Chapter 2

Physical background

2.1 Governing equations

Suppose a compressible fluid (which we may also call a gas) occupies a domain $\Omega \subset \mathbb{R}^3$ named the *flow domain*. The flow domain can vary in time and its position and even shape can depend on the time variable t. In this case we write Ω_t to stress the dependence on t. The state of the fluid is characterized completely by the macroscopic quantities: the *density* $\varrho(x,t)$, the *velocity* $\mathbf{u}(x,t)$, and the *temperature* $\vartheta(x,t)$. These quantities are called *state variables* in the following. The *governing equations* represent three basic principles of fluid mechanics: the *mass balance*

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \quad \text{in } \Omega, \tag{2.1.1a}$$

the *balance of momentum*

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla p = \varrho \mathbf{f} + \operatorname{div} \mathbb{S}(\mathbf{u}) \quad \text{in } \Omega, \tag{2.1.1b}$$

and the *energy conservation law*

$$\partial_t E + \operatorname{div}((E+p)\mathbf{u}) = \operatorname{div}(\mathbb{S}(\mathbf{u})\mathbf{u}) + \operatorname{div}(\kappa \nabla \vartheta) + (\varrho \mathbf{f}) \cdot \mathbf{u} + \varrho Q. \tag{2.1.1c}$$

Here, the vector field $\mathbf{f}$ denotes the density of external mass force, the *heat conduction coefficient* κ is a positive constant, the given function Q is the intensity of the external energy flux, the *viscous stress tensor* $\mathbb{S}$ has the form

$$\mathbb{S}(\mathbf{u}) = \nu_1 \left(\nabla \mathbf{u} + \nabla \mathbf{u}^\top - \frac{2}{3} \operatorname{div} \mathbf{u}\, \mathbb{I} \right) + \nu_2 \operatorname{div} \mathbf{u}\, \mathbb{I}, \tag{2.1.1d}$$

in which the *viscosity coefficients* ν_i, $i = 1,2$, satisfy the inequality $\frac{4}{3}\nu_1 + \nu_2 > 0$, the *energy density* E is given by

$$E = \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e,$$

P. Plotnikov and J. Sokołowski, *Compressible Navier–Stokes Equations: Theory and Shape Optimization*, Monografie Matematyczne 73, DOI 10.1007/978-3-0348-0367-0_2, © Springer Basel 2012

where e is the density of *internal energy*. The physical properties of the fluid are reflected through *constitutive equations* relating the state variables to the pressure p and the internal energy density e. The common point of view is that p and e can be represented as functions of ϱ and ϑ. The functions $p(\varrho,\vartheta)$ and $e(\varrho,\vartheta)$ are not arbitrary but should satisfy the Gibbs equation

$$\frac{1}{\vartheta}de(\varrho,\vartheta) - \frac{p(\varrho,\vartheta)}{\vartheta\varrho^2}d\varrho = ds(\varrho,\vartheta),$$

which means that the left hand side is the exact differential of some function s named *entropy density*. In the classical case of *perfect polytropic* gases, the pressure and the internal energy density are defined by the following formulae, called the *constitutive law*:

$$p = R_m\varrho\vartheta, \quad e = c_v\vartheta. \tag{2.1.1e}$$

Here, R_m is a positive constant inversely proportional to the molecular weight of the gas that is

$$R_m = c_p - c_v, \quad \text{with} \ \ \gamma := c_p/c_v > 1,$$

where c_v is the specific heat capacity at constant volume and c_p is the specific heat capacity at constant pressure; c_v, c_p are positive constants. In this notation the entropy density s takes the form

$$s = \log e + (\gamma - 1)\log\varrho. \tag{2.1.2}$$

The system of differential equations (2.1.1) is called the *compressible Navier-Stokes-Fourier* equations.

It is useful to rewrite the governing equations in dimensionless form which is widely used in applications. To this end, we denote by u_c, ϱ_c, p_c, ϑ_c the typical values of velocity, density, pressure, and temperature, and by l_c and T_c the typical values of length scale and time intervals. For simplicity we assume that $u_c = l_c/T_c$. Under this assumption, the characteristic values form four dimensionless combinations which are named: the Reynolds number, the Pecle number, the Mach number, and the viscosity ratio (see [119]),

$$\mathsf{Re} = \frac{\varrho_c u_c l_c}{\nu_1}, \quad \mathsf{Pe} = \frac{p_c l_c u_c}{\kappa_c\vartheta_c}, \quad \mathsf{Ma}^2 = \frac{\varrho_c u_c^2}{p_c}, \quad \lambda = \frac{1}{3} + \frac{\nu_2}{\nu_1}.$$

