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Infinite Norm Decompositions of C*-algebras

F.N. Arzikulov

Abstract. In the given article the notion of infinite norm decomposition of a C*-algebra is investigated. The infinite norm decomposition is some generalization of Peirce decomposition. It is proved that the infinite norm decomposition of any C*-algebra is a C*-algebra. C*-factors with an infinite and a nonzero finite projection and simple purely infinite C*-algebras are constructed.

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Introduction

In the given article the notion of infinite norm decomposition of a C*-algebra is investigated. It is known that for any projection p of a unital C*-algebra A the next equality is valid $A = pAp \oplus pA(1-p) \oplus (1-p)Ap \oplus (1-p)A(1-p)$, where $\oplus$ is a direct sum of spaces. The infinite norm decomposition is some generalization of Peirce decomposition. First such infinite decompositions were introduced in [1] by the author.

In this article a unital C*-algebra A with an infinite orthogonal set $\{p_i\}$ of equivalent projections such that $\sup_i p_i = 1$, and the set $\sum_{ij}^{o} p_iAp_j = \{\{a_{ij}\}$: for any indexes i, j, $a_{ij} \in p_iAp_j$, and $\|\sum_{k=1,\ldots,i-1}(a_{ki} + a_{ik}) + a_{ii}\| \to 0$ at $i \to \infty\}$ are considered. Note that all infinite sets like $\{p_i\}$ are supposed to be countable.

The main results of the given article are the next:

- For any C*-algebra A with an infinite orthogonal set $\{p_i\}$ of equivalent projections such that $\sup_i p_i = 1$ the set $\sum_{ij}^{o} p_iAp_j$ is a C*-algebra with the componentwise algebraic operations, the associative multiplication and the norm.
- There exist a C*-algebra A and different countable orthogonal sets $\{e_i\}$ and $\{f_i\}$ of equivalent projections in A such that $\sup_i e_i = 1$, $\sup_i f_i = 1$, $\sum_{ij}^{o} e_iAe_j \neq \sum_{ij}^{o} f_iAf_j$.

– If A is a W*-factor of type II$_\infty$, then there exists a countable orthogonal set $\{p_i\}$ of equivalent projections in A such that $\sum_{ij}^o p_iAp_j$ is a C*-factor with a nonzero finite and an infinite projection. In this case $\sum_{ij}^o p_iAp_j$ is not a von Neumann algebra.
– If A is a W*-factor of type III, then for any countable orthogonal set $\{p_i\}$ of equivalent projections in A. The C*-subalgebra $\sum_{ij}^o p_iAp_j$ is simple and purely infinite. In this case $\sum_{ij}^o p_iAp_j$ is not a von Neumann algebra.
– There exists a C*-algebra A with an orthogonal set $\{p_i\}$ of equivalent projections such that $\sum_{ij}^o p_iAp_j$ is not a two-sided ideal of A.

1. Infinite norm decompositions

Lemma 1. *Let A be a C*-algebra, $\{p_i\}$ be an infinite orthogonal set of projections with the least upper bound 1 in the algebra A and let $\mathcal{A} = \{\{p_iap_j\} : a \in A\}$. Then,*

1) *the set $\mathcal{A}$ is a vector space with the next componentwise algebraic operations*

$$\lambda\{p_iap_j\} = \{p_i\lambda ap_j\}, \lambda \in \mathbb{C}$$

$$\{p_iap_j\} + \{p_ibp_j\} = \{p_i(a+b)p_j\}, a, b \in A,$$

2) *the algebra A and the vector space $\mathcal{A}$ can be identified in the sense of the next map*

$$\mathcal{I} : a \in A \to \{p_iap_j\} \in \mathcal{A}.$$

Proof. Item 1) of the lemma can be easily proved.

Proof of item 2): We assert that $\mathcal{I}$ is a one-to-one map. Indeed, it is clear, that for any $a \in A$ there exists a unique set $\{p_iap_j\}$, defined by the element a.

Suppose that there exist different elements a and b in A such that $p_iap_j = p_ibp_j$ for all i, j, i.e., $\mathcal{I}(a) = \mathcal{I}(b)$. Then $p_i(a-b)p_j = 0$ for all i and j. Observe that $p_i((a-b)p_j(a-b)^*) = ((a-b)p_j(a-b)^*)p_i = 0$ and $(a-b)p_j(a-b)^* \geq 0$ for all i, j. Therefore, the element $(a-b)p_j(a-b)^*$ commutes with every projection in $\{p_i\}$.

We prove $(a-b)p_j(a-b)^* = 0$. Indeed, there exists a maximal commutative $*$-subalgebra A_o of the algebra A, containing the set $\{p_i\}$ and the element $(a-b)p_j(a-b)^*$. Since $(a-b)p_j(a-b)^*p_i = p_i(a-b)p_j(a-b)^* = 0$ for any i, then the condition $(a-b)p_j(a-b)^* \neq 0$ contradicts the equality $\sup_i p_i = 1$.

Indeed, in this case $p_i \leq 1 - 1/\|(a-b)p_j(a-b)^*\|(a-b)p_j(a-b)^*$ for any i. Since by $(a-b)p_j(a-b)^* \neq 0$ we have $1 > 1 - 1/\|(a-b)p_j(a-b)^*\|(a-b)p_j(a-b)^*$, then we get a contradiction with $\sup_i p_i = 1$. Therefore $(a-b)p_j(a-b)^* = 0$.

