Chapter 2

Moments

Most results of this chapter are *dual* analogues of those described in Chapter 1. Indeed, the problem of representing polynomials that are positive on a set $\mathbf{K} \subset \mathbb{R}^n$ has a dual facet which is the problem of characterizing sequences of reals that are moment sequences of some finite Borel measure supported on $\mathbf{K}$. Moreover, as we shall see, this beautiful duality is nicely captured by standard duality in convex optimization, applied to some appropriate convex cones of $\mathbb{R}[\mathbf{x}]$. We review basic results in the moment problem and also particularize to some specific important cases like in Chapter 1.

Let $\overline{\mathbf{z}} \in \mathbb{C}^n$ denote the complex conjugate vector of $\mathbf{z} \in \mathbb{C}^n$. Let

$$\left\{\mathbf{z}^{\boldsymbol{\alpha}}\,\overline{\mathbf{z}}^{\boldsymbol{\beta}}\right\}_{\mathbf{z}\in\mathbb{C}^n,\,\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{N}^n}$$

be the basis of monomials for the ring $\mathbb{C}[\mathbf{z},\overline{\mathbf{z}}] = \mathbb{C}[z_1,\dots,z_n,\overline{z}_1,\dots,\overline{z}_n]$ of polynomials of the $2n$ variables $\{z_j,\overline{z}_j\}$, with coefficients in $\mathbb{C}$. Recall that for every $\mathbf{z} \in \mathbb{C}^n$, $\boldsymbol{\alpha} \in \mathbb{N}^n$, the notation $\mathbf{z}^{\boldsymbol{\alpha}}$ stands for the monomial $z_1^{\alpha_1}\cdots z_n^{\alpha_n}$ of $\mathbb{C}[\mathbf{z}]$.

The *support* of a Borel measure μ on $\mathbb{R}^n$ is a closed set, the complement in $\mathbb{R}^n$ of the largest open set $O \subset \mathbb{R}^n$ such that $\mu(O) = 0$ (and $\mathbb{C}^n$ may be identified with $\mathbb{R}^{2n}$).

Definition 2.1.

(a) *The full moment problem.* Let $(g_i)_{i=1}^m \subset \mathbb{C}[\mathbf{z},\overline{\mathbf{z}}]$ be such that each $g_i(\mathbf{z},\overline{\mathbf{z}})$ is real-valued, and let $\mathbf{K} \subset \mathbb{C}^n$ be the set defined by

$$\mathbf{K} = \left\{\mathbf{z} \in \mathbb{C}^n \;:\; g_i(\mathbf{z},\overline{\mathbf{z}}) \geq 0,\ i = 1,\dots,m\right\}.$$

Given an infinite sequence $\mathbf{y} = (y_{\boldsymbol{\alpha\beta}})$ of complex numbers, where $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{N}^n$, is $\mathbf{y}$ a $\mathbf{K}$-moment sequence, i.e., does there exist a measure μ supported on $\mathbf{K}$ such that

$$y_{\boldsymbol{\alpha\beta}} = \int_{\mathbf{K}} \mathbf{z}^{\boldsymbol{\alpha}}\,\overline{\mathbf{z}}^{\boldsymbol{\beta}}\,d\mu, \quad \forall \boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{N}^n\,? \tag{2.1}$$

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(b) *The truncated moment problem.* Given (g_i), $\mathbf{K} \subset \mathbb{C}^n$ as above, a finite subset $\Delta \subset \mathbb{N}^n \times \mathbb{N}^n$, and a finite sequence $\mathbf{y} = (y_{\boldsymbol{\alpha}\boldsymbol{\beta}})_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Delta} \subset \mathbb{C}$ of complex numbers, is $\mathbf{y}$ a $\mathbf{K}$-moment sequence, i.e., does there exist a measure μ supported on $\mathbf{K}$, such that

$$y_{\boldsymbol{\alpha}\boldsymbol{\beta}} = \int_{\mathbf{K}} \mathbf{z}^{\boldsymbol{\alpha}}\, \overline{\mathbf{z}}^{\boldsymbol{\beta}}\, d\mu, \quad \forall(\boldsymbol{\alpha},\boldsymbol{\beta}) \in \Delta? \tag{2.2}$$

Definition 2.2. In both full and truncated cases, a measure μ as in (2.1) or (2.2) is said to be a *representing* measure of the sequence $\mathbf{y}$.

If the representing measure μ is unique, then μ is said to be *determinate* (i.e., determined by its moments), and *indeterminate* otherwise.

Example 2.3. For instance, the probability measure μ on the real line $\mathbb{R}$ with density with respect to the Lebesgue measure given by

$$x \longmapsto f(x) = \begin{cases} \left(x\sqrt{2\pi}\right)^{-1} \exp\left(-\ln(x)^2/2\right) & \text{if } x > 0, \\ 0 \ \text{ otherwise,} \end{cases}$$

called the log-normal distribution, is *not* determinate. Indeed, for each a with $-1 \leq a \leq 1$, the probability measure with density

$$x \longmapsto f_a(x) = f(x)\left[1 + a\sin\left(2\pi \ln x\right)\right]$$

has exactly the same moments as μ.

The above moment problem encompasses all the classical one-dimensional moment problems of the 20th century:

(a) The Hamburger problem refers to $\mathbf{K} = \mathbb{R}$ and $(y_\alpha)_{\alpha\in\mathbb{N}} \subset \mathbb{R}$.

(b) The Stieltjes problem refers to $\mathbf{K} = \mathbb{R}_+$ and $(y_\alpha)_{\alpha\in\mathbb{N}} \subset \mathbb{R}_+$.

(c) The Hausdorff problem refers to $\mathbf{K} = [a, b]$ and $(y_\alpha)_{\alpha\in\mathbb{N}} \subset \mathbb{R}$.

(d) The Toeplitz problem refers to $\mathbf{K}$ being the unit circle in $\mathbb{C}$ and $(y_\alpha)_{\alpha\in\mathbb{Z}} \subset \mathbb{C}$.

In this book, we only consider *real* moment problems, that is, moment problems characterized with

- a set $\mathbf{K} \subset \mathbb{R}^n$, and
- a sequence $\mathbf{y} = (y_{\boldsymbol{\alpha}}) \subset \mathbb{R}$, $\boldsymbol{\alpha} \in \mathbb{N}^n$.

The multi-dimensional moment problem is significantly more difficult than the one-dimensional case, for which the results are fairly complete. This is because, in view of Theorem 2.4 below, obtaining conditions for a sequence to be moments of a representing measure with support on a given subset $\Omega \subseteq \mathbb{R}^n$, is related to characterizing polynomials that are nonnegative on Ω. When the latter characterization is available, it will translate into conditions on the sequence. But as we have seen in Chapter 1, and in contrast to the univariate case, polynomials that are nonnegative on a given set $\Omega \subseteq \mathbb{R}^n$ have no simple characterization, except for compact basic semi-algebraic sets as detailed in Section 1.5. Thus, for instance, the full multi-dimensional $\mathbf{K}$-moment problem is still unsolved for general sets $\mathbf{K} \subset \mathbb{C}^n$, including $\mathbf{K} = \mathbb{C}$.

Before we proceed further, we first state the important Riesz–Haviland theorem. Let $\mathbf{y} = (y_{\boldsymbol{\alpha}}) \subset \mathbb{R}$ be an infinite sequence, and let $L_{\mathbf{y}} \colon \mathbb{R}[\mathbf{x}] \to \mathbb{R}$ be the linear functional

$$f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} f_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \longmapsto L_{\mathbf{y}}(f) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} f_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}}. \tag{2.3}$$

Theorem 2.4 (Riesz–Haviland)**.** *Let $\mathbf{y} = (y_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathbb{N}^n} \subset \mathbb{R}$ and let $\mathbf{K} \subset \mathbb{R}^n$ be closed. There exists a finite Borel measure μ on $\mathbf{K}$ such that*

$$\int_{\mathbf{K}} \mathbf{x}^{\boldsymbol{\alpha}} \, d\mu = y_{\boldsymbol{\alpha}}, \quad \forall \boldsymbol{\alpha} \in \mathbb{N}^n, \tag{2.4}$$

if and only if $L_{\mathbf{y}}(f) \geq 0$ for all polynomials $f \in \mathbb{R}[\mathbf{x}]$ nonnegative on $\mathbf{K}$.

Note that Theorem 2.4 is not very practical, since we do not have any explicit characterization of polynomials that are nonnegative on a general closed set $\mathbf{K} \subset \mathbb{R}^n$. However, we have seen in Chapter 1 some nice representations for the subclass of compact basic semi-algebraic sets $\mathbf{K} \subset \mathbb{R}^n$. Theorem 2.4 will serve as our primary proof tool in the next sections.

