# Chapter 2

# Review of Chains and Cochains

## 2.1 Cell complexes and orientations

Recall (e.g., [Hud]) that a *closed convex linear cell* is the convex hull of finitely many points in Euclidean space. A *convex linear cell complex* $K$ is a finite collection of closed convex linear cells in some $\mathbb{R}^N$ such that if $\sigma \in K$ then every face of $\sigma$ is in $K$, and if $\sigma, \tau \in K$ then the intersection $\sigma \cap \tau$ is in $K$. The underlying closed subset of Euclidean space is denoted $|K|$. Such a complex is a *regular* cell complex, meaning that each (closed) cell is homeomorphic to a closed ball: no identifications occur on its boundary. If $\tau \in K$ is a face of $\sigma \in K$ we write $\tau < \sigma$. A *finite simplicial complex* is a convex linear cell complex, all of whose cells are simplices. Every cell complex admits a simplicial refinement with no extra vertices.

An *orientation* of a finite-dimensional real vector space $V$ is a choice of ordered basis, two being considered equivalent if one can be continuously deformed to the other, through ordered bases. Every (finite-dimensional real) vector space has two orientations. An orientation of a convex linear cell is an orientation of the real affine space that it spans. An orientation of a smooth manifold is a continuously varying choice of orientation of each of its tangent spaces.

Let $K$ be a convex linear cell complex and let $L$ be a (closed) subcomplex. Let $X = |K|$ and let $Y = |K| - |L|$. Although $Y$ is not a union of cells, it is a union of interiors of cells. We refer to this decomposition of $Y$ as a *pseudo cell decomposition* (or a *pseudo-triangulation* if $K$ is a simplicial complex).

Every cell $\sigma \in K$ has two orientations. A choice of orientation for $\sigma$ determines a unique orientation for each codimension 1 face $\tau < \sigma$ such that the orientation of $\tau$ followed by the inward pointing vector $\overrightarrow{\tau\sigma}$ agrees with the orientation of $\sigma$. The complex $K$ is *purely $d$-dimensional* if every cell is the face of some $d$-dimensional cell and there are no cells of dimension greater than $d$.

A *oriented cellular pseudomanifold* is a convex linear cell complex $K$, purely of some dimension $d$, such that every $d-1$-dimensional cell is a face of exactly two $d$-dimensional cells; together with a choice of orientation of each $d$-dimensional cell such that the induced orientations cancel on every $d-1$-dimensional cell.

J. Getz and M. Goresky, *Hilbert Modular Forms with Coefficients in Intersection Homology and Quadratic Base Change*, Progress in Mathematics 298,
DOI 10.1007/978-3-0348-0351-9_2, © Springer Basel 2012

## 2.2 Subanalytic sets and stratifications

Let F = semi-algebraic, semi-analytic, or subanalytic. Any finite union, intersection, or difference of F-subsets of $\mathbb{R}^N$ is again an F-subset of $\mathbb{R}^N$. The closure and the interior of any F-subset of $\mathbb{R}^N$ is again an F-subset of $\mathbb{R}^N$. The image of an F-subset $X \subset \mathbb{R}^N$ by an F-mapping $f : \mathbb{R}^N \to \mathbb{R}^k$ is again an F-set for F= semi-algebraic or subanalytic (but this last statement is false for F= semi-analytic).

Let $X \subset \mathbb{R}^N$ be a set of type F. A *F-Whitney stratification* of $X$ is a locally finite decomposition $X = \bigcup_\alpha X_\alpha$ into disjoint real analytic manifolds or *strata*, such that

- the closure $\overline{X_\alpha}$ of $X_\alpha$ is an F-subset of $\mathbb{R}^N$
- if $X_\alpha \cap \overline{X_\beta} \neq \phi$ then $X_\alpha \subset \overline{X_\beta}$. and the pair $(X_\alpha, X_\beta)$ satisfies Whitney's conditions A and B.

An $F$-stratified space is such a set $X$ together with an $F$-Whitney stratification. Every closed subset $X \subset \mathbb{R}^N$ of type $F$ admits an $F$-Whitney stratification.