Hence, since A is a C*-algebra, than $\|(a-b)p_j(a-b)^*\| = \|((a-b)p_j)((a-b)p_j)^*\| = \|((a-b)p_j)\|\|((a-b)p_j)^*\| = \|(a-b)p_j\|^2 = 0$ for any j. Therefore $(a-b)p_j = 0$, $p_j(a-b)^* = 0$ for any j. Analogously, we can get $p_j(a-b) = 0$, $(a-b)^*p_j = 0$ for any j. Hence the elements $a-b$, $(a-b)^*$ commute with every projection in $\{p_i\}$. Then there exists a maximal commutative $*$-subalgebra A_o of the algebra A, containing the set $\{p_i\}$ and the element $(a-b)(a-b)^*$. Since

$p_i(a-b)(a-b)^* = (a-b)(a-b)^* p_i = 0$ for any i, then the condition $(a-b)(a-b)^* \neq 0$ contradicts the equality $\sup_i p_i = 1$.

Therefore, $(a-b)(a-b)^* = 0$, $a - b = 0$, i.e., $a = b$. Thus the map $\mathcal{I}$ is one-to-one. □

Lemma 2. *Let A be a C*-algebra, $\{p_i\}$ be an infinite orthogonal set of projections with the least upper bound* 1 *in the algebra A and $a \in A$. Then, if $p_i a p_j = 0$ for all i, j, then $a = 0$.*

Proof. Let $p \in \{p_i\}$. Observe that $p_i a p_j a^* = p_i(a p_j a^*) = a p_j a^* p_i = (a p_j a^*) p_i = 0$ for all i, j and $a p_j a^* = a p_j p_j a^* = (a p_j)(p_j a^*) = (a p_j)(a p_j)^* \geq 0$. Therefore, the element $a p_j a^*$ commutes with all projections of the set $\{p_i\}$.

We prove $a p_j a^* = 0$. Indeed, there exists a maximal commutative $*$-subalgebra A_o of the algebra A, containing the set $\{p_i\}$ and the element $a p_j a^*$. Since $p_i(a p_j a^*) = (a p_j a^*) p_i = 0$ for any i, then the condition $a p_j a^* \neq 0$ contradicts the equality $\sup_i p_i = 1$ (see the proof of Lemma 1). Hence $a p_j a^* = 0$.

Hence, since A is a C*-algebra, then

$$\|a p_j a^*\| = \|(a p_j)(a p_j)^*\| = \|(a p_j)\|\|(a p_j)^*\| = \|a p_j\|^2 = 0$$

for any j. Therefore $a p_j = 0$, $p_j a^* = 0$ for any j. Analogously we have $p_j a = 0$, $a^* p_j = 0$ for any j. Hence the elements a, a^* commute with all projections of the set $\{p_i\}$. Then there exists a maximal commutative $*$-subalgebra A_o of the algebra A, containing the set $\{p_i\}$ and the element aa^*. Since $p_i aa^* = aa^* p_j = 0$ for any i, then the condition $aa^* \neq 0$ contradicts the equality $\sup_i p_i = 1$ (see the proof of Lemma 1). Hence $aa^* = 0$ and $a = 0$. □

Lemma 3. *Let A be a C*-algebra on a Hilbert space H, $\{p_i\}$ be an infinite orthogonal set of projections in A with the least upper bound* 1 *in the algebra $B(H)$ and $a \in A$. Then $a \geq 0$ if and only if for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ the inequality $pap \geq 0$ holds, where $p = \sum_{k=1}^n p_k$.*

Proof. By positivity of the operator $T : a \to bab, a \in A$ for any $b \in A$, if $a \geq 0$, then for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ the inequality $pap \geq 0$ holds.

Conversely, let $a \in A$. Suppose that for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ the inequality $pap \geq 0$ holds, where $p = \sum_{k=1}^n p_k$.

Let $a = c + id$ for some nonzero self-adjoint elements c, d in A. Then $(p_i + p_j)(c + id)(p_i + p_j) = (p_i + p_j)c(p_i + p_j) + i(p_i + p_j)d(p_i + p_j) \geq 0$ for all i, j. In this case the elements $(p_i + p_j)c(p_i + p_j)$ and $(p_i + p_j)d(p_i + p_j)$ are self-adjoint. Then $(p_i + p_j)d(p_i + p_j) = 0$ and $p_i d p_j = 0$ for all i, j. Hence by Lemma 2 we have $d = 0$. Therefore $a = c = c^* = a^*$, i.e., $a \in A_{sa}$. Hence, a is a nonzero self-adjoint element in A. Let $b_n^\alpha = \sum_{kl=1}^n p_k^\alpha a p_l^\alpha$ for all natural numbers n and finite subsets $\{p_k^\alpha\}_{k=1}^n \subset \{p_i\}$. Then the set (b_n^α) ultraweakly converges to the element a.

