Chapter 2

Euclidean and Hyperbolic Geometry

X designates again an arbitrary real inner product space containing two linearly independent elements. As throughout the whole book, we do not exclude the case that there exists an infinite and linearly independent subset of X.

A natural and satisfactory definition of hyperbolic geometry over X was already given by Theorem 7 of chapter 1. If T is a separable translation group of X, and d an appropriate distance function of X invariant under T and $O(X)$, then there are, up to isomorphism, exactly two geometries

$$\Big(X, G\left(T, O\left(X\right)\right)\Big).$$

These geometries are called euclidean, hyperbolic geometry over X. Their distance functions are eucl (x, y), hyp (x, y), respectively.

2.1 Metric spaces

A set $S \neq \emptyset$ together with a mapping $d : S \times S \to \mathbb{R}$ is called a *metric space* (S, d) provided

(i) $d(x, y) = 0$ if, and only if, $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$

hold true for all $x, y, z \in S$.

Observe $d(x, y) \geq 0$ for all $x, y \in S$, since (i), (ii), (iii) imply

$$0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y).$$

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(i) is called the *axiom of coincidence*, (ii) the *symmetry axiom* and (iii) the *triangle inequality.*

Proposition 1. *$(X$, eucl), $(X$, hyp) are metric spaces, called the euclidean, hyperbolic metric space, respectively, over X.*

Proof. Axioms (i), (ii) hold true for both structures $(X$, eucl), $(X$, hyp), because of D.c and D.d of step D of the proof of Theorem 7. The triangle inequality of section 1.1 implies

$$\|(x-z)+(z-y)\| \leq \|x-z\|+\|z-y\|$$

for $x, y, z \in X$, i.e. eucl $(x, y) \leq$ eucl $(x, z)+$ eucl (z, y). It remains to prove (iii) for $(X$, hyp). We may assume $x \neq z$. Because of D.a and the invariance of hyp under T and $O\,(X)$, it is sufficient to show (iii) for $z=0$ and $x=\lambda e$ with $\lambda>0$, i.e. to prove

$$L:=\text{hyp }(\lambda e, y) \leq \text{hyp }(\lambda e, 0)+\text{hyp }(0, y)=: R$$

or, equivalently, $\cosh L \leq \cosh R$. Obviously, this latter inequality can be written as

$$\sqrt{1+\lambda^2}\,\sqrt{1+y^2}-(\lambda e)\,y \leq \sqrt{1+\lambda^2}\,\sqrt{1+y^2}+\sqrt{\lambda^2}\,\sqrt{y^2}.$$

So observe finally $-(\lambda e)\,y \leq|(\lambda e)\,y| \leq \sqrt{(\lambda e)^2}\,\sqrt{y^2}$ from the inequality of Cauchy–Schwarz. □

2.2 The lines of L.M. Blumenthal

Let (S, d) be a metric space and $x: \mathbb{R} \rightarrow S$ a function satisfying

$$d\left(x\,(\xi),\, x\,(\eta)\right)=|\xi-\eta| \tag{2.1}$$

for all real ξ, η. Then $\{x\,(\xi) \mid \xi \in \mathbb{R}\}$ is called a (Blumenthal) *line* of (S, d) (L.M. Blumenthal, K. Menger [1], p. 238). Observe that $x: \mathbb{R} \rightarrow S$ must be injective in view of the axiom of coincidence and (2.1).

Lemma 2. *If $\|x+y\|=\|x\|+\|y\|$ holds true for $x, y \in X$, then x, y are linearly dependent.*

Proof. Squaring both sides, we obtain $xy=\|x\|\,\|y\|$. Now apply Lemma 1 of chapter 1. □

We would like to determine all solutions $x: \mathbb{R} \rightarrow X$ of the functional equation (2.1) in the case of $(X$ eucl). Let x be a solution. If $\alpha<\beta<\gamma$ are reals, then, by (2.1),

$$\|x\,(\gamma)-x\,(\alpha)\|=\gamma-\alpha,\ \|x\,(\gamma)-x\,(\beta)\|=\gamma-\beta,\ \|x\,(\beta)-x\,(\alpha)\|=\beta-\alpha,$$

i.e., by Lemma 2, $x(\gamma) - x(\beta),\ x(\beta) - x(\alpha)$ must be linearly dependent. Put

$$p := x(0),\ q := x(1) - x(0).$$

Hence, by (2.1), $\|q\| = 1$. If $0 < 1 < \xi$, we obtain

$$x(\xi) - x(1) = \varrho \cdot \big(x(1) - x(0)\big) = \varrho q$$

for a suitable $\varrho \in \mathbb{R}$. Thus $\xi - 1 = \|x(\xi) - x(1)\| = \|\varrho q\| = |\varrho|$. Moreover,

$$\xi - 0 = \|x(\xi) - p\| = \|x(1) + \varrho q - p\| = |1 + \varrho|.$$

Hence $\varrho = \xi - 1$ and thus $x(\xi) = x(1) + \varrho q = p + \xi q$ for $\xi > 1$, a formula which holds also true for $\xi = 1$, $\xi = 0$, but also in the cases $0 < \xi < 1$, $\xi < 0 < 1$ as similar arguments show. That, on the other hand,

$$x(\xi) := p + \xi q,\ \|q\| = 1,$$

solves (2.1), is obvious. Hence

$$\{(1-\lambda)\,a + \lambda b \mid \lambda \in \mathbb{R}\}$$

with $a, b \in X$ and $a \neq b$ are the *euclidean* lines of $(X,\ \text{eucl})$ by writing $p := a,\ q \cdot \|b - a\| := b - a,\ \xi := \lambda \cdot \|b - a\|$.

Theorem 3. *The (hyperbolic) lines of* (X, hyp) *are given by all sets*

$$\{p \cosh \xi + q \sinh \xi \mid \xi \in \mathbb{R}\},$$

where p, q *are elements of* X *with* $pq = 0$ *and* $q^2 = 1$.

Proof. Let p, q be elements of X satisfying $pq = 0$ and $q^2 = 1$. Define $x : \mathbb{R} \to X$ by

$$x(\xi) = p \cosh \xi + q \sinh \xi \tag{2.2}$$

and observe $\text{hyp}\ \big(x(\xi), x(\eta)\big) = |\xi - \eta|$ for all $\xi, \eta \in \mathbb{R}$. Hence (2.2) is the equation of a line of $(X,\ \text{hyp})$. Suppose now that $x : \mathbb{R} \to X$ solves (2.1) in the case of $(X,\ \text{hyp})$. Since x is injective, choose a real ξ_0 with $x(\xi_0) \neq 0$ and put

$$e := \frac{x(\xi_0)}{\sinh t_0},\ \sinh t_0 := \|x(\xi_0)\|.$$

Define the translation group

$$T_t(h + \sinh \tau \cdot \sqrt{1 + h^2}\, e) = h + \sinh(\tau + t) \cdot \sqrt{1 + h^2}\, e$$

for all $h \in e^{\perp}$ and $\tau, t \in \mathbb{R}$. Since

$$\text{hyp}\ \big(T_t(y), T_t(z)\big) = \text{hyp}\ (y, z) \tag{2.3}$$

holds true for all $y, z \in X$,

$$\xi \to \overline{x}\,(\xi) := T_{-t_0}\big(x\,(\xi + \xi_0)\big)$$

must be a solution of (2.1) as well: by (2.3),

$$\text{hyp }\big(\overline{x}\,(\xi),\, \overline{x}\,(\eta)\big) = \text{hyp }\big(x\,(\xi + \xi_0),\, x\,(\eta + \xi_0)\big) = |\xi - \eta|. \tag{2.4}$$

Notice $T_{t_0}(0) = x\,(\xi_0)$, i.e. $T_{-t_0}\big(x\,(\xi_0)\big) = 0$, i.e. $\overline{x}\,(0) = 0$. By (2.4),

$$\cosh(\xi - \eta) = \sqrt{1 + \overline{x}^2(\xi)}\,\sqrt{1 + \overline{x}^2(\eta)} - \overline{x}\,(\xi)\,\overline{x}\,(\eta).$$

For $\eta = 0$ we get $\cosh\xi = \sqrt{1 + \overline{x}^2(\xi)}$. Thus $\overline{x}^2(\xi) = \sinh^2\xi$ and

$$\overline{x}\,(\xi)\,\overline{x}\,(\eta) = \cosh\xi\cosh\eta - \cosh\,(\xi - \eta) = \sinh\xi\sinh\eta, \tag{2.5}$$

i.e. $[\overline{x}\,(\xi)\,\overline{x}\,(\eta)]^2 = \sinh^2\xi\sinh^2\eta = \overline{x}^2(\xi)\,\overline{x}^2(\eta)$ for all real ξ, η. Hence, by Lemma 1, chapter 1, $\overline{x}\,(\xi), \overline{x}\,(\eta)$ must be linearly dependent. Since $\overline{x}$ is injective and $\overline{x}\,(0) = 0$, we obtain $\overline{x}\,(1) \neq 0$. Put $a \cdot \|\overline{x}\,(1)\| := \overline{x}\,(1)$. Thus

$$\overline{x}\,(\xi) = \varphi\,(\xi) \cdot a \tag{2.6}$$

with a suitable function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $\varphi\,(0) = 0$, in view of the fact that $\overline{x}\,(\xi)$, $\overline{x}\,(1)$ are linearly dependent. (2.5), (2.6) imply

$$\varphi\,(\xi)\,\varphi\,(1) = \sinh\xi\sinh 1$$

for all real ξ, so especially $\varphi^2(1) = \sinh^2 1$, i.e.

$$\begin{aligned} \overline{x}\,(\xi) &= \sinh\xi \cdot \tfrac{\sinh 1}{\varphi\,(1)}\,a \\ &= p \cdot \cosh\xi + q \cdot \sinh\xi \end{aligned}$$

with $p := 0,\ \varphi\,(1)\,q := a\sinh 1$, i.e. $pq = 0$ and $q^2 = 1$. Hence $\overline{x}\,(\xi)$ is of type (2.2), and we finally must show that

$$x\,(\xi) = T_{t_0}\big(\overline{x}\,(\xi - \xi_0)\big) = T_{t_0}\big(q\sinh(\xi - \xi_0)\big)$$

is of type (2.2) as well. This turns out to be a consequence of the following Lemma 4. □

Lemma 4. *Let T be the translation group*

$$T_t(h + \sinh\tau \cdot \sqrt{1 + h^2}\,e) = h + \sinh(\tau + t) \cdot \sqrt{1 + h^2}\,e$$

with axis e, $e^2 = 1$, for all $h \in e^\perp$ and τ, $t \in \mathbb{R}$. If $q \neq 0$ is in X and s in $\mathbb{R}$, there exist $a, b \in X$ with $ab = 0$, $b^2 = 1$ and

$$\{a\cosh\eta + b\sinh\eta \mid \eta \in \mathbb{R}\} = \{T_s(\mu q) \mid \mu \in \mathbb{R}\}.$$

Proof. There is nothing to prove for $q \in \mathbb{R}e$ or $s = 0$. So assume $s \neq 0$, and that q, e are linearly independent. Hence $q \neq (qe)\, e$. Because of

$$\{T_s(\mu q) \mid \mu \in \mathbb{R}\} = \{T_s(\mu \cdot \beta q) \mid \mu \in \mathbb{R}\}$$

for a fixed real $\beta \neq 0$, we may assume $\|q - (qe)\, e\| = 1$, without loss of generality. Put

$$S := \sinh s,\ C := \cosh s,\ j := q - (qe)\, e,\ \alpha := qe,$$

and observe $S \neq 0$, $C > 1$, $j^2 = 1$, $je = 0$, $q = \alpha e + j$, $q^2 = 1 + \alpha^2$. Since (1.8), (1.9) represent the same T_t, we obtain

$$T_s(\mu q) = \mu q + [\mu\alpha\, (C-1) + \sqrt{1 + \mu^2 q^2}\, S]\, e =: x_1 e + x_2 j$$

with $x_1(\mu) = \mu\alpha C + \sqrt{1 + \mu^2(1+\alpha^2)}\, S$ and $x_2(\mu) = \mu$. We hence get

$$x_1^2 - 2\alpha C x_1 x_2 + (\alpha^2 - S^2)\, x_2^2 = S^2$$

with the branch sgn $(x_1 - \mu\alpha C) =$ sgn S, and also

$$\frac{y_1^2}{k} - y_2^2 = 1,\ \text{sgn}\ y_1 = \text{sgn}\ S,$$

$q^2 k := S^2 > 0$, by applying the orthogonal mapping ω of the subspace Σ of X, spanned by e, j, namely

$$\begin{aligned} \delta y_1 &= x_1 C - x_2 \alpha, \\ \delta y_2 &= x_1 \alpha + x_2 C, \end{aligned}$$

$\delta := \sqrt{\alpha^2 + C^2}$. In order to find the interesting branch sgn $y_1 =$ sgn S of the hyperbola

$$\left\{ y_1 e + y_2 j \in \Sigma \;\middle|\; \frac{y_1^2}{k} - y_2^2 = 1 \right\},$$

observe $x_1(\mu) - \mu\alpha C = \sqrt{1 + \mu^2(1+\alpha^2)}\, S,\ x_2(\mu) = \mu$, and hence

$$\begin{aligned} \delta y_1 &= x_1 C - x_2 \alpha = C\,(x_1 - x_2 \alpha C) + x_2 \alpha S^2 \\ &= \big(C\sqrt{1 + \mu^2(1+\alpha^2)} + x_2 \alpha S\big)\, S, \end{aligned}$$

i.e. sgn $y_1 =$ sgn S, if the coefficient of S is positive. But

$$0 < C^2(1 + x_2^2) + x_2^2 \alpha^2 (C^2 - S^2) = C^2(1 + x_2^2) + x_2^2 \alpha^2,$$

i.e. $x_2^2 \alpha^2 S^2 < C^2\big(1 + x_2^2(1+\alpha^2)\big)$, i.e.

$$-x_2 \alpha S \leq |x_2 \alpha S| < C\sqrt{1 + x_2^2(1+\alpha^2)}.$$

Obviously, $l := \{T_s(\mu q) \mid \mu \in \mathbb{R}\} \subset \Sigma$. Hence

$$\omega\,(l) = \{\text{sgn } S \cdot \sqrt{k}\, e \cosh \eta + j \sinh \eta \mid \eta \in \mathbb{R}\},$$

i.e.

$$l = \{\text{sgn } S \cdot \sqrt{k}\, \omega^{-1}(e) \cosh \eta + \omega^{-1}(j) \sinh \eta \mid \eta \in \mathbb{R}\}$$

with

$$\begin{aligned} \delta\omega^{-1}(e) &= Ce - \alpha j, \\ \delta\omega^{-1}(j) &= \alpha e + Cj. \end{aligned}$$

So the line l is given by

$$\{a \cosh \eta + b \sinh \eta \mid \eta \in \mathbb{R}\}$$

with $a := \text{sgn } S \cdot \sqrt{k} \cdot \omega^{-1}(e)$, $b := \omega^{-1}(j)$. Notice $ab = 0$, in view of $\omega^{-1}(e)\omega^{-1}(j) = ej$, and $b^2 = 1$. $\square$

That images of lines under motions are lines follows immediately from the definition of lines. In fact! If $l = \{x\,(\xi) \mid \xi \in \mathbb{R}\}$ is a line and $f : X \to X$ a motion, then, by (2.1),

$$d\Big(f\,\big(x\,(\xi)\big),\, f\,\big(x\,(\eta)\big)\Big) = d\,\big(x\,(\xi),\, x\,(\eta)\big) = |\xi - \eta|$$

for all $\xi, \eta \in \mathbb{R}$. This holds true in euclidean as well as in hyperbolic geometry. In both geometries also holds true the

Proposition 5. *If $a \neq b$ are elements of X, there is exactly one line l through a, b, i.e. with $l \ni a, b$.*

Proof. From D.a (section 1.3) we know that there exists a motion f such that $f\,(a) = 0$ and $f\,(b) = \lambda e$, $\lambda > 0, e$ a fixed element of X with $e^2 = 1$. In the euclidean case there is exactly one line

$$\{(1 - \alpha)\, p + \alpha q \mid \alpha \in \mathbb{R}\},$$

$p \neq q$, through $0, \lambda e$, namely $\{\beta e \mid \beta \in \mathbb{R}\}$. There hence is exactly one line, namely $f^{-1}(\mathbb{R}e)$ through a, b. In the hyperbolic case there is also exactly one line

$$\{v \cosh \xi + w \sinh \xi \mid \xi \in \mathbb{R}\},\ vw = 0,\ w^2 = 1,$$

through $0, \lambda e$, namely $\mathbb{R}e$. This implies that $f^{-1}(\mathbb{R}e)$ is the uniquely determined line through a, b. $\square$

2.3 The lines of Karl Menger

Let (S, d) be a metric space. If $a \neq b$ are elements of S, then

$$[a, b] := \{x \in S \mid d(a, x) + d(x, b) = d(a, b)\}$$

is called the *interval* (the Menger interval) $[a, b]$ (Menger [1], [2]). Observe $a, b \in [a, b] = [b, a]$. Moreover,

$$l(a, b) := \big\{z \in S \backslash \{b\} \mid a \in [z, b]\big\} \cup [a, b] \cup \big\{z \in S \backslash \{a\} \mid b \in [a, z]\big\}$$

is called a (Menger) *line* of (S, d).

In the euclidean case $(X,$ eucl$)$, the interval $[a, b]$ consists of all $x \in X$ with

$$\|(a - x) + (x - b)\| = \|a - b\| = \|a - x\| + \|x - b\|. \tag{2.7}$$

Hence, by Lemma 2, the elements $a - x$ and $x - b$ are linearly dependent. If $x \notin \{a, b\}$, then $x - b = \lambda(a - x)$ with a suitable real $\lambda \notin \{0, -1\}$, i.e.

$$x = \frac{\lambda}{1 + \lambda} a + \frac{1}{1 + \lambda} b = a + \frac{b - a}{1 + \lambda}.$$

For $\lambda > 0$ equation (2.7) holds true, but not for $\lambda \in]-1, 0[$ or $\lambda < -1$. Hence

$$[a, b] = \{a + \mu(b - a) \mid 0 \leq \mu \leq 1\},$$

and $l(a, b) = \{a + \mu(b - a) \mid \mu \in \mathbb{R}\}$. In the case $(X,$ eucl$)$ the Menger lines are thus exactly the previous lines. The same holds true for $(X,$ hyp$)$ as will be proved in Theorem 6.

If $a \neq b$ are elements of X and if

$$\{p \cosh \xi + q \sinh \xi \mid \xi \in \mathbb{R}\}, \tag{2.8}$$

$pq = 0$, $q^2 = 1$, is the hyperbolic line through a, b, then

$$a = p \cosh \alpha + q \sinh \alpha,$$
$$b = p \cosh \beta + q \sinh \beta$$

with uniquely determined reals α, β. If $\beta < \alpha$ we will replace ξ in (2.8) by $\xi' = -\xi$ and q by $q' = -q$. So without loss of generality we may assume $\alpha < \beta$.

Theorem 6. *Let $x(\xi) = p \cosh \xi + q \sinh \xi$ be the equation of the line through $a \neq b$ with $a = x(\alpha)$, $b = x(\beta)$, $\alpha < \beta$. Then*

$$[a, b] = \{x(\xi) \mid \alpha \leq \xi \leq \beta\} \tag{2.9}$$

and $l(a, b) = \{x(\xi) \mid \xi \in \mathbb{R}\}$.

Proof. The right-hand side of (2.9) is a subset of $[a, b]$. This follows from $\alpha \leq \xi \leq \beta$ and

$$\begin{aligned} \text{hyp }\big(x(\alpha),\, x(\beta)\big) &= |\alpha - \beta| = \beta - \alpha, \\ \text{hyp }\big(x(\alpha),\, x(\xi)\big) &= \xi - \alpha, \\ \text{hyp }\big(x(\xi),\, x(\beta)\big) &= \beta - \xi. \end{aligned}$$

Let now z be an element of X with $z \in [a, b]$, i.e. with

$$\beta - \alpha = \text{hyp }\big(x(\alpha)\, x(\beta)\big) = \text{hyp }\big(x(\alpha),\, z\big) + \text{hyp }\big(z, x(\beta)\big).$$

Define $\xi := \alpha + \text{hyp }\big(x(\alpha), z\big)$. Obviously, $\xi - \alpha \geq 0$, and

$$\beta - \xi = \text{hyp }\big(z, x(\beta)\big) \geq 0,$$

i.e. $\alpha \leq \xi \leq \beta$. Hence $x(\xi)$ is an element of the right-hand side of (2.9). Observe

$$\text{hyp }\big(x(\alpha),\, z\big) = \xi - \alpha = \text{hyp }\big(x(\alpha),\, x(\xi)\big), \tag{2.10}$$

$$\text{hyp }\big(z, x(\beta)\big) = \beta - \xi = \text{hyp }\big(x(\xi),\, x(\beta)\big). \tag{2.11}$$

We take a motion f with $f(a) = 0$ and $f(b) = \lambda e$, $\lambda > 0$. Since $x(\xi)$ is on the line through a, b and

$$\text{hyp }(a, b) = \text{hyp }\big(a, x(\xi)\big) + \text{hyp }\big(x(\xi), b\big)$$

holds true, we obtain that $f\big(x(\xi)\big)$ is on the line $\mathbb{R}e$ through 0 and λe, and that

$$\text{hyp }(0, \lambda e) = \text{hyp }\Big(0, f\big(x(\xi)\big)\Big) + \text{hyp }\Big(f\big(x(\xi)\big), \lambda e\Big),$$

i.e. that $f(a) = e \sinh \eta_1$, $f\big(x(\xi)\big) = e \sinh \eta_2$, $f(b) = e \sinh \eta_3$ with $\eta_3 = |\eta_2| + |\eta_3 - \eta_2|$ and $\lambda = \sinh \eta_3$. Hence $0 = \eta_1 \leq \eta_2 \leq \eta_3$ and $f\big(x(\xi)\big) =: \mu e$ with $0 \leq \mu \leq \lambda$. If we take the images of $x(\alpha)$, $z, \dots$ in (2.10), (2.11), we get from these equations with $\overline{z} := f(z)$,

$$\begin{aligned} \sqrt{1 + \overline{z}^2} &= \sqrt{1 + \mu^2}, \\ \sqrt{1 + \overline{z}^2}\,\sqrt{1 + \lambda^2} - \overline{z}\lambda e &= \sqrt{1 + \mu^2}\,\sqrt{1 + \lambda^2} - \mu\lambda, \end{aligned}$$

i.e. $\overline{z}^2 = \mu^2$ and $\overline{z}e = \mu$. Thus $(\overline{z}e)^2 = \overline{z}^2 e^2$, i.e. $\overline{z} \in \mathbb{R}e$, by Lemma 1, chapter 1, i.e. $\overline{z} = \mu e$, by $\overline{z}e = \mu$. Hence $f(z) = \overline{z} = f\big(x(\xi)\big)$, i.e. $z = x(\xi) \in [a, b]$.

