial}{\partial y}(\rho_\theta) = -2\rho\rho_y u_y - \rho^2 u_{yy}$$

which implies

$$\left(\frac{\rho_y}{\rho}\right)_t = -\rho_y u_y - \rho u_{yy}. \tag{2.2.23}$$

Multiplying (2.1.17) by $\lambda\rho_y/\rho$, we have

$$\frac{\lambda\rho_y u_t}{\rho} = \frac{\lambda^2\rho\rho_y u_{yy}}{\rho} + \frac{\lambda^2\rho_y^2 u_y}{\rho} - \frac{\lambda p_\rho \rho_y^2}{\rho} - \frac{\lambda p_\theta\theta_y\rho_y}{\rho}. \tag{2.2.24}$$

We can also get

$$\lambda(-\rho_y u_y - \rho u_{yy})u = -\lambda(\rho u u_y)_y + \lambda\rho u_y^2. \tag{2.2.25}$$

Differentiating $\frac{\lambda^2}{2}(\frac{\rho_y}{\rho})^2 + \frac{\lambda\rho_y u}{\rho}$ with respect to t, we derive from (2.2.23),

$$\left[\frac{\lambda^2}{2}(\frac{\rho_y}{\rho})^2 + \frac{\lambda\rho_y u}{\rho}\right]_t = \lambda^2(\frac{\rho_y}{\rho})(-\rho_y u_y - \rho u_{yy}) + \frac{\lambda\rho_y u_t}{\rho} + \lambda(-\rho_y u_y - \rho u_{yy})u. \tag{2.2.26}$$

By (2.2.25) and (2.2.26), we derive

$$\left[\frac{\lambda^2}{2}(\frac{\rho_y}{\rho})^2 + \frac{\lambda\rho_y u}{\rho}\right]_t + \frac{\lambda p_\rho\rho_y^2}{\rho} + \frac{\lambda p_\theta\theta_y\rho_y}{\rho} - \lambda\rho u_y^2 = -\lambda(\rho u u_y)_y. \tag{2.2.27}$$

Taking the inner product of $\vec{w}$ in $\mathbb{R}^2$ on both sides of (2.1.18), we get

$$\frac{1}{2}\frac{\partial}{\partial t}|\vec{w}|^2 + \mu\rho|\vec{w}_y|^2 = \left(\mu\rho\vec{w}\cdot\vec{w}_y\right)_y. \tag{2.2.28}$$

Multiplying (2.2.22), (2.2.27), (2.2.28) by $e^{\gamma t}, \beta e^{\gamma t}, e^{\gamma t}$ respectively, and then adding the results up, we deduce that

$$\begin{aligned}&\frac{\partial}{\partial t}G(t) + e^{\gamma t}\left[\frac{\bar{\theta}}{\theta}\left(\lambda\rho u_y^2 + \frac{\kappa\rho\theta_y^2}{\theta}\right) + \beta\left(\frac{\lambda p_\rho\rho_y^2}{\rho} + \frac{\lambda p_\theta\theta_y\rho_y}{\rho} - \lambda\rho u_y^2\right) + \mu\rho|\vec{w}_y|^2\right]\\ &\quad = \gamma G(t) + e^{\gamma t}[(1-\beta)\lambda\rho u u_y + 2\mu\rho\vec{w}\cdot\vec{w}_y + \kappa\left(1-\frac{\bar{\theta}}{\theta}\right)\rho\theta_y - (p - p(\bar{\rho},\bar{\theta}))u]_y,\end{aligned} \tag{2.2.29}$$

where

$$G(t) = e^{\gamma t}\left[V\left(\frac{1}{\rho}, u, \vec{w}, \eta\right) + \beta\left(\frac{\lambda^2}{2}\left(\frac{\rho_y}{\rho}\right)^2\right) + \frac{\lambda \rho_y u}{\rho}\right].$$

Integrating (2.2.29) over $[0,1]\times[0,t]$, we get

$$\begin{aligned}
&\int_0^1 G(t)\,dy \\
&+ \int_0^t\int_0^1 e^{\gamma\tau}\left[\frac{\bar{\theta}}{\theta}\left(\lambda\rho u_y^2 + \frac{\kappa\rho\theta_y^2}{\theta}\right) + \beta\left(\frac{\lambda p_\rho \rho_y^2}{\rho} + \frac{\lambda p_\theta \theta_y \rho_y}{\rho} - \lambda\rho u_y^2\right) + \mu\rho|\vec{w}_y|^2\right] dyd\tau \\
&+ \gamma\int_0^t\int_0^1 e^{\gamma t}\left[(1-\beta)\lambda\rho u u_y + 2\mu\rho\vec{w}\cdot\vec{w}_y + \kappa\left(1-\frac{\bar{\theta}}{\theta}\right)\rho\theta_y - (p - p(\bar{\rho},\bar{\theta}))u\right]_y d\tau \\
&= \int_0^1 G(0)\,dy + \gamma\int_0^t\int_0^1 G(\tau)\,dy\mathrm{d}\tau. \qquad (2.2.30)
\end{aligned}$$

Using Cauchy's and Poincaré's inequalities, we deduce that for small $\beta > 0$ and for any $\gamma > 0$,

$$\begin{aligned}
&e^{\gamma t}\Big(\|\rho(t)-\bar{\rho}\|^2 + \|u(t)\|^2 + \|\vec{w}(t)\|^2 + \|\eta(t)-\bar{\eta}\|^2 + \|\rho_y(t)\|^2\Big) \qquad (2.2.31)\\
&\quad + \int_0^t e^{\gamma\tau}\Big(\|u_y\|^2 + \|\vec{w}_y\|^2 + \|\theta_y\|^2 + \|v_y\|^2\Big)(\tau)\,d\tau \\
&\le C_1 + C_1\gamma\int_0^t e^{\gamma\tau}\Big(\|\rho-\bar{\rho}\|^2 + \|u\|^2 + \|\vec{w}\|^2 + \|\eta-\bar{\eta}\|^2 + \|\rho_y\|^2\Big)(\tau)\,d\tau.
\end{aligned}$$

By Lemma 2.2.1, boundary conditions (2.1.21)–(2.1.22) together with the Poincaré inequality, we get

$$\begin{aligned}
\|v(t)-\bar{v}\| \le C_1\|v_y(t)\|, &\qquad \|\vec{w}(t)\| \le C_1\|\vec{w}_y(t)\|, \\
\|u(t)\| \le C_1\|u_y(t)\|, &\quad \|\theta(t)-\bar{\theta}\| \le C_1\|\theta_y(t)\|. \qquad (2.2.32)
\end{aligned}$$

Using the mean value theorem, we infer that

$$\eta(v,\theta) - \bar{\eta} = \eta_v(\tilde{v},\tilde{\theta})(v-\bar{v}) + \eta_\theta(\tilde{v},\tilde{\theta})(\theta-\bar{\theta}).$$

Hence,

$$C_1^{-1}(\|v-\bar{v}\|^2 + \|\theta-\bar{\theta}\|^2) \le \|\eta-\bar{\eta}\|^2 \le C_1(\|v-\bar{v}\|^2 + \|\theta-\bar{\theta}\|^2), \qquad (2.2.33)$$

$$C_1^{-1}\|v-\bar{v}\|^2 \le \|\rho-\bar{\rho}\|^2 \le C_1\|v-\bar{v}\|^2. \qquad (2.2.34)$$

By (2.2.31)–(2.2.34), we get

$$\begin{aligned} &e^{\gamma t}\Big(\|\rho(t)-\bar\rho\|^2+\|u(t)\|^2+\|\vec w(t)\|^2+\|\eta(t)-\bar\eta\|^2+\|\rho_y(t)\|^2\Big)\\ &\quad+\int_0^t e^{\gamma\tau}\Big(\|u_y^2\|+\|\vec w_y\|^2+\|\theta_y\|^2+\|v_y\|^2\Big)(\tau)\,d\tau\\ &\le C_1+C_1\gamma\int_0^t e^{\gamma\tau}\Big(\|u_y^2\|+\|\vec w_y\|^2+\|\theta_y\|^2+\|v_y\|^2\Big)(\tau)\,d\tau \end{aligned}$$

which gives that there exists a constant $\gamma_1'=\gamma_1'(C_1)>0$, such that for any fixed $\gamma\in(0,\gamma']$,

$$\begin{aligned} &e^{\gamma t}\Big(\|u(t)\|^2+\|v(t)-\bar v\|_{H^1}^2+\|\vec w(t)\|^2+\|\theta(t)-\bar\theta\|^2\Big)\\ &\quad+\int_0^t e^{\gamma\tau}\Big(\|u_y\|^2+\|v_y\|^2+\|\vec w_y\|^2+\|\theta_y\|^2\Big)(\tau)\,d\tau\le C_1. \end{aligned}$$

The proof is complete. □

Lemma 2.2.4. *There exist constants $C_1>0$ and $\gamma_1=\gamma_1(C_1)\le\gamma_1'$ such that for any fixed $\gamma\in(0,\gamma_1]$, the global solution $(u(t),v(t),\vec w(t),\theta(t))\in H_+^1$ to problem (2.1.16)–(2.1.21) or (2.1.16)–(2.1.20), (2.1.22) satisfies the following estimates:*

$$\begin{aligned} &e^{\gamma t}\Big(\|u_y(t)\|^2+\|\vec w_y(t)\|^2+\|\theta_y(t)\|^2\Big)+\int_0^t e^{\gamma\tau}\Big(\|u_{yy}\|^2+\|\vec w_{yy}\|^2\\ &\quad+\|\theta_{yy}\|^2+\|u_t\|^2+\|\theta_t\|^2+\|\vec w_t\|^2\Big)(\tau)\,d\tau\le C_1,\quad\forall\ \ t>0. \end{aligned}\tag{2.2.35}$$

Proof. By Lemma 2.2.1's boundary conditions together with the Poincaré inequality, we get

$$\|\vec w_y(t)\|\le C_1\|\vec w_{yy}(t)\|,\quad \|u_y(t)\|\le C_1\|u_{yy}(t)\|,\quad \|\theta_y(t)\|\le C_1\|\theta_{yy}(t)\|.\tag{2.2.36}$$

Multiplying (2.1.17) by $-u_{yy}$ and integrating the resulting equality over $[0,1]$, we have

$$\frac12\frac{d}{dt}\|u_y(t)\|^2+\lambda\int_0^1\frac{u_{yy}^2}{v}\,dy=\int_0^1\left(\frac{\lambda u_y v_y}{v^2}+p_\theta\theta_y+p_v v_y\right)u_{yy}\,dy.\tag{2.2.37}$$

Using the interpolation inequality, we have that for any $\varepsilon>0$,

$$\begin{aligned} \int_0^1\frac{\lambda u_y v_y}{v^2}u_{yy}\,dy&\le C_1\|v_y\|\|u_y\|_{L^\infty}\|u_{yy}\|\\ &\le C_1\|v_y\|\|u_y\|^{\frac12}\|u_{yy}\|^{\frac32}\\ &\le C_1(\|v_y\|^2+\|u_y\|^2)+\varepsilon\|u_{yy}\|^2. \end{aligned}\tag{2.2.38}$$

By Young's inequality and Cauchy's inequality, we have

$$\frac{d}{dt}\|u_y(t)\|^2+C_1^{-1}\|u_{yy}(t)\|^2\le C_1(\|\theta_y(t)\|^2+\|v_y(t)\|^2+\|u_y(t)\|^2).\tag{2.2.39}$$

Multiplying (2.2.39) by $e^{\gamma t}$ and integrating the resulting equality over $[0,t]$, implies that there exists a $\gamma_1 = \gamma_1(C_1) \le \gamma_1'$, such that for any fixed $\gamma \in (0,\gamma_1]$,

$$\begin{aligned} & e^{\gamma t}\|u_y(t)\|^2 + \int_0^t e^{\gamma\tau}\|u_{yy}(\tau)\|^2\,d\tau \\ & \quad \le C_1\int_0^t e^{\gamma\tau}\left(\|\theta_y\|^2 + \|v_y\|^2 + \|u_y\|^2\right)(\tau)\,d\tau + \gamma\int_0^t e^{\gamma\tau}\|u_y(\tau)\|^2\,d\tau \le C_1. \end{aligned} \tag{2.2.40}$$

By (2.1.17),

$$\|u_t(t)\| \le C_1(\|u_{yy}(t)\| + \|\theta_y(t)\| + \|v_y(t)\| + \|u_y(t)\|),$$

which, along with (2.2.40), gives

$$\int_0^t e^{\gamma\tau}\|u_t(\tau)\|^2\,d\tau \le C_1, \quad \forall t>0. \tag{2.2.41}$$

Similarly to (2.2.40)–(2.2.41), we have for $\gamma \in (0,\gamma_1]$ that

$$e^{\gamma t}\|\vec{w}_y(t)\|^2 + \int_0^t e^{\gamma\tau}(\|\vec{w}_{yy}\|^2 + \|\vec{w}_t(\tau)\|^2)(\tau)\,d\tau \le C_1, \forall t>0. \tag{2.2.42}$$

Multiplying (2.1.19) by $-e_\theta\theta_{yy}$ and integrating the resulting equality over [0,1], we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|\theta_y(t)\|^2 + \int_0^1 \frac{\kappa\theta_{yy}^2}{e_\theta v}\,dy \\ & \quad = \int_0^1 \left(\frac{e_v v_t}{e_\theta}\theta_{yy} - \frac{\lambda u_y^2 + \mu\vec{w}_y^{\,2}}{e_\theta v}\theta_{yy} + \frac{p u_y\theta_{yy}}{e_\theta} + \frac{\kappa\theta_y v_y}{e_\theta v^2}\theta_{yy}\right) dy. \end{aligned} \tag{2.2.43}$$

Using the interpolation inequality, we have that for any $\varepsilon > 0$,

$$\begin{aligned} \int_0^1 \frac{\kappa\theta_y v_y}{e_\theta v^2}\theta_{yy}\,dy & \le C_1\|v_y(t)\|\|\theta_y(t)\|_{L^\infty}\|\theta_{yy}(t)\| \\ & \le C_1\|v_y(t)\|\|\theta_y(t)\|^{\frac{1}{2}}\|\theta_{yy}(t)\|^{\frac{3}{2}} \\ & \le C_1(\|v_y(t)\|^2 + \|\theta_y(t)\|^2) + \varepsilon\|\theta_{yy}(t)\|^2, \end{aligned} \tag{2.2.44}$$

$$\begin{aligned} \int_0^1 \frac{\lambda u_y^2}{e_\theta v}\theta_{yy}dy & \le C_1\|u_y(t)\|\|u_y(t)\|_{L^\infty}\|\theta_{yy}(t)\| \\ & \le C_1\|u_y(t)\|\|u_y(t)\|^{\frac{1}{2}}\|u_{yy}(t)\|^{\frac{1}{2}}\|\theta_{yy}(t)\| \\ & \le C_1(\|u_y(t)\|^2 + \|u_{yy}(t)\|^2) + \varepsilon\|\theta_{yy}(t)\|^2, \end{aligned} \tag{2.2.45}$$

$$\begin{aligned} \int_0^1 \frac{\mu|\vec{w}_y|^2}{e_\theta v}\theta_{yy}dy & \le C_1\|\vec{w}_y(t)\|\|\vec{w}_y(t)\|_{L^\infty}\|\theta_{yy}(t)\| \\ & \le C_1\|\vec{w}_y(t)\|\|\vec{w}_y(t)\|^{\frac{1}{2}}\|\vec{w}_{yy}(t)\|^{\frac{1}{2}}\|\theta_{yy}(t)\| \\ & \le C_1(\|\vec{w}_y(t)\|^2 + \|\vec{w}_{yy}(t)\|^2) + \varepsilon\|\theta_{yy}(t)\|^2. \end{aligned} \tag{2.2.46}$$

Inserting (2.2.44)–(2.2.46) into (2.2.43), and using Lemma 2.2.1, we deduce that

$$\frac{d}{dt}\|\theta_y(t)\|^2 + C_1^{-1}\|\theta_{yy}(t)\|^2 \le C_1(\|u_y(t)\|^2 + \|\theta_y(t)\|^2 + \|v_y(t)\|^2 + \|\vec{w}_y(t)\|^2). \tag{2.2.47}$$

Multiplying (2.2.47) by $e^{\gamma t}$, and integrating the resulting equation over $[0,t]$, we have that there exists a constant $\gamma_1 = \gamma_1(C_1) \le \gamma_1'$, such that for any fixed $\gamma \in (0, \gamma_1]$,

$$\begin{aligned} &e^{\gamma t}\|\theta_y(t)\|^2 + \int_0^t e^{\gamma\tau}\|\theta_{yy}(\tau)\|^2 d\tau \\ &\quad \le C_1 \int_0^t e^{\gamma\tau}(\|u_y\|^2 + \|\vec{w}_y\|^2 + \|v_y\|^2)(\tau)d\tau + \gamma\int_0^t e^{\gamma\tau}\|\theta_y(\tau)\|^2 d\tau \le C_1 \end{aligned} \tag{2.2.48}$$

which with

$$\|\theta_t(t)\| \le C_2(\|\theta_{yy}(t)\| + \|\theta_y(t)\| + \|v_y(t)\| + \|u_y(t)\| + \|\vec{w}_y(t)\|)$$

yields

$$\int_0^t e^{\gamma\tau}\|\theta_t(\tau)\|^2\, d\tau \le C_1. \tag{2.2.49}$$

The proof is complete. □

2.3 Proof of Theorem 2.1.2

In this section we shall complete the proof of Theorem 2.1.2 and take that the assumptions in Theorem 2.1.2 to be valid. We begin with the following lemma.