Denote also by f_c and Q_c the characteristic values of mass force and heat influx. They form two dimensionless combinations

$$\mathsf{Fr}_m^2 = \frac{u_c^2}{f_c l_c}, \quad \Theta = \frac{\varrho_c Q_c l_c}{p_c u_c}.$$

Note that here the characteristic quantities ϱ_c, ϑ_c, and p_c should be compatible with the constitutive law. For instance if the pressure is defined by (2.1.1e), then $p_c = R_m\,\varrho_c\,\vartheta_c$. Observe that the specific values of the constants γ, λ, and

Pe depend only on the physical properties of the fluid. For example, for the air under standard conditions, we have $\gamma = 7/5$ and $\mathsf{Pe} = 7/10$. The passage to the dimensionless variables is defined as follows:

$$\begin{aligned} &x \to l_c x, \quad t \to T_c t, \quad \mathbf{u} \to u_c \mathbf{u}, \quad \varrho \to \varrho_c \varrho, \\ &\vartheta \to \vartheta_c \vartheta, \quad \varrho \mathbf{f} \to \varrho_c f_c \varrho \mathbf{f}, \quad \kappa \to \kappa_c \kappa, \end{aligned} \tag{2.1.3}$$

and when performed in (2.1.1), leads to the following system of differential equations for dimensionless quantities in the scaled domain $l_c^{-1}\Omega$, still denoted by Ω:

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\mathsf{Ma}^2} \nabla p = \frac{1}{\mathsf{Re}} \operatorname{div} \mathbb{S}(\mathbf{u}) + \frac{1}{\mathsf{Fr}_m^2} \varrho \mathbf{f} \quad \text{in } \Omega, \tag{2.1.4a}$$

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \quad \text{in } \Omega, \tag{2.1.4b}$$

$$\begin{aligned} &\partial_t(\varrho s) + \operatorname{div}(\varrho s \mathbf{u}) - \frac{1}{\mathsf{Pe}} \operatorname{div} \left(\frac{\kappa}{\vartheta} \nabla \vartheta \right) \\ &\qquad = \frac{1}{\vartheta} \left(\frac{\mathsf{Ma}^2}{\mathsf{Re}} \mathbb{S}(\mathbf{u}) : \nabla \mathbf{u} + \frac{\kappa}{\mathsf{Pe}\vartheta} |\nabla \vartheta|^2 \right) + \Theta \frac{\varrho Q}{\vartheta}, \end{aligned} \tag{2.1.4c}$$

where $\mathbf{u} \otimes \mathbf{u}$ stands for the tensor product of two vectors and the dimensionless viscous stress tensor is defined by

$$\mathbb{S}(\mathbf{u}) = \nabla \mathbf{u} + \nabla \mathbf{u}^\top + (\lambda - 1) \operatorname{div} \mathbf{u}\, \mathbb{I}, \quad \operatorname{div} \mathbb{S}(\mathbf{u}) = \Delta \mathbf{u} + \lambda \nabla \operatorname{div} \mathbf{u}. \tag{2.1.5}$$

If the flow domain Ω_t varies in time, the above equations should be considered in the moving scaled domain.

2.1.1 Isentropic flows. Compressible Navier-Stokes equations

The flow is *barotropic* if the pressure depends only on the density. The most important example of such flows are *isentropic flows*. In order to deduce the governing equations for isentropic flows we note that for a perfect fluid with $\nu_i = \kappa = 0$, $i = 1, 2$, the entropy takes a constant value at each material point. Hence in this case the governing equations have a family of explicit solutions with the entropy $s = \text{const}$. By (2.1.1e) and (2.1.2) in this case we have

$$p(\varrho) = (\gamma - 1) \exp(s_c) \varrho^\gamma,$$

where the positive constant s_c is a characteristic value of the entropy (without loss of generality we can take $(\gamma - 1) \exp(s_c) = 1$). Assuming that this relation holds for $\nu_i \neq 0, i = 1, 2$, we arrive at the system of *compressible Navier-Stokes equations* for isentropic flows of a viscous compressible fluid in dimensionless form:

$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\mathsf{Ma}^2}\nabla p(\varrho) = \frac{1}{\mathsf{Re}}\operatorname{div}\mathbb{S}(\mathbf{u}) + \frac{1}{\mathsf{Fr}_m^2}\varrho\mathbf{f} \quad \text{in } \Omega, \tag{2.1.6a}$$

$$\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u}) = 0 \quad \text{in } \Omega. \tag{2.1.6b}$$

Recall that the exponent γ depends on the physical properties of the fluid. In particular [25], $\gamma = 5/3$ for mono-atomic, $\gamma = 7/5$ for diatomic and $\gamma = 4/3$ for polyatomic gases.