We finally must show that the Menger lines of (X, hyp) are the hyperbolic lines. If $l(a, b)$ is a Menger line, designate by g the hyperbolic line through a, b. If $z \in X \backslash \{b\}$ with $a \in [z, b]$, then the hyperbolic line through z, b must contain a since, by (2.9), intervals are subsets of hyperbolic lines. Hence, by Proposition 5, $z \in g$. Moreover, $z \in X \backslash \{a\}$ with $b \in [a, z]$ belongs also to g, i.e. $l(a, b) \subseteq g$. If $x(\xi) \in g$, we distinguish three cases $\xi < \alpha$, $\alpha \leq \xi \leq \beta$, $\beta < \xi$ with $a = x(\alpha)$, $b = x(\beta)$, $\alpha < \beta$. In the first case we get

$$x(\xi) \in X \backslash \{x(\beta)\} \text{ with } x(\alpha) \in [x(\xi),\, x(\beta)],$$

in the last $x(\xi) \in X \backslash \{x(\alpha)\}$ with $x(\beta) \in [x(\alpha),\, x(\xi)]$. $\square$

2.4 Another definition of lines

We proposed the following definition of a line, W. Benz [1, 6]. Suppose that (S, d) is a metric space and that $c \in S$ and $\varrho \geq 0$ is in $\mathbb{R}$. Then

$$B(c, \varrho) := \{x \in S \mid d(c, x) = \varrho\}$$

is defined to be the *ball* with *center* c and *radius* ϱ. Obviously, $B(c, 0) = \{c\}$. If a, b are distinct elements of S, we will call

$$g(a, b) := \bigl\{x \in S \mid B\bigl(a, d(a, x)\bigr) \cap B\bigl(b, d(b, x)\bigr) = \{x\}\bigr\}$$

a *g-line*. Notice $a, b \in g(a, b) = g(b, a)$.

Let S contain exactly three distinct elements a, b, c and define

$$d(a, b) = 3,\ d(a, c) = 4,\ d(b, c) = 5$$

and $d(x, x) = 0,\ d(x, y) = d(y, x)$ for all $x, y \in S$. Hence (S, d) is a metric space. Of course, (S, d) does not contain a line in the sense of L.M. Blumenthal. The Menger line $l(a, b)$ is given by

$$l(a, b) = \{a, b\},$$

and the g-line $g(a, b)$ by $\{a, b, c\}$.

Define $\Sigma = (X, \text{eucl})$ and $\Sigma' = (X, d)$ with

$$d(x, y) = \frac{\|x - y\|}{1 + \|x - y\|}$$

for all $x, y \in X$. The g-lines of the metric spaces Σ, Σ' coincide. Every Menger line of Σ' contains exactly two distinct elements. There do not exist lines of Σ' in the sense of L.M. Blumenthal, because

$$\frac{\|x(\xi) - x(\eta)\|}{1 + \|x(\xi) - x(\eta)\|} = |\xi - \eta|, \text{for all } \xi, \eta \in \mathbb{R},$$

cannot be true for $\xi = 1$ and $\eta = 0$.

Theorem 7. *Let Σ be one of the metric spaces $(X, \text{eucl}), (X, \text{hyp})$. Then $l(a, b) = g(a, b)$ for all $a \neq b$ of X, where $l(a, b)$ designates the Menger line through a, b.*

Proof. If $g(a, b),\ a \neq b$, is a g-line, then $x \in X$ is in $g(a, b)$ if, and only if,

$$\forall_{z \in X}\ [d(a, z) = d(a, x)] \text{ and } [d(b, z) = d(b, x)] \text{ imply } z = x. \tag{2.12}$$

As a consequence we get

$$f\bigl(g(a, b)\bigr) = g\bigl(f(a), f(b)\bigr),\ a \neq b,$$

for every g-line g and motion f. In order to prove $l(a,b) = g(a,b)$, it is hence sufficient to prove $l(0,\lambda e) = g(0,\lambda e)$ for $\lambda > 0$, i.e. $g(0,\lambda e) = \{\mu e \mid \mu \in \mathbb{R}\}$.

a) Euclidean case. (2.12) has for $a = 0$ and $b = \lambda e$ the form

$$\forall_{z \in X}\ z^2 = x^2 \text{ and } ez = ex \text{ imply } z = x. \tag{2.13}$$

$x = \mu e$ belongs to $g(0,\lambda e)$, because $z^2 = \mu^2$ and $ez = \mu$ imply $(ez)^2 = e^2 z^2$, i.e., by Lemma 1, chapter 1, $z \in \mathbb{R}e$, i.e. $z = \mu e$, by $ez = \mu$. If $x \in g(0,\lambda e)$ put

$$z := -\bigl(x - (xe)\,e\bigr) + (xe)\,e, \tag{2.14}$$

and observe $z^2 = x^2,\ ez = ex$, i.e., by (2.13), $z = x$. Hence, by (2.14), $x = (xe)\,e \in \mathbb{R}e$.

b) Hyperbolic case. (2.12) has for $a = 0$ and $b = \lambda e$ also the form (2.13). So also here we get $g(0,\lambda e) = \mathbb{R}e$. □

2.5 Balls, hyperplanes, subspaces

Proposition 8. *Suppose that $B(c,\varrho),\ B(c',\varrho')$ are balls of (X, eucl) satisfying $\varrho > 0$ and*

$$B(c,\varrho) \subseteq B(c',\varrho').$$

Then $c = c'$ and $\varrho = \varrho'$.

Proof. $c + \frac{\varrho x}{\|x\|} \in B(c,\varrho)$ implies

$$\frac{(c-c')\,x}{\|x\|} = \frac{1}{2\varrho}\left({\varrho'}^2 - \varrho^2 - (c-c')^2\right)$$

for all elements $x \neq 0$ of X. If $c - c'$ were $\neq 0$, the left-hand side of this equation would be 0 for $0 \neq x \perp (c-c')$ and $\neq 0$ for $x = c-c'$ which is impossible, since the right-hand side of the equation does not depend on x. (Notice that $a \perp b$ stands for $ab = 0$.) Hence $c - c' = 0$, and thus

$$0 = {\varrho'}^2 - \varrho^2 - (c-c')^2 = {\varrho'}^2 - \varrho^2.$$ □

Proposition 9. *Let $B(c,\varrho)$, $\varrho > 0$, be a ball of (X, hyp). Then*

$$B(c,\varrho) = \{x \in X \mid \|x-a\| + \|x-b\| = 2\alpha\}$$

with $a := ce^{-\varrho},\ b := ce^{\varrho}$ and $\alpha := \sinh\varrho \cdot \sqrt{1+c^2}$, where e^t denotes the exponential function $\exp(t)$ *for $t \in \mathbb{R}$.*

Proof. Put $S := \sinh\varrho$, $C := \cosh\varrho$ and $p := x - cC$. Observe $C + S = e^{\varrho}$ and $C - S = e^{-\varrho}$.

a) Assume $\|x-a\| = 2\alpha - \|x-b\|$ for a given $x \in X$. Squaring this equation yields

$$S\,(1+c^2) - cp = \sqrt{1+c^2}\,\|x-b\|.$$

Observing $x - b = p - Sc$ and squaring again, we get $(cp)^2 = (p^2 - S^2)(1+c^2)$. This implies

$$|cx + C| = \sqrt{1+c^2}\,\sqrt{1+x^2}, \tag{2.15}$$

since $cx + C = cp + C\,(1+c^2)$. If $-cx - C$ were equal to $\sqrt{1+c^2}\,\sqrt{1+x^2}$, then

$$1 \leq \cosh\,\operatorname{hyp}\,(c,-x) = \sqrt{1+c^2}\,\sqrt{1+x^2} + cx = -C$$

would follow, a contradiction. Hence, by (2.15),

$$\cosh\,\operatorname{hyp}\,(c,x) = C = \cosh\varrho,$$

i.e. $x \in B\,(c,\varrho)$.

b) Assume vice versa $C = \sqrt{1+c^2}\,\sqrt{1+x^2} - cx$, i.e. $x \in B\,(\varrho, c)$, for a given $x \in X$. A similar calculation as in step a), but now in the other direction, leads to

$$\sqrt{(p+cS)^2}\,\sqrt{(p-cS)^2} = |S^2(2+c^2) - p^2|. \tag{2.16}$$

If $S^2(2+c^2) \geq p^2$, then $\|x-a\| + \|x-b\| = 2\alpha$ follows from (2.16). So observe, by the inequality of Cauchy–Schwarz,

$$(cx)^2 \leq c^2x^2 + S^2,$$

i.e., by $(cx+C)^2 = (1+c^2)(1+x^2)$,

$$x^2 - 2\,(cx)\,C + c^2 = (cx)^2 + S^2 - c^2x^2 \leq 2S^2,$$

i.e. $S^2(2+c^2) \geq p^2$. $\square$

Suppose $a, b \in X$ and let γ be a positive real number. Then

$$\{x \in X \mid \|x-a\| + \|x-b\| = \gamma\}$$

is called a *hyperellipsoid* in euclidean geometry, i.e. in (X, eucl). Let now $B\,(c,\varrho)$, $\varrho > 0$, be a hyperbolic ball. If $c = 0$, then, in view of Proposition 9, it is also a euclidean ball with center 0 and radius $\sinh\varrho$. In the case $c \neq 0$, the hyperbolic ball $B\,(c,\varrho)$ is a euclidean hyperellipsoid such that its *foci* $ce^{-\varrho}$, ce^{ϱ} are in

$$\mathbb{R}_{>0}c = \{\lambda c \mid 0 < \lambda \in \mathbb{R}\}.$$

Observe $\tau_0 > 1$ for $ce^{\varrho} =: \tau_0(ce^{-\varrho})$.

Lemma 10. *Let $a \neq 0$ be an element of X and $\tau > 1$ be a real number. Then*

$$\{x \in X \mid \|x - a\| + \|x - \tau a\| = 2\alpha\} \tag{2.17}$$

is the hyperbolic ball $B\,(a\sqrt{\tau}, \ln\,\sqrt{\tau})$ if

$$2\alpha = (\tau - 1)\sqrt{\frac{1}{\tau} + a^2}.$$

Proof. Since $\{x \in X \mid \|x - ce^{-\varrho}\| + \|x - ce^{\varrho}\| = 2\sinh\varrho \cdot \sqrt{1+c^2}\}$ is $B\,(c, \varrho)$, we get with $c := a\sqrt{\tau}$, $\varrho := \ln\,\sqrt{\tau}$, obviously,

$$a = ce^{-\varrho},\ \tau a = ce^{\varrho}, 2\alpha = (e^{\varrho} - e^{-\varrho})\sqrt{1+c^2} = (\tau - 1)\sqrt{\frac{1}{\tau} + a^2}. \qquad \square$$

Proposition 11. *Suppose that $B\,(c, \varrho)$, $B\,(c', \varrho')$ are hyperbolic balls satisfying $\varrho > 0$ and*

$$B\,(c, \varrho) \subseteq B\,(c', \varrho'). \tag{2.18}$$

Then $c = c'$ and $\varrho = \varrho'$.

Proof. Assume that there exist balls $B\,(c, \varrho)$, $B\,(c', \varrho')$ with (2.18), $c \neq c'$ and $\varrho > 0$. If $j \in X$ is given with $j^2 = 1$, there exists, by D.a, a motion μ such that $\mu\,(c) = 0$, $\mu\,(c') = \lambda j$, $\lambda > 0$. Hence $B\,(0, \varrho) \subseteq B\,(\lambda j, \varrho')$, i.e.

$$\text{hyp}\ (0, x) = \varrho \ \text{ implies }\ \text{hyp}\ (\lambda j, x) = \varrho'$$

for all $x \in X$, i.e.

$$\sqrt{1 + x^2} = \cosh\varrho \ \text{ implies }\ \sqrt{1+\lambda^2}\,\sqrt{1+x^2} - \lambda jx = \cosh\varrho'.$$

Applying this implication twice, namely for $x = j\sinh\varrho$ and for $x = i\sinh\varrho$ with $i \in X$, $i^2 = 1$, $ij = 0$ we obtain

$$\sqrt{1+\lambda^2}\cosh\varrho - \lambda\sinh\varrho = \cosh\varrho' = \sqrt{1+\lambda^2}\,\cosh\varrho,$$

a contradiction, since $\lambda > 0$ and $\varrho > 0$. Thus $c = c'$. Take now $j \in X$ with $j^2 = 1$ and $jc = 0$, and observe for $x := \sinh\varrho \cdot j + \cosh\varrho \cdot c$,

$$\text{hyp}\ (c, x) = \varrho,$$

i.e., by (2.18), hyp $(c, x) = \varrho'$. Hence $\varrho = \varrho'$. $\square$

If $a \neq 0$ is in X and $\alpha \in \mathbb{R}$, then we will call

$$H\,(a, \alpha) := \{x \in X \mid ax = \alpha\}$$

a *euclidean hyperplane* of X.

If $e \in X$ satisfies $e^2 = 1$, if $t \in \mathbb{R}$ and $\omega_1, \omega_2 \in O(X)$, then

$$\omega_1 T_t \omega_2(e^\perp) = \{\omega_1 T_t \omega_2(x) \mid x \in e^\perp\}$$

will be called a *hyperbolic hyperplane*, where $\{T_t \mid t \in \mathbb{R}\}$ is based on the axis e and the kernel $\sinh \varrho \cdot \sqrt{1+h^2}$. Of course, mutatis mutandis, also the euclidean hyperplanes can be described this way.

In Proposition 17 parametric representations of hyperbolic hyperplanes will be given.

Proposition 12. *If $H(a,\alpha)$ and $H(b,\beta)$ are euclidean hyperplanes with $H(a,\alpha) \subseteq H(b,\beta)$, then $H(a,\alpha) = H(b,\beta)$ and there exists a real $\lambda \neq 0$ with $b = \lambda a$ and $\beta = \lambda\alpha$.*

Proof. If a, b are linearly dependent, then there exists a real $\lambda \neq 0$ with $b = \lambda a$ since a, b are both unequal to 0. Put $x_0 a^2 := \alpha a$. Hence

$$x_0 \in H(a,\alpha) \subseteq H(b,\beta),$$

i.e. $\beta = bx_0 = \lambda a \cdot x_0 = \lambda\alpha$, and thus $H(a,\alpha) = H(b,\beta)$. If a, b were linearly independent, then

$$q := x_0 + b - \frac{ab}{a^2}\, a \in H(a,\alpha) \subseteq H(b,\beta),$$

i.e. $bx_0 = \beta = bq$, i.e.

$$\left(b - \frac{ab}{a^2}\, a\right)^2 = b^2 - \frac{(ab)^2}{a^2} = b\,(q - x_0) = 0,$$

i.e. $b - \frac{ab}{a^2}\, a = 0$ would hold true. $\square$

If $a \neq 0$ is in X and $a^2 = 1$, then the hyperplanes of (X, hyp) can also be defined by

$$\alpha T_t \beta\,(a^\perp) \text{ with } \alpha, \beta \in O(X) \text{ and } t \in \mathbb{R}:$$

take $\omega \in O(X)$ with $a = \omega(e)$ and observe

$$\alpha T_t \beta\left([\omega(e)]^\perp\right) = \alpha T_t \beta\left(\omega(e^\perp)\right) = \alpha T_t \beta \omega\,(e^\perp).$$

Obviously, $\omega\big(H(a,\alpha)\big) = H\big(\omega(a), \alpha\big)$ for $\omega \in O(X)$, where $H(a,\alpha)$ is a euclidean hyperplane. The image of $H(a,\alpha)$ under $y = x + t$, $t \in X$, is $H(a, at + \alpha)$. Of course, if μ is a hyperbolic motion, then $\mu\,[\omega_1 T_t \omega_2(e^\perp)]$ is again a hyperbolic hyperplane since $\mu \cdot \omega_1 T_t \omega_2$ is also a motion (see I of the proof of Theorem 7 of chapter 1).

A *subspace* of (X, eucl) (or (X, hyp)) is a set $\Gamma \subseteq X$ such that for all $a \neq b$ in Γ the euclidean (hyperbolic) line through a, b is a subset of Γ. Of course, $\emptyset$ and X are subspaces, also every point of X, but lines as well. Since every euclidean (hyperbolic) line is contained in a one- or a two-dimensional subspace of the vector space X, the following proposition must hold true.

Proposition 13. *All euclidean (hyperbolic) subspaces are given by the subspaces of the vector space X and their images under motions.*

A *spherical subspace* of (X, eucl) or (X, hyp) is a set

$$\Gamma \cap B(c, \varrho),$$

where $\Gamma \ni c$ is a subspace and $B(c, \varrho)$ a ball of (X, eucl), (X, hyp), respectively. Without loss of generality we may assume $c = 0$. Hence the following proposition holds true.

Proposition 14. *All spherical subspaces of X are given by the spherical subspaces $\Gamma \cap B(c, \varrho)$ with $c = 0 \in \Gamma$ and their images under motions.*

A subspace V of the vector space X is called *maximal* if, and only if, $V \neq X$ and, moreover, every subspace $W \supseteq V$ of X is equal to X or V. If $0 \neq a \in X$, then $a^\perp$ is a maximal subspace of the vector space X: observe

1) $x, y \in a^\perp$ implies $x + y \in a^\perp$ and $\lambda x \in a^\perp$ for every $\lambda \in \mathbb{R}$,

2) if $W \supseteq a^\perp$ is a subspace of X and $x \in W \backslash a^\perp$, then $xa \neq 0$ and $-x + \frac{xa}{a^2}\, a \in a^\perp \subseteq W$, i.e. $\frac{xa}{a^2}\, a = x + \left(-x + \frac{xa}{a^2}\, a\right) \in W$, i.e. $a \in W$, i.e. $X = a^\perp \oplus \mathbb{R}a \subseteq W$, i.e. $X = W$.

Maximal subspaces of X and their images under euclidean (hyperbolic) motions will be called euclidean (hyperbolic) *quasi–hyperplanes*. Since $a^\perp$ with $0 \neq a \in X$ is maximal, hyperplanes are quasi–hyperplanes. But there are quasi–hyperplanes which are not hyperplanes.

2.6 A special quasi–hyperplane

Let X be the set of all power series with real coefficients and radius of convergence greater than 1,

$$A(\xi) = a_0 + a_1\xi + a_2\xi^2 + \cdots,$$

which will be of interest for us in the interval $[0, 1]$. Define

$$\begin{aligned} \lambda A(\xi) &= \lambda a_0 + \lambda a_1 \xi + \lambda a_2 \xi^2 + \cdots, \\ A(\xi) + B(\xi) &= (a_0 + b_0) + (a_1 + b_1)\,\xi + (a_2 + b_2)\,\xi^2 + \cdots \end{aligned}$$

and $AB = \int_0^1 A(\xi)\, B(\xi)\, d\xi$. Observe that the following set of elements of X, namely

$$e^\xi, 1, \xi, \xi^2, \xi^3, \ldots,$$

$e^\xi := \exp(\xi)$, is linearly independent: if

$$ke^\xi + k_0 \cdot 1 + k_1 \cdot \xi + \cdots + k_n \cdot \xi^n = 0 \tag{2.19}$$

for all $\xi \in [0,1]$ where $k, k_0, \ldots \in \mathbb{R}$, then differentiating (2.19) $(n+1)$-times yields $ke^{\xi} = 0$, i.e. $k = 0$, and differentiating it n-times, $k_n = 0$, and so on, $k_{n-1} = \cdots = k_0 = 0$. Let B be a basis of X which contains the functions $e^{\xi}, 1, \xi, \xi^2, \ldots$. Let V be the subspace of X generated by B' which is defined by B without the function e^{ξ}. Hence V is maximal. Since, of course, $0 \in V$, V must be a euclidean subspace of X. We would like to show that there is no $a \neq 0$ in X such that

$$V = H\,(a, 0), \tag{2.20}$$

i.e. that V is a quasi–hyperplane which is not a hyperplane. Assume that (2.20) holds true for an element $a \neq 0$ in X. Put

$$a\,(\xi) = a_0 + a_1\xi + \cdots$$

and notice

$$0 < \int_0^1 a\,(\xi)\, a\,(\xi)\, d\,(\xi) = \sum_{i=0}^{\infty} \int_0^1 a_i a\,(\xi)\,\xi^i d\xi = 0,$$

since the functions ξ^i, $i = 0, 1, \ldots$, belong to B' and hence to V.