Indeed, we have $A \subseteq B(H)$. Let $\{q_\xi\}$ be a maximal orthogonal set of minimal projections of the algebra $B(H)$ such, that $p_i = \sup_\eta q_\eta$ for some subset $\{q_\eta\} \subset \{q_\xi\}$ for any i. For arbitrary projections q and p in $\{q_\xi\}$ there exists a number $\lambda \in \mathbb{C}$ such, that $qap = \lambda u$, where u is an isometry in $B(H)$, satisfying the conditions $q = uu^*$, $p = u^*u$. Let $q_{\xi\xi} = q_\xi$, $q_{\xi\eta}$ be such element that $q_\xi = q_{\xi\eta}q_{\xi\eta}^*$, $q_\eta = q_{\xi\eta}^* q_{\xi\eta}$ for all different ξ and η. Then, let $\{\lambda_{\xi\eta}\}$ be a set of numbers such that $q_\xi a q_\eta = \lambda_{\xi\eta} q_{\xi\eta}$ for all ξ, η. In this case, since $q_\xi aa^* q_\xi = q_\xi(\sum_\eta \lambda_{\xi\eta}\bar{\lambda}_{\xi\eta})q_\xi < \infty$ we have the quantity of nonzero numbers of the set $\{\lambda_{\xi\eta}\}_\eta$ (ξth string of the infinite-dimensional matrix $\{\lambda_{\xi\eta}\}_{\xi\eta}$) is not greater then the countable cardinal number and the sequence (λ_n^ξ) of all these nonzero numbers converges to zero. Let v_{q_ξ} be a vector of the Hilbert space H which generates the minimal projection q_ξ. Then the set $\{v_{q_\xi}\}$ forms a complete orthonormal system of the space H. Let v be an arbitrary vector of the space H and μ_ξ be a coefficient of Fourier of the vector v, corresponding to v_{q_ξ} in relative to the complete orthonormal system $\{v_{q_\xi}\}$. Then, since $\sum_\xi \mu_\xi \bar{\mu}_\xi < \infty$ we have the quantity of all nonzero elements of the set $\{\mu_\xi\}_\xi$ is not greater then the countable cardinal number and the sequence (μ_n) of all these nonzero numbers converges to zero.

Let ν_ξ be the ξth coefficient of Fourier (corresponding to v_{q_ξ}) of the vector $a(v) \in H$ in relative to the complete orthonormal system $\{v_{q_\xi}\}$. Then $\nu_\xi = \sum_\eta \lambda_{\xi\eta}\mu_\eta$ and the scalar product $< a(v), v >$ is equal to the sum $\sum_\xi \nu_\xi \mu_\xi$. Since the element $a(v)$ belongs to H we have quantity of all nonzero elements in the set $\{\nu_\xi\}_\xi$ is not greater then the countable cardinal number and the sequence (ν_n) of all these nonzero numbers converges to zero.

Let ε be an arbitrary positive number. Then, since quantity of nonzero numbers of the sets $\{\mu_\xi\}_\xi$ and $\{\nu_\xi\}_\xi$ is not greater then the countable cardinal number and $\sum_\xi \nu_\xi \bar{\nu}_\xi < \infty$, $\sum_\xi \mu_\xi \bar{\mu}_\xi < \infty$ we have there exists $\{f_k\}_{k=1}^l \subset \{p_i\}$ such that for the set of indexes $\Omega_1 = \{\xi : \exists p \in \{f_k\}_{k=1}^l, q_\xi \leq p\}$ the next equality holds

$$\left|\sum_\xi \nu_\xi \mu_\xi - \sum_{\xi\in\Omega_1} \nu_\xi \mu_\xi\right| < \varepsilon.$$

Then, since quantity of nonzero numbers of the sets $\{\mu_\xi\}_\xi$ and $\{\lambda_{\xi\eta}\}_\eta$ is not greater then the countable cardinal number, and $\sum_\eta \lambda_{\xi\eta}\bar{\lambda}_{\xi\eta} < \infty$, $\sum_\xi \mu_\xi \bar{\mu}_\xi < \infty$ we have there exists $\{e_k\}_{k=1}^m \subset \{p_i\}$ such that for the set of indexes $\Omega_2 = \{\xi : \exists p \in \{e_k\}_{k=1}^m, q_\xi \leq p\}$ the next equality holds

$$\left|\sum_\eta \lambda_{\xi\eta}\mu_\eta - \sum_{\eta\in\Omega_2} \lambda_{\xi\eta}\mu_\eta\right| < \varepsilon.$$

Hence for the finite set $\{p_k\}_{k=1}^n = \{f_k\}_{k=1}^l \cup \{e_k\}_{k=1}^m$ and the set $\Omega = \{\xi : \exists p \in \{p_k\}_{k=1}^n, q_\xi \leq p\}$ of indexes we have

$$\left|\sum_\xi \nu_\xi \mu_\xi - \sum_{\xi\in\Omega}\left(\sum_{\eta\in\Omega} \lambda_{\xi\eta}\mu_\eta\right)\mu_\xi\right| < \varepsilon.$$

At the same time, $\langle(\sum_{kl=1}^n p_k a p_l)(v), v\rangle = \sum_{\xi\in\Omega}(\sum_{\eta\in\Omega} \lambda_{\xi\eta}\mu_\eta)\mu_\xi$. Therefore,

$$\left|\langle a(v), v\rangle - \left\langle\left(\sum_{kl=1}^n p_k a p_l\right)(v), v\right\rangle\right| < \varepsilon.$$

Hence, since the vector v and the number ε are chosen arbitrarily we have the net (b_n^α) ultraweakly converges to the element a.

Now there exists a maximal orthogonal set $\{e_\xi\}$ of minimal projections of the algebra $B(H)$ of all bounded linear operators on H such that the element a and the set $\{e_\xi\}$ belong to some maximal commutative $*$-subalgebra A_o of the algebra $B(H)$. We have for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ and $e \in \{e_\xi\}$ the inequality $e(\sum_{kl=1}^n p_k a p_l)e \geq 0$ holds by the positivity of the operator $T : b \to ebe, b \in A$.