2.1 One-dimensional moment problems

Given an infinite sequence $\mathbf{y} = (y_j) \subset \mathbb{R}$, we introduce the Hankel matrices $\mathbf{H}_n(\mathbf{y})$, $\mathbf{B}_n(\mathbf{y})$ and $\mathbf{C}_n(\mathbf{y}) \in \mathbb{R}^{(n+1)\times(n+1)}$, defined by

$$\mathbf{H}_n(\mathbf{y})(i,j) = y_{i+j-2}; \quad \mathbf{B}_n(\mathbf{y})(i,j) = y_{i+j-1}; \quad \mathbf{C}_n(\mathbf{y})(i,j) = y_{i+j},$$

for all $i, j \in \mathbb{N}$, with $1 \leq i, j \leq n+1$. The Hankel matrix $\mathbf{H}_n(\mathbf{y})$ is the one-dimensional (or univariate) version of what we later call a *moment* matrix in Subsection 2.2.1.

2.1.1 The full moment problem

Recall that, for any two real-valued square symmetric matrices $\mathbf{A}$, $\mathbf{B}$, the notation $\mathbf{A} \succeq \mathbf{B}$ (resp. $\mathbf{A} \succ \mathbf{B}$) stands for $\mathbf{A} - \mathbf{B}$ being positive semidefinite (resp. $\mathbf{A} - \mathbf{B}$ being positive definite).

For the full Hamburger, Stieltjes, and Hausdorff moment problems, we have:

Theorem 2.5. *Let* $\mathbf{y} = (y_j)_{j\in\mathbb{N}} \subset \mathbb{R}$. *Then:*

(a) (Hamburger) $\mathbf{y}$ *has a representing Borel measure* μ *on* $\mathbb{R}$ *if and only if the quadratic form*

$$\mathbf{x} \longmapsto s_n(\mathbf{x}) = \sum_{i,j=0}^{n} y_{i+j}x_ix_j \tag{2.5}$$

is positive semidefinite for all $n \in \mathbb{N}$. *Equivalently,* $\mathbf{H}_n(\mathbf{y}) \succeq \mathbf{0}$ *for all* $n \in \mathbb{N}$.

(b) (Stieltjes) $\mathbf{y}$ *has a representing Borel measure* μ *on* $\mathbb{R}_+$ *if and only if the quadratic forms* (2.5) *and*

$$\mathbf{x} \longmapsto u_n(\mathbf{x}) = \sum_{i,j=0}^{n} y_{i+j+1}x_ix_j \tag{2.6}$$

are positive semidefinite for $n \in \mathbb{N}$. *Equivalently,* $\mathbf{H}_n(\mathbf{y}) \succeq \mathbf{0}$ *and* $\mathbf{B}_n(\mathbf{y}) \succeq \mathbf{0}$ *for all* $n \in \mathbb{N}$.

(c) (Hausdorff) $\mathbf{y}$ *has a representing Borel measure* μ *on* $[a, b]$ *if and only if the quadratic forms* (2.5) *and*

$$\mathbf{x} \longmapsto v_n(\mathbf{x}) = \sum_{i,j=0}^{n} \left(-y_{i+j+2} + (b+a)\, y_{i+j+1} - ab\, y_{i+j}\right) x_ix_j \tag{2.7}$$

are positive semidefinite for $n \in \mathbb{N}$. *Equivalently,* $\mathbf{H}_n(\mathbf{y}) \succeq \mathbf{0}$ *and* $-\mathbf{C}_n(\mathbf{y}) + (a+b)\,\mathbf{B}_n(\mathbf{y}) - ab\,\mathbf{H}_n(\mathbf{y}) \succeq \mathbf{0}$ *for all* $n \in \mathbb{N}$.

Proof. (a) If $y_n = \int_{\mathbb{R}} z^n \, d\mu$, then

$$s_n(\mathbf{x}) = \sum_{i,j=0}^{n} x_ix_j \int_{\mathbb{R}} z^{i+j} d\mu = \int_{\mathbb{R}} \left(\sum_{i=0}^{n} x_iz^i\right)^2 d\mu \geq 0.$$

Conversely, we assume that (2.5) holds, or, equivalently, $\mathbf{H}_n(\mathbf{y}) \succeq \mathbf{0}$, for all $n \in \mathbb{N}$. Therefore, for every $\mathbf{q} \in \mathbb{R}^{n+1}$ we have $\mathbf{q}'\mathbf{H}_n(\mathbf{y})\mathbf{q} \geq 0$. Let $p \in \mathbb{R}[x]$ be nonnegative on $\mathbb{R}$, so that it is s.o.s. and can be written as $p = \sum_{j=1}^{r} q_j^2$ for some $r \in \mathbb{N}$ and some polynomials $(q_j)_{j=1}^{r} \subset \mathbb{R}[x]$. Then

$$\sum_{k=0}^{2n} p_ky_k = L_{\mathbf{y}}(p) = L_{\mathbf{y}}\left(\sum_{j=1}^{r} q_j^2\right) = \sum_{j=1}^{r} \mathbf{q}_j'\mathbf{H}_n(\mathbf{y})\mathbf{q}_j \geq 0,$$

where $\mathbf{q}_j$ is the vector of coefficients of the polynomial $q_j \in \mathbb{R}[x]$. As $p \geq 0$ was arbitrary, by Theorem 2.4, (2.4) holds with $\mathbf{K} = \mathbb{R}$.

(b) This is similar to part (a).

(c) One direction is immediate. For the converse, we assume that both (2.5) and (2.7) hold. Then, for every $\mathbf{q} \in \mathbb{R}^{n+1}$ we have

$$-\mathbf{q}'\big[\mathbf{C}_n(\mathbf{y}) + \big(a+b\big)\,\mathbf{B}_n(\mathbf{y}) - ab\,\mathbf{H}_n(\mathbf{y})\big]\,\mathbf{q} \geq 0.$$

Let $p \in \mathbb{R}[x]$ be nonnegative on $[a,b]$ of even degree $2n$ and thus, by Theorem 1.7(b), it can be written $x \mapsto p(x) = u(x) + (b-x)(x-a)q(x)$ with both polynomials u, q being s.o.s. with $\deg q \leq 2n-2$, $\deg u \leq 2n$. If $\deg p = 2n-1$ then, again by Theorem 1.7(b), p can be written $x \mapsto p(x) = v(x)(x-a) + w(x)(b-x)$ for some s.o.s. polynomials v, w of degree less than $2n-2$. But then

$$p(x) = ((x-a)p(x) + (b-x)p(x))/(b-a) = u(x) + (b-x)(x-a)q(x)$$

for some s.o.s. polynomials u, q with degree less than $2n$.

Thus, in both even and odd cases, writing $u = \sum_j u_j^2$ and $q = \sum_k q_k^2$ for some polynomials (u_j, q_k) of degree less than n, and with associated vectors of coefficients $(\mathbf{u}_j, \mathbf{q}_k) \subset \mathbb{R}^{n+1}$, one obtains

$$L_{\mathbf{y}}(p) = \sum_j \mathbf{u}_j'\mathbf{H}_n(\mathbf{y})\mathbf{u}_j + \sum_k \left(-\mathbf{q}_k'\left[\mathbf{C}_n(\mathbf{y}) + (a+b)\,\mathbf{B}_n(\mathbf{y}) - ab\,\mathbf{H}_n(\mathbf{y})\right]\mathbf{q}_k\right) \geq 0.$$

Therefore $L_{\mathbf{y}}(p) \geq 0$ for every polynomial p nonnegative on $[a,b]$. By Theorem 2.4, Eq. (2.4) holds with $\mathbf{K} = [a,b]$. ☐

Observe that Theorem 2.5 provides a criterion directly in terms of the sequence (y_n).

2.1.2 The truncated moment problem

We now state the analogue of Theorem 2.5 for the truncated moment problem for a sequence $\mathbf{y} = (y_k)_{k=0}^{2n}$ (even case), and $\mathbf{y} = (y_k)_{k=0}^{2n+1}$ (odd case). We first introduce some notation.