Lemma 2.3.1. *Under the assumptions of Theorem* 2.1.2*, for any* $(v_0, u_0, \vec{w}_0, \theta_0) \in H_+^2$*, the following estimates hold for any* $t > 0$*,*

$$\begin{aligned} &\|u_t(t)\|^2 + \|\vec{w}_t(t)\|^2 + \|\theta_t(t)\|^2 \\ &\qquad + \int_0^t \left(\|u_{ty}(t)\|^2 + \|\vec{w}_{ty}(t)\|^2 + \|\theta_{ty}(t)\|^2\right)(\tau)d\tau \le C_2, \end{aligned} \tag{2.3.1}$$

$$\|u(t)\|_{H^2}^2 + \|v(t) - \bar{v}\|_{H^2}^2 + \|\vec{w}(t)\|_{H^2}^2 + \|\theta(t) - \bar{\theta}\|_{H^2}^2 \le C_2. \tag{2.3.2}$$

Proof. Differentiating (2.1.17) with respect to t, multiplying the result by u_t and integrating over [0,1], we infer that

$$\frac{1}{2}\frac{d}{dt}\|u_t(t)\|^2 = -\int_0^1 \lambda\frac{u_{ty}^2}{v}dy + \int_0^1 \lambda\frac{u_y^2 u_{ty}}{v^2}dy + \int_0^1 p_\theta\theta_t u_{ty}dy + \int_0^1 p_v v_t u_{ty}dy. \tag{2.3.3}$$

Using Cauchy's inequality, Young's inequality and the embedding theorem, we conclude that for any $\varepsilon > 0$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_t(t)\|^2 + C_1^{-1}\|u_{ty}(t)\|^2 &\le \varepsilon\|u_{ty}(t)\|^2 + C_1(\|u_y(t)\|_{L^4}^4 + \|\theta_t(t)\|^2 + \|u_y(t)\|^2)\\ &\le \varepsilon\|u_{ty}(t)\|^2 + C_2(\|u_{yy}(t)\|^2 + \|\theta_t(t)\|^2 + \|u_y(t)\|^2).\end{aligned} \tag{2.3.4}$$

By equations (2.1.16)–(2.1.19), we get

$$\|u_{yy}(t)\| \le C_2(\|u_t(t)\| + \|\theta_y(t)\| + \|v_y(t)\| + \|u_y(t)\|), \|\vec{w}_{yy}(t)\| \le C_2\|\vec{w}_t(t)\|, \tag{2.3.5}$$

$$\|\theta_{yy}(t)\| \le C_2(\|\theta_t(t)\| + \|\theta_y(t)\| + \|v_y(t)\| + \|u_y(t)\| + \|\vec{w}_y(t)\|) \tag{2.3.6}$$

or

$$\|u_t\| \le C_2(\|u_{yy}(t)\| + \|\theta_y(t)\| + \|v_y(t)\| + \|u_y(t)\|), \|\vec{w}_t(t)\| \le C_2\|\vec{w}_{yy}(t)\|, \tag{2.3.7}$$

$$\|\theta_t(t)\| \le C_2(\|\theta_{yy}(t)\| + \|\theta_y(t)\| + \|v_y(t)\| + \|u_y(t)\| + \|\vec{w}_y(t)\|). \tag{2.3.8}$$

Since $(v_0, u_0, \vec{w}_0, \theta_0) \in H_+^2$, we can infer from (2.3.7)–(2.3.8) that

$$\|u_t(y,0)\| \le C_2, \quad \|\vec{w}_t(y,0)\| \le C_2, \quad \|\theta_t(y,0)\| \le C_2. \tag{2.3.9}$$

Integrating (2.3.4) on $[0,t]$, using Lemma 2.2.1 and (2.3.7)–(2.3.9), we get

$$\|u_t(t)\|^2 + \int_0^t \|u_{ty}(\tau)\|^2 d\tau \le C_2, \quad \forall t > 0, \tag{2.3.10}$$

which, together with (2.3.5), implies

$$\|u_{yy}(t)\| \le C_2, \ \forall t > 0. \tag{2.3.11}$$

Differentiating (2.1.18) with respect to t, multiplying the resulting equation by $\vec{w}_t$ and integrating the resulting equality over [0,1], we infer that

$$\frac{1}{2}\frac{d}{dt}\|\vec{w}_t(t)\|^2 = -\int_0^1 \mu\frac{|\vec{w}_{ty}|^2}{v}dy + \int_0^1 \mu\frac{\vec{w}_y \cdot \vec{w}_{ty} v_t}{v^2}dy,$$

which, by using Cauchy's inequality, Young's inequality and the embedding theorem, gives that for $\varepsilon > 0$ small enough,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\vec{w}_t(t)\|^2 + C_1^{-1}\|\vec{w}_{ty}(t)\|^2 &\le \varepsilon\|\vec{w}_{ty}(t)\|^2 + C_1\|v_t\vec{w}_y\|^2\\ &\le \varepsilon\|\vec{w}_{ty}(t)\|^2 + C_1\|v_t(t)\|_{L^\infty}^2\|\vec{w}_y(t)\|^2.\end{aligned} \tag{2.3.12}$$

Thus integrating (2.3.12) over $[0,t]$ and using Lemma 2.2.1, we get

$$\|\vec{w}_t(t)\|^2 + \int_0^t \|\vec{w}_{ty}(\tau)\|^2 d\tau \le C_2\int_0^t \|\vec{w}_y(\tau)\|^2 d\tau + C_1\|\vec{w}_t(y,0)\|^2 \le C_2, \tag{2.3.13}$$

which, along with (2.3.5), gives

$$\|\vec{w}_{yy}(t)\| \le C_2, \forall t > 0. \tag{2.3.14}$$

By (2.1.17)–(2.1.19) and (2.1.23), we get

$$e_\theta \theta_t + e_v v_t = -pu_y + \frac{\lambda u_y^2}{v} + \frac{\mu|\vec{w}_y|^2}{v} + \left(\frac{\kappa\theta_y}{v}\right)_y. \tag{2.3.15}$$

Differentiating (2.3.15) with respect to t, multiplying the result θ_t, and finally integrating the resultant over [0,1], we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\sqrt{e_\theta}\theta_t\|^2 + \int_0^1 \frac{\kappa\theta_{ty}^2}{v}dy = \int_0^1 \Big(& 2\lambda\frac{u_y u_{yt}}{v} - \lambda\frac{u_y^2 v_t}{v^2} + 2\mu\frac{\vec{w}_y\cdot\vec{w}_{yt}}{v} - \mu\frac{|\vec{w}_y|^2 v_t}{v^2} \\ & - u_{yt}p - u_y p_\theta \theta_t - u_y p_v v_t - e_v v_{tt} - e_{vv}v_t^2 \\ & - \frac{1}{2}e_{\theta\theta}\theta_t^2 - \frac{3}{2}e_{\theta v}v_t\theta_t + \frac{\kappa\theta_{ty}\theta_y v_t}{v^2}\Big)\theta_t dy. \end{aligned}\tag{2.3.16}$$

Integrating (2.3.16) over [0,t], using the Cauchy inequality, the Young inequality and the embedding theorem, we conclude that for any $\varepsilon > 0$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\sqrt{e_\theta}\theta_t\|^2 + C_1^{-1}\int_0^t \|\theta_{ty}(\tau)\|^2 d\tau \\ &\le \varepsilon\int_0^t \|\theta_{ty}(\tau)\|^2\, d\tau + C_1\int_0^t \Big(\|\theta_y\|^2 + \|\theta_{yy}\|^2 + \|\theta_t\|^2 \\ &\quad + \|u_y\|^2 + \|u_{ty}\|^2 + \|\vec{w}_y\|^2 + \|\vec{w}_{ty}\|^2 + \|\theta_t(\tau)\|_{L^3}^3\Big)d\tau. \end{aligned}\tag{2.3.17}$$

By the Nirenberg interpolation inequality, we can get for any $\varepsilon > 0$,

$$\begin{aligned}\int_0^t \|\theta_t\|_{L^3}^3(\tau)d\tau &\le C_1\int_0^t \left(\|\theta_t\|^{\frac{5}{2}}\|\theta_{ty}\|^{\frac{1}{2}} + \|\theta_t\|^3\right)(\tau)d\tau \\ &\le C_1\sup_{0\le\tau\le t}\|\theta_t(\tau)\|^{\frac{4}{3}}\int_0^t\left(\|\theta_t\|^2 + \|\theta_t\|^{\frac{5}{3}}\right)(\tau)d\tau + \varepsilon\int_0^t \|\theta_{ty}(\tau)\|^2 d\tau \\ &\le C_1\sup_{0\le\tau\le t}\|\theta_t(\tau)\|^{\frac{4}{3}} + \varepsilon\int_0^t \|\theta_{ty}(\tau)\|^2 d\tau \\ &\le \frac{1}{2}\sup_{0\le\tau\le t}\|\theta_t(\tau)\|^2 + \varepsilon\int_0^t \|\theta_{ty}(\tau)\|^2 d\tau + C_2. \end{aligned}\tag{2.3.18}$$

Hence, inserting (2.3.18) into (2.3.17), and taking $\varepsilon > 0$ small enough, we get

$$\|\theta_t(t)\|^2 + \int_0^t \|\theta_{ty}(\tau)\|^2 d\tau \le C_2, \ \ \forall t > 0. \tag{2.3.19}$$

By (2.3.6) and (2.3.19), we conclude that

$$\|\theta_{yy}(t)\| \leq C_2, \quad \forall t > 0.$$

The proof is complete. $\square$

Lemma 2.3.2. *Under the assumptions of Theorem* 2.1.2*, for any* $(u_0, v_0, \vec{w}_0, \theta_0) \in H_+^2$*, the following estimate holds for any* $t > 0$*:*

$$\|v(t)\|_{H^2}^2 + \int_0^t \|v(\tau)\|_{H^2}^2 d\tau \leq C_2.$$

Proof. Differentiating (2.1.17) with respect to y, by equation (2.1.16),

$$v_{tyy} = u_{yyy},$$

we get

$$\lambda \frac{\partial}{\partial t}\left(\frac{v_{yy}}{v}\right) - p_v v_{yy} = u_{ty} + p_{vv} v_y^2 + 2p_{v\theta} v_y \theta_y + p_\theta \theta_{yy} + p_{\theta\theta}\theta_y^2 + 2\lambda \frac{u_{yy} v_y}{v^2} - 2\lambda \frac{u_y v_y^2}{v^3} \tag{2.3.20}$$

where

$$\lambda \frac{\partial}{\partial t}\left(\frac{v_{yy}}{v}\right) = \lambda \frac{v_{yyt}}{v} - \lambda \frac{v_{yy} v_t}{v^2}.$$

Multiplying (2.3.20) by $\dfrac{v_{yy}}{v}$, we get

$$\begin{aligned} \frac{\lambda}{2}\frac{d}{dt}\left\|\frac{v_{yy}}{v}\right\|^2 + C_1^{-1}\left\|\frac{v_{yy}}{v}\right\|^2 \leq \frac{C_1^{-1}}{4}\left\|\frac{v_{yy}}{v}\right\|^2 &+ C_2\Big(\|u_{yy}v_y\|^2 + \|u_y v_y^2\|^2 + \|\theta_y v_y\|^2 \\ &+ \|v_y\|_{L^4}^4 + \|\theta_y\|_{L^4}^4 + \|\theta_{yy}\|^2 + \|u_{ty}\|^2\Big). \end{aligned} \tag{2.3.21}$$

Integrating (2.3.21) over $[0, t]$ gives

$$\begin{aligned} \|v_{yy}(t)\|^2 + \int_0^t \|v_{yy}(\tau)\|^2 d\tau \leq C_2 \int_0^t &\Big(\|u_{yy}\|^2 + \|u_y\|_{L^\infty}\|v_y\|_{L^4}^4 + \|\theta_y\|_{L^\infty}\|v_y\|^2 \\ &+ \|v_y\|_{L^4}^4 + \|\theta_y\|_{L^4}^4 + \|\theta_{yy}\|^2 + \|u_{ty}\|^2\Big)(\tau) d\tau. \end{aligned}$$

Using the Young inequality, the Poincaré inequality, the Nirenberg interpolation inequality, Lemmas 2.2.1 and 2.3.1, we infer that

$$\begin{aligned} \|v_{yy}(t)\|^2 + \int_0^t \|v_{yy}(\tau)\|^2 d\tau &\leq C_2 \int_0^t \left(\|v_y\|^2 + \|u_{ty}\|^2 + \|u_{yy}\|^2 + \|\theta_{yy}\|^2\right)(\tau) d\tau \\ &\leq C_2. \end{aligned}$$

The proof is complete. $\square$

Lemma 2.3.3. *Under the assumptions of Theorem* 2.1.2, *for any* $(v_0, u_0, \vec{w}_0, \theta_0) \in H_+^2$, *the following estimate holds for any* $t > 0$*:*

$$\int_0^t \left(\|u(\tau)\|_{H^3}^2 + \|\vec{w}(\tau)\|_{H^3}^2 + \|\theta(\tau) - \bar{\theta}\|_{H^3}^2\right) d\tau \leq C_2. \tag{2.3.22}$$

Proof. Differentiating (2.1.17) with respect to y, we infer that

$$\begin{aligned}\frac{\lambda u_{yyy}}{v} &= u_{ty} + p_\theta \theta_{yy} + p_v v_{yy} + 2p_{\theta v}\theta_y v_y + p_{\theta\theta}\theta_y^2 + p_{vv}v_y^2 \\ &\quad + \frac{2\lambda u_{yy}v_y}{v^2} + \frac{\lambda u_y v_{yy}}{v^2} - \frac{2\lambda u_y v_y^2}{v^3}.\end{aligned} \tag{2.3.23}$$

Multiplying (2.3.23) by u_{yyy}, integrating the resultant over [0,1], using Cauchy's inequality and Young's inequality, we get for any $\varepsilon > 0$,

$$\begin{aligned}\|u_{yyy}(t)\|^2 \leq C_2\Big(&\|u_{ty}(t)\|^2 + \|\theta_{yy}(t)\|^2 + \|v_{yy}(t)\|^2 + \|\theta_y v_y\|^2 + \|\theta_y(t)\|_{L^4}^4 \\ &+ \|v_y(t)\|_{L^4}^4 + \|u_{yy}v_y\|^2 + \|v_{yy}u_y\|^2 + \|u_y v_y^2\|^2\Big) + \varepsilon\|u_{yyy}(t)\|^2.\end{aligned} \tag{2.3.24}$$

Integrating (2.3.24) over $[0,t]$, by Lemma 2.2.1 and Lemma 2.3.1, we get

$$\int_0^t \|u_{yyy}(\tau)\|^2 d\tau \leq C_2 \int_0^t \left(\|u_{ty}\|^2 + \|\theta_{yy}\|^2 + \|v_{yy}\|^2 + \|u_{yy}\|^2\right)(\tau) d\tau \leq C_2. \tag{2.3.25}$$

Differentiating (2.1.18) with respect to y, we infer that

$$\frac{\mu \vec{w}_{yyy}}{v} = \vec{w}_{ty} + \frac{2\mu\vec{w}_{yy}v_y}{v^2} + \frac{\mu\vec{w}_y v_{yy}}{v^2} - \frac{2\mu\vec{w}_y v_y^2}{v^3}. \tag{2.3.26}$$

Multiplying (2.3.26) by $\vec{w}_{yyy}$, integrating the resulting equality over $[0,1]$, using Cauchy's inequality and Young's inequality, we get that for any small $\varepsilon > 0$,

$$\|\vec{w}_{yyy}(t)\|^2 \leq C_2\left(\|\vec{w}_{ty}(t)\|^2 + \|v_{yy}(t)\|^2 + \|\vec{w}_{yy}(t)\|^2\right). \tag{2.3.27}$$

Integrating (2.3.27) over $[0,t]$, by Lemma 2.2.1 and Lemma 2.3.1, we arrive at

$$\int_0^t \|\vec{w}_{yyy}(\tau)\|^2 d\tau \leq C_2 \int_0^t (\|\vec{w}_{ty}\|^2 + \|v_{yy}\|^2 + \|\vec{w}_{yy}\|^2)(\tau)\, d\tau \leq C_2. \tag{2.3.28}$$

Differentiating (2.1.19) with respect to y, we obtain

$$\begin{aligned}\frac{\kappa\theta_{yyy}}{v} &= e_\theta\theta_{ty} + e_v v_{ty} + e_{\theta v}\theta_t v_y + e_{\theta v}\theta_y v_t + e_{\theta\theta}\theta_y\theta_t + e_{vv}v_y v_t \\ &\quad - \frac{2\lambda u_{yy}u_y}{v^2} + \frac{\lambda v_y u_y^2}{v^2} - \frac{2\mu\vec{w}_{yy}\cdot\vec{w}_y}{v^2} + \frac{\mu v_y|\vec{w}_y|^2}{v^2} + \frac{2\kappa\theta_{yy}v_y}{v^2} \\ &\quad + \frac{\kappa\theta_y v_{yy}}{v^2} - \frac{2\kappa\theta_y v_y^2}{v^3} + u_{yy}p + p_\theta\theta_y u_y + p_v u_y v_y.\end{aligned} \tag{2.3.29}$$

