

Chapter 2

Pointwise conformal metrics

At the beginning of this chapter we introduce the basic formalism and the derivation of the geometric Yamabe equation. Then, we concentrate on the case where M is compact to illustrate the interplay between geometry and analysis, with a few illuminating examples such as the Kazdan-Warner obstruction, a result of Obata on Einstein manifolds, the far-reaching “generalization” of Bidaut-Véron and Véron and a result of Escobar. Along the way we give a detailed proof, which inspires to P. Petersen’s treatise [Pet06a], of a famous rigidity result of Obata. In this way, we hope to provide some geometrical feeling on the subject of this monograph that will enable us to proceed with the noncompact case: the case of the rest of our investigation.

2.1 The Yamabe equation

2.1.1 The derivation of the Yamabe equation

Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold and consider a pointwise conformal deformation of the metric $\langle \cdot, \cdot \rangle$, that is, a new metric on M of the form

$$\widetilde{\langle \cdot, \cdot \rangle} = \varphi^2 \langle \cdot, \cdot \rangle, \quad (2.1)$$

with φ a strictly positive smooth function. Denoting with \widetilde{R} the curvature tensor of $\widetilde{\langle \cdot, \cdot \rangle}$, we want to determine the relationship between \widetilde{R} and R . Let $\{\theta^i\}$, $i = 1, \dots, m = \dim M$, be a local orthonormal coframe on $(M, \langle \cdot, \cdot \rangle)$ with corresponding Levi-Civita connection forms $\{\theta_j^i\}$. Then, in the new metric $\widetilde{\langle \cdot, \cdot \rangle}$,

$$\widetilde{\theta}^i = \varphi \theta^i, \quad i = 1, \dots, m, \quad (2.2)$$

is a local orthonormal coframe on $(M, \widetilde{\langle \cdot, \cdot \rangle})$. To determine the corresponding connection forms we could use the general theory developed in Chapter 1, but it

is easy to deduce that, if $d\varphi = \varphi_t \theta^t$, the 1-forms

$$\tilde{\theta}_j^i = \theta_j^i + \frac{\varphi_j}{\varphi} \theta^i - \frac{\varphi_i}{\varphi} \theta^j \quad (2.3)$$

are skew-symmetric and satisfy the first structure equation. Thus, they are the desired connection forms relative to the coframe defined in (2.2). In order to determine the curvature forms, we use the structure equations and the expression for the components of the Hessian (see equation (1.41)) to compute

$$\begin{aligned} \tilde{\Theta}_j^i &= d\tilde{\theta}_j^i + \tilde{\theta}_k^i \wedge \tilde{\theta}_j^k \\ &= d\theta_j^i + d\left(\frac{\varphi_j}{\varphi}\right) \wedge \theta^i + \frac{\varphi_j}{\varphi} d\theta^i - d\left(\frac{\varphi_i}{\varphi}\right) \wedge \theta^j - \frac{\varphi_i}{\varphi} d\theta^j + \tilde{\theta}_k^i \wedge \tilde{\theta}_j^k \\ &= -\theta_k^i \wedge \theta_j^k + \Theta_j^i + (-\varphi^{-2} \varphi_k \varphi_j \theta^k + \varphi^{-1} d\varphi_j) \wedge \theta^i - \varphi^{-1} \varphi_j \theta_k^i \wedge \theta^k \\ &\quad - (-\varphi^{-2} \varphi_k \varphi_i \theta^k + \varphi^{-1} d\varphi_i) \wedge \theta^j + \varphi^{-1} \varphi_i \theta_k^j \wedge \theta^k \\ &\quad + \left(\theta_k^i + \frac{\varphi_k}{\varphi} \theta^i - \frac{\varphi_i}{\varphi} \theta^k\right) \wedge \left(\theta_j^k + \frac{\varphi_j}{\varphi} \theta^k - \frac{\varphi_k}{\varphi} \theta^j\right) \\ &= \Theta_j^i + \left(\frac{\varphi_{jk}}{\varphi} - 2\frac{\varphi_j \varphi_k}{\varphi^2}\right) \theta^k \wedge \theta^i - \left(\frac{\varphi_{ik}}{\varphi} - 2\frac{\varphi_i \varphi_k}{\varphi^2}\right) \theta^k \wedge \theta^j - \frac{\varphi_k \varphi_k}{\varphi^2} \theta^i \wedge \theta^j, \end{aligned}$$

that is,

$$\begin{aligned} \tilde{\Theta}_j^i &= \Theta_j^i + \left(\frac{\varphi_{jk}}{\varphi} - 2\frac{\varphi_k \varphi_j}{\varphi^2}\right) \delta_t^i \theta^k \wedge \theta^t - \left(\frac{\varphi_{ik}}{\varphi} - 2\frac{\varphi_i \varphi_k}{\varphi^2}\right) \delta_t^j \theta^k \wedge \theta^t \\ &\quad - \frac{\varphi_l \varphi_l}{\varphi^2} \delta_k^i \delta_t^j \theta^k \wedge \theta^t. \end{aligned}$$

Hence, anti-symmetrizing the coefficients of the wedge products on the right-hand side, and recalling the definition of the curvature tensor, we obtain

$$\begin{aligned} \varphi^2 \tilde{R}_{jkt}^i &= R_{jkt}^i + \left(\frac{\varphi_{jk}}{\varphi} - 2\frac{\varphi_k \varphi_j}{\varphi^2}\right) \delta_t^i - \left(\frac{\varphi_{jt}}{\varphi} - 2\frac{\varphi_t \varphi_j}{\varphi^2}\right) \delta_k^i \\ &\quad - \left(\frac{\varphi_{ik}}{\varphi} - 2\frac{\varphi_i \varphi_k}{\varphi^2}\right) \delta_t^j + \left(\frac{\varphi_{it}}{\varphi} - 2\frac{\varphi_i \varphi_t}{\varphi^2}\right) \delta_k^j \\ &\quad - \frac{\varphi_l \varphi_l}{\varphi^2} (\delta_k^i \delta_t^j - \delta_t^i \delta_k^j). \end{aligned} \quad (2.4)$$

Taking traces with respect to i and k we have

$$\varphi^2 \tilde{R}_{jt} = R_{jt} - (m-2) \frac{\varphi_{jt}}{\varphi} + 2(m-2) \frac{\varphi_j \varphi_t}{\varphi^2} - (m-3) \frac{\varphi_l \varphi_l}{\varphi^2} \delta_t^j - \frac{\varphi_{kk}}{\varphi} \delta_t^j. \quad (2.5)$$

Thus, denoting with $|\nabla\varphi|$, $\text{Hess}(\varphi)$ and $\Delta\varphi$ respectively the length of the gradient, the Hessian and the Laplacian of φ in the metric $\langle \cdot, \cdot \rangle$, and recalling that R_{jt} (resp.

\widetilde{R}_{jt}) are the components of the Ricci tensor Ric with respect to the orthonormal basis θ^i (resp. of $\widetilde{\text{Ric}}$ with respect to $\widetilde{\theta}^i$), we have

$$\begin{aligned} \widetilde{\text{Ric}} = & \text{Ric} - (m-2) \frac{1}{\varphi} \text{Hess}(\varphi) + 2(m-2) \frac{1}{\varphi^2} d\varphi \otimes d\varphi \\ & - (m-3) \frac{|\nabla\varphi|^2}{\varphi^2} \langle \cdot, \cdot \rangle - \frac{\Delta\varphi}{\varphi} \langle \cdot, \cdot \rangle. \end{aligned} \quad (2.6)$$

A further tracing of (2.5) with respect to j and t yields

$$\varphi^2 \widetilde{S} = S - 2(m-1) \frac{\Delta\varphi}{\varphi} - (m-1)(m-4) \frac{|\nabla\varphi|^2}{\varphi^2}. \quad (2.7)$$

In case $m = \dim M \geq 3$, we set

$$\varphi = u^{\frac{2}{m-2}} \quad \text{so that} \quad \widetilde{\langle \cdot, \cdot \rangle} = u^{\frac{4}{m-2}} \langle \cdot, \cdot \rangle. \quad (2.8)$$

In this case, (2.7) immediately gives

$$c_m \Delta u - Su + \widetilde{S} u^{\frac{m+2}{m-2}} = 0, \quad (2.9)$$

with $c_m = 4 \frac{m-1}{m-2}$. In case $m = 2$ we set

$$\varphi = e^u \quad \text{so that} \quad \widetilde{\langle \cdot, \cdot \rangle} = e^{2u} \langle \cdot, \cdot \rangle. \quad (2.10)$$

In this case (2.7) gives

$$2\Delta u - S + \widetilde{S} e^{2u} = 0. \quad (2.11)$$

Equations (2.9) and (2.11) are the classical *Yamabe equations*.

We conclude this section with an immediate application of (2.7) in the compact case, improving on a result of Obata, [Oba62b]. First we need the next simple

Lemma 2.1. *Let $(M, \langle \cdot, \cdot \rangle)$ be a compact Riemannian manifold. Then, every homothetic diffeomorphism is an isometry.*

Proof. Let $\varphi : M \rightarrow M$ be a diffeomorphism such that $\varphi^* \langle \cdot, \cdot \rangle = c^2 \langle \cdot, \cdot \rangle$ for some constant $c > 0$. By contradiction, suppose $c \neq 1$. Without loss of generality, we can assume $c > 1$ (indeed, in case $c < 1$ it suffices to consider φ^{-1}). Now, for every $n \in \mathbb{N}$, let $\varphi^{(n)} : M \rightarrow M$ be the n -th iterate of φ . Then

$$\left(\varphi^{(n)} \right)^* \langle \cdot, \cdot \rangle = c^{2n} \langle \cdot, \cdot \rangle,$$

proving that, for any fixed $p \neq q \in M$,

$$c^{2n} d(p, q) = d\left(\varphi^{(n)}(p), \varphi^{(n)}(q)\right) \leq \text{diam}(M) < +\infty.$$

Since $c > 1$, taking the limit as $n \rightarrow +\infty$, we obtain the desired contradiction. \square

We are now ready to prove the following

Theorem 2.2. *Let $(M, \langle \cdot, \cdot \rangle)$ be a compact manifold of dimension $m \geq 3$ (for the ease of exposition) with scalar curvature $s(x) \leq 0$. Let $\psi : M \rightarrow M$ be a conformal diffeomorphism with scalar curvature $\tilde{S}(x)$. Thus ψ is an isometry if and only if $\tilde{S}(x) = kS(x)$ for some $k \in (0, +\infty)$.*

Proof. If ψ is an isometry, clearly $k = 1$. Vice versa, to simplify notation let $c_m = 4 \frac{m-1}{m-2}$, $\sigma = \frac{m+2}{m-2}$ and $k = a^{1-\sigma}$ for some $a \in (0, +\infty)$. Since ψ is conformal and $m \geq 3$, from equation (2.9) we deduce

$$c_m \Delta u = S(x) [u - a^{1-\sigma} u^\sigma] \quad (2.12)$$

with $u > 0$, $u \in C^\infty(M)$ such that

$$\psi^* \langle \cdot, \cdot \rangle = u^{\frac{4}{m-2}} \langle \cdot, \cdot \rangle.$$

Define the vector field

$$W = c_m \left[\left(\frac{u}{a} \right)^{\sigma-1} - 1 \right] \nabla u;$$

computing its divergence, using (2.12) and the divergence theorem we have

$$\int_M \left[\left(\frac{u}{a} \right)^{\sigma-1} - 1 \right]^2 S(x) u = c_m (\sigma - 1) \int_M \frac{1}{u} \left(\frac{u}{a} \right)^{\sigma-1} |\nabla u|^2.$$

Since $S(x) \leq 0$ it follows that $|\nabla u| \equiv 0$ and $u \equiv a$, that is $\psi : M \rightarrow M$ is a homothety. The result now follows from Lemma 2.1. \square

We shall come back to this kind of problems again in Corollary 2.9 below and later in the complete noncompact case.