A Whitney stratification of a closed $F$-set $X$ implies that the local topological type of $X$ is locally constant along each stratum $S$ in the following sense. Without loss of generality we may assume that $S$ is connected. R. Thom [Th] and J. Mather [Ma1] proved the following:

**Theorem 2.1.** *There exists a compact $F$-Whitney stratified space $\ell$ (the* link of the stratum $S$) *such that every point $x \in S$ has a neighborhood basis in $X$ consisting of neighborhoods $N_x$ homeomorphic to $\mathbb{R}^s \times cone(\ell)$ by a stratum-preserving homeomorphism that is smooth on each stratum and takes $\mathbb{R}^s \times \{pt\}$ to $N_x \cap S$.* □

(Here, $s = \dim(S)$ and $\{pt\}$ denotes the cone point.) Such a neighborhood is called a *basic neighborhood*. The Thom-Mather theorem implies, in particular, that the local homology $H_i(X, X - x; \mathbb{Z})$ of $X$ is finitely generated at every point $x \in X$. An *F-triangulation* of a closed set $X \subset \mathbb{R}^N$ of type F is a locally finite simplicial complex $K$ with $|K| \subset \mathbb{R}^N$ together with an F-isomorphism $f : \mathbb{R}^N \to \mathbb{R}^N$ such that $f(|K|) = X$ and

- For each simplex $\sigma \in K$ the restriction $f|\sigma^o$ of $f$ to its interior is a real analytic isomorphism $\sigma^o \to f(\sigma^o)$.
- Each $f(\sigma)$ is a (closed) set of type $F$ in $\mathbb{R}^N$.

Any two F-triangulations of $X$ have a common refinement. An F-triangulation $f$ of a closed F-set $X \subset \mathbb{R}^N$ is *compatible* with an F-Whitney stratification $X = \bigcup_\alpha X_\alpha$ if, for each $\alpha$ the set $f^{-1}(\overline{X_\alpha})$ is a (closed) subcomplex of $K$. If $X$ is F-Whitney stratified and F-triangulated by a compatible triangulation, and if $x$ is a point in some stratum $S \subset X$ then the *link* $L_x$ of $x$ (in the sense of P.L. topology) is homeomorphic

$$L_x \cong \Sigma^s \ell \tag{2.2.1}$$

to the $s = \dim(S)$-fold suspension of the link $\ell$ of the stratum $S$.

**Theorem 2.2.** *Let $Z \subset X \subset \mathbb{R}^N$ be closed subsets of type F. Then $X$ admits an F-Whitney stratification such that $Z$ is a union of strata. Given any F-Whitney stratification of $X$ there exists an F-triangulation that is subordinate to the stratification.*

The proof of this result has a long history. We list here a few of the important references: [Lo1], [Lo2], [Hardt7], [Hardt4], [Hardt5], [Hardt6], [Hardt3], [Hardt2], [Hardt1], [Hir5], [Hir4], [Hir3], [Hir1], [Hir2], [Gre1], [Joh].

A closed F-subset $X \subset \mathbb{R}^N$ is *purely d-dimensional* if there exists an F-stratification of $X$ such that $X$ is the closure of the union of all of its $d$-dimensional strata. Then $X$ is an *oriented pseudomanifold* if there exists a (simplicial) oriented pseudomanifold $K$ of pure dimension $d$ and an $F$-triangulation $f : |K| \to X$.

For Whitney stratified sets, an oriented pseudomanifold structure may be described without reference to a triangulation. Let $X$ be a purely $d$-dimensional F-set. Then $X$ is a pseudomanifold if it can be Whitney stratified with no strata of dimension $d-1$. In this case an orientation of $X$ is determined by a choice of orientation of each of the $d$-dimensional strata.

