

Introduction

This book was greatly inspired by ideas and results from famous works by M. Kac [67–70]. He showed that various methods of probability theory can be fruitfully applied to important problems of analysis (integral and differential equations). The interconnections between probability and analysis problems play also a central role in the present book.

Our approach is based mainly on the application of analysis methods to probability theory. We widely use the method of operator identities, which was developed in our books [147–149] (see also references therein). The largest chapter of the book is dedicated to Levy processes. Using the method of operator identities we show that, for a broad class of the Levy processes, the Ito representation of the generator L can be written in a convolution type form. Numerous applications follow.

In particular, the exponential asymptotics of the probability $p(t, \Delta)$ (that Levy processes X_τ for $0 \leq \tau \leq t$ stay within the given domain Δ) is proved for the case that t tends to infinity. Thus, an essential generalization of an old problem by M. Kac (for the stable Levy processes) is formulated and solved. Among other important problems treated in this book are, for instance, the principle of imperceptibility of boundary, generalized stationary processes, prediction problems, dual systems, approximation of positive functions, and integrable operators. We note that the formulation and the first results on the principle of imperceptibility of boundary were obtained by M. Kac [67]. The scalar dual differential equations were investigated first by I.S. Kac and M.G. Krein [66] (see also the book by H. Dym and H.P. McKean [34]). The notion of linear positive *polynomial operators*, which are essential for positive approximation, was introduced by P.P. Korovkin [76].

The class of so-called integrable operators was studied and interesting applications of this class to some physics problems and random matrix theory were found in [30, 64] in the 1990s. Initial fundamental results for an even more general class were obtained by the author in 1968 in his paper [136], which was the first paper dedicated to the method of operator identities. Here we discuss further applications of such operators to Riemann–Hilbert problems, random matrix theory, and canonical systems.

The first concrete example of a non-factorable positive operator in Hilbert space is constructed. (The existence of such an operator was proved more than 30

years before in the seminal work by D.R. Larson [87].)

In Chapter 9 we introduce an important *fundamental principle*: solutions of a number of basic problems of physics are given by the functions at which an extremum of the functional $F = \lambda E + S$ is attained, where E stands for energy and S for entropy. In this way, famous Gibbs-type formulas are proved rigorously. Interesting connections with game theory and ideas by J. von Neumann and O. Morgenstern [110] appear here. Correspondingly, energy and entropy may be considered as two players of a kind of cooperative game. We compare pure strategy (classical mechanics) with mixed strategy (quantum mechanics).

Chapter 10 is dedicated to inhomogeneous Boltzmann equations. The cases of the classical and quantum (for Fermi and Bose particles) Boltzmann equations are treated in this chapter. We compare again the corresponding classical and quantum results using a game theoretic point of view. The asymptotics and stability of solutions of Boltzmann equations are also considered.

In the last chapter we investigate the properties of the operator Bezoutiant. In the chapter we omit the assumption that the operator Bezoutiant is normally solvable. We investigate the following problems: to describe the conditions under which the entire functions have no common zeroes, to extend the Schur–Cohn theorem to new classes of entire functions. We apply the general results to the theory of the Bessel and confluent hypergeometric functions.

Let us describe the contents of the book in greater detail. Chapter 1 is dedicated to the theory of Levy processes. During the last 30 years there has been a great revival of interest in Levy processes. New theoretical developments, new approaches, and new applications were obtained (see, e.g., [3, 10, 63, 147, 158, 166, 176, 194] and references therein).

Definition 0.1. A Levy process X_t ($t > 0$) is a stochastic process which satisfies the following conditions:

- 1) X_t has independent and stationary increments.
- 2) $X_0 = 0$, almost surely.
- 3) X_t is stochastic continuous, that is, for all $a > 0$ and for all $s \geq 0$ the relation

$$\lim_{t \rightarrow s} P\{|X_t - X_s| > a\} = 0 \quad (0.1)$$

holds.