It is worth noting that the entropy production for a viscous gas is proportional to $\mathbb{S}(\mathbf{u}) : \nabla\mathbf{u}$. Generally speaking, this means that the total entropy of a viscous gas increases in time and in contrast to the nonviscous case the existence of isentropic solutions for viscous gas dynamics equations is unlikely. In this sense, the compressible Navier-Stokes equations are thermodynamically inconsistent. The equations can be considered as an approximation of the real physical problem. However, this situation is not unusual in mathematical physics. We recall that the standard heat equation is also thermodynamically inconsistent.

Nevertheless, the compressible Navier-Stokes equations play an important role in the theory of compressible fluid dynamics as the only example of physically relevant equations for which we have nonlocal existence results.

2.2 Boundary and initial conditions

Boundary conditions. The governing equations should be supplemented with boundary conditions. The typical boundary conditions for the velocity field are: the first boundary condition (Dirichlet-type condition)

$$\mathbf{u} = \mathbf{U} \quad \text{on } \partial\Omega, \tag{2.2.1}$$

the second boundary condition (Neumann-type condition)

$$(\mathbb{S}(\mathbf{u}) - p\,\mathbb{I})\mathbf{n} = \mathbf{S}_n \quad \text{on } \partial\Omega, \tag{2.2.2}$$

where $\mathbf{n}$ is the outward normal vector to $\partial\Omega$, and $\mathbf{U}$ and $\mathbf{S}_n$ are given vector fields. Important particular cases are the *no-slip boundary condition* with $\mathbf{U} = 0$, and the zero normal stress condition with $\mathbf{S}_n = 0$. A third type of physically and mathematically reasonable condition is the *no-stick boundary condition*

$$\mathbf{u}\cdot\mathbf{n} = 0, \quad ((\mathbb{S}(\mathbf{u}) - p\,\mathbb{I})\mathbf{n})\times\mathbf{n} = 0 \quad \text{on } \partial\Omega,$$

which corresponds to the case of frictionless boundary. The typical boundary conditions for the temperature are the Dirichlet and Neumann boundary conditions.

The formulation of boundary conditions for the density is a more delicate task. Assume that the velocity $\mathbf{u}$ satisfies the first boundary condition (2.2.7), and split the boundary of the flow region into three disjoint sets called the *inlet* $\Sigma_{\rm in}$, the *outgoing set* $\Sigma_{\rm out}$, and the *characteristic set* Σ_0; these sets are defined by

$$\begin{aligned}\Sigma_{\rm in} &= \{x \in \partial\Omega : \mathbf{U}\cdot\mathbf{n} < 0\}, \quad \Sigma_{\rm out} = \{x \in \Sigma : \mathbf{U}\cdot\mathbf{n} > 0\},\\ \Sigma_0 &= \{x \in \partial\Omega : \mathbf{U}\cdot\mathbf{n} = 0\}.\end{aligned} \tag{2.2.3}$$

The density should be given on the inlet:

$$\varrho = \varrho_\infty \quad \text{on } \Sigma_{\rm in}. \tag{2.2.4}$$

The boundary conditions for the density are not needed in the case of $\Sigma_{\rm in} = \emptyset$. In particular, there are no boundary conditions for the density if the velocity satisfies the no-slip or no-stick conditions, i.e., whenever $\Sigma_{\rm in} = \Sigma_{\rm out} = \emptyset$.

If the flow domain depends on t, i.e. $\Omega = \Omega_t$, and material points of $\partial\Omega_t$ are moving with a velocity $\mathbf{V}(x,t)$ then the boundary conditions become

$$\mathbf{u} = \mathbf{U} \quad \text{on } \partial\Omega_t, \quad \varrho = \varrho_\infty \quad \text{on } \Sigma^t_{\rm in}, \tag{2.2.5}$$

where the time dependent inlet is defined by

$$\Sigma^t_{\rm in} = \{x \in \partial\Omega_t : (\mathbf{U}(x,t) - \mathbf{V}(x,t))\cdot\mathbf{n} < 0\}.$$

Initial conditions. At the initial time $t = 0$ the distributions of the velocity field, density and temperature should be prescribed in $\Omega := \Omega_0$ for solutions of the Navier-Stokes-Fourier equations. The velocity field and density should be prescribed for solutions of the compressible Navier-Stokes equations

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \varrho(x,0) = \varrho_0(x) \quad \text{in } \Omega. \tag{2.2.6}$$

Flows around moving bodies. The problem of a gas flow around a moving body is of practical importance. It presents an example of an exterior boundary value problem and is formulated as follows:

Suppose that at time t a body occupies a compact set $S_t \subset \mathbb{R}^d$. Its position and shape can vary in time, but we assume that there is no *mass flux* through its boundary. Let material points of ∂S_t be moving with the velocity $\mathbf{V}(x,t)$. Then the problem is to find the velocity field $\mathbf{u}$ and the density ϱ satisfying the compressible Navier-Stokes equations (2.1.6) along with the boundary and initial conditions

$$\begin{aligned}&\mathbf{u} = \mathbf{V} \quad \text{on } \partial S_t,\\ &\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \varrho(x,0) = \varrho_0(x) \quad \text{in } \Omega_0,\\ &\mathbf{u}(x,t) \to 0, \quad \varrho \to \varrho_\infty \quad \text{as } |x| \to \infty.\end{aligned} \tag{2.2.7}$$

2.3 Power and work of hydrodynamic forces

The *stress tensor* in a viscous compressible flow is defined by

$$\mathbb{T} = \mathbb{S}(\mathbf{u}) - p\,\mathbb{I}, \tag{2.3.1}$$

where the viscous stress tensor $\mathbb{S}(\mathbf{u})$ is given by (2.1.1d), and the force acting from the side of the flow at the boundary point $x \in \partial\Omega$ is equal to

$$\mathbf{R}_f = -\mathbb{T}\,\mathbf{n} = (-\mathbb{S}(\mathbf{u}) + p\,\mathbb{I})\,\mathbf{n}. \tag{2.3.2}$$

Recall that $\mathbf{n}$ is the outward normal vector to $\partial\Omega$. If the flow domain $\Omega = \Omega_t$ varies in time, and material points on its boundary are moving with a given velocity $\mathbf{V}_s(x,t)$, the surface density of the power developed by the hydrodynamic force $\mathbf{R}_f$ is given by

$$J_{\text{dens}} = -\mathbb{T}\,\mathbf{n}\cdot\mathbf{V}_s = (-\mathbb{S}(\mathbf{u}) + p\,\mathbb{I})\,\mathbf{n}\cdot\mathbf{V}_s. \tag{2.3.3}$$

The total hydrodynamic force $\mathbf{R}_\Omega$ and the power J_Ω developed by the hydrodynamic forces are equal to

$$\mathbf{R}_\Omega = \int_{\partial\Omega_t} (-\mathbb{S}(\mathbf{u}) + p\,\mathbb{I})\,\mathbf{n}\,dS, \quad J_\Omega = \int_{\partial\Omega_t} (-\mathbb{S}(\mathbf{u}) + p\,\mathbb{I})\,\mathbf{n}\cdot\mathbf{V}_s\,dS. \tag{2.3.4}$$

Therefore, the total work of the hydrodynamic forces over the time period $[0,T]$ is

$$W_\Omega = \int_0^T \int_{\partial\Omega_t} (-\mathbb{S}(\mathbf{u}) + p\,\mathbb{I})\,\mathbf{n}\cdot\mathbf{V}_s\,dS\,dt. \tag{2.3.5}$$

After scaling (2.1.3) of independent and state variables and the following scaling of the hydrodynamic force, power, and work:

$$\mathbf{R}_\Omega \to \varrho_c u_c^2 l_c^2\,\mathbf{R}_\Omega, \quad J_\Omega \to \varrho_c u_c^3 l_c^2\,J_\Omega, \quad W_\Omega \to T_c \varrho_c u_c^3 l_c^2\,W_\Omega,$$

expressions (2.3.4)–(2.3.5) can be rewritten in dimensionless form as

$$\begin{aligned} \mathbf{R}_\Omega &= -\frac{1}{\mathsf{Re}} \int_{\partial\Omega_t} (\nabla\mathbf{u} + (\nabla\mathbf{u})^\top + (\lambda-1)\operatorname{div}\mathbf{u} - \sigma p\,\mathbb{I})\,\mathbf{n}\,dS, \\ J_\Omega &= -\frac{1}{\mathsf{Re}} \int_{\partial\Omega_t} (\nabla\mathbf{u} + (\nabla\mathbf{u})^\top + (\lambda-1)\operatorname{div}\mathbf{u} - \sigma p\,\mathbb{I})\,\mathbf{n}\cdot\mathbf{V}_s\,dS, \end{aligned} \tag{2.3.6}$$

and

$$W_\Omega = -\frac{1}{\mathsf{Re}} \int_0^T \int_{\partial\Omega_t} (\nabla\mathbf{u} + (\nabla\mathbf{u})^\top + (\lambda-1)\operatorname{div}\mathbf{u} - \sigma p\,\mathbb{I})\,\mathbf{n}\cdot\mathbf{V}_s\,dSdt. \tag{2.3.7}$$

Here, we use the notation

$$\lambda = \frac{1}{3} + \frac{\nu_2}{\nu_1}, \quad \sigma = \frac{\mathsf{Re}}{\mathsf{Ma}^2}. \tag{2.3.8}$$