2.1.2 The Kazdan-Warner obstruction

Let now T denote the traceless Ricci tensor, that is (see equation (1.38))

$$T = \text{Ric} - \frac{S}{m} \langle \cdot, \cdot \rangle.$$

T enables us to immediately find an obstruction to the existence of a conformally deformed metric as in (2.8) or (2.10) with assigned scalar curvature $\tilde{S}(x)$. We shall consider the case $m \geq 3$ so that we shall provide a necessary condition for the existence of a positive solution on M of equation (2.9). Indeed, with respect to a local orthonormal coframe $\{\theta^i\}$ we have

$$T_{ij} = R_{ij} - \frac{S}{m} \delta_{ij}.$$

On the other hand, from (1.36),

$$R_{li,i} = \frac{1}{2}S_l.$$

Thus, tracing the covariant derivative

$$T_{ij,k} = R_{ij,k} - \frac{S_k}{m}\delta_{ij}$$

with respect to j and k yields

$$T_{ik,k} = R_{ik,k} - \frac{S_k}{m}\delta_{ik} = \frac{m-2}{2m}S_i. \quad (2.13)$$

Now, let X be a vector field on M and consider the vector field W associated to the 1-form $T(X, \cdot)$ using the duality induced by the metric, namely,

$$W = T(X, \cdot)^\sharp = X^i T_i^j e_j, \quad T_i^j = T_{ij}, \quad (2.14)$$

where $\{e_j\}$ is the orthonormal frame dual to the coframe $\{\theta^j\}$. Since the covariant derivative satisfies the Leibniz rule, the divergence of W is given by

$$\operatorname{div} W = W_k^k = X_k^i T_{ik} + X^i T_{ik,k}. \quad (2.15)$$

On the other hand, using the symmetry of T_{ik} and (1.20),

$$T_{ik} X_k^i = \frac{1}{2}(X_k^i + X_i^k)T_{ik} = \frac{1}{2}(\mathcal{L}_X \langle \cdot, \cdot \rangle)_{ik} T_{ik}.$$

It follows that

$$\operatorname{div} W = \frac{1}{2}(\mathcal{L}_X \langle \cdot, \cdot \rangle)_{ik} T_{ik} + \frac{m-2}{2m}X(S). \quad (2.16)$$

We now recall that a vector field X is said to be *conformal* if it generates a local 1-parameter group φ_t of local conformal diffeomorphisms, that is

$$(\varphi_t^* \langle \cdot, \cdot \rangle)_p = \rho_t(p) \langle \cdot, \cdot \rangle_p, \quad p \in M$$

for some positive function ρ_t . Using the formula which expresses the Lie derivative \mathcal{L}_X as a derivative along the flow generated by X , namely,

$$\mathcal{L}_X \langle \cdot, \cdot \rangle = \lim_{t \rightarrow 0} \frac{\varphi_t^* \langle \cdot, \cdot \rangle - \langle \cdot, \cdot \rangle}{t},$$

(see e.g. [Lee03], p. 472 ff.) one proves that X is conformal if and only if

$$\mathcal{L}_X \langle \cdot, \cdot \rangle = \lambda \langle \cdot, \cdot \rangle$$

for some function $\lambda = \lambda(p)$, which, using (1.19), is easily seen to be equal to $\frac{2}{m}(\operatorname{div} X)$. Substituting into (2.16) and using the fact that T is traceless, we deduce that

$$\operatorname{div} W = \frac{m-2}{2m}X(S) \quad (2.17)$$

whenever X is a conformal vector field on M . Thus we have the next result which is known as the *Kazdan-Warner obstruction* (see [KW74a] and [BE87]):

Theorem 2.3. *Let $(M, \langle \cdot, \cdot \rangle)$ be a compact manifold of dimension $m \geq 3$ and $\widetilde{\langle \cdot, \cdot \rangle}$ a metric on M conformally related to $\langle \cdot, \cdot \rangle$ with scalar curvature \widetilde{S} . Then, for each conformal vector field X on $(M, \langle \cdot, \cdot \rangle)$ we have*

$$\int_M X(\widetilde{S}) d\widetilde{\text{vol}} = 0. \quad (2.18)$$

Proof. Since X is conformal with respect to the metric $\langle \cdot, \cdot \rangle$, it generates a flow of local diffeomorphisms which are conformal transformations for the metric $\langle \cdot, \cdot \rangle$ and therefore also for the conformally related metric $\widetilde{\langle \cdot, \cdot \rangle}$. It follows that X is conformal also in the metric $\widetilde{\langle \cdot, \cdot \rangle}$. Considering the analogous (2.17) in the metric $\widetilde{\langle \cdot, \cdot \rangle}$ and integrating with the aid of the divergence theorem we obtain (2.18). \square

Remark. As a further application of the method of a moving frame, we give here an explicit formula for $\mathcal{L}_X \widetilde{\langle \cdot, \cdot \rangle}$ in terms of $\mathcal{L}_X \langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle$, from which it can be again deduced that a vector field X conformal w.r.t. the metric $\langle \cdot, \cdot \rangle$ is conformal also in the metric $\widetilde{\langle \cdot, \cdot \rangle}$. Indeed we have the following

Lemma 2.4. *Let $X \in \mathfrak{X}(M)$ be a vector field on the Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$, and let $\widetilde{\langle \cdot, \cdot \rangle} = \varphi^2 \langle \cdot, \cdot \rangle$, $\varphi > 0$, be a conformally deformed metric. Then*

$$\mathcal{L}_X \widetilde{\langle \cdot, \cdot \rangle} = \varphi^2 \mathcal{L}_X \langle \cdot, \cdot \rangle + 2\varphi \langle X, \nabla \varphi \rangle \langle \cdot, \cdot \rangle. \quad (2.19)$$

Proof. We choose a local o.n. coframe $\{\theta^i\}$ with associated frame $\{e_i\}$. First we observe that (2.2) implies that $\widetilde{e}_i = \varphi^{-1} e_i$; then, with the notation of section 1.1.2 we have

$$X = X^i e_i = \widetilde{X}^i \widetilde{e}_i \quad (2.20)$$

and

$$\nabla X = X_k^i \theta^k \otimes e_i, \quad \widetilde{\nabla} X = \widetilde{X}_k^i \widetilde{\theta}^k \otimes \widetilde{e}_i \quad (2.21)$$

with $X_k^i \theta^k = (dX^i + X^j \theta_j^i)$ and $\widetilde{X}_k^i \widetilde{\theta}^k = (d\widetilde{X}^i + \widetilde{X}^j \widetilde{\theta}_j^i)$. From (2.20) we deduce

$$\widetilde{X}^i = \varphi X^i, \quad (2.22)$$

while a computation using (2.21) and (2.3) shows that

$$\widetilde{X}_k^i = X_k^i + \varphi^{-1} (X^i \varphi_k + X^j \varphi_j \delta_{ik} - \varphi_i X^k), \quad (2.23)$$

which implies

$$\widetilde{X}_k^i + \widetilde{X}_k^i = X_k^i + X_k^i + 2\varphi^{-1} X^j \varphi_j \delta_{ik}. \quad (2.24)$$

Equation (2.19) now follows easily from (2.24) and (1.20). \square

Example. The m -dimensional standard sphere \mathbb{S}^m , $m \geq 3$, can be realized, outside the North pole N and the South pole S, as the model manifold (see also Chapter 4, section 4.3)

$$\left((0, \pi) \times \mathbb{S}^{m-1}, dr \otimes dr + (\sin r)^2 d\theta^2 \right),$$

where $d\theta^2$ denotes the standard metric on \mathbb{S}^{m-1} . Using this isometric representation, a conformal vector field X on \mathbb{S}^m is given by

$$X = \sin r \nabla r, \quad X_N = X_S = 0.$$

Now, consider any smooth, radial function $\tilde{s}(r(x))$ on \mathbb{S}^m . Then, by the co-area formula (1.87),

$$\begin{aligned} \int_{\mathbb{S}^m} X(\tilde{s}) d\widetilde{\text{vol}} &= \int_{\mathbb{S}^m} \left\langle X, \frac{d\tilde{s}}{dr} \nabla r \right\rangle d\widetilde{\text{vol}} \\ &= \int_0^\pi \left(\int_{\partial B_r} \frac{d\tilde{s}}{dr} \sin r d\widetilde{\text{vol}} \right) dr \\ &= \int_0^\pi \frac{d\tilde{s}}{dr} \sin r \widetilde{\text{vol}}(\partial B_r) dr. \end{aligned}$$

Accordingly, if $\tilde{s}(r)$ is monotonic and nonconstant, we deduce

$$\int_{\mathbb{S}^m} X(\tilde{s}) d\widetilde{\text{vol}} \neq 0.$$

It follows from Theorem 2.3 that any nonconstant monotonic function $\tilde{s}(r(x))$ cannot be the scalar curvature of a pointwise conformal deformation of the canonical metric of \mathbb{S}^m .

2.1.3 The Weyl and Cotton tensors

We now establish a relation between

$$\widetilde{T} = \widetilde{\text{Ric}} - \frac{\widetilde{S}}{m} \widetilde{\langle, \rangle} \quad \text{and} \quad T = \text{Ric} - \frac{S}{m} \langle, \rangle, \quad (2.25)$$

that will be used below in the proof of Theorem 2.8. Suppose, once again, that (2.1) holds, that is,

$$\widetilde{\langle, \rangle} = \varphi^2 \langle, \rangle.$$

Using (2.6) and (2.7), after some computations we obtain

$$\begin{aligned}
\widetilde{T} &= \widetilde{\text{Ric}} - \frac{\widetilde{S}}{m} \langle \cdot, \cdot \rangle \\
&= T - (m-2) \left\{ \frac{1}{\varphi} \text{Hess}(\varphi) - 2 \frac{1}{\varphi^2} d\varphi \otimes d\varphi \right\} \\
&\quad + \frac{m-2}{m} \left\{ \frac{\Delta\varphi}{\varphi} - 2 \frac{|\nabla\varphi|^2}{\varphi^2} \right\} \langle \cdot, \cdot \rangle \\
&= T + (m-2)\varphi \left\{ \text{Hess}(\varphi^{-1}) - \frac{\Delta(\varphi^{-1})}{m} \langle \cdot, \cdot \rangle \right\}.
\end{aligned} \tag{2.26}$$

Next, we observe that using (2.4) and (2.5) we are able to detect a part of the curvature tensor which is naturally invariant with respect to a conformal change of the metric. Indeed, from (2.5) we have

$$(m-2) \left\{ \frac{\varphi_{jt}}{\varphi} - 2 \frac{\varphi_t \varphi_j}{\varphi^2} \right\} = R_{jt} - \varphi^2 \widetilde{R}_{jt} - \left[(m-3) \frac{\varphi_l \varphi_l}{\varphi^2} + \frac{\varphi_{ll}}{\varphi} \right] \delta_t^j,$$

and inserting into (2.4) gives

$$\begin{aligned}
&\varphi^2 \left[\widetilde{R}_{jkt}^i - \frac{1}{m-2} \left(\widetilde{R}_{ik} \delta_t^j - \widetilde{R}_{jk} \delta_t^i + \widetilde{R}_{jt} \delta_k^i - \widetilde{R}_{it} \delta_k^j \right) \right] \\
&= R_{jkt}^i - \frac{1}{m-2} \left(R_{ik} \delta_t^j - R_{jk} \delta_t^i + R_{jt} \delta_k^i - R_{it} \delta_k^j \right) \\
&\quad + \frac{1}{m-2} \left\{ 2 \frac{\Delta\varphi}{\varphi} + (m-4) \frac{|\nabla\varphi|^2}{\varphi^2} \right\} \left(\delta_k^i \delta_t^j - \delta_t^i \delta_k^j \right).
\end{aligned}$$

On the other hand, by (2.7)

$$2 \frac{\Delta\varphi}{\varphi} + (m-4) \frac{|\nabla\varphi|^2}{\varphi^2} = - \frac{1}{(m-1)} \left(\varphi^2 \widetilde{S} - S \right)$$

and we obtain

$$\begin{aligned}
&\varphi^2 \left[\widetilde{R}_{jkt}^i - \frac{1}{m-2} \left(\widetilde{R}_{ik} \delta_t^j - \widetilde{R}_{jk} \delta_t^i + \widetilde{R}_{jt} \delta_k^i - \widetilde{R}_{it} \delta_k^j \right) \right. \\
&\quad \left. + \frac{\widetilde{S}}{(m-1)(m-2)} \left(\delta_k^i \delta_t^j - \delta_t^i \delta_k^j \right) \right] \\
&= R_{jkt}^i - \frac{1}{m-2} \left(R_{ik} \delta_t^j - R_{jk} \delta_t^i + R_{jt} \delta_k^i - R_{it} \delta_k^j \right) \\
&\quad + \frac{S}{(m-1)(m-2)} \left(\delta_k^i \delta_t^j - \delta_t^i \delta_k^j \right).
\end{aligned}$$

It follows that the $(1,3)$ -tensor, called the *Weyl tensor*, defined as

$$\begin{aligned}
W &= \text{Riem} - \left[\frac{1}{m-2} \left(R_{ik} \delta_t^j - R_{jk} \delta_t^i + R_{jt} \delta_k^i - R_{it} \delta_k^j \right) \right. \\
&\quad \left. - \frac{S}{(m-1)(m-2)} \left(\delta_k^i \delta_t^j - \delta_t^i \delta_k^j \right) \right] \theta^k \otimes \theta^t \otimes \theta^j \otimes e_i
\end{aligned} \tag{2.27}$$

is invariant under a conformal change of the metric.

Remark. It is worth noting that the corresponding $(0, 4)$ -version of W is *not* conformally invariant.

Taking covariant derivatives we obtain

$$\begin{aligned} W_{jks,t}^i &= R_{jks,t}^i - \frac{1}{m-2}(R_{ik,t}\delta_{js} - R_{is,t}\delta_{jk} + R_{js,t}\delta_{ik} - R_{jk,t}\delta_{is}) \\ &\quad + \frac{S_t}{(m-1)(m-2)}(\delta_{ik}\delta_{js} - \delta_{is}\delta_{jk}), \end{aligned}$$

so that taking the divergence with respect to the first index, that is, $W_{jks,t}^t$, using (1.35) and (1.36) we get

$$\begin{aligned} W_{jks,t}^t &= R_{jks,t}^t - \frac{1}{m-2}R_{tk,t}\delta_{js} + \frac{1}{m-2}R_{ts,t}\delta_{jk} - \frac{1}{m-2}R_{js,k} + \frac{1}{m-2}R_{jk,s} \\ &\quad + \frac{S_k}{(m-1)(m-2)}\delta_{sj} - \frac{S_s}{(m-1)(m-2)}\delta_{jk} \\ &= -R_{jk,s} + R_{js,k} + \frac{1}{m-2}R_{jk,s} - \frac{1}{m-2}R_{js,k} \\ &\quad + \frac{1}{m-2}\left(\frac{1}{m-1} - \frac{1}{2}\right)S_k\delta_{js} - \frac{1}{m-2}\left(\frac{1}{m-1} - \frac{1}{2}\right)S_s\delta_{jk} \\ &= \frac{1-m+2}{m-2}R_{jk,s} + \frac{-1+m-2}{m-2}R_{js,k} \\ &\quad + \frac{1}{m-2}\frac{3-m}{2(m-1)}S_k\delta_{js} - \frac{1}{m-2}\frac{3-m}{2(m-1)}S_s\delta_{jk} \\ &= \frac{m-3}{m-2}C_{jsk}, \end{aligned}$$

where C_{jsk} are the components of the *Cotton tensor* C , i.e.,

$$C_{jsk} = R_{js,k} - R_{jk,s} + \frac{1}{2(m-1)}(S_s\delta_{jk} - S_k\delta_{js}).$$

Note that, if $m = 3$, then $\operatorname{div} W \equiv 0$; moreover, it is possible to prove that if $m = 3$, then $W \equiv 0$ (see for instance [Eis49]).