Our approach to Levy processes is based on the following facts. The Levy process X_t defines a strongly continuous semigroup P_t . The generator L of the semigroup P_t is a pseudo-differential operator (Ito formula). We show that for a wide class of Levy processes, the Ito representation of the corresponding generator L can be written in a convolution type form

$$Lf = \frac{d}{dx} S \frac{d}{dx} f, \quad (0.2)$$

where the operator S is given by the relation

$$Sf = \frac{1}{2}\nu f + \int_{-\infty}^{\infty} k(y-x)f(y)dy \quad (\nu = \bar{\nu} \geq 0). \quad (0.3)$$

The obtained representation (0.2), (0.3) enables us to apply the theory of integral equations with difference kernels [147]. We use this representation to study the probability $p(t, \Delta)$ (which was already mentioned above) that the Levy processes X_τ stay within the given domain Δ , see Section 1.5 in Chapter 1. M. Kac obtained the first results of this type for Cauchy processes (see [67]). H. Widom dealt with $p(t, \Delta)$ for symmetric stable processes (see [190]). Note that the stable processes form a special subclass of the Levy processes. We develop further the results by M. Kac and prove them for a wide class of Levy processes. In particular, we obtain the asymptotic formula

$$p(t, \Delta) = e^{-t/\lambda_1} (c_1 + o(1)), \quad \lambda_1 > 0, \quad c_1 > 0, \quad t \rightarrow \infty. \quad (0.4)$$

We separately consider the case, when $\Delta = [-a, a]$, a depends on t and

$$a(t) \rightarrow \infty, \quad t \rightarrow \infty. \quad (0.5)$$

We compare the obtained results with the well-known classical results: the iterated logarithm law, the first hitting time, the most visited sites, and investigate in detail a number of concrete examples of the Levy processes.

In Chapter 2 we consider the stable processes X_t as $t \rightarrow +0$. The principle of imperceptibility of the boundary was formulated by M. Kac in the following dramatic form: “The information that we shall be eaten at the boundary of the domain has not yet reached us” [67]. Here we prove this hypothesis (in the weakened form). We note that the M. Kac principle is closely connected with the asymptotics of the eigenvalues $\lambda_n(\alpha)$ of the quasi-potential operator B_α . (See relations (1.11.2)–(1.11.8) for the definition of B_α .) For symmetric stable processes we proved that

$$\lambda_n(\alpha) = \left(\frac{2a}{n\pi}\right)^\alpha (1 + o(1)), \quad n \rightarrow \infty, \quad 0 < \alpha \leq 2. \quad (0.6)$$

The quasi-potential operator B_α plays an essential role in the problems of Levy processes (Chapters 1 and 2). It is of interest that the same operator B_α plays an important role in certain approximation problems too (Chapter 3).

In Chapter 3 we consider the class Z_α of continuous 2π -periodical functions $f(x)$ which satisfy the inequality

$$|f(x+h) - f(x-h) - 2f(x)| \leq 2|h|^\alpha, \quad 0 < \alpha < 2. \quad (0.7)$$

Korovkin's operators [76] are defined by the relations

$$L_n f = \frac{1}{\pi} \int_{-\pi}^{\pi} U_n(t-x) f(t) dt, \quad f(x) \in Z_\alpha, \quad (0.8)$$

where

$$U_n(t) = \frac{1}{2D_n} \left| \sum_{k=0}^n \varphi\left(\frac{k}{n}\right) e^{ikt} \right|^2, \quad D_n = \sum_{k=0}^n \varphi^2\left(\frac{k}{n}\right), \quad D_n \neq 0. \quad (0.9)$$

We study the method of approximating functions $f(x)$ of the class Z_α by $L_n f$. The measure of this approximation is the value

$$C_n(\varphi, \alpha) = \sup_{f \in Z_\alpha} \|f(x) - L_n f\|, \quad (0.10)$$

where $\|f(x)\| = \max_{|x| \leq \pi} |f(x)|$. Under certain conditions we proved that

$$n^\alpha C_n(\varphi, \alpha) = C(\varphi, \alpha) + o(1), \quad n \rightarrow \infty, \quad 0 < \alpha < 2. \quad (0.11)$$

The explicit formulas for $C(\varphi, \alpha)$ and

$$C^*(\alpha) = \inf_{\varphi \in C_0^{(1)}[0,1]} C(\varphi, \alpha), \quad 0 < \alpha < 2 \quad (0.12)$$

are given. Here $C_0^{(1)}[0, 1]$ stands for the set of functions $\varphi(x)$, which are continuous together with their first derivative $\varphi'(x)$ on the interval $[0, 1]$ and satisfy equalities $\varphi(0) = \varphi(1) = 0$. It is important that

$$g_n(x) = L_n f \geq 0, \quad f(x) \geq 0, \quad x \in [0, 1]. \quad (0.13)$$

Inequalities in (0.13) mean that we approximate the non-negative function $f(x)$ by non-negative functions $g_n(x) = L_n f$. Such a kind of approximation appears in a number of probabilistic problems. (One of the examples is the case that $f(x)$ is a density function.)

