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The Riemann Zeta-function: Approximation of Analytic Functions

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Abstract. In the paper, a short survey on the theory of the Riemann zeta-function is given. The main attention is given to universality-approximation of analytic functions by shifts of the Riemann zeta-function. This includes the effectivization problem, generalization for other zeta-functions, joint universality as well as some applications.

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1. Introduction

We recall that the function $f(z)$ is called analytic at the point z_0 if $f(z)$ has a power series expansion

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m$$

which is convergent in some neighbourhood of the point z_0. The function $f(z)$ is analytic in the set D if it is analytic in each point of D.

A set of points (x, y), where

$$x = f_1(t), \quad y = f_2(t), \quad 0 \leq t \leq 1, \tag{1.1}$$

and $f_1(t)$ and $f_2(t)$ are continuous functions such that, for given x and y the system (1.1) has no more than one solution, is called the Jordan arc.

A set D is said to be connected if any two of its points, arbitrarily chosen, can be connected by a Jordan arc lying in D. A domain is an open connected set.

It is well known that analytic functions can be approximated by polynomials. The ultimate result in this field belongs to S.N. Mergelyan [S.N. Mergelyan (1951)], [S.N. Mergelyan (1952)], see also [J.L. Walsh (1960)].

Theorem 1.1. *Suppose that K is a compact subset on the complex plane with connected complement, and the function $f(z)$ is continuous on K and analytic in the interior of K. Then, for every $\epsilon > 0$, there exists a polynomial $P(z)$ such that*

$$\sup_{z \in K} |f(z) - P(z)| < \epsilon.$$

Note that conditions on the set K and function $f(z)$ are necessary.

In Theorem 1.1, the approximating polynomial depends on the function $f(z)$. It turns out that there exist functions $F(z)$ such that their shifts $F(z+i\tau)$ approximate any analytic function. The simplest of $F(z)$ is the Riemann zeta-function.

In this survey, we give an introduction to the theory of the Riemann zeta-function, state the universality theorem, discuss the effectivization problem of this theorem and present recent results on universality of zeta-functions.

2. The Riemann zeta-function

Let $s = \sigma + it$ be a complex variable. The Riemann zeta-function $\zeta(s)$ is defined, in the half-plane $\sigma > 1$, by the series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}.$$

We recall that the series of the form

$$\sum_{m=1}^{\infty} a_m \mathrm{e}^{-\lambda_m s},$$

where $a_m \in \mathbb{C}$ and $\{\lambda_m\}$ is an increasing sequence of real numbers such that $\lim\limits_{m \to \infty} \lambda_m = +\infty$, are called general Dirichlet series. If $\lambda_m = \log m$, then we have an ordinary Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

The region of convergence as well as of absolute convergence of Dirichlet series is a half-plane. Thus, the Riemann zeta-function is given, for $\sigma > 1$, by ordinary Dirichlet series with coefficients $a_m \equiv 1$.

The function $\zeta(s)$, as a function of a complex variable, was introduced by B. Riemann in 1859. However, the function $\zeta(s)$ with real s earlier was studied by L. Euler.

Denote by $[u]$ the integer part of u. Summing by parts, it it easy to obtain that, for $\sigma > 1$,

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + s \int_1^{\infty} \frac{[x] - x + \frac{1}{2}}{x^{s+1}} \, \mathrm{d}x. \tag{2.1}$$

Clearly, the integral converges absolutely for $\sigma > 0$, and uniformly for $\sigma \geq \varepsilon$ with arbitrary $\varepsilon > 0$. Therefore, it defines a function analytic for $\sigma > 0$. Hence, (2.1) gives analytic continuation for $\zeta(s)$ to the region $\sigma > 0$, except for a simple pole at the point $s = 1$ with residue 1.

Denote, as usual, by $\Gamma(s)$ the Euler gamma-function which is defined, for $\sigma > 0$, by

$$\Gamma(s) = \int_0^\infty e^{-u} u^{s-1} du.$$

Moreover, the function $\Gamma(s)$ is meromorphically continuable over the whole complex plane, the points $s = -m$, $m = 0, 1, 2, \ldots$, are simple poles, and

$$\mathrm{Res}_{s=-m}\Gamma(s) = \frac{(-1)^m}{m!}.$$

The Euler gamma function is involved in the functional equation of the Riemann zeta function

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \tag{2.2}$$

which implies analytic continuation for $\zeta(s)$ to the region $\sigma < \frac{1}{2}$.

Riemann began to study the function $\zeta(s)$ for needs of the distribution of prime numbers, i.e., for the asymptotics for the function

$$\pi(x) = \sum_{p \le x} 1, \quad p \text{ is prime},$$

as $x \to \infty$. A relation of $\zeta(s)$ with prime numbers is clearly seen from the Euler identity

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1, \tag{2.3}$$

which is a simple consequence of the principal theorem of arithmetics and definition of $\zeta(s)$ by Dirichlet series. Riemann proposed [B. Riemann (1859)] an original way to obtain the asymptotic formula

$$\pi(x) \sim \int_2^x \frac{du}{\log u}, \quad x \to \infty,$$

however, his work was not completely correct. Riemann's ideas were realized probably 50 years later independently by C.J. de la Vallée Poussin [C.J. de la Vallée-Poussin (1896)] and J. Hadamard [J. Hadamard (1896)].