2.7 Orthogonality, equidistant surfaces

Let l_1, l_2 be lines through $s \in X$. We will say that l_1 is *orthogonal* to l_2 and write $l_1 \perp l_2$ if, and only if, there exist

$$p_1 \in l_1 \backslash \{s\},\ p_2 \in l_2 \backslash \{s\}$$

such that (see (2.21) for the euclidean and (2.22) for the hyperbolic case)

$$\|p_1 - p_2\|^2 = \|p_1 - s\|^2 + \|s - p_2\|^2, \tag{2.21}$$

$$\cosh \operatorname{hyp}\,(p_1, p_2) = \cosh \operatorname{hyp}\,(p_1, s)\ \cosh \operatorname{hyp}\,(s, p_2). \tag{2.22}$$

Since $(p_1 - p_2)^2 = \big((p_1 - s) + (s - p_2)\big)^2$, we also may write $(p_1 - s)(s - p_2) = 0$ instead of (2.21). Formula (2.22) is the so-called theorem of Pythagoras of hyperbolic geometry (see, for instance, W. Benz [4], p. 153) for the triangle $p_1 s p_2$. If $s = 0$ in (2.22), then this formula reduces to $p_1 p_2 = 0$, i.e. that in 0 euclidean and hyperbolic orthogonality coincide. Observe that $l_1 \perp l_2$ implies $l_2 \perp l_1$. Moreover, there is no line l orthogonal to itself, $l \not\perp l$: if

$$l = \{p + \xi q \mid \xi \in \mathbb{R}\},\ q^2 = 1, \tag{2.23}$$

in the euclidean case or

$$l = \{x\,(\xi) = p \cosh \xi + q \sinh \xi \mid \xi \in \mathbb{R}\},\ pq = 0,\ q^2 = 1, \tag{2.24}$$

in the hyperbolic case, $l \perp l$ would imply

$$(\pi_1 - \sigma)(\sigma - \pi_2) = 0 \text{ for } s = p + \sigma q,\ p_i = p + \pi_i q \neq s\ (i = 1, 2),$$

a contradiction, or for $s = x(\sigma)$, $p_i = x(\pi_i) \neq s\ (i = 1, 2)$, by (2.1),

$$\cosh(\pi_1 - \pi_2) = \cosh(\pi_1 - \sigma)\cosh(\sigma - \pi_2).$$

Put $\alpha := \pi_1 - \sigma$ and $\beta := \sigma - \pi_2$, observe

$$\cosh(\alpha + \beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta,$$

i.e. $\sinh(\pi_1 - \sigma)\sinh(\sigma - \pi_2) = 0$, which is also a contradiction.

Since $l \not\perp l$ holds true for every line l, we obtain, by Proposition 5, that $l_1 \perp l_2$ implies $\#(l_1 \cap l_2) = 1$, i.e. that l_1, l_2 have one single point in common.

If l_1, l_2 are lines with $l_1 \perp l_2$ and μ is a motion, then $\mu(l_1) \perp \mu(l_2)$. This follows from (2.21), (2.22) since distances are invariant under motions.

Let l_1, l_2 be lines through s with $l_1 \perp l_2$. If $a_i \in l_i \backslash \{s\}$, $i = 1, 2$, are arbitrary points, then

$$\|a_1 - a_2\|^2 = \|a_1 - s\|^2 + \|s - a_2\|^2 \tag{2.25}$$

holds true in the euclidean case, and

$$\cosh \mathrm{hyp}\ (a_1, a_2) = \cosh \mathrm{hyp}\ (a_1, s)\cosh \mathrm{hyp}\ (s, a_2) \tag{2.26}$$

in the hyperbolic case.

Equation (2.25) follows from $q_1 q_2 = 0$ (see (2.23)). In order to prove (2.26) we may assume $s = 0$ by applying a suitable motion. As we already know, $l_1 \perp l_2$ is in this case equivalent with $a_1 a_2 = 0$. But (2.26) is given for $s = 0$ by

$$\sqrt{1 + a_1^2}\sqrt{1 + a_2^2} - a_1 a_2 = \sqrt{1 + a_1^2}\sqrt{1 + a_2^2}.$$

Proposition 15. *Let l be a line and $a \notin l$ a point. Then there exists exactly one line g through a with $g \perp l$.*

Proof. Hyperbolic case. Without loss of generality we may assume $a = 0$. Then l is of the form (2.24) with $p \neq 0$. If l_1 is the line through 0 and p, it is trivial to verify $l_1 \perp l$. So assume that there is another line l_2 through 0 and $x(\alpha) \neq p = x(0)$, i.e. $\alpha \neq 0$, with $l_2 \perp l$. This implies

$$\cosh \mathrm{hyp}\ (0, p) = \cosh \mathrm{hyp}\ \big(0, x(\alpha)\big)\cosh \mathrm{hyp}\ \big(x(0),\, x(\alpha)\big),$$

i.e. $\sqrt{1 + p^2} = \cosh\alpha \cdot \sqrt{1 + p^2} \cdot \cosh\alpha$, i.e. $\alpha = 0$, i.e. $x(\alpha) = p$, a contradiction.

Also in the euclidean case we may assume $a = 0$ and that l is of the form (2.23) with $l \not\ni 0$, i.e. that p, q are linearly independent. Obviously, $l \perp \mathbb{R}w$ with $w := p - (pq)\, q$. Moreover, $\mathbb{R}(p + \xi_0 q) \perp l$ implies $(p + \xi_0 q)\, q = 0$. □

If Γ is a subspace of (X, d), where d stands for eucl or hyp, and l a line with $l \cap \Gamma = \{s\}$, then l is called *orthogonal* to Γ, or Γ to l, provided $l \perp g$ holds true for all lines $g \subseteq \Gamma$ passing through s.

Proposition 16. *Let p be a point and H a hyperplane. Then there exists exactly one line $l \ni p$ with $l \perp H$.*

Proof. Case $p \in H$.
Without loss of generality we may assume $p = 0$. Hence in both cases (X, eucl), (X, hyp), H is a euclidean hyperplane $a^\perp$ with $0 \neq a \in X$. The line $\mathbb{R}a$ is orthogonal to every line $g \ni 0$ with $g \subseteq H$: if $g = \mathbb{R}b$, then $g \subseteq a^\perp$ implies $ab = 0$, i.e. $g \perp \mathbb{R}a$. If $\mathbb{R}c$ is orthogonal to every $\mathbb{R}b$ with $ab = 0$, then $b \in a^\perp$ implies $b \in c^\perp$, i.e. $H(a, 0) \subseteq H(c, 0)$, i.e. $\mathbb{R}c = \mathbb{R}a$, in view of Proposition 12.
Case $p \notin H$.
Since a point of H can be transformed into 0 by a motion, we may assume without loss of generality $H = H(a, 0)$ in both cases, i.e. in the euclidean as well in the hyperbolic case. Let

$$p + \mathbb{R}q := \{p + \lambda q \mid \lambda \in \mathbb{R}\}$$

be a euclidean line l orthogonal to H. Hence $l \perp r + \mathbb{R}b$ for all $b \in a^\perp$ where $r \in l \cap H$, i.e. $a^\perp \subseteq q^\perp$, i.e. $l = p + \mathbb{R}a$, by applying Proposition 12. On the other hand, $p + \mathbb{R}a \perp H(a, 0)$. The point of intersection is $r = p - \frac{pa}{a^2}\, a$. It remains to consider $p \notin H$ in the hyperbolic case. Put $H = H(a, 0)$, $a^2 = 1$. If $p - (pa)\, a \neq 0$, we define

$$j := \frac{p - (pa)\, a}{\|p - (pa)\, a\|}.$$

Take $\omega \in O(X)$ with $\omega(e) = j$, where e is the axis of our underlying translation group, and $t \in \mathbb{R}$ with $\omega T_t \omega^{-1}(p) = (pa)\, a$, in view of (T2) for j. Because of

$$\omega T_t \omega^{-1}(x) = x + [(xj)(\cosh t - 1) + \sqrt{1 + x^2}\, \sinh t]\, j$$

for $x \in X$, we obtain $\omega T_t \omega^{-1}(H) = H$ on account of $j \in a^\perp$. There hence exists a motion

$$\mu := \begin{cases} \omega T_t \omega^{-1} & \text{for} \quad p \neq (pa)\, a \\ \text{id} & \text{for} \quad p = (pa)\, a \end{cases}$$

with $\mu(H) = H$ and $\mu(p) \in \mathbb{R}a \backslash H$, by $p \notin H$, i.e. $pa \neq 0$. So we assume, without loss of generality, $H = a^\perp$ and $p = \lambda a$, $\lambda \neq 0$. There hence is a hyperbolic line, namely $l = \mathbb{R}a$ with $p \in l \perp H$. Assume now that there is another hyperbolic line $g \ni p$ with $l \neq g \perp H$. Hence $0 \notin g$ because all hyperbolic lines through 0 are of the form $\mathbb{R}b$. Put $g \cap H =: \{r\}$. Hence

$$\cosh \text{hyp}\ (0, p) = \cosh \text{hyp}\ (r, 0) \cdot \cosh \text{hyp}\ (r, p),$$

i.e. $\sqrt{1 + p^2} = \sqrt{1 + r^2}\, (\sqrt{1 + r^2}\, \sqrt{1 + p^2} - rp)$. But $p \in \mathbb{R}a$, $r \in H$ implies $pr = 0$. Thus $1 + r^2 = 1$, i.e. $r = 0$, a contradiction. □

The distance $d\,(p, H)$ between a point p and a hyperplane H is defined by $d\,(p, r)$, where r is the point of intersection of H and the line $l \ni p$ orthogonal to H. This applies for (X, eucl) as well as for (X, hyp).

Let $\varrho > 0$ be a real number and H be a hyperplane. An interesting set of points is then given by

$$D_\varrho(H) = \{x \in X \mid d\,(x, H) = \varrho\},$$

a so-called *equidistant surface* or *hypercycle* of H. We will look to these sets in the case $0 \in H$. In the euclidean case we get with $a \in X$, $a^2 = 1$,

$$D_\varrho\big(H\,(a, 0)\big) = H\,(a, \varrho) \cup H\,(a, -\varrho),$$

i.e. we obtain the union of two euclidean hyperplanes parallel to $a^\perp$, since the euclidean hyperplanes H_1, H_2 are called *parallel*, $H_1 \parallel H_2$, provided $H_1 = H_2$ or $H_1 \cap H_2 = \emptyset$ hold true. Of course, $H\,(a, \alpha) \parallel H\,(b, \beta)$ is satisfied if, and only if, $\mathbb{R}a = \mathbb{R}b$. In hyperbolic geometry we obtain for $\varrho > 0,\ H = H\,(a, 0),\ a^2 = 1$, as we will show,

$$D_\varrho\big(H\,(a, 0)\big) = H\,(a, \sinh \varrho) \cup H\,(a, -\sinh \varrho).$$

As a matter of fact, this is again the union of two euclidean hyperplanes, and not, say, of two hyperbolic hyperplanes. The point

$$p \in \{a \sinh \varrho,\ -a \sinh \varrho\}$$

has distance ϱ from H. Take $\omega_j \in O\,(X)$ with $\omega_j(e) = j$ for a given $j \in X$ with $j^2 = 1$ and $aj = 0$, i.e. $j \in H$. Now

$$\mu\,(0) = j \sinh t,\ \ \mu\,(p) = p + j \cosh \varrho \sinh t$$

where $\mu := \omega_j T_t \omega_j^{-1}$, $t \in \mathbb{R}$, holds true, and the line through $\mu\,(0)$, $\mu\,(p)$ must be orthogonal to H, in view of $\mathbb{R}p \perp H$. Since $\mu\,(0)$ runs over H by varying j and t, $\mu\,(p)$ runs over $D_\varrho(H)$ on account of

$$\text{hyp }\big(\mu\,(0),\ \mu\,(p)\big) = \text{hyp }(0, p) = \varrho :$$

through $h \in H$ there is exactly one hyperbolic line l orthogonal to H, and on l there are exactly two points of distance ϱ from H. Hence

$$D_\varrho(H) = (a \sinh \varrho + H) \cup (-a \sinh \varrho + H)$$

where $p + H := \{p + h \mid h \in H\}$.

2.8 A parametric representation

Proposition 17. *If H is a hyperbolic hyperplane, there exist $p \in X$ with $p^2 = 1$ and $\gamma \in \mathbb{R}_{\geq 0}$ with $H = \Pi\,(p, \gamma)$, where*

$$\Pi\,(p, \gamma) := \{\gamma p \cosh \xi + y \sinh \xi \mid \xi \in \mathbb{R},\ y \in p^\perp \text{ with } y^2 = 1\}. \tag{2.27}$$

On the other hand, every $\Pi(p,\gamma)$ *is a hyperbolic hyperplane provided* $\gamma \in \mathbb{R}_{\geq 0}$ *and* $p \in X$ *satisfies* $p^2 = 1$*. Moreover,*

$$\Pi(p,0) = p^{\perp} = \{x - (xp)\,p \mid x \in X\}, \tag{2.28}$$

and for $\gamma > 0$*,*

$$\Pi(p,\gamma) = \{\gamma p \cosh\xi + \frac{x-(xp)\,p}{\sqrt{x^2-(xp)^2}} \sinh\xi \mid \xi \in \mathbb{R},\, x \in X\backslash\mathbb{R}p\}. \tag{2.29}$$

Proof. 1. Let $\gamma \geq 0$ and $p \in X$, $p^2 = 1$, be given. Take $\omega \in O(X)$ with $\omega(e) = p$ and $t \in \mathbb{R}$ with $\sinh t = \gamma$. Then $\omega T_t \omega^{-1}(p^{\perp})$ must be a hyperbolic hyperplane. Observe

$$\omega T_t \omega^{-1}(p^{\perp}) = \{h + \sinh t \cdot \sqrt{1+h^2}\, p \mid h \in p^{\perp}\}.$$

With $h = y \sinh\xi$, $y \in p^{\perp}$, $y^2 = 1$, we obtain

$$\omega T_t \omega^{-1}(p^{\perp}) = \{\gamma p \cosh\xi + y \sinh\xi \mid \xi \in \mathbb{R},\, y \in p^{\perp} \text{ with } y^2 = 1\},$$

i.e. $\omega T_t \omega^{-1}(p^{\perp}) = \Pi(p,\gamma)$, i.e. $\Pi(p,\gamma)$ is a hyperbolic hyperplane.

2. *Given a hyperbolic line* l *and a point* $r \in l$*. Then there exists exactly one hyperbolic hyperplane* $H \ni r$ *with* $l \perp H$. In order to prove this statement, take $b \in l\backslash\{r\}$ and a motion μ with $\mu(r) = 0$, $\mu(b) =: a$. Since $r \neq b$ we get $0 \neq a$. There is exactly one hyperbolic hyperplane through 0 which is orthogonal to the line through a and 0, namely $a^{\perp}$. There hence is exactly one hyperbolic hyperplane through r, namely $\mu^{-1}(a^{\perp})$, which is orthogonal to l.

3. Let now H be an arbitrary hyperbolic hyperplane. Because of Proposition 16 there exists exactly one hyperbolic line l through 0 which is orthogonal to H. Let r be the point of intersection of l and H. Let r be the point of intersection of l and H. Because of step 2 we know that H is uniquely determined as the hyperbolic hyperplane through r which is orthogonal to l. But we already know a hyperbolic hyperplane of this kind, namely $a^{\perp}$ for $r = 0$ and $l = \mathbb{R}a$, and $\Pi(p,\gamma)$ for $r \neq 0$, $p := \frac{r}{\|r\|}$, $\gamma := \|r\|$. In fact, $r = \gamma p \in \Pi$ for $\xi = 0$, and $g \perp l$ for all hyperbolic lines g through r and $s := \gamma p \cosh\xi + y \sinh\xi$ with $\xi \neq 0$, $y \in p^{\perp}$, $y^2 = 1$ on account of

$$\cosh \operatorname{hyp}(0,s) = \cosh \operatorname{hyp}(0,r) \cosh \operatorname{hyp}(r,s).$$

Hence $H = \Pi(p,\gamma)$.

4. Since $[x - (xp)\,p]p = 0$, we get $x - (xp)\,p \in p^{\perp}$ for all $x \in X$. If $y \in p^{\perp}$ we obtain $yp = 0$, i.e. $y = y - (yp)\,p$. This proves (2.28). In order to get (2.29) we must show

$$\left\{ \frac{x-(xp)\,p}{\sqrt{x^2-(xp)^2}} \middle| x \in X\backslash\mathbb{R}p \right\} = \{y \in X \mid y \in p^{\perp} \text{ and } y^2 = 1\}.$$

Because of Lemma 1, chapter 1, we have $x^2 = (xp)^2$ if, and only if, $x \in \mathbb{R}p$, in view of $p^2 = 1$. Obviously,

$$y = \frac{x - (xp)\,p}{\sqrt{x^2 - (xp)^2}}$$

satisfies $y \in p^\perp$ and $y^2 = 1$. Given, finally, $y \in X$ with $y \in p^\perp$ and $y^2 = 1$, we obtain

$$y = \frac{y - (yp)\,p}{\sqrt{y^2 - (yp)^2}}$$

with $y \notin \mathbb{R}p$. $\square$

From (2.27) we obtain for $\omega \in O\,(X)$,

$$\omega\left(\Pi\,(p,\gamma)\right) = \{\gamma\omega\,(p)\cosh\xi + z\sinh\xi \mid \xi \in \mathbb{R},\ z \in [\omega\,(p)]^\perp \text{with } z^2 = 1\},$$

i.e.

$$\omega\left(\Pi\,(p,\gamma)\right) = \Pi\left(\omega\,(p),\,\gamma\right).$$

In Theorem 26 we will prove that

$$T_t\left(\Pi\,(p,\gamma)\right) = \Pi\,(\varepsilon p', |\gamma'|)$$

holds true for all $t, \gamma \in \mathbb{R}$ with $\gamma \geq 0$, and all $p \in X$ with $p^2 = 1$, where

$$\begin{aligned} \gamma' &:= \gamma\cosh t + (pe)\sqrt{1+\gamma^2}\sinh t, \\ \varepsilon &:= \operatorname{sgn}\gamma' \text{ for } \gamma' \neq 0 \end{aligned}$$

and $p' \cdot \|A\| := A := p + \left[\frac{\gamma}{\sqrt{1+\gamma^2}}\sinh t + (pe)(\cosh t - 1)\right] e$, by observing $A \neq 0$. In the case $\gamma' = 0$ the value of $\varepsilon \neq 0$ plays no role, since $\Pi\,(\varepsilon p', 0) = (p')^\perp$. In Proposition 27 we will show that $\Pi\,(p,\gamma) \subseteq \Pi\,(q,\delta)$ and $\gamma > 0$ imply $p = q$ and $\gamma = \delta$.

Remark. A parametric representation of euclidean hyperplanes will be given in section 2, chapter 3.