Multiplying (2.3.29) by θ_{yyy}, integrating the resultant over [0,1] and using Cauchy's inequality, we deduce that for any small $\varepsilon > 0$,

$$\|\theta_{yyy}(t)\|^2 \le C_2\Big(\|u_{ty}(t)\|^2 + \|\theta_{ty}(t)\|^2 + \|v_{yy}(t)\|^2 + \|\theta_y v_y\|^2 + \|\theta_y(t)\|_{L^4}^4 + \|v_y(t)\|_{L^4}^4 + \|u_{yy}v_y\|^2 + \|v_{yy}u_y\|^2 + \|u_y v_y^2\|^2\Big) + \varepsilon\|\theta_{yyy}(t)\|^2,$$

i.e.,

$$\|\theta_{yyy}(t)\|^2 \le C_2\Big(\|u_{ty}(t)\|^2 + \|\theta_{ty}(t)\|^2 + \|v_{yy}(t)\|^2 + \|v_{yy}(t)\|^2 + \|u_y(t)\|^2 + \|v_y(t)\|^2 + \|\theta_y(t)\|^2\Big). \tag{2.3.30}$$

Integrating (2.3.30) over $[0,t]$, and using Lemma 2.2.1 and Lemma 2.3.1, we get

$$\int_0^t \|\theta_{yyy}(\tau)\|^2 d\tau \le C_2 \int_0^t \Big(\|u_{ty}\|^2 + \|\theta_{ty}\|^2 + \|v_{yy}\|^2 + \|u_{yy}\|^2 + \|\theta_y\|^2 + \|u_y\|^2 + \|v_y\|^2\Big)(\tau) d\tau \le C_2.$$

The proof is complete. □

Lemma 2.3.4. *There exist constants $C_2 > 0$ and $\gamma_2' = \gamma_2'(C_2) > 0$ such that for any fixed $\gamma \in (0, \gamma_2']$, the global solution $(v(t), u(t), \vec{w}(t), \theta(t)) \in H_+^2$ to problem* (2.1.16)–(2.1.21) *or* (2.1.16)–(2.1.20), (2.1.22) *satisfies that the following estimates:*

$$e^{\gamma t}\left(\|v(t) - \bar{v}\|_{H^2}^2 + \|u(t)\|_{H^2}^2 + \|\vec{w}(t)\|_{H^2}^2 + \|\theta(t) - \bar{\theta}\|_{H^2}^2\right) + \int_0^t e^{\gamma\tau}\left(\|v - \bar{v}\|_{H^2}^2 + \|u\|_{H^3}^2 + \|\vec{w}\|_{H^3}^2 + \|\theta\|_{H^3}^2\right)(\tau) d\tau \le C_2, \quad \forall\ t > 0, \tag{2.3.31}$$

$$e^{\gamma t}\left(\|u_t(t)\|^2 + \|\vec{w}_t(t)\|^2 + \|\theta_t(t)\|^2\right) + \int_0^t e^{\gamma\tau}\left(\|u_{ty}\|^2 + \|\vec{w}_{ty}\|^2 + \|\theta_{ty}\|^2\right)(\tau) d\tau \le C_2, \quad \forall\ t > 0. \tag{2.3.32}$$

Proof. Multiplying (2.3.4) by $e^{\gamma t}$, and integrating the resulting equality over $[0,t]$, we have

$$e^{\gamma t}\|u_t(t)\|^2 + \int_0^t e^{\gamma\tau}\|u_{ty}(\tau)\|^2 d\tau \le C_2 \int_0^t e^{\gamma\tau}(\|u_{yy}\|^2 + \|\theta_t\|^2 + \|u_y\|^2)(\tau) d\tau + \gamma \int_0^t e^{\gamma\tau}\|u_t(\tau)\|^2 d\tau. \tag{2.3.33}$$

By Lemma 2.2.4 and (2.3.33), we can infer that there exists a constant $\gamma_2' = \gamma_2'(C_1) > 0$ such that for any fixed $\gamma \in (0, \gamma_2']$,

$$e^{\gamma t}\|u_t(t)\|^2 + \int_0^t e^{\gamma\tau}\|u_{ty}(\tau)\|^2 d\tau \le C_2. \tag{2.3.34}$$

Multiplying (2.3.24) by $e^{\gamma t}$, and integrating the resulting equality over $[0, t]$, we get

$$\int_0^t e^{\gamma\tau}\|u_{yyy}(\tau)\|^2 d\tau \le C_2. \tag{2.3.35}$$

Multiplying (2.3.12) by $e^{\gamma t}$, and integrating the result over $[0, t]$, we have

$$e^{\gamma t}\|\vec{w}_t(t)\|^2 + \int_0^t e^{\gamma\tau}\|\vec{w}_{ty}(\tau)\|^2 d\tau \le C_2 \int_0^t e^{\gamma\tau}\|\vec{w}_y(\tau)\|^2 d\tau + \gamma \int_0^t e^{\gamma\tau}\|\vec{w}_t(\tau)\|^2 d\tau. \tag{2.3.36}$$

By Lemma 2.2.4 and (2.3.36), we can derive that there exists a constant $\gamma_2' = \gamma_2'(C_1) > 0$ such that for any fixed $\gamma \in (0, \gamma_2']$,

$$e^{\gamma t}\|\vec{w}_t(t)\|^2 + \int_0^t e^{\gamma\tau}\|\vec{w}_{ty}(\tau)\|^2 d\tau \le C_2. \tag{2.3.37}$$

Multiplying (2.3.28) by $e^{\gamma t}$, and integrating the resultant over $[0, t]$, we get

$$\int_0^t e^{\gamma\tau}\|\vec{w}_{yyy}\tau)\|^2 (d\tau \le C_2. \tag{2.3.38}$$

Multiplying (2.3.17) by $e^{\gamma t}$, and integrating the resultant over $[0, t]$, we have

$$\begin{aligned} &e^{\gamma t}\|\theta_t(t)\|^2 + \int_0^t e^{\gamma\tau}\|\theta_{ty}(\tau)\|^2\, d\tau \\ &\quad \le C_2 \int_0^t e^{\gamma\tau}\Big(\|\theta_y\|^2 + \|\theta_{yy}\|^2 + \|\theta_t\|^2 + \|u_y\|^2 + \|u_{ty}\|^2 \\ &\qquad\qquad + \|\vec{w}_y\|^2 + \|\vec{w}_{ty}\|^2 + \|\theta_t\|_{L^3}^3\Big)(\tau)\, d\tau. \end{aligned} \tag{2.3.39}$$

By Lemma 2.2.4 and (2.3.37), we get that there exists a constant $\gamma_2' = \gamma_2'(C_1) > 0$ such that for any fixed $\gamma \in (0, \gamma_2']$,

$$e^{\gamma t}\|\theta_t(t)\|^2 + \int_0^t e^{\gamma\tau}\|\theta_{ty}(\tau)\|^2 d\tau \le C_2. \tag{2.3.40}$$

Multiplying (2.3.30) by $e^{\gamma t}$, and integrating the resultant over $[0, t]$, we get

$$\int_0^t e^{\gamma\tau}\|\theta_{yyy}(\tau)\|^2 d\tau \le C_2, \quad \forall t > 0. \tag{2.3.41}$$

Multiplying (2.3.21) by $e^{\gamma t}$, and integrating over $[0, t]$, we have

$$\begin{aligned} &e^{\gamma t}\|v_{yy}(t)\|^2 + \int_0^t e^{\gamma\tau}\|v_{yy}(\tau)\|^2\, d\tau \\ &\quad \le C_2 \int_0^t e^{\gamma\tau}(\|v_y\|^2 + \|u_{ty}\|^2 + \|u_{yyy}\|^2 + \|\theta_{yy}\|^2)(\tau) d\tau + \gamma \int_0^t e^{\gamma\tau}\|v_t(\tau)\|^2 d\tau. \end{aligned} \tag{2.3.42}$$

By Lemma 2.2.3 and (2.3.42), we can deduce that there exists a constant $\gamma_2' = \gamma_2'(C_1) > 0$ such that for any fixed $\gamma \in (0, \gamma_2']$,

$$e^{\gamma t}\|v_{yy}(t)\|^2 + \int_0^t e^{\gamma\tau}\|v_{yy}(\tau)\|^2 d\tau \le C_2.$$

The proof is complete. □

2.4 Proof of Theorem 2.1.3

Lemma 2.4.1. *Under the assumptions of Theorem* 2.1.3*, for any* $(v_0, u_0, \vec{w}_0, \theta_0) \in H_+^4$ *and for any* $\varepsilon \in (0,1)$ *small enough, we have for any* $t > 0$*,*

$$\|u_{ty}(y,0)\| + \|\vec{w}_{ty}(y,0)\| + \|\theta_{ty}(y,0)\| \le C_3, \tag{2.4.1}$$

$$\begin{aligned} &\|u_{tt}(y,0)\| + \|\vec{w}_{tt}(y,0)\| + \|\theta_{tt}(y,0)\| + \|u_{tyy}(y,0)\| \\ &\quad + \|\vec{w}_{tyy}(y,0)\| + \|\theta_{tyy}(y,0)\| \le C_4, \end{aligned} \tag{2.4.2}$$

$$\|u_{tt}(t)\|^2 + \int_0^t \|u_{tty}(\tau)\|^2\, d\tau \le C_4 + C_4\int_0^t \|\theta_{tty}(\tau)\|^2\, d\tau, \tag{2.4.3}$$

$$\|\vec{w}_{tt}(t)\|^2 + \int_0^t \|\vec{w}_{tty}(\tau)\|^2\, d\tau \le C_4 + C_4\int_0^t \|\vec{w}_{tyy}(\tau)\|^2\, d\tau, \tag{2.4.4}$$

$$\begin{aligned} &\|\theta_{tt}(t)\|^2 + \int_0^t \|\theta_{tty}(\tau)\|^2\, d\tau \\ &\le C_4\varepsilon^{-3} + C_2\varepsilon^{-1}\int_0^t \|\theta_{tyy}(\tau)\|^2\, d\tau + C_1\varepsilon\int_0^t \left(\|u_{tty}\|^2 + \|u_{tyy}\|^2 + \|\vec{w}_{tyy}\|^2\right)(\tau)\, d\tau. \end{aligned} \tag{2.4.5}$$

Proof. By (2.1.17), we can derive

$$\begin{aligned} \|u_t\| &\le C_1(\|u_{yy}\| + \|u_y\|_{L^\infty}\|v_y\| + \|v_y\| + \|\theta_y\|) \\ &\le C_2(\|u_y\|_{H^1} + \|v_y\| + \|\theta_y\|). \end{aligned} \tag{2.4.6}$$

We differentiate (2.1.17) with respect to y, and use Theorems 2.1.1–2.1.2 to derive

$$\begin{aligned} \|u_{ty}\| \le C_1\Big(&\|v_y\|_{L^4}^2 + \|v_y\|_{L^\infty}\|\theta_y\| + \|v_{yy}\| + \|u_y\|_{L^\infty}\|v_y\|_{L^4}^2 + \|\theta_y\|_{L^4}^2 \\ &+ \|u_{yyy}\| + \|\theta_{yy}\| + \|v_y\|_{L^\infty}\|u_{yy}\| + \|v_{yy}\|\|u_y\|_{L^\infty}\|\Big). \end{aligned} \tag{2.4.7}$$

Using the Gagliardo-Nirenberg inequality and the Young inequality, we conclude

$$\begin{aligned} \|u_y\|_{L^\infty} &\le C(\|u_y\|^{\frac12}\|u_{yy}\|^{\frac12} + \|u_{yy}\|) \le C(\|u_y\| + \|u_{yy}\|), \\ \|u_y\|_{L^4}^2 &\le C(\|u_y\|^{\frac34}\|u_{yy}\|^{\frac14} + \|u_y\|)^2 \le C(\|u_y\|^{\frac32}\|u_{yy}\|^{\frac12} + \|u_y\|^2) \\ &\le C(\|u_y\|^2 + \|u_{yy}\|^2). \end{aligned}$$

Finally, using (2.4.7), we can obtain

$$\|u_{ty}\| \le C_2(\|v_y\|_{H^1} + \|u_y\|_{H^2} + \|\theta_y\|_{H^1}) \tag{2.4.8}$$

or

$$\|u_{yyy}\| \le C_2(\|u_{ty}\| + \|v_y\|_{H^1} + \|u_y\|_{H^1} + \|\theta_y\|_{H^1}). \tag{2.4.9}$$

We differentiate (2.1.17) with respect to y twice to derive

$$\begin{aligned}\|u_{tyy}\| \le C_1\Big(&\|v_y\|_{L^6}^3 + \|v_y\|_{L^4}^2\|\theta_y\|_{L^\infty} + \|v_y\|_{L^\infty}\|\|v_{yy}\| + \|\theta_y\|_{L^\infty}\|\|v_{yy}\| \\ &+ \|v_y\|_{L^\infty}\|\theta_y^2\| + \|v_y\|_{L^\infty}\|\|\theta_{yy}\| + \|\theta_y\|_{L^6}^3 + \|v_{yyy}\| + \|\theta_{yyy}\| \\ &+ \|\theta_y\|_{L^\infty}\|\theta_{yy}\| + \|\theta_{yy}\| + \|v_y\|_{L^\infty}\|u_{yyy}\| + \|u_{yy}\|_{L^\infty}\|v_{yy}\| \\ &+ \|u_{yy}\|_{L^\infty}\|v_y\|_{L^4}^2 + \|u_y\|_{L^\infty}\|v_{yyy}\| + \|u_y\|_{L^\infty}\|v_{yy}\|\|v_y\| \\ &+ \|u_{yyyy}\| + \|u_y\|_{L^\infty}\|v_y\|_{L^6}^3\Big).\end{aligned} \tag{2.4.10}$$

Using the Gagliardo-Nirenberg and the Young inequality, we conclude

$$\|u_{tyy}\| \le C_2\left(\|u_y\|_{H^3} + \|v_y\|_{H^2} + \|\theta_y\|_{H^2}\right) \tag{2.4.11}$$

or

$$\|u_{yyyy}\| \le C_2\left(\|u_{tyy}\| + \|u_y\|_{H^2} + \|v_y\|_{H^2} + \|\theta_y\|_{H^2}\right). \tag{2.4.12}$$

By (2.1.18), we can derive

$$\|\vec{w}_t\| \le C_1(\|\vec{w}_{yy}\| + \|\vec{w}_y\|_{L^\infty}\|v_y\|) \le C_2(\|\vec{w}_y\|_{H^1} + \|v_y\|). \tag{2.4.13}$$

We differentiate (2.1.18) with respect to y, and use Theorems 2.1.1–2.1.2 to get

$$\|\vec{w}_{ty}(t)\| \le C_1\left(\|\vec{w}_{yyy}\| + \|\vec{w}_{yy}\|\|v_y\|_{L^\infty} + \|\vec{w}_y\|_{L^\infty}\|v_{yy}\| + \|\vec{w}_y\|_{L^\infty}\|v_y^2\|\right).$$

Using the Gagliardo-Nirenberg inequality and the Young inequality, we conclude

$$\|\vec{w}_{ty}(t)\| \le C_2(\|\vec{w}_y\|_{H^2} + \|v_y\|_{H^1}) \tag{2.4.14}$$

or

$$\|\vec{w}_{yyy}\| \le C_2(\|\vec{w}_y\|_{H^1} + \|v_y\|_{H^1} + \|\vec{w}_{ty}\|). \tag{2.4.15}$$

We differentiate (2.1.18) with respect to y twice to derive

$$\begin{aligned}\|\vec{w}_{tyy}(t)\| &\le C_1\Big(\|\vec{w}_{yyyy}\| + \|v_y\|_{L^\infty}\|\vec{w}_{yyy}\| + \|v_{yy}\|_{L^\infty}\|\vec{w}_{yy}\| + \|\vec{w}_y\|_{L^\infty}\|\vec{w}_{yyy}\| \\ &\qquad + \|\vec{w}_{yy}\|_{L^\infty}\|v_y^2\| + \|\vec{w}_y\|_{L^\infty}\|v_y^3\| + \|\vec{w}_y v_y\|\|v_{yy}\|_{L^\infty}\Big) \\ &\le C_2(\|\vec{w}_y(t)\|_{H^3} + \|v_y(t)\|_{H^2})\end{aligned} \tag{2.4.16}$$

or

$$\|\vec{w}_{yyyy}(t)\| \le C_2(\|\vec{w}_{tyy}(t)\| + \|\vec{w}_y(t)\|_{H^2} + \|v_y(t)\|_{H^2}). \tag{2.4.17}$$

By (2.1.19), we can derive

$$\begin{aligned}\|\theta_t(t)\| &\le C_1\left(\|u_y(t)\| + \|u_y^2(t)\| + \|\vec{w}_y^2(t)\| + \|\theta_{yy}(t)\| + \|\theta_y(t)\|_{L^\infty}\|v_y(t)\|\right)\\ &\le C_2(\|u_y(t)\|_{H^1} + \|\vec{w}_y(t)\|_{H^1} + \|\theta_y(t)\|_{H^1}).\end{aligned} \tag{2.4.18}$$