2.4 Navier-Stokes equations in a moving frame

If the flow region varies in time, then it is convenient, for technical and practical reasons, to reduce the corresponding boundary value problem for fluid dynamics equations to a problem in a fixed domain by a change of independent variables. In this section we describe such a change of variables in the case when Ω_t evolves like an absolutely rigid body. Assume that x and t are dimensionless variables. Recall that a one-parameter family of mappings $y \mapsto x(y, t)$ represents a rigid body motion in Euclidean space $\mathbb{R}^d$ if and only if

$$x = \mathbb{U}(t)y + \mathbf{a}(t), \tag{2.4.1}$$

where $\mathbf{a}(t)$ is an arbitrary vector function and $\mathbb{U}(t)$ is an arbitrary one-parameter family of special orthogonal matrices, i.e., $\mathbb{U}\,\mathbb{U}^\top = \mathbb{I}$ and $\det \mathbb{U} = 1$. For simplicity, we assume that $\mathbf{a}$ and $\mathbb{U}$ are twice continuously differentiable in t and

$$\mathbb{U}(0) = \mathbb{I}, \quad \mathbf{a}(0) = 0.$$

The velocity $\mathbf{V}$ of the rigid body motion (2.4.1) is defined by

$$\mathbf{V}(y, t) = \dot{\mathbb{U}}(t)y + \dot{\mathbf{a}}(t), \tag{2.4.2}$$

where $\dot{\mathbf{a}}(t) = \frac{d}{dt}\mathbf{a}(t)$. Introduce also the vector field

$$\mathbf{W}(y, t) = \mathbb{U}^\top(t)\,\mathbf{V}(y, t) = \mathbb{U}^\top(t)\dot{\mathbb{U}}(t)y + \mathbb{U}^\top(t)\dot{\mathbf{a}}(t). \tag{2.4.3}$$

Next set

$$\mathbf{v}(y, t) = \mathbb{U}^\top(t)\,\mathbf{u}(x(y, t), t) - \mathbf{W}(y, t), \quad \rho(y, t) = \varrho(x(y, t), t). \tag{2.4.4}$$

Lemma 2.4.1. *Let the flow domain be of the form* $\Omega_t = \mathbb{U}(t)\,\Omega_0 + \mathbf{a}(t)$. *Then smooth functions* $(\mathbf{u}, \varrho)$ *satisfy equations* (2.1.6) *in* Ω_t *if and only if the functions* $(\mathbf{v}, \rho)$, *defined by* (2.4.4), *satisfy the equations*

$$\begin{aligned}\partial_t(\rho\,\mathbf{v}) + \operatorname{div}(\rho\,\mathbf{v}\otimes\mathbf{v}) - \frac{1}{\mathsf{Re}}\operatorname{div}\mathbb{S}(\mathbf{v}) \\ + \frac{1}{\mathsf{Ma}^2}\nabla p(\rho) + \mathbb{C}\,\mathbf{v} = \rho\,\mathbf{f}^* \quad \textit{in } \Omega_0,\end{aligned} \tag{2.4.5a}$$

$$\partial_t(\rho) + \operatorname{div}(\rho\mathbf{v}) = 0 \quad \textit{in } \Omega_0, \tag{2.4.5b}$$

where the viscous stress tensor takes the form

$$\mathbb{S}(\mathbf{v}) = \nabla\mathbf{v} + \nabla\mathbf{v}^\top + (\lambda - 1)\operatorname{div}\mathbf{v},$$

and the skew-symmetric matrix $\mathbb{C} = (C_{ij})_{d\times d}$ *and the vector field* $\mathbf{f}^*$ *are defined by*

$$C_{ij} = \frac{\partial W_i}{\partial y_j} - \frac{\partial W_j}{\partial y_i},$$

and

$$\mathbf{f}^* = \frac{1}{\mathsf{Fr}_m^2}\mathbb{U}^\top \mathbf{f}(x(y,t),t) - \partial_t \mathbf{W} + \frac{1}{2}\nabla|\mathbf{W}|^2.$$

Here, W_i are the components of the vector field $\mathbf{W}$ given by (2.4.3)*. The expressions* (2.3.6) *and* (2.3.7) *for the power developed by the hydrodynamic forces and the work produced by these forces read in the new variables*

$$J_\Omega = -\frac{1}{\mathsf{Re}}\int_{\partial\Omega_0} (\nabla\mathbf{v} + (\nabla\mathbf{v})^\top + (\lambda-1)\operatorname{div}\mathbf{v}\,\mathbb{I} - \sigma p(\rho)\,\mathbb{I})\,\mathbf{n}\cdot\mathbf{W}\,dS, \tag{2.4.6}$$

and

$$W_\Omega = -\frac{1}{\mathsf{Re}}\int_0^T\int_{\partial\Omega_0} \big(\nabla\mathbf{v} + (\nabla\mathbf{v})^\top + (\lambda-1)\operatorname{div}\mathbf{v}\,\mathbb{I} - \sigma p(\rho)\,\mathbb{I}\big)\,\mathbf{n}\cdot\mathbf{W}\,dSdt. \tag{2.4.7}$$