The Cotton tensor can also be interpreted in the following way. Let

$$A = \operatorname{Ric} - \frac{S}{2(m-1)} \langle \cdot, \cdot \rangle \quad (2.28)$$

be the *Schouten tensor* of components

$$A_{ij} = R_{ij} - \frac{S}{2(m-1)}\delta_{ij}.$$

Clearly A is symmetric, hence taking covariant derivatives

$$A_{ij,k} = A_{ji,k},$$

but for the last two indices one immediately verifies that

$$A_{ij,k} - A_{ik,j} = C_{ijk}.$$

Hence we can think of the Cotton tensor as the obstruction for the Schouten tensor to be Codazzi; in other words, A is a Codazzi tensor if and only if the Cotton tensor C is identically null. Although we shall not make use of A in what follows, we recall that it enables to write, for $m \geq 3$, the decomposition of the curvature tensor: denoting again with W the $(0, 4)$ version of the Weyl tensor we have

$$R = W + \frac{1}{m-2} A \oslash \langle , \rangle ,$$

where \oslash is the *Kulkarni-Nomizu product*; for the definition of this latter and more information on the Schouten tensor we refer to A. Besse's treatise [Bes08].

The importance of the Weyl and the Cotton tensors will be emphasized in Theorem 2.6 below, which is a classical result due to Weyl (case $m \geq 4$) and to Schouten (case $m = 3$).

We first recall the following

Definition 2.5. A Riemannian manifold (M, \langle , \rangle) of dimension $m \geq 2$ is said to be locally conformally flat if, for every $p \in M$, there exist an open set $U \ni p$ and a function $\varphi \in C^\infty(U)$, $\varphi > 0$, such that the manifold $(U, \varphi^2 \langle , \rangle)$ is flat.

Remark. Recall that, by the classical *Riemann theorem* (see for instance [Spi79]), the flat manifold $(U, \varphi^2 \langle , \rangle)$ is locally isometric to \mathbb{R}^m . It follows that M is locally conformally flat if each point $x \in M$ has a coordinate chart (U, ξ) such that $\xi^* \langle , \rangle_{\mathbb{R}^m}$ is pointwise conformally related to \langle , \rangle . Namely, $\xi : U \rightarrow \mathbb{R}^m$ is a conformal imbedding.

Obviously, flat spaces are locally conformally flat. Here are some more interesting examples of both conformally flat and nonconformally flat manifolds.

Example. Every 2-dimensional Riemannian manifold is locally conformally flat. This fact is known as “the existence of isothermic coordinates” on any smooth surface and was established by Korn and Lichtenstein, [Kor14], [Lic16], assuming that the Riemannian metric at hand has Hölder continuous coefficients. For a conceptually easier proof we refer the reader to the paper [Che55] by Chern. The case of real analytic Riemannian metrics goes back to Gauss. Some regularity condition on the coefficients of the metric is needed as shown by Hartman and Wintner, [HW53]. Now, since every smooth manifold supports a smooth Riemannian metric, using isothermic coordinates one concludes that any orientable smooth surface possesses an underlying complex structure.

Example. The standard sphere \mathbb{S}^m and the standard hyperbolic space \mathbb{H}^m are locally conformally flat. As for the hyperbolic space, we are identifying \mathbb{H}^m with either its Poincaré or its half-space model.

Example. An important class of m -dimensional, locally conformally flat manifolds are those admitting a conformal immersion into the standard sphere \mathbb{S}^m . In general, such an immersion does not exist. For instance, consider the flat (hence conformally flat) torus \mathbb{T}^m . By standard topological arguments, if \mathbb{T}^m were immersed into \mathbb{S}^m , the immersion would be a covering map. Therefore, the fundamental group of \mathbb{T}^m would inject into the fundamental group of \mathbb{S}^m , and this is clearly impossible. A general obstruction to the existence of conformal immersions into spheres is represented by the value of the *Yamabe invariant* of the manifold. Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold of dimension $m \geq 3$ with scalar curvature $S(x)$. The Yamabe invariant of M is the real constant

$$Y(M) = \inf_{v \in C_0^\infty(M) \setminus \{0\}} \frac{\int_M |\nabla v|^2 + \frac{m-2}{4(m-1)} S(x) v^2}{\left(\int_M v^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}}}.$$

Note that, in case $S(x) \equiv 0$, $Y(M)$ reduces to the ordinary Sobolev constant. By a result of Schoen and Yau, if M has a conformal immersion into \mathbb{S}^m , then $Y(M) = Y(\mathbb{S}^m) > 0$; see Chapter 6 in [SY94]. By way of example, it is a simple matter to verify that the Riemannian product $M = \mathbb{T}^{m-1} \times \mathbb{R}$ satisfies $Y(M) = 0$. Indeed, it is a complete, flat manifold with sub-linear volume growth.

Example. Suppose that $m - k \geq 2$. Then, it can be shown that the Riemannian product $\mathbb{S}^{m-k-1} \times \mathbb{H}^{k+1}$ is conformally diffeomorphic to $\mathbb{S}^m \setminus \mathbb{S}^k$, where $\mathbb{S}^k \subset \mathbb{S}^m$ is an equatorial k -sphere. In particular, $\mathbb{S}^{m-k-1} \times \mathbb{H}^{k+1}$ is locally conformally flat.

Example. If M has constant sectional curvature, then the Riemannian products $M \times \mathbb{R}$ and $M \times \mathbb{S}^1$ are locally conformally flat.

Example. In general, the Riemannian product of locally conformally flat manifolds is not locally conformally flat. For instance, the product of standard spheres $\mathbb{S}^m \times \mathbb{S}^m$, $m \geq 2$, is an example of a compact, simply connected, non-locally conformally flat manifold. A direct verification is possible. However, note that this follows directly from Kuiper's Theorem 2.7 below. Indeed, if $\mathbb{S}^m \times \mathbb{S}^m$ were locally conformally flat, by Theorem 2.7 it would be conformally diffeomorphic to the standard sphere \mathbb{S}^{2m} . In particular, the m -th de Rham cohomology groups would satisfy

$$H_{dR}^m(\mathbb{S}^m \times \mathbb{S}^m) \simeq H_{dR}^m(\mathbb{S}^{2m}),$$

but this is impossible. Indeed, for instance, a Mayer-Vietoris argument (see for example [Lee11]) shows that $H_{dR}^m(\mathbb{S}^{2m}) = 0$. On the other hand, by the Künneth formula, (see [GHV72])

$$H_{dR}^m(\mathbb{S}^m \times \mathbb{S}^m) \simeq \bigoplus_{k=0}^m H_{dR}^k(\mathbb{S}^m) \otimes H_{dR}^{m-k}(\mathbb{S}^m) \neq 0,$$

because, by Poincaré duality, $H_{dR}^m(\mathbb{S}^m) = H_{dR}^0(\mathbb{S}^m) = \mathbb{R}$.

One of the most important results in Riemann surfaces theory is the Riemann-Köbe uniformization theorem (see for instance [For91]), according to which every simply connected Riemann surface is bi-holomorphic either to the complex plane \mathbb{C} , to the open unit disk $\mathbb{B}^m \subset \mathbb{C}$ or to the Riemann sphere \mathbb{S}^2 . Note that a bi-holomorphism is a (orientation preserving) conformal diffeomorphism. Therefore, recalling the existence theorem for isothermic coordinates alluded to above, we conclude e.g. that every 2-dimensional, compact, simply connected (hence orientable) Riemannian manifold is conformally diffeomorphic to \mathbb{S}^2 . In general, Riemannian manifolds of dimension $m \geq 3$ do not enjoy any such uniformization property. An obstruction is given in the following (see [Eis49], [Bes08], [SY94])

Theorem 2.6. *Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold, $\dim M = m \geq 3$. A necessary and sufficient condition for $(M, \langle \cdot, \cdot \rangle)$ to be locally conformally flat is that*

$$\begin{cases} C \equiv 0 & \text{if } m = 3, \\ W \equiv 0 & \text{if } m \geq 4. \end{cases}$$

A complete classification of locally conformally flat manifolds is unknown. In general, in higher dimensions, uniformization does require curvature restrictions. However, in the compact setting, we have the following seminal result due to N. Kuiper, [Kui49].

Theorem 2.7. *Let $(M, \langle \cdot, \cdot \rangle)$ be a compact, simply connected, locally conformally flat manifold of dimension $m \geq 3$. Then, M is conformally diffeomorphic to the standard sphere \mathbb{S}^m .*

Proof. The idea of the proof is as follows. Details can be found in Chapter 6 of Schoen-Yau's book [SY94]. Since M is locally conformally flat, every point $x \in M$ has a neighborhood U_α conformally imbedded into \mathbb{R}^m , hence into \mathbb{S}^m . Let ξ_α be such a conformal imbedding. Consider the couple (U_α, ξ_α) . If (U_β, ξ_β) is a second local conformal imbedding, the transition function $\xi_\alpha \circ \xi_\beta^{-1}$ is a local conformal automorphism of \mathbb{S}^m . The classical Liouville theorem then shows that there exists a global conformal transformation $\psi_{(\alpha, \beta)} \in \text{Conf}(\mathbb{S}^m)$ such that

$$\psi_{(\alpha, \beta)} = \xi_\alpha \circ \xi_\beta^{-1}, \text{ on } \xi_\beta(U_\alpha \cap U_\beta).$$

Note that, with the obvious meaning of the symbols, the following cycle conditions are satisfied:

$$\begin{aligned} \psi_{(\alpha, \beta)} \circ \psi_{(\beta, \alpha)} &= id, \\ \psi_{(\alpha, \beta)} \circ \psi_{(\beta, \gamma)} \circ \psi_{(\gamma, \alpha)} &= id. \end{aligned} \tag{2.29}$$

We are now in the position to define a conformal immersion $\Phi : M \rightarrow \mathbb{S}^m$. Indeed, let $x_0 \in M$ and let $(U_{\alpha_0}, \xi_{\alpha_0})$ be a fixed conformal imbedding with $x_0 \in U_{\alpha_0}$. For every $x \in M$ let γ be a chosen path from $\gamma(0) = x_0$ to $\gamma(1) = x$ which is covered

by a finite chain of elements $(U_{\alpha_0}, \xi_{\alpha_0}), \dots, (U_{\alpha_n}, \xi_{\alpha_n})$ such that $U_{\alpha_j} \cap U_{\alpha_{j+1}} \neq \emptyset$. We define

$$\Phi(x) = \xi_{\alpha_0}(x), \text{ on } U_{\alpha_0},$$

and

$$\Phi(x) = \psi_{(a_0, \alpha_1)} \circ \dots \circ \psi_{(\alpha_{n-1}, \alpha_n)} \circ \xi_n(x), \text{ on } U_{\alpha_n}.$$

Since M is simply connected, using the cycle conditions (2.29) together with a monodromy argument shows that Φ is well defined and, by construction, Φ is a conformal immersion. It remains to show that Φ is a diffeomorphism. This follows from standard topological arguments. Indeed, since M is compact and \mathbb{S}^m is connected, Φ is a covering map. But \mathbb{S}^m is simply connected, hence Φ is a bijection. \square

2.2 Some applications in the compact case

2.2.1 A rigidity result of Obata

We now prove a rigidity result for compact Einstein manifolds due to M. Obata, [Oba62a]. Its proof relies on the transformation laws (2.26) and on a rigidity result for complete Riemannian manifolds supporting nontrivial solutions of certain differential inequalities; see Theorem 2.10 below.

Theorem 2.8. *Let $(M, \langle \cdot, \cdot \rangle)$ be a compact, Einstein manifold of dimension $m \geq 3$ and let $\widetilde{\langle \cdot, \cdot \rangle}$ be a pointwise conformal deformation of $\langle \cdot, \cdot \rangle$. If $(M, \widetilde{\langle \cdot, \cdot \rangle})$ has constant scalar curvature \tilde{S} , then $(M, \widetilde{\langle \cdot, \cdot \rangle})$ is Einstein. Furthermore, if $(M, \widetilde{\langle \cdot, \cdot \rangle})$ is not conformally diffeomorphic to the standard sphere \mathbb{S}^m , then $\widetilde{\langle \cdot, \cdot \rangle} = c^2 \langle \cdot, \cdot \rangle$, for some constant $c > 0$.*

Remark. This result has been generalized by J. Escobar, [Esc90], to the case where M has a nonempty boundary $\partial M \neq \emptyset$. In this situation, one also requires that the inclusion $\iota : \partial M \hookrightarrow (M, \langle \cdot, \cdot \rangle)$ is totally geodesic and that $\iota : \partial M \hookrightarrow (M, \widetilde{\langle \cdot, \cdot \rangle})$ is minimal. Thus, in the conclusion of the theorem, the standard sphere \mathbb{S}^m is replaced by the standard hemisphere \mathbb{S}_+^m . Escobar's proof follows closely Obata's original argument. We shall limit ourselves to proving Theorem 2.8, since in any case Escobar's theorem will result as a consequence of Theorem 2.16 of section 2.2.3.

