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On Reciprocal Sequences of Matricial Carathéodory Sequences and Associated Matrix Functions

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Abstract. This paper provides a detailed discussion of reciprocal sequences of finite and infinite matricial Carathéodory sequences, including an examination of the relationship a reciprocal sequence has to its original sequence. The properties of such reciprocal sequences are described. Of particular interest is the fact that the reciprocal sequence of a Carathéodory sequence is again a Carathéodory sequence. Later, these results are applied to matrix-valued functions. The natural focus is, in particular, on the matricial Carathéodory functions associated with a matricial Carathéodory sequence and its reciprocal sequence. These matrix functions are then shown to be Moore-Penrose inverses of each other. The implications of these results for non-negative Hermitian matrix measures on the unit circle are also discussed.

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0. Introduction

Several classes of sequences of complex matrices were discussed in [22]. There, these classes were studied in the context of a generalization of the inversion of power series with matrix coefficients. In particular, each finite or infinite sequence in the set $\mathbb{C}^{p\times q}$ of all complex $p\times q$ matrices was assigned a reciprocal sequence. This was done in such a way (see Definition 1.1) that, for the special case $p=q$ and $\det s_0 \neq 0$, the assigned reciprocal sequence was none other than the usual reciprocal sequence of conventional power series inversion theory.

Our first main objective for this paper is to study the reciprocal sequences of finite and infinite $q\times q$ Carathéodory sequences. We will see that the reciprocal sequence of a Carathéodory sequence is itself a Carathéodory sequence. Furthermore, we will see that the properties of such reciprocal sequences can be fully expressed in terms of the original Carathéodory sequence. Later, starting with an infinite $q \times q$ Carathéodory sequence and its reciprocal $q \times q$ Carathéodory sequence, we consider their respective $q \times q$ Carathéodory functions.

Our second main goal is to show that these functions are Moore-Penrose inverses of one another on the complex open unit disk $\mathbb{D}$. This will, ultimately, yield a new proof for the holomorphicity of the Moore-Penrose inverse of a $q \times q$ Carathéodory function in $\mathbb{D}$. An earlier proof of this result in [5, Theorem 4.5] was based on an approach using the Cayley transform.

This paper is organized as follows. Section 1 offers a brief introduction to reciprocal sequences of finite and infinite sequences of matrices in $\mathbb{C}^{p\times q}$ (see Definition 1.1). We also examine the class of *first term dominant* matrix sequences (see Definition 1.4), which is particularly interesting and useful in the context of reciprocal sequences. These observations are continued from [22]. In this section we also look at *EP sequences*, earlier considered in [22, Section 7]. We build on and expand these results (see Proposition 1.11 and Theorem 1.12).

In Section 2 we present a summary of preparatory results on finite and infinite Toeplitz non-negative definite sequences in $\mathbb{C}^{q\times q}$. The emphasis, here, is on the inner structure of such sequences, which is described using matrix balls (see Proposition 2.9). We also introduce a number of important subclasses of Toeplitz non-negative definite sequences in $\mathbb{C}^{q\times q}$ (see Definition 2.12 and Definition 2.24). Relevant results are drawn from [11] and [8, Section 3.5]. Proposition 2.7 shows that Toeplitz non-negative definite sequences in $\mathbb{C}^{q\times q}$ are first term dominant. This proposition connects Section 2 to Section 1.

Section 3 centers on a discussion of matricial Carathéodory sequences, in light of their relationship to Toeplitz non-negative definite sequences. This relationship shows that a certain duality exists between the two types of sequences and makes it possible to define particular subclasses of matricial Carathéodory sequences using Section 2 (see Definition 3.15 and Definition 3.22).

In Section 4, we focus on reciprocal sequences of matricial $q \times q$ Carathéodory sequences. We show that $q \times q$ Carathéodory sequences are EP sequences and recognize that they are first term dominant (see Proposition 4.1). We then use this fact to apply the EP sequence results of Section 1 to $q \times q$ Carathéodory sequences. This leads directly to our first main result, namely, that the reciprocal sequence of a $q \times q$ Carathéodory sequence is itself a $q \times q$ Carathéodory sequence (see Theorem 4.4). This, in turn, brings us to the question of how a $q \times q$ Carathéodory sequence is related to its reciprocal sequence and to what extent one can be used to describe the other. In the second part of Section 4, we deal mainly with these questions. We will see that for the subclasses introduced in Section 3, both sequences either belong to the relevant subclass or neither of them do (see Theorem 4.8 and Theorem 4.10). Furthermore, we describe the matrix ball structure of reciprocal sequences to Carathéodory sequences in terms of the original Carathéodory sequence (see Lemma 4.9).

The focus of Section 5 is on a particular reciprocal sequence operation for matricial Toeplitz non-negative definite sequences. Because of how closely matricial Toeplitz non-negative definite sequences and matricial Carathéodory sequences are related, we are able to use Section 4 in constructing a new Toeplitz non-negative definite sequence to associate with a given initial Toeplitz non-negative definite sequence. Having established a procedure for constructing new sequences of this type, we proceed with an extensive examination of the relationship between an initial matricial Toeplitz non-negative definite sequence and the Toeplitz non-negative definite sequence generated from it.

Section 6 includes a short introduction to the theory for the matricial Carathéodory class $\mathcal{C}_q(\mathbb{D})$, where $\mathbb{D} := \{ w \in \mathbb{C} : |w| < 1 \}$ is the open unit disk in the complex plane $\mathbb{C}$. The $\mathcal{C}_q(\mathbb{D})$ class is made up of every $q \times q$ matrix function Ω that is holomorphic in $\mathbb{D}$ with non-negative real part at all $w \in \mathbb{D}$. Via Taylor coefficients, we see that there is a well-known one-to-one correspondence between the class $\mathcal{C}_q(\mathbb{D})$ and the set of all infinite $q \times q$ Carathéodory sequences (see Propositions 6.10 and 6.12). This suggests introducing a number of special subclasses of $\mathcal{C}_q(\mathbb{D})$, which we do with the help of Section 3.

Section 7 is dedicated to an analysis of the Moore-Penrose inverse Ω^+ of a function $\Omega \in \mathcal{C}_q(\mathbb{D})$. We show that $\Omega^+ \in \mathcal{C}_q(\mathbb{D})$ and also that the Taylor coefficient sequence for Ω^+ is the reciprocal sequence to Ω's Taylor coefficient sequence (see Theorem 7.3). We furthermore show that for the subclasses of $\mathcal{C}_q(\mathbb{D})$ defined in Section 6, both of the $\mathcal{C}_q(\mathbb{D})$ functions Ω and Ω^+ either belong to the relevant subclass or neither of them do (see Theorem 7.9 and Proposition 7.10).

We are also interested in applying these results to the theory of non-negative Hermitian $q \times q$ measures on the unit circle $\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}$. For this reason, we include a brief review of results in matricial harmonic analysis on $\mathbb{T}$. Because of the matricial version of a classical result by G. Herglotz (see Proposition 8.1), we know that (via Fourier coefficients) there is a one-to-one correspondence between the set of all infinite Toeplitz non-negative definite sequences in $\mathbb{C}^{q \times q}$ and the set of all non-negative Hermitian $q \times q$ measures on the Borel σ-algebra for the unit circle. This one-to-one correspondence makes it possible to generate a reciprocal non-negative Hermitian measure using an operation based on Section 5's method of generating a Toeplitz non-negative definite sequence "by reciprocation". This idea then informs how Section 8 develops. The aforementioned construction was already considered in [8, Section 3.6] for the special case of a non-negative Hermitian measure F with non-singular matrix $F(\mathbb{T})$. There, the focus was on the relationship between associated pairs of orthogonal systems of matrix polynomials.

Our closing Section 9 concentrates on a class of matrix functions that are holomorphic in the upper half-plane $\Pi_+ := \{ z \in \mathbb{C} : \operatorname{Im} z \in (0, +\infty) \}$. This matrix function class is particularly interesting in the context of the matrix version of the Hamburger Moment Problem. The class $\mathcal{R}_q(\Pi_+)$ discussed in Section 9 is comprised of all matrix functions $G : \Pi_+ \longrightarrow \mathbb{C}^{q\times q}$ that are holomorphic in Π_+ and having non-negative Hermitian imaginary part at all $z \in \Pi_+$. There is a one-to-one correspondence between $\mathcal{R}_q(\Pi_+)$ and $\mathcal{C}_q(\mathbb{D})$, which makes it possible to show that the Moore-Penrose inverse G^+ of a function $G \in \mathcal{R}_q(\Pi_+)$ is holomorphic and also that null space and range remain constant (see Theorem 9.4).

To conclude this introduction, we quickly review some notation. Throughout this paper, let p and q be positive integers. We use $\mathbb{N}$ and $\mathbb{N}_0$ to denote the sets of positive and non-negative integers, respectively. The set of all integers is $\mathbb{Z}$. For any $\alpha \in \mathbb{Z}$ and $\varkappa \in \mathbb{Z} \cup \{+\infty\}$, we let $\mathbb{Z}_{\alpha,\varkappa} := \{\ell \in \mathbb{Z} : \alpha \leq \ell \leq \varkappa\}$.

The set of all Hermitian matrices in $\mathbb{C}^{q\times q}$ will be denoted by $\mathbb{C}_{\mathrm{H}}^{q\times q}$, while $\mathbb{C}_{\geq}^{q\times q}$ and $\mathbb{C}_{>}^{q\times q}$ will stand for the sets of all non-negative and positive Hermitian matrices, respectively. The set of all contractive $p\times q$ matrices is defined as $\mathbb{K}_{p\times q} := \{A \in \mathbb{C}^{p\times q} : \|A\|_{\mathrm{S}} \leq 1\}$ and the set of all strictly contractive $p \times q$ matrices as $\mathbb{D}_{p\times q} := \{A \in \mathbb{C}^{p\times q} : \|A\|_{\mathrm{S}} < 1\}$, where $\|\cdot\|_{\mathrm{S}}$ is the operator norm in $\mathbb{C}^{p\times q}$. Suppose that $A \in \mathbb{C}^{p\times q}$, then $\mathcal{N}(A)$ is A's null space and $\mathcal{R}(A)$ is its range. The zero matrix in $\mathbb{C}^{p\times q}$ and the identity matrix in $\mathbb{C}^{q\times q}$ are denoted by $0_{p\times q}$ and I_q, respectively. Suppose $A \in \mathbb{C}^{p\times q}$, then its adjoint will be denoted by A^* and its Moore-Penrose inverse by A^+. For each $A \in \mathbb{C}^{q\times q}$, the matrices $\operatorname{Re} A := \frac{1}{2}(A + A^*)$ and $\operatorname{Im} A := \frac{1}{2i}(A - A^*)$ are, respectively, called the real and imaginary parts of A. The determinant of A will be denoted by $\det A$. For $n \in \mathbb{N}$ and any sequence $(A_j)_{j=1}^n$ of complex $p \times q$ matrices, we also define $\operatorname{row}(A_j)_{j=1}^n := \begin{pmatrix} A_1, & A_2, & \cdots, & A_n \end{pmatrix}$ and $\operatorname{col}(A_j)_{j=1}^n := \left[\operatorname{row}\left(A_j^{\mathrm{T}}\right)_{j=1}^n\right]^{\mathrm{T}}$. For each $q \in \mathbb{N}$, we let

$$\mathcal{R}_{q,\geq} := \left\{A \in \mathbb{C}^{q\times q} : \operatorname{Re} A \in \mathbb{C}_{\geq}^{q\times q}\right\} \quad \text{and} \quad \mathcal{I}_{q,\geq} := \left\{A \in \mathbb{C}^{q\times q} : \operatorname{Im} A \in \mathbb{C}_{\geq}^{q\times q}\right\}.$$

We also define

$$\mathcal{R}_{q,>} := \left\{A \in \mathbb{C}^{q\times q} : \operatorname{Re} A \in \mathbb{C}_{>}^{q\times q}\right\} \quad \text{and} \quad \mathcal{I}_{q,>} := \left\{A \in \mathbb{C}^{q\times q} : \operatorname{Im} A \in \mathbb{C}_{>}^{q\times q}\right\}.$$

Clearly, the set $\mathbb{C}_{\geq}^{q\times q}$ of all non-negative Hermitian complex $q \times q$ matrices is a subset of $\mathcal{R}_{q,\geq}$ and the set $\mathbb{C}_{>}^{q\times q}$ of all positive Hermitian complex $q \times q$ matrices is a subset of $\mathcal{R}_{q,>}$. A complex $q \times q$ matrix A is called range-Hermitian (or an EP matrix) if $\mathcal{R}(A) = \mathcal{R}(A^*)$. The set of all range-Hermitian matrices in $\mathbb{C}^{q\times q}$ will be denoted by $\mathbb{C}_{\mathrm{EP}}^{q\times q}$.

1. Reciprocal sequences

The approach to constructing a special transformation for sequences of complex matrices considered here was introduced in [22]. The main theme of [22] was invertibility for sequences of complex matrices. In this section, we provide a quick review of relevant results from [22], because they will later be important.

Definition 1.1. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^{\varkappa}$ be a sequence of complex $p \times q$ matrices. The sequence $(s_j^\sharp)_{j=0}^{\varkappa}$ of $q \times p$ matrices defined by

$$s_k^\sharp := \begin{cases} s_0^+, & \text{if } k = 0, \\ -s_0^+ \sum_{l=0}^{k-1} s_{k-l} s_l^\sharp, & \text{if } k \in \mathbb{Z}_{1,\varkappa}, \end{cases}$$

is called the reciprocal sequence corresponding to $(s_j)_{j=0}^{\varkappa}$.

Remark 1.2. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^{\varkappa}$ be a sequence of complex $p \times q$-matrices with reciprocal sequence $(s_j^\sharp)_{j=0}^{\varkappa}$. Given an arbitrary $m \in \mathbb{Z}_{0,\varkappa}$, the reciprocal sequence corresponding to $(s_j)_{j=0}^{m}$ is then $(s_j^\sharp)_{j=0}^{m}$.

Example 1.3. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$. Suppose $(s_j)_{j=0}^{\varkappa}$ is a sequence in $\mathbb{C}^{p\times q}$ and that $u \in \mathbb{C}$. For $j \in \mathbb{Z}_{0,\varkappa}$, suppose that $s_{j,u} := u^j s_j$. By induction, we then obtain $(s_{j,u})^\sharp = u^j s_j^\sharp$, for $j \in \mathbb{Z}_{0,\varkappa}$.

We will see that reciprocal sequences are especially interesting when considered for the following class of matricial sequences.

Definition 1.4. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$. A sequence $(s_j)_{j=0}^{\varkappa}$ in $\mathbb{C}^{p\times q}$ is called first term dominant if

$$\mathcal{N}(s_0) \subseteq \bigcap_{j=0}^{\varkappa} \mathcal{N}(s_j) \qquad \text{and} \qquad \bigcup_{j=0}^{\varkappa} \mathcal{R}(s_j) \subseteq \mathcal{R}(s_0).$$

The set of all such sequences $(s_j)_{j=0}^{\varkappa}$ in $\mathbb{C}^{p\times q}$ will be denoted by $\mathcal{D}_{p\times q,\varkappa}$.

Example 1.5. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$. All sequences $(s_j)_{j=0}^{\varkappa}$ in $\mathbb{C}^{q\times q}$ with $\det s_0 \neq 0$ belong to $\mathcal{D}_{q\times q,\varkappa}$.

Remark 1.6. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and $(s_j)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C}^{p\times q}$. Then $(s_j)_{j=0}^{\varkappa} \in \mathcal{D}_{p\times q,\varkappa}$ if and only if $(s_j)_{j=0}^{m} \in \mathcal{D}_{p\times q,m}$, for all $m \in \mathbb{Z}_{0,\varkappa}$.

Remark 1.7. Let $(s_j)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C}^{q\times q}$ and let $(\alpha_j)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C} \setminus \{0\}$. Then $(s_j)_{j=0}^{\varkappa} \in \mathcal{D}_{q\times q,\varkappa}$ if and only if $(\alpha_j s_j)_{j=0}^{\varkappa} \in \mathcal{D}_{q\times q,\varkappa}$.

Given a $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and a sequence $(s_j)_{j=0}^{\varkappa}$ in $\mathbb{C}^{p\times q}$, we define, for each $m \in \mathbb{Z}_{0,\varkappa}$, the lower triangular block Toeplitz matrices $\mathrm{S}_m^{(s)}$ as

$$\mathrm{S}_m^{(s)} := \begin{pmatrix} s_0 & 0 & 0 & \dots & 0 \\ s_1 & s_0 & 0 & \dots & 0 \\ s_2 & s_1 & s_0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ s_m & s_{m-1} & s_{m-2} & \dots & s_0 \end{pmatrix} \tag{1.1}$$

and, whenever it is clear which sequence is meant, we will simply write S_m instead of $\mathrm{S}_m^{(s)}$. For each $m \in \mathbb{Z}_{0,\varkappa}$, using $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$ instead of $(s_j)_{j=0}^{\varkappa}$, we also set

$$\mathrm{S}_m^{\sharp} := \mathrm{S}_m^{(s^{\sharp})}. \tag{1.2}$$

The following result (see [22, Proposition 4.20]) is particularly useful.

Proposition 1.8. *If $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and $(s_j)_{j=0}^{\varkappa} \in \mathcal{D}_{p\times q,\varkappa}$, then $\mathrm{S}_m^+ = \mathrm{S}_m^{\sharp}$ for each $m \in \mathbb{Z}_{0,\varkappa}$.*

If $r, s \in \mathbb{N}$, $A = (a_{jk})_{\substack{j=1,\dots,p\\k=1,\dots,q}} \in \mathbb{C}^{p\times q}$ and $B \in \mathbb{C}^{r\times s}$, then the Kronecker product $A \otimes B$ of the matrices A and B is given by $A \otimes B := (a_{jk}B)_{\substack{j=1,\dots,p\\k=1,\dots,q}}$.

It should be noted that, if $s_0 \in \mathbb{C}^{p\times q}$ and if $m \in \mathbb{N}$, then the complex $(m+1)\,p \times (m+1)\,q$ matrix $\operatorname{diag}(s_0, s_0, \dots, s_0)$ can be expressed as $I_{m+1} \otimes s_0$.

Lemma 1.9. *If $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and $(s_j)_{j=0}^{\varkappa} \in \mathcal{D}_{p\times q,\varkappa}$, then, for each $m \in \mathbb{Z}_{0,\varkappa}$,*

$$\mathrm{S}_m\mathrm{S}_m^+ = I_{m+1} \otimes (s_0 s_0^+) \qquad \textit{and} \qquad \mathrm{S}_m^+\mathrm{S}_m = I_{m+1} \otimes (s_0^+ s_0).$$

If, furthermore, $p = q$ and if $s_0 \in \mathbb{C}_{\mathrm{EP}}^{q\times q}$, then $\mathrm{S}_m\mathrm{S}_m^+ = \mathrm{S}_m^+\mathrm{S}_m$ for every $m \in \mathbb{Z}_{0,\varkappa}$.

Proof. Combine part (a) of [22, Theorem 4.21] with [22, Proposition 3.6]. □

We now consider a special class of sequences of complex square matrices, namely the set of EP sequences (see [22, Section 7]). If $n \in \mathbb{N}_0$ and $(s_j)_{j=0}^{n}$ is a sequence in $\mathbb{C}^{q\times q}$, then $(s_j)_{j=0}^{n}$ is called an EP sequence when $\mathrm{S}_n^{(s)} \in \mathbb{C}_{\mathrm{EP}}^{(n+1)q\times(n+1)q}$, where $\mathrm{S}_n^{(s)}$ is defined by (1.1). If $n \in \mathbb{N}$ and $(s_j)_{j=0}^{n}$ is an EP sequence in $\mathbb{C}^{q\times q}$, then, for any $k \in \mathbb{Z}_{0,n-1}$, it follows that $(s_j)_{j=0}^{k}$ is also an EP sequence (see [22, Remark 7.2]). A sequence $(s_j)_{j=0}^{\infty}$ in $\mathbb{C}^{q\times q}$ is thus called an EP sequence when $(s_j)_{j=0}^{n}$ is an EP sequence for all $n \in \mathbb{N}_0$. If $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$, then $\mathcal{F}_{q,\varkappa}^{\mathrm{EP}}$ will stand for the set of all EP sequences $(s_j)_{j=0}^{\varkappa}$ in $\mathbb{C}^{q\times q}$. The following proposition (see [22, Proposition 7.4]) shows that EP sequences make up a subclass of first term dominant sequences (introduced in Definition 1.4).

Proposition 1.10. *Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and $(s_j)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C}^{q\times q}$.*

(a) *The following two conditions are equivalent to one another:*
   (i) $(s_j)_{j=0}^{\varkappa} \in \mathcal{F}_{q,\varkappa}^{\mathrm{EP}}$.
   (ii) $s_0 \in \mathbb{C}_{\mathrm{EP}}^{q\times q}$ *and* $(s_j)_{j=0}^{\varkappa} \in \mathcal{D}_{q\times q,\varkappa}$.

(b) *Suppose* (i) *is satisfied, then $s_0 s_0^+ = s_0^+ s_0$ and, for each $n \in \mathbb{Z}_{0,\varkappa}$,*

$$\mathrm{S}_n\mathrm{S}_n^+ = I_{n+1} \otimes \left(s_0 s_0^+\right) \qquad \textit{and} \qquad \mathrm{S}_n^+\mathrm{S}_n = I_{n+1} \otimes \left(s_0^+ s_0\right).$$

Proposition 1.11. *If $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and $(s_j)_{j=0}^{\varkappa} \in \mathcal{F}_{q,\varkappa}^{\mathrm{EP}}$, then $(s_j)_{j=0}^{n} \in \mathcal{F}_{q,n}^{\mathrm{EP}}$ and $\mathrm{S}_n^+ = \mathrm{S}_n^{\sharp}$ for each $n \in \mathbb{Z}_{0,\varkappa}$.*

Our next result will offer us further insight into the structure of EP sequences.

Theorem 1.12. *Let $q \geq 2$, $r \in \mathbb{Z}_{1,\,q-1}$ and $\varkappa \in \mathbb{N}_0 \cup \{\,+\infty\,\}$. Suppose, furthermore, that $(\,s_j\,)_{j=0}^{\varkappa} \in \mathcal{F}_{q,\,\varkappa}^{\mathrm{EP}}$ with $\operatorname{rank} s_0 = r$ (and therefore $\dim[\mathcal{N}(\,s_0\,)] = q - r$). Let $(\,u_s\,)_{s=1}^{r}$ be an orthonormal basis in $\mathcal{R}(\,s_0\,)$ and $(\,u_s\,)_{s=r+1}^{q}$ be an orthonormal basis in $\mathcal{N}(\,s_0\,)$. For each $\ell \in \mathbb{Z}_{1,\,q}$, let $U_\ell := \begin{pmatrix} u_1, & u_2, & \ldots, & u_\ell \end{pmatrix}$ and, for each $j \in \mathbb{Z}_{0,\,\varkappa}$, let $\widetilde{s}_j := U_r^* s_j U_r$. Then:*

(a) *$(\,u_s\,)_{s=1}^{q}$ is an orthonormal basis in $\mathbb{C}^{q\times 1}$ and the matrix U_q is unitary.*
(b) *$U_q^* s_j U_q = \operatorname{diag}\big(\widetilde{s}_j,\, 0_{(q-r)\times(q-r)}\big)$ for each $j \in \mathbb{Z}_{0,\,\varkappa}$. The matrix $\widetilde{s}_0$ is, in particular, non-singular.*

Proof. By part (a) of Proposition 1.10, we get $s_0 \in \mathbb{C}_{\mathrm{EP}}^{q\times q}$. Thus, it follows by Proposition A.5 that $\mathcal{N}(\,s_0\,) = \mathcal{N}(\,s_0^*\,)$. Hence, we see that $\mathbb{C}^{q\times 1} = \mathcal{R}(\,s_0\,) \oplus \mathcal{N}(\,s_0\,)$, from which we obtain (a). Furthermore, by part (a) of Proposition 1.10, we have $(\,s_m\,)_{m=0}^{\varkappa} \in \mathcal{D}_{q\times q,\,\varkappa}$. Recalling Definition 1.4, we then obtain $\mathcal{N}(\,s_0\,) \subseteq \mathcal{N}(\,s_j\,)$ and, since $\mathcal{N}(\,s_0\,) = \mathcal{N}(\,s_0^*\,)$, it follows by [22, Proposition 5.1] that

$$\mathcal{N}(\,s_0\,) \subseteq \mathcal{N}\left(\,s_j^*\,\right). \tag{1.3}$$

For each $m \in \mathbb{Z}_{r+1,\,q}$, we have, by assumption, that $u_m \in \mathcal{N}(\,s_0\,)$, which, because of (1.3), implies $s_j u_m = 0_{q\times 1}$ and $s_j^* u_m = 0_{q\times 1}$ for each $j \in \mathbb{Z}_{0,\,\varkappa}$. Consequently,

$$\begin{aligned} U_q^* s_j U_q &= \begin{pmatrix} (\,u_1,\,\cdots,\,u_r\,)^* s_j (\,u_1,\,\cdots,\,u_r\,), & (\,u_1,\,\cdots,\,u_r\,)^* s_j (\,u_{r+1},\,\cdots,\,u_q\,) \\ (\,u_{r+1},\,\cdots,\,u_q\,)^* s_j (\,u_1,\,\cdots,\,u_r\,), & (\,u_{r+1},\,\cdots,\,u_q\,)^* s_j (\,u_{r+1},\,\cdots,\,u_q\,) \end{pmatrix} \\ &= \operatorname{diag}\big(\widetilde{s}_j,\, 0_{(q-r)\times(q-r)}\big) \end{aligned}$$

for each $j \in \mathbb{Z}_{0,\,\varkappa}$. Since U_q is non-singular, it follows that

$$\operatorname{rank} \widetilde{s}_0 = \operatorname{rank}\left[\operatorname{diag}\big(\widetilde{s}_0,\, 0_{(q-r)\times(q-r)}\big)\right] = \operatorname{rank}\left[\,U_q^* s_0 U_q\,\right] = \operatorname{rank} s_0 = r.$$

Thus, the matrix $\widetilde{s}_0$ is non-singular. □

2. Matricial Toeplitz non-negative definite sequences

Our focus in this section will be on a special class of sequences in $\mathbb{C}^{q\times q}$. Before we describe this class, we first introduce some notation. Given a $\varkappa \in \mathbb{N}_0$ and a sequence $(\,C_j\,)_{j=0}^{\varkappa}$ in $\mathbb{C}^{q\times q}$ we define, for each $n \in \mathbb{Z}_{0,\,\varkappa}$, the block Toeplitz matrix

$$\mathrm{T}_n^{(C)} := \begin{pmatrix} C_0 & C_1^* & \ldots & C_n^* \\ C_1 & C_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & C_1^* \\ C_n & \ldots & C_1 & C_0 \end{pmatrix}. \tag{2.1}$$

Whenever it is clear which sequence is meant, we will simply write T_n instead of $\mathrm{T}_n^{(C)}$. If $n \in \mathbb{N}_0$ and $(\,C_j\,)_{j=0}^{n}$ is a sequence in $\mathbb{C}^{q\times q}$, then $(\,C_j\,)_{j=0}^{n}$ is called a Toeplitz non-negative definite sequence (or T-n.n.d. sequence, for short) if T_n is non-negative Hermitian. We call $(\,C_j\,)_{j=0}^{n}$ a Toeplitz positive definite sequence (or T-p.d. sequence) if T_n is positive Hermitian. If $n \in \mathbb{N}$ and $(\,C_j\,)_{j=0}^{n}$ is a T-n.n.d.

sequence in $\mathbb{C}^{q\times q}$, then, for each $m \in \mathbb{Z}_{0,\,n-1}$, so is $(\,C_j\,)_{j=0}^{m}$. Similarly, if $(\,C_j\,)_{j=0}^{n}$ is T-p.d., the same holds for $(\,C_j\,)_{j=0}^{m}$. Thus, we say that a sequence $(\,C_j\,)_{j=0}^{\infty}$ is Toeplitz non-negative definite (T-n.n.d.) if, for each $n \in \mathbb{N}_0$, the sequence $(\,C_j\,)_{j=0}^{n}$ is Toeplitz non-negative definite and Toeplitz positive definite (T-p.d.) if, for each $n \in \mathbb{N}_0$, the sequence $(\,C_j\,)_{j=0}^{n}$ is Toeplitz positive definite. For $\varkappa \in \mathbb{N}_0 \cup \{\,+\infty\,\}$, the set of all T-n.n.d. sequences in $\mathbb{C}^{q\times q}$ will be denoted by $\mathcal{T}_{q,\,\varkappa}$ and the set of all T-p.d. sequences in $\mathbb{C}^{q\times q}$ by $\widetilde{\mathcal{T}}_{q,\,\varkappa}$.

Example 2.1. Suppose $K \in \mathbb{C}^{q\times q}$. For each $j \in \{\,0,\,1\,\}$, let $C_j := K^j$. A well-known characterization of non-negative Hermitian block matrices (see, for instance, [8, Lemma 1.1.9]) shows that $(\,C_j\,)_{j=0}^{1} \in \mathcal{T}_{q,\,1}$ if and only if $K \in \mathbb{K}_{q\times q}$, and also that $(\,C_j\,)_{j=0}^{1} \in \widetilde{\mathcal{T}}_{q,\,1}$ if and only if $K \in \mathbb{D}_{q\times q}$.

Example 2.2. Let $\varkappa \in \mathbb{N}_0 \cup \{\,+\infty\,\}$. If $A \in \mathbb{C}_{\geq}^{q\times q}$ and $C_j := A$ for each $j \in \mathbb{Z}_{0,\,\varkappa}$, then $(\,C_j\,)_{j=0}^{\varkappa} \in \mathcal{T}_{q,\,\varkappa}$, since $\mathrm{T}_n = D^*AD \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$ for all $n \in \mathbb{Z}_{0,\,\varkappa}$, where $D := \begin{pmatrix} I_q, & I_q, & \cdots, & I_q \end{pmatrix} \in \mathbb{C}^{q\times(n+1)q}$.

Remark 2.3. Let $\varkappa \in \mathbb{N}_0 \cup \{\,+\infty\,\}$ and $(\,C_j\,)_{j=0}^{\varkappa} \in \mathcal{T}_{q,\,\varkappa}$. Suppose, furthermore, that $A \in \mathbb{C}^{q\times p}$. Then $(\,A^*C_jA\,)_{j=0}^{\varkappa} \in \mathcal{T}_{p,\,\varkappa}$.