It is worth noting that from the mathematical standpoint, the only difference between equations (2.1.6) and (2.4.5) is the presence of the *Coriolis force* $\mathbb{C}\mathbf{v}$ in (2.4.5).

Remark 2.4.2. In three dimensions the expression for the Coriolis force can be written in the traditional form of the cross product

$$\mathbb{C}\mathbf{v} = 2\boldsymbol{\omega}\times\mathbf{v}, \quad \text{where the angular velocity is} \quad 2\boldsymbol{\omega} = \operatorname{rot}\mathbf{W}. \tag{2.4.8}$$

A similar representation holds in the two-dimensional case if we assume that the angular velocity is orthogonal to the plane of motion.

Stationary flows. In general, the matrix $\mathbb{C}$ and the force $\mathbf{f}^*$ depend on the time variable t and equations (2.4.5) are nonautonomous. The equations become autonomous and lead to stationary solutions only if

$$\partial_t\mathbf{W} = \partial_t\{\mathbb{U}^\top(t)\mathbf{f}(x(y,t),t)\} = 0. \tag{2.4.9}$$

By (2.4.3), the equality $\partial_t\mathbf{W} = 0$ is equivalent to the relations

$$\mathbb{U}(t) = e^{\mathbb{B}t}, \quad \dot{\mathbf{a}}(t) = e^{-\mathbb{B}t}\mathbf{b},$$

in which $\mathbb{B}$ is a skew-symmetric matrix and $\mathbf{b}$ is a constant vector. If this is the case, then the matrix $\mathbb{C} = \mathbb{B}$ and the corresponding angular velocity $\boldsymbol{\omega}$ are constants. In particular, condition (2.4.9) is fulfilled if the vector field $\mathbf{f}$ and the angular velocity $\boldsymbol{\omega}$ are constants such that

$$\mathbf{f}\times\boldsymbol{\omega} = 0.$$

Example 2.4.3. The most important case is the Galileo transformation such that

$$\mathbb{U} = \mathbb{I}, \quad \mathbf{a}(t) = \mathbf{v}_\infty t, \quad \mathbf{v}_\infty = \text{const}.$$

In this case

$$\mathbb{C} = 0, \quad \mathbf{f}^* = \frac{1}{\mathsf{Fr}_m^2}\mathbf{f}(y - \mathbf{v}_\infty t, t).$$

Example 2.4.4. Another important example is the motion of gas on rotating Earth. Assume that the origin of the coordinate system is at the center of Earth and the x_1 axis is directed to the north pole. In this case the dimensionless angular velocity is given by

$$\boldsymbol{\omega} = \omega \mathbf{e}_1, \quad \mathbf{e}_1 = (1,0,0), \quad \omega = \omega_e/T_c,$$

where ω_e is the angular velocity of Earth. If (y,t) is a moving frame connected with Earth, then

$$x = e^{\mathbb{C}_\omega t} y, \quad \text{i.e. } \mathbb{U}(t) = e^{\mathbb{C}_\omega t}, \quad \mathbf{a}(t) = 0.$$

Here, the matrix $\mathbb{C}_\omega$ defined by $\mathbb{C}_\omega y = \boldsymbol{\omega} \times y$ has the form

$$\mathbb{C}_\omega = \omega \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$e^{\mathbb{C}_\omega t} = \cos \omega t \begin{pmatrix} \cos \omega t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \sin \omega t \begin{pmatrix} \sin \omega t & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The gravity force is determined from the Newton law and, after scaling, can be taken in the form

$$\mathbf{f} = -\frac{x}{|x|^3},$$

and $\mathbf{W} = \mathbb{C}_\omega y$. It follows from Lemma 2.4.1 that the functions $\mathbf{v}$ and ρ satisfy (2.4.5) with

$$\mathbf{f}^* = -\frac{1}{\mathsf{Fr}_m^2} \frac{y}{|y|^3} + \frac{\omega^2}{2} \nabla (y_2^2 + y_3^2).$$

This means that in the rotating system of coordinates we have to take into account the Coriolis and centrifugal forces, and the system of governing equations is autonomous, i.e., no terms in the equations are explicitly time dependent, but the solutions are functions of the time and the spatial variables.