Differentiate (2.1.19) with respect to y, and use Theorems 2.1.1–2.1.2 to obtain

$$\|\theta_{ty}(t)\| \le C_2(\|u_y(t)\|_{H^1} + \|\vec{w}_y(t)\|_{H^1} + \|\theta_y(t)\|_{H^2} + \|v_y(t)\|_{H^1}) \tag{2.4.19}$$

or

$$\|\theta_{yyy}(t)\| \le C_2\left(\|\theta_{ty}(t)\| + \|u_y(t)\|_{H^1} + \|\vec{w}_y(t)\|_{H^1} + \|\theta_y(t)\|_{H^1} + \|v_y(t)\|_{H^1}\right). \tag{2.4.20}$$

Differentiate (2.1.18) with respect to y twice to derive

$$\|\theta_{tyy}(t)\| \le C_2\left(\|u_y(t)\|_{H^2} + \|\vec{w}_y(t)\|_{H^2} + \|\theta_y(t)\|_{H^3} + \|v_y(t)\|_{H^2}\right) \tag{2.4.21}$$

or

$$\|\theta_{yyyy}(t)\| \le C_2\left(\|\theta_{tyy}(t)\| + \|u_y(t)\|_{H^2} + \|\vec{w}_y(t)\|_{H^2} + \|\theta_y(t)\|_{H^2} + \|v_y(t)\|_{H^2}\right). \tag{2.4.22}$$

Differentiate (2.1.17) with respect to t to obtain

$$\begin{aligned}\|u_{tt}(t)\| \le C_2\Big(&\|\theta_{ty}(t)\| + \|\theta_t(t)\|_{L^\infty}\|\theta_y(t)\| + \|\theta_y(t)\|_{L^\infty}\|u_y(t)\|\\ &+ \|v_{ty}(t)\| + \|u_{tyy}(t)\| + \|v_y(t)\|_{L^\infty}\|\theta_t(t)\| + \|u_{ty}(t)\|\|v_y(t)\|_{L^\infty}\\ &+ \|v_y(t)\|_{L^\infty}\|u_{yy}(t)\| + \|u_y(t)\|_{L^\infty}\|v_y^2(t)\| + \|u_y(t)\|_{L^\infty}\|v_y(t)\|\Big)\\ \le C_2\Big(&\|\theta_y(t)\|_{H^2} + \|v_y(t)\|_{H^2} + \|u_y(t)\|_{H^3}\Big).\end{aligned} \tag{2.4.23}$$

We differentiate (2.1.18) with respect to t to deduce

$$\begin{aligned}\|\vec{w}_{tt}(t)\| \le C_2\Big(&\|\vec{w}_{tyy}(t)\| + \|u_y(t)\|_{L^\infty}\|\vec{w}_{yy}(t)\| + \|\vec{w}_{ty}(t)\|\|v_y(t)\|_{L^\infty}\\ &+ \|u_{yy}(t)\|\|\vec{w}_y(t)\|_{L^\infty} + \|\vec{w}_y(t)\|_{L^\infty}\|v_y(t)u_y(t)\|\Big)\\ \le C_2\left(\right.&\left.\|\vec{w}_y(t)\|_{H^3} + \|u_y(t)\|_{H^2} + \|v_y(t)\|_{H^2}\right).\end{aligned} \tag{2.4.24}$$

We differentiate (2.1.19) with respect to t to infer

$$\begin{aligned}\|\theta_{tt}(t)\| \le C_2\Big(&\|\theta_t\|_{L^4}^2 + \|u_y\|_{L^4}^2 + \|\theta_t\|\|u_y\|_{L^\infty} + \|u_{ty}\| + \|u_{ty}\|\|u_y\|_{L^\infty}\\ &+ \|\vec{w}_{ty}\|\|\vec{w}_y\|_{L^\infty} + \|u_y\|_{L^\infty}\|\vec{w}_t\|_{L^4}^2 + \|\theta_{tyy}\| + \|u_y\|_{L^\infty}\|\theta_{yy}\|\\ &+ \|\theta_{yt}\|\|v_y\|_{L^\infty} + \|\theta_y\|_{L^\infty}\|v_{yt}\| + \|u_y\|_{L^\infty}\|v_y\theta_y\|\Big)\end{aligned}$$

$$
\begin{aligned}
&\leq C_2\Big(\|u_y\|_{H^1}+\|v_y\|_{H^1}+\|\theta_y\|_{H^2}+\|\theta_{tyy}\| \\
&\qquad +\|u_{ty}\|+\|\theta_{ty}\|+\|\vec{w}_{ty}\|\Big), \qquad (2.4.25)\\
&\leq C_2\Big(\|u_y(t)\|_{H^2}+\|\vec{w}_y(t)\|_{H^2}+\|\theta_y(t)\|_{H^3}+\|v_y(t)\|_{H^2}\Big). \qquad (2.4.26)
\end{aligned}
$$

Thus estimates (2.4.1)–(2.4.2) follow from (2.4.6)–(2.4.25).

Differentiating (2.1.17) with respect to t twice, multiplying the resulting equation by u_{tt} in $L^2(0,1)$, performing an integration by parts, using Theorems 2.1.1–2.1.2, we arrive at

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_{tt}(t)\|^2 &\leq -\lambda\int_0^1\frac{|u_{tty}|^2}{v}\,dy+C_2\Big(\|u_y\|_{L^4}^2+\|\theta_{tt}\|+\|u_{ty}\|+\|\theta_t\|_{L^4}^2\\
&\qquad +\|\theta_t\|_{L^\infty}\|u_y\|+\|u_y\|_{L^\infty}\|u_{ty}\|+\|u_y\|_{L^6}^3\Big)\|u_{tty}\|\\
&\leq -C_1^{-1}\|u_{tty}\|^2+C_2\left(\|u_y\|_{H^2}^2+\|\theta_t\|_{H^1}^2+\|u_{ty}\|^2+\|v_y\|^2+\|\theta_{tt}\|^2\right). \quad (2.4.27)
\end{aligned}
$$

Thus, by Theorems 2.1.1–2.1.2, we deduce

$$
\begin{aligned}
&\|u_{tt}(t)\|^2+\int_0^t\|u_{tty}(\tau)\|^2\,d\tau\\
&\leq \|u_{tt}(y,0)\|^2+C\int_0^t\Big(\|u_y\|_{H^2}^2+\|\theta_t\|_{H^1}^2+\|u_{ty}\|^2+\|v_y\|^2+\|\theta_{tt}\|^2\Big)(\tau)\,d\tau\\
&\leq C_4+C_2\int_0^t\|\theta_{tt}(\tau)\|^2\,d\tau.
\end{aligned}
$$

By (2.4.25), we conclude

$$
\|u_{tt}(t)\|^2+\int_0^t\|u_{tty}(\tau)\|^2\,d\tau\leq C_4+C_2\int_0^t\|\theta_{tyy}(\tau)\|^2\,d\tau
$$

which, along with Theorems 2.1.1–2.1.2, gives estimate (2.4.3).

Differentiating (2.1.18) with respect to t twice, multiplying the resulting equation by $\vec{w}_{tt}$ in $L^2(0,1)$, performing an integration by parts, and using Theorems 2.1.1–2.1.2, we derive

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\vec{w}_{tt}(t)\|^2 \qquad (2.4.28)\\
&\leq -\mu\int_0^1\frac{|\vec{w}_{tty}|^2}{v}\,dy+C_2\Big(\|\vec{w}_{ty}\|_{L^\infty}\|v_y\|+\|\vec{w}_y\|_{L^\infty}\|u_{ty}\|+\|\vec{w}_y\|_{L^\infty}\|u_y\|_{L^4}^2\Big)\|\vec{w}_{tty}\|\\
&\leq -(2C)_1^{-1}\|\vec{w}_{tty}\|^2+C_2\left(\|\vec{w}_{tyy}\|^2+\|v_y\|^2+\|\vec{w}_y\|^2+\|u_y\|_{H^1}^2+\|u_{ty}\|^2\right).
\end{aligned}
$$

By (2.4.28), we can obtain

$$
\|\vec{w}_{tt}(t)\|^2+\int_0^t\|\vec{w}_{tty}(\tau)\|^2\,dy\leq C_4+C_2\int_0^t\|\vec{w}_{tyy}(\tau)\|^2\,d\tau
$$

which, along with Theorems 2.1.1–2.1.2, gives estimate (2.4.4).

Differentiating (2.1.19) with respect to t twice, multiplying the resulting equation by θ_{tt} in $L^2(0,1)$, performing an integration by parts, and using Theorems 2.1.1–2.1.2, we infer that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 e_\theta\theta_{tt}^2 dy\\
&= -\int_0^1 \left(\frac{\kappa\theta_y}{v}\right)_{tt}\theta_{tty}\,dy - \int_0^1 \left(e_{\theta tt}\theta_t + e_{utt}v_t\right)\theta_{tt}\,dy - \frac{3}{2}\int_0^1 e_\theta\theta_{tt}^2\,dy\\
&\quad - 2\int_0^1 \left[e_{vt} - \left(-p + \frac{\lambda u_y}{v}\right)_t\right] u_{ty}\theta_{tt}\,dy - \int_0^1 \left(e_v + p - \frac{\lambda u_y}{v}\right) u_{tty}\theta_{tt}\,dy\\
&\quad + \int_0^1 \left(-p + \frac{\lambda u_y}{v}\right)_{tt} u_y\theta_{tt}\,dy + \mu\int_0^1 \left(\frac{\vec{w}_y\cdot\vec{w}}{v}\right)_{tt}\theta_{tt}\,dy\\
&= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7.
\end{aligned} \tag{2.4.29}$$

By virtue of Theorems 2.1.1–2.1.2, (2.4.1)–(2.4.2) and using the embedding theorem, we deduce that for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
A_1 &= -\kappa\int_0^1 \left(\frac{\theta_y}{v}\right)_{tt}\theta_{tty}\,dy \\
&= -\kappa\int_0^1 \frac{\theta_{tty}^2}{v}\,dy + \kappa\int_0^1 \left(\frac{2\theta_{ty}u_y}{v^2} + \frac{\theta_y u_{ty}}{v^2} - 2\frac{\theta_y u_y^2}{v^3}\right)\theta_{tty}\,dy\\
&\le -C_1^{-1}\|\theta_{tty}\|^2 + C_2\|\theta_{tty}\|\left(\|\theta_{ty}\|_{L^\infty}\|u_y\| + \|\theta_y\|_{L^\infty}\|u_{ty}\| + \|\theta_y\|_{L^\infty}\|u_y\|_{L^4}^2\right)\\
&\le -(2C_1)^{-1}\|\theta_{tty}\|^2 + C_2\left(\|\theta_{ty}\|^2 + \|\theta_y\|_{H^1}^2 + \|u_{ty}\|^2 + \|u_y\|_{H^1}^2 + \|\theta_{tyy}\|^2\right),
\end{aligned} \tag{2.4.30}$$

$$\begin{aligned}
A_2 &= -\int_0^1 \left(e_{\theta tt}\theta_t + e_{utt}v_t\right)\theta_{tt}\,dy \\
&\le C_1\int_0^1 \left[(|v_t| + |\theta_t|)^2 + |v_{tt}| + |\theta_{tt}|\right](|\theta_t| + |v_t|)|\theta_{tt}|\,dy\\
&\le C_1\|\theta_{tt}\|_{L^\infty}(\|\theta_t\| + \|u_y\|)[(\|u_y\|_{L^\infty} + \|\theta_t\|_{L^\infty})(\|\theta_t\| + \|u_y\|) + \|u_{ty}\| + \|\theta_{tt}\|]\\
&\le \varepsilon\|\theta_{tty}\|^2 + C_2\varepsilon^{-1}(\|u_{ty}\|^2 + \|u_y\|_{H^1}^2 + \|\theta_{ty}\|^2 + \|\theta_{tt}\|^2 + \|\theta_t\|^2),
\end{aligned} \tag{2.4.31}$$

$$A_3 = -\frac{3}{2}\int_0^1 e_\theta\theta_{tt}^2\,dy \le C_1\int_0^1 (|\theta_t| + |u_y|)|\theta_{tt}|^2\,dy \le \varepsilon\|\theta_{tty}\|^2 + C_2\varepsilon^{-1}\|\theta_{tt}\|^2, \tag{2.4.32}$$

$$\begin{aligned}
A_4 &= -2\int_0^1 \left[e_{vt} - \left(-p + \frac{\lambda u_y}{v}\right)_t\right] u_{ty}\theta_{tt}\,dy \\
&\le C_1\int_0^1 \left(|\theta_t| + |u_y| + |u_{ty}| + |u_y^2|\right)|u_{ty}||\theta_{tt}|\,dy\\
&\le C_2\|u_{ty}\|_{L^\infty}\left(\|u_y\| + \|\theta_t\| + \|u_{ty}\| + \|u_y\|_{L^4}^2\right)\|\theta_{tt}\|\\
&\le C_2\|u_{ty}\|^{\frac{1}{2}}\|u_{tyy}\|^{\frac{1}{2}}\left(\|u_y\|_{H^1} + \|u_{ty}\| + \|\theta_t\|\right)\|\theta_{tt}\|
\end{aligned} \tag{2.4.33}$$

which implies

$$\begin{aligned}
\int_0^t A_4(\tau)\,d\tau &\le C_2 \sup_{0\le\tau\le t}\|\theta_{tt}(\tau)\| \int_0^t \|u_{ty}\|^{\frac12}\|u_{tyy}\|^{\frac12}\left(\|u_y\|_{H^1}+\|u_{ty}\|+\|\theta_t\|\right)d\tau\\
&\le C_2 \sup_{0\le\tau\le t}\|\theta_{tt}(\tau)\| \left(\int_0^t \|u_{ty}(\tau)\|^2\,d\tau\right)^{\frac14}\left(\int_0^t \|u_{tyy}(\tau)\|^2\,d\tau\right)^{\frac14}\\
&\quad\times\left(\int_0^t (\|u_y\|_{H^1}^2+\|u_{ty}\|^2+\|\theta_t\|^2)(\tau)\,d\tau\right)^{\frac12}\\
&\le \varepsilon\left(\sup_{0\le\tau\le t}\|\theta_{tt}(\tau)\|^2+\int_0^t\|u_{tyy}(\tau)\|^2\,d\tau\right)+C_2\varepsilon^{-3}, \qquad (2.4.34)
\end{aligned}$$

$$A_5 = -\int_0^1\left(e_v+p-\frac{\lambda u_y}{v}\right)u_{tty}\theta_{tt}\,dy \le \varepsilon\|u_{tty}\|^2+C_2\varepsilon^{-1}\|\theta_{tt}\|^2, \qquad (2.4.35)$$

$$\begin{aligned}
A_6 &= \int_0^1\left(-p+\frac{\lambda u_y}{v}\right)_{tt}u_y\theta_{tt}\,dy\\
&\le C_1\int_0^1\left((|v_t|+|\theta_t|)^2+|\theta_{tt}|+|u_{tty}|+|u_{ty}u_y|+|u_y|^3\right)|u_y||\theta_{tt}|\,dy\\
&\le C_1\|u_y\|_{L^\infty}\|\theta_{tt}\|\Big((\|v_t\|_{L^\infty}+\|\theta_t\|_{L^\infty})(\|v_t\|+\|\theta_t\|)+\|u_{tty}\|+\|u_y\|\\
&\quad+\|\theta_{tt}\|+\|u_{ty}\|+\|u_y\|_{L^\infty}+\|u_y\|_{L^6}^3\Big)\\
&\le \varepsilon\|u_{tty}\|^2+C_2\varepsilon^{-1}\left(\|u_{tt}\|^2+\|u_y\|^2+\|\theta_{tt}\|^2+\|\theta_{ty}\|^2+\|\theta_t\|^2\right), \qquad (2.4.36)
\end{aligned}$$

$$\begin{aligned}
A_7 &= \mu\int_0^1\left(\frac{\vec w_y\cdot\vec w}{v}\right)_{tt}\theta_{tt}\,dy\\
&\le C_2\|\theta_{tt}\|\Big(\|\vec w_{ty}\|_{L^4}^2+\|\vec w_{tty}\|_{L^\infty}+\|\vec w_y\|_{L^\infty}\|\vec w_{ty}\|\|u_y\|_{L^\infty}\\
&\quad+\|\vec w_y\|_{L^\infty}^2\|u_{yy}\|+\|u_y\|_{L^\infty}^2\|\vec w_y\|_{L^4}^2\Big)\\
&\le C_2\varepsilon^{-1}\|\theta_{tt}\|^2+\varepsilon\Big(\|\vec w_{tyy}\|^2+\|\vec w_{ty}\|^2\\
&\quad+\|\vec w_{tty}\|^2+\|\vec w_y\|_{H^1}^2+\|u_y\|_{H^1}^2\Big). \qquad (2.4.37)
\end{aligned}$$

Integrating with respect to t, using Theorems 2.1.1–2.1.2 and (2.4.35)–(2.4.37), we can derive

$$\|\sqrt{e_\theta}\theta_{tt}\|^2 \le C_4+\int_0^t (A_1+A_2+A_3+A_4+A_5+A_6+A_7)(\tau)\,d\tau.$$