Remark 2.4. Let $\varkappa \in \mathbb{N}_0 \cup \{\,+\infty\,\}$ and $s \in \mathbb{N}$. For each $r \in \mathbb{Z}_{1,\,s}$, let $\alpha_r \in [\,0,\,+\infty)$ and $\left(C_j^{(r)}\right)_{j=0}^{\varkappa} \in \mathcal{T}_{q,\,\varkappa}$. Then $\left(\sum_{r=1}^{s}\alpha_r C_j^{(r)}\right)_{j=0}^{\varkappa} \in \mathcal{T}_{q,\,\varkappa}$. If, furthermore, there exists an $r_0 \in \mathbb{Z}_{1,\,s}$ such that $\alpha_{r_0} \in (\,0,\,+\infty\,)$ and $\left(C_j^{(r_0)}\right)_{j=0}^{\varkappa} \in \widetilde{\mathcal{T}}_{q,\,\varkappa}$, then $\left(\sum_{r=1}^{s}\alpha_r C_j^{(r)}\right)_{j=0}^{\varkappa} \in \widetilde{\mathcal{T}}_{q,\,\varkappa}$.

Remark 2.5. Let $\varkappa \in \mathbb{N}_0 \cup \{\,+\infty\,\}$ and $s \in \mathbb{Z}_{2,\,+\infty}$. Suppose, furthermore, that $(\,q_j\,)_{j=1}^{s}$ is a sequence in $\mathbb{N}$ and that $q = \sum_{r=1}^{s} q_j$. For each $r \in \mathbb{Z}_{1,\,s}$, let $\left(C_j^{(r)}\right)_{j=0}^{\varkappa}$ be a sequence from $\mathbb{C}^{q_j\times q_j}$. Then:

(a) $\left(\operatorname{diag}\left(C_j^{(1)},\, C_j^{(2)},\, \ldots,\, C_j^{(s)}\right)\right)_{j=0}^{\varkappa} \in \mathcal{T}_{q,\,\varkappa}$ if and only if $\left(C_j^{(r)}\right)_{j=0}^{\varkappa} \in \mathcal{T}_{q_r,\,\varkappa}$ for all $r \in \mathbb{Z}_{1,\,s}$.

(b) $\left(\operatorname{diag}\left(C_j^{(1)},\, C_j^{(2)},\, \ldots,\, C_j^{(s)}\right)\right)_{j=0}^{\varkappa} \in \widetilde{\mathcal{T}}_{q,\,\varkappa}$ if and only if $\left(C_j^{(r)}\right)_{j=0}^{\varkappa} \in \widetilde{\mathcal{T}}_{q_r,\,\varkappa}$ for all $r \in \mathbb{Z}_{1,\,s}$.

Remark 2.6. Let $\varkappa \in \mathbb{N}_0 \cup \{\,+\infty\,\}$.

(a) Suppose $(\,b_j\,)_{j=0}^{\varkappa} \in \mathcal{T}_{1,\,\varkappa}$ and $(\,C_j\,)_{j=0}^{\varkappa} \in \mathcal{T}_{q,\,\varkappa}$, then $(\,b_jC_j\,)_{j=0}^{\varkappa} \in \mathcal{T}_{q,\,\varkappa}$.

(b) Suppose $(\,b_j\,)_{j=0}^{\varkappa} \in \widetilde{\mathcal{T}}_{1,\,\varkappa}$ and $(\,C_j\,)_{j=0}^{\varkappa} \in \widetilde{\mathcal{T}}_{q,\,\varkappa}$, then $(\,b_jC_j\,)_{j=0}^{\varkappa} \in \widetilde{\mathcal{T}}_{q,\,\varkappa}$.

With our next result, we establish a link to Section 1.

Proposition 2.7. *Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$. Then $\mathcal{T}_{q,\varkappa} \subseteq \mathcal{D}_{q\times q,\varkappa}$.*

Proof. The case in which $\varkappa = 0$ is trivial. We thus suppose that $\varkappa \geq 1$. Let $(C_j)_{j=0}^{\varkappa}$ be a T-n.n.d. sequence in $\mathbb{C}^{q\times q}$ and $\ell \in \mathbb{Z}_{1,\varkappa}$. We thus have $\mathrm{T}_\ell \in \mathbb{C}_{\geq}^{(\ell+1)q\times(\ell+1)q}$. In particular, this implies that

$$\begin{pmatrix} C_0 & C_\ell^* \\ C_\ell & C_0 \end{pmatrix} \in \mathbb{C}_{\geq}^{2q\times 2q}.$$

It thus follows that $\mathcal{R}(C_\ell) \subseteq \mathcal{R}(C_0)$, $\mathcal{R}(C_\ell^*) \subseteq \mathcal{R}(C_0)$ and $C_0^* = C_0$ (see, e.g., [8, Lemma 1.1.9]). Hence, we obtain

$$\bigcup_{j=0}^{\varkappa} \mathcal{R}(C_j) \subseteq \mathcal{R}(C_0) \qquad \text{and} \qquad \bigcup_{j=0}^{\varkappa} \mathcal{R}(C_j^*) \subseteq \mathcal{R}(C_0^*).$$

Finally, applying [22, Proposition 5.1] yields $(C_j)_{j=0}^{\varkappa} \in \mathcal{D}_{q\times q,\varkappa}$. □

A direct consequence of Proposition 2.7 is the following well-known fact:

Corollary 2.8. *If $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and $(C_j)_{j=0}^{\varkappa}$ is a T-n.n.d. sequence in $\mathbb{C}^{q\times q}$ with $C_0 = 0_{q\times q}$, then $C_j = 0_{q\times q}$ for all $j \in \mathbb{Z}_{0,\varkappa}$.*

We will now take a more detailed look at the structure of Toeplitz non-negative definite sequences in $\mathbb{C}^{q\times q}$. We draw from [11] and [8, Section 3.4], where the structure of Toeplitz non-negative definite sequences is discussed in detail. This structure is described using special matrices. For each $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and any sequence $(C_j)_{j=0}^{\varkappa}$ in $\mathbb{C}^{q\times q}$, we set

$$M_1 := 0_{q\times q}, \qquad L_1 := C_0 \qquad \text{and} \qquad R_1 := C_0. \tag{2.2}$$

If $\varkappa \geq 1$, then, for each $n \in \mathbb{Z}_{1,\varkappa}$, we furthermore define

$$Z_n := \operatorname{row}(C_{n+1-j})_{j=1}^{n} \qquad \text{and} \qquad Y_n := \operatorname{col}(C_j)_{j=1}^{n} \tag{2.3}$$

as well as

$$L_{n+1} := C_0 - Z_n \mathrm{T}_{n-1}^{+} Z_n^*, \qquad R_{n+1} := C_0 - Y_n^* \mathrm{T}_{n-1}^{+} Y_n \tag{2.4}$$

(where we use the block Toeplitz matrix in (2.1)) and

$$M_{n+1} := Z_n \mathrm{T}_{n-1}^{+} Y_n. \tag{2.5}$$

The following proposition describes the inherent structure of a Toeplitz non-negative definite sequence in $\mathbb{C}^{q\times q}$.

Proposition 2.9. *Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(C_j)_{j=0}^{\varkappa}$ be a T-n.n.d. sequence in $\mathbb{C}^{q\times q}$. Then:*

(a) *$(L_{j+1})_{j=0}^{\varkappa}$ and $(R_{j+1})_{j=0}^{\varkappa}$ are monotonically decreasing sequences, where*

$$\operatorname{rank} L_{j+1} = \operatorname{rank} R_{j+1} \qquad \text{and} \qquad \det L_{j+1} = \det R_{j+1},$$

for each $j \in \mathbb{Z}_{0,\varkappa}$. If $\varkappa = +\infty$, then the sequences $(L_{j+1})_{j=0}^{\infty}$ and $(R_{j+1})_{j=0}^{\infty}$ converge to non-negative Hermitian matrices L and R, respectively. When this is the case, $\det L = \det R$.

(b) *Suppose that* $\varkappa \geq 1$. *For each* $n \in \mathbb{Z}_{0,\varkappa-1}$, *the matrix*

$$K_{n+1} := \left(\sqrt{L_{n+1}} \right)^{+} (C_{n+1} - M_{n+1}) \left(\sqrt{R_{n+1}} \right)^{+}$$

is contractive and

$$C_{n+1} = M_{n+1} + \sqrt{L_{n+1}} K_{n+1} \sqrt{R_{n+1}}.$$

If $n \geq 1$, *then*

$$L_{n+2} = \sqrt{L_{n+1}} \left(I_q - K_{n+1} K_{n+1}^* \right) \sqrt{L_{n+1}},$$

$$R_{n+2} = \sqrt{R_{n+1}} \left(I_q - K_{n+1}^* K_{n+1} \right) \sqrt{R_{n+1}}$$

and all of the following conditions are equivalent:

(i) $L_{n+1} = L_{n+2}$.

(ii) $R_{n+1} = R_{n+2}$.

(iii) $C_{n+1} = M_{n+2}$.

(iv) $K_{n+1} = 0_{q\times q}$.

(c) *If* $(C_j)_{j=0}^{\varkappa}$ *is T-p.d. then, for each* $n \in \mathbb{Z}_{0,\varkappa-1}$, *the matrices* L_{n+1} *and* R_{n+1} *are positive Hermitian and* K_{n+1} *is strictly contractive.*

Proof. See [8, Remark 3.4.1, Theorem 3.4.1 and Remark 3.4.3]. □

If $\varkappa \in \mathbb{N} \cup \{ +\infty \}$ and $(C_j)_{j=0}^{\varkappa}$ is a T-n.n.d. sequence in $\mathbb{C}^{q\times q}$, then the sequence $(K_j)_{j=1}^{\varkappa}$ defined in Proposition 2.9 is called the Schur parameter sequence of $(C_j)_{j=0}^{\varkappa}$. Note that Schur parameter sequences appear, for instance, for Toeplitz positive definite sequences in [7, Definition 2.3], where a sequence of this type is referred to as the *sequence of canonical moments*.

Definition 2.10. A sequence $(C_j)_{j=0}^{\infty} \in \mathcal{T}_{q,\infty}$ is called totally Toeplitz positive definite if the matrix L from part (a) of Proposition 2.9 satisfies $\det L \neq 0$. The set of all totally Toeplitz positive definite sequences in $\mathcal{T}_{q,\infty}$ will be denoted by $\mathcal{T}_{q,\infty}^{\mathrm{t}}$.

Proposition 2.9 shows that

$$\mathcal{T}_{q,\infty}^{\mathrm{t}} \subseteq \widetilde{\mathcal{T}}_{q,\infty}. \tag{2.6}$$

We next concentrate on the extension problem for finite Toeplitz non-negative definite matricial sequences. If M, A, $B \in \mathbb{C}^{q\times q}$, then the set

$$\mathfrak{K}(M; A, B) := \{ M + AKB \ : \ K \in \mathbb{K}_{q\times q} \}$$

is called the (closed) matrix ball with center M, left semi-radius A and right semi-radius B and the set

$$\mathring{\mathfrak{K}}(M; A, B) := \{ M + AKB \ : \ K \in \mathbb{D}_{q\times q} \}$$

is called the open matrix ball with center M, left semi-radius A and right semi-radius B. The following theorem (see [8, Theorem 3.4.1]) gives us a complete answer to the extension problem for finite Toeplitz non-negative definite sequences.

Theorem 2.11. *Let* $n \in \mathbb{N}_0$ *and* $(C_j)_{j=0}^{n}$ *be a sequence in* $\mathbb{C}^{q\times q}$. *Then:*

(a) *The set*

$$\mathcal{C}_{\geq}\left[(C_j)_{j=0}^{n}\right] := \left\{ C_{n+1} \in \mathbb{C}^{q\times q} \;:\; (C_j)_{j=0}^{n+1} \in \mathcal{T}_{q,\,n+1} \right\}$$

is non-empty if and only if $(C_j)_{j=0}^{n} \in \mathcal{T}_{q,\,n}$.

(b) *Suppose* $(C_j)_{j=0}^{n} \in \mathcal{T}_{q,\,n}$. *Then* $L_{n+1}, R_{n+1} \in \mathbb{C}_{\geq}^{q\times q}$ *and*

$$\mathcal{C}_{\geq}\left[(C_j)_{j=0}^{n}\right] = \mathfrak{K}\left(M_{n+1}; \sqrt{L_{n+1}}, \sqrt{R_{n+1}}\right).$$

In particular, $M_{n+1} \in \mathcal{C}_{\geq}\left[(C_j)_{j=0}^{n}\right]$.

(c) *The set*

$$\mathcal{C}_{>}\left[(C_j)_{j=0}^{n}\right] := \left\{ C_{n+1} \in \mathbb{C}^{q\times q} \;:\; (C_j)_{j=0}^{n+1} \in \widetilde{\mathcal{T}}_{q,\,n+1} \right\}$$

is non-empty if and only if $(C_j)_{j=0}^{n} \in \widetilde{\mathcal{T}}_{q,\,n}$.

(d) *Suppose* $(C_j)_{j=0}^{n} \in \widetilde{\mathcal{T}}_{q,\,n}$. *Then* $L_{n+1}, R_{n+1} \in \mathbb{C}_{>}^{q\times q}$ *and*

$$\mathcal{C}_{>}\left[(C_j)_{j=0}^{n}\right] = \mathring{\mathfrak{K}}\left(M_{n+1}; \sqrt{L_{n+1}}, \sqrt{R_{n+1}}\right).$$

In particular, $M_{n+1} \in \mathcal{C}_{>}\left[(C_j)_{j=0}^{n}\right]$.

The following definition is motivated by the role which the matrix M_k (introduced in (2.2) and (2.5)) plays in Theorem 2.11; more precisely, the fact that M_k appears as the center of the matrix balls in parts (b) and (d).

Definition 2.12. Let $\varkappa \in \mathbb{N} \cup \{+\infty\}$ and $(C_j)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C}^{q\times q}$.

(a) Suppose $k \in \mathbb{Z}_{1,\,\varkappa}$. We say that $(C_j)_{j=0}^{\varkappa}$ is a central sequence of order k (or an order k central sequence), if $C_k = M_k$ for all $j \in \mathbb{Z}_{k,\,\varkappa}$, where M_k is given by (2.2) and (2.5).

(b) Let $k \in \mathbb{Z}_{1,\,\varkappa}$. We call $(C_j)_{j=0}^{\varkappa}$ a central sequence of minimal order k (or a minimal order k central sequence) if it has both of the following two properties:
  (i) $(C_j)_{j=0}^{\varkappa}$ is order k central.
  (ii) If $\varkappa \geq 2$ and $k \in \mathbb{Z}_{2,\,\varkappa}$, then $(C_j)_{j=0}^{\varkappa}$ is not order $k-1$ central.

(c) The sequence $(C_j)_{j=0}^{\varkappa}$ is simply called a central sequence, if there exists a $k \in \mathbb{Z}_{1,\,\varkappa}$ such that $(C_j)_{j=0}^{\varkappa}$ is order k central.

Remark 2.13. Let $\varkappa \in \mathbb{N} \cup \{+\infty\}$ and $k \in \mathbb{Z}_{1,\,\varkappa}$. Furthermore, suppose that $(C_j)_{j=0}^{\varkappa}$ is an order k central sequence in $\mathbb{C}^{q\times q}$. For each $\ell \in \mathbb{Z}_{k,\,\varkappa}$, the sequence $(C_j)_{j=0}^{\varkappa}$ is then also order ℓ central.

It is easy to characterize order 1 central sequences in $\mathbb{C}^{q\times q}$.

Remark 2.14. Let $\varkappa \in \mathbb{N} \cup \{+\infty\}$ and $(C_j)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C}^{q\times q}$.

(a) Inductively, we see from (2.4) and (2.5) that $(C_j)_{j=0}^{\varkappa}$ is order 1 central when $C_j = 0_{q\times q}$, for all $j \in \mathbb{Z}_{1,\,\varkappa}$.

(b) From (a) we see that $(C_j)_{j=0}^{\varkappa}$ is T-n.n.d. and order 1 central if and only if $C_0 \in \mathbb{C}_{\geq}^{q\times q}$ and $C_j = 0_{q\times q}$, for all $j \in \mathbb{Z}_{1,\varkappa}$.
(c) From (a) we furthermore see that $(C_j)_{j=0}^{\varkappa}$ is T-p.d. and order 1 central when $C_0 \in \mathbb{C}_{>}^{q\times q}$ and $C_j = 0_{q\times q}$, for all $j \in \mathbb{Z}_{1,\varkappa}$.

Our next steps will be towards characterizing central Toeplitz non-negative definite sequences.

Proposition 2.15. *Let $\varkappa \in \mathbb{N} \cup \{+\infty\}$ and $(C_j)_{j=0}^{\varkappa}$ be a T-n.n.d. sequence in $\mathbb{C}^{q\times q}$ with Schur parameter sequence $(K_j)_{j=1}^{\varkappa}$. Let, furthermore, $k \in \mathbb{Z}_{1,\varkappa}$. All of the following conditions are equivalent:*

(i) *$(C_j)_{j=0}^{\varkappa}$ is order k central.*
(ii) *$L_{j+1} = L_k$ for all $j \in \mathbb{Z}_{k,\varkappa}$.*
(iii) *$R_{j+1} = R_k$ for all $j \in \mathbb{Z}_{k,\varkappa}$.*
(iv) *$K_j = 0_{q\times q}$ for all $j \in \mathbb{Z}_{k,\varkappa}$.*

Proof. The equivalence of (i)–(iv) follows directly from Proposition 2.9. □

Next, we arrive at recursion formulas for the elements of a central Toeplitz non-negative definite sequence. The corresponding result, the following Theorem 2.16, is an immediate consequence of [14, Proposition 1 and Remark 1]. A result equivalent to Theorem 2.16 is found in [15, Theorem 32] as well as in [8, Theorem 3.4.3]. For the special case of a Toeplitz positive definite sequence, an equivalent result to Theorem 2.16 was already included in [12, Theorem 20] with two different proofs. The first of these proofs is based on applying an extension problem result (for the Wiener algebra $W(\mathbb{T})$) by Dym/Gohberg [9, Theorem 6.1]. The second of these proofs uses results for orthogonal matrix polynomials on the unit circle by Delsarte/Genin/Kamp [28]. For the special case of a Toeplitz positive definite sequence, a further equivalent result to Theorem 2.16 can be found in Ellis/Gohberg [10, Theorem 2.2].

Theorem 2.16. *Let $\varkappa \in \mathbb{Z}_{2,\infty} \cup \{+\infty\}$, $k \in \mathbb{Z}_{1,\varkappa-1}$ and $(C_j)_{j=0}^{\varkappa}$ be an order k central T-n.n.d. sequence in $\mathbb{C}^{q\times q}$. Let $s \in \mathbb{Z}_{k+1,\varkappa}$. Suppose that*

$$Z_{s,k} := \operatorname{row}\,(C_{s-j})_{j=1}^{k} \qquad \text{and} \qquad Y_{s,k} := \operatorname{col}\,(C_{s-1-k+j})_{j=1}^{k}$$

and that Z_k and Y_k are given by (2.3). Then

$$C_s = Z_{s,k}\mathrm{T}_{k-1}^{+}Y_k \qquad \text{and} \qquad C_s = Z_k\mathrm{T}_{k-1}^{+}Y_{s,k}.$$

Corollary 2.17. *Let $\varkappa \in \mathbb{Z}_{2,\varkappa} \cup \{+\infty\}$ and $(C_j)_{j=0}^{\varkappa}$ be an order 2 central T-n.n.d. sequence in $\mathbb{C}^{q\times q}$. Suppose that $s \in \mathbb{Z}_{2,\varkappa}$. Then:*

(a) *$C_s = C_{s-1}C_0^{+}C_1$ and $C_s = C_1C_0^{+}C_{s-1}$.*
(b) *$C_s = C_1\left(C_0^{+}C_1\right)^{s-1}$ and $C_s = \left(C_1C_0^{+}\right)^{s-1}C_1$.*

Proof. Part (a) follows directly from Theorem 2.16, while (b) follows from (a). □

Given an $n \in \mathbb{N}_0$ and a sequence $(C_j)_{j=0}^n$ in $\mathbb{C}^{q\times q}$, we next consider the unique extension of $(C_j)_{j=0}^n$ to an order $n+1$ central sequence.

Remark 2.18. Let $n \in \mathbb{N}_0$ and $(C_j)_{j=0}^n$ be a sequence in $\mathbb{C}^{q\times q}$. There exists a unique order $n+1$ central sequence $\left(\widetilde{C}_j\right)_{j=0}^\infty$ in $\mathbb{C}^{q\times q}$ such that $\widetilde{C}_j = C_j$ for each $j \in \mathbb{Z}_{0,n}$. This sequence is called the central sequence corresponding to $(C_j)_{j=0}^n$.

We have now arrived at an idea central to our topic. Specifically, we now consider central sequences for the case that our initial sequence is Toeplitz non-negative or positive definite.

Lemma 2.19. *Let $k \in \mathbb{N}$ and $(C_j)_{j=0}^{k-1}$ be a sequence in $\mathbb{C}^{q\times q}$. Suppose that $\left(\widetilde{C}_j\right)_{j=0}^\infty$ is the central sequence corresponding to $(C_j)_{j=0}^{k-1}$. Then:*

(a) *$\left(\widetilde{C}_j\right)_{j=0}^\infty \in \mathcal{T}_{q,\infty}$ if and only if $(C_j)_{j=0}^{k-1} \in \mathcal{T}_{q,k-1}$.*

(b) *All of the following conditions are equivalent:*

  (i) $\left(\widetilde{C}_j\right)_{j=0}^\infty \in \mathcal{T}_{q,\infty}^{\mathrm{t}}$.

  (ii) $\left(\widetilde{C}_j\right)_{j=0}^\infty \in \widetilde{\mathcal{T}}_{q,\infty}$.

  (iii) $(C_j)_{j=0}^{k-1} \in \widetilde{\mathcal{T}}_{q,k-1}$.

Proof. Part (a) follows immediately from parts (b) and (d) of Theorem 2.11.
(b) "(iii)$\Longrightarrow$(i)". Because of (iii), part (c) of Proposition 2.9 implies $L_k \in \mathbb{C}_>^{q\times q}$. Suppose that $\left(\widetilde{L}_{s+1}\right)_{s=0}^\infty$ is the sequence constructed from $\left(\widetilde{C}_j\right)_{j=0}^\infty$ via (2.2) - (2.4) and that $\widetilde{L} := \lim_{s\longrightarrow\infty} \widetilde{L}_{s+1}$. Since Proposition 2.15 yields $\widetilde{L}_s = L_k$ for each $s \in \mathbb{Z}_{k,\infty}$, we get that $\widetilde{L} = L_k$. Because the matrix L_k is positive Hermitian, it follows that $\det \widetilde{L} \neq 0$ and we have (i).
"(i)$\Longrightarrow$(ii)" follows directly from (2.6).
"(ii)$\Longrightarrow$(iii)" follows from $\left(\widetilde{C}_j\right)_{j=0}^{k-1} = (C_j)_{j=0}^{k-1}$ and the definition of $\widetilde{\mathcal{T}}_{q,\infty}$. □

If $k \in \mathbb{N}$ and $(C_j)_{j=0}^{k-1} \in \mathcal{T}_{q,k-1}$, then Theorem 2.16 gives us a method for recursively constructing the central sequence corresponding to $(C_j)_{j=0}^{k-1}$. In anticipation of later applications, we formulate this result for the special case $k=1$.

Corollary 2.20. *Suppose that $(C_j)_{j=0}^1$ is a T-n.n.d. sequence in $\mathbb{C}^{q\times q}$ and that $\left(\widetilde{C}_j\right)_{j=0}^\infty$ is the central sequence corresponding to $(C_j)_{j=0}^1$. Then*

$$\widetilde{C}_s = C_1\left(C_0^+ C_1\right)^{s-1} \qquad \text{and} \qquad \widetilde{C}_s = \left(C_1 C_0^+\right)^{s-1} C_1$$

for each $s \in \mathbb{Z}_{2,+\infty}$.

Proof. We need only combine Corollary 2.17 with Remark 2.18. □

Example 2.21. Let $K \in \mathbb{K}_{q\times q}$. For each $j \in \mathbb{N}_0$, let $C_j := K^j$. Using Example 2.1, Corollary 2.20 and Lemma 2.19, it is easily verified that:

(a) $(C_j)_{j=0}^{\infty}$ is an order 2 central T-n.n.d. sequence in $\mathbb{C}^{q\times q}$ and if (and only if) $K \neq 0_{q\times q}$, then $(C_j)_{j=0}^{\infty}$ is minimal order 2 central.

(b) $L_1 = I_q$, $R_1 = I_q$ and, for each $k \in \mathbb{N}$, furthermore

$$L_{k+1} = I_q - KK^* \qquad \text{and} \qquad R_{k+1} = I_q - K^*K.$$

(c) $(C_j)_{j=0}^{\infty} \in \mathcal{T}_{q,\infty}^{\mathrm{t}}$ if and only if $K \in \mathbb{D}_{q\times q}$.

Example 2.22. Let $K \in \mathbb{K}_{q\times q} \cap \mathbb{C}_{\geq}^{q\times q}$ and $r \in \mathbb{N}$. For each $j \in \mathbb{N}_0$, suppose $C_j := K^{r+j}$. Then $(C_j)_{j=0}^{\infty}$ is an order 2 central T-n.n.d. sequence in $\mathbb{C}^{q\times q}$. This can be recognized as follows. Since $K \in \mathbb{C}_{\geq}^{q\times q}$, part (b) of Lemma A.4 implies $KK^+ = K^+K$. Thus, by induction, we obtain

$$(K^r)^+ K^{r+1} = K. \tag{2.7}$$

For each $j \in \mathbb{N}_0$, we have $C_j = \left(\sqrt{K^r}\right)^* K^j \sqrt{K^r}$. Combining this with the fact that Example 2.21 yields $\left(K^j\right)_{j=0}^{\infty} \in \mathcal{T}_{q,\infty}$, we see by Remark 2.3 that $(C_j)_{j=0}^{\infty} \in \mathcal{T}_{q,\infty}$. Therefore, $(C_j)_{j=0}^{1} \in \mathcal{T}_{q,1}$. Thus, (2.7) implies

$$C_1\left(C_0^+C_1\right)^{s-1} = K^{r+1}\left[(K^r)^+ K^{r+1}\right]^{s-1} = K^{r+1}K^{s-1} = K^{r+s} = C_s$$

for each $s \in \mathbb{Z}_{2,\infty}$. Thus, Corollary 2.20 shows that $(C_j)_{j=0}^{\infty}$ is the central sequence corresponding to $(C_j)_{j=0}^{1}$. Hence, $(C_j)_{j=0}^{\infty}$ is order 2 central.

Remark 2.23. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$.

(a) If $(C_j)_{j=0}^{\varkappa} \in \mathcal{T}_{q,\varkappa}$ and $w \in \mathbb{D} \cup \mathbb{T}$, then from part (a) of Remark 2.6 and Example 2.21, it follows that $\left(w^j C_j\right)_{j=0}^{\varkappa} \in \mathcal{T}_{q,\varkappa}$.

(b) If $(C_j)_{j=0}^{\varkappa} \in \widetilde{\mathcal{T}}_{q,\varkappa}$ and $w \in \mathbb{D}$, then from part (b) of Remark 2.6 and Example 2.21, it follows that $\left(w^j C_j\right)_{j=0}^{\varkappa} \in \widetilde{\mathcal{T}}_{q,\varkappa}$.

We now introduce another class of matricial sequences which will be particularly interesting when we again look at Toeplitz non-negative definite sequences.

Definition 2.24. Let $\varkappa \in \mathbb{N} \cup \{+\infty\}$ and $(C_j)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C}^{q\times q}$.

(a) Suppose $k \in \mathbb{Z}_{1,\varkappa}$. We will say that $(C_j)_{j=0}^{\varkappa}$ is a canonical sequence of order k (or an order k canonical sequence), if $\operatorname{rank} \mathrm{T}_{k-1} = \operatorname{rank} \mathrm{T}_k$.

(b) Suppose $k \in \mathbb{Z}_{1,\varkappa}$. We call $(C_j)_{j=0}^{\varkappa}$ a canonical sequence of minimal order k (or a minimal order k canonical sequence) if it has the following two properties:
  (i) $(C_j)_{j=0}^{\varkappa}$ is order k canonical.
  (ii) If $\varkappa \geq 2$ and $k \in \mathbb{Z}_{2,\varkappa}$, then for each $\ell \in \mathbb{Z}_{1,k}$, the sequence $(C_j)_{j=0}^{\varkappa}$ is not order ℓ canonical.

(c) $(C_j)_{j=0}^{\varkappa}$ is simply called a canonical sequence, if there exists a $k \in \mathbb{Z}_{1,\varkappa}$ such that $(C_j)_{j=0}^{\varkappa}$ is order k canonical.

We next arrive at a characterization of canonical Toeplitz non-negative definite sequences.

Lemma 2.25. *If* $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ *and* $(C_j)_{j=0}^{\varkappa}$ *is a T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$, *then* $\operatorname{rank} \mathrm{T}_n = \operatorname{rank} \mathrm{T}_{n-1} + \operatorname{rank} L_{n+1}$ *for all* $n \in \mathbb{Z}_{1,\varkappa}$.

Proof. Apply [8, Lemma 1.1.7]. □

Proposition 2.26. *Let* $\varkappa \in \mathbb{N} \cup \{+\infty\}$ *and* $(C_j)_{j=0}^{\varkappa}$ *be a T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$. *Let, furthermore,* $k \in \mathbb{Z}_{1,\varkappa}$. *All of the following conditions are equivalent:*

(i) $(C_j)_{j=0}^{\varkappa}$ *is order* k *canonical.*
(ii) $L_{k+1} = 0_{q\times q}$.
(iii) $R_{k+1} = 0_{q\times q}$.
(iv) $L_{j+1} = 0_{q\times q}$ *for all* $j \in \mathbb{Z}_{k,\varkappa}$.
(v) $R_{j+1} = 0_{q\times q}$ *for all* $j \in \mathbb{Z}_{k,\varkappa}$.