## 2.3 Sheaves and the derived category

Let $X$ be a real or complex algebraic, analytic, semi-analytic or sub-analytic set. Then $X$ is locally compact, Hausdorff, and is homeomorphic to a (locally finite) simplicial complex. Throughout this section we fix a regular, commutative, Noetherian ring $R$ (with unit) of finite cohomological dimension. (A principal ideal domain, for example, is such a ring.) Recall that a *complex of sheaves* of $R$-modules $\mathbf{S}^\bullet$ on $X$ is a collection of sheaves $\mathbf{S}^i$ and differentials $d_i : \mathbf{S}^i \to \mathbf{S}^{i+1}$ with $d^2 = 0$. The associated *cohomology sheaf* of degree $i$ is $\mathbf{H^i}(\mathbf{S}^\bullet) = \ker d_i / \mathit{Im}\, d_{i-1}$. If each $\mathbf{S}^i$ is fine, flabby, soft, or injective, then the cohomology $H^*(X, \mathbf{S}^\bullet)$ (resp. cohomology with compact supports $H^*_c(X, \mathbf{S}^\bullet)$) is given by the cohomology of the complex of global sections (resp. global sections with compact supports).

It is customary to denote by $\mathbf{S}^\bullet[n]$ the shift of $\mathbf{S}^\bullet$ by $n$, that is, $(\mathbf{S}^\bullet[n])^k = \mathbf{S}^{n+k}$. A morphism $\mathbf{S}^\bullet \to \mathbf{T}^\bullet$ is a *quasi-isomorphism* if it induces an isomorphism on the associated cohomology sheaves. In this case, the complex $T^\bullet$ is called an *injective resolution* of $\mathbf{S}^\bullet$ if each $\mathbf{T}^j$ is injective (in the category of sheaves of $R$ modules).

A complex of sheaves $\mathbf{S}^\bullet$ is *cohomologically locally constant* (CLC) if each of the cohomology sheaves $\mathbf{H^i}(\mathbf{S}^\bullet)$ is locally constant. The complex $\mathbf{S}^\bullet$ is *cohomologically constructible* with respect to a given stratification of $X$ if each of the cohomology sheaves $\mathbf{H^i}(\mathbf{S}^\bullet)$ is locally constant on each stratum. Let $D^b_c(X)$ denote the bounded constructible derived category: its objects consist of complexes of sheaves that are bounded from below and are cohomologically constructible with respect to some Whitney $F$-stratification of $X$. In this category, every quasi-isomorphism is invertible. See, for example, [Iv], [GelM], [Gre5]. Many functors $F$

defined on the category $\mathrm{Sh}(X)$ of sheaves on $X$ pass to derived functors $RF$. In particular, we shall use the standard notations $Rf_*$, $Rf_!$, $f^*$, $f^!$ for the derived push-forward, derived push-forward with proper supports, the pull-back and the extraordinary pull-back on sheaves. If $\mathbf{S}^\bullet, \mathbf{T}^\bullet$ are complexes of sheaves on $X$ then $\mathbf{RHom}^\bullet(\mathbf{S}^\bullet, \mathbf{T}^\bullet)$ denotes the complex of sheaves that is obtained from the double complex of pre-sheaves which associates to any open subset $j : U \subset X$ the $R$ module $\mathrm{Hom}(j^*\mathbf{S}^p, j^*\mathbf{I}^q)$ where $\mathbf{T}^\bullet \to \mathbf{I}^\bullet$ is an injective resolution of $\mathbf{T}^\bullet$. In this case

$$\mathrm{Hom}_{D^b_c(X)}(\mathbf{S}^\bullet, \mathbf{T}^\bullet) = H^0(X, \mathbf{RHom}^\bullet(\mathbf{S}^\bullet, \mathbf{T}^\bullet)).$$

If $\mathbf{S}^\bullet$ is cohomologically constructible then it follows from the Thom-Mather theorem (Section 2.2) that the stalk cohomology (or "local cohomology")

$$H^i(j_x^*\mathbf{S}^\bullet) = H^i_x(\mathbf{S}^\bullet) = \mathbf{H^i}(\mathbf{S}^\bullet)_x$$

(of $\mathbf{S}^\bullet$ at the point $x \in X$) coincides with the cohomology $H^i(U_x, \mathbf{S}^\bullet)$ of any basic neighborhood $U_x$ of $x$ in $X$. Here $j_x : \{x\} \to X$ denotes the inclusion. Similarly

$$H^i_c(U_x, \mathbf{S}^\bullet) \cong H^i(j_x^!(\mathbf{S}^\bullet))$$

is the stalk cohomology with compact supports.