By the previous part of the proof the net $(e_\xi b_n^\alpha e_\xi)_{\alpha n}$ ultraweakly converges to the element $e_\xi a e_\xi$ for any index ξ. Then we have $e_\xi b_n^\alpha e_\xi \geq 0$ for all n and α. Therefore, the ultraweak limit $e_\xi a e_\xi$ of the net $(e_\xi b_n^\alpha e_\xi)_{\alpha n}$ is a nonnegative element. Hence $e_\xi a e_\xi \geq 0$. Therefore, since e_ξ is chosen arbitrarily we have $a \geq 0$. □

Lemma 4. *Let A be a C*-algebra on a Hilbert space H, $\{p_i\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$ and $a \in A$. Then*

$$\|a\| = \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}.$$

Proof. The inequality $-\|a\|1 \leq a \leq \|a\|1$ holds. Then $-\|a\|p \leq pap \leq \|a\|p$ for all natural numbers n and finite subsets $\{p_k^\alpha\}_{k=1}^n \subset \{p_i\}$, where $p = \sum_{k=1}^n p_k$. Therefore

$$\|a\| \geq \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}.$$

At the same time, since the finite subset $\{p_k\}_{k=1}^n$ of $\{p_i\}$ is chosen arbitrarily and by Lemma 6 we have

$$\|a\| = \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}.$$

Otherwise, if

$$\|a\| > \lambda = \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}$$

then by Lemma 3 $-\lambda 1 \leq a \leq \lambda 1$. But the last inequality is a contradiction. □

Lemma 5. *Let A be a C*-algebra on a Hilbert space H, $\{p_i\}$ be an infinite orthogonal set of projections in A with the least upper bound 1 in the algebra $B(H)$, and let $\mathcal{A} = \{\{p_i a p_j\} : a \in A\}$. Then,*

1) *the vector space $\mathcal{A}$ is a unit-order space with respect to the order $\{p_i a p_j\} \geq 0$ ($\{p_i a p_j\} \geq 0$ if for any finite subset $\{p_k\}_{k=1}^n \subset \{p_i\}$ the inequality $pap \geq 0$ holds, where $p = \sum_{k=1}^n p_k$) and the norm*

$$\|\{p_i a p_j\}\| = \sup\left\{\left\|\sum_{kl=1}^n p_k a p_l\right\| : n \in N, \{p_k\}_{k=1}^n \subseteq \{p_i\}\right\}.$$

2) *the algebra A and the unit-order space $\mathcal{A}$ can be identified as unit-order spaces in the sense of the map*

$$\mathcal{I} : a \in A \to \{p_i a p_j\} \in \mathcal{A}.$$

Proof. This lemma follows by Lemmas 1, 3 and 4. □

Remark. Observe that by Lemma 4 the order and the norm in the unit-order space $\mathcal{A} = \{\{p_i a p_j\} : a \in A\}$ can be defined as follows to: $\{p_i a p_j\} \geq 0$ if $a \geq 0$; $\|\{p_i a p_j\}\| = \|a\|$. By Lemmas 3 and 4 they are equivalent to the order and the norm, defined in Lemma 5, correspondingly.

Let A be a C*-algebra, $\{p_i\}$ be a countable orthogonal set of equivalent projections in A such that $\sup_i p_i = 1$ and

$$\sum_{ij}^{o} p_i A p_j = \left\{ \{a_{ij}\} : \text{for any indexes}\, i, j,\ a_{ij} \in p_i A p_j, \text{ and } \left\| \sum_{k=1,\ldots,i-1} (a_{ki} + a_{ik}) + a_{ii} \right\| \to 0 \,\text{at}\ i \to \infty \right\}.$$

If we introduce a componentwise algebraic operations in this set then $\sum_{ij}^{o} p_i A p_j$ becomes a vector space. Also, note that $\sum_{ij}^{o} p_i A p_j$ is a vector subspace of $\mathcal{A}$. Observe that $\sum_{ij}^{o} p_i A p_j$ is a normed subspace of the algebra $\mathcal{A}$ and $\| \sum_{i,j=1}^{n} a_{ij} - \sum_{i,j=1}^{n+1} a_{ij} \| \to 0$ at $n \to \infty$ for any $\{a_{ij}\} \in \sum_{ij}^{o} p_i A p_j$.

Let $\sum_{ij}^{o} a_{ij} := \lim_{n\to\infty} \sum_{i,j=1}^{n} a_{ij}$ for any $\{a_{ij}\} \in \sum_{ij}^{o} p_i A p_j$ and

$$C^*(\{p_i A p_j\}_{ij}) := \left\{ \sum_{ij}^{o} a_{ij} : \{a_{ij}\} \in \sum_{ij}^{o} p_i A p_j \right\}.$$

Then $C^*(\{p_i A p_j\}_{ij}) \subseteq A$. By Lemma 5 A and $\mathcal{A}$ can be identified. We observe that, the normed spaces $\sum_{ij}^{o} p_i A p_j$ and $C^*(\{p_i A p_j\}_{ij})$ can also be identified. Further, without loss of generality we will use these identifications.

Theorem 6. *Let A be a unital C^*-algebra, $\{p_i\}$ be a countable orthogonal set of equivalent projections in A and $\sup_i p_i = 1$. Then $\sum_{ij}^{o} p_i A p_j$ is a C^*-subalgebra of A with the componentwise algebraic operations, the associative multiplication and the norm.*

Proof. We have $\sum_{ij}^{o} p_i A p_j$ is a normed subspace of the algebra A.