2.9 Ends, parallelity, measures of angles

The notion of an *end* as introduced by David Hilbert (1862–1943) concerns hyperbolic geometry. If $w \in X\backslash\{0\}$, then we will call

$$\mathbb{R}_{\geq 0}w := \{\lambda w \mid \lambda \in \mathbb{R} \text{ and } \lambda \geq 0\}$$

an *end* of X. Two ends $\mathbb{R}_{\geq 0}w_1$, $\mathbb{R}_{\geq 0}w_2$ are equal if, and only if, there exists $\lambda > 0$ with $w_2 = \lambda w_1$. To every hyperbolic line l there will be associated two ends, the so-called ends of l. For

$$l_{p,q} = l = \{p\cosh\xi + q\sinh\xi \mid \xi \in \mathbb{R}\} \tag{2.30}$$

with $p, q \in X$ and $pq = 0$, $q^2 = 1$, the two ends of l are

$$\mathbb{R}_{\geq 0}(p+q),\ \mathbb{R}_{\geq 0}(p-q). \tag{2.31}$$

Note that p is the only element y in l with $\|y\| = \min_{x\in l}\|x\|$. This implies for $p, p', q, q' \in X$ with $pq = 0 = p'q'$ and $q^2 = 1 = q'^2$ that the lines $l_{p,q}$ and $l_{p',q'}$ coincide if and only if $p = p'$ and $q' \in \{q, -q\}$. If $l = \mathbb{R}q$, then $l = (\mathbb{R}_{\geq 0}q) \cup \big(\mathbb{R}_{\geq 0}(-q)\big)$. In the case $p \neq 0$,

$$\begin{aligned} &\mathbb{R}_{\geq 0}(p+q) \cup \mathbb{R}_{\geq 0}(-p-q),\\ &\mathbb{R}_{\geq 0}(p-q) \cup \mathbb{R}_{\geq 0}(-p+q) \end{aligned}$$

are the two asymptotes of the hyperbola of which (2.30) is a branch. Obviously, (2.31) are the limiting positions of $\mathbb{R}_{\geq 0}(p\cosh\xi + q\sinh\xi)$ for $\xi \to +\infty$, $\xi \to -\infty$, respectively:

$$\mathbb{R}_{\geq 0}\left(p + q\,\frac{\sinh\xi}{\cosh\xi}\right) \to \begin{array}{l} \mathbb{R}_{\geq 0}(p+q) \text{ for } \xi \to +\infty \\ \mathbb{R}_{\geq 0}(p-q) \text{ for } \xi \to -\infty \end{array}.$$

Proposition 18. *Let $E_i = \mathbb{R}_{\geq 0}w_i$, $i = 1, 2$, be distinct ends. Then there is exactly one hyperbolic line, of which E_1, E_2 are the ends.*

Proof. If $E_2 = -E_1$, i.e. if $\mathbb{R}_{\geq 0}w_2 = \mathbb{R}_{\geq 0}(-w_1)$, then $\mathbb{R}w_1$ is the uniquely determined line with the ends E_1, E_2. In the case that w_1, w_2 are linearly independent, we must solve

$$2\lambda_1 w_1 = p + q,\ 2\lambda_2 w_2 = p - q$$

in $\lambda_1, \lambda_2, p, q$ with $\lambda_1 > 0$, $\lambda_2 > 0$, $pq = 0$, $q^2 = 1$. This implies, by assuming $w_1^2 = 1 = w_2^2$, without loss of generality,

$$\begin{aligned} &p = \lambda_1 w_1 + \lambda_2 w_2, && q = \lambda_1 w_1 - \lambda_2 w_2,\\ &2\lambda_1^2(1 - w_1 w_2) = 1, && \lambda_1 = \lambda_2, \end{aligned}$$

with a uniquely determined solution

$$\left\{ \frac{(w_1+w_2)\cosh\xi + (w_1-w_2)\sinh\xi}{\sqrt{2\,(1 - w_1w_2)}} \,\Big|\, \xi \in \mathbb{R} \right\},$$

in view of $w_1w_2 \leq |w_1w_2| < \|w_1\|\,\|w_2\| = 1$ since w_1, w_2 are linearly independent. $\square$

Let E be an end of X and μ be a hyperbolic motion. We would like to define the end $\mu(E)$. If $E = \mathbb{R}_{\geq 0}a$, $a^2 = 1$, put $\omega(E) := \mathbb{R}_{\geq 0}\omega(a)$ for $\omega \in O(X)$. Suppose $t \in \mathbb{R}$ and that T_t is a translation of (X, hyp) with axis e. Then

$$T_t\big(\{\lambda a \mid \lambda \geq 0\}\big) = \big\{\lambda a + [\lambda\,(ae)(\cosh t - 1) + \sqrt{1+\lambda^2}\sinh t]\,e \mid \lambda \geq 0\big\}.$$

We are now interested in the question whether

$$\mathbb{R}_{\geq 0}\big(T_t(\lambda a)\big) \tag{2.32}$$

tends to a limiting position for $0 < \lambda \to +\infty$. Instead of (2.32) we may write

$$\mathbb{R}_{\geq 0}\left(a + \left[(ae)(\cosh t - 1) + \sqrt{\frac{1}{\lambda^2} + 1}\,\sinh t\right] e\right),$$

and we obtain as limiting position

$$\mathbb{R}_{\geq 0}\big(a + [(ae)(\cosh t - 1) + \sinh t]\, e\big) \tag{2.33}$$

which we define as the end $T_t(E) = T_t(\mathbb{R}_{\geq 0}a)$. In the case $0 > \lambda \to -\infty$ we observe

$$\begin{aligned} & \tfrac{1}{\lambda}\big(\lambda a + [\lambda\,(ae)(\cosh t - 1) + \sqrt{1+\lambda^2}\,\sinh t]\, e\big) \\ = \quad & a + \left[(ae)(\cosh t - 1) - \sqrt{\tfrac{1}{\lambda^2} + 1}\,\sinh t\right] e \\ \to \quad & a + [(ae)(\cosh t - 1) - \sinh t]\, e, \end{aligned}$$

a result which corresponds to (2.33), replacing there a by $-a$, i.e. substituting $\mathbb{R}_{\geq 0}\big(T_t[\lambda \cdot (-a)]\big)$, $0 < \lambda \to +\infty$, for $\mathbb{R}_{\geq 0}\big(T_t(\lambda a)\big)$, $0 > \lambda \to -\infty$.

Proposition 19. *If E is an end of the line l and μ a motion, then $\mu\,(E)$ is an end of $\mu\,(l)$.*

Proof. Let $x\,(\xi) = p\cosh\xi + q\sinh\xi$ be the equation of l, and let E be given, say, by $\mathbb{R}_{\geq 0}(p + q)$ thus considering the case $\xi \to +\infty$. If $\mu \in O\,(X)$, we obtain $\mathbb{R}_{\geq 0}\big(\mu\,(p) + \mu\,(q)\big)$ as end of $\mu\,(l)$ for $\xi \to +\infty$, i.e. we get the end

$$\mathbb{R}_{\geq 0}\big(\mu\,(p+q)\big) = \mu\,(E).$$

Suppose now that $\mu = T_t$. We already know, by (2.33), with $\frac{p+q}{\sqrt{p^2+q^2}}$, i.e. $\frac{p+q}{\sqrt{1+p^2}}$ instead of a,

$$T_t(E) = \mathbb{R}_{\geq 0}\left(\frac{p+q}{\sqrt{1+p^2}} + \left[\frac{(p+q)\,e}{\sqrt{1+p^2}}\,(\cosh t - 1) + \sinh t\right] e\right).$$

Moreover, $\mathbb{R}_{\geq 0}\big(T_t(p\cosh\xi + q\sinh\xi)\big)$ is given by

$$\mathbb{R}_{\geq 0}\big(p + q\tanh\xi + [(p + q\tanh\xi)\, e\,(\cosh t - 1) + \sqrt{1+p^2}\,\sinh t]\, e\big),$$

which tends to

$$\mathbb{R}_{\geq 0}\big(p + q + [(p+q)\, e\,(\cosh t - 1) + \sqrt{1+p^2}\,\sinh t]\, e\big)$$

for $\xi \to +\infty$, i.e. which tends to $T_t(E)$. Hence $\mu\,(E)$ is an end of $\mu\,(l)$. $\square$

Two euclidean lines

$$l_i := \{p_i + \xi q_i \mid \xi \in \mathbb{R}\}$$

are called *parallel*, $l_1 \parallel l_2$, provided $\mathbb{R}q_1 = \mathbb{R}q_2$. Parallelity is an equivalence relation on the set of euclidean lines of X. If $l = p + \mathbb{R}q$ is a euclidean line and r a point, there exists exactly one euclidean line, namely $g = r + \mathbb{R}q$ through r, parallel to l.

Two hyperbolic lines of X are called *parallel* provided they have at least one end in common. If l_1, l_2 are hyperbolic lines, of course, $l_1 \parallel l_1$ holds true and also that $l_1 \parallel l_2$ implies $l_2 \parallel l_1$. However, parallelity need not be transitive. In order to verify this statement take elements a, b of X with $a^2 = 1 = b^2$ and $ab = 0$. Define

$$\begin{aligned} l_1 &= \{a \cosh \xi + b \sinh \xi \mid \xi \in \mathbb{R}\}, \\ l_2 &= \{-a \cosh \xi + b \sinh \xi \mid \xi \in \mathbb{R}\}, \end{aligned}$$

and $l = \mathbb{R}\,(a+b)$. We obtain $l_1 \parallel l$, because these lines have $\mathbb{R}_{\geq 0}(a+b)$ in common, moreover, $l \parallel l_2$ since $\mathbb{R}_{\geq 0}(-a-b)$ is an end of both lines. But $l_1 \parallel l_2$ does not hold true: the ends of l_1 are $\mathbb{R}_{\geq 0}(a+b)$, $\mathbb{R}_{\geq 0}(a-b)$, and those of l_2 are

$$\mathbb{R}_{\geq 0}(-a+b),\ \mathbb{R}_{\geq 0}(-a-b).$$

If p is a point and $E := \mathbb{R}_{\geq 0}\, a$ an end, there is exactly one hyperbolic line through p having E as an end. In order to prove this statement take a motion μ with $\mu\,(p) = 0$. Of course, there is exactly one line through 0 having $\mu\,(E) =: \mathbb{R}_{\geq 0} b$ as an end, namely $\mathbb{R}b$. Hence, by Proposition 19, there is exactly one line, namely $\mu^{-1}(\mathbb{R}b)$, through p with E as an end.

If l is a line and $p \notin l$ a point, there are exactly two lines $l_1 \neq l_2$ through p which are parallel to l: take the two distinct ends E_1, E_2 associated with l, and then the lines l_1, l_2 through p with E_1, E_2, respectively, as an end.

Let $l = \{x\,(\xi) = p \cosh \xi + q \sinh \xi \mid \xi \in \mathbb{R}\}$ be a hyperbolic line and $a = x\,(\alpha)$ be a point of l. The two sets

$$\{x\,(\xi) \mid \xi \geq \alpha\},\ \{x\,(\xi) \mid \xi \leq \alpha\} \tag{2.34}$$

are called (hyperbolic) *rays* with *starting point* $x\,(\alpha)$. If $l = \{x\,(\xi) = p + \xi q \mid \xi \in \mathbb{R}\}$ is a euclidean line and $x\,(\alpha) = p + \alpha q$ a point a of l, then (2.34) are said to be (euclidean) *rays* with *starting point* $x\,(\alpha)$. Images $\mu\,(R)$ of rays R under motions μ are rays, and if a is the starting point of R, then $\mu\,(a)$ is the starting point of $\mu\,(R)$.

It is clear how to associate each of the ends of a hyperbolic line l to the two rays $R_1, R_2 \subset l$ of l with the same starting point. In this connection we will speak of *the end of a ray* or of a *ray through an end*.

Let R_1, R_2 be rays with the same starting point v such that $R_1 \cup R_2$ is not a line. The triple (R_1, R_2, v) consisting of the (unordered) pair R_1, R_2 and the point

v will be called an *angle*. If $p_i \in R_i$, $i = 1, 2$, is the point with

$$d\,(v, p_i) = 1,\ i = 1, 2,$$

then the measure $\measuredangle(R_1, R_2, v)$ of the angle (R_1, R_2, v) is defined by $\measuredangle(R_1, R_2, v) \in [0, \pi]$ and

$$1 - \cos\measuredangle(R_1, R_2, v) = \begin{cases} \dfrac{1}{2}\,[\text{eucl}\ (p_1, p_2)]^2 \\ 2\dfrac{\cosh \text{hyp}\ (p_1, p_2) - 1}{\cosh 2 - 1} \end{cases} \tag{2.35}$$

for the euclidean, hyperbolic case, respectively. (For an axiomatic definition of measures of angles in 2-dimensional euclidean or hyperbolic geometry see, for instance, the book [4] of the author.)

If R_1, R_2 are rays both with starting point v and μ a motion, then

$$\measuredangle(R_1, R_2, v) = \measuredangle\big(\mu\,(R_1),\ \mu\,(R_2),\ \mu\,(v)\big).$$

This is clear since distances are preserved under motions.

Let a, b, v be elements of X with $a \neq 0 \neq b$ and R_1, R_2 the rays

$$v + \mathbb{R}_{\geq 0}a,\ v + \mathbb{R}_{\geq 0}b,$$

respectively. Define $p_1 = v + \frac{1}{\|a\|}\,a,\ p_2 = v + \frac{1}{\|b\|}\,b,\ \gamma = \measuredangle(R_1, R_2, v)$. Hence

$$\begin{aligned} ab &= \|a\| \cdot \|b\| \cdot (p_1 - v)(p_2 - v) \\ &= \tfrac{1}{2}\|a\| \cdot \|b\| \cdot \big((p_1 - v)^2 + (p_2 - v)^2 - [(p_1 - v) - (p_2 - v)]^2\big) \\ &= \|a\| \cdot \|b\| \cdot \big(1 - \tfrac{1}{2}\,[p_1 - p_2]^2\big), \end{aligned}$$

i.e. $ab = \|a\| \cdot \|b\| \cdot \cos\gamma$, in view of (2.35). As a consequence we get the so-called cosine theorem:

$$\begin{aligned} [\text{eucl}\ (v + a,\ v + b)]^2 &= [(v + a) - (v + b)]^2 = (a - b)^2 \\ &= a^2 + b^2 - 2\|a\| \cdot \|b\| \cdot \cos\gamma, \end{aligned}$$

i.e., by $A = \text{eucl}\ (v, v + a)$, $B = \text{eucl}\ (v, v + b)$,

$$[\text{eucl}\ (v + a,\ v + b)]^2 = A^2 + B^2 - 2AB\cos\measuredangle(R_1, R_2, v).$$

Similarly, we would like to consider the case of hyperbolic geometry.

Let a, b, v be elements of X with $a \neq v \neq b$. If l_1 is the hyperbolic line through v, a, and l_2 the one through v, b, if R_1, R_2 are the (hyperbolic) rays with starting point v and $a \in R_1$, $b \in R_2$, then the cosine theorem of hyperbolic geometry holds true:

$$\cosh C = \cosh A \cdot \cosh B - \sinh A \cdot \sinh B \cdot \cos\gamma$$

where $C = \text{hyp }(a,b)$, $A = \text{hyp }(v,a)$, $B = \text{hyp }(v,b)$, $\gamma = \measuredangle(R_1, R_2, v)$.

For the proof of this statement we may assume $v = 0$ without loss of generality, since distances and measures of angles are preserved under motions. So put

$$l_i := \{x_i(\xi) = q_i \sinh \xi \mid \xi \in \mathbb{R}\},\ i = 1, 2,$$

with $q_i^2 = 1$, $i = 1, 2$, and with a sign for q_i such that $\xi \geq 0$ describes R_i for $i = 1, 2$. Hence $x_1(0) = v = x_2(0)$ and

$$p_i = x_i(1),\ i = 1, 2,\ a =: x_1(\alpha),\ b =: x_2(\beta),$$

with $\alpha > 0$, $\beta > 0$, and thus $\|a\| = \sinh \alpha$, $\|b\| = \sinh \beta$,

$$\cosh C = \sqrt{1 + a^2}\,\sqrt{1 + b^2} - ab,$$

$\cosh A = \sqrt{1 + a^2}$, $\cosh B = \sqrt{1 + b^2}$, $\cosh \text{hyp }(p_1, p_2) = (\cosh 1)^2 - q_1 q_2 (\sinh 1)^2$. Moreover, $\sinh A = \|a\|$, $\sinh B = \|b\|$,

$$ab = x_1(\alpha)\, x_2(\beta) = q_1 q_2 \sinh \alpha \sinh \beta,$$

and, by (2.35), $\cosh 2 = 1 + 2 \sinh^2 1$,

$$\begin{aligned} \cos \gamma \quad &= 1 - 2\, \frac{\cosh \text{hyp }(p_1, p_2) - 1}{\cosh 2 - 1} \\ &= 1 - 2\, \frac{(1 - q_1 q_2)[\sinh 1]^2}{\cosh 2 - 1} = q_1 q_2. \end{aligned}$$

Hence $\sqrt{1 + a^2}\,\sqrt{1 + b^2} - ab = \cosh A \cdot \cosh B - \sinh A \cdot \sinh B \cdot \cos \gamma$, since $\sinh A = \sinh \alpha$ and $\sinh B = \sinh \beta$, q.e.d.

Remark. Measures of angles $(R_1, R_2, 0)$ coincide in euclidean and hyperbolic geometry because of the previous formulas $\cos \gamma = q_1 q_2$ and $q_1 q_2 = \|q_1\| \cdot \|q_2\| \cos \gamma$. Notice, moreover, that the cosine theorem in both geometries leads for $\gamma = \frac{\pi}{2}$ to (2.21), (2.22), respectively.

2.10 Angles of parallelism, horocycles

Proposition 20. *Let $k \neq l$ be parallel hyperbolic lines with E as common end, $p \in l \backslash k$ a point, $a \ni p$ the line orthogonal to k, and r the point of intersection of k and a. If $R_1 \subset a$ is the ray through r with starting point p, and $R_2 \subset l$ the ray through E, also with starting point p, then*

$$tan\ \frac{1}{2}\left(\measuredangle(R_1, R_2, p)\right) = e^{-\text{hyp }(p,r)}.$$

Proof. Without loss of generality we may assume $p = 0$,

$$k = \{r \cosh \xi + q \sinh \xi \mid \xi \in \mathbb{R}\},\ rq = 0,\ q^2 = 1,$$

$a = \mathbb{R}r$, $R_1 = \mathbb{R}_{\geq 0}r$, $l = \mathbb{R}\,(r + q)$, $R_2 = \mathbb{R}_{\geq 0}(r + q)$. Put $\gamma := \measuredangle(R_1, R_2, p)$. Observe $r \neq 0 = p$, since $r \in k \not\ni p$. From (2.35) we obtain

$$1 - \cos \gamma = \frac{\cosh \text{hyp }(p_1, p_2) - 1}{\sinh^2 1} = 1 - \frac{\|r\|}{\sqrt{1 + r^2}},$$

in view of $rq = 0$, $q^2 = 1$, $p_1 = \frac{r}{\|r\|} \sinh 1$, $p_2 = \frac{r+q}{\sqrt{1+r^2}} \sinh 1$. We hence get

$$\cos \gamma = \frac{\|r\|}{\sqrt{1 + r^2}},$$

i.e. $\gamma \in \,]0, \frac{\pi}{2}[$ because of $0 < \frac{\|r\|}{\sqrt{1+r^2}} < 1$. From

$$\cosh \text{hyp }(p, r) = \sqrt{1 + r^2}$$

we obtain

$$e^{-\text{hyp }(p,r)} = \sqrt{1 + r^2} - \|r\|,\ e^{\text{hyp }(p,r)} = \sqrt{1 + r^2} + \|r\|,$$

i.e.

$$\tan \frac{1}{2}\gamma = \sqrt{\frac{1 - \cos \gamma}{1 + \cos \gamma}} = \sqrt{\frac{\sqrt{1 + r^2} - \|r\|}{\sqrt{1 + r^2} + \|r\|}} = e^{-\text{hyp }(p,r)}. \qquad \square$$

Proposition 21. *Let l be a hyperbolic line and $R \subset l$ a ray with starting point v. There exists a paraboloid as limiting position for the balls $B\big(c, \text{hyp }(c, v)\big)$ with $c \in R$ and* hyp $(c, v) \to \infty$. *This limiting position is called a horocycle.*

Proof. If $l = \{x\,(\xi) := p \cosh \xi + q \sinh \xi \mid \xi \in \mathbb{R}\}$, $pq = 0$, $q^2 = 1$, and $v = x\,(\alpha)$, we may assume $R = \{x\,(\xi) \mid \xi \geq \alpha\}$, without loss of generality. Put $c =: x\,(\alpha + \varrho)$, $\varrho > 0$. Then

$$B_\varrho := B\big(c,\ \text{hyp }(c, v)\big) = B\big(x\,(\alpha + \varrho), \varrho\big) = \big\{x \in X \mid \text{hyp }\big(x\,(\alpha + \varrho), x\big) = \varrho\big\},$$

i.e. $B_\varrho = \{x \in X \mid \cosh(\alpha + \varrho)\sqrt{1 + p^2}\,\sqrt{1 + x^2} - x\,(\alpha + \varrho)\,x = \cosh \varrho\}$ holds true. This implies

$$\sqrt{1 + x^2}\,\sqrt{1 + p^2} - x\big(p + q \tanh(\alpha + \varrho)\big) = \frac{\cosh \varrho}{\cosh(\alpha + \varrho)},$$

i.e. $\sqrt{1 + x^2}\,\sqrt{1 + p^2} - x\,(p + q) = e^{-\alpha}$ for $\varrho \to +\infty$. Hence the limiting position for B_ϱ, $\varrho \to \infty$, is

$$B_\infty := \{x \in X \mid \sqrt{1 + x^2} - xm = \tau\} \tag{2.36}$$

with $m := \frac{p+q}{\sqrt{1+p^2}}$, i.e. $m^2 = 1$, and $\tau := \frac{e^{-\alpha}}{\sqrt{1+p^2}} > 0$.

In view of $X = m^\perp \oplus \mathbb{R}m$, we will write $x =: \overline{x} + x_0 m$ with $\overline{x} \in m^\perp$ and $x_0 \in \mathbb{R}$. Thus

$$B_\infty = \{x \in X \mid \overline{x}^2 - 2\tau x_0 + 1 = \tau^2\}$$

by observing $x_0 + \tau \geq 0$ for an element x of B_∞: assuming $x_0 + \tau < 0$ would lead to

$$\tau^2 = \overline{x}^2 - 2\tau x_0 + 1 > \overline{x}^2 + 2\tau^2 + 1,$$

i.e. to $\overline{x}^2 + \tau^2 + 1 < 0$. The surface S of X,

$$S := \{\xi w + \eta m \mid \xi, \eta \in \mathbb{R},\, w \in m^\perp,\, w^2 = 1,\, \xi^2 = 2\tau\eta + \tau^2 + 1\},$$

is called a *paraboloid*, and $B_\infty = S$ holds true. $\square$

If H^1, H^2 *are horocycles, there exists a hyperbolic motion* μ *with* $\mu\,(H^1) = H^2$. If H^i, $i = 1, 2$, is based on the ray R_i with starting point v_i, we take points $p_i \in R_i$, $i = 1, 2$, satisfying hyp $(v_i, p_i) = 1$. Moreover, we take a motion μ with

$$\mu\,(v_1) = v_2,\ \mu\,(p_1) = p_2.$$

Hence $\mu\,(R_1) = R_2$ and

$$\mu\Big(B\,(c_1, \text{hyp}\ (c_1, v_1))\Big) = B\,\Big(\mu\,(c_1),\ \text{hyp}\ \big(\mu\,(c_1),\, v_2\big)\Big)$$

for all $c_1 \in R_1$ with hyp $(c_1, v_1) \to \infty$, i.e. for all $c_2 := \mu\,(c_1) \in R_2$ with hyp $(c_2, v_2) \to \infty$. Thus $\mu\,(H^1)$ and H^2 coincide.