## 2.4 The sheaf of chains

Let $\mathbf{E}$ be a local coefficient system (= locally constant sheaf) of $R$ modules on a set $X \subset \mathbb{R}^n$ of type F. There are many quasi-isomorphic versions of the sheaf $\mathbf{C}^\bullet(X, \mathbf{E})$ of chains on $X$. We briefly recall the construction of the sheaf of F-chains.

Let $T$ be a (locally finite) F-triangulation of $X$. For each simplex $\sigma$ of $T$ the restriction of $\mathbf{E}$ to $\sigma$ has a canonical trivialisation, so we may unambiguously refer to the *fiber* $\mathbf{E}_\sigma$. An $i$-dimensional ($T$-simplicial) *Borel-Moore chain* with coefficients in $\mathbf{E}$ is a (locally finite) linear combination of oriented simplices $\xi = \sum_t e_t\sigma_t$ with $e_t \in \mathbf{E}_{\sigma_t}$ whose support $|\xi|$ is closed in $X$; we identify $e_t\sigma_t$ with $-e_t\sigma_t'$ where $\sigma'$ is the same simplex as $\sigma$ but with the opposite orientation. The collection of all Borel-Moore $i$-chains with respect to the F-triangulation $T$ forms an $R$ module $C_i^{BM,T}(X, \mathbf{E})$ and the usual boundary map gives a homomorphism $\partial_i : C_i^{BM,T}(X, \mathbf{E}) \to C_{i-1}^{BM,T}(X, \mathbf{E})$. If $T'$ is a refinement of the triangulation $T$ then the natural homomorphism $C_i^{BM,T}(X, \mathbf{E}) \to C_i^{BM,T'}(X, \mathbf{E})$ induces an isomorphism

$$H_i^{BM,T}(X, \mathbf{E}) \cong H_i^{BM,T'}(X, \mathbf{E})$$

on homology. Define the complex of (Borel-Moore) F-chains

$$C_i^{BM}(X, \mathbf{E}) = \varinjlim_T C_i^{BM,T}(X, \mathbf{E})$$

to be the direct limit over all F-triangulations of $X$. The Borel-Moore chains then form a pre-sheaf (with respect to the open subsets of type F). For if $U \subset V$ are open F-subsets of $X$ and if $T$ is a triangulation of $V$ then it is possible to find a triangulation $T'$ of $U$ such that each simplex of $T'$ is contained in a unique simplex of $T$. This procedure gives a homomorphism $C_i^{BM}(U, \mathbf{E}) \to C_i^{BM}(V, \mathbf{E})$. The *sheaf of F-chains* $\mathbf{C}^\bullet(X, \mathbf{E})$ on $X$ is the complex of sheaves whose $R$ module of sections over an open set $U \subset X$ is

$$\Gamma\left(U, \mathbf{C}^{-\mathbf{i}}(X, \mathbf{E})\right) = C_i^{BM}(U, \mathbf{E})$$

with $d_{-i} = \partial_i$ (for $i \geq 0$). (It is placed in negative degrees so that the differentials raise degree.) The sheaf $\mathbf{C}^\bullet(X, \mathbf{E})$ is soft ([Hab] Section II.5), so the sheaf cohomology over any open set $U \subset X$ can be obtained as the cohomology of the complex of sections over $U$. With this in mind, the *Borel-Moore* homology is defined by

$$H_i^{BM}(U, \mathbf{E}) := H^{-i}(U, \mathbf{C}^\bullet(X, \mathbf{E})).$$

The complex of (compact) F-chains on $U \subset X$ is the complex

$$C_i(U, \mathbf{E}) = \Gamma_c(U, \mathbf{C}^{-\mathbf{i}}(X, \mathbf{E}))$$

of sections with compact support. The (local) *homology sheaf*

$$\mathbf{H}^{-i}(\mathbf{C}^\bullet(\mathbf{X}, \mathbf{E}))$$

is the cohomology sheaf of the sheaf of chains. It is a topological invariant and its stalk cohomology is the *local homology*, that is, $H_x^{-i}\left(\mathbf{C}^\bullet(X, \mathbf{E})\right) = H_i(X, X - x; \mathbf{E})$. The cohomology with compact support $H_c^{-i}(X, \mathbf{C}^\bullet(X, \mathbf{E}))$ is the ordinary homology $H_i(X, \mathbf{E})$.