It turned out that a problem of the asymptotics for $\pi(x)$ is closely connected to zeros of the function $\zeta(s)$. From the functional equation (2.2) it follows that $\zeta(s) = 0$ for $s = -2m$, $m \in \mathbb{N}$. These zeros of $\zeta(s)$ are called trivial and, in general, are not interesting. The Euler identity (2.3) shows that $\zeta(s) \neq 0$ for $\sigma > 1$. It is not difficult to show that $\zeta(s) \neq 0$ on the line $\sigma = 1$, and this is already sufficient to obtain the asymptotics for $\pi(x)$. This and (2.2) imply that $\zeta(s) \neq 0$ for $\sigma \leq 0$. Application of elements of the theory of entire functions of order 1 for the function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

allows us to prove that the function $\zeta(s)$ has infinitely many zeros in the strip $0 < \sigma < 1$. These zeros of $\zeta(s)$ are called non-trivial, and play an important role not only in analytic number theory but in mathematics in general. The famous Riemann hypothesis (RH) says that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\sigma = \frac{1}{2}$. RH is included in the list of seven Millennium Prize Problems, for each of which a solution carries a prize of US \$ 1 million, set up by the Clay Mathematics Institute [The Millennium Prize problems (2006)].

There exist results favorable for RH, however some facts do not support it. There are many computations on the location of zeros of $\zeta(s)$. For example, in [J. van de Lune, H.J.J. te Riele, D.T. Winter (1986)] the first 1500000001 zeros of $\zeta(s)$ were found, all lying on the critical line; moreover, they all are simple. Of course, calculations can not prove RH, they can only disprove it.

In applications, it is important to know regions where $\zeta(s) \neq 0$. The best result in this direction is of the form: there exists an absolute constant $c > 0$ such that $\zeta(s) \neq 0$ in the region

$$\sigma \geq 1 - \frac{c}{(\log t)^{2/3}(\log\log t)^{1/3}}, \quad t \geq t_0 > 0.$$

This result is due to H.-E. Richert who never published its proof.

For $T > 0$, let $N(T)$ denote the number of non-trivial zeros of $\zeta(s)$ lying in the rectangle $0 < \sigma < 1$, $0 < t \leq T$. Then the von Mangoldt formula

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \mathrm{O}(\log T), \quad T \to \infty,$$

is true. This formula was conjectured by Riemann and proved in [H. von Mangoldt (1895)] by H. von Mangoldt.

Let $N_0(T)$ denote the number of zeros of $\zeta(s)$ of the form $s = \frac{1}{2}+it$, $0 < t \leq T$. Then RH is equivalent to the assertion that $N_0(T) = N(T)$ for all $T > 0$. We recall some results on the relation between $N(T)$ and $N_0(T)$.

In 1914, G.H. Hardy proved [G.H. Hardy (1914)] that $N_0(T) \to \infty$ as $T \to \infty$. More precisely, he obtained that $N_0(T) > cT$, $c > 0$, for $T \geq T_0$.

In 1942, A. Selberg found [A. Selberg (1942)] that $N_0(T) > cT\log T$, $c > 0$, for $T \geq T_0$, that is, that a positive proportion of non-trivial zeros lies on the critical line.

A very important result belongs to N. Levinson. In 1974, he proved [N. Levinson (1974)] that

$$N_0(T) \geq \frac{1}{3}N(T).$$

In 1983, J.B. Conrey improved [J.B. Conrey (1989)] this result till $N_0(T) \geq \frac{2}{5}N(T)$.

One more important conjecture in the theory of the Riemann zeta-function is the Lindelöf hypothesis (LH). LH asserts that, with arbitrary $\varepsilon > 0$,

$$\zeta\left(\frac{1}{2}+it\right) = \mathrm{O}_\varepsilon(t^\varepsilon), \quad t \geq t_0,$$

or equivalently

$$\zeta\left(\sigma+it\right) = \mathrm{O}_\varepsilon(t^\varepsilon), \quad t \geq t_0,$$

for all $\sigma > \frac{1}{2}$. The classical estimate says that

$$\zeta\left(\frac{1}{2}+it\right) = \mathrm{O}\left(t^{\frac{1}{6}}\right).$$

The best result in this direction belongs to M.N. Huxley [M.N. Huxley (2005)], and is of the form

$$\zeta\left(\frac{1}{2}+it\right) = \mathrm{O}\left(t^{\frac{32}{205}+\varepsilon}\right).$$

It is well known that RH implies LH.

There are several equivalents of LH. One of them is related to the moments of $\zeta(s)$. Namely, LH is equivalent to the estimates: for arbitrary $\varepsilon > 0$

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} \mathrm{d}t = \mathrm{O}_\varepsilon(T^{1+\varepsilon}), \quad k \in \mathbb{N},$$

or

$$\int_1^T \left|\zeta\left(\sigma+it\right)\right|^{2k} \mathrm{d}t = \mathrm{O}_\varepsilon(T^{1+\varepsilon}), \quad k \in \mathbb{N},$$

for all $\sigma > \frac{1}{2}$.

In general, the moment problem is a very important and difficult one in the theory of the Riemann zeta-function.In some applications, individual values of $\zeta(s)$ can be replaced by its mean-value estimates. There exists a conjecture that, as $T \to \infty$,

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} \mathrm{d}t \sim c_k T(\log T)^{k^2}, \quad k > 0.$$

This is proved only for three values of k:

$c_1 = 1$, Hardy-Littlewood (1918);

$c_2 = \frac{1}{2\pi^2}$, Ingham (1926).