2.11 Geometrical subspaces

If $S \neq \emptyset$ is a set of hyperplanes of (X, d), i.e. of $(X,$ eucl$)$ or $(X,$ hyp$)$, the intersection

$$\Sigma = \bigcap_{H \in S} H$$

will be called a *geometrical subspace* of (X, d). In this case we often will write $\Sigma \in \Gamma\,(X, d)$. We also define $X \in \Gamma\,(X, d)$. Let $a \neq 0$ be an element of X. Because of

$$H\,(a, 0) \cap H\,(a, 1) = \emptyset,$$

we obtain $\emptyset \in \Gamma\,(X,\ \text{eucl})$. Similarly,

$$H\,(a, 0) \cap \Pi\,(a, 1) = \emptyset,$$

i.e. $\emptyset \in \Gamma\,(X,\ \text{hyp})$. If $\Sigma \notin \{\emptyset, X\}$ is in $\Gamma\,(X, d)$, let μ be a motion with $\mu\,(p) = 0$ for a fixed element p of Σ. Hence $\mu\,(\Sigma)$ is an intersection

$$\mu\,(\Sigma) = \bigcap_{a \in S} a^\perp$$

with $0 \notin S \subseteq X$. Observing $0^\perp = X$, we obtain

Proposition 22. *All geometrical subspaces of* (X, d) *are given by* $\emptyset$*, moreover by*

$$\bigcap_{a \in S} a^{\perp},\ \emptyset \neq S \subseteq X,$$

and their images under motions.

We would like to show $D := \bigcap_{a \in X} a^{\perp} = \{0\}$. In fact! If $p \neq 0$ were in D, then $p \in a^{\perp}$ for all $a \in X$ would imply $p \in p^{\perp}$, i.e. $p^2 = 0$, i.e. $p = 0$, a contradiction. Hence, by Proposition 22, every set consisting of one single point is in $\Gamma(X, d)$.

Proposition 23. *Let* V, $\dim V \geq 1$, *be a finite-dimensional subspace* $\neq X$ *of the vector space* X*. Then the images of* V *under motions are in* $\Gamma(X, d)$*. So especially the lines of* (X, d) *are geometrical subspaces.*

Proof. Let $I(V)$ be the intersection of all hyperplanes containing the finite-dimensional subspace V, $\dim V \geq 1$, of the vector space X. Of course, we assume $n := \dim V < \dim X$ in the case that X is finite-dimensional. Hence $V \subseteq I(V)$. As a matter of fact, even $V = I(V)$ holds true. So assume there would exist

$$r \in I(V) \backslash V. \tag{2.37}$$

Let $b_1, \dots, b_n$ be a basis of V satisfying

$$b_i b_j = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{cases},$$

$i, j \in \{1, \dots, n\}$. Notice $V = \{\sum_{i=1}^{n} \xi_i b_i \mid \xi_i \in \mathbb{R}\} \ni 0$ and put

$$z := \sum_{i=1}^{n} (r b_i)\, b_i.$$

Obviously, $(r - z)\, b_i = 0$, $i = 1, \dots, n$, and $z \neq r$ since $z \in V$ and $r \notin V$, by (2.37). Hence $V \subseteq H(r - z, 0)$, i.e. $r \in H(r - z, 0)$, by (2.37), and thus $(r - z)\, r = 0$. We obtain, by $z \in V$, i.e. by $z \in H(r - z, 0)$,

$$(r - z)^2 = (r - z)\, r - (r - z)\, z = 0,$$

i.e. $r = z \in V$, a contradiction. Hence $V = I(V)$. Thus V must be a geometrical subspace of (X, d). Now apply Proposition 22. $\square$

The geometrical subspaces as described in Proposition 23 are given in the case $(X,\ \text{hyp})$ as follows. Let $p \in X$, $\gamma \in \mathbb{R}$ satisfy $p^2 = 1$ and $\gamma \geq 0$. Suppose that W, $n := \dim W \geq 1$, is a finite-dimensional subspace of the vector space $p^{\perp}$. Then

$$\{\gamma p \cosh \xi + y \sinh \xi \mid \xi \in \mathbb{R},\ y \in W \text{ with } y^2 = 1\}$$

will be called an n-dimensional (geometrical) subspace of $(X,\ \text{hyp})$.

Not every subspace of (X, d) for $d =$ eucl or $d =$ hyp needs to be a geometrical subspace. Assume that $Q \ni 0$ is a quasi–hyperplane which is not a hyperplane. If

$$Q \subseteq H\,(a, 0), \tag{2.38}$$

$a \neq 0$, holds true, then $Q = H\,(a, 0)$ or $H\,(a, 0) = X$, since Q is a maximal subspace of X. Hence (2.38) is impossible and, as a consequence, Q cannot be a geometrical subspace of (X, d).

Other interesting geometrical subspaces occur in the case that X is not finite-dimensional, in the form

$$\bigcap_{i=1}^{n} a_i^{\perp},$$

where n is a positive integer and where $a_1, \dots, a_n \in X$ are linearly independent, satisfying $a_i^2 = 1$ for $i = 1, \dots, n$. It will be easy to prove:

$$\bigcap_{i=1}^{n} a_i^{\perp} = \left\{ x - \sum_{i=1}^{n} \alpha_i a_i \mid x \in X \text{ and } M \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} xa_1 \\ \vdots \\ xa_n \end{pmatrix} \right\},$$

where M is given by the regular matrix

$$M = \begin{pmatrix} a_1^2 & a_1 a_2 & \dots & a_1 a_n \\ a_2 a_1 & a_2^2 & \dots & a_2 a_n \\ \vdots & & & \\ a_n a_1 & a_n a_2 & \dots & a_n^2 \end{pmatrix}$$

(det M is called Gram's determinant). In fact, take an element $x \in X$ with the described $\alpha_1, \dots, \alpha_n$. Hence

$$\begin{pmatrix} (x - \Sigma\alpha_i a_i) & a_1 \\ \vdots & \\ (x - \Sigma\alpha_i a_i) & a_n \end{pmatrix} = \begin{pmatrix} xa_1 \\ \vdots \\ xa_n \end{pmatrix} - M \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0,$$

i.e. $(x - \Sigma\alpha_i a_i)\, a_j = 0$ for $j = 1, \dots, n$, i.e. $x - \Sigma\alpha_i a_i \in \bigcap_{j=1}^{n} a_j^{\perp}$.

If, on the other hand, $x \in \bigcap_{i=1}^{n} a_i^{\perp}$ holds true, $xa_j = 0$ is satisfied for $j = 1, \dots, n$. From

$$M \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} xa_1 \\ \vdots \\ xa_n \end{pmatrix} = 0,$$

we then obtain $\alpha_1 = \dots = \alpha_n = 0$. Hence x has the required form

$$x - \sum_{i=1}^{n} \alpha_i a_i.$$

2.12 The Cayley–Klein model

The Weierstrass map $w : X \to X$,

$$w\,(x) := \frac{x}{\sqrt{1+x^2}}, \tag{2.39}$$

is a bijection between X and $P := \{x \in X \mid x^2 < 1\}$. In view of

$$[w\,(x)]^2 = \frac{x^2}{1+x^2} < 1,$$

we obtain $w\,(x) \in P$ for $x \in X$. Moreover,

$$w^{-1}(x) = \frac{x}{\sqrt{1-x^2}} \in X$$

is the uniquely determined $y \in X$ satisfying $w\,(y) = x$ for $x \in P$. Defining

$$g\,(x,y) := \text{hyp}\,\left(w^{-1}(x),\, w^{-1}(y)\right)$$

for $x, y \in P$, we get

$$\cosh g\,(x,y) = \frac{1-xy}{\sqrt{1-x^2}\,\sqrt{1-y^2}}. \tag{2.40}$$

If $x \neq y$ are elements of P, then $(x-y)^2 > 0$, $1-x^2 > 0$ and hence

$$D := [x\,(x-y)]^2 + (1-x^2)(x-y)^2 > 0.$$

Put $\{a,b\} := \{x + \xi\,(y-x) \mid \xi \in \mathbb{R}\} \cap \{z \in X \mid z^2 = 1\}$, i.e. put

$$\{a,b\} := \{x + \xi_1(y-x),\, x + \xi_2(y-x)\}$$

with $\{(x-y)^2\xi_1,\, (x-y)^2\xi_2\} = \{x\,(x-y) \pm \sqrt{D}\}$. We now would like to determine

$$|\ln\{a,b;x,y\}|,$$

where $\ln\xi$ for $0 < \xi \in \mathbb{R}$ is defined by the real number η satisfying $\exp\,(\eta) = \xi$, and where

$$\{z_1,z_2;z_3,z_4\} := \frac{\lambda_1-\lambda_3}{\lambda_1-\lambda_4} : \frac{\lambda_2-\lambda_3}{\lambda_2-\lambda_4}$$

designates the *cross ratio* of the ordered quadruple z_1, z_2, z_3, z_4 of four distinct points

$$z_i = p + \lambda_i q,\ i = 1,2,3,4,$$

on the line $p + \mathbb{R}q$, $q \neq 0$, which does not depend on the representation of the line

$$p + \mathbb{R}q = p' + \mathbb{R}q'.$$

Writing the points a, b, x, y of $\{a, b; x, y\}$ in the form

$$x + \xi\,(y - x),$$

we obtain $0 < \{a, b; x, y\} \in \{L, L^{-1}\}$ with

$$L := \frac{\xi_1}{\xi_1 - 1} \cdot \frac{\xi_2 - 1}{\xi_2}.$$

Observe here $\xi_1 < 0$ and $1 < \xi_2$ or $\xi_2 < 0$ and $1 < \xi_1$. The exact value L or L^{-1} of $\{a, b; x, y\}$ depends on how we associate a, b to ξ_1, ξ_2. Put

$$(x - y)^2 \xi_i = x\,(x - y) + \varepsilon_i \sqrt{D},$$

$i = 1, 2$, with $\varepsilon_1 = -\varepsilon_2 = 1$. Then we get

$$\frac{1}{2} \ln L = \ln \frac{1 - xy + \sqrt{D}}{\sqrt{1 - x^2}\,\sqrt{1 - y^2}} =: R$$

by observing $xy \le |xy| \le \sqrt{x^2}\,\sqrt{y^2} < 1$. Because of $|\ln L^{-1}| = |\ln L|$, we obtain

$$\frac{1}{2} |\ln \{a, b; x, y\}| = |R|, \tag{2.41}$$

independent of how we associate a, b to ξ_1, ξ_2. Now, by (2.40),

$$\cosh |R| = \frac{e^R + e^{-R}}{2} = \frac{1 - xy}{\sqrt{1 - x^2}\,\sqrt{1 - y^2}} = \cosh g\,(x, y),$$

since $1 - xy + \sqrt{D} = e^R \sqrt{1 - x^2}\,\sqrt{1 - y^2}$. Hence, by (2.41),

$$g\,(x, y) = \frac{1}{2} |\ln \{a, b; x, y\}|. \tag{2.42}$$

It is certainly more convenient to work with the expression (2.40) than with (2.42), since there the elements a, b must be determined before $g\,(x, y)$ can be calculated.

We now would like to look to different notions like translation and hyperplane as they appear in the Cayley–Klein model.

If ω is a surjective orthogonal mapping, i.e. a bijective orthogonal mapping, we obtain

$$w\,(x) = \frac{x}{\sqrt{1 + x^2}},\; w\big(\omega\,(x)\big) = \frac{\omega\,(x)}{\sqrt{1 + [\omega\,(x)]^2}} = \omega\big(w\,(x)\big),$$

since $[\omega\,(x)]^2 = x^2$. Hence, if $x \in X$ goes over in $\omega\,(x)$, then $w\,(x)$ in $w\omega\,(x) = \omega w\,(x)$. Thus ω remains an orthogonal mapping, however, restricted on P. If $x \in X$ goes over in $T_t(x)$, then $w\,(x)$ in

$$w\big(T_t(x)\big) = wT_t w^{-1}\big(w\,(x)\big).$$

Thus the translation T_t, say, based on $e \in X$, $e^2 = 1$, as axis, corresponds to

$$z \to wT_t w^{-1}(z) =: T'_t(z)$$

for all $z \in P$. This implies for $z \in P$,

$$T'_t(z) = \frac{z + [(ze)(\cosh t - 1) + \sinh t]\, e}{\cosh t + (ze) \sinh t}, \tag{2.43}$$

by observing $\cosh t + (ze) \sinh t > 0$, which holds, since

$$-(ze) \tanh t \geq 1$$

would contradict

$$|-(ze) \tanh t| \leq \sqrt{z^2}\sqrt{e^2} \cdot 1 < 1.$$

Notice

$$T'_t T'_s = wT_t w^{-1} \cdot wT_s w^{-1} = wT_{t+s} w^{-1} = T'_{t+s},$$

and also $T'_0 = \mathrm{id}$ on P, and $T'_t T'_{-t} = T'_0$. If

$$\{p \cosh \xi + q \sinh \xi \mid \xi \in \mathbb{R}\},\ pq = 0,\ q^2 = 1, \tag{2.44}$$

is a line, we obtain its image in P as the set

$$\left\{ \frac{p}{\sqrt{1+p^2}} + \frac{q}{\sqrt{1+p^2}} \tanh \xi \mid \xi \in \mathbb{R} \right\}. \tag{2.45}$$

This is the segment of the euclidean ball $B\,(0,1)$ connecting its points

$$\frac{p}{\sqrt{1+p^2}} - \frac{q}{\sqrt{1+p^2}} \text{ and } \frac{p}{\sqrt{1+p^2}} + \frac{q}{\sqrt{1+p^2}}. \tag{2.46}$$

If $u \neq v$ are points on $B\,(0,1)$, i.e. if they satisfy $u^2 = 1 = v^2$, then $\left(\frac{v+u}{2}\right)^2 < 1$, and (2.44) with

$$p = \frac{v+u}{2k},\ q = \frac{v-u}{2k},\ k = \sqrt{1 - \left(\frac{v+u}{2}\right)^2} = \sqrt{\frac{1-uv}{2}} \tag{2.47}$$

is the inverse image of the segment $\{u + \lambda\,(v-u) \mid 0 < \lambda < 1\}$.

Obviously, the two ends of $\{p \cosh \xi + q \sinh \xi \mid \xi \in \mathbb{R}\}$ can be described by the points (2.46) of $B\,(0,1)$,

$$\frac{p \pm q}{\sqrt{1+p^2}}.$$

Proposition 24. *The image of* $\Pi\,(p,\gamma)$ $\big($*see* (2.27)$\big)$ *under the mapping* $w : X \to P$ *is given by*

$$\Pi'(p,\gamma) := \{x \in P \mid px = \frac{\gamma}{\sqrt{1+\gamma^2}}\}.$$

Proof. Because of

$$z := w\,(\gamma p \cosh \xi + y \sinh \xi) = \frac{\gamma p}{\sqrt{1+\gamma^2}} + \frac{y}{\sqrt{1+\gamma^2}} \tanh \xi$$

for $y \in p^\perp$, $y^2 = 1$, we get

$$pz = \frac{\gamma}{\sqrt{1+\gamma^2}}, \tag{2.48}$$

i.e. $z \in \Pi'(p,\gamma)$. Let now $z \in P$ be given satisfying (2.48). Hence, by $p^2 = 1$,

$$p\left(z - \frac{\gamma p}{\sqrt{1+\gamma^2}}\right) = 0,$$

i.e. $Y := z - \frac{\gamma p}{\sqrt{1+\gamma^2}} \in p^\perp$. For $Y = 0$ we get $w^{-1}(z) = \gamma p \in \Pi\,(p,\gamma)$. Suppose now $Y \neq 0$. We obtain, by $z \in P$ and $Y \perp p$,

$$0 < 1 - z^2 = 1 - \left(Y + \frac{\gamma p}{\sqrt{1+\gamma^2}}\right)^2 = \frac{1 - (1+\gamma^2)\,Y^2}{1+\gamma^2},$$

i.e. $0 < 1 - (1+\gamma^2)\,Y^2 < 1$. There hence exists $\xi > 0$ with

$$\cosh \xi = \frac{1}{\sqrt{1-(1+\gamma^2)\,Y^2}}.$$

This implies $\tanh \xi = \sqrt{1+\gamma^2} \cdot \|Y\|$. Put $y = \frac{1}{\|Y\|}\,Y$ and observe $y \in p^\perp$, $y^2 = 1$ and

$$z = \frac{\gamma p}{\sqrt{1+\gamma^2}} + \|Y\| \cdot y = \frac{\gamma p}{\sqrt{1+\gamma^2}} + \frac{y}{\sqrt{1+\gamma^2}} \tanh \xi,$$

i.e. $w^{-1}(z) = \gamma p \cosh \xi + y \sinh \xi \in \Pi\,(p,\gamma)$. □

Remark. Notice $\gamma p \cosh \xi + y \sinh \xi = \gamma p \cosh(-\xi) + (-y) \sinh(-\xi)$ and that $y \in p^\perp$, $y^2 = 1$ implies $(-y) \in p^\perp$, $(-y)^2 = 1$, so that, for instance, ξ could be chosen always non-negative. But in this case not all $y \in p^\perp$ with $y^2 = 1$ occur in the representation of $\Pi\,(p,\gamma)$.

Let $H\,(p,\alpha)$ be an arbitrary hyperplane of $(X,\ \text{eucl})$ with $p^2 = 1$. We will assume $\alpha \geq 0$, because otherwise we could work with $H\,(-p,-\alpha)$. If there is at least one point a in

$$H\,(p,\alpha) \cap B\,(0,1), \tag{2.49}$$

then $0 \leq \alpha \leq 1$: this follows from $pa = \alpha$ and $a^2 = 1$ by means of $\alpha \geq 0$ and

$$\alpha = pa \leq |pa| \leq \sqrt{p^2}\,\sqrt{a^2} = 1.$$

The intersection (2.49) contains exactly one point if, and only if, $\alpha = 1$. For $\alpha = 1$ we only have p in this intersection, since $px = 1$, $x^2 = 1$ imply

$$1 = px = |px| = \sqrt{p^2}\,\sqrt{x^2},$$

i.e. $x \in \{p, -p\}$, i.e. $x = p$ because of $p\,(-p) = -1$. If $0 \leq \alpha < 1$, take $r \in X$ with $pr = 0$ and $r^2 = 1$. Then

$$p\,(\alpha p \pm \sqrt{1-\alpha^2}\,r) = \alpha,\ (\alpha p \pm \sqrt{1-\alpha^2}\,r)^2 = 1$$

lead to distinct points in (2.49).

Proposition 25. *If $p \in X$ satisfies $p^2 = 1$ and if $0 \leq \alpha < 1$ holds true for $\alpha \in \mathbb{R}$, then $\Pi\,(p,\gamma)$ with*

$$\gamma = \frac{\alpha}{\sqrt{1-\alpha^2}}$$

is the image of $\{x \in P \mid px = \alpha\}$ under $w^{-1} : P \to X$.

Proof. The proof follows from Proposition 24, in view of $\frac{\gamma}{\sqrt{1+\gamma^2}} = \alpha$. □

2.13 Hyperplanes under translations

Theorem 26. *Let $t, \gamma \in \mathbb{R}$ be given with $\gamma \geq 0$ and $p \in X$ with $p^2 = 1$. Suppose that*

$$T_t(x) = x + \big((xe)(\cosh t - 1) + \sqrt{1+x^2}\,\sinh t\big)\,e,$$

$x \in X$, is a hyperbolic translation based on the axis e, $e^2 = 1$. Define

$$\begin{aligned} \gamma' &:= \gamma\cosh t + (pe)\sqrt{1+\gamma^2}\,\sinh t, \\ \varepsilon &:= \operatorname{sgn}\gamma' \text{ for } \gamma' \neq 0,\ \varepsilon \in \mathbb{R}\backslash\{0\} \text{ for } \gamma' = 0, \end{aligned}$$

and $p' \cdot \|A\| := A := p + \left[\frac{\gamma}{\sqrt{1+\gamma^2}}\,\sinh t + (pe)(\cosh t - 1)\right] e$ by observing $A \neq 0$. Then

$$T_t\big(\Pi\,(p,\gamma)\big) = \Pi\,(\varepsilon p',\,|\gamma'|) \tag{2.50}$$

holds true.