A similar construction [Bre] may be made with singular chains, and the resulting complex of sheaves (which is a topological invariant and does not depend on a choice of piecewise linear structure) is canonically quasi-isomorphic to the sheaf of F-chains. We will sometimes refer to "the" sheaf of chains $\mathbf{C}^\bullet(X, E)$ without reference to a particular PL or analytic structure on $X$.

If $R$ is a field and if $\mathbf{E} = \mathbf{R}$ is the constant local system then the sheaf of chains on $X$ is called the *dualizing sheaf* (with coefficients in $\mathbf{R}$) and it is denoted $\mathbb{D}_X^\bullet$. (For an arbitrary regular Noetherian ring $R$ of finite cohomological dimension, the dualizing sheaf is obtained from the sheaf of chains by tensoring with an injective resolution of $R$ [Bo84, Section 7.A].)

We remark that if $\xi = \sum_t a_t \sigma_t \in C_i^{BM}(U, R)$ is a chain with constant coefficients, and if $s \in \Gamma(|\xi|, \mathbf{E})$ is a section of $\mathbf{E}$ over the support of $\xi$ then we obtain, in a natural way a chain $s\xi = \sum_t a_t s(\sigma_t)\sigma_t \in C_i^{BM}(U, \mathbf{E})$.

## 2.5 Homology manifolds

As in the previous section, we assume the coefficient ring $R$ is a regular Noetherian ring of finite cohomological dimension, and we let $F$ refer to semi-algebraic, semi-analytic, or subanalytic. Let $Y$ be a *purely $n$-dimensional* set of type $F$ (so $Y$ is contained in some Euclidean space and its closure is also an $n$-dimensional set of type $F$). The set $Y$ is an *$R$-homology manifold* if

$$H_j(Y, Y - y; R) = \begin{cases} 0 & \text{if } j \neq n \\ R & \text{if } j = n \end{cases}$$

or, equivalently, if the local homology sheaf $\mathbf{H}^{-\mathbf{j}}(\mathbf{C}^\bullet(Y, R))$ is a local system of rank 1 for $j = n$ and vanishes for $j \neq n$. Assume $Y$ is an $R$-homology manifold. The *$R$-orientation sheaf*

$$\mathcal{O}_Y = \mathbf{H}^{-n}(\mathbf{C}^\bullet(Y, R)).$$

is the local system whose fiber at each point $y \in Y$ is $H_n(Y, Y - y; R)$.

If an orientation of $Y$ exists (Section 2.1) then it determines an isomorphism between $\mathcal{O}_Y$ and the trivial local system $\mathbf{R}$. If $Y$ is also connected (but not necessarily compact) then $H_n^{BM}(Y, \mathcal{O}_Y) \cong R$. A choice of generator $[Y]$ of this group is called a *fundamental class.* It can be represented by a (Borel-Moore) chain $\xi \in C_n^{BM}(Y, \mathcal{O}_Y)$ whose support is the union of all the $n$-dimensional simplices in a triangulation of $Y$, that is, $|\xi| = Y$. There is a canonical quasi-isomorphism $\mathcal{P} : \mathcal{O}_Y \to \mathbf{C}^\bullet(Y, R)[-n]$ which assigns to each sufficiently small open F-ball $U \subset Y$ the chain $[U] \in \Gamma\left(U, \mathbf{C}^{-n}(Y, \mathcal{O}_Y)\right)$. It induces a quasi-isomorphism

$$\mathcal{O}_Y \otimes \mathbf{E} \to \mathbf{C}^\bullet(Y, \mathbf{E})[-n] \tag{2.5.1}$$

for any finite-dimensional local system $\mathbf{E}$ of $R$ modules on $Y$. The resulting isomorphisms of cohomology groups are often referred to as *Poincaré duality* isomorphisms,

$$H^i(Y, \mathcal{O}_Y \otimes \mathbf{E}) \cong H_{n-i}^{BM}(Y, \mathbf{E})$$
$$H_c^i(Y, \mathcal{O}_Y \otimes \mathbf{E}) \cong H_{n-i}(Y, \mathbf{E}).$$