From a probabilistic limit theorem for $\left|\zeta\left(\frac{1}{2}+it\right)\right|$ it follows [A. Laurinčikas (1996)] that $c_k = 1$ for $k = c(\log\log T)^{-\frac{1}{2}}$, $c > 0$.

3. Universality

We understand the universal mathematical object as an object having influence for a wide class of other objects. In analysis, this influence often is related with a certain approximation.

The first universal object in analysis was found by Fekete in 1914. He proved that there exists a real power series

$$\sum_{m=1}^{\infty} a_m x^m$$

which is divergent for all $x \neq 0$. Moreover, this divergence is so extreme that,for every continuous function f on $[-1, 1]$, $f(0) = 0$, there exists a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} \sum_{m=1}^{n_k} a_m x^m = f(x)$$

uniformly on $[-1, 1]$.

After Fekete's result, many universal objects were found. We recall one theorem of Birkhoff. He proved [G.D. Birkhoff (1929)] that there exists an entire function $f(z)$ such that, for every entire function $g(z)$, there exists a sequence of complex numbers $\{a_m\}$ such that

$$\lim_{m\to\infty} f(z + a_m) = g(z)$$

uniformly on compact subsets of the complex plane.

The term of universality was used for the first time by J. Marcinkiewicz in [J. Marcinkiewicz (1935)]. He obtained the following result. Let $\{h_n\}$ be a sequence of real numbers and $\lim\limits_{n\to\infty} h_n = 0$. Then he proved that there exists a continuous function $f \in C[0, 1]$ such that, for every continuous function $g \in C[0, 1]$, there exists an increasing sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\lim_{k\to\infty} \frac{f(x + h_{n_k}) - f(x)}{h_{n_k}} = g(x)$$

almost everywhere on $[0, 1]$. Marcinkiewicz called the function f a primitive universal.

However, all of the above and other known universal objects, were not explicitly given, only their existence was proved. As recently as 1975, S.M. Voronin [S.M. Voronin (1975)] found the first explicitly given universal (in a certain sense object). It was not very strange that this object is the famous Riemann zeta-function $\zeta(s)$.

The first version of the Voronin theorem is as follows.

Theorem 3.1 [S.M. Voronin (1975)]. *Let $0 < r < \frac{1}{4}$. Suppose that $f(s)$ is a continuous non-vanishing function on the disc $|s| \leq r$ which is analytic in the interior* *of this disc. Then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that*

$$\max_{|s|\le r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-f(s)\right|<\varepsilon.$$

Roughly speaking, the Voronin theorem asserts that any analytic function is approximated with desired accuracy uniformly on the disc by shifts of the Riemann zeta-function. Voronin himself called his theorem "theorem o kruzhochkakh". Nowadays its name is the Voronin universality theorem.

A modern version of the Voronin theorem has a bit more general form. Denote by meas$\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, see, for example, [A. Laurinčikas (1996)].

Theorem 3.2. *Let K be a compact subset of the strip $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$ with connected complement. Let $f(s)$ be a continuous and non-vanishing on K function which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\right\}>0.$$

The latter theorem shows that the set of shifts $\zeta(s+i\tau)$ whose approximate uniformly on K a given analytic function $f(s)$ is sufficiently rich, it has a positive lower density.

The universality in the Voronin sense of $\zeta(s)$ has a direct connection to RH.It is known, see, for example, [J. Steuding (2007)] that RH is equivalent to the following statement: for every $\varepsilon > 0$,

$$\liminf_{T\to\infty}\frac{1}{T}\text{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau)-\zeta(s)|<\varepsilon\right\}>0,$$

where the set K is the same as in Theorem 3.2.

In Theorem 3.2, the shifts $\zeta(s + i\tau)$ occur, where τ varies continuously in the interval $[0, T]$. Therefore, the universality of $\zeta(s)$ in Theorem 3.2 is called continuous. Also, a discrete universality of $\zeta(s)$ is known. It is included in the following theorem.

Theorem 3.3. *Let $h > 0$ be a fixed number, and K and $f(s)$ be the same as in Theorem 3.2. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty}\frac{1}{N+1}\sharp\left\{0\le m\le N:\sup_{s\in K}|\zeta(s+imh)-f(s)|<\varepsilon\right\}>0.$$

4. Effectivization problem

The universality theorem for $\zeta(s)$ has one very important shortcoming. It is not effective in the sense that we do not know any concrete value $\tau \in \mathbb{R}$ for which

$$\sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon,$$

or $m \in \mathbb{N}$ with

$$\sup_{s\in K} |\zeta(s+imh) - f(s)| < \varepsilon.$$

For applications of the universality theorem, it is sufficient to know at least an interval $[0, T_0]$ containing τ with an approximating property. However, in our opinion, the latter problem is very difficult. The first attempt in this direction was made in [A. Good (1981)] by A. Good, however, his results are too complicated to be given here. Interesting results were obtained by my student R. Garunkštis. Suppose that the function $f(s)$ is analytic on the disc $|s| \le 0.05$ and $\max\limits_{|s|\le 0.05} |f(s)| < 1$. Then Garunkštis proved [R. Garunkštis (2003)] that, for every $0 < \varepsilon < \frac{1}{2}$, there exists τ,

$$0 \le \tau \le \exp\left\{\exp\{10\varepsilon^{-13}\}\right\},$$

such that

$$\max_{|s|\le 0.0001} \left|\log\zeta\left(s+\frac{3}{4}+i\tau\right) - f(s)\right| < \varepsilon,$$

and

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T] : \max_{|s|\le 0.0001}\left|\log\zeta\left(s+\frac{3}{4}+i\tau\right) - f(s)\right| < \varepsilon\right\} \\ \ge \exp\left\{-\varepsilon^{-13}\right\}.$$