Proof. Combining Definition 2.24 and Lemma 2.25 with part (b) of Proposition 2.9 yields the proof. □

Corollary 2.27. *Let* $\varkappa \in \mathbb{N} \cup \{+\infty\}$ *and* $k \in \mathbb{Z}_{1,\varkappa}$. *Furthermore, suppose that* $(C_j)_{j=0}^{\varkappa}$ *is an order* k *canonical T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$. *Then:*

(a) *For each* $\ell \in \mathbb{Z}_{k,\varkappa}$, *the sequence* $(C_j)_{j=0}^{\varkappa}$ *is order* ℓ *canonical.*
(b) *If* $\varkappa \geq 2$ *and* $k \in \mathbb{Z}_{1,\varkappa-1}$, *then, for each* $\ell \in \mathbb{Z}_{k+1,\varkappa}$, *the sequence* $(C_j)_{j=0}^{\varkappa}$ *is order* ℓ *central.*

Proof. Use Propositions 2.26 and 2.15. □

Example 2.28. Let $\varkappa \in \mathbb{N} \cup \{+\infty\}$ and $A \in \mathbb{C}_{\mathrm{H}}^{q\times q}$. For each $j \in \mathbb{Z}_{0,\varkappa}$ let $C_j := A$. Then, for each $n \in \mathbb{Z}_{0,\varkappa}$, it follows that $\operatorname{rank} \mathrm{T}_n = \operatorname{rank} A$. Therefore, $(C_j)_{j=0}^{\varkappa}$ is order 1 canonical. In particular, if $A \in \mathbb{C}_{\geq}^{q\times q}$, then it follows from Example 2.2 that $(C_j)_{j=0}^{\varkappa}$ is an order 1 canonical T-n.n.d. sequence.

Example 2.29. Let $K \in \mathbb{K}_{q\times q}$. For each $j \in \mathbb{N}_0$, let $C_j := K^j$. Then $(C_j)_{j=0}^{\infty}$ is canonical if and only if K is unitary. When this is the case, $(C_j)_{j=0}^{\infty}$ is order 1 canonical. This can be recognized as follows: From part (a) of Example 2.21, we see that $(C_j)_{j=0}^{\infty} \in \mathcal{T}_{1,\infty}$. If K is not unitary, then part (b) of Example 2.21 shows us that $L_{k+1} \neq 0_{q\times q}$, for each $k \in \mathbb{N}_0$. By Proposition 2.26, it thus follows that $(C_j)_{j=0}^{\infty}$ is not canonical. If K is unitary, then part (b) of Example 2.21 shows us that $L_{k+1} = 0_{q\times q}$, for each $k \in \mathbb{N}$. By Proposition 2.26, we thus see that $(C_j)_{j=0}^{\infty}$ is order 1 canonical.

The following lemma demonstrates an important approach to constructing canonical $\mathcal{T}_{q,\infty}$ sequences.

Lemma 2.30. *Suppose* $r \in \mathbb{N}$, $(A_s)_{s=1}^r$ *is a sequence in* $\mathbb{C}_{\geq}^{q\times q}$ *and that* $(z_s)_{s=1}^r$ *is a sequence of pairwise different points in* $\mathbb{T}$. *For each* $j \in \mathbb{N}_0$, *let*

$$C_j := \sum_{s=1}^{r} {z_s}^{-j} A_s.$$

Then $(C_j)_{j=0}^\infty \in \mathcal{T}_{q,\infty}$ *and* $\operatorname{rank} \mathrm{T}_n = \sum_{s=1}^{r} \operatorname{rank} A_s$ *for* $n \in \mathbb{Z}_{r-1,\infty}$. *The sequence* $(C_j)_{j=0}^\infty$ *is, furthermore, order* r *canonical and* $(C_j)_{j=0}^{r-1} \in \widetilde{\mathcal{T}}_{q,r-1}$ *if and only if* $(A_s)_{s=1}^r$ *is a sequence in* $\mathbb{C}_{>}^{q\times q}$.

Proof. Let $n \in \mathbb{N}_0$. Considering the Vandermonde matrix

$$V_{q,n}\big(\,(z_s)_{s=1}^r\,\big) := \begin{pmatrix} z_1^0 I_q & z_1^1 I_q & \cdots & z_1^n I_q \\ z_2^0 I_q & z_2^1 I_q & \cdots & z_2^n I_q \\ \vdots & \vdots & & \vdots \\ z_r^0 I_q & z_r^1 I_q & \cdots & z_r^n I_q \end{pmatrix},$$

we obtain

$$\mathrm{T}_n = \big[V_{q,n}\big(\,(z_s)_{s=1}^r\,\big)\big]^* \left[\operatorname{diag}(A_1, A_2, \ldots, A_r)\right] \big[V_{q,n}\big(\,(z_s)_{s=1}^r\,\big)\big]. \tag{2.8}$$

Since $(A_s)_{s=1}^r$ is a sequence in $\mathbb{C}_{\geq}^{q\times q}$, this implies $\mathrm{T}_n \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$. Hence, $(C_j)_{j=0}^\infty \in \mathcal{T}_{q,\infty}$. Since the elements of $(z_s)_{s=1}^r$ are pairwise different, it follows that

$$\operatorname{rank}\big[V_{q,n}\big(\,(z_s)_{s=1}^r\,\big)\big] = r \cdot q, \tag{2.9}$$

for all $n \in \mathbb{Z}_{r-1,\infty}$ and thus, from (2.8) that

$$\operatorname{rank} \mathrm{T}_n = \operatorname{rank}\left[\operatorname{diag}(A_1, A_2, \ldots, A_r)\right] = \sum_{s=1}^{r} \operatorname{rank} A_s.$$

We thus also see that $(C_j)_{j=0}^\infty$ is order r canonical. For $n = r-1$ it follows from (2.9) that $\det\big[V_{q,r-1}\big(\,(z_s)_{s=1}^r\,\big)\big] \neq 0$. Therefore, from (2.8), we see that $(C_j)_{j=0}^{r-1} \in \widetilde{\mathcal{T}}_{q,r-1}$ if and only if $(A_s)_{s=1}^r$ is a sequence in $\mathbb{C}_{>}^{q\times q}$. □

The next result shows (see [20, Corollary 1.14]) that every canonical sequence $(C_j)_{j=0}^n \in \mathcal{T}_{q,\infty}$ is of the form described in Lemma 2.30.

Theorem 2.31. *Suppose that* $(C_j)_{j=0}^\infty$ *is a sequence in* $\mathbb{C}^{q\times q}$. *Then both of the following two conditions are equivalent:*

(i) $(C_j)_{j=0}^\infty$ *is a canonical T-n.n.d. sequence.*

(ii) *There are some* $r \in \mathbb{N}$, *pairwise different points* $z_1, z_2, \ldots, z_r \in \mathbb{T}$ *and matrices* $A_1, A_2, \ldots, A_r \in \mathbb{C}_{\geq}^{q\times q}$ *such that*

$$C_j = \sum_{m=1}^{r} z_m^{-j} A_m \qquad j \in \mathbb{N}_0.$$

The following proposition can be seen as an addendum to Theorem 2.11.

Proposition 2.32. *Let $n \in \mathbb{N}_0$ and $(C_j)_{j=0}^{n} \in \widetilde{\mathcal{T}}_{q,n}$. Suppose, furthermore, that $C_{n+1} \in \mathbb{C}^{q\times q}$. Then both of the following two conditions are equivalent:*

(i) *The sequence $(C_j)_{j=0}^{n+1}$ is an order $n+1$ canonical T-n.n.d. sequence.*

(ii) *There exists a unitary matrix $U_{n+1} \in \mathbb{C}^{q\times q}$ such that*

$$C_{n+1} = M_{n+1} + \sqrt{L_{n+1}}U_{n+1}\sqrt{R_{n+1}}.$$

Proof. Combine part (b) of Proposition 2.9, part (b) of Theorem 2.11 and Proposition 2.26. □

For a detailed discussion of canonical Toeplitz non-negative definite sequences in $\mathbb{C}^{q\times q}$, we refer the reader to [20, Section 1], where, for arbitrary $n \in \mathbb{N}_0$ and $(C_j)_{j=0}^{n} \in \mathcal{T}_{q,n}$, the set of all $C_{n+1} \in \mathcal{C}_{\geq}\left[(C_j)_{j=0}^{n}\right]$ for which $(C_j)_{j=0}^{n+1}$ is order n canonical is described. This set is never empty.

The canonical extension problem for sequences in $\mathcal{T}_{q,n}$ is a special case of the problem of determining all rank-preserving extensions of such sequences. This more general problem was dealt with in [11, Theorem 3]. For discussions of this topic in the scalar case, we refer the reader to the monograph Iohvidov [24] as well as the article Akimoto/Ito [1].

3. Matricial Carathéodory sequences

In this section, we present some basic facts on matricial Carathéodory sequences and their relationship to Toeplitz non-negative definite sequences of matrices. If $n \in \mathbb{N}_0$, then a sequence $(s_j)_{j=0}^{n}$ in $\mathbb{C}^{q\times q}$ is called a $q\times q$ Carathéodory sequence (or simply a $q\times q$ C-sequence) if the real part $\operatorname{Re}\mathrm{S}_n$ of the matrix S_n given by (1.1) and $\mathrm{S}_n := \mathrm{S}_n^{(s)}$ is non-negative Hermitian, i.e., if $\mathrm{S}_n \in \mathcal{R}_{(n+1)q,\geq}$, and a strict $q\times q$ Carathéodory sequence (or strict $q\times q$ C-sequence) if $\operatorname{Re}\mathrm{S}_n$ is positive Hermitian, i.e., if $\mathrm{S}_n \in \mathcal{R}_{(n+1)q,>}$.

Letting $n \in \mathbb{N}$, $m \in \mathbb{Z}_{0,n-1}$ and $(s_j)_{j=0}^{n}$ be a $q\times q$ C-sequence, we see from (1.1) that S_m is the upper left $(m+1)q \times (m+1)q$ block of S_n. Hence, $(s_j)_{j=0}^{m}$ is also a $q\times q$ C-sequence. Similarly, if $(s_j)_{j=0}^{n}$ is a strict C-sequence, then $(s_j)_{j=0}^{m}$ is also a strict C-sequence. For this reason, we call a sequence $(s_j)_{j=0}^{\infty}$ in $\mathbb{C}^{q\times q}$ a $q\times q$ Carathéodory sequence if, for each $n \in \mathbb{N}_0$, the sequence $(s_j)_{j=0}^{n}$ is a $q\times q$ Carathéodory sequence. A Carathéodory sequence $(s_j)_{j=0}^{\infty}$ is called strict if, for each $n \in \mathbb{N}_0$, the sequence $(s_j)_{j=0}^{n}$ is a strict $q\times q$ Carathéodory sequence. For each $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$, the set of all $q\times q$ Carathéodory sequences will be denoted by $\mathcal{C}_{q,\varkappa}$ and the set of all strict $q\times q$ Carathéodory sequences by $\widetilde{\mathcal{C}}_{q,\varkappa}$.

Remark 3.1. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$. If $(s_j)_{j=0}^{\varkappa}$ is a $q\times q$ C-sequence, then $s_0 \in \mathcal{R}_{q,\geq}$. If $(s_j)_{j=0}^{\varkappa}$ is a strict $q\times q$ C-sequence, then $s_0 \in \mathcal{R}_{q,>}$.

Remark 3.2. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C}^{q\times q}$ with $s_j = 0_{q\times q}$, for each $j \in \mathbb{Z}_{1,\varkappa}$. Then $(s_j)_{j=0}^{\varkappa}$ is a $q \times q$ C-sequence if and only if $s_0 \in \mathcal{R}_{q,\geq}$. Moreover, $(s_j)_{j=0}^{\varkappa}$ is a strict $q\times q$ C-sequence if and only if $s_0 \in \mathcal{R}_{q,>}$.

Remark 3.3. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$. If $(s_j)_{j=0}^{\varkappa}$ is a $q \times q$ C-sequence and $A \in \mathbb{C}^{q\times p}$, then $(A^* s_j A)_{j=0}^{\varkappa}$ is a $p \times p$ C-sequence.

Remark 3.4. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $m \in \mathbb{N}$. For each $r \in \mathbb{Z}_{1,m}$, suppose that $\alpha_r \in [0, +\infty)$ and $\left(s_j^{(r)}\right)_{j=0}^{\varkappa} \in \mathcal{C}_{q,\varkappa}$. Then $\left(\sum_{r=1}^{m} \alpha_r s_j^{(r)}\right)_{j=0}^{\varkappa} \in \mathcal{C}_{q,\varkappa}$. If, furthermore, there exists an $r_0 \in \mathbb{Z}_{1,m}$ such that $\alpha_{r_0} \in (0, +\infty)$ and $\left(s_j^{(r_0)}\right)_{j=0}^{\varkappa} \in \widetilde{\mathcal{C}}_{q,\varkappa}$, then $\left(\sum_{r=1}^{m} \alpha_r s_j^{(r)}\right)_{j=0}^{\varkappa} \in \widetilde{\mathcal{C}}_{q,\varkappa}$.

Remark 3.5. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and $r \in \mathbb{N}$. Suppose that $(q_j)_{j=1}^{r}$ is a sequence in $\mathbb{N}$ such that $q = \sum_{j=1}^{r} q_j$. For each $m \in \mathbb{Z}_{1,r}$, let $\left(s_j^{(m)}\right)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C}^{q_m\times q_m}$. Then $(d_j)_{j=0}^{\varkappa} := \left(\operatorname{diag}\left(s_j^{(1)}, s_j^{(2)}, \ldots, s_j^{(r)}\right)\right)_{j=0}^{\varkappa}$ is a $q \times q$ C-sequence if and only if $\left(s_j^{(m)}\right)_{j=0}^{\varkappa}$ is a $q_m \times q_m$ C-sequence for all $m \in \mathbb{Z}_{1,r}$. The sequence $(d_j)_{j=0}^{\varkappa}$ is a strict $q \times q$ C-sequence if and only if $\left(s_j^{(m)}\right)_{j=0}^{\varkappa}$ is a strict $q_m \times q_m$ C-sequence for all $m \in \mathbb{Z}_{1,r}$.

Matricial Carathéodory sequences are closely related to Toeplitz non-negative definite sequences of matrices.

Remark 3.6. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$. Let $(s_j)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C}^{q\times q}$ and let $(C_j)_{j=0}^{\varkappa}$ be defined by

$$C_\ell := \begin{cases} \operatorname{Re} s_0, & \text{if } \ell = 0, \\ \frac{1}{2} s_\ell, & \text{if } \ell \in \mathbb{Z}_{1,\varkappa}. \end{cases} \tag{3.1}$$

Then $\operatorname{Re} \mathrm{S}_k = \mathrm{T}_k$ for each $k \in \mathbb{Z}_{0,\varkappa}$. Thus, $(s_j)_{j=0}^{\varkappa}$ is a $q \times q$ C-sequence if and only if $(C_j)_{j=0}^{\varkappa}$ is Toeplitz non-negative definite. Furthermore, $(s_j)_{j=0}^{\varkappa}$ is a strict $q \times q$ C-sequence if and only if $(C_j)_{j=0}^{\varkappa}$ is Toeplitz positive definite.

Remark 3.7. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(C_j)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C}^{q\times q}$. Suppose that $s_0 := C_0$ and that $s_j := 2C_j$, for each $j \in \mathbb{Z}_{1,\varkappa}$. For each $k \in \mathbb{Z}_{0,\varkappa}$, it then follows that $\mathrm{T}_k = \operatorname{Re} \mathrm{S}_k$. Thus, $(C_j)_{j=0}^{\varkappa}$ is a T-n.n.d. sequence if and only if $(s_j)_{j=0}^{\varkappa}$ is a $q \times q$ Carathéodory sequence and $s_0 = s_0^*$.

Remark 3.8. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and $(s_j)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C}^{q\times q}$ and let $(\widehat{s}_j)_{j=0}^{\varkappa}$ be defined by $\widehat{s}_0 := \operatorname{Re} s_0$ and $\widehat{s}_\ell := s_\ell$ for each $\ell \in \mathbb{Z}_{1,\varkappa}$. Then Remark 3.6

shows that $(s_j)_{j=0}^{\varkappa}$ is a $q\times q$ C-sequence if and only if $(\widehat{s}_j)_{j=0}^{\varkappa}$ is a $q\times q$ C-sequence. $(s_j)_{j=0}^{\varkappa}$ is a strict $q\times q$ C-sequence if and only if $(\widehat{s}_j)_{j=0}^{\varkappa}$ is a strict $q\times q$ C-sequence.

Remark 3.9. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$.

(a) If $(r_j)_{j=0}^{\varkappa} \in \mathcal{C}_{1,\varkappa}$ and $(s_j)_{j=0}^{\varkappa} \in \mathcal{C}_{q,\varkappa}$, then, by part (a) of Remark 2.6 and Remark 3.6, it follows that $(r_j s_j)_{j=0}^{\varkappa} \in \mathcal{C}_{q,\varkappa}$.

(b) If $(r_j)_{j=0}^{\varkappa} \in \widetilde{\mathcal{C}}_{1,\varkappa}$ and $(s_j)_{j=0}^{\varkappa} \in \widetilde{\mathcal{C}}_{q,\varkappa}$, then, by part (b) of Remark 2.6 and Remark 3.6, it follows that $(r_j s_j)_{j=0}^{\varkappa} \in \widetilde{\mathcal{C}}_{q,\varkappa}$.

Remark 3.10. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$.

(a) Suppose $u \in \mathbb{D} \cup \mathbb{T}$ and $(s_j)_{j=0}^{\varkappa} \in \mathcal{C}_{q,\varkappa}$. By part (a) of Remark 2.23 and part (a) of Remark 3.9 it follows that $\left(u^j s_j\right)_{j=0}^{\varkappa} \in \mathcal{C}_{q,\varkappa}$.

(b) Suppose $u \in \mathbb{D}$ and $(s_j)_{j=0}^{\varkappa} \in \widetilde{\mathcal{C}}_{q,\varkappa}$. By part (b) of Remark 2.23 and part (b) of Remark 3.9 it follows that $\left(u^j s_j\right)_{j=0}^{\varkappa} \in \widetilde{\mathcal{C}}_{q,\varkappa}$.

Lemma 3.11. *Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$. Then $\mathcal{T}_{q,\varkappa} \subseteq \mathcal{C}_{q,\varkappa}$. Furthermore, a sequence $(C_j)_{j=0}^{\varkappa} \in \mathcal{T}_{q,\varkappa}$ belongs to $\widetilde{\mathcal{C}}_{q,\varkappa}$ if and only if $C_0 \in \mathbb{C}_{>}^{q\times q}$.*

Proof. Let $(C_j)_{j=0}^{\varkappa} \in \mathcal{T}_{q,\varkappa}$. Thus, $C_0 \in \mathbb{C}_{\geq}^{q\times q}$ and, in particular, it follows that $C_0 \in \mathcal{R}_{q,\geq}$. Hence, by Remark 3.2, it follows that the sequence $(r_j)_{j=0}^{\varkappa}$ given by $r_j := \delta_{j,0} C_0$, where $\delta_{j,k}$ is the Kronecker delta, belongs to $\mathcal{C}_{q,\varkappa}$. If we define the sequence $(s_j)_{j=0}^{\varkappa}$ as we did in Remark 3.7 using $(C_j)_{j=0}^{\varkappa}$, then by the same remark $(s_j)_{j=0}^{\varkappa}$ belongs to $\mathcal{C}_{q,\varkappa}$. Since $C_j = \frac{1}{2}(r_j + s_j)$ for each $j \in \mathbb{Z}_{0,\varkappa}$, it follows from Remark 3.4 that $(C_j)_{j=0}^{\varkappa} \in \mathcal{C}_{q,\varkappa}$. If $C_0 \in \mathbb{C}_{>}^{q\times q}$, then $C_0 \in \mathcal{R}_{q,>}$. Since $C_0 \in \mathbb{C}_{>}^{q\times q}$, it therefore follows by Remark 3.2 that $(r_j)_{j=0}^{\varkappa} \in \widetilde{\mathcal{C}}_{q,\varkappa}$ and thus by Remark 3.4 that $(C_j)_{j=0}^{\varkappa} \in \widetilde{\mathcal{C}}_{q,\varkappa}$. Conversely, if we suppose that $(C_j)_{j=0}^{\varkappa} \in \widetilde{\mathcal{C}}_{q,\varkappa}$, then it immediately follows that $C_0 = \operatorname{Re} C_0 \in \mathbb{C}_{>}^{q\times q}$. □

Example 3.12. Let $K \in \mathbb{K}_{q\times q}$ and let $C_j := K^j$, for each $j \in \mathbb{N}_0$. We then have $C_0 = I_q \in \mathbb{C}_{>}^{q\times q}$ and see from Example 2.21 that $(C_j)_{j=0}^{\infty} \in \mathcal{T}_{q,\infty}$. Thus, applying Lemma 3.11, we see that $(C_j)_{j=0}^{\infty} \in \widetilde{\mathcal{C}}_{q,\infty}$.

Example 3.13. Let $K \in \mathbb{K}_{q\times q} \cap \mathbb{C}_{\geq}^{q\times q}$, $r \in \mathbb{N}$ and $C_j := K^{r+j}$, for each $j \in \mathbb{N}_0$. Then $(C_j)_{j=0}^{\infty} \in \mathcal{C}_{q,\infty}$. Furthermore, $(C_j)_{j=0}^{\infty} \in \widetilde{\mathcal{C}}_{q,\infty}$ if and only if $K \in \mathbb{C}_{>}^{q\times q}$. This follows by combining Example 2.22 and Lemma 3.11, while observing that $K^r \in \mathbb{C}_{>}^{q\times q}$ if and only if $K \in \mathbb{C}_{>}^{q\times q}$.

Motivated by Remark 3.6, we now go about implementing special modifications of Definitions 2.10, 2.12 and 2.24.

Definition 3.14. A sequence $(s_j)_{j=0}^{\infty}$ in $\mathbb{C}^{q\times q}$ is called a totally strict $q\times q$ Carathéodory sequence if the sequence $(C_j)_{j=0}^{\infty}$ defined by (3.1) with $\varkappa = +\infty$ is a totally

Toeplitz positive definite sequence in $\mathbb{C}^{q\times q}$. The set of all totally strict $q\times q$ Carathéodory sequences will be denoted by $\mathcal{C}^{\,\mathrm{t}}_{q,\infty}$.

Formula (2.6) and Remark 3.6 imply that

$$\mathcal{C}^{\,\mathrm{t}}_{q,\infty}\subseteq\widetilde{\mathcal{C}}_{q,\infty}. \tag{3.2}$$

Definition 3.15. Let $\varkappa\in\mathbb{N}\cup\{+\infty\}$ and $(s_j)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C}^{q\times q}$. Furthermore, let the sequence $(C_j)_{j=0}^{\varkappa}$ be defined by (3.1).

(a) Suppose $k\in\mathbb{Z}_{1,\varkappa}$. We say that $(s_j)_{j=0}^{\varkappa}$ is a $q\times q$ Carathéodory-central sequence of order k (or a $q\times q$ order k C-central sequence), if $(C_j)_{j=0}^{\varkappa}$ is order k central.

(b) Suppose $k\in\mathbb{Z}_{1,\varkappa}$. We say that $(s_j)_{j=0}^{\varkappa}$ is a $q\times q$ Carathéodory-central sequence of minimal order k (or a $q\times q$ minimal order k C-central sequence) if $(C_j)_{j=0}^{\varkappa}$ is central of minimal order k.

(c) The sequence $(s_j)_{j=0}^{\varkappa}$ is simply called a $q\times q$ Carathéodory-central sequence (or a $q\times q$ C-central sequence), if there exists a $k\in\mathbb{Z}_{1,\varkappa}$ such that $(s_j)_{j=0}^{\varkappa}$ is order k central.

Remark 3.16. Let $\varkappa\in\mathbb{N}\cup\{+\infty\}$ and $k\in\mathbb{Z}_{1,\varkappa}$. Furthermore, suppose that $(s_j)_{j=0}^{\varkappa}$ is a $q\times q$ order k C-central sequence. For any $\ell\in\mathbb{Z}_{k,\varkappa}$, Remark 2.13 then shows that $(s_j)_{j=0}^{\varkappa}$ is also ℓ order C-central.

The following is the analogous result to Remark 2.18 for C-centrality.

Remark 3.17. Let $n\in\mathbb{N}_0$ and $(s_j)_{j=0}^{n}$ be a sequence in $\mathbb{C}^{q\times q}$.

(a) There exists a unique order $n+1$ C-central sequence $(\widetilde{s}_j)_{j=0}^{\infty}$ in $\mathbb{C}^{q\times q}$ such that $\widetilde{s}_j=s_j$, for each $j\in\mathbb{Z}_{0,n}$. This sequence $(\widetilde{s}_j)_{j=0}^{\infty}$ is called the C-central sequence corresponding to $(s_j)_{j=0}^{n}$.

(b) Let $(C_j)_{j=0}^{n}$ be defined by (3.1) with $\varkappa=n$. Then the central sequence $(\widetilde{C}_j)_{j=0}^{\infty}$ corresponding to $(C_j)_{j=0}^{n}$ is given by $\widetilde{C}_0=\operatorname{Re}\widetilde{s}_0$ and $\widetilde{C}_\ell=\frac{1}{2}\widetilde{s}_\ell$ for each $\ell\in\mathbb{N}$.

Remark 3.18. Let $k\in\mathbb{N}$ and $(s_j)_{j=0}^{\infty}$ be an order k C-central sequence in $\mathbb{C}^{q\times q}$. By part (a) of Remark 3.17 it then follows that $(s_j)_{j=0}^{\infty}$ is the C-central sequence corresponding to $(s_j)_{j=0}^{k-1}$.

We now consider C-central sequences in more detail.

Lemma 3.19. *Let $k\in\mathbb{N}$ and $(s_j)_{j=0}^{k-1}$ be a sequence in $\mathbb{C}^{q\times q}$. Suppose that $(\widetilde{s}_j)_{j=0}^{\infty}$ is the C-central sequence corresponding to $(s_j)_{j=0}^{k-1}$. Then:*

(a) $(\widetilde{s}_j)_{j=0}^{\infty}\in\mathcal{C}_{q,\infty}$ *if and only if* $(s_j)_{j=0}^{k-1}\in\mathcal{C}_{q,k-1}$.

(b) *The following three conditions are all equivalent:*

(i) $(s_j)_{j=0}^{k-1} \in \widetilde{\mathcal{C}}_{q,\,k-1}$.
(ii) $(\widetilde{s}_j)_{j=0}^{k-1} \in \mathcal{C}_{q,\,k-1}^{\mathrm{t}}$.
(iii) $(\widetilde{s}_j)_{j=0}^{\infty} \in \widetilde{\mathcal{C}}_{q,\,\infty}$.

Proof. Combine Remark 3.6, Remark 3.17 and Lemma 2.19. □

Example 3.20. Let $K \in \mathbb{K}_{q\times q}$ and the sequence $(C_j)_{j=0}^{\infty}$ be given by $s_0 := I_q$ and $s_j := 2K^j$ for each $j \in \mathbb{N}$. From Example 2.21 and Remark 3.6 we see that:

(a) $(s_j)_{j=0}^{\infty}$ is an order 2 C-central $q\times q$ C-sequence and if (and only if) $K \neq 0_{q\times q}$, then $(s_j)_{j=0}^{\infty}$ is minimal order 2 C-central.
(b) $(s_j)_{j=0}^{\infty} \in \mathcal{C}_{q,\,\infty}^{\mathrm{t}}$ if and only if $K \in \mathbb{D}_{q\times q}$.

Example 3.21. Let $K \in \mathbb{K}_{q\times q} \cap \mathbb{C}_{\geq}^{q\times q}$, $r \in \mathbb{N}$ and $(C_j)_{j=0}^{\infty}$ be given by $s_0 := K^r$ and $s_j := 2K^{r+j}$ for each $j \in \mathbb{N}$. From Example 2.22 and Remark 3.6 we see that $(C_j)_{j=0}^{\infty}$ is an order 2 C-central $q \times q$ Carathéodory sequence.

Recalling Remark 3.6, we now see how Definition 2.24 carries over to $q \times q$ Carathéodory sequences.

Definition 3.22. Let $\varkappa \in \mathbb{N} \cup \{+\infty\}$ and $(s_j)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C}^{q\times q}$. Furthermore, let the sequence $(C_j)_{j=0}^{\varkappa}$ be defined by (3.1).

(a) Suppose $k \in \mathbb{Z}_{1,\,\varkappa}$. We say that $(s_j)_{j=0}^{\varkappa}$ is a $q \times q$ Carathéodory-canonical sequence of order k (or $q\times q$ order k C-canonical sequence), if $(C_j)_{j=0}^{\varkappa}$ is order k canonical.
(b) Suppose $k \in \mathbb{Z}_{1,\,\varkappa}$. We say that $(s_j)_{j=0}^{\varkappa}$ is $q \times q$ Carathéodory-canonical of minimal order k (or $q \times q$ minimal order k C-canonical) if $(C_j)_{j=0}^{\varkappa}$ is minimal order k canonical.
(c) $(s_j)_{j=0}^{\varkappa}$ is simply called a $q \times q$ Carathéodory-canonical sequence (or a $q \times q$ C-canonical sequence), if there exists a $k \in \mathbb{Z}_{1,\,\varkappa}$ such that $(s_j)_{j=0}^{\varkappa}$ is order k canonical.

Combining Definition 3.22 and Corollary 2.27 (while recalling Remark 3.6 and Definition 3.15), we obtain the following remark.

Remark 3.23. Let $\varkappa \in \mathbb{N}\cup\{+\infty\}$ and $k \in \mathbb{Z}_{1,\,\varkappa}$. Suppose that $(s_j)_{j=0}^{\varkappa}$ is an order k C-canonical $q \times q$ Carathéodory sequence. Then:

(a) For each $\ell \in \mathbb{Z}_{k,\,\varkappa}$, the sequence $(s_j)_{j=0}^{\varkappa}$ is order ℓ C-canonical.
(b) Let $\varkappa \geq 2$ and $k \in \mathbb{Z}_{1,\,\varkappa-1}$. For any $\ell \in \mathbb{Z}_{k+1,\,\varkappa}$, it then follows that $(s_j)_{j=0}^{\varkappa}$ is order ℓ C-central.

Example 3.24. Let $\varkappa \in \mathbb{N} \cup \{+\infty\}$ and $A \in \mathbb{C}_{\geq}^{q\times q}$. Suppose, furthermore, that $s_0 := A$ and $s_j := 2A$ for each $j \in \mathbb{Z}_{1,\,\varkappa}$. Then Example 2.28 and Remark 3.6 show that $(s_j)_{j=0}^{\varkappa}$ is an order 1 C-central $q \times q$ Carathéodory sequence.