Thus

$$\begin{aligned}
&\|\theta_{tt}(t)\|^2 + \int_0^t \|\theta_{tty}(\tau)\|^2\,d\tau \\
&\quad \le C_1\varepsilon\left\{\sup_{0\le\tau\le t}\|\theta_{tt}(\tau)\|^2 + \int_0^t (\|u_{tty}\|^2 + \|u_{tyy}\|^2)(\tau)\,d\tau\right\} \\
&\qquad + C_4\varepsilon^{-3} + C_2\varepsilon^{-1}\int_0^t (\|\theta_{tt}\|^2 + \|\theta_{tty}\|^2)(\tau)\,d\tau \\
&\qquad + C_1\varepsilon\int_0^t \left(\|\vec{w}_{tyy}\|^2 + \|\vec{w}_{tty}\|^2\right)(\tau)\,d\tau
\end{aligned} \tag{2.4.38}$$

which, along with Theorems 2.1.1–2.1.2 and (2.4.4), (2.4.25), (2.4.38), gives estimate (2.4.5). □

Lemma 2.4.2. *Under assumptions of Theorem* 2.1.3*, for any* $(v_0, u_0, \vec{w}_0, \theta_0) \in H_+^4$ *and for any* $\varepsilon \in (0,1)$*, we have for any* $t > 0$*,*

$$\|u_{ty}(t)\|^2 + \int_0^t \|u_{tyy}(\tau)\|^2\,d\tau \le C_3\varepsilon^{-6} + C_1\varepsilon^2\int_0^t (\|u_{tty}\|^2 + \|\theta_{tyy}\|^2)(\tau)\,d\tau, \tag{2.4.39}$$

$$\|\vec{w}_{ty}(t)\|^2 + \int_0^t \|\vec{w}_{tyy}(\tau)\|^2\,d\tau \le C_3\varepsilon^{-6} + C_1\varepsilon^2\int_0^t \|\vec{w}_{tty}(\tau)\|^2\,d\tau, \tag{2.4.40}$$

$$\begin{aligned}
&\|\theta_{ty}(t)\|^2 + \int_0^t \|\theta_{tyy}(\tau)\|^2\,d\tau \\
&\quad \le C_3\varepsilon^{-6} + C_2\varepsilon^2\int_0^t (\|u_{tyy}\|^2 + \|\vec{w}_{tyy}\|^2 + \|\theta_{tty}\|^2)(\tau)\,d\tau.
\end{aligned} \tag{2.4.41}$$

Proof. Differentiating (2.1.17) with respect to y and t, multiplying the resulting equation by u_{ty} and integrating by parts in $L^2(0,1)$, we arrive at

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_{ty}(t)\|^2 &= \left(-p + \frac{\lambda u_y}{v}\right)_{ty} u_{ty}\Big|_{y=0}^{y=1} - \int_0^1 \left(-p + \frac{\lambda u_y}{v}\right)_{ty} u_{tyy}\,dy \\
&= B_1 + B_2.
\end{aligned} \tag{2.4.42}$$

We employ Theorems 2.1.1–2.1.2 and Lemma 2.4.1, the interpolation inequality and Poincaré's inequality to get

$$\begin{aligned}
B_1 &\le C_1\Big(\|u_{yy}\|_{L^\infty} + (\|u_y\|_{L^\infty} + \|\theta_t\|_{L^\infty})(\|v_y\|_{L^\infty} + \|\theta_y\|_{L^\infty}) \\
&\qquad + \|\theta_{ty}\|_{L^\infty} + \|u_{tyy}\|_{L^\infty} + \|u_{ty}\|_{L^\infty}\|v_y\|_{L^\infty} + \|u_{yy}\|_{L^\infty}\|u_y\|_{L^\infty} \\
&\qquad + \|u_y\|_{L^\infty}^2\|v_y\|_{L^\infty}\Big)\|u_{ty}\|_{L^\infty} \\
&\le C_2(B_{11} + B_{12})\|u_{ty}\|^{\frac{1}{2}}\|u_{tyy}\|^{\frac{1}{2}}
\end{aligned} \tag{2.4.43}$$

where

$$B_{11} = \|v_y\|_{H^2} + \|\theta_t\| + \|\theta_{ty}\|,$$
$$B_{12} = \|u_{ty}\|^{\frac{1}{2}}\|u_{tyy}\|^{\frac{1}{2}} + \|u_{tyyy}\|^{\frac{1}{2}}\|u_{tyy}\|^{\frac{1}{2}} + \|\theta_{ty}\|^{\frac{1}{2}}\|\theta_{tyy}\|^{\frac{1}{2}}.$$

Applying Young's inequality several times, we have that for any $\varepsilon \in (0,1)$,

$$C_2 B_{11}\|u_{ty}\|^{\frac{1}{2}}\|u_{tyy}\|^{\frac{1}{2}} \leq \frac{\varepsilon^2}{2}\|u_{tyy}\|^2 + C_2\varepsilon^{-\frac{2}{3}}\Big(\|u_{ty}\|^2 + \|u_y\|_{H^2}^2 + \|\theta_t\|^2 + \|\theta_{ty}\|^2\Big), \tag{2.4.44}$$

$$\begin{aligned} C_2 B_{11}\|u_{ty}\|^{\frac{1}{2}}\|u_{tyy}\|^{\frac{1}{2}} \leq{}& \frac{\varepsilon^2}{2}\|u_{tyy}\|^2 + \varepsilon^2\left(\|\theta_{tyy}\|^2 + \|u_{tyyy}\|^2\right) \\ &+ C_2\varepsilon^{-6}\left(\|\theta_{ty}\|^2 + \|u_{ty}\|^2\right). \end{aligned} \tag{2.4.45}$$

Thus we infer from (2.4.43)–(2.4.45) and Theorems 2.1.1–2.1.2 and Lemma 2.4.1,

$$B_1 \leq \varepsilon^2(\|u_{tyy}\|^2 + \|\theta_{tyy}\|^2 + \|u_{tyyy}\|^2) + C_2\varepsilon^{-6}(\|u_{ty}\|^2 + \|u_y\|_{H^2}^2 + \|\theta_t\|^2 + \|\theta_{ty}\|^2) \tag{2.4.46}$$

which, together with Lemmas 2.2.1–2.2.3, further leads to

$$\int_0^t B_1(\tau)\,d\tau \leq \varepsilon^2\int_0^t\Big(\|u_{tyy}\|^2 + \|\theta_{tyy}\|^2 + \|u_{tyyy}\|^2\Big)(\tau)\,d\tau + C_2\varepsilon^{-6},\ \forall\, t>0, \tag{2.4.47}$$

$$\begin{aligned} B_2 \leq{}& -\lambda\int_0^1 \frac{u_{tyy}^2}{v}\,dy + C_1\Big[(\|u_y\| + \|\theta_t\|)(\|v_y\|_{L^\infty} + \|\theta_y\|_{L^\infty}) + \|u_{yy}\| + \|\theta_{ty}\| \\ &+ \|v_y\|_{L^\infty}\|u_{ty}\| + \|u_y\|_{L^\infty}\|u_{yy}\| + \|u_y\|_{L^\infty}^2\|v_y\|\Big]\|u_{tyy}\| \\ \leq{}& (-2C_1)^{-1}\|u_{tyy}\|^2 + C_2\varepsilon^2\left(\|u_{ty}\|^2 + \|u_y\|_{H^1}^2 + \|v_y\|_{H^1}^2 + \|\theta_t\|_{H^1}^2\right). \end{aligned} \tag{2.4.48}$$

By (2.4.42), (2.4.47)–(2.4.48) and Theorems 2.1.1–2.1.2 and Lemma 2.4.1, for any $\varepsilon \in (0,1)$ small enough, we have

$$\|u_{ty}(t)\|^2 + \int_0^t\|u_{tyy}(\tau)\|^2\,d\tau \leq C_2\varepsilon^{-6} + C_1\varepsilon^2\int_0^t(\|u_{tyyy}\|^2 + \|\theta_{tyy}\|^2)(\tau)\,d\tau. \tag{2.4.49}$$

On the other hand, differentiating (2.1.17) with respect to y and t, and using Theorems 2.1.1–2.1.2 and Lemma 2.4.1, we derive

$$\|u_{tyyy}(t)\| \leq C_2\left(\|u_{tty}\| + \|u_{tyy}\| + \|\theta_t\|_{H^2} + \|\theta_y\|_{H^1} + \|u_y\|_{H^2} + \|u_y\|_{H^1}\right). \tag{2.4.50}$$

Thus inserting (2.4.50) into (2.4.49) implies estimate (2.4.39).

Differentiating (2.1.18) with respect to y and t, multiplying the resulting equation by $\vec{w}_{ty}$ in $L^2(0,1)$, and integrating by parts, we can obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\vec{w}_{ty}(t)\|^2 &= \mu\left(\frac{\vec{w}_y}{v}\right)_{ty}\vec{w}_{ty}\Big|_{y=0}^{y=1} - \mu\int_0^1\left(\frac{\vec{w}_y}{v}\right)_{ty}\vec{w}_{tyy}\,dy \\ &= D_1 + D_2 \end{aligned} \tag{2.4.51}$$

where

$$\begin{aligned}
D_1 &\le C_1\|\vec{w}_{ty}\|_{L^\infty}\Big(\|\vec{w}_{tyy}\|_{L^\infty} + \|\vec{w}_{ty}\|_{L^\infty}\|v_y\|_{L^\infty} + \|\vec{w}_{yy}\|_{L^\infty}\|u_y\|_{L^\infty} \\
&\qquad + \|\vec{w}_y\|_{L^\infty}\|u_{yy}\|_{L^\infty} + \|\vec{w}_y\|_{L^\infty}\|v_y u_y\|_{L^\infty}\Big) \\
&\le C_1\varepsilon^2(\|\vec{w}_{tyy}\|^2 + \|\vec{w}_{tyyy}\|^2) + C_2\varepsilon^{-6}\Big(\|\vec{w}_{ty}\|^2 + \|\vec{w}_y\|_{H^2}^2 \\
&\qquad + \|u_y\|_{H^2}^2 + \|v_y\|_{H^1}^2\Big), \qquad (2.4.52) \\
D_2 &\le -\mu\int_0^1 \frac{|\vec{w}_{tyy}|^2}{v}\,dy + C_1\|\vec{w}_{tyy}\|\Big(\|\vec{w}_{ty}\|_{L^\infty}\|v_y\| + \|\vec{w}_{yy}\|\|u_y\|_{L^\infty} \\
&\qquad + \|\vec{w}_y\|_{L^\infty}\|u_{yy}\| + \|\vec{w}_{ty}\|_{L^\infty}\|v_y\|\|u_y\|_{L^\infty}\Big) \\
&\le -(2C_1)^{-1}\|\vec{w}_{tyy}\|^2 + C_2\varepsilon^2\Big(\|\vec{w}_{ty}\|^2 + \|v_y\|^2 + \|u_y\|_{H^1}^2 \\
&\qquad + \|\vec{w}_y\|_{H^1}^2 + \|\vec{w}_{tyy}\|^2\Big). \qquad (2.4.53)
\end{aligned}$$

By (2.4.51)–(2.4.53) and Theorems 2.1.1–2.1.2 and Lemma 2.4.1, we have

$$\|\vec{w}_{ty}(t)\|^2 + \int_0^t \|\vec{w}_{tyy}(\tau)\|^2\,d\tau \le C_2\varepsilon^{-6} + C_1\varepsilon^2\int_0^t \|\vec{w}_{tyyy}(\tau)\|^2\,d\tau. \qquad (2.4.54)$$

Differentiating (2.1.18) with respect to y and t, and using Lemmas 2.2.1–2.2.4, we derive

$$\|\vec{w}_{tyyy}(t)\| \le C_2(\|\vec{w}_{tty}\| + \|\vec{w}_{tyy}\| + \|\vec{w}_y\|_{H^2} + \|u_y\|_{H^2} + \|v_y\|_{H^2} + \|\vec{w}_{ty}\|). \quad (2.4.55)$$

Thus inserting (2.4.55) into (2.4.54) implies estimate (2.4.40).

Analogously, we get from (2.1.19),

$$\frac{1}{2}\frac{d}{dt}\int_0^1 e_\theta\theta_{ty}^2\,dy = E_1 + E_2 + E_3 + E_4 + E_5 \qquad (2.4.56)$$

where

$$\begin{aligned}
E_1 &= \kappa\left(\frac{\theta_y}{v}\right)_{ty}\theta_{ty}\Big|_{y=0}^{y=1}, \\
E_2 &= -\kappa\int_0^1 \left(\frac{\theta_y}{v}\right)_{ty}\theta_{tyy}\,dy, \\
E_3 &= -\int_0^1 \left[\left(e_v - p + \frac{\lambda u_y}{v}\right)u_y\right]_{ty}\theta_{ty}\,dy, \\
E_4 &= -\int_0^1 \left(e_{\theta ty}\theta_t + e_{\theta y}\theta_{tt} + \frac{1}{2}e_{\theta t}\theta_{ty}\right)\theta_{ty}\,dy, \\
E_5 &= \mu\int_0^1 \left(\frac{\vec{w}_y\cdot\vec{w}}{v}\right)_{ty}\theta_{ty}\,dy.
\end{aligned}$$

It follows that

$$
\begin{aligned}
E_1 \le{}& C_2\|\theta_{ty}\|_{L^\infty}\Big(\|\theta_{tyy}\|_{L^\infty} + \|\theta_{ty}\|_{L^\infty}\|v_y\|_{L^\infty} + \|\theta_{yy}\|_{L^\infty}\|u_y\|_{L^\infty}\\
&+ \|\theta_y\|_{L^\infty}\|u_{yy}\|_{L^\infty} + \|\theta_y\|_{L^\infty}\|u_y\|_{L^\infty}\|v_y\|_{L^\infty}\Big)\\
\le{}& C_2\|\theta_{ty}\|^{\frac12}\|\theta_{tyy}\|^{\frac12}\Big(\|\theta_{tyy}\|^{\frac12}\|\theta_{tyyy}\|^{\frac12} + \|\theta_{ty}\|^{\frac12}\|\theta_{tyy}\|^{\frac12}\\
&+ \|\theta_y\|_{H^2} + \|u_y\|_{H^2} + \|v_y\|_{H^1}\Big),
\end{aligned}
$$

$$
\|\theta_{ty}\|^{\frac12}\|\theta_{tyy}\|^{\frac12}\|\theta_{tyy}\|^{\frac12}\|\theta_{tyyy}\|^{\frac12} \le \frac{\varepsilon^2}{3}\left(\|\theta_{tyy}\|^2 + \|\theta_{tyyy}\|^2\right) + C_2\varepsilon^{-6}\|\theta_{ty}\|^2,
$$

$$
\Big(\|\theta_{ty}\|^{\frac12}\|\theta_{tyy}\|^{\frac12}\Big)^2 \le \frac{\varepsilon^2}{3}\|\theta_{tyy}\|^2 + C_2\varepsilon^{-2}\|\theta_{ty}\|^2,
$$

$$
\begin{aligned}
&\|\theta_{ty}\|^{\frac12}\|\theta_{tyy}\|^{\frac12}(\|\theta_y\|_{H^2} + \|u_y\|_{H^2} + \|v_y\|_{H^1})\\
&\quad\le \frac{\varepsilon^2}{3}\|\theta_{tyy}\|^2 + C_2\varepsilon^{-2}\Big(\|\theta_{ty}\|^2 + \|\theta_y\|_{H^2}^2 + \|u_y\|_{H^2}^2 + \|v_y\|_{H^1}^2\Big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
E_1 \le{}& \varepsilon^2(\|\theta_{tyy}\|^2 + \|\theta_{tyyy}\|^2) + C_2\varepsilon^{-6}\Big(\|\theta_{ty}\|^2 + \|\theta_y\|_{H^2}^2\\
&+ \|u_y\|_{H^2}^2 + \|v_y\|_{H^1}^2\Big), \qquad (2.4.57)
\end{aligned}
$$

$$
\begin{aligned}
E_2 \le{}& -\kappa\int_0^1 \frac{\theta_{tyy}^2}{v}\,dy + C_2\|\theta_{tyy}\|\Big(\|\theta_{ty}\|\|v_y\|_{L^\infty} + \|\theta_{yy}\|\|u_y\|_{L^\infty}\\
&+ \|u_{yy}\|\|\theta_y\|_{L^\infty} + \|u_yv_y\|\|\theta_y\|_{L^\infty}\Big)\\
\le{}& -(2C_1)^{-1}\|\theta_{tyy}\|^2 + C_2\varepsilon^2\Big(\|\theta_{ty}\|^2 + \|\theta_y\|_{H^1}^2 + \|u_y\|_{H^1}^2\Big). \qquad (2.4.58)
\end{aligned}
$$