Also, an estimate for the upper universality density is known [J. Steuding (2003)]. Suppose that $r \in \left(0, \frac{1}{4}\right)$ and the function $f(s)$ is non-vanishing and analytic on the disc $|s| \le r$. Then, for every $\varepsilon \ge 0$,

$$\limsup_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T] : \max_{|s|\le r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right) - f(s)\right| < \varepsilon\right\} = \mathrm{O}(\varepsilon).$$

The last result in this direction is as follows. For $\underline{b} = (b_0, b_1, \dots, b_{n-1}) \in \mathbb{C}^n$, let

$$\|\underline{b}\| = \sum_{k=0}^{n-1} |b_n|,$$

and

$$A(n, \underline{b}, \varepsilon) = |\log|b_0|| + \left(\frac{\|\underline{b}\|}{\varepsilon}\right)^{n^2}.$$

Then we have the following statement.

Theorem 4.1. [R. Garunkštis, A. Laurinčikas, K. Matsumoto, J. Steuding, R. Steuding (2010)] *Let $s_0 = \sigma_0 + it_0$, $\sigma_0 \in \left(\frac{1}{2}, 1\right)$ and $K = \{s \in \mathbb{C} : |s - s_0| \le r\}$. Moreover, let $g : K \to \mathbb{C}$ be a continuous function, $g(s_0) \ne 0$, which is analytic on the disc $|s - s_0| \le r$. Then, for every $\varepsilon \in (0, |g(s_0)|)$, there exist real numbers $\tau \in [T, 2T]$ and $\delta = \delta(\varepsilon, g, \tau) > 0$, connected by the equality*

$$M(\tau)\frac{\delta^n}{1-\delta} = \frac{\varepsilon}{3}\left(2 - \mathrm{e}^{\delta r}\right)$$

with

$$M(\tau) = \max_{|s-s_0|=r} |\zeta(s+i\tau)|,$$

such that

$$\max_{|s-s_0|\le\delta r} |\zeta(s+i\tau) - g(s)| < \varepsilon.$$

Here $T = T(g, \varepsilon, \sigma_0) > r$ *satisfies the inequality*

$$T \ge C(n,\sigma_0)\exp\left\{\exp\left\{5A\left(n, \underline{g}, \frac{\varepsilon}{3}\right)^{\frac{8}{1-\sigma_0}+\frac{8}{\sigma_0-\frac{1}{2}}}\right\}\right\},$$

where

$$\underline{g} = \left(g(s_0), g'((s_0), \dots, g^{(n-1)}(s_0)\right),$$

and $C(n, \sigma_0)$ *is an effective computable constant depending on* n *and* σ_0*.*

Remark. *The requirement* $g(s_0) \ne 0$ *can be removed if* $A(n, \underline{g}, \frac{\varepsilon}{3})$ *is changed by* $A(n, \underline{g}_\varepsilon, \frac{\varepsilon}{3})$*, where*

$$\underline{g}_\varepsilon = \left(\frac{\varepsilon}{2}, g'(s_0), \dots, g^{(n-1)}(s_0)\right).$$

We note that Theorem 4.1 gives only an approximation to the effectivization problem of the universality theorem, and is far from the full solution of the problem.

5. Other zeta-functions

The Riemann zeta-function is not unique in having the above universality property. There exists the Linnik-Ibragimov conjecture that all functions $Z(s)$ in some half-plane given by Dirichlet series

$$Z(s) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \quad \sigma > \sigma_0,$$

analytically continuable to the half-plane $\sigma > \sigma_1$, $\sigma_1 < \sigma_0$, and satisfying certain natural growth conditions, are universal in the Voronin sense. Usually, in the proof of universality the estimates, for $\sigma_1 < \sigma < \sigma_0$,

$$\int_0^T |Z(\sigma+it)|^2 \mathrm{d}t \ll T$$

and

$$Z(\sigma+it) \ll t^a, \quad a > 0, \quad t > t_0 > 0,$$

are applied. In our opinion, the latter conjecture is very difficult, however, the majority of classical zeta-functions are universal.

On the other hand, there exist non-universal functions given by Dirichlet series. For example, suppose that

$$a_m = \begin{cases} 1 \text{ if } m = m_0^k, \ k \in \mathbb{N}, \\ 0 \text{ if } m \neq m_0^k, \end{cases}$$

where $m_0 \in \mathbb{N}\backslash\{1\}$. Then we have that, for $\sigma > 0$,

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s} = \sum_{k=1}^{\infty} \frac{1}{m_0^{ks}} = \frac{1}{m_0^s - 1}.$$

The function $(m_0^s - 1)^{-1}$ is analytic in the whole complex plane except, for simple poles on the line $\sigma = 0$; it however, obviously, is non-universal. In this section, we discuss some examples of other universal zeta-functions.