For an infinite sequence $\mathbf{y} = (y_j)_{j\in\mathbb{N}}$, write the Hankel moment matrix $\mathbf{H}_n(\mathbf{y})$ in the form

$$\mathbf{H}_n(\mathbf{y}) = [\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n] \tag{2.8}$$

where $(\mathbf{v}_j) \subset \mathbb{R}^{n+1}$ denote the column vectors of $\mathbf{H}_n(\mathbf{y})$. The Hankel rank of $\mathbf{y} = (y_j)_{j=0}^{2n}$, denoted by $\mathrm{rank}(\mathbf{y})$, is the smallest integer $1 \leq i \leq n$ such that $\mathbf{v}_i \in \mathrm{span}\{\mathbf{v}_0, \ldots, \mathbf{v}_{i-1}\}$. If $\mathbf{H}_n(\mathbf{y})$ is nonsingular, then $\mathrm{rank}(\mathbf{y}) = n+1$. Given an $m \times n$ matrix $\mathbf{A}$, $\mathrm{Range}(\mathbf{A})$ denotes the image space of $\mathbf{A}$, i.e., $\mathrm{Range}(\mathbf{A}) = \{\mathbf{A}\mathbf{u},\, \mathbf{u} \in \mathbb{R}^n\}$.

Theorem 2.6 (The even case). *Let $\mathbf{y} = (y_j)_{0\leq j\leq 2n} \subset \mathbb{R}$. Then:*

(a) $\mathbf{y}$ *has a representing Borel measure μ on $\mathbb{R}$ if and only if $\mathbf{H}_n(\mathbf{y}) \succeq \mathbf{0}$ and* $\mathrm{rank}\,(\mathbf{H}_n(\mathbf{y})) = \mathrm{rank}(\mathbf{y})$.

(b) $\mathbf{y}$ *has a representing Borel measure* μ *on* $\mathbb{R}_+$ *if and only if* $\mathbf{H}_n(\mathbf{y}) \succeq 0$, $\mathbf{B}_{n-1}(\mathbf{y}) \succeq \mathbf{0}$, *and the vector* $(y_{n+1}, \ldots, y_{2n})$ *is in* $\mathrm{Range}(\mathbf{B}_{n-1}(\mathbf{y}))$.

(c) $\mathbf{y}$ *has a representing Borel measure* μ *on* $[a, b]$ *if and only if* $\mathbf{H}_n(\mathbf{y}) \succeq \mathbf{0}$ *and* $(a + b)\,\mathbf{B}_{n-1}(\mathbf{y}) \succeq ab\,\mathbf{H}_{n-1}(\mathbf{y}) + \mathbf{C}_{n-1}(\mathbf{y})$.

Theorem 2.7 (The odd case). *Let* $\mathbf{y} = (y_j)_{0 \leq j \leq 2n+1} \subset \mathbb{R}$. *Then:*

(a) $\mathbf{y}$ *has a representing Borel measure* μ *on* $\mathbb{R}$ *if and only if* $\mathbf{H}_n(\mathbf{y}) \succeq \mathbf{0}$ *and* $\mathbf{v}_{n+1} \in \mathrm{Range}(\mathbf{H}_n(\mathbf{y}))$.

(b) $\mathbf{y}$ *has a representing Borel measure* μ *on* $\mathbb{R}_+$ *if and only if* $\mathbf{H}_n(\mathbf{y}) \succeq 0$, $\mathbf{B}_n(\mathbf{y}) \succeq \mathbf{0}$, *and the vector* $(y_{n+1}, \ldots, y_{2n+1})$ *is in* $\mathrm{Range}(\mathbf{H}_n(\mathbf{y}))$.

(c) $\mathbf{y}$ *has a representing Borel measure* μ *on* $[a, b]$ *if and only if* $b\,\mathbf{H}_n(\mathbf{y}) \succeq \mathbf{B}_n(\mathbf{y})$ *and* $\mathbf{B}_n(\mathbf{y}) \succeq a\,\mathbf{H}_n(\mathbf{y})$.

Example 2.8. In the univariate case $n = 1$, let $\mathbf{y} \in \mathbb{R}^5$ be the truncated sequence $\mathbf{y} = (1, 1, 1, 1, 2)$, hence with associated Hankel moment matrix

$$\mathbf{H}_2(\mathbf{y}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

One may easily check that $\mathbf{H}_2(\mathbf{y}) \succeq 0$, but it turns out that $\mathbf{y}$ has no representing Borel measure μ on the real line $\mathbb{R}$. However, observe that, for sufficiently small $\epsilon > 0$, the perturbed sequence $\mathbf{y}_\epsilon = (1, 1, 1 + \epsilon, 1, 1, 2)$ satisfies $\mathbf{H}_2(\mathbf{y}_\epsilon) \succ 0$ and so, by Theorem 2.6(a), $\mathbf{y}_\epsilon$ has a finite Borel representing measure μ_ϵ. But then, necessarily, there is no compact interval $[a, b]$ such that μ_ϵ is supported on $[a, b]$ for every $\epsilon > 0$.

In truncated moment problems, i.e., given a finite sequence $\mathbf{y} = (y_k)_{k=0}^n$, the basic issue is to find conditions under which we may extend the sequence $\mathbf{y}$ so as to be able to build up positive semidefinite moment matrices of higher orders. These higher-order moment matrices are called *positive extensions* or *flat extensions* when their rank does not increase with size. The rank and range conditions in Theorems 2.6–2.7 are such conditions.

2.2 The multi-dimensional moment problem

Most of the applications considered in later chapters of this book refer to real (and not complex) moment problems. Correspondingly, we introduce the basic concepts of moment and localizing matrices in the real case $\mathbb{R}^n$. However, these concepts also have their natural counterparts in $\mathbb{C}^n$, with the usual scalar product $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_j \overline{u}_j v_j$.

As already mentioned, the multi-dimensional case is significantly more difficult because of the lack of a nice characterization of polynomials that are nonnegative on a given subset $\Omega \subseteq \mathbb{R}^n$. Fortunately, we have seen in Section 1.5 that such a characterization exists for the important case of compact basic semi-algebraic sets.

For an integer $r \in \mathbb{N}$, let $\mathbb{N}^n_r = \{\boldsymbol{\alpha} \in \mathbb{N}^n \,:\, |\boldsymbol{\alpha}| \leq r\}$ with $|\boldsymbol{\alpha}| = \sum_{i=1}^n \alpha_i \leq r$. Recall that

$$\mathbf{v}_r(\mathbf{x}) = \big(1, x_1, x_2, \ldots x_n, x_1^2, x_1x_2, \ldots, x_1x_n, \ldots, x_1^r, \ldots, x_n^r\big)' \tag{2.9}$$

denotes the canonical basis of the real vector space $\mathbb{R}[\mathbf{x}]_r$ of real-valued polynomials of degree at most r (and let $s(r) = \binom{n+r}{n}$ denote its dimension). Then, a polynomial $p \in \mathbb{R}[\mathbf{x}]_r$ is written as

$$\mathbf{x} \longmapsto p(\mathbf{x}) = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} p_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} = \langle \mathbf{p}, \mathbf{v}_r(\mathbf{x})\rangle,$$

where $\mathbf{p} = \{p_{\boldsymbol{\alpha}}\} \in \mathbb{R}^{s(r)}$ denotes its vector of coefficients in the basis (2.9). And so we may identify $p \in \mathbb{R}[\mathbf{x}]$ with its vector of coefficients $\mathbf{p} \in \mathbb{R}^{s(r)}$.

2.2.1 Moment and localizing matrix

We next define the important notions of moment and localizing matrices.

Moment matrix

Given a $s(2r)$-sequence $\mathbf{y} = (y_{\boldsymbol{\alpha}})$, let $\mathbf{M}_r(\mathbf{y})$ be the *moment* matrix of dimension $s(r)$, with rows and columns labeled by (2.9), and constructed as follows:

$$\mathbf{M}_r(\mathbf{y})(\boldsymbol{\alpha}, \boldsymbol{\beta}) = L_{\mathbf{y}}\big(\mathbf{x}^{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\beta}}\big) = y_{\boldsymbol{\alpha}+\boldsymbol{\beta}}, \quad \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^n_r, \tag{2.10}$$

with $L_{\mathbf{y}}$ defined in (2.3). Equivalently, $\mathbf{M}_r(\mathbf{y}) = L_{\mathbf{y}}(\mathbf{v}_r(\mathbf{x})\mathbf{v}_r(\mathbf{x})')$, where the latter notation means that we apply $L_{\mathbf{y}}$ to each entry of the matrix $\mathbf{v}_r(\mathbf{x})\mathbf{v}_r(\mathbf{x})'$.