Similarly,

$$
E_3 \le \varepsilon^2\|u_{tyy}\|^2 + C_2\varepsilon^{-2}\left(\|u_y\|_{H^2}^2 + \|u_{ty}\|^2 + \|\theta_t\|_{H^1}^2\right), \qquad (2.4.59)
$$

$$
\begin{aligned}
E_4 \le{}& \varepsilon^2\|\theta_{tyy}\|^2 + C_2\varepsilon^{-2}\Big(\|u_y\|_{H^1}^2 + \|u_{ty}\|^2 + \|\theta_t\|_{H^1}^2\\
&+ \|\theta_y\|_{H^2}^2 + \|u_{ty}\|^2\Big), \qquad (2.4.60)
\end{aligned}
$$

$$
E_5 \le \varepsilon^2\|\vec{w}_{tyy}\|^2 + C_2\varepsilon^2\|\theta_{tyy}\|^2 + C_2\varepsilon^{-2}\left(\|\vec{w}_y\|_{H^2}^2 + \|\vec{w}_{ty}\|^2 + \|u_y\|_{H^1}^2\right). \quad (2.4.61)
$$

By (2.4.56)–(2.4.61) and Theorems 2.1.1–2.1.2 and Lemma 2.4.1, we have

$$
\begin{aligned}
\|\theta_{ty}(t)\|^2 + \int_0^t \|\theta_{tyy}(\tau)\|^2\,d\tau \le{}& C_1\varepsilon^2\int_0^t(\|\theta_{tyyy}\|^2 + \|u_{tyy}\|^2 + \|\vec{w}_{tyy}\|^2)(\tau)\,d\tau\\
&+ C_2\varepsilon^{-6}\int_0^t(\|\theta_{tyy}\|^2 + \|\theta_y\|_{H^2}^2 + \|u_y\|_{H^2}^2 + \|v_y\|_{H^1}^2)(\tau)\,d\tau\\
&+ C_2\varepsilon^{-2}\int_0^t(\|\vec{w}_y\|_{H^2}^2 + \|\vec{w}_{ty}\|^2)(\tau)\,d\tau. \qquad (2.4.62)
\end{aligned}
$$

Differentiating (2.1.19) with respect to y and t, and using Theorems 2.1.1–2.1.2 and Lemma 2.4.1, we derive

$$\begin{aligned}\|\theta_{tyyy}\| \le C_2\Big(&\|u_{ty}\|_{H^1} + \|\theta_t\|_{H^1} + \|\theta_y\|_{H^2} + \|\theta_{tt}\|_{H^1} + \|\vec{w}_{ty}\|_{H^1}\\ &+ \|\vec{w}_y\|_{H^1} + \|v_y\|_{H^1}\Big).\end{aligned} \tag{2.4.63}$$

By (2.4.62)–(2.4.63) and Theorems 2.1.1–2.1.2 and Lemma 2.4.1, we derive estimate (2.4.41). □

Lemma 2.4.3. *Under assumptions of Theorem* 2.1.3, *for any* $(v_0, u_0, \vec{w}_0, \theta_0) \in H_+^4$ *and for any* $\varepsilon \in (0,1)$, *we have for any* $t > 0$,

$$\begin{aligned}&\|u_{tt}(t)\|^2 + \|\vec{w}_{tt}(t)\|^2 + \|\theta_{tt}(t)\|^2 + \|u_{ty}(t)\|^2 + \|\vec{w}_{ty}(t)\|^2 + \|\theta_{ty}(t)\|^2\\ &\quad + \int_0^t \Big(\|u_{tty}\|^2 + \|u_{tyy}\|^2 + \|\vec{w}_{tty}\|^2 + \|\vec{w}_{tyy}\|^2 + \|\theta_{tty}\|^2 + \|\theta_{tyy}\|^2\Big)(\tau)\, d\tau \le C_4,\end{aligned} \tag{2.4.64}$$

$$\begin{aligned}&\|u_{yyy}(t)\|_{H^1}^2 + \|u_{yy}(t)\|_{W^{1,\infty}}^2 + \|\theta_{yyy}(t)\|_{H^1}^2 + \|\theta_{yy}(t)\|_{w^{1,\infty}}^2 + \|\vec{w}_{yyy}(t)\|_{H^1}^2\\ &\quad + \|\vec{w}_{yy}(t)\|_{W^{1,\infty}}^2 + \|v_{tyyy}(t)\|^2 + \|u_{tyy}(t)\|^2 + \|\vec{w}_{tyy}(t)\|^2 + \|\theta_{tyy}(t)\|^2\\ &\quad + \int_0^t \Big(\|u_{tt}\|^2 + \|\vec{w}_{tt}\|^2 + \|\theta_{yy}\|_{w^{1,\infty}}^2 + \|\theta_{tt}\|^2 + \|u_{yy}\|_{W^{1,\infty}}^2 + \|\vec{w}_{yy}\|_{W^{1,\infty}}^2\\ &\quad + \|u_{tyy}\|_{H^1}^2 + \|\theta_{tyy}\|_{H^1}^2 + \|\vec{w}_{tyy}\|_{H^1}^2 + \|u_{ty}\|_{W^{1,\infty}}^2 + \|\theta_{ty}\|_{w^{1,\infty}}^2\\ &\quad + \|\vec{w}_{ty}\|_{W^{1,\infty}}^2 + \|v_{tyyy}\|_{H^1}^2\Big)(\tau)\, d\tau \le C_4,\end{aligned} \tag{2.4.65}$$

$$\|v_{yyy}(t)\|_{H^1}^2 + \|v_{yy}(t)\|_{W^{1,\infty}}^2 + \int_0^t \left(\|v_{yyy}\|_{H^1}^2 + \|v_{yy}\|_{W^{1,\infty}}^2\right)(\tau)\, d\tau \le C_4, \tag{2.4.66}$$

$$\int_0^t \left(\|u_{yyyy}\|_{H^1}^2 + \|\theta_{yyyy}\|_{H^1}^2 + \|\vec{w}_{yyyy}\|_{H^1}^2\right)(\tau)\, d\tau \le C_4. \tag{2.4.67}$$

Proof. Adding up (2.4.39), (2.4.40) and (2.4.41), picking $\varepsilon \in (0,1)$ small enough, we arrive at

$$\begin{aligned}&\|u_{ty}(t)\|^2 + \|\vec{w}_{ty}(t)\|^2 + \|\theta_{ty}(t)\|^2 + \int_0^t \left(\|u_{tyy}\|^2 + \|\vec{w}_{tyy}\|^2 + \|\theta_{tyy}\|^2\right)(\tau)\, d\tau\\ &\quad \le C_3\varepsilon^{-6} + C_1\varepsilon^2 \int_0^t \left(\|u_{tty}\|^2 + \|\vec{w}_{tty}\|^2 + \|\theta_{tty}\|^2\right)(\tau)\, d\tau.\end{aligned} \tag{2.4.68}$$

Now multiplying (2.4.3) and (2.4.4) by ε respectively, multiplying (2.4.5) by $\varepsilon^{3/2}$, then adding the resultant to (2.4.68), and choosing $\varepsilon \in (0,1)$ small enough, we obtain (2.4.64).

Differentiating (2.1.17) with respect to y, and using (2.1.16), we derive

$$\lambda \frac{\partial}{\partial t}\left(\frac{v_{yy}}{v}\right) - p_v v_{yy} = u_{ty} + E(y,t) \tag{2.4.69}$$

where

$$E(y,t) = 2\lambda\frac{u_{yy}u_y}{v^2} - 2\lambda\frac{u_y v_y^2}{v^3} + p_{vv}v_y^2 + 2p_{v\theta}v_y\theta_y + p_{\theta\theta}\theta_y^2 + p_\theta\theta_{yy}.$$

Differentiating (2.4.69) with respect to y, we have

$$\lambda\frac{\partial}{\partial t}\left(\frac{v_{yyy}}{v}\right) - p_v v_{yyy} = E_1(y,t) \tag{2.4.70}$$

where

$$\begin{aligned}
&E_1(y,t) = E_y(y,t) + p_{vy}v_{yy} + u_{tyy} + \lambda\left(\frac{v_{yy}v_y}{v^2}\right)_t,\\
&\|E_1(t)\| \le C_2\left(\|u_{tyy}\| + \|u_y\|_{H^2} + \|v_y\|_{H^1} + \|\theta_y\|_{H^2}\right).
\end{aligned}$$

Thus

$$\int_0^t \|E_1(\tau)\|^2 d\tau \le C_4. \tag{2.4.71}$$

Multiplying (2.4.70) by $\frac{v_{yyy}}{v}$ in $L^2(0,1)$, we can obtain

$$\frac{d}{dt}\left\|\frac{v_{yyy}}{v}\right\|^2 + C_1^{-1}\left\|\frac{v_{yyy}}{v}\right\|^2 \le C_1\|E_1(t)\|^2. \tag{2.4.72}$$

We infer from (2.4.71)–(2.4.72) that

$$\|v_{yyy}(t)\|^2 + \int_0^t \|v_{yyy}(\tau)\|^2\, d\tau \le C_4, \quad \forall\, t > 0 \tag{2.4.73}$$

which, together with (2.4.9), (2.4.15), (2.4.20) and (2.4.64), gives

$$\|u_{yyy}(t)\| + \|\vec{w}_{yyy}(t)\| + \|\theta_{yyy}(t)\| \le C_4. \tag{2.4.74}$$

By (2.4.12), (2.4.17), (2.4.22) and (2.4.64), we have

$$\int_0^t (\|u_{yyy}\|_{H^1}^2 + \|\theta_{yyy}\|_{H^1}^2 + \|\vec{w}_{yyy}\|_{H^1}^2)(\tau)\, d\tau \le C_4. \tag{2.4.75}$$

Using the embedding theorem and (2.4.73)–(2.4.75), we have

$$\begin{aligned}
&\|u_{yyy}(t)\|^2 + \|\vec{w}_{yyy}(t)\|^2 + \|\theta_{yyy}(t)\|^2 + \|u_{yy}(t)\|_{L^\infty}^2 + \|\theta_{yy}(t)\|_{L^\infty}^2 + \|\vec{w}_{yy}(t)\|_{L^\infty}^2\\
&\quad + \int_0^t \left(\|u_{yyy}\|_{H^1}^2 + \|\theta_{yyy}\|_{H^1}^2 + \|\vec{w}_{yyy}\|_{H^1}^2\right)(\tau)\, d\tau \le C_4.
\end{aligned} \tag{2.4.76}$$

Differentiating (2.1.17), (2.1.19) with respect to t respectively, and using Lemmas 2.4.1–2.4.2, we have

$$\begin{aligned}\|u_{tyy}(t)\| &\le C_1\|u_{tt}(t)\| + C_2(\|u_y(t)\|_{H^1} + \|v_y(t)\|_{H^1} + \|u_{ty}(t)\| + \|\theta_t(t)\|)\\ &\le C_4, \qquad (2.4.77)\\ \|\vec{w}_{tyy}(t)\| &\le C_1\|\vec{w}_{tt}(t)\| + C_2(\|\vec{w}_y(t)\|_{H^1} + \|u_y(t)\|_{H^1} + \|v_y(t)\| + \|\vec{w}_{ty}(t)\|)\\ &\le C_4, \qquad (2.4.78)\\ \|\theta_{tyy}(t)\| &\le C_1\|\theta_{tt}(t)\| + C_2(\|\theta_y(t)\|_{H^1} + \|u_y(t)\|_{H^1} + \|u_{ty}(t)\| + \|\theta_t(t)\|_{H^1}\\ &\qquad + \|\vec{w}_{ty}(t)\| + \|\vec{w}_y(t)\|_{H^1}) \le C_4. \qquad (2.4.79)\end{aligned}$$

By (2.4.12), (2.4.17), (2.4.22), (2.4.75)–(2.4.79) and Lemmas 2.4.1–2.4.2, for any $t > 0$, we get

$$\begin{aligned}&\|u_{yyyy}(t)\|^2 + \|\vec{w}_{yyyy}(t)\|^2 + \|\theta_{yyyy}(t)\|^2 \qquad (2.4.80)\\ &+\int_0^t\Big(\|u_{tyy}\|^2 + \|\vec{w}_{tyy}\|^2 + \|\theta_{tyy}\|^2 + \|u_{yyyy}\|^2 + \|\vec{w}_{yyyy}\|^2 + \|\theta_{yyyy}\|^2\Big)(\tau)\,d\tau \le C_4\end{aligned}$$

which, together with (2.4.76) and the interpolation inequality, gives

$$\begin{aligned}&\|u_{yyy}(t)\|^2_{L^\infty} + \|\vec{w}_{yyy}(t)\|^2_{L^\infty} + \|\theta_{yyy}(t)\|^2_{L^\infty} \qquad (2.4.81)\\ &\quad + \int_0^t \left(\|u_{yyy}\|^2_{L^\infty} + \|\vec{w}_{yyy}\|^2_{L^\infty} + \|\theta_{yyy}\|^2_{L^\infty}\right)(\tau)\,d\tau \le C_4.\end{aligned}$$

Differentiating (2.4.70) with respect to y, we get

$$\lambda\frac{\partial}{\partial t}\left(\frac{v_{yyyy}}{v}\right) - p_v v_{yyyy} = E_2(t) \qquad (2.4.82)$$

where

$$E_2(t) = E_{1y}(t) + p_{vy}v_{yyy} + \lambda\left(\frac{v_{yyy}v_y}{v^2}\right)_t.$$

By Lemmas 2.4.1–2.4.2, (2.4.77)–(2.4.80) and the embedding theorem, the interpolation inequality, we deduce

$$\begin{aligned}\|E_{1y}(t)\| &\le C_1\|u_{tyyy}\| + C_4\left(\|u_y\|_{H^3} + \|v_y\|_{H^2} + \|\theta_y\|_{H^3}\right),\\ \|E_2(t)\| &\le C_1\|u_{tyyy}\| + C_4\left(\|u_y\|_{H^3} + \|v_y\|_{H^2} + \|\theta_y\|_{H^3}\right). \qquad (2.4.83)\end{aligned}$$

By (2.4.23), (2.4.24), (2.4.26) and (2.4.80), we have

$$\int_0^t \left(\|u_{tt}\|^2 + \|\vec{w}_{tt}\|^2 + \|\theta_{tt}\|^2\right)(\tau)\,d\tau \le C_4, \quad \forall\, t > 0 \qquad (2.4.84)$$

which, together with (2.4.50), (2.4.55), (2.4.63) and (2.4.64), gives

$$\int_0^t \left(\|u_{tyyy}\|^2 + \|\vec{w}_{tyyy}\|^2 + \|\theta_{tyyy}\|^2\right)(\tau)\,d\tau \le C_4, \quad \forall t > 0. \qquad (2.4.85)$$

Thus

$$\int_0^t \|E_2(\tau)\|^2\,d\tau \le C_4, \quad \forall\, t>0. \tag{2.4.86}$$

Multiplying (2.4.82) by $\frac{v_{yyyy}}{v}$ in $L^2(0,1)$, we can obtain

$$\frac{d}{dt}\left\|\frac{v_{yyyy}}{v}\right\|^2 + C_1^{-1}\left\|\frac{v_{yyyy}}{v}\right\|^2 \le C_1\|E_2(t)\|^2. \tag{2.4.87}$$

Thus

$$\left\|\frac{v_{yyyy}}{v}(t)\right\|^2 + \int_0^t \left\|\frac{v_{yyyy}}{v}(\tau)\right\|^2 d\tau \le C_4, \quad \forall\, t>0. \tag{2.4.88}$$

Differentiating (2.1.17), (2.1.18), (2.1.19) with respect to y three times respectively, using Lemmas 2.4.1–2.4.2 and Poincaré's inequality, we infer

$$\|u_{yyyyy}(t)\| \le C_1\|u_{tyyy}(t)\| + C_2\left(\|u_y(t)\|_{H^3} + \|v_y(t)\|_{H^3} + \|\theta_y(t)\|_{H^3}\right), \tag{2.4.89}$$

$$\|\vec{w}_{yyyyy}(t)\| \le C_1\|\vec{w}_{tyyy}(t)\| + C_2\left(\|\vec{w}_y(t)\|_{H^3} + \|v_y(t)\|_{H^3}\right), \tag{2.4.90}$$

$$\begin{aligned}\|\theta_{yyyyy}(t)\| \le C_1\|\theta_{tyyy}(t)\| + C_2\big(&\|u_y(t)\|_{H^3} + \|\vec{w}_y(t)\|_{H^3} + \|\theta_y(t)\|_{H^3}\\ &+ \|\theta_{tyy}(t)\| + \|v_y(t)\|_{H^3}\big).\end{aligned} \tag{2.4.91}$$

Using (2.1.16), (2.4.76), (2.4.85), (2.4.88)–(2.4.91) and Lemmas 2.4.1–2.4.2, the interpolation inequality, we have

$$\int_0^t \left(\|u_{yyyyy}\|^2 + \|v_{tyyyy}\|^2 + \|\vec{w}_{yyyyy}\|^2 + \|\theta_{yyyyy}\|^2\right)(\tau)\,d\tau \le C_4, \quad \forall t>0, \tag{2.4.92}$$

$$\int_0^t \left(\|u_{yy}\|^2_{W^{2,\infty}} + \|\theta_{yy}\|^2_{W^{2,\infty}} + \|\vec{w}_{yy}\|^2_{W^{2,\infty}}\right)(\tau)\,d\tau \le C_4, \quad \forall t>0. \tag{2.4.93}$$

Finally, using (2.1.16), (2.4.76)–(2.4.81), (2.4.84)–(2.4.85), (2.4.92)–(2.4.93) and Sobolev's interpolation inequality, we can derive the desired estimates (2.4.65)–(2.4.67). □