Let $0 < \alpha \leq 1$ be a fixed parameter. The Hurwitz zeta-function $\zeta(s, \alpha)$ is defined, for $\sigma > 1$, by

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

and can be meromorphically continued to the whole complex plane. The point $s = 1$ is a simple pole with residue 1. Obviously, $\zeta(s, 1) = \zeta(s)$, so $\zeta(s, \alpha)$ is a generalization of the Riemann zeta-function. On the other hand, its properties are governed by the arithmetical nature of the parameter α. The simplest case is of transcendental α, i.e., when α is not a root of any polynomial with rational coefficients. In this case, the set

$$\{\log(m + \alpha) : m \in \mathbb{N}_0\}, \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

is linearly independent over the field $\mathbb{Q}$ of rational numbers. We observe that $\zeta(s, \alpha)$ with transcendental α has no Euler product over primes, therefore, its universality differs from that of the Riemann zeta-function: the approximated function can be not necessarily non-vanishing, see, for example, [A. Laurinčikas, R. Garunkštis (2002)].

Theorem 5.1. *Suppose that α is transcendental. Let K be a compact subset of the strip D with connected complement, and $f(s)$ be a continuous function on K and analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0, T] : \sup_{s \in K} |\zeta(s + i\tau, \alpha) - f(s)| < \varepsilon \right\} > 0.$$

If α is rational, then the function $\zeta(s, \alpha)$ is also universal. However, the case of algebraic irrational α is an open problem.

Some periodic generalizations of the Riemann and Hurwitz zeta-functions are known. Let $\mathfrak{a} = a_m : m \in \mathbb{N}$ be a periodic with a minimal period $k \in \mathbb{N}$ sequence of complex numbers. Then the function

$$\zeta(s;\mathfrak{a}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s}, \quad \sigma > 1,$$

is called a periodic zeta-function. The periodicity of $\mathfrak{a}$ implies, for $\sigma > 1$, the equality

$$\zeta(s;\mathfrak{a}) = \frac{1}{k^s} \sum_{l=1}^{k} a_l \zeta\left(s, \frac{l}{k}\right).$$

Hence, by virtue of the well-known properties of $\zeta(s,\alpha)$, we have that the function $\zeta(s;\mathfrak{a})$ has analytic continuation to the whole complex plane. If

$$a = \frac{1}{k} \sum_{l=1}^{k} a_l \neq 0,$$

then the point $s = 1$ is a simple pole of $\zeta(s;\mathfrak{a})$ with residue a,while if $a \neq 0$, then $\zeta(s;\mathfrak{a})$ is an entire function.

If $\mathfrak{a}$ is a multiplicative sequence, i.e., $a_1 = 1$ and $a_{mn} = a_m a_n$ for all coprimes $m, n \in \mathbb{N}$, then an analogue of Theorem 3.2 is true [A. Laurinčikas, D. Šiaučiūnas (2006)] for the function $\zeta(\alpha, \alpha)$. In the general case, the following result is known [J. Kaczorowski (2009)].

Theorem 5.2. *For every non-zero periodic sequence* $\mathfrak{a}$ *of complex numbers with period* k*, there exists a positive constant* $c_0 = c_0(\mathfrak{a})$ *such that, for every compact subset* $K \subset D$ *with connected complement,*

$$\max_{s\in K} \mathrm{Im} s - \min_{s \in K} \mathrm{Im} s \leq c_0,$$

every continuous non-vanishing function $f(s)$ *on* K *which is analytic in the interior of* K*, and every* $\varepsilon > 0$*,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\mathfrak{a}) - f(s)| < \varepsilon \right\} > 0.$$

Now let $\mathfrak{b} = \{b_m : m \in \mathbb{N}_0\}$ be another periodic with a minimal period $l \in \mathbb{N}$ sequence of complex numbers, and α, $0 < \alpha \leq 1$, be a fixed parameter. The periodic Hurwitz zeta-function $\zeta(s,\alpha,\mathfrak{b})$ is defined, for $\sigma > 1$, by

$$\zeta(s,\alpha;\mathfrak{b}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}.$$

In view of periodicity of $\mathfrak{b}$,for $\sigma > 1$,

$$\zeta(s,\alpha;\mathfrak{b}) = \frac{1}{l^s} \sum_{k=0}^{l-1} a_k \zeta\left(s, \frac{k+l}{l}\right),$$

and this equality gives analytic continuation for $\zeta(s,\alpha;\mathfrak{b})$ to the whole complex plane. The function $\zeta(s,\alpha;\mathfrak{b})$ is entire, if

$$b=\frac{1}{l}\sum_{k=0}^{l-1}a_k=0,$$

and has a simple pole with residue b at $s=1$ if $b\neq 0$.

If α is transcendental, then an analogue of Theorem 5.1 is true [A. Javtokas, A. Laurinčikas (2006)] for the function $\zeta(s,\alpha;\mathfrak{b})$.