Example 3.25. Let $K \in \mathbb{K}_{q\times q}$. Suppose, furthermore, that $s_0 := I_q$ and $s_j := 2K^j$ for $j \in \mathbb{N}$. Then part (a) of Example 3.20 shows that $(s_j)_{j=0}^{\infty}$ is a $q\times q$ Carathéodory sequence. Example 2.29 furthermore shows that $(s_j)_{j=0}^{\infty}$ is C-canonical if and only if K is unitary. When this is the case, $(s_j)_{j=0}^{\infty}$ is order 1 C-canonical.

4. Reciprocal sequences of matricial Carathéodory sequences

We will, in this section, discuss reciprocal sequences of matricial Carathéodory sequences. Our first task will be to show that the reciprocal sequence of a Carathéodory sequence is itself a Carathéodory sequence. We will draw heavily on results established in Section 1. We next show that every Carathéodory sequence belongs to the class of matrices introduced in Definition 1.4, thus establishing the aforementioned connection to Section 1.

Proposition 4.1. *Suppose* $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$.

(a) $\mathcal{C}_{q,\varkappa} \subseteq \mathcal{F}^{\mathrm{EP}}_{q,\varkappa}$.
(b) $\mathcal{C}_{q,\varkappa} \subseteq \mathcal{D}_{q\times q,\varkappa}$.

Proof. Suppose that $(s_j)_{j=0}^{\varkappa}$ is in $\mathcal{C}_{q,\varkappa}$. It then follows, for each $n \in \mathbb{N}_0$, that $\mathrm{S}_n \in \mathcal{R}_{(n+1)q,\geq}$. Thus, by Lemma A.8, we have $\mathrm{S}_n \in \mathbb{C}^{(n+1)q\times(n+1)q}_{\mathrm{EP}}$, for each $n \in \mathbb{N}_0$, and therefore $(s_j)_{j=0}^{\varkappa} \in \mathcal{F}^{\mathrm{EP}}_{q,\varkappa}$. The proof of part (a) is complete. Part (b) follows by part (a) of Proposition 1.10 from the part already proved. □

Corollary 4.2. *Suppose* $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$*. Then* $\mathcal{T}_{q,\varkappa} \subseteq \mathcal{F}^{\mathrm{EP}}_{q,\varkappa}$.

Proof. Use part (a) of Proposition 4.1 and Lemma 3.11. □

From Corollary 4.2 and part (a) of Proposition 1.10 we furthermore obtain $\mathcal{T}_{q,\varkappa} \subseteq \mathcal{D}_{q\times q,\varkappa}$, as was earlier shown in Proposition 2.7. We next consider the reciprocal sequence to the reciprocal sequence of a Carathéodory sequence $(s_j)_{j=0}^{\varkappa}$, i.e., its second reciprocal sequence $\left((s_j^{\sharp})^{\sharp}\right)_{j=0}^{\varkappa}$. This sequence must coincide with the original Carathéodory sequence.

Corollary 4.3. *If* $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ *and* $(s_j)_{j=0}^{\varkappa}$ *is a* $q \times q$ *Carathéodory sequence, then* $\left((s_j^{\sharp})^{\sharp}\right)_{j=0}^{\varkappa} = (s_j)_{j=0}^{\varkappa}$.

Proof. Combine Proposition 4.1 with [22, Proposition 5.13, Remark 4.7]. □

We now come to the first main result of this section.

Theorem 4.4. *Let* $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ *and* $(s_j)_{j=0}^{\varkappa}$ *be a* $q \times q$ *Carathéodory sequence. The reciprocal sequence* $(s_j^{\sharp})_{j=0}^{\varkappa}$ *is then also a* $q \times q$ *Carathéodory sequence. If* $(s_j)_{j=0}^{\varkappa}$ *is a strict* $q \times q$ *Carathéodory sequence, then* $(s_j^{\sharp})_{j=0}^{\varkappa}$ *is also a strict* $q \times q$ *Carathéodory sequence.*

Proof. By Proposition 4.1, it follows that $(s_j)_{j=0}^{\varkappa} \in \mathcal{F}_{q,\varkappa}^{\mathrm{EP}}$. Let $n \in \mathbb{Z}_{0,\varkappa}$. Propositions A.6 and 1.11 now yield

$$\operatorname{Re} \mathrm{S}_n^{\sharp} = \left(\mathrm{S}_n^{\sharp}\right)^* (\operatorname{Re} \mathrm{S}_n)\, \mathrm{S}_n^{\sharp}. \tag{4.1}$$

Thus, since $\operatorname{Re} \mathrm{S}_n \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$, we see from (4.1) that $\operatorname{Re} \mathrm{S}_n^{\sharp} \in \mathbb{C}_{\geq}^{(n+1)q\times(n+1)q}$. Therefore, $\mathrm{S}_n^{\sharp} \in \mathcal{R}_{(n+1)q,\geq}$. Thus, $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$ is a $q \times q$ C-sequence.

Suppose $(s_j)_{j=0}^{\varkappa}$ is a strict $q \times q$ C-sequence. By Remark 3.1, it follows that $s_0 \in \mathcal{R}_{q,>}$. Thus, by part (a) of Lemma A.12, we have $\det s_0 \neq 0$. Using Definition 1.1, we see that $s_0^{\sharp} = s_0^{-1}$. Thus, $\det \mathrm{S}_n^{\sharp} = (\det s_0)^{-(n+1)} \neq 0$. Because $\operatorname{Re} \mathrm{S}_n \in \mathbb{C}_{>}^{(n+1)q\times(n+1)q}$, we see from (4.1) that $\operatorname{Re} \mathrm{S}_n^{\sharp} \in \mathbb{C}_{>}^{(n+1)q\times(n+1)q}$. Hence, $\mathrm{S}_n^{\sharp} \in \mathcal{R}_{(n+1)q,>}$ and $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$ is a strict $q \times q$ C-sequence. □

Corollary 4.5. *Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^{\varkappa}$ be a sequence in $\mathbb{C}^{q\times q}$. The following two conditions are equivalent to one another:*

(i) *$(s_j)_{j=0}^{\varkappa}$ is a $q \times q$ Carathéodory sequence.*

(ii) *$(s_j)_{j=0}^{\varkappa} \in \mathcal{D}_{q\times q,\varkappa}$ and its reciprocal sequence $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$ is a $q \times q$ Carathéodory sequence.*

Proof. "(i)$\Longrightarrow$(ii)". Because of (i), Proposition 4.1 implies $(s_j)_{j=0}^{\varkappa} \in \mathcal{D}_{q\times q,\varkappa}$, while it follows by Theorem 4.4 that $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$ is a $q \times q$ Carathéodory sequence.

"(ii)$\Longrightarrow$(i)". Because of (ii), it follows by Theorem 4.4 and Corollary 4.3 that $(s_j)_{j=0}^{\varkappa}$ is a $q \times q$ Carathéodory sequence. □

Together, Proposition 4.1 and Theorem 1.12 will give us a better picture of the structure of a $q \times q$ Carathéodory sequence.

Theorem 4.6. *Let $q \geq 2$, $r \in \mathbb{Z}_{1,q-1}$ and $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$. Furthermore, let $(s_j)_{j=0}^{\varkappa}$ be a $q \times q$ Carathéodory sequence with $\operatorname{rank} s_0 = r$. Suppose $(\widetilde{s}_j)_{j=0}^{\varkappa}$ is defined as in Theorem* 1.12. *Then, parts* (a) *and* (b) *of Theorem* 1.12 *both hold true and $(\widetilde{s}_j)_{j=0}^{\varkappa}$ is an $r \times r$ Carathéodory sequence.*

Proof. By Proposition 4.1, we have $(s_j)_{j=0}^{\varkappa} \in \mathcal{F}_{q,\varkappa}^{\mathrm{EP}}$. Parts (a) and (b) of Theorem 1.12 therefore both hold true. From Remark 3.3 it, furthermore, follows that $(\widetilde{s}_j)_{j=0}^{\varkappa}$ is an $r \times r$ Carathéodory sequence. □

We now study the relationship a Carathéodory sequence has to its the reciprocal sequence.

Lemma 4.7. *Let $\varkappa \in \mathbb{N} \cup \{+\infty\}$ and let $(s_j)_{j=0}^{\varkappa}$ be a $q \times q$ Carathéodory sequence. Furthermore, let $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$ be the reciprocal sequence to $(s_j)_{j=0}^{\varkappa}$. Suppose that*

$$C_j := \begin{cases} \operatorname{Re} s_0 & \text{if } j = 0, \\ \frac{1}{2} s_j & \text{if } j \in \mathbb{Z}_{1,\varkappa} \end{cases} \qquad \text{and} \qquad C_j^{[\sharp]} := \begin{cases} \operatorname{Re} s_0^{\sharp} & \text{if } j = 0, \\ \frac{1}{2} s_j^{\sharp} & \text{if } j \in \mathbb{Z}_{1,\varkappa} \end{cases} \tag{4.2}$$

for each $j \in \mathbb{Z}_{0,\varkappa}$. Then:

(a) $(C_j)_{j=0}^{\varkappa}$ *and* $\left(C_j^{[\sharp]}\right)_{j=0}^{\varkappa}$ *are both T-n.n.d. sequences in* $\mathbb{C}^{q\times q}$.

(b) *Let* $k \in \mathbb{Z}_{0,\varkappa}$. *Recalling* (2.1), *let*

$$\mathrm{T}_k := \mathrm{T}_k^{(C)} \qquad \textit{and} \qquad \mathrm{T}_k^{[\sharp]} := \mathrm{T}_k^{(C^{[\sharp]})}.$$

Then $\operatorname{rank} \mathrm{T}_k = \operatorname{rank} \mathrm{T}_k^{[\sharp]}$,

$$\mathrm{T}_k = \operatorname{Re} \mathrm{S}_k, \qquad \mathrm{T}_k^{[\sharp]} = \operatorname{Re} \mathrm{S}_k^{\sharp}, \tag{4.3}$$

$$\mathrm{T}_k^{[\sharp]}\mathrm{S}_k = \left(\mathrm{S}_k^{+}\right)^{*}\mathrm{T}_k, \qquad \mathrm{S}_k\mathrm{T}_k^{[\sharp]} = \mathrm{T}_k\left(\mathrm{S}_k^{+}\right)^{*}, \tag{4.4}$$

$$\mathrm{S}_k^{*}\mathrm{T}_k^{[\sharp]} = \mathrm{T}_k\mathrm{S}_k^{+} \qquad \textit{and} \qquad \mathrm{T}_k^{[\sharp]}\mathrm{S}_k^{*} = \mathrm{S}_k^{+}\mathrm{T}_k. \tag{4.5}$$

(c) *For each* $k \in \mathbb{Z}_{1,\varkappa}$, *the matrices* Z_k *and* Y_k *defined by* (2.3) *satisfy*

$$Z_k\mathrm{T}_{k-1}^{+}\mathrm{T}_{k-1} = Z_k \qquad \textit{and} \qquad \mathrm{T}_{k-1}\mathrm{T}_{k-1}^{+}Y_k = Y_k. \tag{4.6}$$

(d) *For each* $k \in \mathbb{Z}_{1,\varkappa}$, *the matrices* $Z_k^{[\sharp]}$ *and* $Y_k^{[\sharp]}$ *defined by*

$$Z_k^{[\sharp]} := \operatorname{row}\left(C_{k+1-j}^{[\sharp]}\right)_{j=1}^{k} \qquad \textit{and} \qquad Y_k^{[\sharp]} := \operatorname{col}\left(C_j^{[\sharp]}\right)_{j=1}^{k}$$

satisfy

$$Z_k^{[\sharp]} = -s_0^{+}Z_k\mathrm{S}_{k-1}^{+}, \qquad Y_k^{[\sharp]} = -\mathrm{S}_{k-1}^{+}Y_k s_0^{+}, \tag{4.7}$$

$$Z_k^{[\sharp]} = -s_0^{+}Z_k\mathrm{T}_{k-1}^{+}\mathrm{S}_{k-1}^{*}\mathrm{T}_{k-1}^{[\sharp]} \quad \textit{and} \quad Y_k^{[\sharp]} = -\mathrm{T}_{k-1}^{[\sharp]}\mathrm{S}_{k-1}^{*}\mathrm{T}_{k-1}^{+}Y_k s_0^{+}. \tag{4.8}$$

Proof. Part (a) follows from Remark 3.6 using Theorem 4.4. The equations in (4.3) follow directly from the definitions of the relevant matrices. From Proposition 4.1, we see that $(s_j)_{j=0}^{\varkappa} \in \mathcal{F}_{q,\varkappa}^{\mathrm{EP}}$. Combining this with (4.3), we need only apply Proposition 1.11 along with parts (g), (i), (j), (h) and (k) of Corollary A.7 to obtain the remaining equations of part (b). Let $k \in \mathbb{Z}_{1,\varkappa}$. The matrix T_k is non-negative Hermitian and admits the block-partitions

$$\mathrm{T}_k = \begin{pmatrix} \mathrm{T}_{k-1} & Z_k^{*} \\ Z_k & C_0 \end{pmatrix} \qquad \text{and} \qquad \mathrm{T}_k = \begin{pmatrix} C_0 & Y_k^{*} \\ Y_k & \mathrm{T}_{k-1} \end{pmatrix}. \tag{4.9}$$

Using well-known properties of non-negative Hermitian block matrices (see, for instance, [8, Lemma 1.1.9]), we obtain both equations in (4.6). Because of Proposition 4.1, the equations in (4.7) follow directly from [22, Corollary 4.23].
Using (b) and (c), we then see that

$$-s_0^{+}Z_k\mathrm{T}_{k-1}^{+}\mathrm{S}_{k-1}^{*}\mathrm{T}_{k-1}^{[\sharp]} = -s_0^{+}Z_k\mathrm{T}_{k-1}^{+}\mathrm{T}_{k-1}\mathrm{S}_{k-1}^{+} = -s_0^{+}Z_k\mathrm{S}_{k-1}^{+} = Z_k^{[\sharp]},$$

and similarly,

$$-\mathrm{T}_{k-1}^{[\sharp]}\mathrm{S}_{k-1}^{*}\mathrm{T}_{k-1}^{+}Y_k s_0^{+} = -\mathrm{S}_{k-1}^{+}\mathrm{T}_{k-1}\mathrm{T}_{k-1}^{+}Y_k s_0^{+} = -\mathrm{S}_{k-1}^{+}Y_k s_0^{+} = Y_k^{[\sharp]}.$$

Thus, the proof is complete. □

Using part (b) of Lemma 4.7, we will see that the $q\times q$ Carathéodory sequence properties introduced in Definition 3.22 carry over to reciprocal sequences.

Theorem 4.8. *Let $\varkappa \in \mathbb{N} \cup \{+\infty\}$ and let $(s_j)_{j=0}^{\varkappa}$ be a $q \times q$ Carathéodory sequence. Furthermore, let $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$ be the reciprocal sequence to $(s_j)_{j=0}^{\varkappa}$. Then:*

(a) *$(s_j)_{j=0}^{\varkappa}$ is C-canonical if and only if $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$ is C-canonical.*

(b) *Let $k \in \mathbb{Z}_{1,\varkappa}$. Then:*

  (b1) *$(s_j)_{j=0}^{\varkappa}$ is order k C-canonical if and only if $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$ is order k C-canonical.*

  (b2) *$(s_j)_{j=0}^{\varkappa}$ is minimal order k C-canonical if and only if $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$ is minimal order k C-canonical.*

Proof. Part (b) follows directly from Definition 3.22 and part (b) of Lemma 4.7, while part (a) follows immediately from (b). □

We more closely analysed the structure of Toeplitz non-negative definite sequences in Proposition 2.9. In particular, we reviewed how this structure could be described by certain matrix balls. We now suppose that we have a $q \times q$ Carathéodory sequence $(s_j)_{j=0}^{\varkappa}$. In Lemma 4.7, we introduced two Toeplitz non-negative sequences $(C_j)_{j=0}^{\varkappa}$ and $\left(C_j^{[\sharp]}\right)_{j=0}^{\varkappa}$, which we defined with the help of $(s_j)_{j=0}^{\varkappa}$ and $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$. Our next objective is to express the matrices which describe the inner structure of $\left(s_j^{[\sharp]}\right)_{j=0}^{\varkappa}$ in terms of the sequences $(s_j)_{j=0}^{\varkappa}$ and $(C_j)_{j=0}^{\varkappa}$. For this reason, we recall that for each $A \in \mathbb{C}^{q\times q}$, well-known results on left and right polar decompositions of square matrices say that there exist unitary $q \times q$ matrices U and V, such that

$$A = \sqrt{AA^*}U \qquad \text{and} \qquad A = V\sqrt{A^*A}.$$

Lemma 4.9. *Let $\varkappa \in \mathbb{N} \cup \{+\infty\}$. Suppose $(s_j)_{j=0}^{\varkappa}$ is a $q \times q$ Carathéodory sequence with reciprocal sequence $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$. Suppose, furthermore, that $(C_j)_{j=0}^{\varkappa}$ and $\left(C_j^{[\sharp]}\right)_{j=0}^{\varkappa}$ are the matricial T-n.n.d. sequences introduced in Lemma 4.7. Then:*

(a) *Let $L_1^{[\sharp]} := C_0^{[\sharp]}$, $R_1^{[\sharp]} := C_0^{[\sharp]}$ and, for each $k \in \mathbb{Z}_{1,\varkappa}$, let*

$$L_{k+1}^{[\sharp]} := C_0^{[\sharp]} - Z_{k-1}^{[\sharp]}\left[\mathrm{T}_{k-1}^{[\sharp]}\right]^{+}\left(Z_{k-1}^{[\sharp]}\right)^{*}$$

and

$$R_{k+1}^{[\sharp]} := C_0^{[\sharp]} - \left(Y_{k-1}^{[\sharp]}\right)^{*}\left[\mathrm{T}_{k-1}^{[\sharp]}\right]^{+}Y_{k-1}^{[\sharp]}.$$

Then the matrices L_{k+1}, R_{k+1}, $L_{k+1}^{[\sharp]}$ and $R_{k+1}^{[\sharp]}$ are all non-negative Hermitian and satisfy

$$L_{k+1}^{[\sharp]} = s_0^{+}L_{k+1}\left(s_0^{+}\right)^{*}, \qquad R_{k+1}^{[\sharp]} = \left(s_0^{+}\right)^{*}R_{k+1}s_0^{+}, \tag{4.10}$$

$$L_{k+1} = s_0L_{k+1}^{[\sharp]}s_0^{*}, \qquad R_{k+1} = s_0^{*}R_{k+1}^{[\sharp]}s_0 \tag{4.11}$$

and, in particular,

$$\operatorname{rank} L_{k+1}^{[\sharp]} = \operatorname{rank} L_{k+1} = \operatorname{rank} R_{k+1} = \operatorname{rank} R_{k+1}^{[\sharp]}. \tag{4.12}$$

If U_{k+1} and V_{k+1} are unitary $q \times q$ matrices such that U_{k+1} produces a left polar decomposition of $s_0^+ \sqrt{L_{k+1}}$ and V_{k+1} a right polar decomposition of $\sqrt{R_{k+1}} s_0^+$, then

$$\sqrt{L_{k+1}^{[\sharp]}} U_{k+1} = s_0^+ \sqrt{L_{k+1}} \qquad \textit{and} \qquad V_{k+1} \sqrt{R_{k+1}^{[\sharp]}} = \sqrt{R_{k+1}} s_0^+. \tag{4.13}$$

(b) *Let $M_1^{[\sharp]} := 0_{q \times q}$ and, for each $m \in \mathbb{Z}_{1,\varkappa}$, let*

$$M_{m+1}^{[\sharp]} := Z_m^{[\sharp]} \left(\mathrm{T}_{m-1}^{[\sharp]} \right)^+ Y_m^{[\sharp]}.$$

Suppose that $k \in \mathbb{Z}_{1,\varkappa}$. Then:

$$M_{k+1}^{[\sharp]} = s_0^+ Z_k \mathrm{S}_{k-1}^+ \mathrm{S}_{k-1}^* \mathrm{T}_{k-1}^+ Y_k s_0^+, \tag{4.14}$$

$$M_{k+1}^{[\sharp]} = s_0^+ Z_k \mathrm{T}_{k-1}^+ \mathrm{S}_{k-1}^* \mathrm{S}_{k-1}^+ Y_k s_0^+, \tag{4.15}$$

$$M_{k+1}^{[\sharp]} = s_0^+ \left(2 Z_k \mathrm{S}_{k-1}^+ Y_k - M_{k+1} \right) s_0^+, \tag{4.16}$$

$$C_k^{[\sharp]} - M_k^{[\sharp]} = -s_0^+ \left(C_k - M_k \right) s_0^+ \tag{4.17}$$

and

$$C_k - M_k = -s_0 \left(C_k^{[\sharp]} - M_k^{[\sharp]} \right) s_0. \tag{4.18}$$

(c) *Let $(K_j)_{j=1}^{\varkappa}$ be the Schur parameter sequence for $(C_j)_{j=0}^{\varkappa}$ and $\left(K_j^{[\sharp]}\right)_{j=1}^{\varkappa}$ the Schur parameter sequence for $\left(C_j^{[\sharp]}\right)_{j=0}^{\varkappa}$. Suppose $s \in \mathbb{Z}_{1,\varkappa}$ and that U_s and V_s are unitary $q \times q$ matrices such that U_s produces a left polar decomposition of $s_0^+ \sqrt{L_s}$ and V_s a right polar decomposition of $\sqrt{R_s} s_0^+$. Then:*

$$\left(\sqrt{L_s^{[\sharp]}} \right)^+ \sqrt{L_s^{[\sharp]}} U_s K_s = U_s K_s, \tag{4.19}$$

$$K_s V_s \left(\sqrt{R_s^{[\sharp]}} \right)^+ \sqrt{R_s^{[\sharp]}} = K_s V_s \tag{4.20}$$

and

$$K_s^{[\sharp]} = -U_s K_s V_s. \tag{4.21}$$

Proof. (a) By Remark 3.1, we see that $s_0 \in \mathcal{R}_{q,\geq}$. By part (c) of Lemma A.8 and Proposition A.6, it then follows that

$$\operatorname{Re}\left(s_0^+ \right) = s_0^+ \left(\operatorname{Re} s_0 \right) \left(s_0^+ \right)^* \qquad \text{and} \qquad \operatorname{Re}\left(s_0^+ \right) = \left(s_0^+ \right)^* \left(\operatorname{Re} s_0 \right) s_0^+.$$

By Definition 1.1 and (4.2), we obtain

$$C_0^{[\sharp]} = \operatorname{Re} s_0^{\sharp} = \operatorname{Re}\left(s_0^+ \right) = s_0^+ \left(\operatorname{Re} s_0 \right) \left(s_0^+ \right)^* = s_0^+ C_0 \left(s_0^+ \right)^* \tag{4.22}$$

and, similarly,

$$C_0^{[\sharp]} = \left(s_0^+ \right)^* C_0 s_0^+. \tag{4.23}$$

Recalling (2.2), we see that this implies

$$L_1^{[\sharp]} = s_0^+ L_1 \left(s_0^+ \right)^* \qquad \text{and} \qquad R_1^{[\sharp]} = \left(s_0^+ \right)^* R_1 s_0^+. \tag{4.24}$$

Suppose now that $k \in \mathbb{Z}_{1,\varkappa}$. Part (d) of Lemma 4.7 yields that the equations in (4.8) hold true. From part (a) of Lemma 4.7, it follows that T_{k-1} and $\mathrm{T}_{k-1}^{[\sharp]}$ are both Hermitian matrices. Thus, from equations (4.8) we obtain

$$\left(Z_k^{[\sharp]} \right)^* = -\mathrm{T}_{k-1}^{[\sharp]} \mathrm{S}_{k-1} \mathrm{T}_{k-1}^+ Z_k^* \left(s_0^+ \right)^* \tag{4.25}$$

and

$$\left(Y_k^{[\sharp]} \right)^* = -\left(s_0^+ \right)^* Y_k^* \mathrm{T}_{k-1}^+ \mathrm{S}_{k-1} \mathrm{T}_{k-1}^{[\sharp]}. \tag{4.26}$$

Part (b) of Lemma 4.7 give us

$$\mathrm{S}_{k-1}^* \mathrm{T}_{k-1}^{[\sharp]} = \mathrm{T}_{k-1} \mathrm{S}_{k-1}^+ \qquad \text{and} \qquad \mathrm{T}_{k-1}^{[\sharp]} \mathrm{S}_{k-1}^* = \mathrm{S}_{k-1}^+ \mathrm{T}_{k-1}. \tag{4.27}$$

By Proposition 4.1, we have $(s_j)_{j=0}^{\varkappa} \in \mathcal{F}_{q,\varkappa}^{\mathrm{EP}}$. Parts (b) and (d) of Corollary A.7 along with Proposition 1.11 give us

$$(\operatorname{Re} \mathrm{S}_{k-1}) \mathrm{S}_{k-1}^+ \mathrm{S}_{k-1} = \operatorname{Re} \mathrm{S}_{k-1} \qquad \text{and} \qquad \mathrm{S}_{k-1} \mathrm{S}_{k-1}^+ (\operatorname{Re} \mathrm{S}_{k-1}) = \operatorname{Re} \mathrm{S}_{k-1}.$$

Thus, it follows by Remark 1.2 and part (b) of Lemma 4.7 that

$$\mathrm{T}_{k-1} \mathrm{S}_{k-1}^+ \mathrm{S}_{k-1} = \mathrm{T}_{k-1} \qquad \text{and} \qquad \mathrm{S}_{k-1} \mathrm{S}_{k-1}^+ \mathrm{T}_{k-1} = \mathrm{T}_{k-1}. \tag{4.28}$$

Using (4.8), (4.25), (4.27) and (4.28), we see that

$$\begin{aligned}
&Z_k^{[\sharp]} \left(\mathrm{T}_{k-1}^{[\sharp]} \right)^+ \left(Z_k^{[\sharp]} \right)^* \\
&\quad = \left[-s_0^+ Z_k \mathrm{T}_{k-1}^+ \mathrm{S}_{k-1}^* \mathrm{T}_{k-1}^{[\sharp]} \right] \left(\mathrm{T}_{k-1}^{[\sharp]} \right)^+ \left[-\mathrm{T}_{k-1}^{[\sharp]} \mathrm{S}_{k-1} \mathrm{T}_{k-1}^+ Z_k^* \left(s_0^+ \right)^* \right] \\
&\quad = s_0^+ Z_k \mathrm{T}_{k-1}^+ \mathrm{S}_{k-1}^* \mathrm{T}_{k-1}^{[\sharp]} \mathrm{S}_{k-1} \mathrm{T}_{k-1}^+ Z_k^* \left(s_0^+ \right)^* \\
&\quad = s_0^+ Z_k \mathrm{T}_{k-1}^+ \mathrm{T}_{k-1} \mathrm{S}_{k-1}^+ \mathrm{S}_{k-1} \mathrm{T}_{k-1}^+ Z_k^* \left(s_0^+ \right)^* \\
&\quad = s_0^+ Z_k \mathrm{T}_{k-1}^+ \mathrm{T}_{k-1} \mathrm{T}_{k-1}^+ Z_k^* \left(s_0^+ \right)^* \\
&\quad = s_0^+ Z_k \mathrm{T}_{k-1}^+ Z_k^* \left(s_0^+ \right)^*
\end{aligned} \tag{4.29}$$

and, similarly, using (4.26), (4.8), (4.27) and (4.28) also that

$$\left(Y_k^{[\sharp]} \right)^* \left(\mathrm{T}_{k-1}^{[\sharp]} \right)^+ Y_k^{[\sharp]} = \left(s_0^+ \right)^* Y_k^* \mathrm{T}_{k-1}^+ Y_k s_0^+. \tag{4.30}$$

From (4.22), (4.29) and (2.4) we see that

$$\begin{aligned}
L_{k+1}^{[\sharp]} &= C_0^{[\sharp]} - Z_k^{[\sharp]} \left(\mathrm{T}_{k-1}^{[\sharp]} \right)^+ \left(Z_k^{[\sharp]} \right)^* \\
&= s_0^+ C_0 \left(s_0^+ \right)^* - s_0^+ Z_k \mathrm{T}_{k-1}^+ Z_k^* \left(s_0^+ \right)^* \\
&= s_0^+ L_{k+1} \left(s_0^+ \right)^*.
\end{aligned} \tag{4.31}$$

Similarly, from (4.23), (4.30) and (2.4) we also see that

$$R_{k+1}^{[\sharp]} = \left(s_0^+ \right)^* R_{k+1} s_0^+. \tag{4.32}$$

The sequences $(L_{j+1})_{j=0}^{\varkappa}$, $(R_{j+1})_{j=0}^{\varkappa}$, $\left(L_{j+1}^{[\sharp]}\right)_{j=0}^{\varkappa}$ and $\left(R_{j+1}^{[\sharp]}\right)_{j=0}^{\varkappa}$ are monotonically decreasing sequences of non-negative Hermitian matrices, by part (a) of Lemma 4.7 and [8, Remark 3.4.3]. Therefore,

$$\mathcal{R}(L_{k+1}) \subseteq \mathcal{R}(L_1) = \mathcal{R}(C_0) = \mathcal{R}(\operatorname{Re} s_0)$$

and, similarly, $\mathcal{R}(R_{k+1}) \subseteq \mathcal{R}(\operatorname{Re} s_0)$. Because of $s_0 \in \mathcal{R}_{q,\geq}$ and part (b) of Lemma A.8, we have $\mathcal{R}(\operatorname{Re} s_0) \subseteq \mathcal{R}(s_0)$. Thus,

$$\mathcal{R}(L_{k+1}) \subseteq \mathcal{R}(s_0) \qquad \text{and} \qquad \mathcal{R}(R_{k+1}) \subseteq \mathcal{R}(s_0). \tag{4.33}$$

Consequently, part (a) of Lemma A.2 implies

$$s_0 s_0^+ L_{k+1} = L_{k+1} \qquad \text{and} \qquad s_0 s_0^+ R_{k+1} = R_{k+1}. \tag{4.34}$$

Since L_{k+1} and R_{k+1} are non-negative Hermitian, we have, in particular,

$$L_{k+1}^* = L_{k+1} \qquad \text{and} \qquad R_{k+1}^* = R_{k+1}. \tag{4.35}$$

By considering orthogonal complements, we see from (4.33) and (4.35) that

$$\mathcal{N}(s_0^*) \subseteq \mathcal{N}(L_{k+1}) \qquad \text{and} \qquad \mathcal{N}(s_0^*) \subseteq \mathcal{N}(R_{k+1}). \tag{4.36}$$

Because of $s_0 \in \mathcal{R}_{q,\geq}$, part (c) of Lemma A.8 yields $s_0 \in \mathbb{C}_{\mathrm{EP}}^{q\times q}$. It thus follows by Proposition A.5 that $\mathcal{N}(s_0) = \mathcal{N}(s_0^*)$. Along with (4.36), this implies $\mathcal{N}(s_0) \subseteq \mathcal{N}(L_{k+1})$ and $\mathcal{N}(s_0) \subseteq \mathcal{N}(R_{k+1})$. Hence, part (b) of Lemma A.2 implies

$$L_{k+1} s_0^+ s_0 = L_{k+1} \qquad \text{and} \qquad R_{k+1} s_0^+ s_0 = R_{k+1}. \tag{4.37}$$

Using (4.10), (4.34) and (4.35), it follows that

$$\begin{aligned} s_0 L_{k+1}^{[\sharp]} s_0^* &= s_0 s_0^+ L_{k+1} \left(s_0^+\right)^* s_0^* = L_{k+1} \left(s_0^+\right)^* s_0^* = L_{k+1}^* \left(s_0^+\right)^* s_0^* \\ &= \left(s_0 s_0^+ L_{k+1}\right)^* = L_{k+1}^* = L_{k+1}. \end{aligned}$$

Similarly, (4.10), (4.37) and (4.35) imply $s_0^* R_{k+1}^{[\sharp]} s_0 = R_{k+1}$. From (4.10) and (4.11), we see that $\operatorname{rank} L_{k+1} = \operatorname{rank} L_{k+1}^{[\sharp]}$ and $\operatorname{rank} R_{k+1} = \operatorname{rank} R_{k+1}^{[\sharp]}$. Because part (a) of Lemma 4.7 and part (a) of Proposition 2.9 yield $\operatorname{rank} L_{k+1} = \operatorname{rank} R_{k+1}$, we get (4.12). From (4.10), we see that

$$\left(s_0^+ \sqrt{L_{k+1}}\right)\left(s_0^+ \sqrt{L_{k+1}}\right)^* = s_0^+ L_{k+1} \left(s_0^+\right)^* = L_{k+1}^{[\sharp]}$$

and

$$\left(\sqrt{R_{k+1}}\, s_0^+\right)^* \left(\sqrt{R_{k+1}}\, s_0^+\right) = \left(s_0^+\right)^* R_{k+1} s_0^+ = R_{k+1}^{[\sharp]}.$$

By our choice of U_{k+1} and V_{k+1}, it thus follows that

$$s_0^+ \sqrt{L_{k+1}} = \sqrt{\left(s_0^+ \sqrt{L_{k+1}}\right)\left(s_0^+ \sqrt{L_{k+1}}\right)^*}\, U_{k+1} = \sqrt{L_{k+1}^{[\sharp]}}\, U_{k+1}$$

and

$$\sqrt{R_{k+1}}\, s_0^+ = V_{k+1} \sqrt{\left(\sqrt{R_{k+1}}\, s_0^+\right)^* \left(\sqrt{R_{k+1}}\, s_0^+\right)} = V_{k+1} \sqrt{R_{k+1}^{[\sharp]}}.$$

This completes the proof of (a).