Lemma 2.4.4. *Under assumptions of Theorem* 2.1.3*, for any* $(v_0, u_0, \vec{w}_0, \theta_0) \in H^4_+$ *and for any* $\varepsilon \in (0,1)$*, we have for any* $t>0$*,*

$$\begin{aligned}&\|v(t)-\bar{v}\|^2_{H^4} + \|v_t(t)\|^2_{H^3} + \|v_{tt}(t)\|^2_{H^1} + \|u(t)\|^2_{H^4} + \|u_t(t)\|^2_{H^2} + \|u_{tt}(t)\|^2\\ &\quad + \|\vec{w}(t)\|^2_{H^4} + \|\vec{w}_t(t)\|^2_{H^2} + \|\vec{w}_{tt}(t)\|^2 + \|\theta(t)-\bar{\theta}\|^2_{H^4} + \|\theta_t(t)\|^2_{H^2} + \|\theta_{tt}(t)\|^2\\ &\quad + \int_0^t \Big(\|v-\bar{v}\|^2_{H^4} + \|u\|^2_{H^5} + \|u_t\|^2_{H^3} + \|u_{tt}\|^2_{H^1} + \|\vec{w}\|^2_{H^5} + \|\vec{w}_t\|^2_{H^3} + \|\vec{w}_{tt}\|^2\\ &\quad + \|\theta-\bar{\theta}\|^2_{H^5} + \|\theta_t\|^2_{H^3} + \|\theta_{tt}\|^2_{H^1}\Big)(\tau)\,d\tau \le C_4,\end{aligned} \tag{2.4.94}$$

$$\int_0^t \left(\|v_t\|^2_{H^4} + \|v_{tt}\|^2_{H^2} + \|v_{ttt}\|^2\right)(\tau)\,d\tau \le C_4. \tag{2.4.95}$$

Proof. Exploiting (2.1.16) and Lemmas 2.4.1–2.4.3, we easily obtain Lemma 2.4.4. The proof is complete. □

Lemma 2.4.5. *Under assumptions of Theorem* 2.1.3*, for any* $(v_0, u_0, \vec{w}_0, \theta_0) \in H_+^4$ *, there exists a positive constant* $\gamma_4^{(1)} = \gamma_4^{(1)}(C_4) \le \gamma_2(C_2)$ *such that for any fixed* $\gamma \in (0, \gamma_4^{(1)}]$*, the following estimates hold for any* $t > 0$*, and* $\varepsilon \in (0,1)$ *small enough,*

$$e^{\gamma t}\|u_{tt}(t)\|^2 + \int_0^t e^{\gamma\tau}\|u_{tty}(\tau)\|^2\,d\tau \le C_4 + C_4\int_0^t e^{\gamma\tau}\|\theta_{tyy}(\tau)\|^2\,d\tau, \tag{2.4.96}$$

$$e^{\gamma t}\|\vec{w}_{tt}(t)\|^2 + \int_0^t e^{\gamma\tau}\|\vec{w}_{tty}(\tau)\|^2\,d\tau \le C_4 + C_4\int_0^t e^{\gamma\tau}\|\vec{w}_{tyy}(\tau)\|^2\,d\tau, \tag{2.4.97}$$

$$\begin{aligned} e^{\gamma t}\|\theta_{tt}(t)\|^2 + \int_0^t e^{\gamma\tau}\|\theta_{tty}(\tau)\|^2\,d\tau &\le C_4\varepsilon^{-3} + C_2\varepsilon^{-1}\int_0^t e^{\gamma\tau}\|\theta_{tyy}(\tau)\|^2\,d\tau \\ &+ C_1\varepsilon\int_0^t e^{\gamma\tau}\left(\|u_{tty}\|^2 + \|u_{tyy}\|^2 + \|\vec{w}_{tty}\|^2 + \|\vec{w}_{tyy}\|^2\right)(\tau)\,d\tau. \end{aligned} \tag{2.4.98}$$

Proof. Multiplying (2.4.27) by $e^{\gamma t}$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(e^{\gamma t}\|u_{tt}(t)\|^2\right) \le \frac{\gamma}{2}e^{\gamma t}\|u_{tt}(t)\|^2 - C_1^{-1}e^{\gamma t}\|u_{tty}(t)\|^2 + C_2e^{\gamma t}\Big(\|u_y(t)\|_{H^2}^2 \\ + \|\theta_t(t)\|^2 + \|\theta_{ty}(t)\|^2 + \|u_{ty}(t)\|^2 + \|v_y(t)\|^2 + \|\theta_{tt}(t)\|^2\Big). \end{aligned} \tag{2.4.99}$$

Using (2.1.12) and Poincaré's inequality, we can derive

$$\|u_{tt}(t)\| \le C_1\|u_{tty}(t)\|.$$

Exploiting (2.4.25) and Lemmas 2.2.3–2.2.4, we arrive at

$$\begin{aligned} e^{\gamma t}\|u_{tt}(t)\|^2 &\le C_4 - (C_1^{-1} - C_1\gamma)\int_0^t e^{\gamma\tau}\|u_{tty}(\tau)\|^2\,d\tau + C_2\int_0^t e^{\gamma\tau}\|\theta_{tt}(\tau)\|^2\,d\tau \\ &\le C_4 - (C_1^{-1} - C_1\gamma)\int_0^t e^{\gamma\tau}\|u_{tty}(\tau)\|^2\,d\tau + C_4\int_0^t e^{\gamma\tau}\|\theta_{tyy}(\tau)\|^2\,d\tau \end{aligned}$$

which gives (2.4.96) if we take $\gamma > 0$ so small that $0 < \gamma \le \min\big[\frac{1}{4C_1^2}, \gamma_2(C_2)\big]$. Multiplying (2.4.28) by $e^{\gamma t}$, using (2.4.24), Theorems 2.1.1–2.1.3 and Poincaré's inequality, we have

$$e^{\gamma t}\|\vec{w}_{tt}(t)\|^2 \le C_4 - (C_1^{-1} - C_1\gamma)\int_0^t e^{\gamma\tau}\|\vec{w}_{tty}(\tau)\|^2\,d\tau + C_4\int_0^t e^{\gamma\tau}\|\vec{w}_{tyy}(\tau)\|^2\,d\tau$$

which gives (2.4.97) if we take $\gamma > 0$ so small that $0 < \gamma \le \min\big[\frac{1}{4C_1^2}, \gamma_2(C_2)\big]$.

Multiplying (2.4.29) by $e^{\gamma\tau}$, using (2.4.30)–(2.4.37), (2.4.25), Theorems 2.1.1–2.1.3 and Poincaré's inequality, for any $\varepsilon \in (0,1)$ small enough, we have

$$\begin{aligned}
&\frac{1}{2}e^{\gamma t}\|\sqrt{e_\theta}\theta_{tt}\|^2 \le C_4 + \frac{\gamma}{2}\int_0^t e^{\gamma\tau}\|\sqrt{e_\theta}\theta_{tt}\|^2(\tau)\,d\tau \\
&\quad + \int_0^t e^{\gamma\tau}(A_1+A_2+A_3+A_4+A_5+A_6+A_7)(\tau)\,d\tau \\
&\le C_4\varepsilon^{-3} + \frac{\gamma}{2}\int_0^t e^{\gamma\tau}\|\sqrt{e_\theta}\theta_{tt}(\tau)\|^2\,d\tau - (2C_1)^{-1}\int_0^t e^{\gamma\tau}\|\theta_{tty}(\tau)\|^2\,d\tau \\
&\quad + C_2\int_0^t \left[e^{\gamma\tau}\|\theta_{tyy}(\tau)\|^2 + 2\varepsilon e^{\gamma\tau}\|\theta_{tty(\tau)}\|^2\right]\,d\tau \\
&\quad + C_2\varepsilon^{-1}\int_0^t e^{\gamma\tau}\|\theta_{tt}(\tau)\|^2\,d\tau + \sup_{0\le\tau\le t}\|\theta_{tt}(\tau)\|\left(\int_0^t e^{\gamma\tau}\|u_{ty}(\tau)\|^2\,d\tau\right)^{1/4} \\
&\quad \times\left(\int_0^t e^{\gamma\tau}\|u_{tyy}(\tau)\|^2\,d\tau\right)^{1/4}\left[\int_0^t e^{\gamma\tau}(\|u_y\|_{H^1}^2 + \|\theta_y\|_{H^1}^2 + \|u_{ty}\|^2)(\tau)\,d\tau\right]^{1/2} \\
&\quad + \varepsilon\int_0^t e^{\gamma\tau}(\|u_{tty}\|^2 + \|\vec{w}_{tty}\|^2 + \|\vec{w}_{tyy}\|^2)(\tau)\,d\tau \\
&\le C_4\varepsilon^{-3} - \left(\frac{1}{2C_1} - 2\varepsilon - C_1\gamma\right)\int_0^t e^{\gamma\tau}\|\theta_{tty}(\tau)\|^2\,d\tau + C_2\varepsilon^{-1}\int_0^t e^{\gamma\tau}\|\theta_{tyy}(\tau)\|^2\,d\tau \\
&\quad + \varepsilon\int_0^t e^{\gamma\tau}\left(\|u_{tty}\|^2 + \|u_{tyy}\|^2 + \|\vec{w}_{tty}\|^2 + \|\vec{w}_{tyy}\|^2\right)(\tau)\,d\tau + \varepsilon\sup_{0\le\tau\le t}\|\theta_{tt}(\tau)\|^2
\end{aligned}$$

which gives (2.4.98) if we take $\gamma > 0$ small enough. □

Lemma 2.4.6. *Under assumptions of Theorem* 2.1.3, *for any* $(v_0, u_0, \vec{w}_0, \theta_0) \in H_+^4$, *there exists a positive constant* $\gamma_4^{(2)} \le \gamma_4^{(1)}$ *such that for any fixed* $\gamma \in (0, \gamma_4^{(2)}]$, *the following estimates hold for any* $t > 0$, *and* $\varepsilon \in (0,1)$ *small enough,*

$$\begin{aligned}
&e^{\gamma t}\|u_{ty}(t)\|^2 + \int_0^t e^{\gamma\tau}\|u_{tyy}(\tau)\|^2\,d\tau \\
&\qquad \le C_2\varepsilon^{-6} + C_1\varepsilon^2\int_0^t e^{\gamma\tau}\left(\|u_{tty}\|^2 + \|\theta_{tyy}\|^2\right)(\tau)\,d\tau, \qquad (2.4.100)
\end{aligned}$$

$$\begin{aligned}
&e^{\gamma t}\|\vec{w}_{ty}(t)\|^2 + \int_0^t e^{\gamma\tau}\|\vec{w}_{tyy}(\tau)\|^2\,d\tau \\
&\qquad \le C_2\varepsilon^{-6} + C_1\varepsilon^2\int_0^t e^{\gamma\tau}\|\vec{w}_{tty}(\tau)\|^2\,d\tau, \qquad (2.4.101)
\end{aligned}$$

$$
\begin{aligned}
&e^{\gamma t}\|\theta_{ty}(t)\|^2 + \int_0^t e^{\gamma\tau}\|\theta_{tyy}(\tau)\|^2\,d\tau \\
&\qquad \le C_2\varepsilon^{-6} + C_1\varepsilon^2\int_0^t e^{\gamma\tau}\left(\|u_{tyy}\|^2 + \|\theta_{tty}\|^2 + \|\vec{w}_{tyy}\|^2\right)(\tau)\,d\tau, \qquad (2.4.102)\\
&e^{\gamma t}\left(\|u_{ty}(t)\|^2 + \|\vec{w}_{ty}(t)\|^2 + \|\theta_{ty}(t)\|^2\right) \\
&\qquad + \int_0^t e^{\gamma\tau}\left(\|u_{tyy}\|^2 + \|\vec{w}_{tyy}\|^2 + \|\theta_{tyy}\|^2\right)(\tau)\,d\tau \\
&\qquad \le C_2\varepsilon^{-6} + C_1\varepsilon^2\int_0^t e^{\gamma\tau}(\|u_{tty}\|^2 + \|\vec{w}_{tty}\|^2 + \|\theta_{tty}\|^2)(\tau)\,d\tau. \qquad (2.4.103)
\end{aligned}
$$

Proof. Adding up (2.4.100), (2.4.101) and (2.4.102), picking $\varepsilon \in (0,1)$ small enough, we obtain (2.4.103).

Multiplying (2.4.42) by $e^{\gamma t}$, using (2.4.47)–(2.4.48) and Lemmas 2.2.3–2.2.4, for any $\varepsilon \in (0,1)$ small enough, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(e^{\gamma t}\|u_{ty}\|^2\right) \\
&\quad \le \frac{1}{2}\gamma e^{\gamma t}\|u_{ty}\|^2 + e^{\gamma t}\Big[\varepsilon^2(\|u_{tyy}\|^2 + \|\theta_{tyy}\|^2 + \|u_{tyyy}\|^2) \\
&\qquad + C_2\varepsilon^{-6}\left(\|u_y\|_{H^2}^2 + \|\theta_t\|^2 + \|\theta_{ty}\|^2 + \|u_{ty}\|^2\right)\Big] - (2C_1)^{-1}e^{\gamma t}\|u_{tyy}\|^2 \\
&\qquad + C_2 e^{\gamma t}\left(\|u_y\|_{H^1}^2 + \|\theta_t\|_{H^1}^2 + \|u_{ty}\|^2 + \|v_y\|_{H^1}^2\right). \qquad (2.4.104)
\end{aligned}
$$

Integrating (2.4.104) with respect to t and using Poincaré's inequality, we have

$$
\begin{aligned}
e^{\gamma t}\|u_{ty}(t)\|^2 \le C_2\varepsilon^{-6} - [(2C_1)^{-1} - 2\varepsilon^2 - C_1\gamma]\int_0^t e^{\gamma\tau}\|u_{tyy}(\tau)\|^2\,d\tau \\
+ C_1\varepsilon^2\int_0^t e^{\gamma\tau}(\|u_{tty}\|^2 + \|\theta_{tyy}\|^2)(\tau)\,d\tau
\end{aligned}
$$

which gives (2.4.100) if we take $\gamma > 0$ and $\varepsilon \in (0,1)$ so small that $0 < \varepsilon < \min\left[1, \frac{1}{8C_1}\right]$ and $0 < \gamma < \min\left[\gamma_4^{(1)}, \frac{1}{8C_1{}^2}\right] \equiv \gamma_4^{(2)}$.

Similarly, multiplying (2.4.51) by $e^{\gamma t}$, using (2.4.52)–(2.4.53), (2.4.55) and Theorems 2.1.1–2.1.3, for any $\varepsilon \in (0,1)$ small enough , we obtain

$$
\begin{aligned}
e^{\gamma t}\|\vec{w}_{ty}(t)\|^2 \le C_2\varepsilon^{-6} - [(2C_1)^{-1} - C_1\varepsilon^2]\int_0^t e^{\gamma\tau}\|\vec{w}_{tyy}(\tau)\|^2\,d\tau \\
+ C_1\varepsilon^2\int_0^t e^{\gamma\tau}\|\vec{w}_{tyyy}(\tau)\|^2\,d\tau
\end{aligned}
$$

which gives (2.4.101) if we take $\varepsilon \in (0,1)$ small enough.