Let us consider an example with $n = r = 2$. In this case, $\mathbf{M}_2(\mathbf{y})$ becomes

$$\mathbf{M}_2(\mathbf{y}) = \left[\begin{array}{c|cc|ccc}
y_{00} & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\
\hline
y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\
y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\
\hline
y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\
y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\
y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04}
\end{array}\right].$$

In general, $\mathbf{M}_r(\mathbf{y})$ defines a bilinear form $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathbf{y}}$ on $\mathbb{R}[\mathbf{x}]_r$ as follows:

$$\big\langle p, q\big\rangle_{\mathbf{y}} = L_{\mathbf{y}}(pq) = \big\langle \mathbf{p}, \mathbf{M}_r(\mathbf{y})\mathbf{q}\big\rangle = \mathbf{p}'\mathbf{M}_r(\mathbf{y})\mathbf{q}, \quad \forall \mathbf{p}, \mathbf{q} \in \mathbb{R}^{s(r)},$$

where again $p, q \in \mathbb{R}[\mathbf{x}]_r$, and $\mathbf{p}, \mathbf{q} \in \mathbb{R}^{s(r)}$ denote their vector of coefficients.

Recall from Definition 2.2 that, if $\mathbf{y}$ is a sequence of moments for some measure μ, then μ is called a representing measure for $\mathbf{y}$, and if unique, then μ is said to be determinate, and indeterminate otherwise, whereas $\mathbf{y}$ is called a determinate (resp. indeterminate) moment sequence. In addition, for every $q \in \mathbb{R}[\mathbf{x}]$,

$$\langle \mathbf{q}, \mathbf{M}_r(\mathbf{y})\mathbf{q} \rangle = L_{\mathbf{y}}(q^2) = \int q^2 \, d\mu \geq 0, \tag{2.11}$$

so that $\mathbf{M}_r(\mathbf{y}) \succeq \mathbf{0}$. It is also immediate to check that if the polynomial q^2 is expanded as $q(\mathbf{x})^2 = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} h_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$, then

$$\langle \mathbf{q}, \mathbf{M}_r(\mathbf{y})\mathbf{q} \rangle = L_{\mathbf{y}}(q^2) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} h_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}}.$$

- Every measure with *compact* support (say $\mathbf{K} \subset \mathbb{R}^n$) is determinate, because by the Stone–Weierstrass theorem, the space of polynomials is dense (for the sup-norm) in the space of continuous functions on $\mathbf{K}$.
- Not every sequence $\mathbf{y}$ that satisfies $\mathbf{M}_i(\mathbf{y}) \succeq \mathbf{0}$ has a representing measure μ on $\mathbb{R}^n$. This is in contrast with the one-dimensional case where, by Theorem 2.5(a), a full sequence $\mathbf{y}$ such that $\mathbf{H}_n(\mathbf{y}) \succeq \mathbf{0}$ for all $n = 0, 1, \ldots$, has a representing measure. (Recall that, in the one-dimensional case, the moment matrix is just the Hankel matrix $\mathbf{H}_n(\mathbf{y})$ in (2.8).) However, we have the following useful result:

Proposition 2.9. *Let $\mathbf{y}$ be a sequence indexed in the basis $\mathbf{v}_\infty(\mathbf{x})$, which satisfies $\mathbf{M}_i(\mathbf{y}) \succeq \mathbf{0}$, for all $i = 0, 1, \ldots$.*

(a) *If the sequence $\mathbf{y}$ satisfies*

$$\sum_{k=1}^{\infty} \left[L_{\mathbf{y}}\left(x_i^{2k}\right) \right]^{-1/2k} = +\infty, \quad i = 1, \ldots, n, \tag{2.12}$$

then $\mathbf{y}$ has a determinate representing measure on $\mathbb{R}^n$.

(b) *If there exist $c, a > 0$ such that*

$$|\, y_{\boldsymbol{\alpha}} \,| \leq c\, a^{|\boldsymbol{\alpha}|}, \quad \forall \boldsymbol{\alpha} \in \mathbb{N}^n, \tag{2.13}$$

then $\mathbf{y}$ has a determinate representing measure with support contained in the box $[-a, a]^n$.

In the one-dimensional case (2.12) is called Carleman's condition. The moment matrix also has the following properties:

Proposition 2.10. *Let $d \geq 1$, and let $\mathbf{y} = (y_{\boldsymbol{\alpha}}) \subset \mathbb{R}$ be such that $M_d(\mathbf{y}) \succeq 0$. Then*

$$|\, y_{\boldsymbol{\alpha}} \,| \leq \max \left[y_0, \max_{i=1,\ldots,n} L_{\mathbf{y}}\left(x_i^{2d}\right) \right], \quad \forall \boldsymbol{\alpha} \in \mathbb{N}^n_{2d}.$$

In addition, rescaling $\mathbf{y}$ *so that* $y_0 = 1$*, and letting* $\tau_d = \max_{i=1,\ldots,n} L_{\mathbf{y}}(x_i^{2d})$*,*

$$|\, y_{\boldsymbol{\alpha}} \,|^{\frac{1}{|\boldsymbol{\alpha}|}} \leq \tau_d^{\frac{1}{2d}}, \quad \forall \boldsymbol{\alpha} \in \mathbb{N}^n_{2d}.$$

Localizing matrix

Given a polynomial $u \in \mathbb{R}[\mathbf{x}]$ with coefficient vector $\mathbf{u} = \{u_{\boldsymbol{\gamma}}\}$, we define the *localizing* matrix with respect to $\mathbf{y}$ and u to be the matrix $\mathbf{M}_r(u\,\mathbf{y})$ with rows and columns indexed by (2.9), and obtained from $\mathbf{M}_r(\mathbf{y})$ by:

$$\mathbf{M}_r(u\,\mathbf{y})(\boldsymbol{\alpha},\boldsymbol{\beta}) = L_{\mathbf{y}}\big(u(\mathbf{x})\mathbf{x}^{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\beta}}\big) = \sum_{\boldsymbol{\gamma}\in\mathbb{N}^n} u_{\boldsymbol{\gamma}} y_{\boldsymbol{\gamma}+\boldsymbol{\alpha}+\boldsymbol{\beta}}, \quad \forall \boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{N}^n_r. \tag{2.14}$$

Equivalently, $\mathbf{M}_r(u\,\mathbf{y}) = L_{\mathbf{y}}(u\,\mathbf{v}_r(\mathbf{x})\mathbf{v}_r(\mathbf{x})')$, where the previous notation means that $L_{\mathbf{y}}$ is applied entrywise. For instance, when $n = 2$, with

$$\mathbf{M}_1(\mathbf{y}) = \begin{bmatrix} y_{00} & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{bmatrix} \quad \text{and} \quad \mathbf{x} \longmapsto u(\mathbf{x}) = a - x_1^2 - x_2^2,$$

we obtain

$$\mathbf{M}_1(u\,\mathbf{y}) = \begin{bmatrix} ay_{00} - y_{20} - y_{02} & ay_{10} - y_{30} - y_{12} & ay_{01} - y_{21} - y_{03} \\ ay_{10} - y_{30} - y_{12} & ay_{20} - y_{40} - y_{22} & ay_{11} - y_{31} - y_{13} \\ ay_{01} - y_{21} - y_{03} & ay_{11} - y_{31} - y_{13} & ay_{02} - y_{22} - y_{04} \end{bmatrix}.$$

Similarly to (2.11), we have

$$\big\langle \mathbf{p}, \mathbf{M}_r(u\,\mathbf{y})\mathbf{q} \big\rangle = L_{\mathbf{y}}(u\,pq)$$

for all polynomials $p, q \in \mathbb{R}[\mathbf{x}]_r$ with coefficient vectors $\mathbf{p}, \mathbf{q} \in \mathbb{R}^{s(r)}$. In particular, if $\mathbf{y}$ has a representing measure μ, then

$$\big\langle \mathbf{q}, \mathbf{M}_r(u\,\mathbf{y})\mathbf{q} \big\rangle = L_{\mathbf{y}}\big(u\,q^2\big) = \int u\,q^2\,d\mu \tag{2.15}$$

for each polynomial $q \in \mathbb{R}[\mathbf{x}]$ with coefficient vector $\mathbf{q} \in \mathbb{R}^{s(r)}$. Hence, $\mathbf{M}_r(u\,\mathbf{y}) \succeq \mathbf{0}$ whenever μ has its support contained in the set $\{\mathbf{x} \in \mathbb{R}^n \,:\, u(\mathbf{x}) \geq 0\}$.

It is also immediate to check that if the polynomial uq^2 is expanded as $u(\mathbf{x})\,q(\mathbf{x})^2 = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} h_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}$ then

$$\big\langle \mathbf{q}, \mathbf{M}_r(u\,\mathbf{y})\mathbf{q} \big\rangle = \sum_{\boldsymbol{\alpha}\in\mathbb{N}^n} h_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}} = L_{\mathbf{y}}\big(u\,q^2\big). \tag{2.16}$$

2.2.2 Positive and flat extensions of moment matrices

We next discuss the notion of positive extension for moment matrices.