Before proving the theorem we point out the following simple, interesting, consequence, to be compared with Theorem 2.2. Later on, in the setting of complete, non-Einstein manifolds, we shall give a version of the next result assuming that the conformal diffeomorphism at hand preserves the scalar curvature (see Theorem 5.9).

Corollary 2.9. *Let $(M, \langle \cdot, \cdot \rangle)$ be a compact Einstein manifold of dimension $m \geq 3$ which is not conformally diffeomorphic to \mathbb{S}^m . Let $\varphi \in \text{Conf}(M)$ be a conformal*

diffeomorphism such that $\varphi^* \langle , \rangle$ has constant scalar curvature. Then $\varphi \in \text{Iso}(M)$, that is, it is a Riemannian isometry.

Proof. By Theorem 2.8, $\varphi^* \langle , \rangle = c^2 \langle , \rangle$, for some constant $c > 0$. Therefore, the result follows from Lemma 2.1. \square

Now we give the

Proof of Theorem 2.8. For the sake of convenience, we set $\widetilde{\langle , \rangle} = \varphi^{-2} \langle , \rangle$ and denote with \sim quantities that refer to the metric $\widetilde{\langle , \rangle}$. According to (2.26),

$$T = \widetilde{T} + (m-2) \varphi \left\{ \widetilde{\text{Hess}}(\varphi^{-1}) - \frac{\widetilde{\Delta}(\varphi^{-1})}{m} \widetilde{\langle , \rangle} \right\}.$$

Since \langle , \rangle is Einstein we have $T \equiv 0$, and we deduce that, in components with respect to a local orthonormal coframe $\{\widetilde{\theta}^j\}$ for $\widetilde{\langle , \rangle}$,

$$-\widetilde{T}_{ij} = (m-2) \varphi \left\{ (\varphi^{-1})_{ij} - \frac{\widetilde{\Delta}(\varphi^{-1})}{m} \delta_{ij} \right\}. \quad (2.30)$$

Since \widetilde{T} is a traceless tensor,

$$\varphi^{-1} \left| \widetilde{T} \right|_{\widetilde{\langle , \rangle}}^2 = -(m-2) (\varphi^{-1})_{ij} \widetilde{T}_{ij}.$$

On the other hand, according to equation (2.13),

$$\widetilde{T}_{ik,k} = \frac{m-2}{2m} \widetilde{S}_i$$

and since \widetilde{S} is constant, we obtain

$$\widetilde{T}_{ik,k} = 0. \quad (2.31)$$

Thus, if we let $\{\widetilde{e}_j\}$ be the dual frame of $\{\widetilde{\theta}^j\}$, and define the vector field \widetilde{W} by the formula

$$\widetilde{W} = (\varphi^{-1})_i \widetilde{T}_{ij} \widetilde{e}_j, \quad (2.32)$$

where $d(\varphi^{-1}) = (\varphi^{-1})_i \widetilde{\theta}^i$, a computation that uses (2.31) shows that

$$\widetilde{\text{div}} \widetilde{W} = (\varphi^{-1})_{ij} \widetilde{T}_{ij} + (\varphi^{-1})_i \widetilde{T}_{ik,k} = (\varphi^{-1})_{ij} \widetilde{T}_{ij},$$

and we conclude that

$$\varphi^{-1} \left| \widetilde{T} \right|_{\widetilde{\langle , \rangle}}^2 = -(m-2) \widetilde{\text{div}} \widetilde{W}.$$

Integrating on $(M, \langle \cdot, \cdot \rangle)$ and using the divergence theorem we deduce that

$$\int_M \varphi^{-1} \left| \widetilde{T} \right|_{\langle \cdot, \cdot \rangle}^2 d\text{vol} = 0.$$

Accordingly, $\widetilde{T} \equiv 0$ and $(M, \langle \cdot, \cdot \rangle)$ is an Einstein manifold.

Suppose now that the m -dimensional manifold $(M, \langle \cdot, \cdot \rangle)$ is not conformally diffeomorphic to \mathbb{S}^m , $m \geq 3$. Having set as usual $\langle \cdot, \cdot \rangle = u^{\frac{4}{m-2}} \langle \cdot, \cdot \rangle$, we shall prove that u is identically equal to a positive constant c^2 . Let S be the scalar curvature of $\langle \cdot, \cdot \rangle$, which is constant since $(M, \langle \cdot, \cdot \rangle)$ is Einstein. We first claim that either $S = \widetilde{S} = 0$ or $S \cdot \widetilde{S} > 0$. To see this, we recall that S and \widetilde{S} are related by the Yamabe equation

$$c_m \Delta u - Su + \widetilde{S} u^{\frac{m+2}{m-2}} = 0, \quad c_m = 4 \frac{m-1}{m-2}.$$

Suppose that $S \leq 0$ and $\widetilde{S} \geq 0$. Then,

$$\Delta u \geq 0,$$

and since $(M, \langle \cdot, \cdot \rangle)$ is compact we deduce that u is a positive constant. Therefore, inserting this information into the Yamabe equation, we conclude that

$$0 \geq S = \widetilde{S} u^{\frac{4}{m-2}} \geq 0,$$

proving that $S = \widetilde{S} = 0$, and the claim follows. The same conclusion holds if we assume instead that $S \geq 0$ and $\widetilde{S} \leq 0$. Thus, we need to consider three possible cases.

First case: $S = \widetilde{S} = 0$. Using the Yamabe equation once more gives $\Delta u = 0$ and, therefore, u is a positive constant.

Second case: $S < 0$ and $\widetilde{S} < 0$. Up to rescaling $\langle \cdot, \cdot \rangle$ by a positive constant, we can assume that $S = \widetilde{S} < 0$. Let

$$u(x_0) = \max_M u, \quad u(x_1) = \min_M u.$$

Then, by the usual maximum principle, $\Delta u(x_0) \leq 0$ which implies, according to the Yamabe equation,

$$Su(x_0) \left(1 - u^{\frac{4}{m-2}}(x_0) \right) \leq 0.$$

As a consequence,

$$u(x) \leq u(x_0) \leq 1.$$

Similarly, since u achieves its minimum at x_1 we have $\Delta u(x_1) \geq 0$. This latter, in turn, implies

$$1 \leq u(x_1) \leq u(x),$$

and, therefore, $u \equiv 1$.

Third case: $S > 0$ and $\widetilde{S} > 0$. Let us observe that, since $\widetilde{\langle \cdot, \cdot \rangle} = u^{\frac{4}{m-2}} \langle \cdot, \cdot \rangle$ and both the metrics are Einstein, setting

$$v = u^{-\frac{2}{m-2}},$$

by (2.26) we have

$$\text{Hess}(v) - \frac{\Delta v}{m} \langle \cdot, \cdot \rangle = 0 \quad \text{on } M. \quad (2.33)$$

We define the vector field $X = \nabla v$ so that $\text{div } X = \Delta v$ and

$$\frac{1}{2} \mathcal{L}_X \langle \cdot, \cdot \rangle = \text{Hess}(v).$$

From the above, we deduce

$$\frac{1}{2} \mathcal{L}_X \langle \cdot, \cdot \rangle = \frac{\text{div } X}{m} \langle \cdot, \cdot \rangle,$$

that is, X is a conformal vector field. We now show, using again the moving frame formalism, that since M is Einstein, $f = \text{div } X = \Delta v$ satisfies the equation

$$\text{Hess}(f) + \frac{S}{m(m-1)} f \langle \cdot, \cdot \rangle = 0, \quad (2.34)$$

where, we recall, $s > 0$. First we observe that the previous equation can be rewritten as

$$\text{Hess}(\Delta v) + \Delta v \frac{S}{m(m-1)} \langle \cdot, \cdot \rangle = 0,$$

or, in components,

$$(\Delta v)_{kt} = v_{iikt} = -\Delta v \frac{S}{m(m-1)} \delta_{kt};$$

then we note that, by (2.33), we have

$$v_{ij} = \frac{v_{tt}}{m} \delta_{ij}, \quad (2.35)$$

from which we deduce

$$v_{ijk} = \frac{v_{ttk}}{m} \delta_{ij} \quad (2.36)$$

and

$$v_{ijkl} = \frac{v_{ttkl}}{m} \delta_{ij}. \quad (2.37)$$

By (1.54) applied to v we deduce

$$\begin{aligned} v_{iikl} &= (\Delta v)_{kl} = v_{ikil} + v_{sl} R_{s i i k} + v_s R_{s i i k, l} \\ &= v_{k i i l} - v_{s l} R_{s k} - v_s R_{s k, l} = v_{k i i l} - v_{s l} R_{s k}, \end{aligned}$$

since $R_{sk,l} = 0$. Then, using (2.35), (2.36) and (2.37) we obtain

$$(\Delta v)_{kl} = v_{kii} - \frac{v_{ii}}{m} \delta_{sl} R_{sk} = \frac{v_{ssil} \delta_{ki}}{m} - \frac{v_{ii}}{m} R_{kl},$$

which easily implies (2.34).

In case f is constant, we get $f = \Delta v = 0$ and, since M is compact, v itself must be constant. This implies that also u is constant, as desired. Finally, in case f is nonconstant, we can apply the next result to conclude that $(M, \langle \cdot, \cdot \rangle)$ is isometric to the sphere $\mathbb{S}_{k^2}^m$ of constant curvature $k^2 = S/m(m-1)$. But this implies that $(M, \langle \cdot, \cdot \rangle)$ is conformally diffeomorphic to the standard sphere \mathbb{S}^m , against our initial assumption. \square

The following result is again due to Obata, [Oba62a] (for the analogous statement for \mathbb{S}_+^m , the upper hemisphere, see Theorem 2.14 below); see also [PR].

Theorem 2.10. *Let $(M, \langle \cdot, \cdot \rangle)$ be a complete Riemannian manifold of dimension m . Suppose that there exists a nonconstant, smooth function $f : M \rightarrow \mathbb{R}$ such that, for some constant $k > 0$,*

$$\text{Hess}(f) + k^2 f \langle \cdot, \cdot \rangle = 0. \quad (2.38)$$

Then $(M, \langle \cdot, \cdot \rangle)$ is isometric to the sphere $\mathbb{S}_{k^2}^m$ of constant sectional curvature k^2 .

Proof. First, we show that f has a critical point. Indeed, by contradiction, suppose that $\nabla f \neq 0$ on M . Consider the smooth vector field $X = \nabla f / |\nabla f|$ on M . Since $|X| \in L^\infty(M)$ and $(M, \langle \cdot, \cdot \rangle)$ is geodesically complete, then X is complete (see e.g. [Lee03], Chapter 12). Let $\gamma : \mathbb{R} \rightarrow M$ be an integral curve of X , namely $X_{\gamma(s)} = \dot{\gamma}(s)$, for every $s \in \mathbb{R}$. A direct computation that uses (2.38) shows that, for every vector field Y ,

$$\langle D_{\dot{\gamma}} \dot{\gamma}, Y \rangle = \frac{1}{|\nabla f|} \text{Hess}(f)(\dot{\gamma}, Y) - \frac{1}{|\nabla f|} \text{Hess}(f)(\dot{\gamma}, \dot{\gamma}) \langle \dot{\gamma}, Y \rangle = 0.$$

Therefore, γ is a unit speed geodesic of M . Evaluating (2.38) along γ we deduce that the function $y(s) = f \circ \gamma(s)$ satisfies

$$y'' = -k^2 y$$

which is oscillatory. Let $s_0 \in \mathbb{R}$ be such that $y'(s_0) = 0$. Then, recalling that γ is an integral curve of X , we conclude

$$0 = y'(s_0) = \langle \nabla f(\gamma(s_0)), \dot{\gamma}(s_0) \rangle = |\nabla f(\gamma(s_0))| \neq 0,$$

a contradiction.