We will present one more example of universal zeta-functions with the Euler product. Let $SL(2,\mathbb{Z})$ denote the full modular group, i.e.,

$$SL(2,\mathbb{Z})=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}: a,b,c,d\in\mathbb{Z}, ad-bc=1\right\}.$$

Suppose that the function $F(z)$ is analytic in the upper half-plane $\mathrm{Im}z>0$ and, for all $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{Z})$, the functional equation

$$F\left(\frac{az+b}{cz+d}\right)=(cz+d)^{\kappa}F(z)$$

with a certain $\kappa\in 2\mathbb{N}$ is satisfied. Then $F(s)$ has the Fourier series expansion

$$F(z)=\sum_{m=-\infty}^{\infty}c_m\mathrm{e}^{2\pi imz}.$$

In the case when $c_m=0$ for $m\leq 0$, the function is called a cusp form of weight κ. Additionally, suppose that $F(z)$ is an eigenform of all Hecke operators

$$(T_nf)(z)=n^{\kappa-1}\sum_{d|n}d^{-\kappa}\sum_{b=0}^{d-1}f\left(\frac{nz+bd}{d^2}\right).$$

Then it is proved that $c_m\neq 0$, and, after normalization, we have that

$$F(z)=\sum_{m=1}^{\infty}c_m\mathrm{e}^{2\pi imz}\quad\text{with } c_1=1. \tag{5.1}$$

To the cusp form (5.1), we attach the zeta-function

$$\varphi(s,F)=\sum_{m=1}^{\infty}\frac{c_m}{m^s},\quad \sigma>\frac{\kappa+1}{2}.$$

Since the coefficients c_m are multiplicative, $\varphi(s,F)$ has the Euler product representation

$$\varphi(s,F)=\prod_p\left(1-\frac{\alpha(p)}{p^s}\right)^{-1}\left(1-\frac{\beta(p)}{p^s}\right)^{-1},\quad \sigma>\frac{\kappa+1}{2}.$$

Here $\alpha(p)$ and $\beta(p)$ are complex numbers, $\beta(p)=\overline{\alpha}(p)$, $\alpha(p)\beta(p)=1$ and $\alpha(p)+\beta(p)=c(p)$. Moreover, the function $\varphi(s,F)$ is analytically continued to an entire function. The universality of $\varphi(s,F)$ has been proved in [A. Laurinčikas, K. Matsumoto (2001)].

Theorem 5.3 [A. Laurinčikas, K. Matsumoto (2001)]. *Let K be a compact subset of the strip $\{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\}$ with connected complement, and $f(s)$ be a continuous non-vanishing on K function which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{s\in K} |\varphi(s+i\tau, F) - f(s)| < \varepsilon \right\} > 0.$$

6. Joint universality

A more complicated and interesting problem is a simultaneous approximation of a collection of analytic functions by shifts of zeta-functions. The first result in this direction also belongs to S.M. Voronin: in [S.M. Voronin (1975)], he obtained the joint universality for Dirichlet L-functions. For the definition of Dirichlet L-functions, a notion of a Dirichlet character is needed. A full definition is rather complicated, therefore, we only observe that every arithmetical function $g(m) \neq 0$ satisfies the following conditions:

1° $g(m)$ is a completely multiplicative function ($g(mn) = g(m)g(n)$) for all $m, n \in \mathbb{N}$;
2° $g(m)$ is periodic with period k;
3° $g(m) = 0$ if $(m,k) > 1$, and $g(m) \neq 0$ if $(m,k) = 1$ coincides with one of the Dirichlet characters modulo k.

Let χ be a Dirichlet character. Then the corresponding Dirichlet L-function $L(s,\chi)$ is defined, for $\sigma > 1$, by

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

A character

$$\chi_0(m) = \begin{cases} 1, & \text{if } (m,k) = 1, \\ 0, & \text{if } (m,k) > 0, \end{cases}$$

is called the principal character modulo k. It is not difficult to see that, for $\sigma > 1$,

$$L(s,\chi_0) = \zeta(s) \prod_{p|k} \left(1 - \frac{1}{p^s}\right),$$

thus $L(s,\chi_0)$ has a simple pole at $s = 1$ with residue $\prod_{p|k} \left(1 - \frac{1}{p^s}\right)$. If $\chi \neq \chi_0$, then the function $L(s,\chi)$ is entire.

Let $l, k \in \mathbb{N}$, $(l,k) = 1$. Define

$$\pi(x,k,l) = \sum_{\substack{p \le x \\ p \equiv l (\text{mod} k)}} 1.$$

Dirichlet L-functions are applied for the investigation of prime numbers in arithmetical progressions, i.e., for the asymptotics of the function $\pi(x,k,l)$ as $x \to \infty$. It has been proved that

$$\pi(x,k,l) \sim \frac{x}{\varphi(k)\log x}, \quad x \to \infty,$$

where $\varphi(k)$ is the Euler function: $\varphi(k) = \sharp\{1 \le m \le k : (m,k) = 1\}$.

Let $\chi_1(\mathrm{mod}k_1)$ and $\chi_2(\mathrm{mod}k_2)$ be two Dirichlet characters, and $k = [k_1, k_2]$ denote the least common multiple. The characters χ_1 and χ_2 are called equivalent if, for $(m,k) = 1$,

$$\chi_1(m) = \chi_2(m).$$

Each Dirichlet L-function is also universal in the Voronin sense. Moreover, the first example of the joint universality is related to Dirichlet L-functions.

