Operator Theory:
Advances and Applications, Vol. 224, 11–24

Continuity of Spectra in Rieffel's Pseudodifferential Calculus

F. Belmonte and M. Măntoiu

Abstract. Using the fact that Rieffel's quantization sends covariant continuous fields of C^*-algebras in continuous fields of C^*-algebras, we prove spectral continuity results for families of Rieffel-type pseudodifferential operators.

Mathematics Subject Classification (2010). Primary 35S05, 81Q10, Secondary 46L55, 47C15.

Keywords. Pseudodifferential operator, spectrum, Rieffel quantization, C^*-algebra, continuous field, dynamical system.

Introduction

One naturally expects that topics or tools coming from the standard pseudodifferential theory could make sense and even work in the more general setting of Rieffel's calculus. In [16, 17], some C^*-algebraic techniques of spectral analysis ([3, 4, 10, 15, 18] and references therein) were tuned with Rieffel quantization [24], getting results on spectra and essential spectra of certain self-adjoint operators that seemed to be out of reach by other methods. In the present article we continue the project by studying *spectral continuity.*

Pioneering work on applying C^*-algebraic methods to spectral continuity problems and applications to discrete physical systems may be found in [3, 5, 8]. Results on continuity of spectra for unbounded Schrödinger-like Hamiltonians (especially with magnetic fields) appear in [1, 2, 13, 20] and references therein.

Roughly, the abstract problem can be stated as follows: For each point t of the locally compact space T we are given a self-adjoint element (a classical observable) $f(t)$ of a C^*-algebra $\mathcal{A}(t)$, which is Abelian for most of the applications, and we assume some simple-minded continuity property in the variable t for this family. By quantization, $f(t)$ is turned into a quantum observable $\mathfrak{f}(t)$ belonging to a new, non-commutative C^*-algebra $\mathfrak{A}(t)$ (in spite of the notation, rather often $\mathfrak{f}(t)$ is just $f(t)$ with a new interpretation). We inquire if the family $S(t) := \mathrm{sp}\,[\mathfrak{f}(t)]$

of spectra computed in these new algebras vary continuously with t. Intuitively, outer continuity says that the family cannot suddenly expand: if for some t_0 there is a gap in the spectrum of $f(t_0)$ around a point $\lambda_0 \in \mathbb{R}$, then for t close to t_0 all the spectra $S(t)$ will have gaps around λ_0. On the other hand, inner continuity insures that if $f(t_0)$ has some spectrum in a non-trivial interval of $\mathbb{R}$, this interval will contain spectral points of all the elements $f(t)$ for t close to t_0. Although traditionally $\mathfrak{A}(t)$ is thought to be a C^*-algebra of bounded operators in some Hilbert space, the abstract situation is both natural and fruitful. One can work with abstract C^*-algebras $\mathfrak{A}(t)$ and then, if necessarily, represent them faithfully in Hilbert spaces; the spectrum will be preserved under representation.

It is well known (see Theorem 3.2) that spectral continuity can be obtained from corresponding continuity properties of resolvent families of the elements $\mathfrak{f}(t)$ but this involves both inversion and norm in each complicated C^*-algebra $\mathfrak{A}(t)$. Things are smoothed out if the family $\{\mathfrak{A}(t) \mid t \in T\}$ has a priori continuity properties, that may be connected to the concept of (upper or lower semi)-continuous C^*-bundle, cf. [25, 26] and Definition 1.1. We are going to investigate the case in which $\{\mathfrak{A}(t) \mid t \in T\}$ in obtained from another family $\{\mathcal{A}(t) \mid t \in T\}$ of simpler (classical) C^*-algebras by Rieffel quantization.

Rieffel's calculus [24, 14] is a method that transforms "simpler" C^*-algebras and morphisms into more complicated ones. The ingredients to do this are an action of the vector group $\Xi := \mathbb{R}^d$ by automorphisms of the "simple" algebra as well as a skew symmetric linear operator of Ξ. The initial data are naturally defining a Poisson structure, regarded as a mathematical modelization of the observables of a classical physical system. After applying the machine to this classical data one gets a C^*-algebra seen as the family of observables of the same system, but written in the language of Quantum Mechanics.

In simple situations the multiplication in the initial C^*-algebra is just pointwise multiplication of functions defined on some locally compact topological space Σ, on which Ξ acts by homeomorphisms. The non-commutative product in the quantized algebra can be interpreted as a symbol composition of a pseudodifferential type. Actually the concrete formulae generalize and are motivated by the usual Weyl calculus [9].

The basic technical fact is that *by Rieffel quantization an upper semi-continuous fields of C^*-algebras is turned into an upper semi-continuous fields of C^*-algebras and the same is true if upper semi-continuity is replaced by lower semi-continuity.* This is shown in [6]; a partial result without proof is announced in [11] (see also [12]). For the convenience of the readers, we are going to sketch a new proof in Section 1, relying on results from [7, 23].

As said before the most interesting cases, those which are closer to the initial spirit of Weyl quantization, involve Abelian initial algebras $\mathcal{A}$. In this situation the information is encoded in a topological dynamical system with locally compact space Σ and the upper semi-continuous field property can be read in the existence of a continuous covariant surjection $q : \Sigma \to T$; if this one is open, then lower semi-continuity also holds. This is explained in Section 2.

Using these facts, in the final sections, we prove spectral continuity. We start with families of elements belonging to the abstract Rieffel algebras. Then we outline a setting in which these algebras admit interesting faithful representations in a unique Hilbert space, thus getting spectral continuity for families of pseudodifferential-like operators. Making suitable adaptations of the dynamical system, we also include an outer continuity result for *essential spectra* of Rieffel pseudodifferential operators. As an example, we are going to show that our results cover families of zero order standard pseudodifferential operators and this is new up to or knowledge. Spectral continuity for families of elliptic strictly positive order Hamiltonians (even including variable magnetic fields) is known; see [1, 2, 13, 20]. But the methods of these articles do not extend in some obvious way to zero-order operators. The resolvent of an elliptic operator of order $m > 0$ is a pseudodifferential operator of strictly negative order and this helps a lot. In the framework of [1] for instance, it allows using a certain form of crossed product C^*-algebras, which form semi-continuous fields by well-known results [19, 22, 23]; this is not available if $m = 0$. Continuity in Planck's constant $\hbar$, treated in [24] and in [16], is also special case of our general results but we shall not repeat this here.

The full strength of these spectral techniques would require an extension of Rieffel's calculus to suitable families of unbounded elements. Hopefully this will be achieved in the future and this would be the right opportunity to present detailed examples, which could include non-elliptic positive order pseudodiferential operators with variable magnetic fields.

1. Families of Rieffel quantized C^*-algebras

Let T be a locally compact space (always supposed Hausdorff); we denote dy $\mathcal{C}(T)$ the space of all complex continuous functions defined on T and vanishing at infinity.