Let $o \in M$ be a critical point of f and set $r(x) = d(x, o)$. Note that $f(o) \neq 0$ for otherwise, having fixed any unit speed geodesic γ issuing from o , we would have

that $y(s) = f \circ \gamma(s)$ solves the Cauchy problem

$$\begin{cases} y'' = -k^2 y, \\ y(0) = 0, \\ y'(0) = 0, \end{cases}$$

and, hence, $y(s) \equiv 0$. Since this would be true for every geodesic γ we should conclude that $f \equiv 0$, a contradiction. Thus, without loss of generality, we assume that $f(o) = 1$. We claim that, for every $x \in M$, it holds that

$$f(x) = \cos(kr(x)). \quad (2.39)$$

Indeed, consider a unit speed, minimizing geodesic $\gamma : [0, r(x)] \rightarrow M$ from $\gamma(0) = o$ to $\gamma(r(x)) = x$. As noted above, the smooth function $y(s) = f \circ \gamma(s)$ is the solution of the Cauchy problem

$$\begin{cases} y'' = -k^2 y, \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$$

Therefore, $y(s) = \cos(ks)$. Evaluating at $s = r(x)$ we conclude the validity of (2.39). Now, observe that the function $\cos(ks)$ is strictly decreasing on $(0, \pi/k)$. It follows from (2.39) that

$$r(x) = k^{-1} \arccos(f(x))$$

is smooth on the geodesic ball $B_{\pi/k}(o) \setminus \{o\}$. Applying the Bishop density result of Chapter 1 we therefore conclude that

$$\text{cut}(o) \cap B_{\pi/k}(o) = \emptyset. \quad (2.40)$$

Therefore, the exponential map $\exp_o : \mathbb{B}_{\pi/k}^m(0) \subset T_o M \rightarrow B_{\pi/k}(o)$ is a diffeomorphism. Let us introduce geodesic polar coordinates (r, θ) on $T_o M$. Furthermore, let us consider a local orthonormal coframe $\{\theta^\alpha\}$ on \mathbb{S}^{m-1} with dual frame $\{E_\alpha\}$. Thus, the standard metric of \mathbb{S}^{m-1} is written as $d\theta^2 = \sum \theta^\alpha \otimes \theta^\alpha$. We extend both $\{\theta^\alpha\}$ and $\{E_\alpha\}$ radially. Then, by Gauss' lemma,

$$\langle \cdot, \cdot \rangle = dr \otimes dr + \sigma_{\alpha\beta}(r, \theta) \theta^\alpha \otimes \theta^\beta.$$

Furthermore, since $\langle \cdot, \cdot \rangle$ is infinitesimally Euclidean and the standard metric of $\mathbb{R}^m \approx T_o M$ is written as

$$\langle \cdot, \cdot \rangle_{\mathbb{R}^m} = dr \otimes dr + r^2 \delta_{\alpha\beta} \theta^\alpha \otimes \theta^\beta,$$

we have the further condition

$$\sigma_{\alpha\beta}(r, \theta) = \delta_{\alpha\beta} r^2 + o(r^2), \text{ as } r \searrow 0. \quad (2.41)$$

Now we use the fact that

$$\mathcal{L}_{\nabla r} \langle \cdot, \cdot \rangle = 2\text{Hess}(r), \text{ on } B_{\pi/k}(o) \setminus \{o\}$$

where $\mathcal{L}_{\nabla r}$ is the Lie derivative in the radial direction ∇r . Thus, by the definition of Lie derivative we deduce that

$$2\text{Hess}(r)(E_\alpha, E_\beta) = \mathcal{L}_{\nabla r} \langle \cdot, \cdot \rangle(E_\alpha, E_\beta) = \nabla r(\sigma_{\alpha\beta}). \quad (2.42)$$

Since

$$\nabla r = -\frac{\nabla f}{|\nabla f|},$$

the Hessian of the distance function r can be expressed as

$$\begin{aligned} \text{Hess}(r)(E_\alpha, E_\beta) &= -\left\langle D_{E_\alpha} \frac{\nabla f}{|\nabla f|}, E_\beta \right\rangle \\ &= -\frac{1}{|\nabla f|} \text{Hess}(f)(E_\alpha, E_\beta) \\ &= \frac{k^2 \cos(kr)}{k \sin(kr)} \langle E_\alpha, E_\beta \rangle \\ &= k \cot(kr) \sigma_{\alpha\beta}. \end{aligned}$$

Inserting into (2.42), and recalling (2.41) we obtain that $\sigma_{\alpha\beta}$ are the (unique) solutions of the asymptotic Cauchy problems

$$\begin{cases} \partial_r \sigma_{\alpha\beta} = 2k \cot(kr) \sigma_{\alpha\beta}, \text{ on } B_{\pi/k}(o) \setminus \{o\}, \\ \sigma_{\alpha\beta}(r, \theta) = \delta_{\alpha\beta} r^2 + o(r^2), \text{ as } r \searrow 0. \end{cases}$$

Integrating, finally gives

$$\sigma_{\alpha\beta} = k^{-2} \sin^2(kr) \delta_{\alpha\beta}.$$

We have thus established that, in polar coordinates of $\mathbb{B}_{\pi/k}^m(0) \setminus \{o\} = (0, \pi/k) \times \mathbb{S}^{m-1} \subset T_o M$, it holds that

$$\langle \cdot, \cdot \rangle = dr \otimes dr + k^{-2} \sin^2(kr) d\theta^2.$$

Since

$$((0, \pi/k) \times \mathbb{S}^{m-1}, dr \otimes dr + k^{-2} \sin^2(kr) d\theta^2) \quad (2.43)$$

is isometric to the m -dimensional 2-punctured sphere $\mathbb{S}_{k^2}^m \setminus \{2 \text{ points}\}$ of constant curvature k^2 , we conclude that the geodesic ball $B_{\pi/k}(o) \subset M$ is isometric to $\mathbb{S}_{k^2}^m \setminus \{\text{point}\}$. To complete the proof it suffices to show that

$$B_{\pi/k}(o) = M \setminus \{\text{point}\}. \quad (2.44)$$

Indeed, suppose we have already proved this fact. Then, by Seifert-Van Kampen's theorem, (see [Lee11]) M is simply connected. Moreover, $M \setminus \{\text{point}\}$ has constant curvature k^2 because it is isometric to $S_{k^2}^m \setminus \{\text{point}\}$. By continuity, M itself has constant curvature k^2 . Therefore, the Hopf classification theorem (see [Pet06b]) tells us that M must be isometric to $S_{k^2}^m$, as desired. The proof of (2.44) combines Morse theoretic and cut-locus arguments. First of all, we observe that $\partial B_{\pi/k}(o)$ is made up entirely by nondegenerate critical points of f . Therefore, by Morse's lemma, (see [Mil63], [Pet06b]) $\partial B_{\pi/k}(o)$ is a discrete, compact (hence finite) set. Indeed, for every $x \in \partial B_{\pi/k}(o)$, by continuity we have

$$|\nabla f|(x) = -k \sin(\pi) = 0,$$

and, by assumption,

$$\text{Hess}(f)(x) = -k^2 f(x) \langle \cdot, \cdot \rangle_x,$$

where, $f(x) = \cos(\pi) = -1$. In particular, $\text{Hess}(f)$ is (strictly) negative definite on $\partial B_{\pi/k}(o)$, as claimed. To conclude, we note that $\partial B_{\pi/k}(o)$ is connected. In fact,

$$\partial B_{\pi/k}(o) = \exp_o \left(\partial \mathbb{B}_{\pi/k}^m(0) \right). \quad (2.45)$$

To see this, we recall that

$$\exp_o \left(\partial \mathbb{B}_{\pi/k}^m(0) \cap T_o M \setminus c(o) \right) = \partial B_{\pi/k}(o) \cap M \setminus \text{cut}(o), \quad (2.46)$$

where $c(o) \subset T_o M$ is the tangential cut-locus of $o \in M$. Let $\bar{v} \in \partial \mathbb{B}_{\pi/k}^m(0) \cap c(o)$. By definition, $\bar{x} = \exp_o \bar{v} \in \text{cut}(o)$. Since, by (2.40),

$$B_{\pi/k}(o) \subset M \setminus \text{cut}(o),$$

and, on a generic complete Riemannian manifold,

$$\exp_o \left(\overline{\mathbb{B}_{\pi/k}^m(0)} \right) \subseteq \overline{B_{\pi/k}(o)},$$

we must conclude that $\bar{x} \notin B_{\pi/k}(o)$, that is,

$$\exp_o \left(\partial \mathbb{B}_{\pi/k}^m(0) \cap c(o) \right) \subseteq \partial B_{\pi/k}(o) \cap \text{cut}(o).$$

Conversely, let $x \in \partial B_{\pi/k}(o) \cap \text{cut}(o)$. By definition, there is a unit speed geodesic $\gamma(t) = \exp_o(vt) : [0, +\infty) \rightarrow M$ from $\gamma(0) = o$ to $\gamma(\pi/k) = x$ that does not minimize distances past π/k . Then, $v\pi/k \in \partial \mathbb{B}_{\pi/k}^m(0) \cap c(o)$ and $\exp_o(v\pi/k) = x$. Summarizing,

$$\exp_o \left(\partial \mathbb{B}_{\pi/k}^m(0) \cap c(o) \right) = \partial B_{\pi/k}(o) \cap \text{cut}(o). \quad (2.47)$$

From (2.46) and (2.47) we conclude the validity of (2.45). \square

2.2.2 A result by M. F. Bidaut-Véron and L. Véron

The vector field \widetilde{W} defined in (2.32), with \widetilde{T}_{ij} given in (2.30), will suggest to us how to proceed to provide a proof of Theorem 2.12 below. First we give an intrinsic definition: by direct manipulation of the quantities involved we have

$$\begin{aligned}
 \widetilde{W} &= (\varphi^{-1})_i \left\{ -(\varphi^{-1})_{ij} + \frac{(\varphi^{-1})_{kk}}{m} \delta_{ij} \right\} (m-2) \varphi \widetilde{e}_j \\
 &= \frac{m-2}{m} \varphi (\varphi^{-1})_{kk} (\varphi^{-1})_i \widetilde{e}_i - (m-2) \varphi (\varphi^{-1})_{ij} (\varphi^{-1})_i \widetilde{e}_j \\
 &= \frac{m-2}{m} \varphi (\widetilde{\Delta} \varphi^{-1}) \widetilde{\nabla} \varphi^{-1} - (m-2) \varphi \widetilde{\text{Hess}}(\varphi^{-1}) \left(\widetilde{\nabla} \varphi^{-1}, \cdot \right)^\# \\
 &= (m-2) \varphi \left\{ \frac{\widetilde{\Delta} \varphi^{-1}}{m} \widetilde{\nabla} \varphi^{-1} - \widetilde{\text{Hess}}(\varphi^{-1}) \left(\widetilde{\nabla} \varphi^{-1}, \cdot \right)^\# \right\} \\
 &= (m-2) \varphi \left\{ \frac{\widetilde{\Delta} \varphi^{-1}}{m} \widetilde{\nabla} \varphi^{-1} - \frac{1}{2} \widetilde{\nabla} \left| \widetilde{\nabla} \varphi^{-1} \right|_{\langle \cdot, \cdot \rangle}^2 \right\}.
 \end{aligned}$$

We set the following general

Definition 2.11. Given $u \in C^2(M)$, $u > 0$, the vector field

$$Z = u^\alpha \left\{ \frac{1}{2} \nabla |\nabla u|^2 - \frac{\Delta u}{m} \nabla u \right\}, \quad \alpha \in \mathbb{R}, \quad (2.48)$$

is called an Obata type vector field.

Thus \widetilde{W} above is an Obata vector field (modulo the multiplicative constant $2-m$) with respect to the metric $\widetilde{\langle \cdot, \cdot \rangle}$. Furthermore, the function involving φ satisfies equation (2.7) with the roles of $\langle \cdot, \cdot \rangle$ and $\widetilde{\langle \cdot, \cdot \rangle}$ interchanged, that is

$$\varphi^2 S = \widetilde{S} - 2(m-1) \frac{\widetilde{\Delta} \varphi}{\varphi} - (m-1)(m-4) \frac{\left| \widetilde{\nabla} \varphi \right|_{\widetilde{\langle \cdot, \cdot \rangle}}^2}{\varphi^2}.$$

This, in turn, implies

$$\begin{aligned}
 \widetilde{\Delta} \varphi^{-1} &= -\frac{1}{\varphi^2} \widetilde{\Delta} \varphi + \frac{2}{\varphi^3} \left| \widetilde{\nabla} \varphi \right|_{\widetilde{\langle \cdot, \cdot \rangle}}^2 \\
 &= \frac{S}{2(m-1)} \varphi - \frac{\widetilde{S}}{2(m-1)} \varphi^{-1} + \frac{m}{2} \frac{\left| \widetilde{\nabla} \varphi \right|_{\widetilde{\langle \cdot, \cdot \rangle}}^2}{\varphi^3} \\
 &= -\frac{\widetilde{S}}{2(m-1)} \varphi^{-1} + \frac{S}{2(m-1)} (\varphi^{-1})^{-1} + \frac{m}{2} \frac{\left| \widetilde{\nabla} \varphi^{-1} \right|_{\widetilde{\langle \cdot, \cdot \rangle}}^2}{\varphi^{-1}}.
 \end{aligned}$$

Therefore, $u = \varphi^{-1}$ is a positive solution of the differential equation

$$\tilde{\Delta}u = \frac{m}{2} \frac{|\tilde{\nabla}u|^2}{u} + \frac{S}{2(m-1)} \left(u^{-1} - \frac{\tilde{S}}{S} u \right).$$

We have thus obtained the further suggestion to consider the vector field Z defined in (2.48), with u a positive solution of a differential equation of the type

$$\Delta u = (\beta + 1) \frac{|\nabla u|^2}{u} + \frac{1}{\beta} \left(u^{1+\beta(1-\sigma)} - \lambda u \right), \quad \beta \in \mathbb{R} \setminus \{0\}, \sigma > 1, \lambda \in \mathbb{R}. \quad (2.49)$$

Note that, if

$$\Delta u = -f(u, |\nabla u|), \quad f = f(u, t) \in C^1(\mathbb{R}^2),$$

a straightforward computation which exploits the Bochner-Weitzenböck formula (1.48) gives

$$\begin{aligned} \operatorname{div} Z &= u^\alpha \left\{ |\operatorname{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \right\} \\ &+ u^{\alpha-1} \left\{ \frac{\alpha}{m} f(u, |\nabla u|) + \frac{1-m}{m} u \frac{\partial f}{\partial u}(u, |\nabla u|) \right\} |\nabla u|^2 \\ &+ u^\alpha \operatorname{Ric}(\nabla u, \nabla u) + \alpha u^{\alpha-1} \operatorname{Hess}(u)(\nabla u, \nabla u) \\ &+ \frac{1-m}{m} u^\alpha \frac{\partial f}{\partial t}(u, |\nabla u|) |\nabla u|^{-1} \operatorname{Hess}(u)(\nabla u, \nabla u). \end{aligned}$$

Hence, if f has the special form

$$f(u, |\nabla u|) = \delta \frac{|\nabla u|^2}{u} + g(u), \quad \delta \in \mathbb{R},$$

the previous formula becomes

$$\begin{aligned} \operatorname{div} Z &= u^\alpha \left\{ |\operatorname{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \right\} \\ &+ u^{\alpha-1} \left\{ \frac{\alpha}{m} g(u) + \frac{1}{m} (\alpha + m - 1) \delta \frac{|\nabla u|^2}{u} + \frac{1-m}{m} u g'(u) \right\} |\nabla u|^2 \\ &+ u^\alpha \operatorname{Ric}(\nabla u, \nabla u) \\ &+ u^{\alpha-1} \left\{ \alpha + 2\delta \frac{1-m}{m} \right\} \operatorname{Hess}(u)(\nabla u, \nabla u). \end{aligned}$$