The morphism $\mathcal{P}$ may also be viewed as a quasi-isomorphism $\mathcal{P} : \mathbf{E} \to \mathbf{C}^\bullet(Y, \mathcal{O}_Y \otimes \mathbf{E})[-n]$ with resulting Poincaré duality isomorphisms

$$H^i(Y, \mathbf{E}) \cong H_{n-i}^{BM}(Y, \mathcal{O}_Y \otimes \mathbf{E})$$
$$H_c^i(Y, \mathbf{E}) \cong H_{n-i}(Y, \mathcal{O}_X \otimes \mathbf{E}).$$

## 2.6 Cellular Borel-Moore chains

In Chapters 8 and 9 we will need to integrate differential forms (defined on the non-compact top stratum $Y$ of a modular variety $X$) over chains (which are themselves non-compact), with coefficients in local systems on $Y$ that may not extend over its compactification $X$. Integration of non-compact chains on non-compact manifolds leads to a host of potential pathological difficulties, none of which (fortunately) occur in the setting of modular cycles on modular varieties. The purpose of this section is to provide a few standard but not previously easily referenceable technical tools which will be used to guarantee that the integrals we will eventually consider are well behaved. The main point is that the manifold $Y$ and the modular cycles in $Y$ are *compactifiable*.

Let $X$ be a set of type $F$ (= semi-algebraic, semi-analytic, or subanalytic) and let $f : |K| \to X$ be an $F$-triangulation of $X$, where $K$ is a locally finite simplicial complex. Let $L \subset K$ be a closed subcomplex such that the open set $Y = f(|K| - |L|)$ is dense in $X$. The resulting decomposition of $Y$ is a *pseudo cell decomposition* in the sense of Section 2.1.

Let $\mathbf{E}$ be a local coefficient system of $R$ modules on $Y$. If $\sigma$ is a cell of $K$ whose interior $\sigma^o$ is contained in $Y$, then the fibers $E_x, E_y$ of $E$ over any two points $x, y \in \sigma \cap Y$ are canonically isomorphic. Therefore we may refer unambiguously to the fiber $E_\sigma$. If $\tau < \sigma$ and $\tau^o \subset Y$ then there is a canonical isomorphism $\Phi_{\sigma\tau} : E_\sigma \to E_\tau$.

An $r$-dimensional *elementary Borel-Moore cellular chain* (on $Y$ with coefficients in $\mathbf{E}$) is an equivalence class of formal products $a_\sigma \sigma$ where $\sigma$ is an oriented $r$-dimensional cell of $K$ such that $\sigma^o \subset Y$ and where $a_\sigma \in \mathbf{E}_\sigma$; modulo the identification

$$a_\sigma \sigma \sim (-a_\sigma)\sigma'$$

where $\sigma'$ is the same cell but with the opposite orientation.

The boundary $\partial a_\sigma \sigma$ of an elementary $r$-dimensional chain is defined to be

$$\partial a_\sigma \sigma = \sum_\tau \Phi_{\sigma\tau}(a_\sigma)\tau$$

where the sum is taken over those $r-1$-dimensional faces $\tau < \sigma$ such that $\tau^o \subset Y$.

The R-module of *cellular Borel-Moore chains* $\widehat{C}_r^K(Y, \mathbf{E})$ (with respect to the pseudo-cell decomposition $K$) is the module of finite formal linear combinations of elementary $r$ chains. Let $\xi = \sum_i a_i \sigma_i \in \widehat{C}_r^K(Y, \mathbf{E})$ be a cellular Borel-Moore chain. Its *support* $|\xi|$ is the intersection of $Y$ with the union of those cells $\sigma_i$ such that $a_i \neq 0$. If $K'$ is a (finite) refinement of $K$ (and we write $K' < K$) there is a canonical injection $\widehat{C}_r^K(Y, \mathbf{E}) \to \widehat{C}_r^{K'}(Y, \mathbf{E})$ which preserves supports. The proof of the following Lemma will appear in Appendix A below.