Definition 1.1. (*see* [19, 23, 26] *and references therein*) *By* upper semi-continuous field of C^*-algebras *we mean a family* $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ *of epimorphisms of C^*-algebras indexed by the locally compact topological space T and satisfying:*

1. *For every $b \in \mathcal{B}$ one has $\| b \|_{\mathcal{B}} = \sup_{t\in T} \| \mathcal{P}(t)b \|_{\mathcal{B}(t)}$.*
2. *For every $b \in \mathcal{B}$ the map $T \ni t \mapsto \| \mathcal{P}(t)b \|_{\mathcal{B}(t)}$ is upper semi-continuous and vanishes at infinity.*
3. *There is a multiplication $\mathcal{C}(T) \times \mathcal{B} \ni (\varphi, b) \to \varphi * b \in \mathcal{B}$ such that*

$$\mathcal{P}(t)[\varphi * b] = \varphi(t)\,\mathcal{P}(t)b\,, \qquad \forall\, t \in T\,,\ \varphi \in \mathcal{C}(T)\,,\ b \in \mathcal{B}\,.$$

If the map $t \mapsto \| \mathcal{P}(t)b \|_{\mathcal{B}(t)}$ is upper semi-continuous for every $b \in \mathcal{B}$, we say that $\left\{\mathcal{B} \xrightarrow{\mathcal{P}(t)} \mathcal{B}(t) \mid t \in T\right\}$ *is* an upper semi-continuous field of C^*-algebras.

One can identify $\mathcal{B}$ with a C^*-algebra of sections of the field. It will always be assumed that $\mathcal{B}(t) \neq \{0\}$ for all $t \in T$.

We go on by describing briefly Rieffel quantization [24]. Let $(\Xi, [\![\cdot,\cdot]\!])$ be a $2n$-dimensional symplectic vector space and $(\mathcal{A}, \Theta, \Xi)$ a *C^*-dynamical system*, meaning that Ξ acts strongly continuously by automorphisms of the C^*-algebra $\mathcal{A}$. We denote by $\mathcal{A}^\infty$ the family of elements f such that the mapping $\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{A}$ is C^∞; it is a dense *-algebra of $\mathcal{A}$. Inspired by Weyl's pseudodifferential calculus, one keeps the involution unchanged but introduce on $\mathcal{A}^\infty$ the product

$$f \# g := \pi^{-2n} \int_\Xi \int_\Xi dY dZ\, e^{2i[\![Y,Z]\!]}\, \Theta_Y(f)\, \Theta_Z(g)\,, \tag{1.1}$$

defined by oscillatory integral techniques. One gets a *-algebra $(\mathcal{A}^\infty, \#, ^*)$, which admits a C^*-completion $\mathfrak{A}$ in a C^*-norm $\|\cdot\|_{\mathfrak{A}}$ as described in [24]. The action Θ leaves $\mathcal{A}^\infty$ invariant and extends [24, Prop. 5.11] to a strongly continuous action of the C^*-algebra $\mathfrak{A}$, that will also be denoted by Θ. The space $\mathfrak{A}^\infty$ of C^∞-vectors coincide with $\mathcal{A}^\infty$, cf [24, Th. 7.1].

Let $(\mathcal{A}_j, \Theta_j, \Xi, [\![\cdot,\cdot]\!])$, $j = 1, 2$, be two data as above and let $\mathcal{R} : \mathcal{A}_1 \to \mathcal{A}_2$ be a Ξ-morphism, *i.e.*, a (C^*-)morphism intertwining the two actions Θ_1, Θ_2. Then $\mathcal{R}$ sends $\mathcal{A}_1^\infty$ into $\mathcal{A}_2^\infty$ and extends to a morphism $\mathfrak{R} : \mathfrak{A}_1 \to \mathfrak{A}_2$ that also intertwines the corresponding actions.

Let now T be a locally compact Hausdorff space and let $\left\{\mathcal{A} \xrightarrow{\mathcal{P}(t)} \mathcal{A}(t) \mid t \in T\right\}$ be a field of C^*-algebras. We are given actions Θ of Ξ on $\mathcal{A}$ and $\Theta(t)$ of Ξ on $\mathcal{A}(t)$ satisfying $\Theta(t)_X \circ \mathcal{P}(t) = \mathcal{P}(t) \circ \Theta_X$ for each $t \in T$ and $X \in \Xi$. One can say that $\left\{\mathcal{A} \xrightarrow{\mathcal{P}(t)} \mathcal{A}(t) \mid t \in T\right\}$ is *a covariant field of C^*-algebras.* Then, by Rieffel quantization, one constructs the new covariant field $\left\{\mathfrak{A} \xrightarrow{\mathfrak{P}(t)} \mathfrak{A}(t) \mid t \in T\right\}$.

Theorem 1.2. *Rieffel quantization transforms covariant semi-continuous fields of C^*-algebras into covariant semi-continuous fields of C^*-algebras.*

It is understood that the statement holds separately for upper and for lower semi-continuity. In the remaining part of this section we are going to present a proof of this result, different from that of [6].

First define

$$\kappa : \Xi \times \Xi \to \mathbb{T} := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}\,, \qquad \kappa(X,Y) := \exp\left(-\frac{i}{2}\, [\![X,Y]\!]\right) \tag{1.2}$$

and notice that it is a group 2-cocycle, i.e., for all $X, Y, Z \in \Xi$ one has

$$\kappa(X,Y)\, \kappa(X+Y,Z) = \kappa(Y,Z)\, \kappa(X,Y+Z)\,, \qquad \kappa(X,0) = 1 = \kappa(0,X)\,.$$

Thus the initial data is converted into $(\mathcal{A}, \Theta, \Xi, \kappa)$, a very particular case of *twisted C^*-dynamical system* [21, 22]. To any twisted C^*-dynamical system one associates canonically a C^*-algebra $\mathcal{A} \rtimes_\Theta^\kappa \Xi$ (called *twisted crossed product*). This is the enveloping C^*-algebra of the Banach *-algebra $\left(L^1(\Xi;\mathcal{A}), \diamond, ^\diamond, \|\cdot\|_1\right)$, where

$$\| F \|_1 := \int_\Xi dX \, \| F(X) \|_{\mathcal{A}}, \qquad F^\diamond(X) := F(-X)^*$$

and (symmetrized version of the standard form)

$$(F_1 \diamond F_2)(X) := \int_\Xi dY \, \kappa(X,Y) \, \Theta_{(Y-X)/2} \left[F_1(Y)\right] \, \Theta_{Y/2} \left[F_2(X-Y)\right]. \tag{1.3}$$