Definition 2.11. Given a finite sequence $\mathbf{y} = (y_{\boldsymbol{\alpha}})_{|\boldsymbol{\alpha}|\leq 2r}$ with $\mathbf{M}_r(\mathbf{y}) \succeq \mathbf{0}$, the *moment extension problem* is defined as follows: extend the sequence $\mathbf{y}$ with new scalars $y_{\boldsymbol{\beta}}$, $2r < |\boldsymbol{\beta}| \leq 2(r+1)$, so as to obtain a new finite sequence $(y_{\boldsymbol{\alpha}})_{|\boldsymbol{\alpha}|\leq 2(r+1)}$ such that $\mathbf{M}_{r+1}(\mathbf{y}) \succeq \mathbf{0}$.

If such an extension $\mathbf{M}_{r+1}(\mathbf{y})$ is possible, it is called a *positive extension* of $\mathbf{M}_r(\mathbf{y})$. If, in addition, rank $\mathbf{M}_{r+1}(\mathbf{y})$ = rank $\mathbf{M}_r(\mathbf{y})$, then $\mathbf{M}_{r+1}(\mathbf{y})$ is called a *flat extension* of $\mathbf{M}_r(\mathbf{y})$.

For truncated moment problems, flat extensions play an important role. We first introduce the notion of an atomic measure. An s-atomic measure is a measure with s *atoms*, that is, a linear positive combination of s Dirac measures.

Theorem 2.12 (Flat extension). *Let $\mathbf{y} = (y_{\boldsymbol{\alpha}})_{|\boldsymbol{\alpha}|\leq 2r}$. Then the sequence $\mathbf{y}$ admits a* rank $\mathbf{M}_r(\mathbf{y})$*-atomic representing measure μ on $\mathbb{R}^n$ if and only if $\mathbf{M}_r(\mathbf{y}) \succeq \mathbf{0}$ and $\mathbf{M}_r(\mathbf{y})$ admits a flat extension $\mathbf{M}_{r+1}(\mathbf{y}) \succeq 0$.*

Theorem 2.12 is useful as it provides a simple numerical means to check whether a finite sequence has a representing measure.

Example 2.13. Let μ be the measure on $\mathbb{R}$ defined to be $\mu = \delta_0 + \delta_1$, that is, μ is the sum of two Dirac measures at the points $\{0\}$ and $\{1\}$. Then

$$\mathbf{M}_1(\mathbf{y}) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{M}_2(\mathbf{y}) = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

and, obviously, rank $\mathbf{M}_2(\mathbf{y})$ = rank $\mathbf{M}_1(\mathbf{y}) = 2$.

2.3 The K-moment problem

The (real) $\mathbf{K}$-moment problem identifies those sequences $\mathbf{y}$ that are moment sequences of a measure with support contained in a set $\mathbf{K} \subset \mathbb{R}^n$.

Given m polynomials $g_i \in \mathbb{R}[\mathbf{x}]$, $i = 1, \ldots, m$, let $\mathbf{K} \subset \mathbb{R}^n$ be the basic semi-algebraic set

$$\mathbf{K} = \left\{\mathbf{x} \in \mathbb{R}^n \; : \; g_i(\mathbf{x}) \geq 0, \; i = 1, \ldots, m\right\}. \tag{2.17}$$

For notational convenience, we also define $g_0 \in \mathbb{R}[\mathbf{x}]$ to be the constant polynomial with value 1 (i.e., $g_0 = 1$).

Recall that, given a family $(g_j)_{j=1}^m \subset \mathbb{R}[\mathbf{x}]$, we denote by g_J, $J \subseteq \{1, \ldots, m\}$, the polynomial $\mathbf{x} \mapsto g_J(\mathbf{x}) = \prod_{j\in J} g_j(\mathbf{x})$. In particular, when $J = \emptyset$, $g_\emptyset = 1$.

Let $\mathbf{y} = (y_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathbb{N}^n}$ be a given infinite sequence. For every $r \in \mathbb{N}$ and every $J \subseteq \{1, \ldots, m\}$, let $\mathbf{M}_r(g_J\, \mathbf{y})$ be the localizing matrix of order r with respect to the polynomial $g_J = \prod_{j \in J} g_j$ (so that, with $J = \emptyset$, $\mathbf{M}_r(g_\emptyset\, \mathbf{y}) = \mathbf{M}_r(\mathbf{y})$ is the moment matrix (of order r) associated with $\mathbf{y}$).

As we have already seen, there is a duality between the theory of moments and the representation of positive polynomials. The following important theorem, which is the dual facet of Theorem 1.15 and Theorem 1.17, makes this statement more precise.

Theorem 2.14. *Let* $\mathbf{y} = (y_{\boldsymbol{\alpha}}) \subset \mathbb{R}$, $\boldsymbol{\alpha} \in \mathbb{N}^n$, *be a given infinite sequence in* $\mathbb{R}$, $L_{\mathbf{y}} : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$ *be the Riesz-functional introduced in* (2.3), *and let* $\mathbf{K}$ *be as in* (2.17), *assumed to be compact. Then:*

(a) *The sequence* $\mathbf{y}$ *has a finite Borel representing measure with support contained in* $\mathbf{K}$ *if and only if*

$$L_{\mathbf{y}}\big(f^2\, g_J\big) \geq 0, \quad \forall J \subseteq \big\{1, \ldots, m\big\}, \quad \forall f \in \mathbb{R}[\mathbf{x}], \tag{2.18}$$

or, equivalently, if and only if

$$\mathbf{M}_r\big(g_J\, \mathbf{y}\big) \succeq \mathbf{0}, \quad \forall J \subseteq \big\{1, \ldots, m\big\}, \quad \forall r \in \mathbb{N}. \tag{2.19}$$

(b) *Assume that there exists* $u \in \mathbb{R}[\mathbf{x}]$ *of the form*

$$u = u_0 + \sum_{j=1}^{m} u_i g_i, \quad u_i \in \Sigma[\mathbf{x}], \quad i = 0, 1, \ldots, m,$$

and such that the level set $\{\mathbf{x} \in \mathbb{R}^n \ : \ u(\mathbf{x}) \geq 0\}$ *is compact. Then* $\mathbf{y}$ *has a finite Borel representing measure with support contained in* $\mathbf{K}$ *if and only if*

$$L_{\mathbf{y}}\big(f^2\, g_j\big) \geq 0, \quad \forall j = 0, 1, \ldots, m, \quad \forall f \in \mathbb{R}[\mathbf{x}], \tag{2.20}$$

or, equivalently, if and only if

$$\mathbf{M}_r(g_j\, \mathbf{y}) \succeq \mathbf{0}, \quad \forall j = 0, 1, \ldots, m, \quad \forall r \in \mathbb{N}. \tag{2.21}$$

Proof. (a) For every $J \subseteq \{1, \ldots, m\}$ and $f \in \mathbb{R}[\mathbf{x}]_r$, the polynomial $f^2 g_J$ is nonnegative on $\mathbf{K}$. Therefore, if $\mathbf{y}$ is the sequence of moments of a measure μ supported on $\mathbf{K}$, then $\int f^2 g_J\, d\mu \geq 0$. Equivalently, $L_{\mathbf{y}}(f^2\, g_J) \geq 0$, or, in view of (2.16), $\mathbf{M}_r(g_J\, \mathbf{y}) \succeq \mathbf{0}$. Hence (2.18)–(2.19) hold.

Conversely, assume that (2.18) or equivalently (2.19) holds. As $\mathbf{K}$ is compact, it follows from Theorem 2.4 that $\mathbf{y}$ is the moment sequence of a measure with support contained in $\mathbf{K}$ if and only if $\sum_{\boldsymbol{\alpha} \in \mathbb{N}^n} f_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}} \geq 0$ for all polynomials $f \geq 0$ on $\mathbf{K}$. Let $f > 0$ on $\mathbf{K}$, so that, by Theorem 1.15,

$$f = \sum_{J \subseteq \{1, \ldots, m\}} p_J\, g_J \tag{2.22}$$

for some polynomials $\{p_J\} \subset \mathbb{R}[\mathbf{x}]$, all sums of squares. Hence, since $p_J \in \Sigma[\mathbf{x}]$, from (2.18) and from the linearity of $L_{\mathbf{y}}$ we have $L_{\mathbf{y}}(f) \geq 0$. Hence, for all polynomials $f > 0$ on $\mathbf{K}$, we have $L_{\mathbf{y}}(f) = \sum_{\boldsymbol{\alpha}} f_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}} \geq 0$. Next, let $f \in \mathbb{R}[\mathbf{x}]$ be nonnegative on $\mathbf{K}$. Then for arbitrary $\epsilon > 0$, $f + \epsilon > 0$ on $\mathbf{K}$, and thus, $L_{\mathbf{y}}(f + \epsilon) = L_{\mathbf{y}}(f) + \epsilon y_0 \geq 0$. As $\epsilon > 0$ was arbitrary, $L_{\mathbf{y}}(f) \geq 0$ follows. Therefore, $L_{\mathbf{y}}(f) \geq 0$ for all $f \in \mathbb{R}[\mathbf{x}]$ nonnegative on $\mathbf{K}$, which, by Theorem 2.4, implies that $\mathbf{y}$ is the moment sequence of some measure with support contained in $\mathbf{K}$.