(b) Recalling part (d) of Lemma 4.7, we see that

$$\begin{aligned} M_{k+1}^{[\sharp]} &= Z_k^{[\sharp]} \left(\mathrm{T}_{k-1}^{[\sharp]} \right)^{+} Y_k^{[\sharp]} \\ &= \left(-s_0^{+} Z_k \mathrm{T}_{k-1}^{+} \mathrm{S}_{k-1}^{*} \mathrm{T}_{k-1}^{[\sharp]} \right) \left(\mathrm{T}_{k-1}^{[\sharp]} \right)^{+} \left(-\mathrm{T}_{k-1}^{[\sharp]} \mathrm{S}_{k-1}^{*} \mathrm{T}_{k-1}^{+} Y_k s_0^{+} \right) \\ &= s_0^{+} Z_k \mathrm{T}_{k-1}^{+} \mathrm{S}_{k-1}^{*} \mathrm{T}_{k-1}^{[\sharp]} \mathrm{S}_{k-1}^{*} \mathrm{T}_{k-1}^{+} Y_k s_0^{+} \end{aligned} \tag{4.38}$$

for $k \in \mathbb{Z}_{1,\varkappa}$. By part (c) of Lemma 4.7, it follows that (4.6) holds true. From (4.38), the first equation in (4.27) and (4.6), we obtain

$$\begin{aligned} M_{k+1}^{[\sharp]} &= s_0^{+} Z_k \mathrm{T}_{k-1}^{+} \mathrm{S}_{k-1}^{*} \mathrm{T}_{k-1}^{[\sharp]} \mathrm{S}_{k-1}^{*} \mathrm{T}_{k-1}^{+} Y_k s_0^{+} \\ &= s_0^{+} Z_k \mathrm{T}_{k-1}^{+} \mathrm{T}_{k-1} \mathrm{S}_{k-1}^{+} \mathrm{S}_{k-1}^{*} \mathrm{T}_{k-1}^{+} Y_k s_0^{+} \\ &= s_0^{+} Z_k \mathrm{S}_{k-1}^{+} \mathrm{S}_{k-1}^{*} \mathrm{T}_{k-1}^{+} Y_k s_0^{+}. \end{aligned}$$

Thus, (4.14) is proved. From (4.38), the second equation in (4.27) and the second equation in (4.6), we conclude that

$$M_{k+1}^{[\sharp]} = s_0^{+} Z_k \mathrm{T}_{k-1}^{+} \mathrm{S}_{k-1}^{*} \mathrm{S}_{k-1}^{+} Y_k s_0^{+}.$$

By part (b) of Lemma 4.7, it follows that

$$\mathrm{T}_{k-1} = \operatorname{Re} \mathrm{S}_{k-1} \tag{4.39}$$

and hence

$$2\mathrm{T}_{k-1} = \mathrm{S}_{k-1} + \mathrm{S}_{k-1}^{*}. \tag{4.40}$$

Using the second equation in (4.6) and (4.40), we now see that

$$2Y_k = 2\mathrm{T}_{k-1}\mathrm{T}_{k-1}^{+} Y_k = \mathrm{S}_{k-1}\mathrm{T}_{k-1}^{+} Y_k + \mathrm{S}_{k-1}^{*}\mathrm{T}_{k-1}^{+} Y_k,$$

i.e.,

$$\mathrm{S}_{k-1}^{*}\mathrm{T}_{k-1}^{+} Y_k = 2Y_k - \mathrm{S}_{k-1}\mathrm{T}_{k-1}^{+} Y_k. \tag{4.41}$$

From (4.14) and (4.41), we obtain

$$\begin{aligned} M_{k+1}^{[\sharp]} &= s_0^{+} Z_k \mathrm{S}_{k-1}^{+} \left(2Y_k - \mathrm{S}_{k-1}\mathrm{T}_{k-1}^{+} Y_k \right) s_0^{+} \\ &= s_0^{+} \left(2Z_k \mathrm{S}_{k-1}^{+} Y_k - Z_k \mathrm{S}_{k-1}^{+} \mathrm{S}_{k-1} \mathrm{T}_{k-1}^{+} Y_k \right) s_0^{+}. \end{aligned} \tag{4.42}$$

Since $(s_j)_{j=0}^{\varkappa}$ is a $q \times q$ Carathéodory sequence, we have $\mathrm{S}_{k-1} \in \mathcal{R}_{kq,\geq}$. Thus, it follows by part (d) of Lemma A.8 that

$$\mathrm{S}_{k-1}\mathrm{S}_{k-1}^{+} = \mathrm{S}_{k-1}^{+}\mathrm{S}_{k-1} \tag{4.43}$$

and from part (b) of Lemma A.8 that $\mathcal{R}(\operatorname{Re}\mathrm{S}_{k-1}) \subseteq \mathcal{R}(\mathrm{S}_{k-1})$. Therefore, (4.39) implies $\mathcal{R}(\mathrm{T}_{k-1}) \subseteq \mathcal{R}(\mathrm{S}_{k-1})$. Because $\mathrm{T}_{k-1}^{*} = \mathrm{T}_{k-1}$, it follows from Lemma A.3 that $\mathcal{R}(\mathrm{T}_{k-1}) = \mathcal{R}\left(\mathrm{T}_{k-1}^{+}\right)$. Hence, $\mathcal{R}\left(\mathrm{T}_{k-1}^{+}\right) \subseteq \mathcal{R}(\mathrm{S}_{k-1})$. Thus, by part (a) of Lemma A.2, we have

$$\mathrm{S}_{k-1}\mathrm{S}_{k-1}^{+}\mathrm{T}_{k-1}^{+} = \mathrm{T}_{k-1}^{+}.$$

This, along with (4.43) and (2.5), implies that

$$Z_k \mathrm{S}_{k-1}^{+} \mathrm{S}_{k-1} \mathrm{T}_{k-1}^{+} Y_k = Z_k \mathrm{S}_{k-1} \mathrm{S}_{k-1}^{+} \mathrm{T}_{k-1}^{+} Y_k = M_{k+1}.$$

Thus, using (4.42), we obtain (4.16).

In order to prove (4.17), we first consider the case $\varkappa = 1$. From Definition 1.1, we see that $s_1^{\sharp} = -s_0^{+} s_1 s_0^{+}$. Recalling that $M_1 = 0_{q\times q}$, $M_1^{[\sharp]} = 0_{q\times q}$ and $\frac{1}{2} s_1 = C_1$ (which follows from (4.2)) we now see that

$$C_1^{[\sharp]} - M_1^{[\sharp]} = \frac{1}{2} s_1^{\sharp} = -s_0^{+} \left(\frac{1}{2} s_1 \right) s_0^{+} = -s_0^{+} C_1 s_0^{+} = -s_0^{+} \left(C_1 - M_1 \right) s_0^{+}.$$

Thus, (4.17) is proved for $\varkappa = 1$. Suppose now that $\varkappa \geq 2$ and $k \in \mathbb{Z}_{2,\varkappa}$. Since (4.2) shows that $s_j = 2C_j$ for each $j \in \mathbb{Z}_{1,\varkappa}$, it follows by [22, Corollary 4.24] that

$$C_k^{[\sharp]} = \frac{1}{2} s_0^{\sharp} = s_0^{+} \left(-C_k + 2 Z_{k-1} \mathrm{S}_{k-2}^{+} Y_{k-1} \right) s_0^{+}.$$

Equation (4.16) yields

$$M_k^{[\sharp]} = s_0^{+} \left(2 Z_{k-1} \mathrm{S}_{k-2}^{+} Y_{k-1} - M_k \right) s_0^{+}.$$

Therefore, we also obtain (4.17) for $\varkappa \geq 2$ and $k \in \mathbb{Z}_{2,\varkappa}$.

Now we again consider an arbitrary $\varkappa \in \mathbb{N} \cup \{+\infty\}$ and an arbitrary $m \in \mathbb{Z}_{1,\varkappa}$. We know from Proposition 4.1 that $(s_j)_{j=0}^{\varkappa} \in \mathcal{D}_{q\times q,\varkappa}$. Thus,

$$\mathcal{N}(s_0) \subseteq \mathcal{N}(s_m) \qquad \text{and} \qquad \mathcal{R}(s_m) \subseteq \mathcal{R}(s_0). \tag{4.44}$$

Since (4.2) shows that $C_m = \frac{1}{2} s_m$, we have $\mathcal{N}(s_m) = \mathcal{N}(C_m)$ and $\mathcal{R}(s_m) = \mathcal{R}(C_m)$. Therefore, (4.44) yields $\mathcal{N}(s_0) \subseteq \mathcal{N}(C_m)$ and $\mathcal{R}(C_m) \subseteq \mathcal{R}(s_0)$. By Lemma A.2, it thus follows that

$$C_m s_0^{+} s_0 = C_m \qquad \text{and} \qquad s_0 s_0^{+} C_m = C_m. \tag{4.45}$$

From (4.17), $M_1 = 0_{q\times q}$ and (4.45), we see that

$$\begin{aligned} -s_0 \left(C_1^{[\sharp]} - M_1^{[\sharp]} \right) s_0 &= -s_0 \left[-s_0^{+} \left(C_1 - M_1 \right) s_0^{+} \right] s_0 \\ &= s_0 s_0^{+} C_1 s_0^{+} s_0 = C_1 = C_1 - M_1. \end{aligned}$$

We have thus shown (4.18) for $\varkappa = 1$. Suppose now that $\varkappa \geq 2$ and that $k \in \mathbb{Z}_{2,\varkappa}$. It follows from (4.45) that

$$s_0 s_0^{+} Z_{k-1} = Z_{k-1} \qquad \text{and} \qquad Y_{k-1} s_0^{+} s_0 = Y_{k-1}.$$

Because of (2.5), we then get

$$s_0 s_0^{+} M_k s_0^{+} s_0 = s_0 s_0^{+} Z_{k-1} \mathrm{T}_{k-2}^{+} Y_{k-1} s_0^{+} s_0 = Z_{k-1} \mathrm{T}_{k-2}^{+} Y_{k-1} = M_k.$$

Thus, using (4.17) and (4.45), we now obtain

$$\begin{aligned} -s_0 \left(C_k^{\sharp} - M_k^{\sharp} \right) s_0 &= -s_0 \left[-s_0^{+} \left(C_k - M_k \right) s_0^{+} \right] s_0 \\ &= s_0 s_0^{+} C_k s_0^{+} s_0 - s_0 s_0^{+} M_k s_0^{+} s_0 = C_k - M_k. \end{aligned}$$

The proof of (4.18) is thus complete.

(c) From part (a) of Lemma A.4 and (4.33), we see that

$$\mathcal{R}\left(\sqrt{L_s}\right) = \mathcal{R}\left(L_s\right) \subseteq \mathcal{R}\left(s_0\right).$$

Part (a) of Lemma A.2, therefore implies that

$$s_0 s_0^+ \sqrt{L_s} = \sqrt{L_s}.$$

Thus, it now follows from (4.13) that

$$s_0 \sqrt{L_s^{[\sharp]}}\, U_s = s_0 s_0^+ \sqrt{L_s} = \sqrt{L_s}.$$

Considering adjoints of these matrices, we obtain

$$\sqrt{L_s} = U_s^* \sqrt{L_s^{[\sharp]}} s_0^*. \tag{4.46}$$

By definition of the Schur parameters for $(C_j)_{j=0}^{\varkappa}$ and part (b) of Lemma A.4, we have

$$\begin{aligned}
\sqrt{L_s}\left(\sqrt{L_s}\right)^+ K_s &= \sqrt{L_s}\left(\sqrt{L_s}\right)^+ \left(\sqrt{L_s}\right)^+ (C_s - M_s)\left(\sqrt{R_s}\right)^+ \\
&= \left(\sqrt{L_s}\right)^+ \sqrt{L_s}\left(\sqrt{L_s}\right)^+ (C_s - M_s)\left(\sqrt{R_s}\right)^+ \\
&= K_s.
\end{aligned}$$

From (4.46), we thus obtain

$$K_s = U_s^* \sqrt{L_s^{[\sharp]}}\, s_0^* \left(\sqrt{L_s}\right)^+ K_s. \tag{4.47}$$

Equation (4.47), the unitarity of the matrix U_s, part (c) of Lemma A.4 and (4.47) again imply that

$$\begin{aligned}
\left(\sqrt{L_s^{[\sharp]}}\right)^+ \sqrt{L_s^{[\sharp]}}\, U_s K_s &= \left(\sqrt{L_s^{[\sharp]}}\right)^+ \sqrt{L_s^{[\sharp]}}\, U_s U_s^* \sqrt{L_s^{[\sharp]}}\, s_0^* \left(\sqrt{L_s}\right)^+ K_s \\
&= \left(\sqrt{L_s^{[\sharp]}}\right)^+ L_s^{[\sharp]} s_0^* \left(\sqrt{L_s}\right)^+ K_s \\
&= \sqrt{L_s^{[\sharp]}}\, s_0^* \left(\sqrt{L_s}\right)^+ K_s \\
&= U_s K_s.
\end{aligned}$$

This completes the proof of (4.19). Similarly, we obtain (4.20).
By part (a) of Lemma 4.7 and Proposition 2.9, we have

$$C_s - M_s = \sqrt{L_s} K_s \sqrt{R_s}.$$

Thus, using the definition of the Schur parameters for $(C_j)_{j=0}^{\varkappa}$ and $\left(C_j^{[\sharp]}\right)_{j=0}^{\varkappa}$ as well as formulas (4.17), (4.13), (4.19) and (4.20), we obtain

$$\begin{aligned}
K_s^{[\sharp]} &= \left(\sqrt{L_s^{[\sharp]}}\right)^+ \left(C_s^{[\sharp]} - M_s^{[\sharp]}\right) \left(\sqrt{R_s^{[\sharp]}}\right)^+ \\
&= \left(\sqrt{L_s^{[\sharp]}}\right)^+ \left[-s_0^+ (C_s - M_s) s_0^+\right] \left(\sqrt{R_s^{[\sharp]}}\right)^+ \\
&= -\left(\sqrt{L_s^{[\sharp]}}\right)^+ s_0^+ \sqrt{L_s} K_s \sqrt{R_s} s_0^+ \left(\sqrt{R_s^{[\sharp]}}\right)^+ \\
&= -\left(\sqrt{L_s^{[\sharp]}}\right)^+ \sqrt{L_s^{[\sharp]}} U_s K_s V_s \sqrt{R_s^{[\sharp]}} \left(\sqrt{R_s^{[\sharp]}}\right)^+ \\
&= -U_s K_s V_s \sqrt{R_s^{[\sharp]}} \left(\sqrt{R_s^{[\sharp]}}\right)^+ = -U_s K_s V_s.
\end{aligned}$$

Thus, we have shown (4.21). This completes the proof. □

Lemma 4.9 leads us to our next result. We will see that the reciprocal sequences of Carathéodory sequences inherit the C-centrality properties of their generating sequences.

Theorem 4.10. *Let $\varkappa \in \mathbb{N} \cup \{+\infty\}$ and $(s_j)_{j=0}^{\varkappa}$ be a $q \times q$ Carathéodory sequence. Furthermore, suppose $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$ is the reciprocal sequence to $(s_j)_{j=0}^{\varkappa}$. Then:*

(a) *$(s_j)_{j=0}^{\varkappa}$ is C-central if and only if $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$ is C-central.*

(b) *Let $k \in \mathbb{Z}_{1,\varkappa}$. Then:*
   - (b1) *$(s_j)_{j=0}^{\varkappa}$ is order k C-central if and only if $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$ is order k C-central.*
   - (b2) *$(s_j)_{j=0}^{\varkappa}$ is minimal order k C-central if and only if $\left(s_j^{\sharp}\right)_{j=0}^{\varkappa}$ is minimal order k C-central.*

Proof. (b) follows directly from Definition 3.15 and part (b) of Lemma 4.9 while (a) follows immediately from (b). □

Our next result shows that generating reciprocal sequences, as an operation, is compatible with the operation of generating the central $q \times q$ Carathéodory sequence of a finite $q \times q$ Carathéodory sequence.

Theorem 4.11. *Let $n \in \mathbb{N}_0$, let $(s_j)_{j=0}^{n}$ be a $q \times q$ Carathéodory sequence and let $\left(s_j^{\sharp}\right)_{j=0}^{n}$ be the reciprocal sequence to $(s_j)_{j=0}^{n}$. Recalling Theorem 4.4, let $(\widetilde{s}_j)_{j=0}^{\infty}$ and $(\widetilde{s}_{j,\sharp})_{j=0}^{\infty}$ be the C-central sequences for $(s_j)_{j=0}^{n}$ and $\left(s_j^{\sharp}\right)_{j=0}^{n}$, respectively. Then $(\widetilde{s}_{j,\sharp})_{j=0}^{\infty}$ is the reciprocal sequence to $(\widetilde{s}_j)_{j=0}^{\infty}$.*

Proof. By Remark 3.17, we have $\widetilde{s}_j = s_j$ for each $j \in \mathbb{Z}_{0,n}$ and the sequence $(\widetilde{s}_j)_{j=0}^{\infty}$ is order $n+1$ C-central. Suppose that $\left(\widetilde{s}_j^{\sharp}\right)_{j=0}^{\infty}$ is the reciprocal sequence to $(\widetilde{s}_j)_{j=0}^{\infty}$. Because of Remark 1.2, we have $\left(\widetilde{s}_j^{\sharp}\right)_{j=0}^{n} = \left(s_j^{\sharp}\right)_{j=0}^{n}$ and part (a) of

Lemma 3.19, Theorem 4.4 and part (a) of Theorem 4.10 we see that $\left(\widetilde{s}_j^{\sharp}\right)_{j=0}^{\infty}$ is a $q \times q$ order $n+1$ C-central Carathéodory sequence. It thus follows by Remark 3.17 that $\left(\widetilde{s}_j^{\sharp}\right)_{j=0}^{\infty}$ is the C-central $q \times q$ sequence for $\left(s_j^{\sharp}\right)_{j=0}^{n}$. We therefore have $\left(\widetilde{s}_j^{\sharp}\right)_{j=0}^{\infty} = \left(\widetilde{s}_{j,\sharp}\right)_{j=0}^{n}$. □

We next consider reciprocal sequences of totally strict Carathéodory sequences.

Lemma 4.12. *Let* $(s_j)_{j=0}^{\infty} \in \mathcal{C}_{q,\infty}$ *and suppose that* $\left(s_j^{\sharp}\right)_{j=0}^{\infty}$ *is the reciprocal sequence to* $(s_j)_{j=0}^{\infty}$. *Then the sequences* $(L_{k+1})_{k=0}^{\infty}$, $(R_{k+1})_{k=0}^{\infty}$, $\left(L_{k+1}^{[\sharp]}\right)_{k=0}^{\infty}$ *and* $\left(R_{k+1}^{[\sharp]}\right)_{k=0}^{\infty}$ *are monotonically decreasing and convergent. Their limits*

$$L := \lim_{k\longrightarrow\infty} L_{k+1}, \qquad R := \lim_{k\longrightarrow\infty} R_{k+1},$$

$$L^{[\sharp]} := \lim_{k\longrightarrow\infty} L_{k+1}^{[\sharp]} \qquad \text{and} \qquad R^{[\sharp]} := \lim_{k\longrightarrow\infty} R_{k+1}^{[\sharp]}$$

are, furthermore, all non-negative Hermitian. Moreover,

$$L^{[\sharp]} = s_0^{+} L \left(s_0^{+}\right)^{*}, \qquad R^{[\sharp]} = \left(s_0^{+}\right)^{*} R s_0^{+},$$

$$L = s_0 L^{[\sharp]} s_0^{*}, \qquad \text{and} \qquad R = s_0^{*} R^{[\sharp]} s_0.$$

In particular, $\operatorname{rank} L = \operatorname{rank} L^{[\sharp]}$ *and* $\operatorname{rank} R = \operatorname{rank} R^{[\sharp]}$.

Proof. Use Proposition 2.9, Theorem 4.4 and Lemma 4.9. □

Theorem 4.13. *Let* $(s_j)_{j=0}^{\infty} \in \mathcal{C}_{q,\infty}$ *and suppose that* $\left(s_j^{\sharp}\right)_{j=0}^{\infty}$ *is the reciprocal sequence to* $(s_j)_{j=0}^{\infty}$. *Then* $(s_j)_{j=0}^{\infty} \in \mathcal{C}_{q,\infty}^{\mathrm{t}}$ *if and only if* $\left(s_j^{\sharp}\right)_{j=0}^{\infty} \in \mathcal{C}_{q,\infty}^{\mathrm{t}}$.

Proof. Use Lemma 4.12. □

5. Matricial Toeplitz non-negative definite sequences generated by reciprocation

In this section, we will soon arrive at a particular operation, which, given a Toeplitz non-negative definite sequence $(C_j)_{j=0}^{\varkappa}$ in $\mathbb{C}^{q\times q}$, will make it possible to construct a new Toeplitz non-negative definite sequence $\left(C_j^{\square}\right)_{j=0}^{\varkappa}$ from $(C_j)_{j=0}^{\varkappa}$. We next provide a detailed look into this construction.

Proposition 5.1. *Let* $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ *and* $(C_j)_{j=0}^{\varkappa}$ *be a T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$. *Suppose that*

$$s_j := \begin{cases} C_0 & \text{if } j = 0, \\ 2C_j & \text{if } j \in \mathbb{Z}_{1,\varkappa}. \end{cases} \tag{5.1}$$

Then $(s_j)_{j=0}^{\varkappa}$ *is a* $q \times q$ *Carathéodory sequence with* $s_0 = s_0^*$ *and this sequence's reciprocal sequence* $\left(s_j^\sharp\right)_{j=0}^{\varkappa}$ *is also a* $q \times q$ *Carathéodory sequence with* $s_0^\sharp = (s_0^\sharp)^*$. *Furthermore, if* $\left(C_j^\square\right)_{j=0}^{\varkappa}$ *is defined as*

$$C_j^\square := \begin{cases} s_0^\sharp & \text{if } j = 0, \\ \frac{1}{2} s_j^\sharp & \text{if } j \in \mathbb{Z}_{1,\varkappa}, \end{cases} \tag{5.2}$$

then $\left(C_j^\square\right)_{j=0}^{\varkappa}$ *is a* $q \times q$ *T-n.n.d. sequence.*

Proof. By Remark 3.7, it follows that $(s_j)_{j=0}^{\varkappa} \in \mathcal{C}_{q,\varkappa}$ and $s_0 = s_0^*$. Theorem 4.4 thus implies $\left(s_j^\sharp\right)_{j=0}^{\varkappa} \in \mathcal{C}_{q,\varkappa}$, where $s_0 = s_0^*$ leads to $s_0^\sharp = (s_0^\sharp)^*$ in view of Definition 1.1 (note also Lemma A.3). It therefore follows by Remark 3.7 that $\left(C_j^\square\right)_{j=0}^{\varkappa}$ is a $q \times q$ T-n.n.d. sequence. □

Proposition 5.1 leads us to the following definition.

Definition 5.2. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(C_j)_{j=0}^{\varkappa}$ be a T-n.n.d. sequence in $\mathbb{C}^{q\times q}$. We will call the sequence $\left(C_j^\square\right)_{j=0}^{\varkappa}$ defined in (formula (5.2) of) Proposition 5.1, the T-n.n.d. sequence in $\mathbb{C}^{q\times q}$ generated from $(C_j)_{j=0}^{\varkappa}$ by reciprocation.

A T-n.n.d. sequence generated by reciprocation is not the same as a reciprocal sequence, as the following example demonstrates.

Example 5.3. Let $K \in \mathbb{K}_{q\times q}$. Suppose, for $j \in \mathbb{N}_0$, that $C_j := K^j$. Example 2.21 yields $(C_j)_{j=0}^{\infty} \in \mathcal{T}_{q,\infty}$. On the one hand, it follows by induction that the reciprocal sequence $\left(C_j^\sharp\right)_{j=0}^{\infty}$ to $(C_j)_{j=0}^{\infty}$ is

$$C_j^\sharp = \begin{cases} I_q, & \text{if } j = 0, \\ -K, & \text{if } j = 1, \\ 0_{q\times q}, & \text{if } j \in \mathbb{Z}_{2,+\infty}. \end{cases}$$

On the other hand, it also follows by induction that the sequence $\left(C_j^\square\right)_{j=0}^{\infty}$ generated from $(C_j)_{j=0}^{\infty}$ by reciprocation is given by

$$C_j^\square = (-K)^j, \qquad j \in \mathbb{N}_0.$$

Proposition 5.4. *Let* $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$ *and let* $(C_j)_{j=0}^{\varkappa}$ *be a T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$. *Suppose that* $\left(C_j^\square\right)_{j=0}^{\varkappa}$ *is the T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$ *generated from* $(C_j)_{j=0}^{\varkappa}$ *by reciprocation. If* $\left(\left(C_j^\square\right)^\square\right)_{j=0}^{\varkappa}$ *is the T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$ *generated from* $\left(C_j^\square\right)_{j=0}^{\varkappa}$ *by reciprocation, then*

$$\left(\left(C_j^\square\right)^\square\right)_{j=0}^{\varkappa} = (C_j)_{j=0}^{\varkappa}.$$

Proof. For each $j \in \mathbb{Z}_{0,\varkappa}$, let

$$s_j := \begin{cases} C_0 & \text{if } j = 0, \\ 2C_j & \text{if } j \in \mathbb{Z}_{1,\varkappa}. \end{cases} \tag{5.3}$$

Because of (5.3) and (5.2), for each $j \in \mathbb{Z}_{0,\varkappa}$, we have,

$$s_j^\sharp = \begin{cases} C_0^\square & \text{if } j = 0, \\ 2C_j^\square & \text{if } j \in \mathbb{Z}_{1,\varkappa}. \end{cases}$$

For each $j \in \mathbb{Z}_{0,\varkappa}$, we thus see, by the definition of $\left(\left(C_j^\square \right)^\square \right)_{j=0}^{\varkappa}$, that

$$\left(C_j^\square \right)^\square = \begin{cases} \left(s_0^\sharp \right)^\sharp & \text{if } j = 0, \\ \frac{1}{2} \left(s_j^\sharp \right)^\sharp & \text{if } j \in \mathbb{Z}_{1,\varkappa}. \end{cases} \tag{5.4}$$

From (5.1), it follows by Remark 3.7 that $(s_j)_{j=0}^{\varkappa}$ is a $q \times q$ Carathéodory sequence. Therefore, by Corollary 4.3, we have

$$\left(\left(s_j^\sharp \right)^\sharp \right)_{j=0}^{\varkappa} = (s_j)_{j=0}^{\varkappa} . \tag{5.5}$$

Finally, from (5.4), (5.5) and (5.3), we obtain

$$\left(\left(C_j^\square \right)^\square \right)_{j=0}^{\varkappa} = (C_j)_{j=0}^{\varkappa} ,$$

which completes the proof. □

We now look at how Definition 5.2 relates to the previous section.