Let (a_n) be a sequence of elements in $\sum_{ij}^{o} p_i A p_j$ such that (a_n) norm converges to some element $a \in A$. We have $p_i a_n p_j \to p_i a p_j$ at $n \to \infty$ for all i and j. Hence $p_i a p_j \in p_i A p_j$ for all i, j. Let $b^n = \sum_{k=1}^{n} (p_{n-1} a p_k + p_k a p_{n-1}) + p_n a p_n$ and $c_m^n = \sum_{k=1}^{n} (p_{n-1} a_m p_k + p_k a_m p_{n-1}) + p_n a_m p_n$ for any n. Then $c_m^n \to b^n$ at $m \to \infty$. It should be proven that $\|b_n\| \to 0$ at $n \to \infty$.

Let $\varepsilon \in \mathbb{R}_+$. Then there exists m_o such that $\|a - a_m\| < \varepsilon$ for any $m > m_o$. Also for all n and $\{p_k\}_{k=1}^{n} \subset \{p_i\}$ $\|(\sum_{k=1}^{n} p_k)(a - a_m)(\sum_{k=1}^{n} p_k)\| < \varepsilon$. Hence $\|b^n - c_m^n\| < 2\varepsilon$ for any $m > m_o$. At the same time, $\|b^n - c_{m_1}^n\| < 2\varepsilon$, $\|b^n - c_{m_2}^n\| < 2\varepsilon$

for all $m_o < m_1$, m_2. Since $(a_n) \subset \sum_{ij}^o p_iAp_j$ then for any m $\|c_m^n\| \to 0$ at $n \to \infty$. Hence, since $\|c_{m_1}^n\| \to 0$ and $\|c_{m_2}^n\| \to 0$ at $n \to \infty$ we have there exists n_o such that $\|c_{m_1}^n\| < \varepsilon$, $\|c_{m_2}^n\| < \varepsilon$ and $\|c_{m_1}^n + c_{m_2}^n\| < 2\varepsilon$ for any $n > n_o$. Then $\|2b_n\| = \|b^n - c_{m_1}^n + c_{m_1}^n + c_{m_2}^n + b^n - c_{m_2}^n\| \leq \|b^n - c_{m_1}^n\| + \|c_{m_1}^n + c_{m_2}^n\| + \|b^n - c_{m_2}^n\| < 2\varepsilon + 2\varepsilon + 2\varepsilon = 6\varepsilon$ for any $n > n_o$, i.e., $\|b_n\| < 3\varepsilon$ for any $n > n_o$. Since ε is chosen arbitrarily we have $\|b_n\| \to 0$ at $n \to \infty$. Therefore $a \in \sum_{ij}^o p_iAp_j$. Since the sequence (a_n) is chosen arbitrarily we have $\sum_{ij}^o p_iAp_j$ is a Banach space.

Let $\{a_{ij}\}$, $\{b_{ij}\}$ be arbitrary elements of the Banach space $\sum_{ij}^o p_iAp_j$. Let $a_m = \sum_{kl=1}^m a_{kl}$, $b_m = \sum_{kl=1}^m b_{kl}$ for all natural numbers m. We have the sequence (a_m) converges to $\{a_{ij}\}$ and the sequence (b_m) converges to $\{b_{ij}\}$ in $\sum_{ij}^o p_iAp_j$. Also for all n and m $a_mb_n \in \sum_{ij}^o p_iAp_j$. Then for any n the sequence (a_mb_n) converges to $\{a_{ij}\}b_n$ at $m \to \infty$. Hence $\{a_{ij}\}b_n \in \sum_{ij}^o p_iAp_j$. Note that $\sum_{ij}^o p_iAp_j \subseteq A$. Therefore for any $\varepsilon \in \mathbb{R}_+$ there exists n_o such that $\|\{a_{ij}\}b_{n+1} - \{a_{ij}\}b_n\| \leq \|\{a_{ij}\}\|\|b_{n+1} - b_n\| \leq \varepsilon$ for any $n > n_o$. Hence the sequence $(\{a_{ij}\}b_n)$ converges to $\{a_{ij}\}\{b_{ij}\}$ at $n \to \infty$. Since $\sum_{ij}^o p_iAp_j$ is a Banach space then $\{a_{ij}\}\{b_{ij}\} \in \sum_{ij}^o p_iAp_j$. Since $\sum_{ij}^o p_iAp_j \subseteq A$ we have $\sum_{ij}^o p_iAp_j$ is a C*-algebra. □

Let H be an infinite-dimensional Hilbert space, $B(H)$ be the algebra of all bounded linear operators. Let $\{p_i\}$ be a countable orthogonal set of equivalent projections in $B(H)$ and $\sup_i p_i = 1$. Let $\{\{p_j^i\}_j\}_i$ be the set of infinite subsets of $\{p_i\}$ such that for all distinct ξ and η $\{p_j^\xi\}_j \cap \{p_j^\eta\}_j = \oslash$, $|\{p_j^\xi\}_j| = |\{p_j^\eta\}_j|$ and $\{p_i\} = \cup_i\{p_j^i\}_j$. Then let $q_i = \sup_j p_j^i$ in $B(H)$ for all i. Then $\sup_i q_i = 1$ and $\{q_i\}$ be a countable orthogonal set of equivalent projections. Then we say that the countable orthogonal set $\{q_i\}$ of equivalent projections *is defined by the set* $\{p_i\}$ in $B(H)$. We have the next corollary.