To get rid of the last term containing $\operatorname{Hess}(u)(\nabla u, \nabla u)$ we observe that, given $\gamma \in \mathbb{R}$,

$$\operatorname{div} (\gamma u^\beta |\nabla u|^2 \nabla u) = 2\gamma u^\beta \operatorname{Hess}(u)(\nabla u, \nabla u) + \gamma u^\beta |\nabla u|^2 \Delta u + \beta \gamma u^{\beta-1} |\nabla u|^4.$$

We choose $\beta = \alpha - 1$, $\gamma = \frac{(m-1)\delta}{m} - \frac{\alpha}{2}$ so that, for the vector field

$$V = Z + \left(\frac{m-1}{m}\delta - \frac{\alpha}{2} \right) u^{\alpha-1} |\nabla u|^2 \nabla u, \quad (2.50)$$

we have

$$\begin{aligned} \operatorname{div} V &= u^\alpha \left\{ |\operatorname{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \right\} \\ &+ u^{\alpha-1} \left\{ \frac{\alpha(m+2) - 2(m-1)\delta}{2m} g(u) - \frac{m-1}{m} u g'(u) \right\} |\nabla u|^2 \\ &+ \frac{u^{\alpha-2}}{2m} \{ 3m\alpha\delta - m\alpha(\alpha-1) - 2\delta^2(m-1) \} |\nabla u|^4 \\ &+ u^\alpha \operatorname{Ric}(\nabla u, \nabla u). \end{aligned}$$

Now, if $\delta = -(\beta + 1)$ and $g(u) = \frac{\lambda}{\beta} u - \frac{1}{\beta} u^{1+\beta(1-\sigma)}$ from the previous relation we obtain

$$\begin{aligned} \operatorname{div} V &= u^\alpha \left\{ |\operatorname{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \right\} \\ &- \frac{1}{2m\beta} [2(m-1)\beta\sigma + (m+2)\alpha] u^{\alpha+\beta(1-\sigma)} |\nabla u|^2 \\ &- \frac{1}{2m} [m\alpha^2 + (2+3\beta)m\alpha + 2(m-1)(\beta+1)^2] u^{\alpha-2} |\nabla u|^4 \\ &+ \frac{1}{2m} \left\{ 2m \operatorname{Ric}(\nabla u, \nabla u) + \frac{\lambda}{\beta} [\alpha(m+2) + 2(m-1)\beta] |\nabla u|^2 \right\} u^\alpha. \end{aligned} \quad (2.51)$$

We are now ready to prove the following beautiful result first due to Bidaut-Véron and Véron, [BVV91].

Theorem 2.12. *Let $(M, \langle \cdot, \cdot \rangle)$ be a compact manifold of dimension $m \geq 2$ and Ricci curvature satisfying*

$$\operatorname{Ric} \geq k > 0.$$

Let φ be a positive solution of

$$\Delta \varphi - \lambda \varphi + \varphi^\sigma = 0 \quad \text{on } M \quad (2.52)$$

for some constant $\sigma > 1$, $\lambda > 0$. Then φ is constant provided

- (i) $m = 2$ and $\lambda \leq 2k/(\sigma - 1)$;
- (ii) $m \geq 3$, $\lambda \leq mk/(m-1)(\sigma - 1)$, $\sigma \leq (m+2)/(m-2)$ and
 - (A) either at least one of the last two inequalities is strict or
 - (B) $(M, \langle \cdot, \cdot \rangle)$ has constant scalar curvature S and it is not isometric to $S_{k^2}^m$, the sphere of constant curvature $k^2 = s/(m-1)m$.

Proof. We set $u = \varphi^{-1/\beta}$, $\beta \neq 0$. Then u is positive and satisfies equation (2.49), hence (2.51) holds. With the aid of the divergence theorem we obtain

$$\begin{aligned} 0 = \int_M 2mu^\alpha \left\{ |\text{Hess}(u)|^2 - \frac{1}{m}(\Delta u)^2 \right\} - A \int_M u^{\alpha-2} |\nabla u|^4 \\ + \int_M u^\alpha \left\{ 2m \text{Ric}(\nabla u, \nabla u) + D |\nabla u|^2 \right\} - B \int_M u^{\alpha-\beta(\sigma-1)} |\nabla u|^2, \end{aligned} \quad (2.53)$$

where, for ease of notation, we have set

$$\begin{aligned} A &= m\alpha^2 + (3\beta + 2)m\alpha + 2(m-1)(1+\beta)^2, \\ B &= \frac{1}{\beta}[2(m-1)\beta\sigma + (m+2)\alpha], \quad D = \frac{\lambda}{\beta}[(m+2)\alpha + 2(m-1)\beta]. \end{aligned}$$

Next we observe that, by Newton's inequality

$$|\text{Hess}(u)|^2 \geq \frac{1}{m}(\Delta u)^2,$$

the first integral on the right-hand side of the above is nonnegative. The idea of the proof is to find, under the conditions listed in (i) and (ii), $\beta \neq 0$ and $\alpha \in \mathbb{R}$, such that

$$A \leq 0, \quad B \leq 0, \quad \text{and} \quad \text{Ric} + \frac{D}{2m} \geq 0 \quad (2.54)$$

and at least one of the above inequalities is strict. Once this is achieved, then (2.53) implies that $\nabla u \equiv 0$; thus u and therefore φ are constant. Let $y = 1 + 1/\beta$, $\delta = -\alpha/\beta$, so that $y, \delta \in \mathbb{R}$, $y \neq 1$. Rewriting A, B and D in terms of y and δ , the inequalities to be established become

$$\begin{aligned} \text{(a)} \quad & 2\frac{m-1}{m}y^2 - 2\delta y + \delta^2 - \delta \leq 0, \\ \text{(b)} \quad & 2\sigma\frac{m-1}{m+2} \leq \delta, \\ \text{(c)} \quad & 2\frac{m}{m+2} \text{Ric} \geq \lambda\left(\delta - 2\frac{m-1}{m+2}\right) \langle \cdot, \cdot \rangle, \end{aligned} \quad (2.55)$$

with at least one strict inequality. If either (i) or (ii) holds, then, for $m \geq 2$,

$$2\frac{m}{m+2} \text{Ric} \geq \lambda\left(2\frac{m-1}{m+2}\sigma - 2\frac{m-1}{m+2}\right) \langle \cdot, \cdot \rangle \quad (2.56)$$

and setting

$$\delta = 2\sigma\frac{m-1}{m+2} > 0,$$

inequalities (2.55) (b) and (c) are satisfied. In order to find a value $y \neq 1$ which satisfies (2.55) (a) with strict inequality, it suffices that the quadratic polynomial in y on the left-hand side has two distinct solutions, which, taking into account our choice of σ , in turn amounts to the validity of the inequality $(m+1) - (m-2)\sigma > 0$.

This inequality being always trivially satisfied if $m = 2$, it remains to analyze the case where $m > 2$, $\sigma = (m + 2) / (m - 2)$. We assume that $(M, \langle \cdot, \cdot \rangle)$ has constant scalar curvature S and show that if φ is a positive nonconstant solution of (2.49), then $(M, \langle \cdot, \cdot \rangle)$ is isometric to the sphere $\mathbb{S}_{k^2}^m$ of constant sectional curvature $k^2 = S / (m - 1)m$. Our assumption that φ is not constant implies that u is positive, and nonconstant; by the condition on σ , equality holds in (2.55) (b) and (a) with $y = \frac{m}{m-2}$, so that $A = B = 0$. Since the third and the fourth summands in (2.53) vanish, while the integrands in the first and third integral are nonnegative, they must vanish identically. In particular we must have

$$|\text{Hess}(u)|^2 = \frac{(\Delta u)^2}{m}, \text{ on } M$$

and therefore, by the equality case in the Cauchy-Schwarz inequality,

$$\text{Hess}(u) = \frac{\Delta u}{m} \langle \cdot, \cdot \rangle, \text{ on } M.$$

Now we proceed as in the proof of Theorem 2.8. We set $X = \nabla u$ and we observe that $f = \text{div } X$ satisfies

$$\text{Hess}(f) + \frac{S}{m(m-1)} f \langle \cdot, \cdot \rangle = 0.$$

Note that f is not constant, for otherwise the above would imply $Sf \equiv 0$ and since $S \geq mk > 0$ by the assumption on the Ricci curvature, we would conclude that

$$f = \Delta u = 0$$

so that u is constant on M . Obata's Theorem 2.10 implies the conclusion. \square

Remark. 1. Equation (2.52) is, of course, a normalized version of the more general case

$$\Delta \varphi - \lambda \varphi + \mu \varphi^\sigma = 0 \quad \text{on } M, \quad (2.57)$$

with $\lambda, \mu > 0$ and $\varphi > 0$. In this situation the required condition on λ depends on μ .

2. Similarly to what happened in the proof of Theorem 2.8, if we consider equation (2.57) with $\lambda \leq 0$, $\mu \geq 0$, then $\Delta \varphi \leq 0$ on the compact manifold $(M, \langle \cdot, \cdot \rangle)$, and hence φ is a positive constant by the maximum principle. For $\lambda \geq 0$, $\mu \leq 0$, then $\Delta \varphi - \lambda \varphi \geq 0$ and we obtain the same conclusion. Finally, for $\lambda \leq 0$, $\mu \leq 0$, $\Delta \varphi + (\mu \varphi^{\sigma-1}) \varphi \leq 0$ and again φ is constant.
3. The above observation also shows that Theorem 2.8 is a consequence of Theorem 2.12 for $m \geq 3$. Indeed, if S is the scalar curvature of $(M, \langle \cdot, \cdot \rangle)$ and $\widetilde{\langle \cdot, \cdot \rangle} = \varphi^{\frac{4}{m-2}} \langle \cdot, \cdot \rangle$, $\varphi > 0$, then $c_m \Delta \varphi - S \varphi + \widetilde{S} \varphi = 0$, with \widetilde{S} the constant scalar curvature of $\widetilde{\langle \cdot, \cdot \rangle}$. However, the Einstein condition is not required for $(M, \langle \cdot, \cdot \rangle)$ in Theorem 2.12.

2.2.3 A version of Theorem 2.12 on manifolds with boundary

We shall now give a version of Theorem 2.12 in case M is compact with nonempty boundary ∂M . For simplicity we consider the case that M is oriented. In the next result we explicate the boundary term in applying the divergence theorem to (2.51).

Lemma 2.13. *Let $(M, \langle \cdot, \cdot \rangle)$ be an m -dimensional oriented compact Riemannian manifold with boundary ∂M . Let Π and h denote respectively the second fundamental tensor and the mean curvature of the embedding $\partial M \hookrightarrow M$ in the direction of the outward unit normal vector field ν . Let $u \in C^3(M)$ and let $\tilde{u} = u|_{\partial M}$. If V is the vector field defined in (2.50), that is*

$$V = u^\alpha \left\{ \frac{1}{2} \nabla |\nabla u|^2 - \frac{\Delta u}{m} \nabla u \right\} - \left(\frac{m-1}{m} (\beta + 1) + \frac{\alpha}{2} \right) u^{\alpha-1} |\nabla u|^2 \nabla u,$$

we have

$$\begin{aligned} 2m \int_M \operatorname{div} V &= \int_{\partial M} 2\tilde{u}^\alpha [(m-1) \operatorname{Hess}(u)(\nu, \nu) - (m+1) \Delta \tilde{u}] \frac{\partial u}{\partial \nu} \\ &\quad - \int_{\partial M} [2(m-1)(\beta+1) + 3m\alpha] \tilde{u}^{\alpha-1} |\nabla \tilde{u}|^2 \frac{\partial u}{\partial \nu} \\ &\quad - \int_{\partial M} [2(m-1)(\beta+1) + m\alpha] \tilde{u}^{\alpha-1} \left(\frac{\partial u}{\partial \nu} \right)^3 \\ &\quad + \int_{\partial M} 2(m-1) h \tilde{u}^\alpha \left(\frac{\partial u}{\partial \nu} \right)^2 + 2m \tilde{u}^\alpha \langle \Pi(\nabla \tilde{u}, \nabla \tilde{u}), \nu \rangle. \end{aligned} \quad (2.58)$$

Proof. Let $\{e_1, \dots, e_{m-1}, e_m = \nu\}$ be a Darboux frame along $\partial M \hookrightarrow M$. Set

$$h_{ij} = \langle \Pi(e_i, e_j), \nu \rangle,$$

so that

$$h = \frac{1}{m-1} h_{kk},$$

where $1 \leq i, j, k \leq m-1$. Then, with obvious meaning of the notation we have

$$\tilde{u}_i = u_i, \quad u_m = \frac{\partial u}{\partial \nu} = \langle \nabla u, \nu \rangle.$$

It therefore follows that

$$\begin{cases} \tilde{u}_{ij} = u_{ij} + u_m h_{ij}, \\ \left(\frac{\partial u}{\partial \nu} \right)_i = u_{mi} - \tilde{u}_j h_{ij}. \end{cases}$$