In Chapter 4 we consider generalized stationary processes. Similar to the previous Chapters 1-3, our approach in Chapter 4 is based on the theory of operators with difference kernels [147]. The notion of generalized stationary processes was introduced by I.M. Gelfand and N.Ya. Vilenkin [45]. Note that any device has a certain "inertia" and, hence, it measures not a classical, but a generalized process. We study an important class of the generalized processes: S_j -generalized stationary processes (see [149, Ch. 6]). These processes are associated with the bounded operators with difference kernels:

$$S_j \varphi = \frac{d}{dt} \int_a^b s(t-u) \varphi(u) du. \quad (0.14)$$

Following [115, 116], we solve in Chapter 4 the optimal filtering and prediction problems for the S_j -generalized stationary processes. We introduce and investigate also some interesting subclasses of the S_j -generalized stationary processes: white noise type processes, power-law noises.

Problems of triangular factorization are discussed in Chapter 5. We stress that the triangular factorization plays an essential role in a number of problems: integral equations [144, 147], inverse problems [148, 149], non-linear differential equations [148]. It is well-known that the positive definite and invertible $m \times m$ matrices admit triangular factorization. D. Larson [87] proved the *existence* of a positive definite and invertible but non-factorable operator. In Chapter 5 we construct *concrete examples* of such operators. In particular, the operators

$$Sf = f(x) - \mu \int_0^\infty \frac{\sin(\pi(x-t))}{\pi(x-t)} f(t) dt, \quad f(x) \in L^2(0, \infty), \quad 0 < \mu < 1 \quad (0.15)$$

are positive definite and invertible but non-factorable. Such operators are used in a number of theoretical and applied problems: in optics, in random matrices theory [105], generalized stationary processes (see Chapter 4), Bose gas theory [105]. Using positive definite and invertible but non-factorable operators we could substitute pure existence theorems [87] by concrete examples in the well-known problems posed by J.R. Ringrose [126], and R.V. Kadison and I.M. Singer [71]. We note that the Kadison–Singer problem was stated independently by I. Gohberg and M.G. Krein [52].

In Chapter 6 we compare the thermodynamics characteristics of quantum and classical approaches. E. Wigner and J.G. Kirkwood (see [69]) showed that the quantum statistical sum

$$Z_q(\beta, h) = \sum_{n=1}^{\infty} e^{-\beta E_n(h)}, \quad \beta = 1/kT \quad (0.16)$$

and the classical statistical sum

$$Z_c(\beta) = \iint e^{-\beta H(p,q)} dp dq \quad (0.17)$$

are connected by the relation

$$\lim_{h \rightarrow 0} (2\pi h)^N Z_q(\beta, h) = Z_c(\beta), \quad (0.18)$$

where N is the dimension of the corresponding coordinate space, k is the Boltzmann constant, T is the temperature and

$$H(p, q) = \frac{1}{2m} \sum_{j=1}^N p_j^2 + V(q). \quad (0.19)$$

Here $E_n(h)$ are the eigenvalues of the corresponding energy operator L . We stress that relation (0.19) holds when $h \rightarrow 0$. However, the comparison of the quantum and classical approaches without the demand for h to be small is of essential scientific and methodological interest. To do it we consider the quantum mean energy

$$E_q(\beta, h) = \sum_{n=1}^{\infty} E_n(h) e^{-\beta E_n(h)} / Z_q(\beta, h) \quad (0.20)$$

and the classical mean energy

$$E_c(\beta) = \iint H(p, q) e^{-\beta H(p, q)} dp dq / Z_c(\beta) \quad (0.21)$$

of the same system. In Chapter 6 we discuss the following conjectures:

1) The inequality

$$(2\pi h)^N Z_q(\beta, h) \leq Z_c(\beta) \quad (0.22)$$

holds for all $h > 0$ and $\beta > 0$.