Proof. Notice $A^2 = \frac{1}{1+\gamma^2} + \left(\frac{\gamma}{\sqrt{1+\gamma^2}}\cosh t + (pe)\sinh t\right)^2 > 0$, i.e. $A \neq 0$ and, moreover,

$$\sqrt{1+\gamma'^2} = \|A\| \cdot \sqrt{1+\gamma^2}. \tag{2.51}$$

Instead of (2.50) we prove

$$w\left[\Pi\left(\varepsilon p',\, |\gamma'|\right)\right] = wT_t w^{-1}\left[\left\{x \in P \mid px = \frac{\gamma}{\sqrt{1+\gamma^2}}\right\}\right], \tag{2.52}$$

since, by Proposition 25,

$$w^{-1}\left(\left\{x \in P \mid px = \frac{\gamma}{\sqrt{1+\gamma^2}}\right\}\right) = \Pi\left(p, \gamma\right).$$

From Proposition 24 we obtain

$$w\left[\Pi\left(\varepsilon p',\, |\gamma'|\right)\right] = \left\{x \in P \mid p'x = \frac{\gamma'}{\sqrt{1+\gamma'^2}}\right\}, \tag{2.53}$$

since $\varepsilon p'x = \frac{|\gamma'|}{\sqrt{1+\gamma'^2}}$ can be rewritten as $p'x = \frac{\gamma'}{\sqrt{1+\gamma'^2}}$. So in order to prove (2.50), we show, with (2.43),

$$\left\{x \in P \mid p'x = \frac{\gamma'}{\sqrt{1+\gamma'^2}}\right\} = T_t'\left[\left\{z \in P \mid pz = \frac{\gamma}{\sqrt{1+\gamma^2}}\right\}\right]. \tag{2.54}$$

Applying the decomposition $X = p^\perp \oplus \mathbb{R}p$, we will write

$$z = \overline{z} + z_0 p,\ \overline{z} \in p^\perp,\ z_0 \in \mathbb{R},$$

for $z \in X$. Hence

$$\left\{z \in P \mid pz = \frac{\gamma}{\sqrt{1+\gamma^2}}\right\} = \left\{\overline{z} + \frac{\gamma}{\sqrt{1+\gamma^2}}\, p \mid \overline{z} \in p^\perp,\ \|\overline{z}\| < \frac{1}{\sqrt{1+\gamma^2}}\right\}.$$

T_t' is a bijection of P. With $\alpha := \frac{\gamma}{\sqrt{1+\gamma^2}}$, we obtain

$$T_t'\left[\left\{\overline{z} + \alpha p \mid \overline{z} \in p^\perp,\ \|\overline{z}\| < \sqrt{1-\alpha^2}\right\}\right],$$

by (2.43), as the set of all points

$$u\left(z\right) := \frac{\overline{z} + \alpha p + \left[(\overline{z}e + \alpha pe)(\cosh t - 1) + \sinh t\right] e}{\cosh t + (\overline{z}e + \alpha pe)\sinh t} \tag{2.55}$$

with $\|\overline{z}\|^2 < 1 - \alpha^2$. In view of (2.54), we will show

$$p' \cdot u\left(z\right) = \frac{\gamma'}{\sqrt{1+\gamma'^2}},$$

i.e. by (2.51),

$$A \cdot u(z) = \frac{\gamma' \cdot \|A\|}{\sqrt{1+\gamma'^2}} = \frac{\gamma'}{\sqrt{1+\gamma^2}}.$$

Calling the nominator, denominator of the right-hand side of (2.55) $N(z)$, $D(z)$, respectively, the equation

$$A \cdot N(z) = D(z) \cdot (\alpha \cosh t + (pe) \sinh t)$$

must be verified, which can be accomplished easily. Observe, finally, that T_t maps hyperbolic hyperplanes onto such hyperplanes, and that consequently T_t' maps images (under w) of hyperbolic hyperplanes of X onto images of such hyperplanes. □

Proposition 27. *Let $p, q \in X$ and $\gamma, \delta \in \mathbb{R}$ be given with $p^2 = 1 = q^2$, $\gamma \geq 0$ and $\delta \geq 0$. If*

$$\Pi(p, \gamma) \subseteq \Pi(q, \delta) \tag{2.56}$$

and $\gamma > 0$ hold true, then $p = q$ and $\gamma = \delta$. If (2.56) *and $\gamma = 0$ hold true, then $p = \pm q$ and $\delta = 0$.*

Proof. Instead of (2.56) we will consider $w\big(\Pi(p,\gamma)\big) \subseteq w\big(\Pi(q,\delta)\big)$, i.e.

$$L := \left\{ x \in P \mid px = \frac{\gamma}{\sqrt{1+\gamma^2}} \right\} \subseteq \left\{ x \in P \mid qx = \frac{\delta}{\sqrt{1+\delta^2}} \right\} =: R.$$

If $v \neq 0$ is in $p^\perp$, we obtain, by $\frac{1/2}{1+\gamma^2} + \frac{\gamma^2}{1+\gamma^2} < 1$,

$$\pm \frac{v}{\|v\| \sqrt{2(1+\gamma^2)}} + \frac{\gamma p}{\sqrt{1+\gamma^2}} \in L.$$

These two points x_1, x_2 must hence be elements of R, i.e. $q \cdot (x_1 - x_2) = 0$, i.e. $v \in q^\perp$. Thus $p^\perp \subseteq q^\perp$, i.e. $H(p,0) \subseteq H(q,0)$, i.e., by Proposition 12 we get $p = q$ or $p = -q$. Now

$$\frac{\gamma p}{\sqrt{1+\gamma^2}} \in L \subseteq R \text{ implies } \frac{\gamma pq}{\sqrt{1+\gamma^2}} = \frac{\delta}{\sqrt{1+\delta^2}}.$$

Hence, if $\gamma > 0$, we obtain $p = q$, i.e. $\gamma = \delta$, and if $\gamma = 0$, $p = \pm q$ and $\delta = 0$ is the consequence. □

2.14 Lines under translations

Let $\{p \cosh \xi + q \sinh \xi \mid \xi \in \mathbb{R}\}$ be a hyperbolic line l with elements $p, q \in X$ satisfying $pq = 0$, $q^2 = 1$. For $\omega \in O(X)$ we obtain

$$\omega(l) = \{\omega(p) \cosh \xi + \omega(q) \sinh \xi \mid \xi \in \mathbb{R}\} \tag{2.57}$$

with $0 = pq = \omega(p)\,\omega(q)$, $1 = qq = \omega(q)\,\omega(q)$. Suppose that $e \in X$, $t \in \mathbb{R}$ are given with $e^2 = 1$. We are then interested in the image $T_t(l)$ of l under the hyperbolic translation T_t with axis e.

Theorem 28. *Define*

$$\begin{aligned} p' &:= p + [pe\,(\cosh t - 1) + \sqrt{1+p^2}\,\sinh t]\,e, \\ q' &:= q + (qe)(\cosh t - 1)\,e, \\ A &:= qe \sinh t, \\ B &:= pe \sinh t + \sqrt{1+p^2}\,\cosh t. \end{aligned}$$

Then $|A| < B$. *Define* $\alpha \in \mathbb{R}$ *by* $B \cdot \tanh \alpha := -A$. *Then*

$$T_t(l) = \{p^* \cosh \eta + q^* \sinh \eta \mid \eta \in \mathbb{R}\} \tag{2.58}$$

with

$$\begin{aligned} p^* &:= p' \cosh \alpha + q' \sinh \alpha, \\ q^* &:= p' \sinh \alpha + q' \cosh \alpha \end{aligned}$$

and $p^* q^* = 0$, $(q^*)^2 = 1$.

Proof. Observe

$$p'^2 = B^2 - 1,$$

i.e. $B^2 \geq 1$ because of $p'^2 \geq 0$, and $q'^2 = 1 + A^2$, $p'q' = AB$. We now would like to prove

$$|A| < B.$$

Case $A \geq 0$. Here we get

$$(q-p)\,e \sinh t \leq |(q-p)\,e \sinh t| \leq \sqrt{(q-p)^2 e^2}\,\sinh |t|,$$

i.e. $(q-p)\,e \sinh t \leq \sqrt{1+p^2}\,\sinh |t| < \sqrt{1+p^2}\,\cosh t$.
Case $A < 0$. We must prove $-A < B$. Observe

$$(-q-p)\,e \sinh t \leq |(q+p)\,e \sinh t| \leq \sqrt{1+p^2}\,\sinh |t| < \sqrt{1+p^2}\,\cosh t.$$

Because of $B^2 \geq 1$ and $|A| < B$, we obtain $B \geq 1$ and

$$\left| -\frac{A}{B} \right| < 1,$$

i.e. $\tanh \alpha = -\frac{A}{B}$ determines $\alpha \in \mathbb{R}$ uniquely. Hence

$$\sinh \alpha = -\frac{A}{\sqrt{B^2 - A^2}},\ \cosh \alpha = \frac{B}{\sqrt{B^2 - A^2}}.$$

In view of $p'^2 = B^2 - 1$, $p'q' = AB$, $q'^2 = A^2 + 1$, we thus obtain

$$\begin{aligned} (q^*)^2 &= (B^2-1)\tfrac{A^2}{B^2-A^2} - 2AB\,\tfrac{AB}{B^2-A^2} + (A^2+1)\tfrac{B^2}{B^2-A^2} = 1, \\ p^*q^* &= -(A^2+B^2)\tfrac{AB}{B^2-A^2} + AB\,\tfrac{A^2+B^2}{B^2-A^2} = 0. \end{aligned}$$

Notice

$$T_t(p\cosh\xi + q\sinh\xi) = p'\cosh\xi + q'\sinh\xi,$$

and put $\eta := \xi - \alpha$. Then

$$p'\cosh\xi + q'\sinh\xi = p^*\cosh\eta + q^*\sinh\eta,$$

by $\cosh\xi = \cosh\alpha\cosh\eta + \sinh\alpha\sinh\eta$ and $\sinh\xi = \sinh\alpha\cosh\eta + \dots$. Hence

$$T_t(l) = \{p^*\cosh\eta + q^*\sinh\eta \mid \eta \in \mathbb{R}\}. \qquad \square$$

2.15 Hyperbolic coordinates

Let $n \geq 2$ be an integer and suppose that V is a subspace of dimension n of the vector space X. Let $b_1, \dots, b_n$ be a basis of V satisfying $b_ib_j = 0$ for $i \neq j$ and $b_i^2 = 1$ for all $i, j \in \{1, \dots, n\}$. If $p \in V$ and if

$$p = p_1b_1 + \cdots + p_nb_n$$

holds true with $p_1, \dots, p_n \in \mathbb{R}$, then $(p_1, \dots, p_n)$ will be called the *cartesian coordinates* of p, and $(x_1, \dots, x_n)$ with

$$\begin{aligned} \sqrt{1+p_2^2+\cdots+p_n^2}\,\sinh x_1 &= p_1, \\ \sqrt{1+p_3^2+\cdots+p_n^2}\,\sinh x_2 &= p_2, \\ &\vdots \\ \sqrt{1+p_n^2}\,\sinh x_{n-1} &= p_{n-1}, \\ \sinh x_n &= p_n, \end{aligned}$$

its *hyperbolic coordinates*. Designate by π the mapping which associates for every $p \in V$ to the cartesian coordinates $(p_1, \dots, p_n)$ of p its hyperbolic coordinates

$$\pi\,(p_1, \dots, p_n) = (x_1, \dots, x_n).$$

The mapping π is bijective, because $\pi^{-1}(x_1, \dots, x_n)$ is given by $(p_1, \dots, p_n)$ with

$$\begin{aligned} p_n &= \sinh x_n, \\ p_{n-1} &= \sinh x_{n-1}\cdot\cosh x_n, \\ p_{n-2} &= \sinh x_{n-2}\cdot\cosh x_{n-1}\cdot\cosh x_n, \\ &\vdots \\ p_1 &= \sinh x_1\cdot\cosh x_2\cdot\cosh x_3\cdots\cosh x_n. \end{aligned}$$

Proposition 29. *Let $e \in V$ satisfy $e^2 = 1$. Extend $e =: b_1$ to a basis $b_1, \dots, b_n$ of V, again with*

$$b_i b_j = \begin{cases} 0 & \text{for} \quad i \neq j \\ 1 & \text{for} \quad i = j \end{cases}$$

for all $i, j \in \{1, \dots, n\}$. Representing then the points of V by hyperbolic coordinates $(x_1, \dots, x_n)$,

$$T_t(x_1, \dots, x_n) = (x_1 + t, x_2, \dots, x_n)$$

holds true for all $t \in \mathbb{R}$ and for all points of V in hyperbolic coordinates $(x_1, \dots, x_n)$, where T_t are hyperbolic translations with axis e.

Proof. Put $S_1 := \sinh x_1, C_1 := \cosh x_1$, and so on. Then

$$\begin{aligned} &T_t(x_1, \dots, x_n) = T_t(S_1 C_2 \dots C_n b_1 + S_2 C_3 \dots C_n b_2 + \dots + S_n b_n) \\ &= p + [pe\,(\cosh t - 1) + C_1 C_2 \dots C_n \sinh t]\, e \end{aligned}$$

with $p := S_1 C_2 \dots C_n b_1 + \dots + S_n b_n$. Put $S_t := \sinh t$, $C_t := \cosh t$. Then

$$\begin{aligned} T_t(x_1, \dots, x_n) \quad &= p + [S_1 C_2 \dots C_n (C_t - 1) + C_1 C_2 \dots C_n S_t]\, b_1 \\ &= p + (S_1 C_t - S_1 + C_1 S_t)\, C_2 C_3 \dots C_n b_1 \\ &= \sinh(x_1 + t)\, C_2 \dots C_n b_1 + S_2 C_3 \dots C_n b_2 + \dots + S_n b_n \\ &= (x_1 + t, x_2, x_3, \dots, x_n). \end{aligned}$$

□

2.16 All isometries of $(X,\ \text{eucl})$, $(X,\ \text{hyp})$

The mapping $f : S \to S$ of a metric space (S, d) will be called an *isometry* of (S, d) provided

$$d\big(f(x),\ f(y)\big) = d(x, y) \tag{2.59}$$

holds true for all $x, y \in S$.

Isometries are injective mappings since $x \neq y$ for $x, y \in S$ implies

$$0 \neq d(x, y) = d\big(f(x),\ f(y)\big),$$

i.e. $f(x) \neq f(y)$. However, isometries need not be surjective. In chapter 1 we presented an example of an orthogonal mapping ω which is not surjective. Because of

$$d\big(\omega(x),\ \omega(y)\big) = d(x, y),$$

$d(x, y) := \|x - y\|$, for all x, y of the underlying real inner product space X, this ω hence represents an isometry of the metric space $(X,\ \text{eucl})$ which is not surjective.

Surjective isometries of (S, d) are called *motions* of (S, d). Of course, the set of all motions of the metric space (S, d) is a group $M\,(S, d)$ under the permutation product. In view of I (see the proof of Theorem 7, chapter 1) we already know

$$M\,(X, d) = \{\alpha T_t \beta \mid \alpha, \beta \in O\,(X),\ t \in \mathbb{R}\} \tag{2.60}$$

for $(X,\ \text{eucl})$ or $(X,\ \text{hyp})$. Here and throughout section 2.3 T is the euclidean or hyperbolic translation group with a given axis $e \in X$, $e^2 = 1$, i.e.

$$T_t(x) = x + te \tag{2.61}$$

in the euclidean and

$$T_t(x) = x + [(xe)(\cosh t - 1) + \sqrt{1 + x^2}\,\sinh t]\,e \tag{2.62}$$

in the hyperbolic case for all $x \in X$.

The following statement now presents the set of all isometries of (X, d) in the euclidean or hyperbolic case.

Proposition 30. *The set of all isometries of (X, d) is given by*

$$I\,(X, d) = \{\alpha T_t \beta \mid \alpha \in O\,(X),\ \beta \in \tilde{O}\,(X),\ t \in \mathbb{R}\}, \tag{2.63}$$

where $\tilde{O}\,(X)$ designates the set of all orthogonal mappings of X.

Proof. Suppose that δ is an isometry of (X, d) and that $\delta\,(0) =: p$. Because of A (see the proof of Theorem 7 in chapter 1) there exists $\gamma \in O\,(X)$ with

$$\gamma\delta\,(0) = \|p\|e.$$

In view of property (T 2) of a translation group, there exists $t \in \mathbb{R}$ satisfying

$$T_t\gamma\delta\,(0) = 0.$$

The mapping $\varphi := T_t\gamma\delta$ preserves distances and it satisfies $\varphi\,(0) = 0$.

Euclidean case. Hence for all $x, y \in X$,

$$\|x - y\| = \|\varphi\,(x) - \varphi\,(y)\|.$$

Thus $\varphi \in \tilde{O}\,(X)$, in view of Proposition 3 of chapter 1. This implies

$$\delta = \gamma^{-1} T_{-t}\varphi$$

with $\gamma^{-1} \in O\,(X)$.

Hyperbolic case. $\text{hyp}\,(x, y) = \text{hyp}\,\big(\varphi\,(x), \varphi\,(y)\big)$ for all $x, y \in X$ implies

$$\sqrt{1 + x^2}\,\sqrt{1 + y^2} - xy = \sqrt{1 + \xi^2}\,\sqrt{1 + \eta^2} - \xi\eta \tag{2.64}$$

with $\xi := \varphi\,(x)$, $\eta := \varphi\,(y)$ and, especially for $x = 0$, $y = z$,

$$z^2 = [\varphi\,(z)]^2$$

for all $z \in X$, i.e., by (2.64), $xy = \varphi\,(x)\,\varphi\,(y)$ for all $x, y \in X$. Hence

$$\|x - y\| = \|\varphi\,(x) - \varphi\,(y)\|,$$

and thus $\varphi \in \tilde{O}\,(X)$, by Proposition 3, chapter 1. $\square$

2.17 Isometries preserving a direction

Let T be a translation group with axis $e \in X$, $e^2 = 1$. The following three statements hold true for hyperbolic as well as for euclidean geometry. The given proof of Lemma 31 is based on $(X,\ \mathrm{hyp})$.

Lemma 31. *Given $\alpha \in O(X)$ with $\alpha(e) = \varepsilon e$, $\varepsilon \in \mathbb{R}$. Then $\alpha T_t \alpha^{-1}(x) = T_{\varepsilon t}(x)$ for all $x \in X$ and $t \in \mathbb{R}$.*

Proof. $[\alpha(e)]^2 = e^2$ implies $\varepsilon^2 = 1$. With $\alpha^{-1}(e) = \varepsilon e$ and

$$x = h + x_0 e,\ h \in e^\perp,\ x_0 \in \mathbb{R},$$

we obtain $\alpha^{-1}(h)\,\alpha^{-1}(e) = he = 0$ and $\alpha^{-1}(h) \in e^\perp$, and hence

$$\begin{aligned}
\alpha T_t \alpha^{-1}(x) &= \alpha T_t(\alpha^{-1}(h) + x_0 \varepsilon e)\\
&= \alpha\left(\alpha^{-1}(h) + [x_0 \varepsilon \cosh t + \sqrt{1 + h^2 + x_0^2}\,\sinh t]\, e\right)\\
&= x + [(xe)(\cosh \varepsilon t - 1) + \sqrt{1 + x^2}\,\sinh \varepsilon t]\, e = T_{\varepsilon t}(x),
\end{aligned}$$

by $\alpha^{-1}(h)\,\alpha^{-1}(h) = h^2$ and (2.62). □

Corollary. *Define $\chi(x) = h - x_0 e$ for $x = h + x_0 e$ with $h \in e^\perp$ and $x_0 \in \mathbb{R}$. Then*

$$\chi\, T_t = T_{-t}\, \chi$$

for all $t \in \mathbb{R}$.

Proof. Notice $\chi \in O(X)$ and $\chi(e) = -e$. □

Theorem 32. *Suppose that $f : X \to X$ is an isometry. Then*

$$f(x) - x \in \mathbb{R}e \text{ for all } x \in X \tag{2.65}$$

holds true if, and only if, $f \in T \cup T\chi$.

Proof. Obviously, $f \in T \cup T\chi$ satisfies (2.65). Let now $f : X \to X$ be an isometry satisfaying (2.65).
Case 1: $f \in \tilde{O}(X)$. *Here $f = \mathrm{id}$ or $f = \chi$ holds true.* In order to prove this statement observe

$$f(e) - e \in \mathbb{R}e,$$

i.e. $f(e) = \lambda e$ with a suitable $\lambda \in \mathbb{R}$. Hence $e^2 = [f(e)]^2$, i.e. $\lambda^2 = 1$. Because of

$$0 = he = f(h)\, f(e) = f(h) \cdot \lambda e$$

for $h \in e^\perp$, we obtain $f(h) \in e^\perp$, i.e.

$$f(h + x_0 e) = f(h) + x_0 \lambda e,\ f(h) \in e^\perp, \tag{2.66}$$

for $x = h + x_0 e$, $h \in e^\perp$, $x_0 \in \mathbb{R}$. By (2.65)

$$f\,(h + x_0 e) = h + x_0 e + \mu e \tag{2.67}$$

with a suitable $\mu \in \mathbb{R}$. Hence, by (2.66), (2.67), $f\,(h) = h$, i.e. by (2.66),

$$f\,(h + x_0 e) = h + x_0 \lambda e.$$

Thus $f = \mathrm{id}$ for $\lambda = 1$, and $f = \chi$ for $\lambda = -1$.
Case 2: $f \notin \tilde{O}\,(X)$. Since $f \in I\,(X, d)$,

$$f = \alpha T_t \beta$$

with $\alpha \in O\,(X)$, $\beta \in \tilde{O}\,(X)$ and $t \in \mathbb{R}$. Because of $f \notin \tilde{O}\,(X)$, we obtain $t \neq 0$. Hence $T_t(0) \neq 0$. By (2.65), $\alpha T_t \beta\,(0) = \lambda e$ with a suitable $\lambda \in \mathbb{R}$. Thus

$$0 \neq T_t(0) = \lambda \alpha^{-1}(e),$$

which implies $\alpha^{-1}(e) = \varepsilon e$, $\varepsilon \in \mathbb{R}$, in view of $T_t(0) \in \mathbb{R}\,e$. So we obtain $\alpha\,(e) = \varepsilon e$ with $\varepsilon^2 = 1$, i.e. by Lemma 31,

$$f = \alpha T_t \alpha^{-1} \cdot \alpha\beta = T_{\varepsilon t} \cdot \gamma$$

with $\gamma := \alpha\beta \in \tilde{O}\,(X)$. Since $T_{-\varepsilon t}$ and $f = T_{\varepsilon t} \cdot \gamma$ have property (2.65), hence also their product γ. This implies, by Case 1, $\gamma = \mathrm{id}$ or $\gamma = \chi$. Thus $f \in T \cup T\chi$. □

2.18 A characterization of translations

The following Theorem 33 is essentially a corollary of Theorem 32.