2.5 Flow around a moving body

Let us return to the flow of a viscous gas around a moving body. Assume that the body is rigid and its configuration S_t at time t is determined in dimensionless variables by the relations

$$S_t = \mathbb{U}(t) S + \mathbf{a}(t), \quad \mathbb{U}^\top \mathbb{U} = \mathbb{I}, \quad \mathbb{U}(0) = \mathbb{I}, \quad \mathbf{a}(0) = 0, \tag{2.5.1}$$

where the initial configuration S is a fixed compact subset of $\mathbb{R}^d$. In particular, a point of S evolves with the dimensionless velocity $\mathbf{V}(y,t) = \dot{\mathbb{U}}(t)\, y + \dot{\mathbf{a}}(t)$. Then the modified velocity field

$$\mathbf{v}(y,t) = \mathbb{U}^\top(t) \mathbf{u}(x(y,t),t) - \mathbf{W}(y,t) \quad \text{with} \quad \mathbf{W} = \mathbb{U}^\top(t) \mathbf{V}(y,t)$$

and the density $\rho(y,t) = \varrho(x(y,t),t)$ satisfy equations (2.4.5) in the flow domain $\Omega_0 = \mathbb{R}^d \setminus S$. It follows from (2.2.7), (2.4.2), and (2.4.3) that in addition ρ and $\mathbf{v}$ satisfy the following boundary and initial conditions:

$$\begin{aligned} &\mathbf{v} = 0 \quad \text{on } \partial S, \\ &\mathbf{v}(y,0) = \mathbf{v}_0(y), \quad \rho(y,0) = \varrho_0(y) \quad \text{in } \mathbb{R}^d \setminus S, \\ &\mathbf{v}(y,t) + \mathbf{W}(y,t) \to 0, \quad \rho \to \varrho_\infty \quad \text{as } |y| \to \infty, \end{aligned} \tag{2.5.2}$$

where the initial vector field is $\mathbf{v}_0(y) = \mathbf{u}_0(y) - (\dot{\mathbb{U}}(0)y - \dot{\mathbf{a}}(0))$.

For the purposes of numerical simulation, the boundary value problems in unbounded domains should be replaced by modified problems in bounded domains. To this end, we choose an arbitrary hold-all domain $B \subset \mathbb{R}^3$, for instance, a sufficiently large ball, such that $S \Subset B$. Next, we transfer the boundary conditions from infinity to ∂B and arrive at the following boundary value problem for $\mathbf{v}$ and ρ, for a given $T > 0$.

Problem 2.5.1. *Find functions* $(\mathbf{v}, \rho)$ *satisfying*

$$\begin{aligned} \partial_t(\rho\,\mathbf{v}) + \operatorname{div}(\rho\,\mathbf{v} \otimes \mathbf{v}) - \frac{1}{\mathsf{Re}} \operatorname{div} \mathbb{S}(\mathbf{v}) & \\ + \frac{1}{\mathsf{Ma}^2} \nabla p(\rho) + \mathbb{C}\,\mathbf{v} = \rho\,\mathbf{f}^* \quad & \textit{in } \Omega \times (0,T), \end{aligned} \tag{2.5.3a}$$

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0 \quad \textit{in } \Omega \times (0,T), \tag{2.5.3b}$$

$$\begin{aligned} &\mathbf{v} = 0 \quad \textit{on } \partial S \times (0,T), \quad \mathbf{v} = -\mathbf{W} \quad \textit{on } \partial B \times (0,T), \\ &\rho = \varrho_\infty \quad \textit{on } \Sigma_{\text{in}}, \\ &\mathbf{v}(y,0) = \mathbf{v}_0(y) \quad \textit{in } \Omega, \quad \rho(y,0) = \varrho_0(y) \quad \textit{in } \Omega, \end{aligned} \tag{2.5.3c}$$

where

$$\Omega = B \setminus S, \quad \Sigma_{\text{in}} = \{(y,t) \in \partial B \times (0,T) : \mathbf{W}(y,t) \cdot \mathbf{n}(y) > 0\}.$$

Now we recalculate the work W_S of the hydrodynamic forces acting on the moving body. To this end we can use (2.4.7), which gives

$$W_S = -\frac{1}{\mathsf{Re}} \int_0^T \int_{\partial S} \big(\nabla \mathbf{v} + (\nabla \mathbf{v})^\top + (\lambda - 1) \operatorname{div} \mathbf{v}\ \mathbb{I} - \sigma p(\rho)\,\mathbb{I}\big)\,\mathbf{n} \cdot \mathbf{W}\,dSdt. \tag{2.5.4}$$

But this formula is inconvenient since it involves the surface integral of $\nabla \mathbf{v}$ and p which is not defined for weak solutions. We replace (2.5.4) by a more complicated but robust formula which only contains volume integrals.