Hence, using the identity $\langle \nabla |\nabla u|^2, X \rangle = 2 \operatorname{Hess}(u)(X, \nabla u)$, we get

$$\begin{aligned}
2m \int_{\partial M} \langle V, \nu \rangle &= \int_{\partial M} -2\tilde{u}^\alpha \Delta u \frac{\partial u}{\partial \nu} + 2m\tilde{u}^\alpha \text{Hess}(u)(\nabla u, \nu) \\
&\quad - \int_{\partial M} [2(m-1)(\beta+1) + m\alpha] \tilde{u}^{\alpha-1} |\nabla u|^2 \frac{\partial u}{\partial \nu} \\
&= \int_{\partial M} 2\tilde{u}^\alpha [(m-1) \text{Hess}(u)(\nu, \nu) - \Delta \tilde{u}] \frac{\partial u}{\partial \nu} \\
&\quad + \int_{\partial M} 2(m-1) h\tilde{u}^\alpha \left(\frac{\partial u}{\partial \nu} \right)^2 \\
&\quad - \int_{\partial M} [2(m-1)(\beta+1) + m\alpha] \tilde{u}^{\alpha-1} \left[|\nabla \tilde{u}|^2 + \left(\frac{\partial u}{\partial \nu} \right)^2 \right] \frac{\partial u}{\partial \nu} \\
&\quad + \int_{\partial M} 2m\tilde{u}^\alpha \langle \Pi(\nabla \tilde{u}, \nabla \tilde{u}), \nu \rangle + 2m\tilde{u}^\alpha \left\langle \nabla \frac{\partial u}{\partial \nu}, \nabla \tilde{u} \right\rangle.
\end{aligned}$$

We use the first Green formula (see e.g. [GT01]) to deal with the last integrand of the above formula, that is,

$$\int_{\partial M} 2m\tilde{u}^\alpha \left\langle \nabla \frac{\partial u}{\partial \nu}, \nabla \tilde{u} \right\rangle = - \left(\int_{\partial M} 2m\tilde{u}^\alpha \Delta \tilde{u} \frac{\partial u}{\partial \nu} + 2m\alpha \tilde{u}^{\alpha-1} |\nabla \tilde{u}|^2 \frac{\partial u}{\partial \nu} \right).$$

Formula (2.58) now follows at once. \square

In the next result we shall make use of the following extension of Obata's theorem 2.10 due to Escobar [Esc90]. Its proof is similar to that of Theorem 2.10 and it will therefore be omitted.

Theorem 2.14. *Let $(M, \langle \cdot, \cdot \rangle)$ be an m -dimensional Riemannian compact manifold with boundary ∂M . Assume that there exists a nonconstant function f on M satisfying*

$$\begin{cases} \text{Hess}(f) + kf \langle \cdot, \cdot \rangle = 0 & \text{on } M, \\ \frac{\partial f}{\partial \nu} \equiv 0 & \text{on } \partial M \end{cases} \quad (2.59)$$

with k a positive constant. Then $(M, \langle \cdot, \cdot \rangle)$ is isometric to $\mathbb{S}_+^m(\sqrt{k})$, the upper hemisphere of radius $k^{-1/2}$ with the induced metric from \mathbb{R}^{m+1} .

We are now ready for the next

Lemma 2.15. *Let $(M, \langle \cdot, \cdot \rangle)$ be an m -dimensional, $m \geq 3$, compact manifold with boundary ∂M such that the embedding $\partial M \hookrightarrow M$ is totally geodesic. Assume that $(M, \langle \cdot, \cdot \rangle)$ is Einstein but not Ricci flat. Let v be a nonconstant function satisfying*

$$\begin{cases} \text{Hess}(v) - \frac{\Delta v}{m} \langle \cdot, \cdot \rangle = 0 & \text{on } M, \\ \frac{\partial v}{\partial \nu} \equiv 0 & \text{on } \partial M \end{cases} \quad (2.60)$$

with ν the outward unit normal to ∂M . Then the scalar curvature S is positive and $(M, \langle \cdot, \cdot \rangle)$ is isometric to $\mathbb{S}_+^m\left(\sqrt{\frac{m(m-1)}{S}}\right)$.

Proof. Let $X = \nabla v$. Then, because of the first equation of (2.60) X is a conformal vector field and hence, since S is constant, $f = \operatorname{div} X = \Delta v$ satisfies (2.34), that is

$$\operatorname{Hess}(f) + \frac{S}{m(m-1)} f \langle \cdot, \cdot \rangle \quad \text{on } M. \quad (2.61)$$

We shall now show that

$$\frac{\partial f}{\partial \nu} \equiv 0 \quad \text{on } M. \quad (2.62)$$

Towards this end we observe that, with respect to a Darboux frame $\{e_i, e_m = \nu\}$, with $1 \leq i \leq m-1$ along the embedding $\partial M \hookrightarrow M$, from conformality of X we have

$$X_{im} + X_{mi} = \frac{2}{m} (\operatorname{div} X) \delta_{im} = 0. \quad (2.63)$$

Thus differentiating in the e_k direction, $1 \leq k \leq m-1$,

$$X_{imk} + X_{mik} = 0. \quad (2.64)$$

Now $X_m = \langle X, \nu \rangle = \frac{\partial v}{\partial \nu} \equiv 0$ on ∂M because of (2.60), thus $X_{mik} = 0$ on ∂M for $1 \leq i, k \leq m-1$. From (2.64) we then deduce

$$X_{imk} \equiv 0 \quad \text{on } \partial M. \quad (2.65)$$

Now, since X is conformal

$$X_{ab} + X_{ba} = \frac{2}{m} (\operatorname{div} X) \delta_{ab}, \quad 1 \leq a, b \leq m$$

and for $1 \leq k \leq m-1$

$$X_{kk} = \frac{\operatorname{div} X}{m} \quad (\text{no sum over } k).$$

Hence

$$X_{kkm} = \frac{1}{m} (\operatorname{div} X)_m \quad (\text{no sum over } k). \quad (2.66)$$

From the Ricci identities (1.47),

$$X_{kkm} - X_{kkm} = X_s R_{skkm} \quad (\text{no sum over } k).$$

Thus using (2.65)

$$\sum_{k=1}^{m-1} X_{kkm} = -X_t R_{tm} \quad \text{on } \partial M.$$

But $(M, \langle \cdot, \cdot \rangle)$ is Einstein and therefore $R_{sm} = 0$ for $1 \leq s \leq m-1$. Furthermore $X_m = 0$, hence

$$\sum_{k=1}^{m-1} X_{kkm} = 0. \quad (2.67)$$

Using (2.66) and (2.67) we find

$$\frac{m}{m-1}(\operatorname{div} X)_m = \sum_{k=1}^{m-1} X_{kkm} = 0 \quad \text{on } \partial M$$

that is, the validity of (2.62). Thus (2.60) is satisfied by $f = \Delta v$ with $k = \frac{S}{m(m-1)}$ and the conclusion follows at once from Theorem 2.14 once we prove that f is nonconstant and $S > 0$. We reason by contradiction. If f were constant, by (2.61) since $S \neq 0$ we would have $f \equiv 0$ so that $X = \nabla v$ would be a Killing field and $\Delta v \equiv 0$. Since $\operatorname{div}(v\nabla v) = v\Delta v + |\nabla v|^2 = |\nabla v|^2$ and $\frac{\partial v}{\partial \nu} \equiv 0$ the divergence theorem yields

$$\int_M |\nabla v|^2 \equiv 0,$$

that is, v is constant contradicting the assumption of the lemma. Thus f is nonconstant. Next, to show that $S > 0$ we trace (2.61) to obtain

$$\begin{cases} \Delta f + \frac{S}{m(m-1)}f = 0 & \text{on } M, \\ \frac{\partial f}{\partial \nu} \equiv 0 & \text{on } \partial M. \end{cases}$$

From the divergence theorem

$$0 = \int_{\partial M} f \frac{\partial f}{\partial \nu} = \int_M \operatorname{div}(f\nabla f) = \int_M f\Delta f + |\nabla f|^2 = - \int_M f^2 \frac{S}{m(m-1)} + |\nabla f|^2$$

and thus

$$S = m(m-1) \frac{\int_M |\nabla f|^2}{\int_M f^2} > 0. \quad \square$$

We are now ready to prove

Theorem 2.16. *Let $(M, \langle \cdot, \cdot \rangle)$ be a compact manifold of dimension $m \geq 2$ with boundary ∂M . Assume that the embedding $\partial M \hookrightarrow M$ is totally geodesic and that*

$$\operatorname{Ric} \geq k > 0. \quad (2.68)$$

Let $\sigma > 1$, $\lambda > 0$ and φ a positive solution of

$$\begin{cases} \Delta \varphi - \lambda \varphi + \varphi^\sigma = 0 & \text{on } M, \\ \frac{\partial \varphi}{\partial \nu} \equiv 0 & \text{on } \partial M \end{cases} \quad (2.69)$$

with ν the outward unit normal to ∂M . Then φ is constant provided

- (i) $m = 2$ and $\lambda \leq \frac{2k}{\sigma-1}$;
- (ii) $m \geq 3$, $\lambda \leq \frac{m}{m-1} \frac{k}{\sigma-1}$, $\sigma \leq \frac{m+2}{m-2}$ and
(A) either at least one of the last two inequalities is strict or

(B) $(M, \langle \cdot, \cdot \rangle)$ is Einstein and it is not isometric to an upper hemisphere of constant curvature.

Proof. We set $u = \varphi^{-1/\beta}$ with $\beta \neq 0$. Then u satisfies (2.49) and $\frac{\partial u}{\partial \nu} \equiv 0$ on ∂M . Using (2.51), (2.58), $\frac{\partial u}{\partial \nu} \equiv 0$ and the fact that $\partial M \hookrightarrow M$ is totally geodesic we get again (2.53). Now the proof proceeds exactly as in Theorem 2.12 up to

$$\text{Hess}(u) = \frac{\Delta u}{m} \langle \cdot, \cdot \rangle \quad \text{on } M.$$

Since $\frac{\partial u}{\partial \nu} \equiv 0$ on ∂M we can now apply Lemma 2.15 and we reach a contradiction. \square

Remark. As it is apparent from the proof of Lemma 2.15, we may substitute the assumption that $(M, \langle \cdot, \cdot \rangle)$ is Einstein in (B) with $(M, \langle \cdot, \cdot \rangle)$ has constant scalar curvature and $\text{Ric}(\nu, X) = 0$ for every $X \in T\partial M^\perp \subseteq TM$ on ∂M .

Remark. 1. An observation similar to the Remark on page 61 holds also here. Point 1. applies *verbatim* to the more general equation

$$\Delta \varphi - \lambda \varphi + \mu \varphi^\sigma = 0 \quad \text{on } M \tag{2.70}$$

with $\lambda, \mu > 0$ and $\varphi > 0$.

2. If λ and μ have different sign, that is, $\lambda \leq 0$ and $\mu \geq 0$ or $\lambda \geq 0$ and $\mu \leq 0$, $\varphi > 0$, then from (2.69)

$$0 = \int_{\partial M} \frac{\partial \varphi}{\partial \nu} = \int_M \Delta \varphi = \int_M \lambda \varphi - \mu \varphi^\sigma,$$

so that

$$\varphi(\lambda - \mu \varphi^{\sigma-1}) \equiv 0.$$

From this latter we easily deduce $\lambda = \mu = 0$ and φ is constant from (2.69) and compactness of M , without requiring any further assumption.

3. Finally, if $\lambda, \mu < 0$ we have

$$\Delta \varphi - \lambda \varphi = -\mu \varphi^\sigma \geq 0.$$

Suppose first that φ assumes its maximum at a point $x_0 \in \partial M$. Then, by the boundary point Lemma (see [PW67] for more details) one has $\frac{\partial \varphi}{\partial \nu}(x_0) > 0$, contradicting $\frac{\partial \varphi}{\partial \nu}(x_0) \leq 0$ on ∂M . Hence φ has to assume its maximum at $x_0 \in M \setminus \partial M$. Similarly φ attains its minimum at $x_1 \in M \setminus \partial M$. Then

$$0 \geq \Delta \varphi(x_0) = \lambda \varphi(x_0) - \mu \varphi^\sigma(x_0),$$

and therefore

$$\frac{\mu}{\lambda} \varphi^{\sigma-1}(x_0) \leq 1.$$

Analogously, since $\Delta\varphi(x_1) \geq 0$ one gets

$$\frac{\mu}{\lambda}\varphi^{\sigma-1}(x_1) \geq 1.$$

Since $\frac{\mu}{\lambda} > 0$ and φ is positive we conclude that $\varphi(x_0) = \varphi(x_1)$ and therefore φ is constant.

Remark. In Theorem 2.16 we can substitute the assumption that the embedding of the boundary in M is totally geodesic with that of a convex boundary (see for instance [PRS03a]). It seems also worth mentioning that using this result Ilias obtained some sharp Sobolev constants on M (see [Ili96] and [PRS03a], Theorem 3.5).

With the above observation and with the aid of Lemma 2.19, from Theorem 2.16 we deduce the following result of Escobar (see [Esc90], Theorem 4.1):

Theorem 2.17. *Let $(M, \langle \cdot, \cdot \rangle)$ be a compact Einstein manifold of dimension $m \geq 3$ and with boundary ∂M totally geodesic in $(M, \langle \cdot, \cdot \rangle)$. Let $\widetilde{\langle \cdot, \cdot \rangle} = \varphi^{\frac{4}{m-2}}$ be a conformal change of metric with constant scalar curvature \widetilde{S} and with $\partial M \hookrightarrow (M, \widetilde{\langle \cdot, \cdot \rangle})$ minimal. Then $\widetilde{\langle \cdot, \cdot \rangle}$ is Einstein and if $(M, \langle \cdot, \cdot \rangle)$ is not isometric to an upper hemisphere of constant curvature, φ is constant, that is, the conformal change is an homothety.*

2.2.4 A rigidity result of Escobar

In this section we prove a nice rigidity result due to Escobar, [Esc90]. Towards this aim we need some preliminary facts contained in the next two lemmas.