In the same way, for each $t \in T$, to $(\mathcal{A}(t), \Theta(t), \Xi, \kappa)$ one associates the twisted crossed product $\mathcal{A}(t) \rtimes_{\Theta(t)}^\kappa \Xi$. Let us use the abbreviations $\mathfrak{C} := \mathcal{A} \rtimes_\Theta^\kappa \Xi$ and $\mathfrak{C}(t) := \mathcal{A}(t) \rtimes_{\Theta(t)}^\kappa \Xi$. The epimorphism $\mathcal{P}(t) : \mathcal{A} \to \mathcal{A}(t)$ raises canonically to an epimorphism $\mathcal{P}(t)^\rtimes : \mathfrak{C} \to \mathfrak{C}(t)$. As a consequence of results in [19, 22, 23] (see [23, Sect. 3] for instance), if $\left\{ \mathcal{A} \xrightarrow{\mathcal{P}(t)} \mathcal{A}(t) \mid t \in T \right\}$ is an upper (or lower, respectively) semi-continuous field, then $\left\{ \mathfrak{C} \xrightarrow{\mathcal{P}(t)^\rtimes} \mathfrak{C}(t) \mid t \in T \right\}$ is also an upper (resp. lower) semi-continuous field.

Thus it would be enough to have an efficient connection between Rieffel quantized C^*-algebras and twisted crossed products. We present some consequences of results from [7]. We recall first that the Schwartz space $\mathcal{S}(\Xi)$ is a *-algebra under complex conjugation and the Weyl product

$$(h \sharp k)(X) := \pi^{-2n} \int_\Xi \int_\Xi dY dZ \, e^{2i[\![Y,Z]\!]} \, h(X+Y) \, k(X+Z). \tag{1.4}$$

Fix now an element $h \in \mathcal{S}(\Xi) \setminus \{0\}$ satisfying $h \sharp h = h = \overline{h}$ and define for each $f \in \mathcal{A}^\infty = \mathfrak{A}^\infty$ and any $X \in \Xi$

$$\left[M_h(f)\right](X) := \int_\Xi dY \, e^{-i[\![X,Y]\!]} h(Y) \Theta_Y(f). \tag{1.5}$$

It is shown in [7] that M_h can be extended as an injective C^*-morphism $M_h : \mathfrak{A} \to \mathfrak{C} \equiv \mathcal{A} \rtimes_\Theta^\kappa \Xi$. We recall that injective C^*-morphisms are isometric. The construction can be repeated with $(\mathcal{A}, \Theta)$ replaced by $(\mathcal{A}(t), \Theta(t))$, so for each $t \in T$ one gets an isometry $M(t)_h : \mathfrak{A}(t) \to \mathfrak{C}(t) \equiv \mathcal{A}(t) \rtimes_{\Theta(t)}^\kappa \Xi$. In addition, by [7] one has $M(t)_h \circ \mathfrak{P}(t) = \mathcal{P}(t)^\rtimes \circ M_h$. Then, for any $f \in \mathfrak{A}$

$$\| \mathfrak{P}(t) f \|_{\mathfrak{A}(t)} = \| M(t)_h[\mathfrak{P}(t) f] \|_{\mathfrak{C}(t)} = \| \mathcal{P}(t)^\rtimes \left[M_h(f)\right] \|_{\mathfrak{C}(t)} .$$

Therefore, under the right assumption, the mapping $t \mapsto \| \mathfrak{P}(t) f \|_{\mathfrak{A}(t)}$ has the desired semi-continuity properties. The first condition in the definition of a semi-continuous field of C^*-algebras is checked analogously:

$$\begin{aligned} \| f \|_{\mathfrak{A}} = \| M_h(f) \|_{\mathfrak{C}} &= \sup_t \| \mathcal{P}(t)^\rtimes \left[M_h(f)\right] \|_{\mathfrak{C}(t)} \\ &= \sup_t \| M(t)_h[\mathfrak{P}(t) f] \|_{\mathfrak{C}(t)} = \sup_t \| \mathfrak{P}(t) f \|_{\mathfrak{A}(t)} . \end{aligned}$$

Finally, one must define the mapping $\star : \mathcal{C}(T) \times \mathfrak{A} \to \mathfrak{A}$ that should be deduced from the already existing $* : \mathcal{C}(T) \times \mathcal{A} \to \mathcal{A}$. Let $\varphi \in \mathcal{C}(T)$ and $f \in \mathfrak{A}$. There is a sequence $(f_n)_{n\in\mathbb{N}} \in \mathfrak{A}^\infty = \mathcal{A}^\infty$ with $\| f - f_n \|_{\mathfrak{A}} \to 0$ for $n \to \infty$. One sets $\varphi \star f := \lim_n \varphi * f_n$. We leave to the reader the easy task to check that this limit exists in $\mathfrak{A}$ and that the identity $\mathfrak{P}(t)[\varphi \star f] = \varphi(t)\mathfrak{P}(t)f$ holds for every $t \in T$.

2. The Abelian case

We denote by $\mathcal{C}(\Sigma)$ the Abelian C^*-algebra of all complex continuous functions on the locally compact Hausdorff space Σ that are arbitrarily small outside large compact subsets. When Σ is compact, $\mathcal{C}(\Sigma)$ is unital. We indicate a framework leading naturally to fields of C^*-algebras.

We assume given a continuous surjection $q : \Sigma \to T$. Then we have the disjoint decomposition of Σ in closed subsets $\Sigma = \sqcup_{t\in T}\Sigma_t$, where $\Sigma_t := q^{-1}(\{t\})$. One has the canonical injections $j_t : \Sigma_t \to \Sigma$ and the restriction epimorphisms $\mathcal{R}(t) : \mathcal{C}(\Sigma) \to \mathcal{C}(\Sigma_t)$, with $\mathcal{R}(t)f := f|_{\Sigma_t} = f \circ j_t$, $\forall\, t \in T$. This is the right setting to get semi-continuous fields of Abelian C^*-algebras.

Proposition 2.1. *In the setting above* $\left\{\mathcal{C}(\Sigma) \xrightarrow{\mathcal{R}(t)} \mathcal{C}(\Sigma_t) \mid t \in T\right\}$ *is an upper semi-continuous field of commutative* C^**-algebras. If* q *is also open, the field is continuous.*

Proof. Obviously $\cap_{t\in T} \ker[\mathcal{R}(t)] = \{0\}$, since $f|_{\Sigma_t} = 0$, $\forall\, t \in T$ implies $f = 0$. On the other hand, setting

$$\varphi * f := (\varphi \circ q)f\,, \qquad \forall\, \varphi \in \mathcal{C}(T)\,,\ f \in \mathcal{C}(\Sigma)\,, \tag{2.1}$$

we get immediately $\mathcal{R}(t)(\varphi * f) = \varphi(t)\,\mathcal{R}(t)f$, $\forall\, t \in T$.