Theorem 6.1 [S.M. Voronin (1975)]. *Suppose that $\chi_1, \dots, \chi_n$ are pairwise non-equivalent Dirichlet characters, and $L(s,\chi_1), \dots, L(s,\chi_n)$ are the corresponding Dirichlet L-functions. Let $K_1, \dots, K_n$ be compact subsets of the strip $\{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ with connected complements, and let functions $f_1(s), \dots, f_n(s)$ be continuous non-vanishing on $K_1, \dots, K_n$ and analytic in the interior of K_1, ..., K_n, respectively. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0,T] : \sup_{1<j\le n} \sup_{s\in K_j} |L(s+i\tau, \chi_j) - f(s)| < \varepsilon \right\} > 0.$$

Now let $\alpha_j \in \mathbb{R}$, $0 < \alpha_j \le 1$, $j = 1, \dots, r$, and

$$L(\alpha_1, \dots, \alpha_r) = \{\log(m + x_j) : m \in \mathbb{N}_0,\ j = 1, \dots, r\}.$$

A joint universality theorem for Hurwitz zeta-functions is of the form.

Theorem 6.2. *Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over $\mathbb{Q}$. For $j = 1, \dots, r$, let K_j be a compact subset of the strip D with connected complement, and let $f_j(s)$ be a continuous function on K_j which is analytic in the interior of K_j. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0,T] : \sup_{1<j\le r} \sup_{s\in K_j} |\zeta(s+i\tau, \alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Note that, differently from Theorem 6.1, the approximated analytic functions in Theorem 6.2 can have zeros on the set K_j.

There exist other results on joint universality, however, some of them are conditional. We present a recent theorem on joint universality of zeta-functions with periodic coefficients.

Let $\mathfrak{a}_j = \{a_{jm} : m \in \mathbb{N}\}$ be a periodic sequence of complex numbers with minimal period $k_j \in \mathbb{N}$, and $\zeta(s; \mathfrak{a}_j)$ denote the corresponding periodic zeta-function, $j = 1, \dots, r_1$, $r_1 > 1$. Let $\mathfrak{b}_j = \{b_{jm} : m \in \mathbb{N}_0\}$ be another periodic sequence of complex numbers with minimal period $l_j \in \mathbb{N}$, $0 < \alpha_j \le 1$, and $\zeta(s, \alpha_j; \mathfrak{b}_j)$ be the corresponding periodic Hurwitz zeta-function, $j = 1, \dots, r_2$, $r_2 > 1$.

Denote by $k = [k_1, \ldots, k_2]$ the least common multiple of the periods $k_1, \ldots, k_r$, and let $\eta_1, \ldots, \eta_{\varphi(k)}$ be the reduced residue system modulo k. Define the matrix

$$A = \begin{pmatrix} a_{1\eta_1} & a_{2\eta_1} & \ldots & a_{r_1\eta_1} \\ a_{1\eta_2} & a_{2\eta_2} & \ldots & a_{r_1\eta_2} \\ \ldots & \ldots & \ldots & \ldots. \\ a_{1\eta_{\varphi(k)}} & a_{2\eta_{\varphi(k)}} & \ldots & a_{r_1\eta_{\varphi(k)}} \end{pmatrix}.$$

Theorem 6.3 [A. Laurinčikas (2010)]. *Suppose that $\mathfrak{a}_1, \ldots, \mathfrak{a}_{r_1}$ are multiplicative,* $\operatorname{rank}(A) = r_1$, *and the numbers $\alpha_1, \ldots, \alpha_{r_2}$ are algebraically independent over $\mathbb{Q}$. Let $K_1, \ldots, K_{r_1}$ be compact subsets of the strip $D = \{s \in \mathbb{C} : 1/2 < \sigma < 1\}$ with connected complements, and let the functions $f_1(s), \ldots, f_r(s)$ be continuous non-vanishing on $K_1, \ldots, K_{r_1}$ and analytic in the interior of $K_1, \ldots, K_{r_1}$, respectively. Let $\hat{K}_1, \ldots, \hat{K}_{r_1}$ also be compact subsets of D with connected complements, and let the functions $\hat{f}_1(s), \ldots, \hat{f}_{r_1}(s)$ be continuous on $\hat{K}_1, \ldots, \hat{K}_{r_2}$ and analytic in the interior of $\hat{K}_1, \ldots, \hat{K}_{r_2}$, respectively. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\Big\{\tau \in [0,T] : \sup_{1<j\le r_1} \sup_{s\in K_j} |\zeta(s+i\tau, \mathfrak{a}_j) - f_j(s)| < \varepsilon,$$
$$\sup_{1<j\le r_2} \sup_{s\in \hat{K}_j} |\zeta(s+i\tau, \alpha_j; \mathfrak{b}_j) - \hat{f}_j(s)| < \varepsilon\Big\} > 0.$$

7. Proof of universality theorems

The original proof of the Voronin universality theorem is based on an analogue of the Riemann theorem on rearrangement of terms of series in Hilbert spaces. However, a more convenient and universal approach uses probabilistic limit theorems in the sense of weak convergence of probability measures in the space of analytic functions.

Let S be a metric space, and let $\mathcal{B}(S)$ denote the class of Borel sets of the space S, i.e., the σ-field generated by open sets of S. Let P_n, $n \in \mathbb{N}$, and P be probability measures on $(S, \mathcal{B}(S))$. We recall that P_n converges weakly to P as $n \to \infty$ if

$$\lim_{n\to\infty} \int_S f \mathrm{d}P_n = \int_S f \mathrm{d}P$$

for every real continuous bounded function f on S.