Corollary 7. *Let A be a unital C*-algebra on a Hilbert space H, $\{p_i\}$ be a countable orthogonal set of equivalent projections in A and* $\sup_i p_i = 1$. *Let $\{q_i\}$ be a countable orthogonal set of equivalent projections in $B(H)$ defined by the set $\{p_i\}$ in $B(H)$. Then $\sum_{ij}^o q_iAq_j$ is a C*-subalgebra of the algebra A.*

Proof. Let $\{\{p_j^i\}_j\}_i$ be the set of infinite subsets of $\{p_i\}$ such that for all distinct ξ and η $\{p_j^\xi\}_j \cap \{p_j^\eta\}_j = \oslash$, $|\{p_j^\xi\}_j| = |\{p_j^\eta\}_j|$ and $\{p_i\} = \cup_i\{p_j^i\}_j$. Then let $q_i = \sup_j p_j^i$ in $B(H)$ for all i. Then we have for all i and j $q_iAq_j = \{\{p_\xi^i a p_\eta^j\}_{\xi\eta} : a \in A\}$. Hence $q_iAq_j \subset A$ for all i and j.

The rest part of the proof is the repeating of the proof of Theorem 6. □

Example. 1. Let $\mathcal{M}$ be the closure on the norm of the inductive limit $\mathcal{M}_o$ of the inductive system

$$C \to M_2(C) \to M_3(C) \to M_4(C) \to \cdots ,$$

where $M_n(C)$ is mapped into the upper left corner of $M_{n+1}(C)$. Then $\mathcal{M}$ is a C*-algebra ([1]). The algebra $\mathcal{M}$ contains the minimal projections of the form e_{ii}, where e_{ij} is an infinite-dimensional matrix, whose (i,i)th component is 1 and the rest components are zeros. These projections form the countable orthogonal set $\{e_{ii}\}_{i=1}^{\infty}$ of minimal projections. Let

$$M_n^o(\mathbb{C}) = \Big\{ \sum_{ij} \lambda_{ij} e_{ij} : \lambda_{ij} \in \mathbb{C}\, \text{for any indexes}\, i, j\, \text{and} \\ \Big\| \sum_{k=1,\dots,i-1} (\lambda_{ki} e_{ki} + \lambda_{ik} e_{ik}) + \lambda_{ii} e_{ii} \Big\| \to 0\, \text{at}\; i \to \infty \Big\}.$$

Then $\mathbb{C} \cdot 1 + M_n^o(\mathbb{C}) = \mathcal{M}$ (see [2]) and by Theorem 6 $M_n^o(\mathbb{C})$ is a simple C*-algebra. Note that there exists a mistake in the formulation of Theorem 3 in [2]. $\mathbb{C} \cdot 1 + M_n^o(\mathbb{C})$ is a C*-algebra. But the algebra $\mathbb{C} \cdot 1 + M_n^o(\mathbb{C})$ is not simple. Because $\mathbb{C} \cdot 1 + M_n^o(\mathbb{C}) \neq M_n^o(\mathbb{C})$ and $M_n^o(\mathbb{C})$ is an ideal of the algebra $\mathbb{C} \cdot 1 + M_n^o(\mathbb{C})$, i.e., $[\mathbb{C} \cdot 1 + M_n^o(\mathbb{C})] \cdot M_n^o(\mathbb{C}) \subseteq M_n^o(\mathbb{C})$.

2. There exist a C*-algebra A and different countable orthogonal sets $\{e_i\}$ and $\{f_i\}$ of equivalent projections in A such that $\sup_i e_i = 1$, $\sup_i f_i = 1$, $\sum_{ij}^o e_i A e_j \neq \sum_{ij}^o f_i A f_j$. Indeed, let H be an infinite-dimensional Hilbert space, $B(H)$ be the algebra of all bounded linear operators. Let $\{p_i\}$ be a countable orthogonal set of equivalent projections in $B(H)$ and $\sup_i p_i = 1$. Then $\sum_{ij}^o p_i B(H) p_j \subset B(H)$. Let $\{\{p_j^i\}_j\}_i$ be the set of infinite subsets of $\{p_i\}$ such that for all distinct ξ and η $\{p_j^\xi\}_j \cap \{p_j^\eta\}_j = \oslash$, $|\{p_j^\xi\}_j| = |\{p_j^\eta\}_j|$ and $\{p_i\} = \cup_i \{p_j^i\}_j$. Then let $q_i = \sup_j p_j^i$ for all i. Then $\sup_i q_i = 1$ and $\{q_i\}$ be a countable orthogonal set of equivalent projections. We assert that $\sum_{ij}^o p_i B(H) p_j \neq \sum_{ij}^o q_i B(H) q_j$. Indeed, let $\{x_{ij}\}$ be a set of matrix units constructed by the infinite set $\{p_j^1\}_j \in \{\{p_j^i\}_j\}_i$, i.e., for all i, j, $x_{ij} x_{ij}^* = p_i^1$, $x_{ij}^* x_{ij} = p_j^1$, $x_{ii} = p_i^1$. Then the von Neumann algebra $\mathcal{N}$ generated by the set $\{x_{ij}\}$ is isometrically isomorphic to $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. We note that $\mathcal{N}$ is not a subset of $\sum_{ij}^o p_i B(H) p_j$. At the same time, $\mathcal{N} \subseteq \sum_{ij}^o q_i B(H) q_j$ and $\sum_{ij}^o p_i^1 \mathcal{N} p_j^1 \subseteq \sum_{ij}^o p_i B(H) p_j$.