We need to study continuity properties of the mapping

$$\begin{aligned} T \ni t \mapsto n_f(t) := \|\mathcal{R}(t)f\|_{\mathcal{C}(\Sigma_t)} &= \sup_{\sigma\in\Sigma_t} |f(\sigma)| \\ &= \inf\left\{\|f + h\|_{\mathcal{C}(\Sigma)} \ \mid\ h \in \mathcal{C}(\Sigma),\ h|_{\Sigma_t} = 0\right\} \in \mathbb{R}_+ \,. \end{aligned}$$

The last expression for the norm can be justified directly easily, but it also follows from the canonical isomorphism $\mathcal{C}(\Sigma_t) \cong \mathcal{C}(\Sigma)/\mathcal{C}_{\Sigma_t}(\Sigma)$, where $\mathcal{C}_{\Sigma_t}(\Sigma)$ is the ideal of functions $h \in \mathcal{C}(\Sigma)$ such that $h|_{\Sigma_t} = 0$.

We first assume that q is only continuous. For every $S \subset T$ we set $\Sigma_S := q^{-1}(S)$. It is easy to see by the Stone-Weierstrass Theorem that

$$\mathcal{C}_{(t)}(\Sigma) := \{h \in \mathcal{C}(\Sigma) \mid \exists \text{ an open neighborhood } U \text{ of } t \text{ such that } h|_{\Sigma_{\overline{U}}} = 0\}$$

is a self-adjoint 2-sided ideal dense in $\mathcal{C}_{\Sigma_t}(\Sigma)$. Let $t_0 \in T$ and $\varepsilon > 0$; by density and the definition of inf

$$\exists\, h \in \mathcal{C}_{(t_0)}(\Sigma) \ \text{ such that } \ n_f(t_0) + \varepsilon \ \geq\ \|f + h\|_{\mathcal{C}(\Sigma)}\,.$$

Let U be the open neighborhood of t_0 for which $h|_{\Sigma_{\overline{U}}} = 0$. For any $t \in U$ one also has $h \in \mathcal{C}_{(t)}(\Sigma)$, so

$$n_f(t) = \inf\left\{\|f+g\|_{\mathcal{C}(\Sigma)} \ \mid g \in \mathcal{C}_{(t)}(\Sigma)\right\} \le \|f+h\|_{\mathcal{C}(\Sigma)} \le n_f(t_0) + \varepsilon$$

and this is upper semi-continuity.

Let us also suppose q open, let $t_0 \in T$ and $\varepsilon > 0$. By the definition of sup, there exists $\sigma_0 \in \Sigma_{t_0}$ such that $|f(\sigma_0)| \ge n_f(t_0) - \varepsilon/2$. Since f is continuous, there is a neighborhood V of σ_0 in Σ such that

$$|f(\sigma)| \ge |f(\sigma_0)| - \varepsilon/2 \ge n_f(t_0) - \varepsilon, \qquad \forall\, \sigma \in V.$$

Since q is open, $U := q(V)$ is a neighborhood of t_0 . For every $t \in U$ we have $\Sigma_t \cap V \ne \emptyset$, so for such t

$$n_f(t) \ge \sup\{|f(\sigma)| \mid \sigma \in \Sigma_t \cap V\} \ge n_f(t_0) - \varepsilon$$

and this is lower semi-continuity. □

Suppose now that a continuous action Θ of Ξ by homeomorphisms of Σ is also given. For $(\sigma, X) \in \Sigma \times \Xi$ we are going to use all the notations

$$\Theta(\sigma, X) = \Theta_X(\sigma) = \Theta_\sigma(X) \in \Sigma \tag{2.2}$$

for the X-transformed of the point σ. The function Θ is continuous and the homeomorphisms Θ_X, Θ_Y satisfy $\Theta_X \circ \Theta_Y = \Theta_{X+Y}$ for every $X, Y \in \Xi$.

The action Θ of Ξ on Σ induces an action of Ξ on $\mathcal{C}(\Sigma)$ (also denoted by Θ) given by $\Theta_X(f) := f \circ \Theta_X$. This action is strongly continuous, *i.e.*, for any $f \in \mathcal{C}(\Sigma)$ the mapping

$$\Xi \ni X \mapsto \Theta_X(f) \in \mathcal{C}(\Sigma) \tag{2.3}$$

is continuous; thus we are placed in the setting presented in the first section. We denote by $\mathcal{C}(\Sigma)^\infty \equiv \mathcal{C}^\infty(\Sigma)$ the set of elements $f \in \mathcal{C}(\Sigma)$ such that the mapping (2.3) is C^∞; it is a dense *-algebra of $\mathcal{C}(\Sigma)$. The general theory supplies a non-commutative C^*-algebra $\mathfrak{A} \equiv \mathfrak{C}(\Sigma)$, acted continuously by the group Ξ , with smooth vectors $\mathfrak{C}^\infty(\Sigma) = \mathcal{C}^\infty(\Sigma)$.

To insure covariance for the emerging families of C^*-algebras, we impose a condition of compatibility between the action Θ and the mapping q .

Definition 2.2. *We say that the surjection q is* Θ-covariant *if it satisfies the equivalent conditions:*

1. *Each Σ_t is Θ-invariant.* 2. *For each $X \in \Xi$ one has $q \circ \Theta_X = q$.*

Recall now the Rieffel-quantized C^*-algebras $\mathfrak{C}(\Sigma)$ and $\mathfrak{C}(\Sigma_t)$ as well as the epimorphisms $\mathfrak{R}(t) : \mathfrak{C}(\Sigma) \to \mathfrak{C}(\Sigma_t)$. Applying Theorem 1.2 and Proposition 2.1, one gets

Corollary 2.3. *Assume that $q : \Sigma \to T$ is a Θ-covariant continuous surjection. Then the covariant field of non-commutative C^*-algebras $\left\{\mathfrak{C}(\Sigma) \xrightarrow{\mathfrak{R}(t)} \mathfrak{C}(\Sigma_t) \,|\, t \in T\right\}$ is upper semi-continuous.*

If q is also open, then the field is continuous.

3. Spectral continuity for symbols

Let us introduce the concept of continuity for families of sets that will be useful below.

Definition 3.1. *Let T be a Hausdorff locally compact topological space and let $\{S(t) \mid t \in T\}$ be a family of compact subsets of $\mathbb{R}$.*

1. *The family is called* outer continuous *if for any $t_0 \in T$ and any compact subset K of $\mathbb{R}$ such that $K \cap S(t_0) = \emptyset$, there exists a neighborhood V of t_0 with $K \cap S(t) = \emptyset$, $\forall t \in V$.*
2. *The family $\{S(t) \mid t \in T\}$ is called* inner continuous *if for any $t_0 \in T$ and any open subset A of $\mathbb{R}$ such that $A \cap S(t_0) \neq \emptyset$, there exists a neighborhood W of t_0 with $A \cap S(t) \neq \emptyset$, $\forall t \in W$.*
3. *If the family is both inner and outer continuous, we say simply that it is* continuous.