Lemma 5.5. *Let $\varkappa \in \mathbb{N}_0 \cup \{ +\infty \}$ and let $(C_j)_{j=0}^{\varkappa}$ be a T-n.n.d. sequence in $\mathbb{C}^{q\times q}$. Suppose that $\left(C_j^\square \right)_{j=0}^{\varkappa}$ is the T-n.n.d. sequence in $\mathbb{C}^{q\times q}$ generated from $(C_j)_{j=0}^{\varkappa}$ by reciprocation. For $j \in \mathbb{Z}_{0,\varkappa}$, let the matrix s_j be given by* (5.3)*. Then $(s_j)_{j=0}^{\varkappa}$ is a $q \times q$ Carathéodory sequence and*

$$C_j = \begin{cases} \operatorname{Re} s_0 & \text{if } j = 0, \\ \frac{1}{2} s_j & \text{if } j \in \mathbb{Z}_{1,\varkappa} \end{cases} \qquad \text{and} \qquad C_j^\square = \begin{cases} \operatorname{Re} s_0^\sharp & \text{if } j = 0, \\ \frac{1}{2} s_j^\sharp & \text{if } j \in \mathbb{Z}_{1,\varkappa}. \end{cases}$$

Proof. By Proposition 5.1, $(s_j)_{j=0}^{\varkappa}$ is a $q \times q$ Carathéodory sequence. By definition of $(C_j)_{j=0}^{\varkappa}$, we have $C_0^* = C_0$. Thus, $s_0^* = s_0$. Recalling Definition 1.1, we therefore have $\left(s_0^\sharp \right)^* = \left(s_0^+ \right)^* = (s_0^*)^+ = s_0^+ = s_0^\sharp$. We thus see that $\operatorname{Re} s_0 = s_0$ and $\operatorname{Re} s_0^\sharp = s_0^\sharp$. Along with Definition 5.2, this completes the proof. □

Using Lemma 5.5, we can now translate Theorems 4.8, 4.10, 4.11 and 4.13 so that they apply to Definition 5.2.

Theorem 5.6. *Let $\varkappa \in \mathbb{N} \cup \{ +\infty \}$ and let $(C_j)_{j=0}^{\varkappa}$ be a T-n.n.d. sequence in $\mathbb{C}^{q\times q}$. Let, furthermore, $\left(C_j^\square \right)_{j=0}^{\varkappa}$ be the T-n.n.d. sequence in $\mathbb{C}^{q\times q}$ generated from $(C_j)_{j=0}^{\varkappa}$ by reciprocation. Then:*

(a) $(C_j)_{j=0}^{\varkappa}$ *is canonical if and only if* $\left(C_j^{\square}\right)_{j=0}^{\varkappa}$ *is canonical.*
(b) *Let* $k \in \mathbb{Z}_{1,\varkappa}$. *Then:*
 (b1) $(C_j)_{j=0}^{\varkappa}$ *is order* k *canonical if and only if* $\left(C_j^{\square}\right)_{j=0}^{\varkappa}$ *is order* k *canonical.*
 (b2) $(C_j)_{j=0}^{\varkappa}$ *is minimal order* k *canonical if and only if* $\left(C_j^{\square}\right)_{j=0}^{\varkappa}$ *is minimal order* k *canonical.*

Proof. Combine Lemma 5.5 with Theorem 4.8, recalling Definition 3.22. □

Theorem 5.7. *Let* $\varkappa \in \mathbb{N} \cup \{+\infty\}$ *and let* $(C_j)_{j=0}^{\varkappa}$ *be a T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$. *Let, furthermore,* $\left(C_j^{\square}\right)_{j=0}^{\varkappa}$ *be the T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$ *generated from* $(C_j)_{j=0}^{\varkappa}$ *by reciprocation. Then:*

(a) $(C_j)_{j=0}^{\varkappa}$ *is central if and only if* $\left(C_j^{\square}\right)_{j=0}^{\varkappa}$ *is central.*
(b) *Let* $k \in \mathbb{Z}_{1,\varkappa}$. *Then:*
 (b1) $(C_j)_{j=0}^{\varkappa}$ *is order* k *central if and only if* $\left(C_j^{\square}\right)_{j=0}^{\varkappa}$ *is order* k *central.*
 (b2) $(C_j)_{j=0}^{\varkappa}$ *is minimal order* k *central if and only if* $\left(C_j^{\square}\right)_{j=0}^{\varkappa}$ *is minimal order* k *central.*

Proof. Combine Lemma 5.5 with Theorem 4.10, recalling Definition 3.15. □

Theorem 5.8. *Let* $n \in \mathbb{N}_0$ *and* $(C_j)_{j=0}^{n}$ *be a T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$. *Suppose that* $\left(C_j^{\square}\right)_{j=0}^{n}$ *is the T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$ *generated from* $(C_j)_{j=0}^{n}$ *by reciprocation. Let, furthermore,* $(D_j)_{j=0}^{\infty}$ *and* $(E_j)_{j=0}^{\infty}$ *be the central T-n.n.d. sequences for* $(C_j)_{j=0}^{n}$ *and* $\left(C_j^{\square}\right)_{j=0}^{n}$, *respectively. Suppose, also, that* $\left(D_j^{\square}\right)_{j=0}^{\infty}$ *is the T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$ *generated from* $(D_j)_{j=0}^{\infty}$ *by reciprocation. Then*

$$\left(D_j^{\square}\right)_{j=0}^{\infty} = (E_j)_{j=0}^{\infty}.$$

Proof. Combine Lemma 5.5 with Theorem 4.11. □

Theorem 5.9. *Let* $(C_j)_{j=0}^{\infty}$ *be a T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$. *Suppose, furthermore, that* $\left(C_j^{\square}\right)_{j=0}^{\infty}$ *is the T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$ *generated from* $(C_j)_{j=0}^{\infty}$ *by reciprocation. Then* $(C_j)_{j=0}^{\infty} \in \mathcal{C}_{q,\infty}^{\mathrm{t}}$ *if and only if* $\left(C_j^{\square}\right)_{j=0}^{\infty} \in \mathcal{C}_{q,\infty}^{\mathrm{t}}$.

Proof. Combine Lemma 5.5 and Theorem 4.13, recalling Definition 3.14. □

6. Matricial Carathéodory functions

The set of $q \times q$ Carathéodory sequences is closely related to the following class of holomorphic $q \times q$ matrix functions. A function $\Omega : \mathbb{D} \longrightarrow \mathbb{C}^{q\times q}$ is called a $q \times q$ Carathéodory function in $\mathbb{D}$ if it is holomorphic in $\mathbb{D}$ and if $\Omega(w) \in \mathcal{R}_{q,\geq}$ for all $w \in \mathbb{D}$. The set of all $q \times q$ Carathéodory functions in $\mathbb{D}$ will be denoted by $\mathcal{C}_q(\mathbb{D})$.

Remark 6.1. If $\Omega \in \mathcal{C}_q(\mathbb{D})$ and $A \in \mathbb{C}^{q\times p}$, then $A^*\Omega A \in \mathcal{C}_p(\mathbb{D})$.

Remark 6.2. If $n \in \mathbb{N}$, $(\alpha_j)_{j=1}^n$ is a sequence in $[0, +\infty)$ and $(\Omega_j)_{j=1}^n$ is a sequence in $\mathcal{C}_q(\mathbb{D})$, then $\sum_{j=1}^n \alpha_j\Omega_j \in \mathcal{C}_q(\mathbb{D})$.

The class $\mathcal{C}_q(\mathbb{D})$ is closely related to the *Schur* class. A matrix function $S : \mathbb{D} \longrightarrow \mathbb{C}^{p\times q}$ is called a $p \times q$ Schur function in $\mathbb{D}$ if it is holomorphic in $\mathbb{D}$ and if $S(w)$ is contractive for all $w \in \mathbb{D}$. The set of all $p \times q$ Schur functions in $\mathbb{D}$ will be denoted by $\mathcal{S}_{p\times q}(\mathbb{D})$. We furthermore define

$$\mathcal{S}'_{p\times q}(\mathbb{D}) := \{ f \in \mathcal{S}_{p\times q}(\mathbb{D}) \ : \ f(w) \in \mathbb{D}_{p\times q} \text{ for all } w \in \mathbb{D} \},$$

as well as $\mathcal{S}(\mathbb{D}) := \mathcal{S}_{1\times 1}(\mathbb{D})$ and $\mathcal{S}'(\mathbb{D}) := \mathcal{S}'_{1\times 1}(\mathbb{D})$. We next review a well-known relationship that exists between the classes $\mathcal{C}_q(\mathbb{D})$ and $\mathcal{S}_{q\times q}(\mathbb{D})$ (see [8, Proposition 2.1.3]).

Lemma 6.3. *Let $\Omega \in \mathcal{C}_q(\mathbb{D})$. Then* $\det[I_q + \Omega(w)] \neq 0$ *for each $w \in \mathbb{D}$, and the function*

$$S := (I_q - \Omega)(I_q + \Omega)^{-1} \tag{6.1}$$

belongs to $\mathcal{S}_{q\times q}(\mathbb{D})$. Furthermore, $\det[I_q + S(w)] \neq 0$ *for each $w \in \mathbb{D}$, and*

$$\Omega = (I_q - S)(I_q + S)^{-1}. \tag{6.2}$$

For each $\Omega \in \mathcal{C}_q(\mathbb{D})$, the matrix-valued function in (6.1) is called the Cayley transform of Ω. Lemma A.13 immediately leads us to another well-known relationship for the sets $\mathcal{S}_{q\times q}(\mathbb{D})$ and $\mathcal{C}_q(\mathbb{D})$.

Lemma 6.4. *Suppose that $S \in \mathcal{S}_{q\times q}(\mathbb{D})$, then $I_q + S \in \mathcal{C}_q(\mathbb{D})$.*

Example 6.5. Suppose that $K \in \mathbb{K}_{q\times q}$ and, furthermore, that $\Omega : \mathbb{D} \longrightarrow \mathbb{C}^{q\times q}$ is defined as $w \longmapsto I_q + wK$. Then $\Omega \in \mathcal{C}_q(\mathbb{D})$.

One of our goals will later be to generalize the following well-known result for the class $\mathcal{C}_q(\mathbb{D})$. Because it relates to our later approach, we include a proof.

Proposition 6.6. *Let $\Omega \in \mathcal{C}_q(\mathbb{D})$ be such that* $\det\Omega$ *does not vanish identically in $\mathbb{D}$. Then* $\det[\Omega(w)] \neq 0$ *for any $w \in \mathbb{D}$, and $\Omega^{-1} \in \mathcal{C}_q(\mathbb{D})$.*

Proof. Let S be the Cayley transform of Ω. By Lemma 6.3, it follows that

$$S \in \mathcal{S}_{q\times q}(\mathbb{D}) \tag{6.3}$$

and we obtain (6.2). Since $\det\Omega$ is not the zero function in $\mathbb{D}$, we see from (6.2) that $\det(I_q - S)$ is not the zero function in $\mathbb{D}$. Combining this with (6.3), we see from [8, Lemma 2.1.7] that $\det(I_q - S)$ does not vanish anywhere in $\mathbb{D}$. Thus, it follows from (6.2) that Ω does not vanish anywhere in $\mathbb{D}$. Therefore (note also the formulas after Remark A.1), we see that $\Omega^{-1} \in \mathcal{C}_q(\mathbb{D})$. □

Remark 6.7. Let $\Omega \in \mathcal{C}_q(\mathbb{D})$ and $f \in \mathcal{S}'(\mathbb{D})$. The properties of holomorphic functions yield $\Omega \circ f \in \mathcal{C}_q(\mathbb{D})$.

Example 6.8. Let $\Omega \in \mathcal{C}_q(\mathbb{D})$ and $u \in \mathbb{D} \cup \mathbb{T}$. Suppose that $\Omega_u : \mathbb{D} \longrightarrow \mathbb{C}^{q\times q}$ is defined as $\Omega_u : w \longmapsto \Omega(uw)$. From Remark 6.7, we see that $\Omega_u \in \mathcal{C}_q(\mathbb{D})$.

Example 6.9. Let $\Omega \in \mathcal{C}_q(\mathbb{D})$ and $m \in \mathbb{Z}_{2,\infty}$. Suppose that $\Omega_{[m]} : \mathbb{D} \longrightarrow \mathbb{C}^{q\times q}$ is defined as $\Omega_{[m]} : w \longmapsto \Omega(w^m)$. From Remark 6.7, we see that $\Omega_{[m]} \in \mathcal{C}_q(\mathbb{D})$.

For more clarity later on, we next describe the well-known relationship that exists between $q \times q$ Carathéodory functions and $q \times q$ Carathéodory sequences (see, for instance, [11]).

Proposition 6.10. *Let $\Omega \in \mathcal{C}_q(\mathbb{D})$ and, for each $w \in \mathbb{D}$, let*

$$\Omega(w) = \sum_{j=0}^{\infty} s_j w^j \tag{6.4}$$

be the Taylor series representation of Ω. Then $(s_j)_{j=0}^{\infty}$ is a $q \times q$ Carathéodory sequence.

Example 6.11. Let $K \in \mathbb{K}_{q\times q}$. The sequence $(s_j)_{j=0}^{\infty}$, given by $s_j := \delta_{j,0} I_q + \delta_{j,1} K$, for each $j \in \mathbb{N}_0$, is then a $q\times q$ Carathéodory sequence. Indeed, if $\Omega : \mathbb{D} \longrightarrow \mathbb{C}^{q\times q}$ is defined as $\Omega(w) := I_q + wK$, then Example 6.5 yields $\Omega \in \mathcal{C}_q(\mathbb{D})$ and $(s_j)_{j=0}^{\infty}$ is the Taylor coefficient sequence for Ω. Proposition 6.10 thus shows that $(s_j)_{j=0}^{\infty} \in \mathcal{C}_{q,\infty}$.

Proposition 6.12. *Let $(s_j)_{j=0}^{\infty}$ be a $q \times q$ Carathéodory sequence. Then:*

(a) *The sequence $\left(\sum_{j=0}^{n} s_j w^j \right)_{n=0}^{\infty}$ converges for any $w \in \mathbb{D}$.*

(b) *If $\Omega : \mathbb{D} \longrightarrow \mathbb{C}^{q\times q}$ is defined as $\Omega(w) := \lim_{n\longrightarrow\infty} \sum_{j=0}^{n} s_j w^j$, then $\Omega \in \mathcal{C}_q(\mathbb{D})$.*

If $(s_j)_{j=0}^{\infty}$ is a $q \times q$ Carathéodory sequence, then the function $\Omega \in \mathcal{C}_q(\mathbb{D})$ defined in part (b) of Proposition 6.12 will be referred to as the function in $\mathcal{C}_q(\mathbb{D})$ corresponding to $(s_j)_{j=0}^{\infty}$.

Example 6.13. Let $(s_j)_{j=0}^{\infty} \in \mathcal{C}_{q,\infty}$ and $u \in \mathbb{D} \cup \mathbb{T}$. Suppose, for each $j \in \mathbb{N}_0$, that $s_{j,u} := u^j s_j$. Part (a) of Remark 3.10 then yields $(s_{j,u})_{j=0}^{\infty} \in \mathcal{C}_{q,\infty}$. If Ω is the function in $\mathcal{C}_q(\mathbb{D})$ corresponding to $(s_j)_{j=0}^{\infty}$, then we see, for the function Ω_u defined in Example 6.8, that

$$\Omega_u(w) = \sum_{j=0}^{\infty} s_{j,u} w^j .$$

Thus, Ω_u is the function in $\mathcal{C}_q(\mathbb{D})$ corresponding to $(s_{j,u})_{j=0}^{\infty}$.

Example 6.14. Let $(s_j)_{j=0}^{\infty} \in \mathcal{C}_{q,\infty}$ and Ω be the function in $\mathcal{C}_q(\mathbb{D})$ corresponding to $(s_j)_{j=0}^{\infty}$. Suppose, furthermore, that $m \in \mathbb{Z}_{2,+\infty}$ and that

$$t_j := \begin{cases} s_{\frac{j}{m}}, & \text{if there exists a } k \in \mathbb{N}_0 \text{ such that } j = k \cdot m, \\ 0_{q\times q}, & \text{if } j \in \mathbb{N}_0 \setminus \{ k \cdot m \ : \ k \in \mathbb{N}_0 \}. \end{cases}$$

Let $\Omega_{[m]} \in \mathcal{C}_q(\mathbb{D})$ be the function defined in Example 6.9. For each $w \in \mathbb{D}$, then

$$\Omega_{[m]}(w) = \sum_{j=0}^{\infty} t_j w^j.$$

Thus, $(t_j)_{j=0}^{\infty} \in \mathcal{C}_{q,\infty}$ and $\Omega_{[m]}$ is the function in $\mathcal{C}_q(\mathbb{D})$ corresponding to $(t_j)_{j=0}^{\infty}$.

Propositions 6.10 and 6.12 suggest introducing a number of special subclasses for $\mathcal{C}_q(\mathbb{D})$ via Taylor coefficient sequences.

Definition 6.15. Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$. Suppose $\Omega \in \mathcal{C}_q(\mathbb{D})$ with Taylor series representation (6.4) for each $w \in \mathbb{D}$. If $(s_j)_{j=0}^{\varkappa} \in \widetilde{\mathcal{C}}_{q,\varkappa}$, then Ω is called a strict $q \times q$ Carathéodory function of order $\varkappa$ (or strict order $\varkappa$ Carathéodory function). The set of all strict Carathéodory functions of order $\varkappa$ will be denoted by $\mathcal{C}_q^{(\varkappa)}(\mathbb{D})$.

Definition 6.16. Suppose $\Omega \in \mathcal{C}_q(\mathbb{D})$ with Taylor series representation (6.4), for each $w \in \mathbb{D}$. If $(s_j)_{j=0}^{\infty} \in \mathcal{C}_{q,\infty}^{\mathrm{t}}$, then Ω is called a totally strict $q \times q$ Carathéodory function in $\mathbb{D}$. The set of all totally strict $q \times q$ Carathéodory functions in $\mathbb{D}$ will be denoted by $\mathcal{C}_q^{\mathrm{t}}(\mathbb{D})$.

From the above two Definitions 6.15 and 6.16, it follows, via (3.2), that

$$\mathcal{C}_q^{\mathrm{t}}(\mathbb{D}) \subseteq \mathcal{C}_q^{(\infty)}(\mathbb{D}) = \bigcap_{\varkappa=0}^{\infty} \mathcal{C}_q^{(\varkappa)}(\mathbb{D}).$$

Definition 6.17. Let $\Omega \in \mathcal{C}_q(\mathbb{D})$ with Taylor series representation (6.4) for $w \in \mathbb{D}$.

(a) Let $k \in \mathbb{N}$. If the sequence $(s_j)_{j=0}^{\infty}$ is C-central of order k, then the function Ω is called central of order k (or order k central).

(b) Let $k \in \mathbb{N}$. If the sequence $(s_j)_{j=0}^{\infty}$ is C-central of minimal order k, then the function Ω is called central of minimal order k (or minimal order k central).

(c) If there exists a $k \in \mathbb{N}$ such that Ω is order k central, then Ω is simply called a central function.

Remark 6.18. Let $k \in \mathbb{N}$ and $\ell \in \mathbb{Z}_{k+1,\infty}$. Suppose that $\Omega \in \mathcal{C}_q(\mathbb{D})$ is central of order k. Remark 3.16 implies that Ω is also central of order ℓ.

Recalling Proposition 6.12 and Lemma 3.19, we are led to the following.

Definition 6.19. Let $n \in \mathbb{N}$ and $(s_j)_{j=0}^{n}$ be a $q \times q$ Carathéodory sequence. Furthermore, let $(\widetilde{s}_j)_{j=0}^{\infty}$ be the C-central sequence corresponding to $(s_j)_{j=0}^{n}$. The function $\Omega_{(s_j)_{j=0}^{n}}$ in $\mathcal{C}_q(\mathbb{D})$ corresponding to $(\widetilde{s}_j)_{j=0}^{\infty}$ is then called the central $q \times q$ Carathéodory function for $(s_j)_{j=0}^{n}$.

Remark 6.20. Let $k \in \mathbb{N}$. Suppose that $\Omega \in \mathcal{C}_q(\mathbb{D})$ is order k central with Taylor series representation (6.4) for each $w \in \mathbb{D}$. Because of Definition 6.19 and Remark 3.18, it follows that $(s_j)_{j=0}^{k-1} \in \mathcal{C}_{q,k-1}$ and Ω is the central $q \times q$ Carathéodory function $\Omega_{(s_j)_{j=0}^{k-1}}$ for $(s_j)_{j=0}^{k-1}$.

The function defined in Definition 6.19 is discussed extensively in [16]. In particular, it was there shown that $\Omega_{(s_j)_{j=0}^{n}}$ can always be interpreted as the restriction of a rational $q \times q$ matrix function to $\mathbb{D}$. Using the $q \times q$ Carathéodory sequence $(s_j)_{j=0}^{n}$, particular quadruples of $q \times q$ matrix polynomials were constructed. These quadruples were then used to obtain left and right quotient representations of $\Omega_{(s_j)_{j=0}^{n}}$. These representations were, in turn, used in [18] to parametrize the solution set of the Carathéodory problem associated with $(s_j)_{j=0}^{n}$.

Proposition 6.21. *Let $n \in \mathbb{N}_0$ and $(s_j)_{j=0}^{n} \in \mathcal{C}_{q,n}$. Then all of the following conditions are equivalent:*

(i) $(s_j)_{j=0}^{n} \in \widetilde{\mathcal{C}}_{q,n}$.
(ii) $\Omega_{(s_j)_{j=0}^{n}} \in \mathcal{C}_q^{\mathrm{t}}(\mathbb{D})$.
(iii) $\Omega_{(s_j)_{j=0}^{n}} \in \mathcal{C}_q^{(\infty)}(\mathbb{D})$.

Proof. Applying part (b) of Lemma 3.19 yields the proof. □

Definition 6.22. Let $\Omega \in \mathcal{C}_q(\mathbb{D})$ with Taylor series representation (6.4) for $w \in \mathbb{D}$.

(a) Let $k \in \mathbb{N}$. If the sequence $(s_j)_{j=0}^{\infty}$ is order k C-canonical, then the function Ω is also called canonical of order k (or order k canonical).
(b) Let $k \in \mathbb{N}$. If the sequence $(s_j)_{j=0}^{\infty}$ is minimal order k C-canonical, then the function Ω is also called canonical of minimal order k (or minimal order k canonical).
(c) If there exists a $k \in \mathbb{N}$ such that Ω is order k canonical, then Ω is simply called a canonical function.

Remark 6.23. Let $k \in \mathbb{N}$, $\ell \in \mathbb{Z}_{k+1,\infty}$ and $\Omega \in \mathcal{C}_q(\mathbb{D})$ be order k canonical.

(a) Part (a) of Corollary 2.27 implies that Ω is also order ℓ canonical.
(b) It follows, by part (b) of Corollary 2.27, that Ω is also central of order ℓ.

For the $q \times q$ Carathéodory sequences in Examples 3.20 and 3.21, we now determine the corresponding functions in $\mathcal{C}_q(\mathbb{D})$.

Example 6.24. Let $K \in \mathbb{K}_{q \times q}$. Then:

(a) It follows for each $w \in \mathbb{D}$ that $-wK \in \mathbb{D}_{q \times q}$ and that $\det(I_q - wK) \neq 0$.
(b) The function $\Omega_K : \mathbb{D} \longrightarrow \mathbb{C}^{q \times q}$, defined via

$$w \longmapsto (I_q + wK)(I_q - wK)^{-1} \tag{6.5}$$

belongs to $\mathcal{C}_q(\mathbb{D})$.

(c) Since $wK \in \mathbb{D}_{q\times q}$, the Neumann series representation yields

$$\Omega_K(w) = (\, I_q + wK\,) \Big[\sum\nolimits_{j=0}^{\infty} (\, wK\,)^j \Big] = I_q + 2\sum\nolimits_{j=1}^{\infty} K^j w^j. \tag{6.6}$$

Example 3.20 shows that the function Ω_K is order 2 central and, if (and only if) $K \neq 0_{q\times q}$, then Ω_K is minimal order 2 central. Furthermore, because of Remark 6.20, Example 3.20 and Proposition 6.21, the following three conditions are all equivalent:

(i) $K \in \mathbb{D}_{q\times q}$.
(ii) $\Omega_K \in \mathcal{C}_q^{\,\mathrm{t}}\,(\,\mathbb{D}\,)$.
(iii) $\Omega_K \in \mathcal{C}_q^{(\infty)}(\,\mathbb{D}\,)$.

(d) For each $w \in \mathbb{D}$, we have $\det\,[\,\Omega_K(w)\,] \neq 0$, $-K \in \mathbb{K}_{q\times q}$ and ${\Omega_K}^{-1} = \Omega_{-K}$.

(e) Example 3.25 shows that the function Ω_K is canonical if and only if K is unitary. When this is the case, Ω_K is order 1 canonical.

Example 6.25. Let $K \in \mathbb{K}_{q\times q} \cap \mathbb{C}_{\geq}^{q\times q}$ and $r \in \mathbb{N}$. Then:

(a) Recalling part (a) of Example 6.24, let $\Omega_{K,\,r} : \mathbb{D} \longrightarrow \mathbb{C}^{q\times q}$ be defined as

$$w \longmapsto \big(\sqrt{K}\big)^r\,(\,I_q + wK\,)\,(\,I_q - wK\,)^{-1}\,\big(\sqrt{K}\big)^r. \tag{6.7}$$

(b) Then

$$\Omega_{K,\,r} = \Big[\big(\sqrt{K}\big)^r\Big]^*\,\Omega_K\big(\sqrt{K}\big)^r, \tag{6.8}$$

where Ω_K is the $q\times q$ Carathéodory function given in Example 6.24. It follows from Example 6.24 and (6.8) that, for each $w \in \mathbb{D}$, the function $\Omega_{K,\,r}$ belongs to $\mathcal{C}_q\,(\,\mathbb{D}\,)$ and admits the Taylor series representation

$$\Omega_{K,\,r}(w) = K^r + 2\sum\nolimits_{j=0}^{\infty} K^{r+j} w^j.$$

(c) Example 6.24 shows that the function $\Omega_{K,\,r}$ is order 2 central.

For the $q \times q$ Carathéodory sequences in Examples 3.12 and 3.13, we now determine the corresponding functions in $\mathcal{C}_q\,(\,\mathbb{D}\,)$.

Example 6.26. Let $K \in \mathbb{K}_{q\times q}$ and $(\,C_j\,)_{j=0}^{\infty}$ be the $q \times q$ C-sequence defined in Example 3.12, i.e., $C_j := K^j$ for each $j \in \mathbb{N}_0$. Suppose that H_K is the function in $\mathcal{C}_q\,(\,\mathbb{D}\,)$ corresponding to $(\,C_j\,)_{j=0}^{\infty}$. Then, for each $w \in \mathbb{D}$, we see from the Neumann series representation that $\det\,(\,I_q - wK\,) \neq 0$ and

$$H_K(w) = (\,I_q - wK\,)^{-1}.$$

Example 6.27. Let $K \in \mathbb{K}_{q\times q} \cap \mathbb{C}_{\geq}^{q\times q}$ and $r \in \mathbb{N}$. Let $(\,C_j\,)_{j=0}^{\infty}$ be the $q \times q$ C-sequence defined in Example 3.13, i.e., $C_j := K^{r+j}$ for each $j \in \mathbb{N}_0$. Suppose that $H_{K,\,r}$ is the function in $\mathcal{C}_q\,(\,\mathbb{D}\,)$ corresponding to $(\,C_j\,)_{j=0}^{\infty}$. It follows that $C_j = \big(\sqrt{K}\big)^r K^j \big(\sqrt{K}\big)^r$ for any $j \in \mathbb{N}_0$. Therefore, it follows from Example 6.26, for each $w \in \mathbb{D}$, that $\det\,(\,I_q - wK\,) \neq 0$ and

$$H_{K,\,r}(w) = \big(\sqrt{K}\big)^r\,(\,I_q - wK\,)^{-1}\,\big(\sqrt{K}\big)^r.$$

7. Moore-Penrose Inverses of Matricial Carathéodory Functions

Our next steps will be towards showing that the Moore-Penrose inverse Ω^+ of a function $\Omega \in \mathcal{C}_q(\mathbb{D})$ is itself a member of $\mathcal{C}_q(\mathbb{D})$. We will, furthermore, show that the sequence of Ω^+'s Taylor coefficients is the reciprocal sequence to Ω's Taylor coefficient sequence. Before we show that the Moore-Penrose inverse of a $q \times q$ Carathéodory function is holomorphic, we first formulate and prove a result which will help us reach our aforementioned goal and which is also of interest on its own.

Lemma 7.1. *Let $(s_j)_{j=0}^{\infty}$ be a sequence in $\mathbb{C}^{q\times q}$ and $\left(s_j^{\sharp}\right)_{j=0}^{\infty}$ be its reciprocal sequence. Suppose that the following conditions are met:*

(I) $(s_j)_{j=0}^{\infty} \in \mathcal{D}_{q\times q,\infty}$.