Lemma 2.5.2. *Let* $(\mathbf{v}, \rho)$ *be a classical solution to problem* (2.5.3) *in a cylinder* $(B \setminus S) \times (0,T)$ *and suppose a function* $\eta \in C^\infty(B \setminus S)$ *satisfies the condition*

$$\eta(x) = 0 \textit{ in a neighborhood of } \partial B, \quad \eta(x) = 1 \textit{ in a neighborhood of } \partial S. \tag{2.5.5}$$

Then

$$W_S = -\int_{B\setminus S} \eta\big\{\rho(\cdot,T)\mathbf{v}(\cdot,T)\cdot\mathbf{W}(\cdot,T) - \rho_0\mathbf{v}_0\cdot\mathbf{W}(\cdot,0)\big\}\,dx +$$
$$\int_0^T\int_{B\setminus S}\left\{\rho\eta\mathbf{v}\cdot\partial_t\mathbf{W} + \left(\rho(\mathbf{v}\otimes\mathbf{v}) - \frac{1}{\mathsf{Re}}\mathbb{T}\right) : \nabla(\eta\mathbf{W}) + \eta(\rho\mathbf{f}^* - \mathbb{C}\mathbf{v})\cdot\mathbf{W}\right\}dxdt, \tag{2.5.6}$$

where $\mathbb{T} = \nabla\mathbf{v} + (\nabla\mathbf{v})^\top + (\lambda-1)\,\mathrm{div}\,\mathbf{v}\,\mathbb{I} - \sigma p(\rho)\,\mathbb{I}$.

Proof. Since η vanishes on ∂B and equals 1 on ∂S, we can integrate by parts to obtain

$$\begin{aligned} -\int_{\partial S}\mathbb{T}\mathbf{n}\cdot\mathbf{W}\,dS &= -\int_{\partial\Omega}\eta\mathbb{T}\mathbf{W}\cdot\mathbf{n}\,dS \\ &= -\int_{B\setminus S}\mathrm{div}\,\mathbb{T}\cdot(\eta\mathbf{W})\,dx - \int_{B\setminus S}\mathbb{T} : \nabla(\eta\mathbf{W})\,dx. \end{aligned}$$

From this, the identity

$$\frac{1}{\mathsf{Re}}\mathbb{S}(\mathbf{v}) - \frac{1}{\mathsf{Ma}^2}p(\rho)\mathbb{I} = \frac{1}{\mathsf{Re}}\mathbb{T},$$

and (2.5.3a) we conclude that

$$\begin{aligned} W_S = -\int_0^T\int_{B\setminus S}\Big\{&\eta\partial_t(\rho\mathbf{v})\cdot\mathbf{W} + \mathrm{div}(\rho\mathbf{v}\otimes\mathbf{v})\cdot(\eta\mathbf{W}) + \eta\mathbb{C}\mathbf{v}\cdot\mathbf{W} \\ &- \eta\rho\mathbf{f}^*\cdot\mathbf{W} + \frac{1}{\mathsf{Re}}\mathbb{T} : \nabla(\eta\nabla\mathbf{W})\Big\}\,dxdt. \end{aligned} \tag{2.5.7}$$

Since $\eta\mathbf{v}\otimes\mathbf{v} = 0$ on $\partial B\cup\partial S$, we can integrate by parts to obtain

$$\begin{aligned} -\int_0^T\int_{B\setminus S}&\big\{\eta\partial_t(\rho\mathbf{v})\cdot\mathbf{W} + \mathrm{div}(\rho\mathbf{v}\otimes\mathbf{v})\cdot(\eta\mathbf{W})\big\}\,dxdt \\ &= \int_0^T\int_{B\setminus S}\big\{\rho\eta\mathbf{v}\cdot\partial_t\mathbf{W} + \rho(\mathbf{v}\otimes\mathbf{v}) : \nabla(\eta\mathbf{W})\big\}dxdt \\ &\quad - \int_{B\setminus S}\eta\big\{\rho(\cdot,T)\mathbf{v}(\cdot,T)\cdot\mathbf{W}(\cdot,T) - \rho_0\mathbf{v}_0\cdot\mathbf{W}(\cdot,0)\big\}\,dx. \end{aligned}$$

Inserting this into (2.5.7) we arrive at (2.5.6). □

http://www.springer.com/978-3-0348-0366-3

Compressible Navier-Stokes Equations
Theory and Shape Optimization
Plotnikov, P.; Sokołowski, J.
2012, XVI, 464 p., Hardcover
ISBN: 978-3-0348-0366-3
A product of Birkhäuser Basel