Denote by $H(D)$ the space of analytic functions on D equipped with the topology of uniform convergence on compacta. On $(H(D), \mathcal{B}(H(D)))$, define the probability measure

$$P_T(A) = \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \zeta(s+i\tau) \in A\right\}.$$

To state a limit theorem for the measure P_T, we need some notation. Let

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \{s \in \mathbb{C} : |s| = 1\}$ for each prime p. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and this gives the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p, and on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$, define the $H(D)$-valued random element $\zeta(s,\omega)$ by the formula

$$\zeta(s,\omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

Note that the latter product converges uniformly on compact subset of the half-plane $\sigma > \frac{1}{2}$ for almost all $\omega \in \Omega$.Denote by P_ζ the distribution of the random element $\zeta(s,\omega)$, i.e.,

$$P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s,\omega) \in A), \quad A \in \mathcal{B}(H(D)).$$

Theorem 7.1. *The probability measure P_T converges weakly to P_ζ as $T \to \infty$.*

The next ingredient of the proof of universality for $\zeta(s)$ is the support of the measure P_ζ.We recall that the support of P_ζ is a minimal closed set S_ζ such that $P_\zeta(S_\zeta) = 1$. The support S_ζ consists of elements $x \in H(D)$ such that, for every neighbourhood G of x, the inequality $P_\zeta(G) > 0$ is satisfied.

Theorem 7.2. *The support of the measure P_ζ is the set*

$$\{g \in H(D) : g(s) \neq 0 \ \textit{or} \ g(s) \equiv 0\}.$$

Proof of Theorem 3.2. First suppose that the function $f(s)$ has a non-vanishing analytic continuation to the strip D. Then, by Theorem 7.2, $f(s) \in S_\zeta$, therefore, defining an open set G by

$$G = \{g \in H(D) : \sup_{s \in K} |g(s) - f(s)| < \varepsilon\},$$

we obtain that

$$\liminf_{T \to \infty} \frac{1}{T} \text{meas} \left\{ \tau \in [0,T] : \sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right\} \geq P_\zeta(G) > 0.$$

Now let $f(s)$ be as in Theorem 3.2. Then, by Theorem 1.1, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} |f(s) - p(s)| < \frac{\varepsilon}{4}.$$

We have that $p(s) \neq 0$ on K. Therefore, we can choose a continuous branch of $\log p(s)$ which is analytic in some region containing K. By Theorem 1.1 again, we can find a polynomial $q(s)$ such that

$$\sup_{s\in K} |p(s) - \mathrm{e}^{q(s)}| < \frac{\varepsilon}{4}. \tag{7.1}$$

This and (7.1) show that

$$\sup_{s\in K} |f(s) - \mathrm{e}^{q(s)}| < \frac{\varepsilon}{2}. \tag{7.2}$$

However, $e^{q(s)}$ is a non-vanishing analytic function on D. Thus, by the first part of the proof

$$\liminf_{T\to\infty} \frac{1}{T}\mathrm{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau) - \mathrm{e}^{q(s)}| < \frac{\varepsilon}{2}\right\} > 0.$$

In view of (7.2),

$$\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau)-f(s)|<\varepsilon\right\}\supset\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau)-\mathrm{e}^{q(s)}|<\frac{\varepsilon}{2}\right\},$$

hence the theorem follows.

8. Some applications

Universality theorems for zeta- and L-functions have theoretical and practical applications. One of the theoretical applications is related to the functional independence of functions.

In 1887, O. Hölder proved [O. Hölder (1887)] that the Euler gamma-function $\Gamma(s)$ does not satisfy any algebraic-differential equation, i.e., there are no polynomials $P \not\equiv$ such that

$$P(\Gamma(s), \Gamma'(s), \ldots, \Gamma^{(n-1)}) \equiv 0.$$

In 1900, Hilbert observed that the algebraic-differential independence of the Riemann zeta-function can be proved by using the above Hölder's result and the functional equation for $\zeta(s)$. S.M. Voronin, using a universality theorem, obtained [S.M. Voronin (1973)] the functional independence of $\zeta(s)$.

Theorem 8.1. *Suppose that the functions* $F_j : \mathbb{C}^N \to \mathbb{C}$ *are continuous,* $j = 0, \ldots, r$, *and*

$$\sum_{j=0}^{r} s^j F_j(\zeta(s), \ldots, \zeta^{(N-1)}(s)) \equiv 0.$$

Then $F_j \equiv 0$ *for* $j = 0, \ldots, r$.

The functional independence also follows for other zeta- and L-functions that are universal in the above sense.

The universality also can be used for approximate computations with analytic functions. Usually, zeta-functions satisfy approximate functional equations.

For example, for the function $\zeta(s)$, the following equation is true [A. Ivič(1985)]. Suppose that $0 \le \sigma \le 1$, $x, y, t \ge c > 0$ and $2\pi xy = t$. Then uniformly in σ,

$$\zeta(s) = \sum_{m \le x} \frac{1}{m^s} + \chi(s) \sum_{m \le y} \frac{1}{m^{1-s}} + \mathrm{O}(x^{-\sigma}) + \mathrm{O}\left(t^{1/2-\sigma} y^{\sigma-1}\right),$$

where

$$\chi(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin \frac{\pi s}{2}.$$

Therefore, first we can evaluate $\zeta(s + i\tau)$, and then, using Theorem 3.2, we can obtain the desired information on a given analytic function $f(s)$.

An application of universality in physics is given in [K.M. Bitar, N.N. Khuri, H.C. Ren (1991)].

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