(II) *There exists a (in $\mathbb{D}$) holomorphic matrix function $\Omega : \mathbb{D} \longrightarrow \mathbb{C}^{q\times q}$ with Taylor expansion (6.4) for each $w \in \mathbb{D}$.*

(III) *There exists a (in $\mathbb{D}$) holomorphic matrix function $\Omega^{\sharp} : \mathbb{D} \longrightarrow \mathbb{C}^{q\times q}$, which can be expressed, for each $w \in \mathbb{D}$, as*

$$\Omega^{\sharp}(w) = \sum_{j=0}^{\infty} s_j^{\sharp} w^j. \tag{7.1}$$

Then $\Omega^+ = \Omega^{\sharp}$ and, for each $w \in \mathbb{D}$

$$\left(\Omega\Omega^+\right)(w) = \left(\Omega\Omega^+\right)(0), \qquad \left(\Omega^+\Omega\right)(w) = \left(\Omega^+\Omega\right)(0), \tag{7.2}$$

$$\mathcal{R}(\Omega(w)) = \mathcal{R}(\Omega(0)) \qquad \textit{and} \qquad \mathcal{N}(\Omega(w)) = \mathcal{N}(\Omega(0)). \tag{7.3}$$

Proof. Because of (I), it follows by Proposition 1.8 that $\mathrm{S}_n^+ = \mathrm{S}_n^{\sharp}$, for each $n \in \mathbb{N}_0$, where S_n and $\mathrm{S}_n^{\sharp}$ are given by (1.1) and (1.2). Thus, for each $n \in \mathbb{N}_0$, we obtain

$$\mathrm{S}_n\mathrm{S}_n^{\sharp}\mathrm{S}_n = \mathrm{S}_n \qquad \text{and} \qquad \mathrm{S}_n^{\sharp}\mathrm{S}_n\mathrm{S}_n^{\sharp} = \mathrm{S}_n^{\sharp}. \tag{7.4}$$

Also, recalling (I) and using Lemma 1.9, for each $n \in \mathbb{N}_0$, we get

$$\mathrm{S}_n\mathrm{S}_n^{\sharp} = I_{n+1} \otimes \left(s_0 s_0^+\right) \qquad \text{and} \qquad \mathrm{S}_n^{\sharp}\mathrm{S}_n = I_{n+1} \otimes \left(s_0^+ s_0\right). \tag{7.5}$$

Let $w \in \mathbb{D}$. Because of (7.4), (7.5), (II) and (III), it therefore follows by [8, Lemmas 1.1.19 and 1.1.21] that

$$\Omega(w)\Omega^{\sharp}(w)\Omega(w) = \Omega(w) \qquad \text{and} \qquad \Omega^{\sharp}(w)\Omega(w)\Omega^{\sharp}(w) = \Omega^{\sharp}(w) \tag{7.6}$$

and also that

$$\Omega(w)\Omega^{\sharp}(w) = s_0 s_0^+ \qquad \text{and} \qquad \Omega^{\sharp}(w)\Omega(w) = s_0^+ s_0. \tag{7.7}$$

In particular, (7.7) implies

$$\left(\Omega(w)\Omega^{\sharp}(w)\right)^* = \Omega(w)\Omega^{\sharp}(w) \qquad \text{and} \qquad \left(\Omega^{\sharp}(w)\Omega(w)\right)^* = \Omega^{\sharp}(w)\Omega(w). \tag{7.8}$$

From (7.6) and (7.8), we obtain $\Omega^+ = \Omega^{\sharp}$. Furthermore, [22, Proposition 8.4] yields (7.2) and (7.3). □

We continue with a first observation on the ranges and null spaces of a function $\Omega \in \mathcal{C}_q(\mathbb{D})$.

Lemma 7.2. *Let $\Omega \in \mathcal{C}_q(\mathbb{D})$ and $w \in \mathbb{D}$. Then*

$$\mathcal{R}\left([\Omega(w)]^*\right) = \mathcal{R}(\Omega(w)) = \mathcal{R}\left([\Omega(w)]^+\right)$$

and

$$\mathcal{N}\left([\Omega(w)]^+\right) = \mathcal{N}(\Omega(w)) = \mathcal{N}\left([\Omega(w)]^*\right).$$

Proof. Since $\Omega \in \mathcal{C}_q(\mathbb{D})$, we see that $\Omega(w) \in \mathcal{R}_{q,\geq}$. By Lemma A.8, it therefore follows that $\Omega(w) \in \mathbb{C}^{q\times q}_{\mathrm{EP}}$. The lemma thus follows from Proposition A.5. □

Our next theorem is also the first main result of this section. It includes the earlier-mentioned and quite versatile generalization of Proposition 6.6. The proof of the theorem, moreover, demonstrates an alternative approach to Proposition 6.6.

Theorem 7.3. *Let $\Omega \in \mathcal{C}_q(\mathbb{D})$ with Taylor series representation (6.4) for each $w \in \mathbb{D}$. Furthermore, let $\left(s_j^\sharp\right)_{j=0}^\infty$ be the reciprocal sequence to $(s_j)_{j=0}^\infty$. Then Ω^+ belongs to $\mathcal{C}_q(\mathbb{D})$ and Ω^+ admits the Taylor series representation*

$$\Omega^+(w) = \sum_{j=0}^\infty s_j^\sharp w^j. \tag{7.9}$$

Furthermore (7.2), (7.3),

$$\mathcal{R}\left(\Omega^+(w)\right) = \mathcal{R}(\Omega(0)) \quad \text{and} \quad \mathcal{N}\left(\Omega^+(w)\right) = \mathcal{N}(\Omega(0)) \tag{7.10}$$

all hold, for each $w \in \mathbb{D}$.

Proof. By assumption, we have (6.4) for each $w \in \mathbb{D}$. Since $\Omega \in \mathcal{C}_q(\mathbb{D})$, we see by Proposition 6.10 that $(s_j)_{j=0}^\infty$ is a $q \times q$ Carathéodory sequence. Thus, it follows by Theorem 4.4 that $\left(s_j^\sharp\right)_{j=0}^\infty$ is also a $q \times q$ Carathéodory sequence. From Proposition 6.12, we now see that there is a function $\Omega^\sharp \in \mathcal{C}_q(\mathbb{D})$ which admits the representation (7.1) for each $w \in \mathbb{D}$. By Proposition 4.1, it follows that $(s_j)_{j=0}^\infty \in \mathcal{D}_{q\times q,\infty}$, since $(s_j)_{j=0}^\infty$ is a $q \times q$ Carathéodory sequence. Thus, using (7.9), (7.10) and Lemma 7.1, we obtain $\Omega^+ = \Omega^\sharp$, as well as (7.2) and (7.3). Equations (7.10) follow by Lemma 7.2. □

In [5, Theorem 4.5] it was already shown, using an alternate approach, that if $\Omega \in \mathcal{C}_q(\mathbb{D})$, then $\Omega^+ \in \mathcal{C}_q(\mathbb{D})$. Likewise, parts of Theorem 7.3 were also proved as part of [5, Theorem 4.5]. The approach used for [5, Theorem 4.5] was largely based on specific properties of the Schur class $\mathcal{S}_{q\times q}(\mathbb{D})$ and how these properties carry over to the class $\mathcal{C}_q(\mathbb{D})$ via Cayley transform.

Corollary 7.4. *Let $(s_j)_{j=0}^\infty$ be a $q \times q$ Carathéodory sequence and Ω be the function in $\mathcal{C}_q(\mathbb{D})$ associated with $(s_j)_{j=0}^\infty$. Furthermore, let $\left(s_j^\sharp\right)_{j=0}^\infty$ be the reciprocal sequence to $(s_j)_{j=0}^\infty$. Then, $\left(s_j^\sharp\right)_{j=0}^\infty$ is a $q \times q$ Carathéodory sequence and, if $\Omega^\sharp$ is the function in $\mathcal{C}_q(\mathbb{D})$ associated with $\left(s_j^\sharp\right)_{j=0}^\infty$, then $\Omega^\sharp = \Omega^+$.*

Proof. Proposition 6.12 shows that $\Omega \in \mathcal{C}_q(\mathbb{D})$ and that (6.4) holds true for each $w \in \mathbb{D}$. Thus, the corollary follows directly from Theorem 7.3. □

We next consider several applications of Theorem 7.3.

Corollary 7.5. *Suppose $n \in \mathbb{N}$, $(\alpha_j)_{j=1}^n$ is a sequence in $[0, +\infty)$ and $(\Omega_j)_{j=1}^n$ is a sequence in $\mathcal{C}_q(\mathbb{D})$. Then*

$$\Big(\sum_{j=1}^n \alpha_j \Omega_j^+\Big)^+ \in \mathcal{C}_q(\mathbb{D}).$$

Proof. By Theorem 7.3, it follows that $(\Omega_j^+)_{j=1}^n$ is a sequence in $\mathcal{C}_q(\mathbb{D})$. Because of Remark 6.2, we therefore have

$$\sum_{j=1}^n \alpha_j \Omega_j^+ \in \mathcal{C}_q(\mathbb{D}).$$

Hence, applying Theorem 7.3 completes the proof. □

Corollary 7.6. *Suppose $S \in \mathcal{S}_{q\times q}(\mathbb{D})$. Then $(I_q + S)^+ \in \mathcal{C}_q(\mathbb{D})$ and the equations*

$$[I_q + S(w)][I_q + S(w)]^+ = [I_q + S(0)][I_q + S(0)]^+$$

and

$$[I_q + S(w)]^+[I_q + S(w)] = [I_q + S(0)]^+[I_q + S(0)]$$

hold true for each $w \in \mathbb{D}$. Furthermore,

$$\mathcal{R}(I_q + S(w)) = \mathcal{R}(I_q + S(0)), \qquad \mathcal{N}(I_q + S(w)) = \mathcal{N}(I_q + S(0)),$$

$$\mathcal{R}\Big([I_q + S(w)]^+\Big) = \mathcal{R}(I_q + S(0)) \quad \text{and} \quad \mathcal{N}\Big([I_q + S(w)]^+\Big) = \mathcal{N}(I_q + S(0))$$

hold true for each $w \in \mathbb{D}$.

Proof. By Lemma 6.4, we see that $I_q + S$ belongs to $\mathcal{C}_q(\mathbb{D})$. Thus, applying Theorem 7.3 completes the proof. □

Our next result tells us more about the structure of Carathéodory functions.

Theorem 7.7. *Let $q \in \mathbb{Z}_{2,+\infty}$ and $r \in \mathbb{Z}_{1,q-1}$. Suppose that $\Omega \in \mathcal{C}_q(\mathbb{D})$ with* rank $[\Omega(0)] = r$. *Furthermore, let $(u_s)_{s=1}^r$ be an orthonormal basis in $\mathcal{R}(\Omega(0))$ and let $(u_s)_{s=r+1}^q$ be an orthonormal basis in $\mathcal{N}(\Omega(0))$. For each $\ell \in \mathbb{Z}_{1,q}$, let $U_\ell := \big(u_1, \; u_2, \; \ldots, \; u_\ell\big)$. Finally, let $\widetilde{\Omega} := U_r^* \Omega U_r$. Then:*

(a) *$(u_s)_{s=1}^q$ is an orthonormal basis in $\mathbb{C}^{q\times 1}$ and the matrix U_q is unitary.*

(b) *The function $\widetilde{\Omega}$ belongs to $\mathcal{C}_r(\mathbb{D})$ and $\widetilde{\Omega}(w)$ is non-singular for all $w \in \mathbb{D}$. Furthermore,*

$$\Omega = U_q^* \Big[\operatorname{diag}\Big(\widetilde{\Omega}, \, 0_{(q-r)\times(q-r)}\Big)\Big] U_q$$

and

$$\Omega^+ = U_q^* \Big[\operatorname{diag}\Big(\widetilde{\Omega}^{-1}, \, 0_{(q-r)\times(q-r)}\Big)\Big] U_q.$$

Proof. Let $(s_j)_{j=0}^{\infty}$ be the Taylor coefficient sequence for Ω. Then, for each $w \in \mathbb{D}$, let (6.4) be the Taylor series representation of Ω. Then $\Omega(0) = s_0$, and, in particular, $\operatorname{rank} s_0 = r$. Since $\Omega \in \mathcal{C}_q(\mathbb{D})$, it follows by Proposition 6.10 that $(s_j)_{j=0}^{\infty} \in \mathcal{C}_{q,\infty}$. Thus, Proposition 4.1 and part (a) of Theorem 1.12 yield part (a). By Remark 6.1, it follows that $\widetilde{\Omega} \in \mathcal{C}_r(\mathbb{D})$. For each $j \in \mathbb{N}_0$, we set

$$\widetilde{s}_j := U_r^* s_j U_r. \tag{7.11}$$

Since $\operatorname{rank} s_0 = r$, we then see from Theorem 4.6 that $\det \widetilde{s}_0 \neq 0$. Let $w \in \mathbb{D}$. Using (6.4) and (7.11), we then obtain

$$\widetilde{\Omega}(w) = \sum_{j=0}^{\infty} U_r^* s_j U_r w^j = \sum_{j=0}^{\infty} \widetilde{s}_j w^j. \tag{7.12}$$

In particular, $\widetilde{\Omega}(0) = \widetilde{s}_0$. Thus, $\det \widetilde{s}_0 \neq 0$ implies $\det \widetilde{\Omega}(0) \neq 0$. Since $\widetilde{\Omega}$ belongs to $\mathcal{C}_r(\mathbb{D})$ and from Proposition 6.6 we now obtain $\det \widetilde{\Omega}(w) \neq 0$ for any $w \in \mathbb{D}$. Because of (7.11) and Theorem 4.6, it follows that

$$s_j = U_q^* \left[\operatorname{diag}\left(\widetilde{s}_j, 0_{(q-r)\times(q-r)}\right) \right] U_q \tag{7.13}$$

for each $j \in \mathbb{N}_0$. Using (6.4), (7.13) and (7.12), we obtain

$$\begin{aligned}
\Omega(w) = \sum_{j=0}^{\infty} s_j w^j &= \sum_{j=0}^{\infty} U_q^* \left[\operatorname{diag}\left(\widetilde{s}_j, 0_{(q-r)\times(q-r)}\right) \right] U_q w^j \\
&= U_q^* \Big(\sum_{j=0}^{\infty} \left[\operatorname{diag}\left(\widetilde{s}_j, 0_{(q-r)\times(q-r)}\right) \right] w^j \Big) U_q \\
&= U_q^* \Big(\operatorname{diag}\Big(\sum_{j=0}^{\infty} \widetilde{s}_j w^j, 0_{(q-r)\times(q-r)} \Big) \Big) U_q \\
&= U_q^* \left[\operatorname{diag}\left(\widetilde{\Omega}(w), 0_{(q-r)\times(q-r)} \right) \right] U_q
\end{aligned}$$

for each $w \in \mathbb{D}$. The remainder of the theorem is a direct consequence of a well-known property of the Moore-Penrose inverse (see, e.g., [8, Lemma 1.1.3]). $\square$

A result similar to Theorem 7.7 was obtained in [5, Theorem 3.5]. The approach used in [5] was based on the Cayley transform and is thus entirely different from the approach used in the above proof of Theorem 4.4, which relies mainly on the structural properties of $q \times q$ Carathéodory sequences described in Theorem 4.6. It should be noted that the approach used in [5, Theorem 4.5] (to show that the Moore-Penrose inverse of a function $\Omega \in \mathcal{C}_q(\mathbb{D})$ is holomorphic) relies on [5, Theorem 3.5]. A closer look at this approach reveals that this can also be shown using the above Theorem 7.7.

Let $\Omega \in \mathcal{C}_q(\mathbb{D})$ and, for each $w \in \mathbb{D}$, let (6.4) be the Taylor series representation of Ω. Theorem 7.3 implies that $\Omega^+ \in \mathcal{C}_q(\mathbb{D})$ and that Ω^+ has Taylor series representation (7.9) for each $w \in \mathbb{D}$. We next explore the relationship between Ω and Ω^+. Since $(s_j)_{j=0}^{\infty}$ and $\left(s_j^{\sharp}\right)_{j=0}^{\infty}$ are, by Proposition 6.10, Carathéodory sequences, we can use the results of Section 3. Our next task will be to find subclasses $\mathcal{C}_q(\mathbb{D})$ which are invariant with respect to the Moore-Penrose inverse, i.e., if a function Ω belongs to our subclass, so should its Moore-Penrose inverse Ω^+.

Proposition 7.8. *Let $\Omega \in \mathcal{C}_q(\mathbb{D})$. Then:*

(a) *Let $\varkappa \in \mathbb{N}_0 \cup \{+\infty\}$. Then $\Omega \in \mathcal{C}_q^{(\varkappa)}(\mathbb{D})$ if and only if $\Omega^+ \in \mathcal{C}_q^{(\varkappa)}(\mathbb{D})$.*
(b) *$\Omega \in \mathcal{C}_q^{\mathrm{t}}(\mathbb{D})$ if and only if $\Omega^+ \in \mathcal{C}_q^{\mathrm{t}}(\mathbb{D})$.*

Proof. Suppose that Ω has the Taylor series representation (6.4) for each $w \in \mathbb{D}$. By Proposition 6.10, it follows that $(s_j)_{j=0}^{\infty}$ is a $q \times q$ C-sequence. Theorem 7.3 thus implies that $\Omega^+ \in \mathcal{C}_q(\mathbb{D})$ and also that Ω^+'s Taylor coefficient sequence is $(s_j^\sharp)_{j=0}^{\infty}$. Recalling Definition 6.15 and Definition 6.16, we see that (a) and (b) follow from Lemma 4.7 and Theorem 4.13, respectively. □

We will next see that for any $\Omega \in \mathcal{C}_q(\mathbb{D})$, the Moore-Penrose inverse Ω^+ will have the same centrality properties.

Theorem 7.9. *Let $\Omega \in \mathcal{C}_q(\mathbb{D})$.*

(a) *Ω is a central function if and only if Ω^+ is central.*
(b) *Let $k \in \mathbb{N}$.*
  (b1) *The function Ω is order k central if and only if Ω^+ is order k central.*
  (b2) *The function Ω is minimal order k central if and only if Ω^+ is minimal order k central.*

Proof. We first proceed as in the proof of Proposition 7.8. Recalling Definition 6.17, we then see that parts (a), (b1) and (b2) follow, respectively, from parts (a), (b1) and (b2) of Theorem 4.10. □

Proposition 7.10. *Let $n \in \mathbb{N}_0$ and $(s_j)_{j=0}^{n}$ be a $q \times q$ Carathéodory sequence. Furthermore, let $\Omega_{(s_j)_{j=0}^n}$ be the central $q \times q$ Carathéodory function for $(s_j)_{j=0}^{n}$. Then, $(s_j^\sharp)_{j=0}^{n}$ is a $q \times q$ Carathéodory sequence and, if $\Omega_{(s_j^\sharp)_{j=0}^n}$ is the central $q \times q$ Carathéodory function for $(s_j^\sharp)_{j=0}^{n}$, then*

$$\Omega_{(s_j^\sharp)_{j=0}^n} = \left[\Omega_{(s_j)_{j=0}^n}\right]^+. \tag{7.14}$$

Proof. By Theorem 4.4, it follows that $(s_j^\sharp)_{j=0}^{n}$ is a $q \times q$ C-sequence. Suppose that $(t_j)_{j=0}^{\infty}$ and $(t_{j,\sharp})_{j=0}^{\infty}$ are the central $q \times q$ C-sequences associated with $(s_j)_{j=0}^{n}$ and $(s_j^\sharp)_{j=0}^{n}$, respectively. Because of Definition 6.19, we see that $(t_j)_{j=0}^{\infty}$ and $(t_{j,\sharp})_{j=0}^{\infty}$ are the Taylor coefficient sequences for $\Omega_{(s_j)_{j=0}^n}$ and $\Omega_{(s_j^\sharp)_{j=0}^n}$, respectively. By Theorem 4.11, it follows that $(t_j^\sharp)_{j=0}^{\infty} = (t_{j,\sharp})_{j=0}^{\infty}$, where $(t_j^\sharp)_{j=0}^{\infty}$ is the reciprocal sequence to $(t_j)_{j=0}^{\infty}$. Therefore, (7.14) follows by Theorem 7.3. □

Proposition 7.11. *Let $\Omega \in \mathcal{C}_q(\mathbb{D})$.*

(a) *Ω is a canonical function if and only if Ω^+ is canonical.*

(b) *Let $k \in \mathbb{N}$.*

  (b1) *The function Ω is minimal order k canonical if and only if Ω^+ is minimal order k canonical.*

  (b2) *The function Ω is order k canonical if and only if Ω^+ is order k canonical.*

Proof. We first proceed as in the proof of Proposition 7.8. Recalling Definition 6.22, we then see that parts (a), (b1) and (b2) follow, respectively, from parts (a), (b1) and (b2) of Theorem 4.8. □

8. An approach to constructing the reciprocal of a non-negative Hermitian $q \times q$ measure

Using the results of previous sections, we now consider applications in non-negative Hermitian $q \times q$ measures on the unit circle $\mathbb{T}$. In order to establish a connection to previous sections, we review the relationships that exist between non-negative Hermitian $q \times q$ measures on $\mathbb{T}$, Toeplitz non-negative definite sequences in $\mathbb{C}^{q \times q}$ and $q \times q$ Carathéodory functions.

The σ-algebra of all Borel subsets of the unit circle $\mathbb{T}$ will be denoted by $\mathfrak{B}_\mathbb{T}$ and the set of all non-negative Hermitian $q \times q$ measures on $(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ by $\mathcal{M}^q_\geq(\mathbb{T}, \mathfrak{B}_\mathbb{T})$. If $F \in \mathcal{M}^q_\geq(\mathbb{T}, \mathfrak{B}_\mathbb{T})$, then, for each $j \in \mathbb{Z}$, the matrix

$$C_j^{(F)} := \int_\mathbb{T} z^{-j} F(dz)$$

is called the jth Fourier coefficient of F. The Fourier coefficients allow us to construct a bijection between the set $\mathcal{M}^q_\geq(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ and the set $\mathcal{T}_{q,\infty}$ of all infinite Toeplitz non-negative definite sequences in $\mathbb{C}^{q \times q}$. This is expressed in the following well-known matricial version of a classical result by Herglotz (see, for instance, [8, Theorem 2.2.1]).

Proposition 8.1. *A sequence $(C_j)_{j=0}^\infty$ in $\mathbb{C}^{q \times q}$ is Toeplitz non-negative definite if and only if there exists an $F \in \mathcal{M}^q_\geq(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ such that*

$$\left(C_j^{(F)} \right)_{j=0}^\infty = (C_j)_{j=0}^\infty . \tag{8.1}$$

If such an $F \in \mathcal{M}^q_\geq(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ exists, then it is unique.

If $(C_j)_{j=0}^\infty$ is a T-n.n.d. sequence in $\mathbb{C}^{q \times q}$, then, by Proposition 8.1, the unique $F \in \mathcal{M}^q_\geq(\mathbb{T}, \mathfrak{B}_\mathbb{T})$ such that (8.1) is met is called the spectral measure for $(C_j)_{j=0}^\infty$. Proposition 8.1 suggests paying particular attention to the subclass

$$\left\{ F \in \mathcal{M}^q_\geq(\mathbb{T}, \mathfrak{B}_\mathbb{T}) : \left(C_j^{(F)} \right)_{j=0}^\infty \in \widetilde{\mathcal{T}}_{q,\infty} \right\} . \tag{8.2}$$

This subclass is particularly interesting for its role in methods of constructing orthogonal matrix polynomials (see Delsarte/Genin/Kamp [6], [8, Section 3.6], [11]). Let $F \in \mathcal{M}^q_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Because of Proposition 8.1, Definition 2.12 can be carried over to non-negative Hermitian measures.

Definition 8.2. Let $F \in \mathcal{M}^q_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ with Fourier coefficient sequence $\left(C_j^{(F)}\right)_{j=0}^{\infty}$.

(a) Let $k \in \mathbb{N}$. We say that the measure F is central of order k (or order k central) if $\left(C_j^{(F)}\right)_{j=0}^{\infty}$ is a central sequence of order k.
(b) Let $k \in \mathbb{N}$. We say that the measure F is central of minimal order k (or minimal order k central) if the sequence $\left(C_j^{(F)}\right)_{j=0}^{\infty}$ is central of minimal order k.
(c) We say that F is a central measure if there exists a $k \in \mathbb{N}$ such that F is central of minimal order k.

Remark 8.3. Suppose that $k \in \mathbb{N}$ and that $F \in \mathcal{M}^q_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ is order k central. For each $\ell \in \mathbb{Z}_{k+1,\infty}$, Remark 2.13 shows that F is also order ℓ central.

A discussion of central measures belonging to the set in (8.2) can be found in [8, Section 3.6]. There it was shown that such measures are, in particular, absolutely continuous with respect to Lebesgue–Borel measure on the unit circle. The density functions for these measures were, furthermore, determined and expressed in terms of special matrix polynomials which play an important role in the theory of orthogonal matrix polynomials on the unit circle.

A standard result for stationary sequences in Hilbert modules says that every $F \in \mathcal{M}^q_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ can be interpreted as a non-stochastic spectral measure of a stationary sequence in a Hilbert module. In [12, Theorem 9], it was shown that F is order $k \in \mathbb{N}$ central if and only if the stationary sequence associated with F is autoregressive of order k. Central sequences and Proposition 8.1 next lead us to consider central measures.

Remark 8.4. Let $n \in \mathbb{N}_0$. Suppose $(C_j)_{j=0}^n$ is a T-n.n.d. sequence in $\mathbb{C}^{q\times q}$ and that $(\widetilde{C}_j)_{j=0}^{\infty}$ is the central sequence corresponding to $(C_j)_{j=0}^n$. Part (a) of Lemma 2.19 then implies that $(\widetilde{C}_j)_{j=0}^{\infty} \in \mathcal{T}_{q,\infty}$. The spectral measure $F_{(C_j)_{j=0}^n} \in \mathcal{M}^q_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ for $(\widetilde{C}_j)_{j=0}^{\infty}$ is then called the central measure for $(C_j)_{j=0}^n$.

Remark 8.5. Let $k \in \mathbb{N}$ and $F \in \mathcal{M}^q_{\geq}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ be order k central. It then, clearly, follows that $\left(C_j^{(F)}\right)_{j=0}^{k-1} \in \mathcal{T}_{q,k-1}$ and $F = F_{\left(C_j^{(F)}\right)_{j=0}^{k-1}}$.

Proposition 8.6. *Let $n \in \mathbb{N}_0$ and $(C_j)_{j=0}^n \in \mathcal{T}_{q,n}$. Then $F_{(C_j)_{j=0}^n}$ belongs to the set in* (8.2) *if and only if $(C_j)_{j=0}^n \in \widetilde{\mathcal{T}}_{q,n}$.*

Proof. The theorem follows from Remark 8.4 and Lemma 2.19. □

Definition 8.7. Let $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ with Fourier coefficient sequence $(C_j^{(F)})_{j=0}^{\infty}$.

(a) Let $k \in \mathbb{N}$. We say that F is canonical of order k (or order k canonical) if $(C_j^{(F)})_{j=0}^{\infty}$ is order k canonical.

(b) Let $k \in \mathbb{N}$. We say that F is canonical of minimal order k (or minimal order k canonical) if $(C_j^{(F)})_{j=0}^{\infty}$ is minimal order k canonical.

(c) We say that F is canonical if there is a $k \in \mathbb{N}$ such that F is order k canonical.

Remark 8.8. Let $k \in \mathbb{N}$ and $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ be order k canonical. Corollary 2.27 then implies:

(a) For each $\ell \in \mathbb{Z}_{k+1,\infty}$, the measure F is order ℓ canonical.

(b) For each $\ell \in \mathbb{Z}_{k+1,\infty}$, the measure F is order ℓ central.

For a given $z \in \mathbb{T}$, we will denote the Dirac measure on $(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ with unit mass in z by ε_z. Combining Theorem 2.31 and Proposition 8.1 leads us to the following characterization of canonical measures in $\mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$.

Theorem 8.9. *Let $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. All of the following conditions are equivalent:*

(i) *F is a canonical measure.*

(ii) *There exists an $r \in \mathbb{N}$, a sequence $(A_s)_{s=1}^{r}$ in $\mathbb{C}_{\geq}^{q\times q}$ and a sequence $(z_s)_{s=1}^{r}$ of pairwise different points in $\mathbb{T}$ such that*

$$F = \sum\nolimits_{s=1}^{r} \varepsilon_{z_s} A_s.$$

For a detailed discussion of canonical measures in $\mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$, we refer the reader to [19], [20] and [21]. It is shown, in these articles, that, given a finite Toeplitz positive definite sequence $(C_j)_{j=0}^{n}$ in $\mathbb{C}^{q\times q}$ and a $u \in \mathbb{T}$, there exists a unique solution F_u of the trigonometric matricial moment problem corresponding to $(C_j)_{j=0}^{n}$ which has maximal mass with respect to the singleton $\{u\}$. More precisely, this solution is such that, for any F from the solution set, the inequality $F(\{u\}) \leq F_u(\{u\})$ holds. This F_u is canonical and can be constructed explicity from the sequence $(C_j)_{j=0}^{n}$.

The class $\mathcal{C}_q(\mathbb{D})$ will often feature prominently. We therefore provide a quick survey of results from [8, Section 2.2] on the relationship between $\mathcal{C}_q(\mathbb{D})$ and $\mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$.

Proposition 8.10. *Let $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. The function $\Omega_F : \mathbb{D} \longrightarrow \mathbb{C}^{q\times q}$ given by*

$$\Omega_F(w) := \int_{\mathbb{T}} \frac{z+w}{z-w} F(dz)$$

belongs to $\mathcal{C}_q(\mathbb{D})$ and, for any $w \in \mathbb{D}$, furthermore

$$\Omega_F(w) = C_0^{(F)} + 2\sum\nolimits_{j=1}^{\infty} C_j^{(F)} w^j. \tag{8.3}$$

If $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and Ω_F is defined as in Proposition 8.10, then Ω_F is called the Riesz-Herglotz transform of F. The close relationship that exists between

$\mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and $\mathcal{C}_q(\mathbb{D})$ is evidenced by the following matricial generalization of a classical theorem by F. Riesz and G. Herglotz.

Theorem 8.11.

(a) *If* $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ *and* $H \in \mathbb{C}_{\mathrm{H}}^{q\times q}$, *then* $\Omega := \Omega_F + iH$ *belongs to* $\mathcal{C}_q(\mathbb{D})$ *and* $\operatorname{Im}[\Omega(0)] = H$.

(b) *If* $\Omega \in \mathcal{C}_q(\mathbb{D})$, *then there exists a unique* $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ *such that*

$$\Omega = \Omega_F + i\operatorname{Im}[\Omega(0)]. \tag{8.4}$$

If $\Omega \in \mathcal{C}_q(\mathbb{D})$, then the unique $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ such that condition (8.4) is met is called the Riesz-Herglotz measure of Ω.

Suppose $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. Let $\left((C_j^{(F)})^{\square}\right)_{j=0}^{\infty}$ be the Toeplitz non-negative definite sequence in $\mathbb{C}^{q\times q}$ generated from $\left(C_j^{(F)}\right)_{j=0}^{\infty}$ by reciprocation (see Definition 5.2 and Proposition 8.1). We are next most interested in describing the properties of the spectral measure $F^{\square}$ belonging to the sequence $\left((C_j^{(F)})^{\square}\right)_{j=0}^{\infty}$. It should, at this juncture, be mentioned that [8, Section 3.6] includes a discussion of this subject matter for the case in which F belongs to the set in (8.2). We now come to the main result of this section.