(b) The proof is similar to part (a) and is left as an exercise. $\square$

Note that the conditions (2.19) and (2.21) of Theorem 2.14 are stated in terms of positive semidefiniteness of the localizing matrices associated with the polynomials g_J and g_j involved in the definition (2.17) of the compact set $\mathbf{K}$. Alternatively, we also have:

Theorem 2.15. *Let $\mathbf{y} = (y_{\boldsymbol{\alpha}}) \subset \mathbb{R}$, $\boldsymbol{\alpha} \in \mathbb{N}^n$, be an infinite sequence, $L_{\mathbf{y}} \colon \mathbb{R}[\mathbf{x}] \to \mathbb{R}$ be the Riesz-functional introduced in* (2.3)*, and let $\mathbf{K}$ be as in* (2.17)*, assumed to be compact. Let $C_G \subset \mathbb{R}[\mathbf{x}]$ be the convex cone defined in* (1.18)*, and let Assumption* 1.23 *hold. Then $\mathbf{y}$ has a representing measure μ with support contained in $\mathbf{K}$ if and only if*

$$L_{\mathbf{y}}(f) \geq 0, \quad \forall f \in C_G, \tag{2.23}$$

or, equivalently,

$$L_{\mathbf{y}}\big(\widehat{g}^{\boldsymbol{\alpha}}\,(1-\widehat{g})^{\boldsymbol{\beta}}\big) \geq 0, \quad \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^m \tag{2.24}$$

with $\widehat{g}$ as in (1.17)*.*

Proof. If $\mathbf{y}$ is the moment sequence of some measure with support contained in $\mathbf{K}$, then (2.24) follows directly from Theorem 2.4, because $\widehat{g}_j \geq 0$ and $1 - \widehat{g}_j \geq 0$ on $\mathbf{K}$, for all $j = 1, \ldots, m$.

Conversely, let (2.24) (and so (2.23)) hold, and let $f \in \mathbb{R}[\mathbf{x}]$, with $f > 0$ on $\mathbf{K}$. By Theorem 1.24, $f \in C_G$, and so f can be written as in (1.18), from which $L_{\mathbf{y}}(f) \geq 0$ follows. Finally, let $f \geq 0$ on $\mathbf{K}$, so that $f + \epsilon > 0$ on $\mathbf{K}$ for every $\epsilon > 0$. Therefore, $0 \leq L_{\mathbf{y}}(f+\epsilon) = L_{\mathbf{y}}(f) + \epsilon y_0$ because $f + \epsilon \in C_G$. As $\epsilon > 0$ was arbitrary, we obtain $L_{\mathbf{y}}(f) \geq 0$. Therefore, $L_{\mathbf{y}}(f) \geq 0$ for all $f \in \mathbb{R}[\mathbf{x}]$, nonnegative on $\mathbf{K}$, which by Theorem 2.4, implies that $\mathbf{y}$ is the moment sequence of some measure with support contained in $\mathbf{K}$. $\square$

Exactly as Theorem 2.14 was the dual facet of Theorems 1.15 and 1.17, Theorem 2.15 is the dual facet of Theorem 1.24.

Note that Eqs. (2.24) reduce to countably many *linear* conditions on the sequence $\mathbf{y}$. Indeed, for fixed $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^m$, we write

$$\widehat{g}^{\boldsymbol{\alpha}}\big(1-\widehat{g}\big)^{\boldsymbol{\beta}} = \sum_{\boldsymbol{\gamma} \in \mathbb{N}^m} q_{\boldsymbol{\gamma}}(\boldsymbol{\alpha}, \boldsymbol{\beta})\, \mathbf{x}^{\boldsymbol{\gamma}},$$

for finitely many coefficients $(q_{\boldsymbol{\gamma}}(\boldsymbol{\alpha},\boldsymbol{\beta}))$. Then, (2.24) becomes

$$\sum_{\boldsymbol{\gamma}\in\mathbb{N}^m} q_{\boldsymbol{\gamma}}\big(\boldsymbol{\alpha},\boldsymbol{\beta}\big)\, y_{\boldsymbol{\gamma}} \geq 0, \quad \forall \boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{N}^m. \tag{2.25}$$

Eq. (2.25) is to be contrasted with the positive semidefiniteness conditions (2.20) of Theorem 2.14.

In the case where all the g_j's in (2.17) are affine (so that $\mathbf{K}$ is a convex polytope), we also have a specialized version of Theorem 2.15.

Theorem 2.16. *Let $\mathbf{y} = (y_{\boldsymbol{\alpha}}) \subset \mathbb{R}$, $\boldsymbol{\alpha}\in\mathbb{N}^n$, be a given infinite sequence, and let $L_{\mathbf{y}}\colon \mathbb{R}[\mathbf{x}] \to \mathbb{R}$ be the Riesz-functional introduced in (2.3). Assume that $\mathbf{K}$ is compact with nonempty interior, and all the g_j's in (2.17) are affine, so that $\mathbf{K}$ is a convex polytope. Then $\mathbf{y}$ has a finite Borel representing measure with support contained in $\mathbf{K}$ if and only if*

$$L_{\mathbf{y}}\big(g^{\boldsymbol{\alpha}}\big) \geq 0, \quad \forall \boldsymbol{\alpha}\in\mathbb{N}^m. \tag{2.26}$$

A sufficient condition for the truncated K-moment problem

Finally, we present a very important sufficient condition for the truncated $\mathbf{K}$-moment problem. That is, we provide a condition on a finite sequence $\mathbf{y} = (y_{\boldsymbol{\alpha}})$ to admit a finite Borel representing measure supported on $\mathbf{K}$. Moreover, this condition can be checked numerically by standard techniques from linear algebra.

Theorem 2.17. *Let $\mathbf{K}\subset\mathbb{R}^n$ be the basic semi-algebraic set*

$$\mathbf{K} = \big\{\mathbf{x}\in\mathbb{R}^n \;:\; g_j(\mathbf{x}) \geq 0,\; j = 1,\dots,m\big\},$$

for some polynomials $g_j\in\mathbb{R}[\mathbf{x}]$ of degree $2v_j$ or $2v_j - 1$, for all $j = 1,\dots,m$. Let $\mathbf{y} = (y_{\boldsymbol{\alpha}})$ be a finite sequence with $|\alpha| \leq 2r$, and let $v = \max_j v_j$. Then $\mathbf{y}$ has a rank $\mathbf{M}_{r-v}(\mathbf{y})$*-atomic representing measure μ with support contained in $\mathbf{K}$ if and only if:*

(a) $\mathbf{M}_r(\mathbf{y}) \succeq \mathbf{0}$, $\mathbf{M}_{r-v}(g_j\mathbf{y}) \succeq \mathbf{0}$, $j = 1,\dots,m$, *and*
(b) rank $\mathbf{M}_r(\mathbf{y}) =$ rank $\mathbf{M}_{r-v}(\mathbf{y})$.

In addition, μ has rank $\mathbf{M}_r(\mathbf{y}) -$ rank $\mathbf{M}_{r-v}(g_i\mathbf{y})$ *atoms $\mathbf{x}\in\mathbb{R}^n$ that satisfy $g_i(\mathbf{x}) = 0$, for all $i = 1,\dots,m$.*

Note that, in Theorem 2.17, the set $\mathbf{K}$ is *not* required to be compact. The rank condition can be checked by standard techniques from numerical linear algebra. However, it is also important to remember that computing the rank is sensitive to numerical imprecisions.