Theorem 33. *An isometry f of (X, d) is a translation $\neq \mathrm{id}$ with axis e if, and only if,*

$$0 \neq f\,(x) - x \in \mathbb{R}e \tag{2.68}$$

holds true for all $x \in X$.

Proof. Suppose that the isometry $f : X \to X$ satisfies (2.68). Hence, by Theorem 32, $f \in T \cup T\chi$. We will show that $f = \mathrm{id}$ and also $f \in T\chi$ have at least one fixpoint, i.e. a point x with $f\,(x) = x$, i.e. with $0 = f\,(x) - x$, so that f must be a translation $\neq \mathrm{id}$ with axis e.
Hyperbolic case: Here $h + e\sqrt{1 + h^2}\,\sinh\frac{t}{2}$ with $h \in e^\perp$ is a fixpoint of $T_t\chi$.
Euclidean case: Here $h + \frac{t}{2}\,e$ is a fixpoint of $T_t\chi$.
Suppose, vice versa, that the translation $T_t \neq \mathrm{id}$ has axis e. Then, of course, $f\,(x) - x \in \mathbb{R}e$ holds true for all $x \in X$. Property (T 2) of a translation group implies that $t = 0$ is a consequence of $T_t(x_0) = x_0$ for a point x_0. □

2.19 Different representations of isometries

Let again $e \in X$ be given with $e^2 = 1$ and suppose that T is the euclidean or hyperbolic translation group with axis e. Given isometries

$$\alpha T_t \beta,\ \gamma T_s \delta$$

of (X, d) with $\alpha, \gamma \in O\,(X)$, $\beta, \delta \in \tilde{O}\,(X)$, $t, s \in \mathbb{R}$, we would like to answer the question, when and only when $\alpha T_t \beta$ and $\gamma T_s \delta$ represent the same isometry.

Theorem 34. *Given* $(X, d) \in \{(X,\ \text{eucl}),\ (X,\ \text{hyp})\}$ *and*

$$\alpha, \gamma \in O\,(X),\ \beta, \delta \in \tilde{O}\,(X),\ t, s \in \mathbb{R}.$$

Then

$$\alpha T_t \beta = \gamma T_s \delta \tag{2.69}$$

holds true if, and only if,

$$t = s = 0,\ \alpha\beta = \gamma\delta \textit{ for } ts = 0,$$

or

$$0 \neq t = \varepsilon s,\ \varepsilon^2 = 1,\ \alpha\beta = \gamma\delta,\ \alpha\,(e) = \varepsilon\gamma\,(e) \textit{ for } ts \neq 0.$$

Proof. Of course, (2.69) holds true for $t = s = 0$, $\alpha\beta = \gamma\delta$, but also if all the presented conditions for $ts \neq 0$ are satisfied, by observing Lemma 31 and

$$\gamma^{-1}\alpha T_{\varepsilon s}\beta = \gamma^{-1}\alpha T_{\varepsilon s}(\gamma^{-1}\alpha)^{-1} \cdot \gamma^{-1}\alpha\beta = T_{\varepsilon^2 s} \cdot \delta.$$

Assume now (2.69), i.e. $\xi T_t \beta = T_s \delta$ with $\xi := \gamma^{-1}\alpha$. Because of

$$\xi T_t \beta\,(0) = T_s \delta\,(0)$$

we obtain

$$\xi\,(e) \cdot \sinh t = e \cdot \sinh s \tag{2.70}$$

and $\xi\,(e)\,\xi\,(e) = ee = 1$.

Case $ts = 0$. Hence, by (2.70), $t = s = 0$, and thus $\alpha\beta = \gamma\delta$, by (2.69).

Case $ts \neq 0$. Because of (2.70) and $\left(\xi\,(e)\right)^2 = 1$, we obtain $\xi\,(e) = \varepsilon e$, $\varepsilon^2 = 1$, $0 \neq t = \varepsilon s$. A consequence of $\xi = \gamma^{-1}\alpha$ then is $\alpha\,(e) = \varepsilon\gamma\,(e)$. Finally observe, by (2.69) and Lemma 31,

$$T_s\delta = \xi T_t \beta = \xi T_t \xi^{-1} \cdot \xi\beta = T_{\varepsilon t} \cdot \xi\beta,$$

i.e. $T_s\delta = T_s\xi\beta$, i.e. $\delta = \xi\beta$. $\square$

2.20 A characterization of isometries

Suppose that (X, d) is one of the metric spaces $(X,\ \text{eucl})$ or $(X,\ \text{hyp})$. Then the following theorem holds true.

Theorem 35. *Let $\varrho > 0$ be a fixed real number and $N > 1$ be a fixed integer. If $f : X \to X$ is a mapping satisfying*

$$d\big(f(x),\ f(y)\big) \leq \varrho \text{ for all } x, y \in X \text{ with } d(x,y) = \varrho, \tag{2.71}$$

$$d\big(f(x),\ f(y)\big) \geq N\varrho \text{ for all } x, y \in X \text{ with } d(x,y) = N\varrho, \tag{2.72}$$

then f must be an isometry of (X, d).

Proof. Euclidean case: $d(x,y) = \|x-y\|$ for all $x, y \in X$. In view of Theorem 4, chapter 1, we obtain

$$f(x) = \omega(x) + t$$

for all $x \in X$, where $\omega \in \tilde{O}(X)$, and t a fixed element of X. Obviously, f satisfies (2.59).

Hyperbolic case: $d(x,y) = \text{hyp}\ (x,y)$ for all $x, y \in X$.

1. *The mapping f preserves hyperbolic distances ϱ and 2ϱ.*

Proof. Let p, q be points of (hyperbolic) distance ϱ, and

$$x(\xi) = a\cosh\xi + b\sinh\xi,\ \xi \in \mathbb{R},$$

with $a, b \in X$, $ab = 0$, $b^2 = 1$, be the line through p, q. If $p = x(\alpha)$, $q = x(\beta)$, then $|\beta - \alpha| = \varrho$. We may assume $\beta - \alpha = \varrho$, since otherwise we would work with $y(\xi) := x(-\xi)$ instead of $x(\xi)$, and $\alpha' := -\alpha$, $\beta' := -\beta$ instead of α, β. Hence

$$p = x(\alpha) \text{ and } q = x(\alpha + \varrho).$$

Define

$$x_\lambda := x(\alpha + \lambda\varrho),\ \lambda \in \{0, 1, \ldots, N\}.$$

Since hyp $(x_0, x_N) = N\varrho$, (2.72) implies

$$\text{hyp}\ (x'_0, x'_N) \geq N\varrho$$

with $x' := f(x)$ for $x \in X$. Observe

$$\text{hyp}\ (x'_\lambda,\ x'_{\lambda+1}) \leq \varrho$$

for $\lambda = 0, \ldots, N-1$, in view of hyp $(x_\lambda, x_{\lambda+1}) = \varrho$. Hence

$$\begin{aligned} N\varrho &\leq \text{hyp}\ (x'_0, x'_N) \leq \text{hyp}\ (x'_0, x'_2) + \sum_{\lambda=2}^{N-1} \text{hyp}\ (x'_\lambda, x'_{\lambda+1}) \\ &\leq \text{hyp}\ (x'_0, x'_1) + \text{hyp}\ (x'_1, x'_2) + \sum_{\lambda=2}^{N-1} \text{hyp}\ (x'_\lambda, x'_{\lambda+1}) \leq N\varrho. \end{aligned}$$

This yields hyp $(x'_\lambda, x'_{\lambda+1}) = \varrho$ and hyp $(x'_0, x'_2) = 2\varrho$. Hence hyp $(p', q') = \varrho$. If p, r are points of distance 2ϱ, we may write

$$p = x(\alpha) \text{ and } r = x(\alpha + 2\varrho).$$

Working now with $q := x(\alpha + \varrho)$, the proof above leads to $2\varrho = \text{hyp}\,(x'_0, x'_2) = \text{hyp}\,(p', r')$. □

2. *If a, b, m are points with $a \neq b$ and*

$$\text{hyp}\,(a, m) = \text{hyp}\,(m, b) = \frac{1}{2}\,\text{hyp}\,(a, b), \tag{2.73}$$

then m must be the hyperbolic midpoint of a, b.

Proof. If $x(\xi) = p\cosh\xi + q\sinh\xi$, $\xi \in \mathbb{R}$, with $pq = 0$, $q^2 = 1$, contains a, b, we may write $\alpha < \beta$,

$$a = x(\alpha) \text{ and } b = x(\beta).$$

Equation (2.73) implies

$$\text{hyp}\,(a, b) = \text{hyp}\,(a, m) + \text{hyp}\,(m, b),$$

and hence that a, b, m are collinear, i.e. on a common line (see the notion of a Menger line). Put $m = x(\gamma)$. By (2.73), we obtain

$$|\gamma - \alpha| = |\beta - \gamma|,$$

i.e. $\gamma = \frac{1}{2}(\alpha + \beta)$, in view of $\alpha < \beta$. □

3. *Given points p, q of distance ϱ, we will write*

$$p = x(\alpha) \textit{ and } q = x(\alpha + \varrho).$$

If $y(\eta)$, $\eta \in \mathbb{R}$, is the line through $p' := f(p)$, $q' := f(q)$, we may write, by step 1,

$$p' = y(\beta),\ q' = y(\beta + \varrho).$$

Then

$$f\big(x(\alpha + \lambda\varrho)\big) = y(\beta + \lambda\varrho) \tag{2.74}$$

holds true for all integers $\lambda \geq 0$.

Proof. Clear for $\lambda \in \{0, 1\}$. Put $p_\lambda := x(\alpha + \lambda\varrho)$. Now

$$\varrho = \text{hyp}\,(p_{\lambda-1}, p_\lambda) = \text{hyp}\,(p_\lambda, p_{\lambda+1}) = \frac{1}{2}\,\text{hyp}\,(p_{\lambda-1}, p_{\lambda+1})$$

and step 1 imply

$$\varrho = \text{hyp}\,(p'_{\lambda-1}, p'_\lambda) = \text{hyp}\,(p'_\lambda, p'_{\lambda+1}) = \frac{1}{2}\,\text{hyp}\,(p'_{\lambda-1}, p'_{\lambda+1}) \tag{2.75}$$

for $\lambda = 1, 2, 3, \ldots$. Assume that (2.74) is proved up to $\lambda \geq 1$. Because of (2.75), the points $p'_{\lambda-1}, p'_\lambda, p'_{\lambda+1}$ must be collinear. Put $p'_{\lambda+1} = y(\gamma)$. Hence

$$\beta + \lambda\varrho = \frac{1}{2}\left(\beta + (\lambda - 1)\,\varrho + \gamma\right)$$

by (2.75) and step 2. Thus $\gamma = \beta + (\lambda+1)\,\varrho$, i.e. (2.74) holds true also for $\lambda+1$. $\square$

A consequence of step 3 is that f preserves all (hyperbolic) distances $\lambda\varrho$ with $\lambda \in \{1, 2, 3, \ldots\}$.

4. *There exists a sequence $\alpha_1, \alpha_2, \alpha_3, \ldots$ of positive real numbers tending to* 0 *such that f preserves all hyperbolic distances α_i.*

Proof. Let $\mu > 1$ be an integer and A, B, C be points with

$$\text{hyp}\ (A, B) = \mu\varrho = \text{hyp}\ (A, C)$$

and hyp $(B, C) = 2\varrho$. Such a triangle exists because of

$$\text{hyp}\ (B, C) < \text{hyp}\ (B, A) + \text{hyp}\ (A, C).$$

We are now interested in the uniquely determined points B_μ, C_μ with

$$\text{hyp}\ (A, B_\mu) = \varrho, \quad \text{hyp}\ (B_\mu, B) = (\mu - 1)\,\varrho,$$
$$\text{hyp}\ (A, C_\mu) = \varrho, \quad \text{hyp}\ (C_\mu, C) = (\mu - 1)\,\varrho.$$

Since this configuration remains unaltered in its lengths under f, also the hyperbolic distance hyp (B_μ, C_μ) is preserved under f. Applying the hyperbolic cosine theorem twice, we get

$$\cosh \text{hyp}\ (B_\mu, C_\mu) = \cosh^2 \varrho - \sinh^2 \varrho \cdot \frac{\cosh^2 \mu\varrho - \cosh 2\varrho}{\sinh^2 \mu\varrho},$$

i.e. $\sinh \mu\varrho \cdot \sinh \frac{1}{2}$ hyp $(B_\mu, C_\mu) = \sinh^2 \varrho$. The sequence

$$\alpha_{\mu-1} := \text{hyp}\ (B_\mu, C_\mu) > 0,\ \mu = 2, 3, \ldots,$$

hence tends to 0. All hyperbolic distances α_μ are preserved under f. $\square$

5. *If $\alpha > 0$ and $x, y \in X$ satisfy* hyp $(x, y) = \alpha$, *then*

$$\text{hyp}\ \big(f(x),\ f(y)\big) \leq \alpha.$$

Proof. This is proved as soon as

$$\forall_{\varepsilon>0}\ \forall_{x,y\in X}\ \text{hyp}\ (x, y) = \alpha \Rightarrow \text{hyp}\ \big(f(x),\ f(y)\big) < \alpha + \varepsilon$$

is shown. Let x, y be elements of X with hyp $(x, y) = \alpha$ and let $x(\xi)$, $\xi \in \mathbb{R}$, be the line joining x, y with $x = x(\sigma)$ and $y = x(\sigma + \alpha)$ for a suitable σ. Suppose that $\varepsilon > 0$ is given. Take an element γ of the sequence $\alpha_1, \alpha_2, \alpha_3, \dots$ of step 4 with $2\gamma < \varepsilon$ and elements $\gamma_1, \dots, \gamma_n$ of $\{\alpha_1, \alpha_2, \dots\}$ satisfying

$$0 < \alpha - (\gamma_1 + \cdots + \gamma_n) < 2\gamma.$$

The γ_i's need not be pairwise distinct. Then

$$0 < \alpha - (\gamma_1 + \cdots + \gamma_n) < 2\gamma < [\alpha - (\gamma_1 + \cdots + \gamma_n)] + \varepsilon \tag{2.76}$$

holds true. Define $x_1 = x\,(\sigma + \gamma_1), \dots, x_n = x\,(\sigma + \gamma_1 + \cdots + \gamma_n)$. Take $p \in X$ with

$$\text{hyp}\,(x_n, p) = \gamma = \text{hyp}\,(p, y).$$

The triangle x_n, p, y exists, because of

$$\text{hyp}\,(x_n, y) < \text{hyp}\,(x_n, p) + \text{hyp}\,(p, y),$$

i.e. because of $\text{hyp}\,(x_n, y) = \alpha - (\gamma_1 + \cdots + \gamma_n) < 2\gamma$. If we designate $f\,(z)$ by z' for $z \in X$, then the triangle inequality implies

$$\text{hyp}\,(x', y') \le \text{hyp}\,(x', x_1') + \cdots + \text{hyp}\,(x_{n-1}', x_n') + \text{hyp}\,(x_n', p') + \text{hyp}\,(p', y').$$

Since distances $\gamma_1, \dots, \gamma_n, \gamma$ are preserved under f, we get

$$\gamma_1 = \text{hyp}\,(x, x_1) = \text{hyp}\,(x', x_1'), \dots, \gamma_n = \text{hyp}\,(x_{n-1}, x_n) = \text{hyp}\,(x_{n-1}', x_n')$$

and $\gamma = \text{hyp}\,(x_n, p) = \text{hyp}\,(x_n', p'), \dots$. Hence

$$\text{hyp}\,(x', y') \le \gamma_1 + \cdots + \gamma_n + \gamma + \gamma < \alpha + \varepsilon,$$

in view of (2.76). $\square$

6. *If r is a positive rational number, then f preserves the hyperbolic distance $r\varrho$.*

Proof. Let $n > 1$ be an integer. Then step 5 implies

$$\forall_{x,y \in X}\ \text{hyp}\,(x, y) = \frac{\varrho}{n} \Rightarrow \text{hyp}\,\big(f\,(x),\, f\,(y)\big) \le \frac{\varrho}{n}.$$

Since distance ϱ is preserved, we get

$$\forall_{x,y \in X}\ \text{hyp}\,(x, y) = n \cdot \frac{\varrho}{n} \Rightarrow \text{hyp}\,\big(f\,(x),\, f\,(y)\big) = n \cdot \frac{\varrho}{n},$$

i.e. we get (2.71), (2.72) for $\frac{\varrho}{n}$ instead of ϱ and for n instead of N. Hence steps 1 and 3, carried out for the present values $\frac{\varrho}{n}$ and n, imply that all distances $\lambda \cdot \frac{\varrho}{n}$ with $\lambda \in \{1, 2, 3, \dots\}$ are preserved. $\square$

7. *If t is a positive rational number and if x, y are points satisfying* $\text{hyp}\,(x, y) < t\varrho$, *then* $\text{hyp}\,\big(f\,(x),\, f\,(y)\big) \le t\varrho$.

Proof. We shall write again $v' := f(v)$ for $v \in X$. Take $z \in X$ with

$$\text{hyp}\,(x, z) = \frac{1}{2}\, t\varrho = \text{hyp}\,(z, y).$$

Step 6 implies $\text{hyp}\,(x', z') = \frac{1}{2}\, t\varrho = \text{hyp}\,(z', y')$. Hence

$$\text{hyp}\,(x', y') \leq \text{hyp}\,(x', z') + \text{hyp}\,(z', y') = t\varrho. \qquad \square$$

8. *If* r, s *are positive rational numbers and* x, y *are points satisfying*

$$r\varrho < \text{hyp}\,(x, y) < s\varrho,$$

then $r\varrho \leq \text{hyp}\,(x', y') \leq s\varrho$ *holds true.*

Proof. Let $x(\tau)$, $\tau \in \mathbb{R}$, be the line joining x, y with $x = x(\xi)$, $y = x(\eta)$, $\xi < \eta$. Hence $\text{hyp}\,(x, y) = \eta - \xi$ and thus $r\varrho < \eta - \xi < s\varrho$. Notice $\text{hyp}\,(x', y') \leq s\varrho$, by step 7. Define $p := x(\xi + s\varrho)$. Then

$$\text{hyp}\,(y, p) = \xi + s\varrho - \eta,$$

i.e. $\text{hyp}\,(y, p) = s\varrho - (\eta - \xi) < (s - r)\,\varrho$. Hence $\text{hyp}\,(y', p') \leq (s - r)\,\varrho$, by step 7. Moreover, $\text{hyp}\,(x, p) = s\varrho$ implies $\text{hyp}\,(x', p') = s\varrho$, on account of step 6. Hence, by the triangle inequality,

$$\text{hyp}\,(x', y') \geq \text{hyp}\,(x', p') - \text{hyp}\,(y', p') \geq s\varrho - (s - r)\,\varrho = r\varrho. \qquad \square$$

9. $\text{hyp}\,(x, y) = \text{hyp}\,\big(f(x),\, f(y)\big)$ *holds true for all* $x, y \in X$.

Proof. If $\text{hyp}\,(x, y) > 0$, take sequences $r_\nu, s_\nu\, (\nu = 1, 2, \ldots)$ of positive rational numbers satisfying

$$r_\nu \varrho < \text{hyp}\,(x, y) < s_\nu \varrho,\ \nu = 1, 2, \ldots,$$

and $\lim r_\nu = \frac{1}{\varrho}\, \text{hyp}\,(x, y) = \lim s_\nu$. Hence

$$r_\nu \varrho \leq \text{hyp}\,\big(f(x),\, f(y)\big) \leq s_\nu \varrho,\ \nu = 1, 2, \ldots,$$

by step 8, i.e. $\text{hyp}\,(x, y) = \text{hyp}\,\big(f(x),\, f(y)\big)$. $\square$

Because of step 9 the mapping $f : X \to X$ must be a hyperbolic isometry. This finally proves Theorem 35. $\square$

Remark. If the dimension of X is finite, then

$$\forall_{x,y \in X}\ d(x, y) = \varrho \Rightarrow d\big(f(x),\, f(y)\big) = \varrho, \tag{2.77}$$

d the euclidean or the hyperbolic distance function, for a fixed $\varrho > 0$ characterizes the isometries (F.S. Beckman, D.A. Quarles [1], B. Farrahi [1], A.V. Kuz'minyh [1]). In other words, if X is finite dimensional, then also $N = 1$ is allowed in Theorem 35. The euclidean part of Theorem 35 was proved in the context of strictly convex linear spaces by W. Benz, H. Berens [1], in the context of a more general $N \in \mathbb{R}$ by F. Radó, D. Andreescu, D. Valcán [1]. The theory beyond the Beckman–Quarles result started with the important contribution of E.M. Schröder [1]. The hyperbolic part of Theorem 35 was proved by W. Benz [8].

2.21 A counterexample

The following examples show that (2.77) does not generally characterize the isometries in the infinite dimensional case. The special example $(X,\ \text{eucl})$ was given by Beckman, Quarles [1], the one concerning hyperbolic geometry by W. Benz [8].