Multiplying (2.4.56) by $e^{\gamma t}$, using (2.4.57)–(2.4.61), (2.4.63) and Theorems 2.1.1–2.1.3, we derive

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(e^{\gamma t}\|\sqrt{e_\theta}\theta_{ty}\|^2\right) &\leq \frac{1}{2}\gamma e^{\gamma t}\|\sqrt{e_\theta}\theta_{ty}\|^2 + \varepsilon^2 e^{\gamma t}(\|\theta_{tyy}\|^2 + \|\theta_{tyyy}\|^2)\\ &+ C_2\varepsilon^{-6}e^{\gamma t}\left(\|\theta_{ty}\|^2 + \|\theta_y\|^2_{H^2} + \|u_y\|^2_{H^2} + \|v_y\|^2_{H^1}\right) - (2C_1)^{-1}e^{\gamma t}\|\theta_{tyy}\|^2\\ &+ C_2\varepsilon^{-2}e^{\gamma t}\Big(\|u_y\|^2_{H^2} + \|\theta_t\|^2_{H^1} + \|u_{ty}\|^2 + \|\theta_y\|^2_{H^2} + \|v_y\|^2\\ &+ \|\vec{w}_y\|^2_{H^2} + \|\vec{w}_{ty}\|^2\Big) + \varepsilon^2 e^{\gamma t}\|\theta_{tyy}\|^2 + C_2\varepsilon^2 e^{\gamma t}\|v_{ty}\|^2. \end{aligned}\tag{2.4.105}$$

Integrating (2.4.105) with respect to t, we can deduce

$$\begin{aligned}e^{\gamma t}\|\theta_{ty}(t)\|^2 \leq{}& \varepsilon^2\int_0^t e^{\gamma\tau}\left(\|\vec{w}_{tyy}\|^2 + \|\theta_{tyyy}\|^2\right)(\tau)\,d\tau\\ &+ C_2\varepsilon^{-6} - \left((2C_1)^{-1} - \varepsilon^2\right)\int_0^t e^{\gamma\tau}\|\theta_{tyy}(\tau)\|^2\,d\tau\end{aligned}$$

which gives (2.4.102) for $\varepsilon \in (0,1)$ small enough. □

Lemma 2.4.7. *Under assumptions of Theorem* 2.1.3*, for any* $(v_0, u_0, \vec{w}_0, \theta_0) \in H^4_+$*, there exists a positive constant* $\gamma_4 \leq \gamma_4^{(2)}$ *such that for any fixed* $\gamma \in (0, \gamma_4]$*, the following estimates hold for any* $t > 0$*, and* $\varepsilon \in (0,1)$*,*

$$\begin{aligned}&e^{\gamma t}(\|u_{tt}(t)\|^2 + \|\theta_{tt}(t)\|^2 + \|\vec{w}_{tt}(t)\|^2 + \|u_{ty}(t)\|^2 + \|\vec{w}_{ty}(t)\|^2 + \|\theta_{ty}(t)\|^2)\\ &\quad+ \int_0^t e^{\gamma\tau}\Big(\|u_{tty}\|^2 + \|u_{tyy}\|^2 + \|\theta_{tty}\|^2 + \|\theta_{tyy}\|^2 + \|\vec{w}_{tty}\|^2 + \|\vec{w}_{tyy}\|^2\Big)(\tau)\,d\tau\\ &\quad\leq C_4,\end{aligned}\tag{2.4.106}$$

$$\begin{aligned}&e^{\gamma t}(\|v_{yyy}(t)\|^2_{H^1} + \|v_{yy}(t)\|^2_{W^{1,\infty}}) + \int_0^t e^{\gamma\tau}(\|v_{yyy}\|^2_{H^1} + \|v_{yy}\|^2_{W^{1,\infty}})(\tau)\,d\tau\\ &\quad\leq C_4,\end{aligned}\tag{2.4.107}$$

$$\begin{aligned}&e^{\gamma t}(\|u_{yyy}(t)\|^2_{H^1} + \|u_{yy}(t)\|^2_{W^{1,\infty}} + \|\vec{w}_{yyy}(t)\|^2_{H^1} + \|\vec{w}_{yy}(t)\|^2_{W^{1,\infty}} + \|\theta_{yyy}(t)\|^2_{H^1}\\ &\quad+ \|\theta_{yy}(t)\|^2_{W^{1,\infty}} + \|v_{tyyy}(t)\|^2 + \|u_{tyy}(t)\|^2 + \|\vec{w}_{tyy}(t)\|^2 + \|\theta_{tyy}(t)\|^2)\\ &\quad+ \int_0^t e^{\gamma\tau}\Big(\|u_{tt}\|^2 + \|\theta_{tt}\|^2 + \|\vec{w}_{tt}\|^2 + \|u_{yyyy}\|^2_{H^1} + \|u_{tyy}\|^2_{H^1} + \|\vec{w}_{yyyy}\|^2_{H^1}\\ &\quad+ \|\vec{w}_{tyy}\|^2_{H^1} + \|\theta_{yyyy}\|^2_{H^1} + \|\theta_{tyy}\|^2_{H^1} + \|u_{yy}\|^2_{W^{2,\infty}} + \|u_{ty}\|^2_{W^{2,\infty}}\\ &\quad+ \|\vec{w}_{yy}\|^2_{W^{2,\infty}} + \|\vec{w}_{ty}\|^2_{W^{2,\infty}} + \|\theta_{yy}\|^2_{W^{2,\infty}} + \|\theta_{ty}\|^2_{W^{2,\infty}} + \|v_{tyyy}\|^2_{H^1}\Big)(\tau)\,d\tau\\ &\quad\leq C_4.\end{aligned}\tag{2.4.108}$$

Proof. (2.4.96) $\times \varepsilon +$ (2.4.97) $\times \varepsilon +$ (2.4.98) $\times \varepsilon^{\frac{3}{2}} +$ (2.4.103), and taking $\varepsilon \in (0,1)$ small enough, we derive (2.4.106). Multiplying (2.4.72) by $e^{\gamma t}$, we can calculate

$$\frac{d}{dt}\left(e^{\gamma t}\left\|\frac{v_{yyyy}}{v}\right\|^2\right) + \left(C_1^{-1} - \gamma\right) e^{\gamma t}\left\|\frac{v_{yyy}}{v}\right\|^2 \le C_1 e^{\gamma t}\|E_1(t)\|^2. \tag{2.4.109}$$

Choose $\gamma > 0$ so small that $0 < \gamma < \gamma_4 \equiv \min\left[\frac{1}{2C_1}, \gamma_4^{(2)}\right]$.

Integrating (2.4.109) with respect to t, we have

$$e^{\gamma t}\left\|\frac{v_{yyy}}{v}(t)\right\|^2 + \frac{1}{2C_1}\int_0^t e^{\gamma\tau}\left\|\frac{v_{yyy}}{v}(\tau)\right\|^2 d\tau \le C_3 + C_1\int_0^t e^{\gamma\tau}\|E_1(\tau)\|^2\, d\tau, \quad \forall\, t > 0.$$

Using (2.4.106), we have

$$e^{\gamma t}\|v_{yyy}(t)\|^2 + \int_0^t e^{\gamma\tau}\|v_{yyy}(\tau)\|^2\, d\tau \le C_4, \qquad \forall\, t > 0. \tag{2.4.110}$$

By (2.4.9), (2.4.15), (2.4.20), (2.4.106) and Theorems 2.1.1–2.1.3, we have

$$e^{\gamma t}(\|u_{yyy}(t)\|^2 + \|\theta_{yyy}(t)\|^2 + \|\vec{w}_{yyy}(t)\|^2) \le C_4. \tag{2.4.111}$$

By (2.4.12), (2.4.17), (2.4.22), (2.4.106) and Theorems 2.1.1–2.1.3, we get

$$\int_0^t e^{\gamma\tau}\left(\|u_{yyy}\|_{H^1}^2 + \|\theta_{yyy}\|_{H^1}^2 + \|\vec{w}_{yyy}\|_{H^1}^2\right)(\tau)\, d\tau \le C_4. \tag{2.4.112}$$

Using the embedding theorem, the interpolation inequality and (2.4.109)–(2.4.112), we conclude

$$\begin{aligned} &e^{\gamma t}\Big(\|u_{yyy}(t)\|^2 + \|\theta_{yyy}(t)\|^2 + \|\vec{w}_{yyy}(t)\|^2 \\ &\qquad + \|u_{yy}(t)\|_{L^\infty}^2 + \|\theta_{yy}(t)\|_{L^\infty}^2 + \|\vec{w}_{yy}(t)\|_{L^\infty}^2\Big) \\ &\qquad + \int_0^t e^{\gamma\tau}\left(\|u_{yyy}\|_{H^1}^2 + \|\theta_{yyy}\|_{H^1}^2 + \|\vec{w}_{yyy}\|_{H^1}^2\right)(\tau)\, d\tau \le C_4. \end{aligned} \tag{2.4.113}$$

Using (2.4.77)–(2.4.79), (2.4.106) and Theorems 2.1.1–2.1.3, we derive

$$\begin{aligned} &e^{\gamma t}(\|u_{tyy}(t)\|^2 + \|\theta_{tyy}(t)\|^2 + \|\vec{w}_{tyy}(t)\|^2) \le e^{\gamma t}\left(\|u_{tt}\|^2 + \|\vec{w}_{tt}\|^2 + \|\theta_{tt}\|^2\right) \\ &\quad + e^{\gamma t}\left(\|u_y\|_{H^1}^2 + \|v_y\|_{H^1}^2 + \|\vec{w}_y\|_{H^1}^2 + \|\theta_t\|^2 + \|\theta_{ty}\|^2 + \|u_{ty}\|^2 + \|\vec{w}_{ty}\|^2\right) \le C_4. \end{aligned} \tag{2.4.114}$$

By (2.4.12), (2.4.17), (2.4.22), (2.4.111)–(2.4.114), we conclude

$$\begin{aligned} &e^{\gamma t}\Big(\|u_{yyy}(t)\|_{H^1}^2 + \|\theta_{yyy}(t)\|_{H^1}^2 + \|\vec{w}_{yyy}(t)\|_{H^1}^2 + \|u_{yy}(t)\|_{W^{1,\infty}}^2 + \|\vec{w}_{yy}(t)\|_{W^{1,\infty}}^2 \\ &\quad + \|\theta_{yy}(t)\|_{W^{1,\infty}}^2 + \|u_{tyy}(t)\|^2 + \|\vec{w}_{tyy}(t)\|^2 + \|\theta_{tyy}(t)\|^2\Big) \\ &\quad + \int_0^t e^{\gamma\tau}\Big(\|u_{yyy}\|_{H^1}^2 + \|\theta_{yyy}\|_{H^1}^2 + \|\vec{w}_{yyy}\|_{H^1}^2 + \|u_{yy}\|_{W^{1,\infty}}^2 + \|\vec{w}_{yy}\|_{W^{1,\infty}}^2 \\ &\quad + \|\theta_{yy}\|_{W^{1,\infty}}^2 + \|u_{tyy}\|^2 + \|\vec{w}_{tyy}\|^2 + \|\theta_{tyy}\|^2\Big)(\tau)\, d\tau \le C_4, \quad \forall t > 0. \end{aligned} \tag{2.4.115}$$

Using (2.4.23)–(2.4.24), (2.4.26), (2.4.115) and Theorem 2.1.3, we can deduce

$$\int_0^t e^{\gamma\tau}\left(\|u_{tt}\|^2 + \|\vec{w}_{tt}\|^2 + \|\theta_{tt}\|^2\right)(\tau)\,d\tau \le C_4, \quad \forall t > 0. \tag{2.4.116}$$

Analogously, by (2.4.49), (2.4.55), (2.4.63), (2.4.106), (2.4.115) and Theorem 2.1.3, we derive

$$\int_0^t e^{\gamma\tau}(\|u_{tyyy}\|^2 + \|\vec{w}_{tyyy}\|^2 + \|\theta_{tyyy}\|^2)(\tau)\,d\tau \le C_4, \quad \forall\, t > 0. \tag{2.4.117}$$

Multiplying (2.4.87) by $e^{\gamma t}$, using (2.4.83), (2.4.106), (2.4.110)–(2.4.117) and Theorem 2.1.3, for any fixed $\gamma \in (0, \gamma_4)$, we have

$$\begin{aligned} e^{\gamma t}\left\|\frac{v_{yyyy}}{v}\right\|^2 + \frac{1}{2C_1}\int_0^t e^{\gamma\tau}\left\|\frac{v_{yyyy}}{v}\right\|^2(\tau)\,d\tau &\le C_4 + C_1\int_0^t e^{\gamma\tau}\|E_2(\tau)\|^2\,d\tau \\ &\le C_4, \quad \forall t > 0. \end{aligned}$$

Thus

$$e^{\gamma t}\|v_{yyyy}(t)\|^2 + \int_0^t e^{\gamma\tau}\|v_{yyyy}(\tau)\|^2\,d\tau \le C_4, \qquad \forall t > 0 \tag{2.4.118}$$

which, combined with (2.4.110), (2.4.118) and the embedding theorem, gives (2.4.107).

On the other hand, by (2.4.89)–(2.4.91), (2.4.115), (2.4.117) and Theorem 2.1.3, we have

$$\begin{aligned} \int_0^t e^{\gamma\tau}\Big(&\|u_{yyyyy}\|^2 + \|\theta_{yyyyy}\|^2 + \|\vec{w}_{yyyyy}\|^2 + \|u_{yy}\|^2_{W^{2,\infty}} \\ &+ \|\vec{w}_{yy}\|^2_{W^{2,\infty}} + \|\theta_{yy}\|^2_{W^{2,\infty}}\Big)(\tau)\,d\tau \le C_4 \end{aligned} \tag{2.4.119}$$

which, combined with (2.4.115)–(2.4.119), gives (2.4.108). □

Lemma 2.4.8. *Under assumptions of Theorem* 2.1.3*, for any* $(v_0, u_0, \vec{w}_0, \theta_0) \in H^4_+$*, for any fixed* $\gamma \in (0, \gamma_4]$*, the following estimates hold for any* $t > 0$*,*

$$\begin{aligned} e^{\gamma t}\Big(&\|v(t) - \bar{v}\|^2_{H^4} + \|u(t)\|^2_{H^4} + \|\theta(t) - \bar{\theta}\|^2_{H^4} + \|v_t\|^2_{H^3} \\ &+ \|v_{tt}(t)\|^2_{H^1} + \|u_t(t)\|^2_{H^2} + \|u_{tt}(t)\|^2 + \|\theta_t(t)\|^2_{H^2} \\ &+ \|\theta_{tt}(t)\|^2 + \|\vec{w}_t(t)\|^2_{H^3} + \|\vec{w}_t(t)\|^2_{H^2} + \|\vec{w}_{tt}(t)\|^2\Big) \le C_4, \end{aligned} \tag{2.4.120}$$

$$\begin{aligned} \int_0^t e^{\gamma\tau}\Big(&\|v(t) - \bar{v}\|^2_{H^4} + \|u\|^2_{H^5} + \|\theta(t) - \bar{\theta}\|^2_{H^5} + \|\vec{w}\|^2_{H^5} + \|u_t\|^2_{H^3} \\ &+ \|u_{tt}\|^2_{H^1} + \|\vec{w}_t\|^2_{H^3} + \|\vec{w}_{tt}\|^2_{H^1} + \|\theta_t\|^2_{H^3} + \|\theta_{tt}\|^2_{H^1} \\ &+ \|v_t\|^2_{H^4} + \|v_{tt}\|^2_{H^2} + \|v_{ttt}\|^2\Big)(\tau)\,d\tau \le C_4. \end{aligned} \tag{2.4.121}$$

Proof. Using (2.1.16), Theorems 2.1.1–2.1.2 and Lemmas 2.4.1–2.4.7, we can prove the conclusion. □

Proof of Theorem 2.1.3. Exploiting Lemmas 2.4.1–2.4.8, we complete the proof of Theorem 2.1.3. □

2.5 Bibliographic Comments

In this section, we shall recall some related known results in this direction.

For the case $\vec{w} = 0$, there are many results on the global existence and asymptotic behavior of solutions to problem (2.1.16)–(2.1.17), (2.1.19)–(2.1.22) with different constitutive assumptions; we refer to Jiang [26, 27, 28, 29], Kawashima and Nishida [31], Kawohl [32], Kazhikhov [34], Kazhikhov and Shelukhin [36], Okada and Kawashima [47], Qin [49, 50, 52, 55, 54], Qin and Muñoz Rivera [70], and Wang [76]. Among these cases we would like to mention two classes of models: an ideal gas and a real viscous gas. For the former case, i.e., for the case of $\vec{w} = 0$ and an ideal gas whose constitutive relations take the following form,

$$e = C_V\theta, \quad E = C_V\theta + \frac{1}{2}(u^2 + |\vec{w}|^2), \quad p = R\frac{\theta}{v}, \tag{2.5.122}$$

with suitable positive constants C_V, R, the global existence and asymptotic behavior of smooth (generalized) solutions to the system (2.1.16), (2.1.17), (2.1.19) were established by many authors; we refer to Jiang [28, 29], Kawashima and Nishida [31], Kazhikhov [34], Kazhikhov and Shelukhin [36], Okada and Kawashima [47], Qin [49, 50, 51, 52, 55, 54, 56, 57], Qin, Wu and Liu [73] on the initial boundary value problems and the Cauchy problem. In detail, Qin [49, 50] established the existence and asymptotic behavior solutions in H^1 to (2.1.16), (2.1.17), (2.1.19)–(2.1.21) for a viscous ideal gas (2.5.122) in bounded domain in $\mathbb{R}$, for which Zheng and Qin [83] obtained the existence of maximal attractors (see also for a viscous ideal gas (2.5.122) in bounded annular domains $G_n = \{x \in \mathbb{R}^n | 0 < a < |x| < b\}$ $(n = 2,\ 3)$ in $\mathbb{R}^n$ for a viscous spherically symmetric ideal gas). For the latter case, i.e., for the case of $\vec{w} = 0$ and a real gas with the same assumptions as those in (2.1.25)–(2.1.36), Qin [55] (see also, Qin [51, 52, 54] with some stronger growth assumptions) established the existence and exponential stability of a C_0-semigroup generated by the solutions to (2.1.16), (2.1.17), (2.1.19)–(2.1.21) in the subspace of $H^i \times H^i \times H^i$ $(i = 1, 2)$ for a viscous ideal gas (2.5.122) in a bounded domain in $\mathbb{R}$.

For the case of $\vec{w} \neq 0$, an ideal flow (2.5.122) which is the special case of $q = r = 0$ of the problem (2.1.16)–(2.1.21), Qin [70] proved the exponential stability and existence of attractors; Wang [76] investigated the global existence, uniqueness, regularity in H^1. In this chapter, under more general assumptions (2.1.25)–(2.1.36) on the constitutive relations than those in [73], we establish the global existence uniqueness and asymptotic behavior of solutions in H^1 and H^2.

The novelties of this chapter consist of the following aspects: (1) the more general constitutive relations and growth assumptions (2.1.25)–(2.1.36) are studied, the related results in H^1 in this chapter have improved and extended those in [73]; (2) the global existence and exponential stability of solutions in H^1 and H^2 are established for the model under consideration; (3) the results in H^2 and $H^4$are obtained first for the model under consideration.

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