2.4 Moment conditions for bounded density

Let $\mathbf{K} \subset \mathbb{R}^n$ be a Borel set with associated Borel σ-field $\mathcal{B}$, and let μ be a Borel measure on $\mathbf{K}$; so $(\mathbf{K}, \mathcal{B}, \mu)$ is a measure space. With f being a Borel measurable function on $(\mathbf{K}, \mathcal{B}, \mu)$, the notation $\|f\|_\infty$ stands for the *essential supremum* of f on $\mathbf{K}$, that is,

$$\|f\|_\infty = \inf \left\{ c \in \overline{\mathbb{R}} \,:\, \mu\{\omega : f(\omega) > c\} = 0 \right\}.$$

The (Lebesgue) space $L_\infty(\mathbf{K}, \mu)$ is the space of functions f such that $\|f\|_\infty < \infty$; it is a Banach space. And so $f \in L_\infty(\mathbf{K}, \mu)$ if f is essentially bounded, that is, bounded outside of a set of measure 0. (See, e.g., [7, p. 89].)

In this section, we consider the $\mathbf{K}$-moment problem with bounded density. That is, given a finite Borel measure μ on $\mathbf{K} \subseteq \mathbb{R}^n$ with moment sequence $\mathbf{y} = (y_{\boldsymbol{\alpha}})$, $\boldsymbol{\alpha} \in \mathbb{N}^n$, under what conditions on $\mathbf{y}$ do we have

$$y_{\boldsymbol{\alpha}} = \int_{\mathbf{K}} \mathbf{x}^{\boldsymbol{\alpha}}\, h\, d\mu, \quad \forall \boldsymbol{\alpha} \in \mathbb{N}^n, \tag{2.27}$$

for some bounded density $0 \leq h \in L_\infty(\mathbf{K}, \mu)$? (The measure μ is called the *reference* measure.)

The measure $d\nu = h\, d\mu$ is said to be *uniformly absolutely continuous* with respect to μ (denoted $\nu \ll \mu$) and h is called the Radon–Nikodym derivative of ν with respect to μ. This is a refinement of the general $\mathbf{K}$-moment problem, where one only asks for existence of some finite Borel representing measure on $\mathbf{K}$ (not necessarily with a density with respect to some reference measure μ).

Recall that, for two finite measures μ, ν on a σ-algebra $\mathcal{B}$, one has the natural partial order $\nu \leq \mu$ if and only if $\nu(B) \leq \mu(B)$ for every $B \in \mathcal{B}$; and observe that $\nu \leq \mu$ obviously implies $\nu \ll \mu$ but the converse does not hold.

2.4.1 The compact case

We first consider the case where the support of the reference measure μ is a compact basic semi-algebraic set $\mathbf{K} \subset \mathbb{R}^n$.

Theorem 2.18. *Let $\mathbf{K} \subset \mathbb{R}^n$ be compact and defined as in* (2.17)*. Let $\mathbf{z} = (z_\alpha)$ be the moment sequence of a finite Borel measure μ on $\mathbf{K}$. Then:*

(a) *A sequence $\mathbf{y} = (y_\alpha)$ has a finite Borel representing measure on $\mathbf{K}$, uniformly absolutely continuous with respect to μ, if and only if there is some scalar κ such that*

$$0 \leq L_{\mathbf{y}}\big(f^2 g_J\big) \leq \kappa\, L_{\mathbf{z}}\big(f^2 g_J\big), \quad \forall f \in \mathbb{R}[\mathbf{x}], \quad \forall J \subseteq \big\{1, \ldots, m\big\}. \tag{2.28}$$

(b) *In addition, if the polynomial $N - \|\mathbf{x}\|^2$ belongs to the quadratic module $Q(g)$ then one may replace* (2.28) *with the weaker condition*

$$0 \leq L_{\mathbf{y}}\big(f^2 g_j\big) \leq \kappa\, L_{\mathbf{z}}\big(f^2 g_j\big), \quad \forall f \in \mathbb{R}[\mathbf{x}], \quad \forall j = 0, \ldots, m \tag{2.29}$$

(*with the convention $g_0 = 1$*).

(c) *Suppose that the* g_j*'s are normalized so that*

$$0 \le g_j \le 1 \text{ on } \mathbf{K}, \quad \forall j = 1, \dots, m,$$

and that the family $(0, 1, \{g_j\})$ *generates the algebra* $\mathbb{R}[\mathbf{x}]$. *Then a sequence* $\mathbf{y} = (y_\alpha)$ *has a finite Borel representing measure on* $\mathbf{K}$, *uniformly absolutely continuous with respect to* μ, *if and only if there is some scalar* κ *such that*

$$0 \le L_{\mathbf{y}}\big(g^{\boldsymbol{\alpha}}\,(1-g)^{\boldsymbol{\beta}}\big) \le \kappa\, L_{\mathbf{z}}\big(g^{\boldsymbol{\alpha}}(1-g)^{\boldsymbol{\beta}}\big), \quad \forall \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}^m. \tag{2.30}$$

Proof. We only prove (a) as (b) and (c) can be proved with very similar arguments. The *only if* part: Let $d\nu = h\, d\mu$ for some $0 \le h \in L_\infty(\mathbf{K}, \mu)$, and let $\kappa = \|h\|_\infty$. Observe that $g_J \ge 0$ on $\mathbf{K}$ for all $J \subseteq \{1, \dots, m\}$. Therefore, for every $J \subseteq \{1, \dots, m\}$ and all $f \in \mathbb{R}[\mathbf{x}]$,

$$L_{\mathbf{y}}\big(f^2\, g_J\big) = \int_{\mathbf{K}} f^2 g_J\, d\nu = \int_{\mathbf{K}} f^2 g_J h\, d\mu \le \kappa \int_{\mathbf{K}} f^2 g_J\, d\mu = \kappa\, L_{\mathbf{z}}\big(f^2 g_J\big),$$

and so (2.28) is satisfied.

The *if* part: Let $\mathbf{y}$ and $\mathbf{z}$ be such that (2.28) holds true. Then, by Theorem 2.14, $\mathbf{y}$ has a finite Borel representing measure ν on $\mathbf{K}$. In addition, let $\boldsymbol{\gamma} = (\gamma_{\boldsymbol{\alpha}})$ with $\gamma_{\boldsymbol{\alpha}} = \kappa z_{\boldsymbol{\alpha}} - y_{\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha} \in \mathbb{N}^n$. From (2.28), one has

$$L_{\boldsymbol{\gamma}}\big(f^2 g_J\big) \ge 0, \quad \forall f \in \mathbb{R}[\mathbf{x}], \quad \forall J \subseteq \{1, \dots, m\},$$

and so, by Theorem 2.14 again, $\boldsymbol{\gamma}$ has a finite Borel representing measure ψ on $\mathbf{K}$. Moreover, from the definition of the sequence $\boldsymbol{\gamma}$,

$$\int f\, d\big(\psi + \nu\big) = \int f\, \kappa\, d\mu, \quad \forall f \in \mathbb{R}[\mathbf{x}],$$

and therefore, as measures on compact sets are moment determinate, $\psi + \nu = \kappa\mu$. Hence $\kappa\mu \ge \nu$, which shows that $\nu \ll \mu$ and so one may write $d\nu = h\, d\mu$ for some $0 \le h \in L_1(\mathbf{K}, \mu)$. From $\nu \le \kappa\mu$ one obtains

$$\int_A (h - \kappa)\, d\mu \le 0, \quad \forall A \in \mathcal{B}(\mathbf{K})$$

(where $\mathcal{B}(\mathbf{K})$ is the Borel σ-algebra associated with $\mathbf{K}$). So $0 \le h \le \kappa$, μ-almost everywhere on $\mathbf{K}$. Equivalently, $\|h\|_\infty \le \kappa$, the desired result. $\square$

Notice that, using moment and localizing matrices,

$$(2.28) \Longleftrightarrow 0 \preceq \mathbf{M}_i\big(g_J\, \mathbf{y}\big) \preceq \kappa\, \mathbf{M}_i(g_J\, \mathbf{z}), \quad i = 1, \dots; \quad J \subseteq \big\{1, \dots, m\big\};$$

$$(2.29) \Longleftrightarrow 0 \preceq \mathbf{M}_i\big(g_j\, \mathbf{y}\big) \preceq \kappa\, \mathbf{M}_i(g_j\, \mathbf{z}), \quad i = 1, \dots; \quad j = 0, 1, \dots, m.$$

2.4.2 The non-compact case

Let the reference measure μ be a finite Borel measure on $\mathbb{R}^n$, not supported on a compact set. As one wishes to find moment conditions, it is natural to consider the case where all the moments $\mathbf{z} = (z_{\boldsymbol{\alpha}})$ of μ are finite, and a simple sufficient condition is that μ satisfies the generalized Carleman condition (2.12) of Nussbaum.