2) The inequality

$$E_q(\beta, h) \geq E_c(\beta) \quad (0.23)$$

holds for all $h > 0$ and $\beta > 0$.

3) The asymptotic relations

$$(2\pi h)^N Z_q(\beta, h) = Z_c(\beta) (1 + o(1)), \quad \beta \rightarrow 0, \quad (0.24)$$

$$E_q(\beta, h) = E_c(\beta) (1 + o(1)), \quad \beta \rightarrow 0 \quad (0.25)$$

are valid.

Recall that $\beta = 1/kT$. Hence, the relation $\beta \rightarrow 0$ is equivalent to the relation $T \rightarrow \infty$. By proving inequality (0.22) we use D. Ray's results [125]. It is interesting that inequalities (0.18) and (0.22) can be interpreted in terms of the principle of imperceptibility of the boundary (see Chapter 2).

In Chapter 7 we generalize the Kac–Krein notion of dual string equations for some classes of canonical continuous and discrete systems. In the last section of the chapter the obtained results are illustrated by a number of concrete examples.

In Chapter 8 we consider the class of generalized integrable operators

$$Sf = L(x)f(x) + P.V. \int_a^b \frac{D(x, t)}{x - t} f(t) dt, \quad (0.26)$$

where $f(x) \in L_k(a, b)$, the $k \times k$ matrix functions $L(x)$ and $D(x, t)$ are such that

$$L(x) = L^*(x), \quad D(x, t) = -D^*(t, x), \quad (0.27)$$

and the symbol *P.V.* indicates that the corresponding integral is understood as the principal value. Here L^* denotes the matrix that is adjoint to L . We assume that the kernel $D(x, t)$ is degenerate, that is,

$$D(x, t) = iA(x)JA^*(t), \quad (0.28)$$

where $A(x)$ is a $k \times m$ matrix function ($k \leq m$) and J is a constant $m \times m$ matrix such that

$$J = J^*, \quad J^2 = I_m. \quad (0.29)$$

We describe interconnections of these operators with the Riemann–Hilbert problems, canonical systems, and various applications.

Operators (0.26) were introduced and studied in our paper [136]. A special important subclass of operators S , where

$$k = 1, \quad L(x) = 1, \quad D(x, x) = 0, \quad (0.30)$$

was dealt with later in the work [64]. The operator identity

$$(QS - SQ)f = \int_a^b D(x, t)f(t)dt, \quad Qf = xf(x) \quad (0.31)$$

plays an essential role in our approach. If the operator S is invertible, then, according to (0.31), the operator $T = S^{-1}$ has the form

$$Tf = M(x)f(x) + P.V. \int_a^b \frac{E(x, t)}{x - t} f(t)dt, \quad (0.32)$$

where

$$E(x, t) = iB(x)JB^*(t), \quad (0.33)$$

$B(x)$ is a $k \times m$ matrix function, $M(x)$ is an $m \times m$ matrix function, and $M(x) = M^*(x)$. We associate [148], with the operators S and T , the canonical differential system

$$\frac{d}{dx}W(x, z) = i \frac{JH(x)}{x - t} W(x, z), \quad W(a, z) = I_m. \quad (0.34)$$

The monodromy matrix $W(z) = W(b, z)$ of system (0.34) coincides with the solution of the Riemann–Hilbert problem

$$W_+(\sigma) = W_-(\sigma)R^2(\sigma), \quad a \leq \sigma \leq b, \quad (0.35)$$

where

$$W_{\pm}(\sigma) = \lim_{y \rightarrow \pm 0} W(z), \quad z = \sigma + iy. \quad (0.36)$$

Here $R(\sigma)$ is the J -module of $W_+(\sigma)$ (see [122]). We note that in the formulated Riemann–Hilbert problem the matrix function $R(\sigma)$ is given.