In applications the sets $S(t)$ are spectra of some self-adjoint elements $f(t)$ of (non-commutative) C^*-algebras $\mathfrak{A}(t)$. The next result states technical conditions under which one gets continuity of such families of spectra. It is taken from [1] and it has been inspired by the treatment in [3]. We include the proof for the convenience of the reader.

Proposition 3.2. *For any $t \in T$ let $f(t)$ be a self-adjoint element in a C^*-algebra $\mathfrak{A}(t)$ with norm $\| \cdot \|_{\mathfrak{A}(t)}$ and inversion $g \mapsto g^{(-1)_{\mathfrak{A}(t)}}$. We denote by $S(t) \subset \mathbb{R}$ the spectrum of $f(t)$ in $\mathfrak{A}(t)$.*

1. *Assume that for any $z \in \mathbb{C} \setminus \mathbb{R}$ the mapping*
$$T \ni t \mapsto \left\| (f(t) - z)^{(-1)_{\mathfrak{A}(t)}} \right\|_{\mathfrak{A}(t)} \in \mathbb{R}_+ \tag{3.1}$$
is upper semi-continuous. Then the family $\{S(t) \mid t \in T\}$ is outer continuous.
2. *Assume that for any $z \in \mathbb{C} \setminus \mathbb{R}$ the mapping* (3.1) *is lower semi-continuous. Then the family $\{S(t) \mid t \in T\}$ is inner continuous.*

Proof. We use the functional calculus for self-adjoint elements in the C^*-algebra $\mathfrak{A}(t)$ to define $\chi[f(t)]$ for every continuous function $\chi : \mathbb{R} \to \mathbb{C}$ vanishing at infinity. Notice that
$$(f(t) - z)^{(-1)_{\mathfrak{A}(t)}} = \chi_z[f(t)], \quad \text{with} \quad \chi_z(\lambda) := (\lambda - z)^{-1}.$$
By a standard argument relying on Stone-Weierstrass Theorem, one deduces that the map $t \mapsto \|\chi[f(t)]\|_{\mathfrak{A}(t)}$ has the same continuity properties (upper or lower semi-continuity, respectively) as (3.1).

Let us suppose now upper semi-continuity in t_0 and assume that $S(t_0) \cap K = \emptyset$ for some compact set K. By Urysohn's Lemma, there exists $\chi \in \mathcal{C}(\mathbb{R})_+$ with $\chi|_K = 1$ and $\chi|_{S(t_0)} = 0$, so $\chi\,[f(t_0)] = 0$. Choose a neighborhood V of t_0 such that for $t \in V$
$$\| \chi[f(t)] \|_{\mathfrak{A}(t)} \leq \| \chi[f(t_0)] \|_{\mathfrak{A}(t_0)} + \frac{1}{2} = \frac{1}{2}.$$

If for some $t \in V$ there exists $\lambda \in K \cap S(t)$, then

$$1 = \chi(\lambda) \leq \sup_{\mu \in S(t)} \chi(\mu) = \| \chi[f(t)] \|_{\mathfrak{A}(t)} \leq \frac{1}{2},$$

which is absurd.

Let us assume now lower semi-continuity in t_0. Pick an open set $A \subset \mathbb{R}$ such that $S(t_0) \cap A \neq \emptyset$ and let $\lambda \in S(t) \cap A$. By Urysohn's Lemma there exist a positive function $\chi \in \mathcal{C}(\mathbb{R})$ with $\chi(\lambda) = 1$ and $\operatorname{supp}(\chi) \subset A$; thus $\| \chi [f(t_0)] \|_{\mathfrak{A}(t_0)} \geq 1$. Suppose moreover that for any neighborhood $W \subset T$ of t_0 there exists $t \in W$ such that $S(t) \cap A = \emptyset$ and thus $\chi [f(t)] = 0$. This clearly contradicts the lower semi-continuity of $t \mapsto \| \chi [f(t)] \|_{\mathfrak{A}(t)}$. We conclude thus the inner continuity property for the family $S(t)$. □

Proving these properties of the resolvents is a priori a difficult task, since this involves working both with norms and composition laws that depend on t. But putting together the information obtained until now, we get our abstract result concerning spectral continuity:

Theorem 3.3. *Let* $\left\{ \mathcal{A} \xrightarrow{\mathcal{P}(t)} \mathcal{A}(t) \mid t \in T \right\}$ *be a covariant upper semi-continuous field of* C^**-algebras indexed by a Hausdorff locally compact space* T *and let* f *be a smooth self-adjoint element of* $\mathcal{A}$. *For any* $t \in T$ *we denote by* $\mathfrak{A}(t)$ *the Rieffel quantization of* $\mathcal{A}(t)$ *and consider* $f(t) := \mathcal{P}(t)f$ *as an element of* $\mathcal{A}(t)^\infty = \mathfrak{A}(t)^\infty \subset \mathfrak{A}(t)$, *with spectrum* $S(t)$ *computed in* $\mathfrak{A}(t)$. *Then the family* $\{S(t) \mid t \in T\}$ *is outer continuous.*

If the field is continuous, the family of subsets will also be continuous.

Proof. Theorem 1.2 allows us to conclude that the quantized field

$$\left\{ \mathfrak{A} \xrightarrow{\mathfrak{P}(t)} \mathfrak{A}(t) \mid t \in T \right\}$$

has the same continuity properties as the original one.

For any $z \in \mathbb{C} \setminus \mathbb{R}$ one has $(f - z)^{(-1)_{\mathfrak{A}}} \in \mathfrak{A}$ and $(f(t) - z)^{(-1)_{\mathfrak{A}(t)}} = \mathfrak{P}(t) \left[(f - z)^{(-1)_{\mathfrak{A}}} \right]$. Therefore the assumptions of Proposition 3.2 are fulfilled both in the upper semi-continuous and in the lower semi-continuous case, so we obtain the desired continuity properties for the family of sets $\{S(t) \mid t \in T\}$. □

Of course, the conclusion also holds for non-smooth self-adjoint elements $f \in \mathfrak{A}$. Very often they are much less "accessible" than the smooth elements, being obtained by an abstract completion procedure, so we only make the statements for C^∞ vectors.

Specializing to the Abelian case and using the notations of Section 2, one gets

Corollary 3.4. *Assume that* $q : \Sigma \to T$ *is a* Θ*-covariant continuous surjection. Let* $f \in \mathcal{C}^\infty(\Sigma)$ *a real function and for each* $t \in T$ *denote by* $S(t)$ *the spectrum of* $f(t) := f|_{\Sigma_t} \in \mathcal{C}^\infty(\Sigma_t) = \mathfrak{C}^\infty(\Sigma_t)$ *seen as an element of the non-commutative* C^**-algebra* $\mathfrak{C}(\Sigma_t)$.