Theorem 8. *Let A be a unital simple C*-algebra on a Hilbert space H, $\{p_i\}$ be a countable orthogonal set of equivalent projections in A and $\sup_i p_i = 1$. Let $\{q_i\}$ be a countable orthogonal set of equivalent projections in $B(H)$ defined by the set $\{p_i\}$ in $B(H)$. Then $\sum_{ij}^o q_i A q_j$ is a simple C*-algebra.*

Proof. By Theorem 6 $\sum_{ij}^o p_i A p_j$ is a C*-algebra. Let $\{\{p_j^i\}_j\}_i$ be the set of infinite subsets of $\{p_i\}$ such that for all distinct ξ and η $\{p_j^\xi\}_j \cap \{p_j^\eta\}_j = \oslash$, $|\{p_j^\xi\}_j| = |\{p_j^\eta\}_j|$ and $\{p_i\} = \cup_i \{p_j^i\}_j$. Then let $q_i = \sup_j p_j^i$ in $B(H)$, for all i. Then we have $q_i A q_j = \{\{p_\xi^i a p_\xi^j\} : a \in A\}$ for all i and j. Hence $q_i A q_j \subset A$ for all i and j. By Corollary 7 $\sum_{ij}^o q_i A q_j$ is a C*-algebra.

Since projections of the set $\{p_i\}$ are pairwise equivalent we have the projection q_i is equivalent to $1 \in A$ for any i. Hence $q_i A q_i \cong A$ and $q_i A q_i$ is a simple C*-algebra for any i.

Let q be an arbitrary projection in $\{q_i\}$. Then qAq is a C*-subalgebra of $\sum_{ij}^{o} q_iAq_j$. Let I be a closed two-sided ideal of the algebra $\sum_{ij}^{o} q_iAq_j$. Then $IqAq \subset I$ and $Iq \cdot qAq \subset Iq$. Therefore $qIqqAq \subseteq qIq$, that is qIq is a closed two-sided ideal of the subalgebra qAq. Since qAq is simple then $qIq = qAq$.

Let q_1, q_2 be arbitrary projections in $\{q_i\}$. We assert that $q_1Iq_2 = q_1Aq_2$ and $q_2Iq_1 = q_2Aq_1$. Indeed, we have the projection $q_1 + q_2$ is equivalent to $1 \in A$. Let $e = q_1 + q_2$. Then $eAe \cong A$ and eAe is a simple C*-algebra. At the same time we have eAe is a subalgebra of $\sum_{ij}^{o} q_iAq_j$ and I is a two-sided ideal of $\sum_{ij}^{o} q_iAq_j$. Hence $IeAe \subset I$ and $Ie \cdot eAe \subset Ie$. Therefore $eIeeAe \subseteq eIe$, that is eIe is a closed two-sided ideal of the subalgebra eAe. Since eAe is simple then $eIe = eAe$. Hence $q_1Iq_2 = q_1Aq_2$ and $q_2Iq_1 = q_2Aq_1$. Therefore $q_iIq_j = q_iAq_j$ for all i and j. We have I is norm closed. Hence $I = \sum_{ij}^{o} q_iAq_j$, i.e., $\sum_{ij}^{o} q_iAq_j$ is a simple C*-algebra. □

2. Applications

Definition. A C*-algebra is called a C*-factor, if it does not have nonzero proper two-sided ideals I and J such that $I \cdot J = \{0\}$, where $I \cdot J = \{ab : a \in I, b \in J\}$.

Theorem 9. *Let $\mathcal{N}$ be a W*-factor of type II_∞ on a Hilbert space H, $\{p_i\}$ be a countable orthogonal set of equivalent projections in $\mathcal{N}$ and $\sup_i p_i = 1$. Then for any countable orthogonal set $\{q_i\}$ of equivalent projections in $B(H)$ defined by the set $\{p_i\}$ in $B(H)$ the C*-algebra $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is a C*-factor with a nonzero finite and an infinite projection. In this case $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is not a von Neumann algebra.*

Proof. By the definition of the set $\{q_i\}$ we have $\sup_i q_i = 1$ and $\{q_i\}$ be a countable orthogonal set of equivalent *infinite* projections. By Theorem 6 we have $\sum_{ij}^{o} q_i\mathcal{N}p_j$ is a C*-subalgebra of $\mathcal{N}$. Let q be a nonzero finite projection of $\mathcal{N}$. Then there exists a projection $p \in \{q_i\}$ such that $qp \neq 0$. We have $q\mathcal{N}q$ is a finite von Neumann algebra. Let $x = pq$. Then $x\mathcal{N}x^*$ is a weakly closed C*-subalgebra. Note that the algebra $x\mathcal{N}x^*$ has a center-valued faithful trace. Let e be a nonzero projection of the algebra $x\mathcal{N}x^*$. Then $ep = e$ and $e \in p\mathcal{N}p$. Hence $e \in \sum_{ij}^{o} q_i\mathcal{N}q_j$. We have the weak closure of $\sum_{ij}^{o} q_i\mathcal{N}q_j$ in the algebra $\mathcal{N}$ coincides with this algebra $\mathcal{N}$. Then by the weak continuity of the multiplication $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is a C*-factor. Note since $1 \notin \sum_{ij}^{o} q_i\mathcal{N}q_j$ then $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is not weakly closed in $\mathcal{N}$. Hence the C*-factor $\sum_{ij}^{o} q_i\mathcal{N}q_j$ is not a von Neumann algebra. □

Remark. Note that, in the article [3] a simple C*-algebra with an infinite and a nonzero finite projection have been constructed by M.Rørdam. In the next corollary we construct a simple purely infinite C*-algebra. Note that simple purely infinite C*-algebras are considered and investigated, in particular, in [4] and [5].