It follows from (0.34) that $W(x, z)$ in the neighborhood of $z = \infty$ admits the representation

$$W(x, z) = I_m + \frac{M_1(x)}{z} + \frac{M_2(x)}{z^2} + \cdots, \quad (0.37)$$

where

$$M_1(x) = i \int_a^x JH(t)dt. \quad (0.38)$$

In Chapter 8, we give a procedure to recover the matrix function $M_1(x)$, which can be used in random matrix theory [36, 106]. Let us note that the monodromy matrix $W(z) = W(b, z)$ is the M.S. Livshits characteristic matrix function of the operator

$$Af = xf(x) + i \int_a^x \beta(x)J\beta^*(t)f(t)dt, \quad f(x) \in L_k^2(a, b), \quad (0.39)$$

(see [30, 106, 178]). Here the $k \times m$ matrix function β and the Hamiltonian H are connected by the relation

$$H(x) = \beta^*(x)\beta(x). \quad (0.40)$$

In terms of $W(z)$, we obtain a sufficient condition of the linear similarity of the operator A to the self-adjoint operator

$$Qf = xf(x), \quad f(x) \in L_k^2(a, b). \quad (0.41)$$

The corresponding result is essentially stronger than our old theorem [136]. We treat in detail the case that

$$\beta(x)J\beta^*(x) = 0. \quad (0.42)$$

The inverse problem to recover the Hamiltonian $H(x)$ of system (0.34) from the given J -module $R(\sigma)$ is solved. In the last section of Chapter 8 we consider a number of examples, both new and classic.

In Chapter 9 we consider the mean energy E and entropy S together. For that purpose we introduce the functional

$$F = \lambda E + S,$$

where $\lambda = -1/(kT)$, k is the Boltzmann constant, and T is temperature. We formulate an important fundamental principle.

Fundamental principle. *The functional F defines the game between the mean energy E and entropy S .*

Using this fundamental principle, we derive rigorously the well-known Gibbs formulas. In game theory [109], the transition from deterministic to probability strategy leads to a gain for players. Similar to game theory, the transition from

classical to quantum mechanics leads also to a gain for both players, that is, for both energy and entropy (see formula (6.0.7) and Theorems 6.7 and 6.10).

The necessity of the game theoretic approach can be explained in the following way. According to the second law of thermodynamics, a physical system in equilibrium has maximal entropy among all states with the same energy. So entropy depends on the value of energy and we have the game theory situation. We note that, according to definition, “game theory models the situations in which an individual success in making choices depends on the choices of others”.

In Chapter 9 we apply the game theoretic approach to the following important problems: quantum and classical mechanics (Gibbs-type formulas), non-extensive statistical mechanics, and algorithmic information theory.

The classical and quantum versions of the Boltzmann equation are investigated in Chapter 10. We note that the quantum version of the Boltzmann equation contains both the fermion and boson cases. The important notion of Kullback–Leibler distance is essentially generalized and new conventional extremal problems, which appear in this way, are solved. The solution $f(t, x, \zeta)$ of the Boltzmann equation is studied in the bounded domain Ω of the x -space. Such an approach essentially changes the usual situation, that is, the total energy depends on t and the notion of distance between a stationary solution and an arbitrary solution of the Boltzmann equation includes the x -space. Thus, the notion of distance remains well-defined in the spatially inhomogeneous case too. (We recall that the Kullback–Leibler distance is defined only in the spatially homogeneous case.) The comparison of the classical and quantum mechanics, which was treated in [153, 159], is generalized here for the case of the Boltzmann equations. It is especially interesting for applications that the fermion and boson cases are essentially different from this point of view. In the last section of the chapter we introduce dissipative and conservative solutions and find the conditions under which stationary solutions of the classical Boltzmann equation are stable.

In the last Chapter 11 we consider the operator version of Bezoutiant. The matrix Bezoutiant is used in order to define the number of common zeroes of two polynomials and describe the distribution of the zeroes of polynomials with respect to the circle $|z| = 1$. M.G. Krein extended the notion of Bezoutiant to entire functions. Various important and interesting results were published as a further development of Bezoutiant theory. In Chapter 11 we introduce main notions of Bezoutiant theory. We omit the assumption that the operator Bezoutiant is normally solvable. This result allows us to apply the general theory to a number of important examples. We would like to emphasize that these examples are the first specific non-trivial examples in the operator Bezoutiant theory.

The book is devoted to important problems on the frontier between analysis (integral and differential equations, spectral theory, and operator theory), probability theory and applications (stable processes, Levy processes, prediction

theory, and positive approximation), and statistical physics (entropy, Gibbs-type formulas, laws of thermodynamics, Boltzmann equations, extremal problems, and game theoretic interpretation). It could be of interest to the specialists in all those domains.

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