Then the family $\{S(t) \mid t \in T\}$ *of compact subsets of* $\mathbb{R}$ *is outer continuous. If* q *is also open, the family of subsets is continuous.*

Remark 3.5. One can use [24, Ex. 10.2] to identify quantum tori as Rieffel-type quantizations of usual tori. One is naturally placed in the setting above and can reproduce some known spectral continuity results [8, 3] on generalized Harper operators.

4. Spectral continuity for operators

The standard approach of Quantum Mechanics asks for Hilbert space operators. This can be achieved by representing faithfully the C^*-algebras $\mathfrak{A}(t)$ in a Hilbert space of L^2-functions in a way that generalizes the Schrödinger representation. We are going to get continuity results for both spectra and essential spectra of the emerging self-adjoint operators. We work in the following

Framework

1. $(\mathcal{C}(\Sigma), \Theta, \Xi)$ is an Abelian C^*-dynamical system, with Σ compact.
2. Ξ is symplectic, given in a Lagrangean decomposition $\Xi = \mathscr{X} \times \mathscr{X}^* \ni X = (x, \xi)\,,\, Y = (y, \eta)\,$, where $\mathscr{X}$ is a n-dimensional real vector space, $\mathscr{X}^*$ is its dual and the symplectic form on Ξ is given in terms of the duality between $\mathscr{X}$ and $\mathscr{X}^*$ by $[\![(x,\xi),(y,\eta)]\!] := y\cdot\xi - x\cdot\eta\,$.
3. $q : \Sigma \to T$ is a Θ-covariant continuous surjection. We also assume that each $\Sigma_t := q^{-1}(\{t\})$ is *a quasi-orbit*, i.e., there is a point $\sigma \in \Sigma_t$ such that the orbit $\mathcal{O}_\sigma := \Theta_\Xi(\sigma)$ is dense in Σ_t (we say that σ *generates the quasi-orbit* Σ_t).
4. We fix a real element $f \in \mathcal{C}^\infty(\Sigma)\,$. For each $t \in T$ and for any point σ generating the quasi-orbit Σ_t we define $f(t) := f|_{\Sigma_t}$ and $f_\sigma(t) := f(t) \circ \Theta_\sigma : \Xi \to \mathbb{R}\,$.
5. We set $H_\sigma(t) := \mathfrak{Op}\,[f_\sigma(t)]$ (self-adjoint operator in the Hilbert space $\mathcal{H} := L^2(\mathscr{X})$), by applying to $f_\sigma(t)$ the usual Weyl pseudodifferential calculus. We denote by $S(t)$ the spectrum of $H_\sigma(t)\,$.

Some explanations are needed. It is easy to see that each $f_\sigma(t)$ belongs to $BC^\infty(\Xi)$, i.e., it is a smooth function with bounded derivatives of any order. Therefore, using oscillatory integrals, one can define the self-adjoint operator in $L^2(\mathscr{X}) \ni u$

$$
\begin{aligned}
[H_\sigma(t)u]\,(x) &\equiv \big[\mathfrak{Op}(f_\sigma(t))u\big](x) \\
&:= (2\pi)^{-n}\int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} \mathrm{d}\xi\, e^{i(x-y)\cdot\xi}\; [f_\sigma(t)]\left(\frac{x+y}{2}, \xi\right) u(y)\,.
\end{aligned}
$$

This operator is bounded by the Calderón-Vaillancourt Theorem [9]. Using the notation (2.2), we see that for every $X \in \Xi$ one has $[f_\sigma(t)]\,(X) := f\,[\Theta_X(\sigma)]\,$; this depends on $t \in T$ through σ and only involves the values of f on the dense subset $\mathcal{O}_\sigma$ of Σ_t. The same is true about $H_\sigma(t)$, which can be written

$$[H_\sigma(t)u](x) = (2\pi)^{-n} \int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} \mathrm{d}\xi \, e^{i(x-y)\cdot\xi} f\left[\Theta_{\left(\frac{x+y}{2},\xi\right)}(\sigma)\right] u(y) \,. \tag{4.1}$$

It is shown in [16] that if σ and σ' are both generating the same quasi-orbit Σ_t, then the operators $H_\sigma(t)$ and $H_{\sigma'}(t)$ are isospectral (but not unitarily equivalent in general). Thus the compact set $S(t)$ only depends on t and not on the choice of the generating element σ.

Theorem 4.1. *Assume the Framework above. Then the family $\{S(t) \mid t \in T\}$ is outer continuous.*

If q is also open, than the family is continuous.

Proof. By Corollary 3.4, it would be enough to show for every t that $S(t)$ coincides with the spectrum of $f(t) \in \mathfrak{C}(\Sigma_t)$. For this we define

$$\mathcal{N}_\sigma : \mathcal{C}^\infty(\Sigma_t) \to BC^\infty(\Xi), \qquad \mathcal{N}_\sigma(g) := g \circ \Theta_\sigma$$

and then set

$$\mathfrak{Op}_\sigma := \mathfrak{Op} \circ \mathcal{N}_\sigma : \mathcal{C}^\infty(\Sigma_t) \to \mathbb{B}(\mathcal{H}) \,.$$

Then one has $H_\sigma(t) := \mathfrak{Op}[f_\sigma(t)] = \mathfrak{Op}_\sigma[f(t)]$. It is not quite trivial, but it has been shown in [16], that $\mathfrak{Op}_\sigma$ extends to a faithful representation of the Rieffel quantized C^*-algebra $\mathfrak{C}(\Sigma_t)$ in $\mathcal{H}$. Faithfulness is implied by the fact that σ generates the quasi-orbit Σ_t, which results in the injectivity of $\mathcal{N}_\sigma$, conveniently extended to $\mathfrak{C}(\Sigma_t)$. It follows then that $\mathrm{sp}[H_\sigma(t)] = \mathrm{sp}[f(t)]$, as required, so the family $\{S(t) \mid t \in T\}$ has the desired continuity properties. □

We recall that *the essential spectrum* of an operator is the part of the spectrum composed of accumulation points or infinitely-degenerated eigenvalues. Let us denote by $S^{\mathrm{ess}}(t)$ the essential spectrum of $H_\sigma(t)$; once again this only depends on t. To discuss the continuity properties of this family of sets we are going to need some preparations relying mainly on results from [16].