Theorem 2.19. *Let $\mathbf{z} = (z_{\boldsymbol{\alpha}})$ be the moment sequence of a finite Borel measure μ on $\mathbb{R}^n$ which satisfies the generalized Carleman condition, i.e.,*

$$\sum_{k=1}^{\infty} L_{\mathbf{z}}\big(x_i^{2k}\big)^{-1/2k} = +\infty, \quad \forall i = 1, \ldots, n. \tag{2.31}$$

A sequence $\mathbf{y} = (y_{\boldsymbol{\alpha}})$ has a finite Borel representing measure ν on $\mathbb{R}^n$, uniformly absolutely continuous with respect to μ, if there exists a scalar $0 < \kappa$ such that, for all $i = 1, \ldots, n$,

$$0 \le L_{\mathbf{y}}\big(f^2\big) \le \kappa\, L_{\mathbf{z}}\big(f^2\big), \quad \forall f \in \mathbb{R}[\mathbf{x}]. \tag{2.32}$$

Proof. For every $i = 1, \ldots, n$, (2.32) with $\mathbf{x} \mapsto f(\mathbf{x}) = x_i^k$ yields

$$L_{\mathbf{y}}\big(x_i^{2k}\big)^{-1/2k} \ge \kappa^{-1/2k}\, L_{\mathbf{z}}\big(x_i^{2k}\big)^{-1/2k}, \quad \forall k = 1, \ldots,$$

and so, using (2.31), one obtains

$$\sum_{k=1}^{\infty} L_{\mathbf{y}}\big(x_i^{2k}\big)^{-1/2k} \ge \sum_{k=1}^{\infty} \kappa^{-1/2k}\, L_{\mathbf{z}}\big(x_i^{2k}\big)^{-1/2k} = +\infty,$$

for every $i = 1, \ldots, n$, i.e., the generalized Carleman condition (2.12) holds for the sequence $\mathbf{y}$. Combining this with the first inequality in (2.32) yields that $\mathbf{y}$ has a unique finite Borel representing measure ν on $\mathbb{R}^n$. It remains to prove that $\nu \ll \mu$ and its density h is in $L_\infty(\mathbb{R}^n, \mu)$.

Let $\boldsymbol{\gamma} = (\gamma_{\boldsymbol{\alpha}})$ with $\gamma_{\boldsymbol{\alpha}} = \kappa z_{\boldsymbol{\alpha}} - y_{\boldsymbol{\alpha}}$ for all $\boldsymbol{\alpha} \in \mathbb{N}^n$. Then the second inequality in (2.32) yields

$$L_{\boldsymbol{\gamma}}\big(f^2\big) \ge 0, \quad \forall f \in \mathbb{R}[\mathbf{x}]. \tag{2.33}$$

Next, observe that, from (2.32), for every $i = 1, \ldots, n$ and every $k = 0, 1, \ldots,$

$$L_{\boldsymbol{\gamma}}\big(x_i^{2k}\big) \le \kappa\, L_{\mathbf{z}}\big(x_i^{2k}\big),$$

which implies that

$$L_{\boldsymbol{\gamma}}\big(x_i^{2k}\big)^{-1/2k} \ge \kappa^{-1/2k}\, L_{\mathbf{z}}\big(x_i^{2k}\big)^{-1/2k}, \tag{2.34}$$

and so, for every $i = 1, \ldots, n$,

$$\sum_{k=1}^{\infty} L_{\boldsymbol{\gamma}}\big(x_i^{2k}\big)^{-1/2k} \ge \sum_{k=1}^{\infty} \kappa^{-1/2k}\, L_{\mathbf{z}}\big(x_i^{2k}\big)^{-1/2k} = +\infty,$$

i.e., $\boldsymbol{\gamma}$ satisfies the generalized Carleman condition. In view of (2.33), $\boldsymbol{\gamma}$ has a (unique) finite Borel representing measure ψ on $\mathbb{R}^n$. Next, from the definition of $\boldsymbol{\gamma}$, one has

$$\int f\, d(\psi + \nu) = \kappa \int f\, d\mu, \quad \forall f \in \mathbb{R}[\mathbf{x}].$$

But as μ (and so $\kappa\mu$) satisfies the Carleman condition (2.31), $\kappa\mu$ is moment determinate and therefore $\kappa\mu = \psi + \nu$.

Hence $\nu \ll \mu$ follows from $\nu \leq \kappa\mu$. Finally, writing $d\nu = h\, d\mu$ for some nonnegative $h \in L_1(\mathbb{R}^n, \mu)$, and using $\nu \leq \kappa\mu$, one obtains

$$\int_A (h - \kappa)\, d\mu \leq 0, \quad \forall A \in \mathcal{B}(\mathbb{R}^n),$$

and so $0 \leq h \leq \kappa$, μ-almost everywhere on $\mathbb{R}^n$. Equivalently, $\|h\|_\infty \leq \kappa$, the desired result. □

Observe that (2.32) is extremely simple as it is equivalent to stating that

$$\kappa\, \mathbf{M}_r(\mathbf{z}) \succeq \mathbf{M}_r(\mathbf{y}) \succeq 0, \quad \forall r = 0, 1, \ldots$$

Checking whether (2.31) holds is not easy in general. However, the condition

$$\int \exp |x_i|\, d\mu < \infty, \quad \forall i = 1, \ldots, n,$$

which is simpler to verify, ensures that (2.31) holds.

2.5 Notes and sources

Moment problems have a long and rich history. For historical remarks and details on various approaches for the moment problem, the interested reader is referred to Landau (1987) [61]. See also Akhiezer (1965) [2], Curto and Fialkow (2000) [27], and Simon (1998) [121]. Example 2.3 is from Feller (1966) [34, p. 227] whereas Example 2.8 is from Laurent (2008) [83]. Theorem 2.4 was first proved by M. Riesz for closed sets $\mathbf{K} \subset \mathbb{R}$, and subsequently generalized to closed sets $\mathbf{K} \subset \mathbb{R}^n$ by Haviland (1935, 1936) [41, 42].

2.1 Most of this section is from Curto and Fialkow (1991) [24]. Theorem 2.6(c) and Theorem 2.7 were proved by Krein and Nudel'man [55], who also gave the sufficient conditions $\mathbf{H}_n(\boldsymbol{\gamma}), \mathbf{B}_{n-1}(\boldsymbol{\gamma}) \succ \mathbf{0}$ for Theorem 2.6(b) and the sufficient condition $\mathbf{H}_n(\boldsymbol{\gamma}), \mathbf{B}_n(\boldsymbol{\gamma}) \succ 0$ for Theorem 2.7(b).

2.2 The localizing matrix was introduced in Curto and Fialkow (2000) [27] and Berg (1987) [12]. The multivariate condition in Proposition 2.9, that generalizes an earlier result of Carleman (1926) [20] in one dimension, is stated in Berg (1987) [12], and was proved by Nussbaum (1966) [95]. Proposition 2.10

is taken from Lasserre (2007) [73]. The infinite and truncated moment matrices (and in particular their kernel) have a lot of very interesting properties. For more details, the interested reader is referred to Laurent (2008) [83].

2.3 Concerning the solution of the **K**-moment problem, Theorem 2.14(a) was proved by Schmüdgen (1991) [110] with a nice interplay between real algebraic geometry and functional analysis. Indeed, the proof uses Stengle's Positivstellensatz (Theorem 1.14) and the spectral theory of self-adjoint operators in Hilbert spaces. Its refinement (b) is due to Putinar (1993) [103], and Jacobi and Prestel (2001) [48]. Incidentally, in Schmüdgen (1991) [110] the Positivstellensatz Theorem 1.15 appears as a corollary of Theorem 2.14(a). Theorem 1.25 is due to Cassier (1984) [21] and Handelman (1988) [38], and appears prior to the more general Theorem 2.15 due to Vasilescu (2003) [127]. Theorems 2.12 and 2.17 are due to Curto and Fialkow (1991, 1996, 1998) [24, 25, 26], where the results are stated for the complex plane $\mathbb{C}$, but generalize to $\mathbb{C}^n$ and $\mathbb{R}^n$. An alternative proof of some of these results can be found in Laurent (2005) [80]; for instance, Theorem 2.17 follows from Laurent (2005) [80, Theorem 5.23].

2.4 This section is from Lasserre (2006) [71]. The moment problem with bounded density was initially studied by Markov on the interval $[0, 1]$ with μ the Lebesgue measure, a refinement of the Hausdorff moment problem where one only asks for existence of some finite Borel representing measure ν on $[0, 1]$. For an interesting discussion with historical details, the interested reader is referred to Diaconis and Freedman (2006) [31], where, in particular, the authors have proposed a simplified proof as well as conditions for existence of density in $L_p([0, 1], \mu)$ with a similar flavor.

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