Theorem 10. *Let $\mathcal{N}$ be a W*-factor of type III on a Hilbert space H. Then for any countable orthogonal set $\{p_i\}$ of equivalent projections in $\mathcal{N}$ such that $\sup_i p_i = 1$, $\sum_{ij}^{o} p_i\mathcal{N}p_j$ is a simple purely infinite C*-algebra. In this case $\sum_{ij}^{o} p_i\mathcal{N}p_j$ is not a von Neumann algebra.*

Proof. Let p_{i_o} be a projection in $\{p_i\}$. We have the projection p_{i_o} can be exhibited as a least upper bound of a countable orthogonal set $\{p^j_{i_o}\}_j$ of equivalent projections in $\mathcal{N}$. Then for any i the projection p_i has a countable orthogonal set $\{p^j_i\}_j$ of equivalent projections in $\mathcal{N}$ such that the set $\bigcup_i \{p^j_i\}_j$ is a countable orthogonal set of equivalent projections in $\mathcal{N}$. Hence the set $\{p_i\}$ is defined by the set $\bigcup_i \{p^j_i\}_j$ in $B(H)$ (in $\mathcal{N}$). Hence by Theorem 8 $\sum^o_{ij} p_i \mathcal{N} p_j$ is a simple C*-algebra. Note, since $1 \notin \sum^o_{ij} p_i \mathcal{N} p_j$ we have $\sum^o_{ij} p_i \mathcal{N} p_j$ is not weakly closed in $\mathcal{N}$. Hence $\sum^o_{ij} p_i \mathcal{N} p_j$ is not a von Neumann algebra.

Suppose there exists a nonzero finite projection q in $\sum^o_{ij} p_i \mathcal{N} p_j$. Then there exists a projection $p \in \{p_i\}$ such that $qp \neq 0$. We have $q(\sum^o_{ij} p_i \mathcal{N} p_j)q$ is a finite C*-algebra. Let $x = pq$. Then $x\mathcal{N}x^*$ is a C*-subalgebra. Moreover $x\mathcal{N}x^*$ is weakly closed and $x\mathcal{N}x^* \subset p\mathcal{N}p$. Hence $x\mathcal{N}x^*$ has a center-valued faithful trace. Then $x\mathcal{N}x^*$ is a finite von Neumann algebra with a center-valued faithful normal trace. Let e be a nonzero projection of the algebra $x\mathcal{N}x^*$. Then $ep = e$ and $e \in p\mathcal{N}p$. Hence $e \in \mathcal{N}$. This is a contradiction. □

Example. Let H be a separable Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. Let $\{q_i\}$ be a maximal orthogonal set of equivalent minimal projections in $B(H)$. Then $\sum^o_{ij} q_i B(H) q_j$ is a two-sided closed ideal of the algebra $B(H)$. Using the set $\{q_i\}$ we construct a countable orthogonal set $\{p_i\}$ of equivalent infinite projections such that $\sup_i p_i = 1$. Let $\{\{q^i_j\}_j\}_i$ be the countable set of countable subsets of $\{q_i\}$ such that for all distinct i_1 and i_2 $\{q^{i_1}_j\}_j \cap \{p^{i_2}_j\}_j = \oslash$ and $\{q_i\} = \cup_i \{q^i_j\}_j$. Then let $p_i = \sup_j q^i_j$ for all i. Then $\sup_i p_i = 1$ and $\{p_i\}$ is a countable orthogonal set of equivalent infinite projections in $B(H)$ defined by $\{q_i\}$ in $B(H)$.

Let $\{q^{ij}_{nm}\}$ be the set of matrix units constructed by the set $\{\{q^i_j\}_j\}_i$, i.e., $q^{ij}_{nm} {q^{ij}_{nm}}^* = q^i_n$, ${q^{ij}_{nm}}^* q^{ij}_{nm} = q^j_m$, $q^{ii}_{nn} = q^i_n$ for all i, j,n,m. Then let $a = \{a^{ij}_{nm} q^{ij}_{nm}\}$ be the decomposition of the element $a \in B(H)$, where the components a^{ij}_{nm} are defined as follows

$$a^{11}_{11} = \lambda, a^{21}_{12} = \lambda, a^{31}_{13} = \lambda, \ldots, a^{n1}_{1n} = \lambda, \ldots,$$

and the rest components a^{ij}_{nm} are zero, i.e., $a^{ij}_{nm} = 0$. Then $p_1 a = a$. Then, since $a \notin \sum^o_{ij} p_i B(H) p_j$ and $p_1 \in \sum^o_{ij} p_i B(H) p_j$ we have $\sum^o_{ij} p_i B(H) p_j$ is not a two-sided ideal of $B(H)$. But by theorem 6 $\sum^o_{ij} p_i B(H) p_j$ is a C*-algebra. Hence there exists a C*-algebra A with an orthogonal set $\{p_i\}$ of equivalent projections such that $\sum^o_{ij} p_i A p_j$ is not a two-sided ideal of A.

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F.N. Arzikulov
Institute of Mathematics and Information Technologies
of the National Academy of Sciences of Uzbekistan
700143 Tashkent, Usbekistan
e-mail: arzikulovfn@rambler.ru

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