Lemma 2.18. *Let $(M, \langle \cdot, \cdot \rangle)$ be a manifold with boundary, constant scalar curvature S and trace-free Ricci tensor T . Let ν be the outward unit normal to $i : \partial M \hookrightarrow M$ and let X be a conformal vector field on M such that, for some $\psi \in C^\infty(\partial M)$, $X|_{\partial M} = \psi\nu$. Then*

$$\int_{\partial M} \psi T(\nu, \nu) = 0. \quad (2.71)$$

Proof. We let $W = T(X, \cdot)^\sharp$ as in (2.14) so that, according to (2.15),

$$\operatorname{div} W = X_{i,k} T_{ik} + X_i T_{ik,k}. \quad (2.72)$$

Since X is conformal,

$$\mathcal{L}_X \langle \cdot, \cdot \rangle = \frac{2}{m} \operatorname{div} X \langle \cdot, \cdot \rangle,$$

or, in other words,

$$X_{ik} + X_{ki} = \left(\frac{2}{m} \operatorname{div} X \right) \delta_{ik}.$$

Hence

$$X_{ik}T_{ik} = \frac{1}{2}(X_{ik} + X_{ki})T_{ik} = \left(\frac{1}{m} \operatorname{div} X\right)T_{kk} = 0,$$

since T is trace-free. Thus using (2.72) and the divergence theorem,

$$\int_{\partial M} T(X, \nu) = \int_M (\operatorname{div} T)(X).$$

However, since the scalar curvature S is constant, from (2.13) $\operatorname{div} T \equiv 0$ and from the above we immediately deduce (2.71). \square

Remark. For $m = \dim M = 2$ there is no need to assume S constant, see equation (2.13).

Lemma 2.19. *Let $(M, \langle \cdot, \cdot \rangle)$ be a manifold with boundary ∂M and $\widetilde{\langle \cdot, \cdot \rangle} = \varphi^2 \langle \cdot, \cdot \rangle$ for some $\varphi > 0$, $\varphi \in C^\infty(M)$. Let ν be the unit outward normal of $\partial M \rightarrow (M, \langle \cdot, \cdot \rangle)$ and h_{ij} the coefficient of the second fundamental form in the direction of ν . Set \tilde{h}_{ij} for the analogous quantities for $\partial M \rightarrow (M, \widetilde{\langle \cdot, \cdot \rangle})$ with respect to $\tilde{\nu} = \varphi^{-1}\nu$. Then, for $1 \leq i, j \leq m-1$,*

$$\tilde{h}_{ij} = \varphi^{-1}h_{ij} + \varphi^{-2}\frac{\partial \varphi}{\partial \nu}\delta_{ij}, \quad (2.73)$$

and for the mean curvature

$$\tilde{h} = \varphi^{-1}\left(h + \frac{\partial \log \varphi}{\partial \nu}\right). \quad (2.74)$$

In particular, if $\partial M \rightarrow (M, \langle \cdot, \cdot \rangle)$ and $\partial M \rightarrow (M, \widetilde{\langle \cdot, \cdot \rangle})$ are both minimal, then $\frac{\partial \varphi}{\partial \nu} \equiv 0$ on ∂M .

Proof. A simple computation. Indeed, with the notation of Chapter 1 and section 2.1.1 we let $\{\theta^i\}$ be a Darboux frame along the inclusion map $\iota : \partial M \rightarrow (M, \langle \cdot, \cdot \rangle)$. Thus $\theta^m = 0$ on ∂M and h_{ij} are defined by the requirement

$$\theta_i^m = h_{ij}\theta^j, \quad 1 \leq i, j, \dots \leq m-1.$$

Similarly, $\tilde{\theta}^i = \varphi\theta^i$ is a Darboux frame along the inclusion $\tilde{\iota} : \partial M \rightarrow (M, \widetilde{\langle \cdot, \cdot \rangle})$.

Thus $\tilde{\theta}^m = 0$ and

$$\tilde{\theta}_i^m = \tilde{h}_{ij}\tilde{\theta}^j, \quad 1 \leq i, j, \dots \leq m-1.$$

To relate h_{ij} with \tilde{h}_{ij} we recall from (2.3) that

$$\tilde{\theta}_i^m = \theta_i^m + \frac{\varphi_i}{\varphi}\theta^m - \frac{\varphi_m}{\varphi}\theta^i, \quad d\varphi = \varphi_j\theta^j + \varphi_m\theta^m.$$

Hence

$$\varphi \tilde{h}_{ij} \theta^j = h_{ij} \theta^j - \frac{\varphi_m}{\varphi} \delta_j^i \theta^j$$

and it follows that

$$\tilde{h}_{ij} = \varphi^{-1} h_{ij} - \varphi^{-2} \varphi_m \delta_{ij},$$

from which (2.73), and thus (2.74), follow immediately. \square

Lemma 2.20. *Let (M, g_0) be a compact Einstein manifold such that $\partial M \hookrightarrow (M, g_0)$ is totally geodesic and let $g = \varphi^{-2} g_0$, $\varphi > 0$, $\varphi \in C^\infty(M)$ be a conformally related metric with constant scalar curvature S and such that $\partial M \hookrightarrow (M, g)$ has constant mean curvature h with respect to the outward unit (with respect to g) normal ν . Let T be the trace-free Ricci tensor of g . Then*

$$\int_M \varphi^{-1} |T|^2 = -(m-2) \int_{\partial M} h \varphi^{-1} T(\nu, \nu) \quad (2.75)$$

(where of course we are integrating with respect to the volume element of g on M and ∂M).

Proof. We set \tilde{g} for g_0 so that $\tilde{g} = \varphi^2 g$. From (2.26)

$$\tilde{T}_{ij} = T_{ij} + (m-2)\varphi \left\{ (\varphi^{-1})_{ij} - \frac{\Delta(\varphi^{-1})}{m} \delta_{ij} \right\},$$

and since \tilde{g} by assumption is Einstein we get

$$-T_{ij} = (m-2)\varphi \left\{ (\varphi^{-1})_{ij} - \frac{\Delta(\varphi^{-1})}{m} \delta_{ij} \right\}.$$

The fact that T is traceless yields

$$-|T|^2 = (m-2)\varphi (\varphi^{-1})_{ij} T_{ij}.$$

Integrating the above on (M, g) we obtain

$$-\int_M \varphi^{-1} |T|^2 = (m-2) \int_M (\varphi^{-1})_{ij} T_{ij}. \quad (2.76)$$

Again from (2.13), since (M, g) has constant scalar curvature

$$T_{ik,k} = 0. \quad (2.77)$$

Similarly to what we did in the proof of Lemma 2.18 we consider the vector field

$$W = T(\nabla \varphi^{-1})^\sharp.$$

Taking its divergence in the metric g we have (see (2.72))

$$\operatorname{div} W = (\varphi^{-1})_{ij} T_{ij} + (\varphi^{-1})_i T_{ik,k}.$$

Hence, using (2.77) and the divergence theorem,

$$\int_{\partial M} T(\nabla(\varphi^{-1}), \nu) = \int_M (\varphi^{-1})_{ij} T_{ij}. \quad (2.78)$$

We need now to write the left-hand side of (2.78) appropriately. First we show that if $Y \in T\partial M$, then $T(Y, \nu) = 0$. Since $T = \operatorname{Ric} - \frac{S}{m} \langle \cdot, \cdot \rangle$, this is clearly equivalent to showing that

$$\operatorname{Ric}_{(M,g)}(Y, \nu) = 0 \quad (2.79)$$

(and this is what we expect if we want to show that g is Einstein, see Theorem 2.21). By assumption $\partial M \hookrightarrow (M, \tilde{g})$ is totally geodesic and therefore it follows from Lemma 2.19 that, since g and \tilde{g} are conformally related, $\partial M \hookrightarrow (M, \tilde{g})$ is totally umbilical. Moreover, from (2.74)

$$0 = h + \varphi^{-1} \frac{\partial \varphi}{\partial \nu}, \quad (2.80)$$

with h the constant mean curvature of $\partial M \hookrightarrow (M, g)$. In other words

$$\varphi h = -\frac{\partial \varphi}{\partial \nu}.$$

In this case we can rewrite (2.73) of Lemma 2.19, since $\partial M \hookrightarrow (M, \tilde{g})$ is totally geodesic, as

$$h_{ij} = h \delta_{ij}$$

for $1 \leq i, j, \dots, \leq m-1$. Since h is constant, from the above we deduce

$$h_{ijk} = 0.$$

On the other hand, from the Codazzi equations,

$$h_{ijk} - h_{ikj} = -R_{ijk}^m.$$

Tracing with respect to i and k we deduce

$$R_{mj} = 0,$$

where R_{ij} are the components of the Ricci tensor of (M, g) . This proves (2.79). It follows that (2.78) becomes

$$\int_{\partial M} \frac{\partial(\varphi^{-1})}{\partial \nu} T(\nu, \nu) = \int_M (\varphi^{-1})_{ij} T_{ij}.$$

On the other hand, we can rewrite (2.80) as

$$\frac{\partial(\varphi^{-1})}{\partial\nu} = \varphi^{-1}h.$$

Substituting into the above we get

$$\int_{\partial M} \varphi^{-1}hT(\nu, \nu) = \int_M (\varphi^{-1})_{ij}T_{ij}. \quad (2.81)$$

Hence (2.75) follows immediately from (2.81) and (2.76). \square

We are now ready to prove Escobar's result ([Esc90]).

Theorem 2.21. *Let \mathbb{B}^m be the (open) unit ball in \mathbb{R}^m and $\overline{\mathbb{B}^m}$ the closed unit ball of \mathbb{R}^m ; let g be a metric on \mathbb{B}^m conformally related to the Euclidean metric $\langle \cdot, \cdot \rangle$. Assume that S , the scalar curvature of $(\overline{\mathbb{B}^m}, g)$, is constant and that $\partial\overline{\mathbb{B}^m} \hookrightarrow (\overline{\mathbb{B}^m}, g)$ has constant mean curvature h with respect to the outward unit (w.r.t. g) normal ν . Then $(\overline{\mathbb{B}^m}, g)$ has constant sectional curvature and it is therefore isometric to a geodesic ball in a space form with an appropriate radius depending on h .*

Proof. We shall prove that $(\overline{\mathbb{B}^m}, g)$ has constant sectional curvature: the remaining part of the conclusion is a well-known result of É. Cartan (see e.g. [Car88]). We introduce on $\overline{\mathbb{B}^m}$ the auxiliary metric

$$g_0 = \frac{4}{(1 + |x|^2)^2} \langle \cdot, \cdot \rangle.$$

Note that g_0 is obtained from the upper hemisphere $(\mathbb{S}_+^m, \bar{g})$, \bar{g} denoting the standard metric induced on \mathbb{S}^m by the inclusion $\mathbb{S}^m \hookrightarrow \mathbb{R}^{m+1}$, by stereographic projection π from the South pole $\pi : \overline{\mathbb{B}^m} \rightarrow \mathbb{S}_+^m$

$$\pi : x \mapsto \left(\frac{2x}{1 + |x|^2}, \frac{1 - |x|^2}{1 + |x|^2} \right).$$

It follows that $(\overline{\mathbb{B}^m}, g_0)$ is Einstein and $\partial\overline{\mathbb{B}^m} \hookrightarrow (\overline{\mathbb{B}^m}, g_0)$ is totally geodesic. Since the position vector field X is conformal on $(\overline{\mathbb{B}^m}, \langle \cdot, \cdot \rangle)$, it is conformal on $(\overline{\mathbb{B}^m}, g_0)$ and, since $g_0|_{\partial\overline{\mathbb{B}^m}} = \langle \cdot, \cdot \rangle_{\partial\overline{\mathbb{B}^m}}$, $X|_{\partial\overline{\mathbb{B}^m}}$ is the outward unit normal with respect to g_0 .

Since g is conformally related to $\langle \cdot, \cdot \rangle$, it is conformally related to g_0 . Let $g = \varphi^{-2}g_0$, $\varphi > 0$, $\varphi \in C^\infty(\overline{\mathbb{B}^m})$. First we show that g is Einstein. Indeed, we are in the assumptions of Lemma 2.20 so that, with the same notation, we have the validity of (2.75). On the other hand, on $\partial\overline{\mathbb{B}^m}$,

$$X = \varphi^{-1}\nu$$

where ν is the outward unit normal with respect to g . From Lemma 2.18,

$$\int_{\overline{\mathbb{B}^m}} \varphi^{-1} T(\nu, \nu) = 0,$$

and constancy of h and (2.75) give

$$\int_M \varphi^{-1} |T|^2 \equiv 0,$$

that is, $T \equiv 0$ on $(\overline{\mathbb{B}^m}, g)$ or, in other words, g is an Einstein metric. If $m = 2$ we are done; for $m \geq 3$, to complete the proof we consider the decomposition of the curvature tensor R_{jkl}^i given in (2.27) that we rewrite as

$$\begin{aligned} R_{jkl}^i &= W_{jkl}^i + \frac{1}{m-2} (T_{ik}\delta_{jl} - T_{il}\delta_{jk} + T_{jl}\delta_{ik} - T_{jk}\delta_{il}) \\ &\quad + \frac{S}{m(m-1)} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \end{aligned}$$

with respect to a local o.n. coframe for g . But g is conformally related to the Euclidean metric and therefore it is conformally flat. Furthermore, g is Einstein and hence $T \equiv 0$. Thus the above formula becomes

$$R_{jkl}^i = \frac{S}{m(m-1)} (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

and $(\overline{\mathbb{B}^m}, g)$ has constant sectional curvature. □

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