First we write each Σ_t as a disjoint Θ-invariant union $\Sigma_t = \Sigma_t^{\mathrm{g}} \sqcup \Sigma_t^{\mathrm{n}}$. The elements σ_1 of Σ_t^{g} are *generic points for* Σ_t, meaning that each of them is generating Σ_t. The points $\sigma_2 \in \Sigma_t^{\mathrm{n}}$ are *non-generic*, i.e., the closure of the orbit $\mathcal{O}_{\sigma_2}$ is strictly contained in Σ_t.

Let us now fix a point $t \in T$ and a generating element $\sigma \in \Sigma_t$. The monomorphism $\mathcal{N}_\sigma$ extends to an isomorphism between $\mathcal{C}(\Sigma_t)$ and a C^*-subalgebra $\mathcal{B}_\sigma(t)$ of the C^*-algebra $BC_{\mathrm{u}}(\Xi)$ of all the bounded uniformly continuous complex functions on Ξ. It is shown in Lemma 2.2 from [16] that only two possibilities can occur, and this is independent of σ: either $\mathcal{C}(\Xi) \subset \mathcal{B}_\sigma(t)$ (and then t is called *of the first type*), or $\mathcal{C}(\Xi) \cap \mathcal{B}_\sigma(t) = \{0\}$ (and then we say that t is *of the second type*). Correspondingly, one has the disjoint decomposition $T = T_I \sqcup T_{II}$.

Theorem 4.2. *Assume the Framework above. Then the family $\{S^{\mathrm{ess}}(t) \mid t \in T\}$ is outer continuous.*

Proof. One must rephrase the essential spectrum $S^{\rm ess}(t):={\rm sp}_{\rm ess}[H_\sigma(t)]$ in convenient C^*-algebraic terms. Assume first that t is of the second type. By [16, Prop. 3.4], the discrete spectrum of $H_\sigma(t)$ is void, thus one has $S^{\rm ess}(t)=S(t)$. If t is of the first type, the subset $\Sigma_t^{\rm n}$ is invariant under the action Θ and it is also closed by [16, Prop. 2.5]. Denoting by $f^{\rm n}(t)$ the restriction of $f(t)$ to $\Sigma_t^{\rm n}$, one gets an element of $\mathcal C^\infty(\Sigma_t^{\rm n})\subset\mathfrak C(\Sigma_t^{\rm n})$ with spectrum $S^{\rm n}(t)$. But [16, Th. 3.7] states among others that $S^{\rm n}(t)$ coincides with $S^{\rm ess}(t)$.

We need to construct now a suitable restricted dynamical system. Let us consider the decomposition

$$\Sigma=\Big(\bigsqcup_{t\in T_I}\Sigma_t\Big)\sqcup\Big(\bigsqcup_{t\in T_{II}}\Sigma_t\Big)=\Big(\bigsqcup_{t\in T_I}\Sigma_t^{\rm g}\Big)\sqcup\Big\{\Big(\bigsqcup_{t\in T_I}\Sigma_t^{\rm n}\Big)\sqcup\Big(\bigsqcup_{t\in T_{II}}\Sigma_t\Big)\Big\}=:\Sigma^{\rm d}\sqcup\Sigma^{\rm ess}\,.$$

One might set $\Sigma_t^{\rm ess}:=\Sigma_t^{\rm n}$ if $t\in T_I$ and $\Sigma_t^{\rm ess}:=\Sigma_t$ if $t\in T_{II}$. Notice that each $\Sigma_t^{\rm ess}$ is not void. This is clear for $t\in T_{II}$, since q has been supposed surjective. If $t\in T_I$ and $\Sigma_t^{\rm n}=\emptyset$, then $\Sigma_t=\Sigma_t^{\rm g}$ is minimal and compact, so $t\in T_{II}$ by Lemma 2.3 in [16], which is absurd. The disjoint union $\Sigma^{\rm ess}:=\sqcup_{t\in T}\Sigma_t^{\rm ess}$ (with the topology induced from Σ) is a compact dynamical system under the restriction of the action Θ of Ξ and $q^{\rm ess}:=q|_{\Sigma^{\rm ess}}:\Sigma^{\rm ess}\to T$ is a covariant continuous surjection. Thus we can apply the previous results and conclude that $\{\mathfrak C(\Sigma^{\rm ess})\to\mathfrak C(\Sigma_t^{\rm ess})\mid t\in T\}$ is an upper semi-continuous field of C^*-algebras; the arrows are Rieffel quantizations of obvious restriction maps.

From all these applied to $f|_{\Sigma^{\rm ess}}\in\mathfrak C^\infty(\Sigma^{\rm ess})$ it follows that

$$\big\{S^{\rm ess}(t)={\rm sp}\,\big[f(t)|_{\Sigma^{\rm ess}(t)}\big]\mid t\in T\big\}$$

is outer continuous. □

Remark 4.3. Even in simple situations, the surjective restriction of a continuous open surjection may not be open. So $q^{\rm ess}$ may fail to be open and in general we don't obtain inner continuity for the family of essential spectra. On the other hand, if openness of the restriction $q^{\rm ess}$ is required explicitly, one clearly gets the inner continuity. Since only the dynamical system $(\Sigma^{\rm ess},\Theta,\Xi)$ is involved in controlling the family of essential spectra, some assumptions weaker than those above would suffice.

Example 4.4. To show that our results cover families of zero-order standard pseudodifferential operators, one has to make some simple choices. Recall that $BC_{\rm u}(\Xi)$, the unital C^*-algebra of all bounded uniformly continuous functions on Ξ, is stable under the action θ of Ξ by translations. The *-algebra of smooth elements is $BC^\infty(\Xi)$, formed of smooth functions with bounded derivatives; such functions can be regarded as zero-order symbols in the sense of Hörmander. By Gelfand theory, $BC_{\rm u}(\Xi)$ is isomorphic to $\mathcal C(\Omega)$ for a compactification Ω of Ξ. Actually, if $g\in BC_{\rm u}(\Xi)$, its image $\hat g$ in $\mathcal C(\Omega)$ is just the extension of g from the dense subset Ξ to the entire Ω. The action by translations θ of Ξ on itself extends to an action by homeomorphisms of Ω.

Let now T be a locally compact space and for each $t \in T$ let $g(t)$ be a real element of $BC^\infty(\Xi)$ and $\hat{g}(t)$ its continuous extension to Ω. We are also requiring that the map $T \ni t \mapsto g(t) \in BC_{\rm u}(\Xi)$ be continuous. Denote by $H(t) := \mathfrak{Op}[g(t)]$ the zero-order pseudodifferential operator obtained by Weyl quantization. We claim that both the spectra $S(t)$ and the essential spectra $S^{\rm ess}(t)$ of these operators form inner and outer continuous families of compact subsets of $\mathbb{R}$.