Let X be the set of all sequences

$$a = (a_1, a_2, a_3, \ldots)$$

of real numbers $a_1, a_2, a_3, \ldots$ such that almost all a_i are zero. Define

$$\begin{aligned} a + b &:= (a_1 + b_1, a_2 + b_2, \ldots), \\ \lambda a &:= (\lambda a_1, \lambda a_2, \ldots), \\ a \cdot b &:= a_1 b_1 + a_2 b_2 + \cdots \end{aligned}$$

for all a, b in X and all real λ. This is a real inner product space which, in other terms, we already introduced in chapter 1. Let X_{rat} be the set of all $a \in X$ such that the a_i's of a are rational. Since X_{rat} is countable, let

$$\omega : \mathbb{N} \to X_{\text{rat}}$$

with $\mathbb{N} = \{1, 2, 3, \ldots\}$ be a fixed bijection. Moreover, suppose that $\varrho > 0$ is a fixed real number. Define

$$\psi\bigl(\omega(i)\bigr) := (x_{i1}, x_{i2}, \ldots) \text{ for } i = 1, 2, \ldots$$

with

(*euclidean case*) $\quad x_{ii} = \frac{\varrho}{\sqrt{2}}$ and $x_{ij} = 0$ for $i \neq j$,

(*hyperbolic case*) $\quad x_{ii} = \sqrt{2}\sinh\frac{\varrho}{2}$ and $x_{ij} = 0$ for $i \neq j$.

We hence get a mapping $\psi : X_{\text{rat}} \to X$. Another mapping $\varphi : X \to X_{\text{rat}}$ will play a role: For every $a \in X$ choose an element $\varphi(a)$ in X_{rat} such that

$$d\bigl(a, \varphi(a)\bigr) < \frac{\varrho}{2}. \tag{2.78}$$

It is now easy to show that

$$f : X \to X$$

with $f(x) := \psi\bigl(\varphi(x)\bigr)$ for $x \in X$ preserves distance ϱ, but no other positive distance. In fact, if $\varphi(x) = \varphi(y)$ for $x, y \in X$, we then obtain $d\bigl(f(x),\ f(y)\bigr) = 0$. If $\varphi(x) \neq \varphi(y)$ for $x, y \in X$, then $d\bigl(f(x),\ f(y)\bigr) = \varrho$. What we finally have to show is that $d(x, y) = \varrho$ implies $\varphi(x) \neq \varphi(y)$. But $\varphi(x) = \varphi(y)$ would lead, in the case $d(x, y) = \varrho$, to the contradiction

$$\varrho = d(x, y) \leq d\bigl(x, \varphi(x)\bigr) + d\bigl(\varphi(y), y\bigr) < \frac{\varrho}{2} + \frac{\varrho}{2},$$

in view of (2.78).

2.22 An extension problem

Let again (X, d) be one of the metric spaces (X, eucl), (X, hyp).

Lemma 36. *Let $a_1 \neq a_2$ and $b_1 \neq b_2$ be points with*

$$d(a_1, a_2) = d(b_1, b_2). \tag{2.79}$$

Then there exists a motion $\mu \in M(X, d)$ satisfying

$$\mu(a_1) = b_1 \text{ and } \mu(a_2) = b_2.$$

Proof. Because of step D.a of the proof of Theorem 7, chapter 1, there exist motions μ_1, μ_2 with $\mu_1(a_1) = 0$, $\mu_1(a_2) = \lambda_1 e$ and $\mu_2(b_1) = 0$, $\mu_2(b_2) = \lambda_2 e$ where λ_1, λ_2 are suitable positive real numbers. Now

$$\begin{aligned} d(a_1, a_2) &= d\big(\mu_1(a_1), \mu_1(a_2)\big) &&= d(0, \lambda_1 e), \\ d(b_1, b_2) &= d\big(\mu_2(b_1), \mu_2(b_2)\big) &&= d(0, \lambda_2 e) \end{aligned}$$

and (2.79) imply $\lambda_1 = \lambda_2$, in view of $\lambda_1, \lambda_2 > 0$. Hence

$$\mu_2^{-1}\mu_1(a_i) = b_i \text{ for } i = 1, 2.$$

□

Lemma 37. *Let m be a positive integer, b an element of X, and suppose that $a_1, \ldots, a_m, a_{m+1}$ are $m+1$ linearly independent elements of X. If*

$$a_{m+1}^2 = b^2 \text{ and } a_{m+1}a_i = ba_i \tag{2.80}$$

hold true for $i = 1, \ldots, m$, there exists $\omega \in O(X)$ with $\omega = \omega^{-1}$,

$$\omega(a_{m+1}) = b \text{ and } \omega(a_i) = a_i \tag{2.81}$$

for $i = 1, \ldots, m$.

Proof. Take an orthogonal basis $c_1, \ldots, c_m$ with $c_i^2 = 1$, $i = 1, \ldots, m$, of the vector space V spanned by $a_1, \ldots, a_m$. Two cases are now important. If $a_{m+1} + b \in V$, put $c_{m+1} = 0$, and if $a_{m+1} + b \notin V$ put

$$r := \frac{a_{m+1} + b}{2} - \sum_{i=1}^{m} \left(\frac{a_{m+1} + b}{2} c_i \right) c_i, \tag{2.82}$$

and, moreover, since $r \neq 0$,

$$c_{m+1} := \frac{r}{\|r\|}.$$

In the second case $c_1, \ldots, c_{m+1}$ must be an orthogonal basis of the vector space spanned by $a_1, \ldots, a_m, a_{m+1} + b$. Define $\omega : X \to X$ by

$$\omega(x) = -x + 2 \sum_{i=1}^{m+1} (xc_i)\, c_i. \tag{2.83}$$

Since a_j is in V, we obtain $a_j c_{m+1} = 0$, and hence

$$\omega(a_j) = -a_j + 2\sum_{i=1}^{m}(a_j c_i)\, c_i = a_j$$

for $j = 1, \ldots, m$. If we write $c_i = \varrho_{i1} a_1 + \cdots + \varrho_{im} a_m$ for suitable real numbers ϱ_{ij}, we get, by (2.80),

$$a_{m+1} c_i = b c_i \tag{2.84}$$

for $i = 1, \ldots, m$. If $a_{m+1} + b \in V$, then

$$\frac{a_{m+1} + b}{2} = \sum_{i=1}^{m}\left(\frac{a_{m+1} + b}{2}\, c_i\right) c_i$$

holds true, i.e., by (2.84), $a_{m+1} + b = 2\sum(a_{m+1} c_i)\, c_i$, i.e., by (2.83), $\omega(a_{m+1}) = b$. If $a_{m+1} + b \notin V$, we obtain, by (2.82) and (2.80),

$$(a_{m+1} - b)\, r = -\sum_{i=1}^{m}\left(\frac{a_{m+1} + b}{2}\, c_i\right)(a_{m+1} c_i - b c_i),$$

i.e., (by (2.84), $(a_{m+1} - b)\, c_{m+1} = 0$. Hence from (2.84)

$$\frac{a_{m+1} + b}{2} = \sum_{i=1}^{m+1}\left(\frac{a_{m+1} + b}{2}\, c_i\right) c_i = \sum_{i=1}^{m+1}(a_{m+1} c_i)\, c_i,$$

i.e. $\omega(a_{m+1}) = b$.

Since ω is linear and an involution, and since it satisfies $[\omega(x)]^2 = x^2$ for all $x \in X$, it must be in $O(X)$. □

Remark. If one of the elements a_{m+1}, b is in the vector space W spanned by $a_1, \ldots, a_m, a_{m+1} + b$, then also is the other one. In this case $V \neq W$ holds true. We then get

$$a_{m+1} = \sum_{i=1}^{m+1}(a_{m+1} c_i)\, c_i = \sum_{i=1}^{m+1}(b c_i)\, c_i = b.$$

The subspaces of (X, d) are given, by Proposition 13, by the subspaces Y of the vector space X and their images under motions of (X, d). If Y has dimension $n \in \{0, 1, 2, \ldots\}$, then the dimension of $\mu(Y)$ for every $\mu \in M(X, d)$ will also be defined by n. In order to show that this dimension of $\mu(Y)$ is well-defined we consider another subspace Y' of the vector space X such that there exists a motion ν with $\mu(Y) = \nu(Y')$. Observe $Y' = \sigma(Y)$ for the motion $\sigma := \nu^{-1}\mu$ which also can be written, in view of Proposition 30, in the form $\sigma = \alpha T_t \beta$ with suitable $\alpha, \beta \in O(X)$ and a suitable translation with respect to an axis e. Hence

$$Y' = \alpha T_t \beta(Y),\ R' := \alpha^{-1}(Y') = T_t(R)$$

with $R := \beta(Y)$. Since α, β are linear and bijective, R and R' are subspaces of the vector space X with $n = \dim Y = \dim R$. We will show that the equation $R' = T_t(R)$ implies $R' = R$, i.e. $\dim Y' = \dim R' = \dim R = n$. There is nothing to prove for $t = 0$. So assume $t \neq 0$. As subspaces of the vector space X, both spaces R and R' contain $0 \in X$. Hence $0, T_t(0) \in R'$ implies $\mathbb{R}e \subseteq R'$, and $0, T_{-t}(0) \in R$, obviously, $\mathbb{R}e \subseteq R$. Assume now $z \in R \backslash \mathbb{R}e$. Hence $T_t(z) \in R'$ and R' contains the subspace W_z of X spanned by $0, e, T_t(z)$. Thus $z \in W_z \subseteq R'$. Similarly $R' \subseteq R$, because of $R = T_{-t}(R')$.

The following theorem will now be proved.

Theorem 38. *Let $S \neq \emptyset$ be a (finite or infinite) subset of a finite-dimensional subspace of (X, d), and let $f : S \to X$ satisfy*

$$d(x, y) = d\big(f(x),\, f(y)\big)$$

for all $x, y \in S$. Then there exists $\varphi \in M(X, d)$ with $f(x) = \varphi(x)$ for all $x \in S$.

Proof. 1. If $S = \{a_1, a_2\}$ contains exactly two elements, define $b_i := f(a_i)$, $i = 1, 2$. Then Lemma 36 proves our theorem in this special case. If $S = \{a\}$, put $b := f(a)$. Because of D.a (see the proof of Theorem 7 in chapter 1), there exists a motion μ_1 such that $\mu_1(a) = 0$, and also a motion μ_2 with $\mu_2\big(f(a)\big) = 0$. Hence $\varphi = \mu_2^{-1}\mu_1$ is a motion transforming a into $f(a)$. So we may assume that S contains at least three distinct points. Let $a \neq p$ be elements of S and take $a_1 \in X$ with

$$d(0, a_1) = d(a, p),$$

and, in view of step 1, $\alpha \in M(X, d)$ such that $\alpha(a) = 0$, $\alpha(p) = a_1$. Because of

$$d(0, a_1) = d(a, p) = d\big(f(a),\, f(p)\big),$$

take $\beta \in M(X, d)$ satisfying $\beta\big(f(a)\big) = 0$, $\beta\big(f(p)\big) = a_1$. Instead of S we would like to work with $\alpha(S)$ containing $0, a_1$, and instead of f with

$$\beta f \alpha^{-1} : \alpha(S) \to X.$$

Notice that $\alpha(S) \ni 0$ implies that $\alpha(S)$ is a subspace of the vector space X. It is hence sufficient to prove Theorem 38 in the following form.

2. *Let $S \ni 0$, a_1 with $a_1 \neq 0$ be a subset of a finite-dimensional subspace Σ of the vector space X, and let $f : S \to X$ satisfy $f(0) = 0$, $f(a_1) = a_1$ and*

$$d(x, y) = d\big(f(x),\, f(y)\big)$$

for all $x, y \in S$. Then there exists $\varphi \in M(X, d)$ with $f(x) = \varphi(x)$ for all $x \in S$.

3. *The euclidean case.* If $\dim \Sigma = 1$, $\Sigma = \mathbb{R}a_1$ holds true. Since

$$\|x - y\| = \|f(x) - f(y)\| \tag{2.85}$$

must be satisfied for all $x, y \in S$, we obtain, by $f(0) = 0$, $f(a_1) = a_1$,

$$\|\lambda a_1 - 0\| = \|f(\lambda a_1) - 0\|$$

and $\|\lambda a_1 - a_1\| = \|f(\lambda a_1) - a_1\|$ for all real λ with $\lambda a_1 \in S$. If

$$\{0, a_1, \lambda a_1\} =: \{x, y, z\},$$

we get for a suitable order

$$\|z - x\| = \|z - y\| + \|y - x\|.$$

This carries over to the f-images, and these must hence be collinear. Moreover, we obtain $f(\lambda a_1) = \lambda a_1$, i.e. $f(s) = s$ for all $s \in S$. Put $\varphi = \mathrm{id}$.

Assume $\dim \Sigma \geq 2$ and that statement 2 holds true for all subspaces Π of X with $\dim \Pi \leq m$ where m is a positive integer. We will show that then statement 2 holds true also in the case $\dim \Pi = m + 1$, provided $\dim X \geq m + 1$. Besides a_1 take elements $a_2, \dots, a_{m+1}$ in S such that $a_1, \dots, a_{m+1}$ are linearly independent. If they do not exist, S must already be contained in a subspace Π_0 with $\dim \Pi_0 \leq m$, and there is nothing to prove. Apply 2 for $S_0 = \{0, a_1, \dots, a_m\}$, and there hence exists $\varphi_1 \in M(X, \text{ eucl})$ with $\varphi_1(0) = 0 = f(0)$, i.e. $\varphi_1 \in O(X)$, and

$$\varphi_1(a_i) = f(a_i),\ i = 1, \dots, m.$$

Instead of f we will work with $f_1 := \varphi_1^{-1} f : S \to X$. Observe

$$\|x - y\| = \|f_1(x) - f_1(y)\| \tag{2.86}$$

for all $x, y \in S$, and $f_1(a_i) = a_i$, $i = 1, \dots, m$. Put $b := f_1(a_{m+1})$. If we apply (2.86) for $x = 0$, $y = a_{m+1}$, and also for $x = a_i$, $y = a_{m+1}$, $i = 1, \dots, m$, we get (2.80). In view of Lemma 37, there hence exists $\varphi_2 \in O(X)$ with $\varphi_2(a_i) = a_i, \varphi_2(a_{m+1}) = b$ for $i = 1, \dots, m$. Put $f_2 := \varphi_2^{-1} f_1$ and observe $f_2(a_i) = a_i$ for $i = 1, \dots, m + 1$, and

$$\|x - y\| = \|f_2(x) - f_2(y)\| \tag{2.87}$$

for all $x, y \in S$.

Let now s be an arbitrary element of S and define $t := f_2(s)$. Suppose that $e_1, \dots, e_{m+1}$ is an orthogonal basis of the vector space V spanned by $a_1, \dots, a_{m+1}$ with $e_i^2 = 1$, $i = 1, \dots, m + 1$. Define $e = 0$ for $t \in V$, and otherwise such that $e_1, \dots, e_{m+1}, e$ is an orthogonal basis, $e^2 = 1$, for the vector space W spanned by $a_1, \dots, a_{m+1}, t$. From (2.87) we get, by $f_2(0) = 0 \in S$,

$$s^2 = t^2 \text{ and } sa_i = ta_i,\ i = 1, \dots, m + 1.$$

If $e_i = \varrho_{i,1} a_1 + \cdots + \varrho_{i,m+1} a_{m+1}$, we hence obtain $se_i = te_i$, $i = 1, \dots, m + 1$. Observe $s \in S \subseteq \Pi$ and therefore

$$s = (se_1)\, e_1 + \cdots + (se_{m+1})\, e_{m+1}$$

and $t = (te_1)\, e_1 + \cdots + (te_{m+1})\, e_{m+1} + (te)\, e$, i.e. $t = s + (te)\, e$. Since $se = 0$, we obtain

$$s^2 = t^2 = s^2 + (te)^2,$$

i.e. $te = 0$, i.e. $s = t$. Hence $f_2 = \text{id}$ on S, and the identity mapping of $O\,(X)$ extends f_2 on X. Thus $\varphi_1\varphi_2(x) = f\,(x)$ for all $x \in S$. This implies that if statement 2 holds true for all $\Pi, \dim \Pi \leq m$, then also for Π with $\dim \Pi = m+1$ provided $\dim X \geq m+1$.

4. *The hyperbolic case.* If $\dim \Sigma = 1$, again $\Sigma = \mathbb{R}a_1$ holds true. Since

$$\text{hyp}\ (x, y) = \text{hyp}\ \big(f\,(x),\ f\,(y)\big) \tag{2.88}$$

must be satisfied for all $x, y \in S$, we obtain, by $f\,(0) = 0,\ f\,(a_1) = a_1$,

$$(\lambda a_1)^2 = \big(f\,(\lambda a_1)\big)^2$$

and $\lambda a_1^2 = a_1 f\,(\lambda a_1)$ for all real λ with $\lambda a_1 \in S$. If

$$\{0, a_1, \lambda a_1\} =: \{x, y, z\},$$

we get for a suitable order

$$\text{hyp}\ (z, x) = \text{hyp}\ (z, y) + \text{hyp}\ (y, x), \tag{2.89}$$

since $0, a_1, \lambda a_1$ are on a common hyperbolic line. This carries over to the f-images implying collinearity for the image points, i.e. for $0, a_1,\ f\,(\lambda a_1)$. Since (2.89) holds also true for the image points

$$f\,(x),\ f\,(y),\ f\,(z),$$

we obtain $f\,(\lambda a_1) = \lambda a_1$, i.e. $f\,(s) = s$ for all $s \in S$.

Assume $\dim \Sigma \geq 2$ and that statement 2 holds true for all subspaces Π of X with $\dim \Pi \leq m$ where m is a positive integer. We now will proceed as in step 3 up till formula (2.86), which must be replaced by

$$\text{hyp}\ (x, y) = \text{hyp}\ \big(f_1(x),\ f_1(y)\big) \tag{2.90}$$

for all $x, y \in S$. It is important to note that the stabilizer of $M\,(X, \text{hyp})$ in the point 0 is given by $O\,(X)$, i.e. that $\gamma \in M\,(X, \text{hyp})$ and $\gamma\,(0) = 0$ imply $\gamma \in O\,(X)$, so that $\varphi_1 \in M\,(X, \text{hyp})$ must be in $O\,(X)$, because of $\varphi_1(0) = 0$. As in step 3 we put $b := f_1(a_{m+1})$. Applying (2.90) for $x = 0,\ y = a_{m+1}$, and also for $x = a_i, y = a_{m+1},\ i = 1, \ldots, m$, we get $a_{m+1}^2 = b^2$ and

$$\sqrt{1 + a_i^2}\,\sqrt{1 + a_{m+1}^2} - a_i a_{m+1} = \sqrt{1 + a_i^2}\,\sqrt{1 + b^2} - a_i b,$$

i.e. (2.81). Proceeding as in step 3, we arrive at

$$\text{hyp}\ (x, y) = \text{hyp}\ \big(f_2(x),\ f_2(y)\big) \tag{2.91}$$

for all $x, y \in S$, instead of (2.87), with $f_2(a_i) = a_i$, $i = 1, \dots, m+1$. With the further definitions of step 3, we obtain from (2.91), by $f_2(0) = 0 \in S$,

$$\text{hyp }(0, s) = \text{hyp }(0, t),\ \text{hyp }(a_i, s) = \text{hyp }(a_i, t)$$

for $i = 1, \dots, m+1$, i.e.

$$s^2 = t^2 \text{ and } sa_i = ta_i,\ i = 1, \dots, m+1.$$

This leads to $s = t$ as in step 3, and finally to $\varphi_1\varphi_2(x) = f(x)$ for all $x \in S$.

This finishes the proof of Theorem 38. □

2.23 A mapping which cannot be extended

We already know an example of an orthogonal mapping $\omega : X \to X$ which is not surjective (see chapter 1 where orthogonal mappings are defined). Of course, in this special case there cannot exist $\varphi \in M(X, d)$ with $\omega(x) = \varphi(x)$ for all $x \in S := X$, since $\varphi : X \to X$ is bijective. Here S is not contained in a finite-dimensional subspace of X.

In order to present a mapping $f : S \to X$ which cannot be extended and where S is a proper subset of X, take as X all sequences

$$(a_1, a_2, a_3, \dots)$$

of real numbers such that almost all of the a_i's are 0. Define, as usual,

$$\begin{aligned} a + b &:= (a_1 + b_1, a_2 + b_2, \dots), \\ \lambda a &:= (\lambda a_1, \lambda a_2, \dots), \\ a \cdot b &:= \sum_{i=1}^{\infty} a_i b_i \end{aligned}$$

for $a, b \in X$, $\lambda \in \mathbb{R}$. A basis of this real inner product space is

$$e_1 = (1, 0, 0, \dots),\ e_2 = (0, 1, 0, \dots),\ \dots .$$

Define $S = \{0, e_2, e_3, e_4, \dots\}$, $f(0) = 0$ and $f(e_i) = e_{i-1}$ for $i = 2, 3, \dots$. Then

$$\|x - y\| = \|f(x) - f(y)\| \text{ and hyp }(x, y) = \text{hyp }\big(f(x), f(y)\big)$$

hold true for all $x, y \in S$. The smallest subspace Σ of X containing S is spanned by $e_2, e_3, \dots$. Hence Σ is infinite-dimensional. If there existed $\varphi \in M(X, d)$ with $\varphi(s) = f(s)$ for all $s \in S$, we would obtain $\varphi \in O(X)$, in view of $\varphi(0) = 0$. Assuming now

$$\varphi(e_1) =: \lambda_1 e_{i_1} + \dots + \lambda_n e_{i_n}$$

with $\lambda_j \in \mathbb{R}$ and $1 \leq i_1 < i_2 < \cdots < i_n$, would imply

$$0 = e_1 e_{i_j+1} = \varphi(e_1)\,\varphi(e_{i_j+1}) = \varphi(e_1)\,e_{i_j} = \lambda_j$$

for $j = 1, \ldots, n$, i.e. $\varphi(e_1) = 0$, contradicting

$$1 = e_1 e_1 = \varphi(e_1)\,\varphi(e_1).$$

There hence does not exist $\varphi \in M(X, d)$ extending f.

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