**Lemma 2.3.** *If $K'$ is a finite refinement of $K$ then the induced mapping on homology*

$$\widehat{H}_r^K(Y, \mathbf{E}) \to \widehat{H}_r^{K'}(Y, \mathbf{E})$$

*is an isomorphism.*

Now let $T$ be a (piecewise linear) triangulation of $Y$ that is subordinate to $K$, that is, a triangulation such that every (closed) simplex in $T$ is contained in a (closed) cell of $K$ as a convex linear subset. (If $Y$ is not compact then $T$ will consist of infinitely many simplices.) Then we obtain a canonical injection

$$\widehat{C}_r^K(Y,\mathbf{E}) \to C_r^{BM}(Y,\mathbf{E}). \tag{2.6.1}$$

**Proposition 2.4.** *The mapping* (2.6.1) *induces an isomorphism on homology,*

$$\widehat{H}_r^K(Y,\mathbf{E}) \to H_r^{BM,T}(Y,\mathbf{E}) \cong H_r^{BM}(Y,\mathbf{E}).$$

In summary, *any pseudo cell decomposition of $Y$ may be used to compute its Borel-Moore homology.* The proof, which is standard but surprisingly messy, is in Appendix A below.

## 2.7 Algebraic cycles

Let $X$ be a nonsingular complex algebraic variety with a local coefficient system $\mathbf{E}$ and let $Y \subset X$ be a complex algebraic closed subvariety of complex dimension $n$. Suppose the local system $E$ has the underlying structure of a (real or complex) vector bundle with a flat connection. (See Section 6.1.) When does $Y$ determine a homology class in $H_{2n}^{BM}(X,\mathbf{E})$?

**Proposition 2.5.** *Let $Z \subset Y$ be a proper complex algebraic subvariety containing the singularities of $Y$ and let $S$ be a* flat *section of $\mathbf{E}|(Y-Z)$. Then the pair $(Y,S)$ determines a Borel-Moore homology class $[Y,S] \in H_{2n}^{BM}(X,\mathbf{E})$.*

*Proof.* According to the above, in order to make $(Y,S)$ into a (Borel-Moore) cycle, one needs a triangulation or a cell decomposition or a pseudo-cell decomposition of $X$ so that $Y$ is a union of cells; plus the data of a cellular chain on $Y$ whose boundary is zero. In other words, one needs an assignment, to each $2n$-dimensional simplex $\sigma \subset Y$, of an element $a_\sigma \in \mathbf{E}_\sigma$ such that the boundary cancels on every $2n-1$-dimensional simplex. Here, $E_\sigma$ is the fiber of $\mathbf{E}$ over any point in $\sigma$. Since $\mathbf{E}$ has a flat connection, it is possible to find a trivialization of $\mathbf{E}|\sigma$ such that the constant sections are flat, so we may interpret the element $a_\sigma$ as a flat section over $\sigma$. Since $Y-Z$ is a manifold, each $2n-1$-dimensional simplex $\tau$ is a face of exactly two $n$-dimensional simplices, say, $\sigma,\sigma'$. The requirement that the boundaries cancel is the same as saying that, in a flat local trivialization of $\mathbf{E}$ on a neighborhood of $\tau$, the two flat (= constant) sections $a_\sigma$ and $a_{\sigma'}$ agree.

But this is exactly what is given in the proposition. It is possible (Theorem 2.2) to find a (smooth) triangulation of $X$ so that $Z$ and $Y$ are unions of simplices, in which case $Z$ consists entirely of simplices with dimension $2n-2$ or less. So all of the $2n$-dimensional and $2n-1$-dimensional simplices of $Y$ are contained in $Y-Z$ on which the flat section $S$ therefore defines a chain whose boundary vanishes. $\square$

http://www.springer.com/978-3-0348-0350-2

Hilbert Modular Forms with Coefficients in Intersection Homology and Quadratic Base Change

Getz, J.; Goresky, M.

2012, XIV, 258 p., Hardcover
ISBN: 978-3-0348-0350-2
A product of Birkhäuser Basel