To see this, construct the locally compact space $\Sigma := \Omega \times T$ and endow it with the action $\Theta_X(\omega, t) := (\theta_X(\omega), t)$. Of course, $q : \Sigma \to T,\ q(\omega, t) := t$ is a Θ-covariant continuous open surjection. Each quasi-orbit $\Sigma_t := q^{-1}(\{t\}) = \Omega \times \{t\}$ is of the first type and can be identified with Ω. The generic points of Ω are the elements of Ξ, while the non-generic ones are those of $\Omega \setminus \Xi$. The restriction of q to $\Sigma^{\rm ess} = (\Omega \setminus \Xi) \times T$ is still (continuous and) open.

We use the family $\{g(t) \mid t \in T\}$ to define a function $f : \Sigma \to \mathbb{R}$ by $f(\omega, t) := [\hat{g}(t)](\omega)$. One checks easily that $f \in \mathcal{C}^\infty(\Sigma)$. Choosing the generating point $\sigma := (0, t) \in \Sigma_t$, one gets $f_{(0,t)} = g(t)$, so $H_\sigma(t) := \mathfrak{Op}[f_\sigma(t)] = \mathfrak{Op}[g(t)] =: H(t)$. So the continuity of the families $\{S(t) \mid t \in T\}$ and $\{S^{\rm ess}(t) \mid t \in T\}$ follows from Theorems 4.1 and 4.2. Obviously, the result for essential spectra still holds if to each Hamiltonian $H(t)$ one adds a compact perturbation $V(t)$.

Acknowledgement

The authors are partially supported by *Núcleo Científico ICM P07-027-F "Mathematical Theory of Quantum and Classical Magnetic Systems"*. They are grateful to an anonymous referee for many useful remarks.

References

[1] N. Athmouni, M. Măntoiu and R. Purice: *On the Continuity of Spectra for Families of Magnetic Pseudodifferential Operators*, J. Math. Phys. **51** (1), (2010).

[2] J. Avron and B. Simon: *Stability of Gaps for Periodic Potentials under a Variation of the Magnetic Field*, J. Phys. A **18**, 2199, (1985).

[3] J. Bellissard: *Lipschitz Continuity of Gap Boundaries for Hofstadter-like Spectra*, Commun. Math. Phys. **160**, 599–613, (1994).

[4] J. Bellissard, D.J. Herrmann and M. Zarrouati: *Hull of Aperiodic Solids and Gap Labelling Theorems*, in Directions in Mathematical Quasicrystals, CIRM Monograph Series, **13**, 207–259, (2000).

[5] J. Bellissard, B. Iochum and D. Testard: *Continuity Properties of the Electronic Spectrum of 1D Quasicristals*, Commun. Math. Phys. **141**, 353–380, (1991).

[6] F. Belmonte and M. Măntoiu: *Covariant Fields of C^*-Algebras under Rieffel Deformation*, submitted.

[7] I. Beltiţă and M. Măntoiu: *Rieffel Quantization and Twisted Crossed Products*, submitted.

[8] G. Elliott: *Gaps in the Spectrum of an Almost Periodic Schrödinger Operator*, C. R. Math. Rep. Acad. Sci. Canada, **4**, 255–259, (1982).

[9] G.B. Folland, *Harmonic Analysis in Phase Space*, Annals of Mathematics Studies, 122. Princeton University Press, Princeton, NJ, 1989.

[10] V. Georgescu and A. Iftimovici: *Crossed Products of C^*-Algebras and Spectral Analysis of Quantum Hamiltonians*, Commun. Math. Phys. **228**, 519–560, (2002).

[11] K. Hannabuss and V. Mathai: *Noncommutative Principal Torus Bundles via Parametrized Strict Deformation Quantization*, AMS Proceedings of Symposia of Pure Mathematics, **81**, 133–148, (2010).

[12] K. Hannabuss and V. Mathai: *Parametrized Strict Deformation Quantization of C^*-Bundles and Hilbert Modules*, J. of the Australian Math. Soc. **90**, 25–38, (2011).

[13] V. Iftimie: *Opérateurs differentiels magnétiques: Stabilité des trous dans le spectre, invariance du spectre essentiel et applications*, Commun. in P.D.E. **18**, 651–686, (1993).

[14] P. Kasprzak: *Rieffel Deformation via Crossed Products*, J. Funct. Anal. **257**, 1288–1332, (2009).

[15] M. Măntoiu: *Compactifications, Dynamical Systems at Infinity and the Essential Spectrum of Generalized Schrödinger Operators*, J. reine angew. Math. **550** (2002), 211–229.

[16] M. Măntoiu: *Rieffel's Pseudodifferential Calculus and Spectral Analysis for Quantum Hamiltonians*, preprint ArXiV and to appear in Ann. Inst. Fourier.

[17] M. Măntoiu: *On the Essential Spectrum of Phase-Space Anisotropic Pseudodifferential Operators*, preprint ArXiV and submitted.

[18] M. Măntoiu, R. Purice and S. Richard, *Spectral and Propagation Results for Magnetic Schrödinger Operators; a C^*-Algebraic Framework*, J. Funct. Anal. **250** (2007), 42–67.

[19] M. Nilsen, *C^*-Bundles and $C_0(X)$-Algebras*, Indiana University Mathematics Journal **45** (2) (1996), 463–477.

[20] G. Nenciu: *Stability of Energy Gaps Under Variations of the Magnetic Field*, Letters in Mat. Phys. **11**, 127–132, (1986).

[21] J. Packer and I. Raeburn: *Twisted Crossed Products of C^*-algebras*, Math. Proc. Camb. Phyl. Soc. **106**, 293–311, (1989).

[22] J. Packer and I. Raeburn: *Twisted Crossed Products of C^*-Algebras II*, Math. Ann. **287**, 595–612, (1990).

[23] M.A. Rieffel: *Continuous Fields of C^*-Algebras Coming from Group Cocycles and Actions*, Math. Ann. **283**, 631–643, (1989).

[24] M.A. Rieffel: *Deformation Quantization for Actions of $\mathbb{R}^d$*, Memoirs of the AMS, **506**, 1993.

[25] J. Tomiyama: *Topological Representation of C^*-Algebras*, Tohoku Math. J. **14**, 187, (1962).

[26] D. Williams: *Crossed Products of C^*-Algebras*, Mathematical Surveys and Monographs, **134**, American Mathematical Society, 2007.

F. Belmonte and M. Măntoiu
Departamento de Matemáticas
Universidad de Chile
Las Palmeras 3425
Casilla 653, Santiago, Chile
e-mail: fabianbelmonte@gmail.com
Marius.Mantoiu@imar.ro

Springer

http://www.springer.com/978-3-0348-0413-4

Spectral Analysis of Quantum Hamiltonians
Spectral Days 2010
Benguria, R.; Friedman, E.; Mantoiu, M. (Eds.)
2012, VII, 338 p., Hardcover
ISBN: 978-3-0348-0413-4
A product of Birkhäuser Basel