Theorem 8.12. *Let* $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ *and* Ω_F *be the Riesz-Herglotz transform of* F. *Suppose that* $\left((C_j^{(F)})^{\square}\right)_{j=0}^{\infty}$ *is the T-n.n.d. sequence in* $\mathbb{C}^{q\times q}$ *generated from* $(C_j^{(F)})_{j=0}^{\infty}$ *by reciprocation. Let* $F^{\square}$ *be the spectral measure for* $\left((C_j^{(F)})^{\square}\right)_{j=0}^{\infty}$. *Then* $F^{\square} \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ *and* $\Omega_{F^{\square}} = \Omega_F^{+}$.

Proof. Let $w \in \mathbb{D}$. By Proposition 8.10, we have (8.3) and

$$\Omega_{F^{\square}}(w) = C_0^{(F^{\square})} + 2\sum\nolimits_{j=1}^{\infty} C_j^{(F^{\square})} w^j. \tag{8.5}$$

Since $F^{\square}$ is the spectral measure for $\left((C_j^{(F)})^{\square}\right)_{j=0}^{\infty}$, we have $C_j^{(F^{\square})} = \left(C_j^{(F)}\right)^{\square}$ for each $j \in \mathbb{N}_0$. From (8.5) it therefore follows that

$$\Omega_{F^{\square}}(w) = \left(C_0^{(F)}\right)^{\square} + 2\sum\nolimits_{j=1}^{\infty}\left[\left(C_j^{(F)}\right)^{\square} w^j\right]. \tag{8.6}$$

We now define $(s_j)_{j=0}^{\infty}$ as $s_0 := C_0^{(F)}$ and, for each $j \in \mathbb{N}$, as $s_j := 2C_j^{(F)}$. From (8.3) we then see that

$$\Omega_F(w) = \sum\nolimits_{j=0}^{\infty} s_j w^j. \tag{8.7}$$

Part (a) of Theorem 8.11 yields $\Omega_F \in \mathcal{C}_q(\mathbb{D})$. Because of (8.7) and Theorem 7.3, we have

$$\Omega_F^{+}(w) = \sum\nolimits_{j=0}^{\infty} s_j^{\sharp} w^j. \tag{8.8}$$

By definition of $\left(\left(C_j^{(F)} \right)^{\square} \right)_{j=0}^{\infty}$, we have

$$s_0^{\sharp} := \left(C_0^{(F)} \right)^{\square} \quad \text{and} \quad s_j^{\sharp} := 2 \left(C_j^{(F)} \right)^{\square}$$

for each $j \in \mathbb{N}$ (see Proposition 5.1). Because of (8.6) and (8.8), this implies $\Omega_{F^{\square}} = \Omega_F^{+}$, which completes the proof. $\square$

Theorem 8.12 shows us that Ω_F leads to a generalization of the classical concept of a reciprocal measure $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ with $\det [F(\mathbb{T})] \neq 0$ (see, for instance, [8, Definition 3.6.10]). Thus, we arrive at the following definition.

Definition 8.13. Let $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$. The measure $F^{\square} \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ defined in Theorem 8.12 is called the reciprocal measure to F.

Theorem 8.14. *Suppose that $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and let $F^{\square}$ be the reciprocal measure to F. Then:*

(a) *The measure F is central if and only if $F^{\square}$ is central.*
(b) *Let $k \in \mathbb{N}$.*
  (b1) *F is order k central if and only if $F^{\square}$ is order k central.*
  (b2) *F is minimal order k central if and only if $F^{\square}$ is minimal order k central.*

Proof. Recalling Definition 8.13 and Definition 8.2, we see that all parts of the theorem follow from Theorem 5.7. $\square$

Proposition 8.15. *Let $n \in \mathbb{N}_0$. Furthermore, let $(C_j)_{j=0}^{n}$ be a T-n.n.d. sequence in $\mathbb{C}^{q\times q}$ and $\left(C_j^{\square}\right)_{j=0}^{n}$ be the T-n.n.d. sequence in $\mathbb{C}^{q\times q}$ generated from $(C_j)_{j=0}^{n}$ by reciprocation. Let $F_{(C_j)_{j=0}^{n}}$ and $F_{\left(C_j^{\square}\right)_{j=0}^{n}}$ be the central measures in $\mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ for $(C_j)_{j=0}^{n}$ and $\left(C_j^{\square}\right)_{j=0}^{n}$, respectively. If $\left[F_{(C_j)_{j=0}^{n}}\right]^{\square}$ is the reciprocal measure to $F_{(C_j)_{j=0}^{n}}$, then*

$$\left[F_{(C_j)_{j=0}^{n}}\right]^{\square} = F_{\left(C_j^{\square}\right)_{j=0}^{n}}.$$

Proof. From Remark 8.4, we see that applying Theorem 5.8 and Theorem 8.12 completes the proof. $\square$

Proposition 8.16. *Suppose that $F \in \mathcal{M}_{\geq}^{q}(\mathbb{T}, \mathfrak{B}_{\mathbb{T}})$ and that $F^{\square}$ is the reciprocal measure to F. Then:*

(a) *The measure F is canonical if and only if $F^{\square}$ is canonical.*
(b) *Let $k \in \mathbb{N}$.*
  (b1) *F is order k canonical if and only if $F^{\square}$ is order k canonical.*
  (b2) *F is minimal order k canonical if and only if $F^{\square}$ is minimal order k canonical.*

Proof. Definition 8.7, Proposition 8.1 and Theorem 5.6 yield the proof. $\square$

9. Matricial R-functions in the open upper half-plane

In this section, we turn our attention to another class of matrix functions. More specifically, we are interested in a special subclass of $q \times q$ matrix functions that are holomorphic in $\Pi_+ := \{ z \in \mathbb{C} \ : \ \operatorname{Im} z \in (0, +\infty) \}$. This class of matrix functions is particularly interesting in the context of the matricial version of the Hamburger Moment Problem and is closely related to $\mathcal{C}_q(\mathbb{D})$, as we will later see.

A function $G \ : \ \Pi_+ \longrightarrow \mathbb{C}^{q \times q}$ is called a $q \times q$ R-function, if G is holomorphic in Π_+ and if the imaginary part $\operatorname{Im} G(z)$ of $G(z)$ is non-negative Hermitian, i.e., if $G(z) \in \mathcal{I}_{q,\geq}$ for all $z \in \Pi_+$. The set of all $q \times q$ R-functions will be denoted by $\mathcal{R}_q(\Pi_+)$. The name *R-function* is here adopted from M.G. Krein. In the literature, it is not uncommon to see functions of this type referred to instead as *Nevanlinna*, *Pick* or *Herglotz functions*. We now proceed with some preliminary observations on the ranges and null spaces of functions in $\mathcal{R}_q(\Pi_+)$.

Lemma 9.1. *Let $G \in \mathcal{R}_q(\Pi_+)$. For each $z \in \Pi_+$,*

$$\mathcal{R}(G(z)) = \mathcal{R}\left([G(z)]^*\right), \qquad \mathcal{R}(G(z)) = \mathcal{R}\left([G(z)]^+\right),$$
$$\mathcal{N}(G(z)) = \mathcal{N}\left([G(z)]^*\right) \qquad \text{and} \qquad \mathcal{N}(G(z)) = \mathcal{N}\left([G(z)]^+\right).$$

Proof. By assumption, $G(z) \in \mathcal{I}_{q,\geq}$. For each $z \in \Pi_+$, Lemma A.10 thus implies $G(z) \in \mathbb{C}^{q \times q}_{\mathrm{EP}}$. Applying Proposition A.5 thus completes the proof. □

We now take a closer look at how the classes $\mathcal{R}_q(\Pi_+)$ and $\mathcal{C}_q(\mathbb{D})$ are related.

Remark 9.2. The function $\gamma : \mathbb{D} \longrightarrow \mathbb{C}$, defined as

$$\gamma(w) := i\frac{1-w}{1+w}$$

is a bijection between $\mathbb{D}$ and Π_+. The inverse function $\delta : \Pi_+ \longrightarrow \mathbb{C}$ to γ is given by

$$\delta(v) = \frac{1+iv}{1-iv}.$$

Lemma 9.3. *Let the mappings γ and δ be defined as in Remark* 9.2.

(a) *If Ω belongs to $\mathcal{C}_q(\mathbb{D})$ and $G := i(\Omega \circ \delta)$, then G belongs to $\mathcal{R}_q(\Pi_+)$ and $\Omega = (-i)(G \circ \gamma)$.*
(b) *If $G \in \mathcal{R}_q(\Pi_+)$ and $\Omega := (-i)(G \circ \gamma)$, then Ω belongs to $\mathcal{C}_q(\mathbb{D})$ and $G = i(\Omega \circ \delta)$.*

Proof. (a) Recalling Remark 9.2, we see that G is holomorphic in Π_+. Let $v \in \Pi_+$. For any $A \in \mathbb{C}^{q \times q}$, we have $\operatorname{Im}(iA) = \operatorname{Re} A$. Thus, it follows that

$$\operatorname{Im}[G(v)] = \operatorname{Im}[i(\Omega \circ \delta)(v)] = \operatorname{Im}[i\Omega(\delta(v))] = \operatorname{Re}[\Omega(\delta(v))].$$

Thus, $G(v) \in \mathcal{I}_{q,\geq}$, since $\Omega(\delta(v)) \in \mathcal{R}_{q,\geq}$. From Remark 9.2 and the definition of G, we thus see that $\Omega = (-i)(G \circ \gamma)$, which completes the proof of (a).
(b) Using Remark 9.2 and the fact that $\operatorname{Re}(-iA) = \operatorname{Im} A$ for any $A \in \mathbb{C}^{q \times q}$, we obtain (b) in much the same way as we did (a) in the first part of the proof. □

Our next theorem is the $\mathcal{R}_q(\Pi_+)$ counterpart to Theorem 7.3.

Theorem 9.4. *Let $G \in \mathcal{R}_q(\Pi_+)$. Then:*

(a) $-G^+ \in \mathcal{R}_q(\Pi_+)$.
(b) $(GG^+)(v) = (GG^+)(i)$ *and* $(G^+G)(v) = (G^+G)(i)$, *for* $v \in \Pi_+$.
(c) $\mathcal{R}(G(v)) = \mathcal{R}(G(i))$ *and* $\mathcal{N}(G(v)) = \mathcal{N}(G(i))$, *for* $v \in \Pi_+$.
(d) $\mathcal{R}(G^+(v)) = \mathcal{R}(G(i))$ *and* $\mathcal{N}(G^+(v)) = \mathcal{N}(G(i))$, *for* $v \in \Pi_+$.

Proof. (a) Let $\Omega := (-i)(G \circ \gamma)$. By part (b) of Lemma 9.3, it thus follows that $\Omega \in \mathcal{C}_q(\mathbb{D})$ and that $G = i(\Omega \circ \delta)$ and, consequently, that

$$G^+ = (-i)(\Omega \circ \delta)^+ = (-i)\left(\Omega^+ \circ \delta\right). \tag{9.1}$$

By Theorem 7.3, it follows that $\Omega^+ \in \mathcal{C}_q(\mathbb{D})$. Therefore, by part (a) of Lemma 9.3, we have

$$i\left(\Omega^+ \circ \delta\right) \in \mathcal{R}_q(\Pi_+). \tag{9.2}$$

Finally, from (9.1) and (9.2) we obtain $-G^+ \in \mathcal{R}_q(\Pi_+)$, which completes the proof of (a).
(b)–(c) By assumption, both G and G^+ are holomorphic in the non-empty open and connected set Π_+. Parts (b) and (c) therefore follow by [22, Proposition 8.4].
(d) Because of (c), part (d) follows immediately by Lemma 9.1. □

For a comprehensive survey on the class $\mathcal{R}_q(\Pi_+)$, we refer the reader to the paper Gesztesy/Tsekanovskii [23].

Appendix A. Some facts from matrix theory

Remark A.1. Let $A \in \mathbb{C}^{q\times q}$. Since $\operatorname{Re}(A^*) = \operatorname{Re} A$, it follows that $A \in \mathcal{R}_{q,\geq}$ if and only if $A^* \in \mathcal{R}_{q,\geq}$. Similarly, since $\operatorname{Im}(A^*) = -\operatorname{Im} A$, it follows that $A \in \mathcal{I}_{q,\geq}$ if and only if $-A^* \in \mathcal{I}_{q,\geq}$.

If $A \in \mathbb{C}^{q\times q}$ is a non-singular matrix, then it is well known that

$$\operatorname{Re}\left(A^{-1}\right) = A^{-1}(\operatorname{Re} A)\left(A^{-1}\right)^*, \qquad \operatorname{Re}\left(A^{-1}\right) = \left(A^{-1}\right)^*(\operatorname{Re} A)A^{-1},$$

$$\operatorname{Im}\left(A^{-1}\right) = -A^{-1}(\operatorname{Im} A)\left(A^{-1}\right)^* \quad \text{and} \quad \operatorname{Im}\left(A^{-1}\right) = -\left(A^{-1}\right)^*(\operatorname{Im} A)A^{-1},$$

so that $A \in \mathcal{R}_{q,\geq}$ implies $A^{-1} \in \mathcal{R}_{q,\geq}$ and $A \in \mathcal{I}_{q,\geq}$ implies $-A^{-1} \in \mathcal{I}_{q,\geq}$. In order to generalize these results for arbitrary complex $q \times q$ matrices, we give a very brief overview of standard results for Moore-Penrose inverses. We will often use the properties of the Moore-Penrose inverse described in these lemmas. For a detailed discussion of the Moore-Penrose inverse and its properties, we refer the reader to [2] and [8, Section 1.1].

Lemma A.2. *Suppose $A \in \mathbb{C}^{p\times q}$.*

(a) *Let $B \in \mathbb{C}^{p\times r}$. The following are all equivalent:*
   (i) $\mathcal{R}(B) \subseteq \mathcal{R}(A)$.
   (ii) $AA^+B = B$.
   (iii) *There exists an $X \in \mathbb{C}^{q\times r}$ such that $AX = B$.*

(b) *Let* $C \in \mathbb{C}^{r \times q}$. *The following are all equivalent:*
  (iv) $\mathcal{N}(A) \subseteq \mathcal{N}(C)$.
  (v) $CA^+A = C$.
  (vi) *There exists a* $Y \in \mathbb{C}^{r \times p}$ *such that* $YA = C$.

Lemma A.3. *Suppose* $A \in \mathbb{C}_{\mathrm{H}}^{q \times q}$, *i.e., that* A *is a Hermitian matrix. Then the Moore-Penrose inverse* A^+ *of* A *is also Hermitian and the following equations hold true:*

$$\mathcal{R}(A) = \mathcal{R}\left(A^+\right), \qquad \mathcal{N}(A) = \mathcal{N}\left(A^+\right) \qquad \text{and} \qquad AA^+ = A^+A.$$

Lemma A.4. *Suppose* $A \in \mathbb{C}_{\geq}^{q \times q}$. *Then* $A^+ \in \mathbb{C}_{\geq}^{q \times q}$ *and:*

(a) $\mathcal{R}(A) = \mathcal{R}(\sqrt{A})$ *and* $\mathcal{N}(A) = \mathcal{N}(\sqrt{A})$.
(b) $AA^+ = A^+A = \sqrt{A}(\sqrt{A})^+ = (\sqrt{A})^+\sqrt{A}$.
(c) $\sqrt{A} = A(\sqrt{A})^+ = (\sqrt{A})^+A$ *and* $(\sqrt{A})^+ = A^+\sqrt{A} = \sqrt{A}A^+$.

We recall that a complex $q \times q$ matrix A is an EP matrix if $\mathcal{R}(A) = \mathcal{R}(A^*)$. We also recall that $\mathbb{C}_{\mathrm{EP}}^{q \times q}$ denotes the set of all EP matrices in $\mathbb{C}^{q \times q}$. Schwerdtfeger [28] first introduced the class $\mathbb{C}_{\mathrm{EP}}^{q \times q}$ (EP stands for "Equal Projectors"). This class of matrices is important in the theory of generalized inverses (see, e.g., the monographs Campbell/Meyer [3, Chapter 4, Section 3] and Ben-Israel/Greville [2, Chapter 4, Section 4], and the papers Pearl [27] and Meyer [26]). We use the following characterizations of the class $\mathbb{C}_{\mathrm{EP}}^{q \times q}$, which can be found in [4] and [29].

Proposition A.5. *If* $A \in \mathbb{C}^{q \times q}$, *then all of the following conditions are equivalent:*

(i) $A \in \mathbb{C}_{\mathrm{EP}}^{q \times q}$.
(ii) $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$.
(iii) $\mathcal{R}(A^*) \subseteq \mathcal{R}(A)$.
(iv) $\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$.
(v) $\mathcal{N}(A^*) \subseteq \mathcal{N}(A)$.
(vi) $\mathcal{N}(A^*) = \mathcal{N}(A)$.
(vii) $AA^+ = A^+A$.
(viii) $A^+ \in \mathbb{C}_{\mathrm{EP}}^{q \times q}$.
(ix) $\mathcal{R}(A) = \mathcal{R}(A^+)$.
(x) $\mathcal{N}(A) = \mathcal{N}(A^+)$.
(xi) $A^+A^2 = A$.
(xii) $A^2A^+ = A$.
(xiii) $A(A^+)^2 = A^+$.
(xiv) $(A^+)^2A = A^+$.

We also require a few additional ways of characterizing the class $\mathbb{C}_{\mathrm{EP}}^{q \times q}$. We will need matricial real and imaginary parts for these characterizations.

Proposition A.6. *Let* $A \in \mathbb{C}^{q \times q}$. *The following conditions are all equivalent:*

(i) $A \in \mathbb{C}_{\mathrm{EP}}^{q \times q}$.
(ii) $\alpha A^+ + \beta(A^+)^* = A^+(\beta A + \alpha A^*)(A^+)^*$ *for all* $\alpha, \beta \in \mathbb{C}$.
(iii) $\mathrm{Re}(A^+) = A^+(\mathrm{Re}\, A)(A^+)^*$ *and* $\mathrm{Im}(A^+) = -A^+(\mathrm{Im}\, A)(A^+)^*$.
(iv) $\alpha A^+ + \beta(A^+)^* = (A^+)^*(\beta A + \alpha A^*)A^+$ *for all* $\alpha, \beta \in \mathbb{C}$.
(v) $\mathrm{Re}(A^+) = (A^+)^*(\mathrm{Re}\, A)A^+$ *and* $\mathrm{Im}(A^+) = -(A^+)^*(\mathrm{Im}\, A)A^+$.
(vi) $\alpha A + \beta A^* = A\left(\beta A^+ + \alpha(A^+)^*\right)A^*$ *for all* $\alpha, \beta \in \mathbb{C}$.
(vii) $\mathrm{Re}\, A = A \cdot \mathrm{Re}(A^+) \cdot A^*$ *and* $\mathrm{Im}\, A = -A \cdot \mathrm{Im}(A^+) \cdot A^*$.
(viii) $\alpha A + \beta A^* = A^*\left(\beta A^+ + \alpha(A^+)^*\right)A$ *for all* $\alpha, \beta \in \mathbb{C}$.
(ix) $\mathrm{Re}\, A = A^* \cdot \mathrm{Re}(A^+) \cdot A$ *and* $\mathrm{Im}\, A = -A^* \cdot \mathrm{Im}(A^+) \cdot A$.

Proof. "(i)$\Longrightarrow$(ii)". It follows from (i), by Proposition A.5, that $AA^+ = A^+A$. It now follows that

$$A^+A\left(A^+\right)^* = \left(AA^+\right)^*\left(A^+\right)^* = \left(A^+AA^+\right)^* = \left(A^+\right)^*$$

and

$$A^+A^*\left(A^+\right)^* = A^+\left(A^+A\right)^* = A^+AA^+ = A^+.$$

For any α, $\beta \in \mathbb{C}$, we therefore have

$$\alpha A^+ + \beta\left(A^+\right)^* = \alpha A^+A^*\left(A^+\right)^* + \beta A^+A\left(A^+\right)^* = A^+\left(\beta A + \alpha A^*\right)\left(A^+\right)^*.$$

"(ii)$\Longrightarrow$(iii)". For $\alpha = \frac{1}{2} = \beta$, part (ii) yields

$$\operatorname{Re}\left(A^+\right) = \frac{1}{2}A^+ + \frac{1}{2}\left(A^+\right)^* = A^+ \cdot \frac{1}{2}\left(A + A^*\right)\left(A^+\right)^* = A^+\left(\operatorname{Re} A\right)\left(A^+\right)^*.$$

Similarly, for $\alpha = \frac{1}{2i}$ and $\beta = -\frac{1}{2i}$, part (ii) also yields the second equation in (iii). "(iii)$\Longrightarrow$(i)". Since $A = \operatorname{Re} A + i\operatorname{Im} A$, $\operatorname{Re}(A^*) = \operatorname{Re} A$ and $\operatorname{Im}(A^*) = -\operatorname{Im} A$, it follows that $A^* = \operatorname{Re} A - i\operatorname{Im} A$. Using $A = \operatorname{Re} A + i\operatorname{Im} A$ and (iii), we thus obtain

$$\begin{aligned} A^+ &= \operatorname{Re}\left(A^+\right) + i\operatorname{Im}\left(A^+\right) \\ &= A^+\left(\operatorname{Re} A\right)\left(A^+\right)^* - iA^+\left(\operatorname{Im} A\right)\left(A^+\right)^* \\ &= A^+\left(\operatorname{Re} A - i\operatorname{Im} A\right)\left(A^+\right)^* \\ &= A^+A^*\left(A^+\right)^* = A^+\left(A^+A\right)^* = \left(A^+\right)^2 A. \end{aligned}$$

Therefore, by Proposition A.5 we obtain (i). Conditions (i), (ii) and (iii) are therefore all equivalent. Similarly, the same holds for conditions (i), (iv) and (v). We next consider the following condition:

(x) $A^+ \in \mathbb{C}^{q\times q}_{\mathrm{EP}}$.

Using the identity $(A^+)^+ = A$, we see from the first part of the proof that (vi)–(x) are all equivalent. By Proposition A.5, it follows that (i) and (x) are equivalent to one another. Thus, (i)–(ix) are all equivalent. □

Corollary A.7. *Suppose* $A \in \mathbb{C}^{q\times q}_{\mathrm{EP}}$, *then:*

(a) $\mathcal{R}(\operatorname{Re} A) \subseteq \mathcal{R}(A)$.
(b) $AA^+ \cdot \operatorname{Re} A = \operatorname{Re} A$.
(c) $\mathcal{N}(A) \subseteq \mathcal{N}(\operatorname{Re} A)$.
(d) $\operatorname{Re} A \cdot A^+A = \operatorname{Re} A$.
(e) $A^*(A^+)^*\operatorname{Re} A = \operatorname{Re} A$.
(f) $\operatorname{Re} A \cdot (A^+)^* A^* = \operatorname{Re} A$.
(g) $\operatorname{rank}(\operatorname{Re} A) = \operatorname{rank}[\operatorname{Re}(A^+)]$.
(h) $A^* \cdot \operatorname{Re}(A^+) = \operatorname{Re} A \cdot A^+$.
(i) $\operatorname{Re}(A^+) \cdot A = (A^+)^* \cdot \operatorname{Re} A$.
(j) $A \cdot \operatorname{Re}(A^+) = \operatorname{Re} A \cdot (A^+)^*$.
(k) $\operatorname{Re}(A^+) \cdot A^* = A^+ \cdot \operatorname{Re} A$.
(l) $\mathcal{R}(\operatorname{Im} A) \subseteq \mathcal{R}(A)$.
(m) $AA^+ \cdot \operatorname{Im} A = \operatorname{Im} A$.
(n) $\mathcal{N}(A) \subseteq \mathcal{N}(\operatorname{Im} A)$.
(o) $\operatorname{Im} A \cdot A^+A = \operatorname{Im} A$.
(p) $A^*(A^+)^*\operatorname{Im} A = \operatorname{Im} A$.
(q) $\operatorname{Im} A \cdot (A^+)^* A^* = \operatorname{Im} A$.
(r) $\operatorname{rank}(\operatorname{Im} A) = \operatorname{rank}[\operatorname{Im}(A^+)]$.

Proof. By Proposition A.6, we have

$$\operatorname{Re} A = A \left[\operatorname{Re} \left(A^+ \right) \right] A^* \qquad \text{and} \qquad \operatorname{Re} A = A^* \left[\operatorname{Re} \left(A^+ \right) \right] A. \tag{A.1}$$

Because of (A.1), parts (a), (b) and (c) thus follow by Lemma A.2. Recalling that $(\operatorname{Re} A)^* = \operatorname{Re} A$ and using parts (d) and (b), we see that

$$A^* \left(A^+ \right)^* \operatorname{Re} A = A^* \left(A^+ \right)^* (\operatorname{Re} A)^* = \left(\operatorname{Re} A \cdot A^+ A \right)^* = (\operatorname{Re} A)^* = \operatorname{Re} A$$

and, similarly, that $\operatorname{Re} A \cdot (A^+)^* A^* = \operatorname{Re} A$. Thus, the proof of (e) and (f) is complete. Proposition A.6 gives us $\operatorname{Re}(A^+) = A^+ [\operatorname{Re} A] (A^+)^*$. From (A.1), we thus obtain (g).
(h) Using (A.1), the identity $(A^+)^+ = A$ and (e), we see that

$$A^* \left(\operatorname{Re} A^+ \right) = A^* \left(A^+ \right)^* \cdot \operatorname{Re} A \cdot A^+ = \operatorname{Re} A \cdot A^+.$$

Similarly, (A.1) and (d) imply (i), whereas (A.1) and (b) show that (j) is true, whereas (A.1) and (f) yield (k). The proofs of (l)–(r) are similar. □

One of our next goals is to show that the sets $\mathcal{R}_{q,\geq}$ and $\mathcal{I}_{q,\geq}$ are subsets of $\mathbb{C}_{\mathrm{EP}}^{q\times q}$. This will serve as part of our motivation for the next few results. Once we have proved that these inclusions are true, we will be able to apply all of our results for $\mathbb{C}_{\mathrm{EP}}^{q\times q}$ to the classes $\mathcal{R}_{q,\geq}$ and $\mathcal{I}_{q,\geq}$.

Lemma A.8. *Let $A \in \mathcal{R}_{q,\geq}$. Then:*

(a) $\mathcal{N}(A) \subseteq \mathcal{N}(\operatorname{Re} A)$.
(b) $\mathcal{R}(\operatorname{Re} A) \subseteq \mathcal{R}(A)$.
(c) $A \in \mathbb{C}_{\mathrm{EP}}^{q\times q}$.
(d) $AA^+ = A^+A$.

Proof. (a) Suppose $x \in \mathcal{N}(A)$. Then $Ax = 0_{q\times 1}$. Since $\operatorname{Re} A \in \mathbb{C}_{\geq}^{q\times q}$, this implies

$$\left(\sqrt{\operatorname{Re} A} x \right)^* \sqrt{\operatorname{Re} A} x = x^* (\operatorname{Re} A) x = \frac{1}{2} \left(x^* A x + (Ax)^* x \right) = 0.$$

By part (a) of Lemma A.4 we then get $x \in \mathcal{N}(\operatorname{Re} A)$.
(b) Since $A \in \mathcal{R}_{q,\geq}$, it follows by Remark A.1 that $A^* \in \mathcal{R}_{q,\geq}$. Thus, (a) implies $\mathcal{N}(A^*) \subseteq \mathcal{N}(\operatorname{Re}(A^*))$. Considering orthogonal complements now yields $\mathcal{R}\left([\operatorname{Re}(A^*)]^* \right) \subseteq \mathcal{R}(A)$. Because of $\operatorname{Re}(A^*) = \operatorname{Re} A$ and $\operatorname{Re}(A^*) \in \mathbb{C}_{\mathrm{H}}^{q\times q}$, we have $[\operatorname{Re}(A^*)]^* = \operatorname{Re}(A^*) = \operatorname{Re} A$. Thus, we finally obtain $\mathcal{R}(\operatorname{Re} A) \subseteq \mathcal{R}(A)$.
(c) Let $x \in \mathcal{N}(A)$. Then $Ax = 0_{q\times 1}$ and, by (a) also $(\operatorname{Re} A) x = 0_{q\times 1}$. Thus,

$$A^* x = Ax + A^* x = 2 (\operatorname{Re} A) x = 0_{q\times 1}.$$

Hence, $\mathcal{N}(A) \subseteq \mathcal{N}(A^*)$. Consequently, Proposition A.5 implies $A \in \mathbb{C}_{\mathrm{EP}}^{q\times q}$.
(d) Because of (c), part (d) follows from Proposition A.5. □

Proposition A.9. *Let $A \in \mathbb{C}^{q\times q}$. Then $A \in \mathcal{R}_{q,\geq}$ if and only if $A^+ \in \mathcal{R}_{q,\geq}$.*

Proof. Use part (c) of Lemma A.8 and Proposition A.6. □

The next two results are the $\mathcal{I}_{q,\geq}$ counterparts to Lemma A.8 and Proposition A.9 and their proofs are thus quite similar to those for the corresponding $\mathcal{R}_{q,\geq}$ results.

Lemma A.10. *If* $A \in \mathcal{I}_{q,\geq}$, *then*

(a) $\mathcal{N}(A) \subseteq \mathcal{N}(\operatorname{Im} A)$.
(b) $\mathcal{R}(\operatorname{Im} A) \subseteq \mathcal{R}(A)$.
(c) $A \in \mathbb{C}^{q\times q}_{\mathrm{EP}}$.
(d) $AA^+ = A^+A$.

Proposition A.11. *Let* $A \in \mathbb{C}^{q\times q}$. *Then* $A \in \mathcal{I}_{q,\geq}$ *if and only if* $-A^+ \in \mathcal{I}_{q,\geq}$.

Lemma A.12. (a) *Let* $A \in \mathcal{R}_{q,>}$, *then* $\det A \neq 0$ *and* $A^{-1} \in \mathcal{R}_{q,>}$.
(b) *Let* $A \in \mathcal{I}_{q,>}$, *then* $\det A \neq 0$ *and* $-A^{-1} \in \mathcal{I}_{q,>}$.

Proof. Since $A \in \mathcal{R}_{q,>}$, it follows that $\mathcal{N}(\operatorname{Re} A) = \{0_{q\times 1}\}$. By part (a) of Lemma A.8, we thus obtain $\mathcal{N}(A) = \{0_{q\times 1}\}$. Therefore, $\det A \neq 0$. Finally, part (c) of Lemma A.8 and Proposition A.6 yield $A^{-1} \in \mathcal{R}_{q,>}$. The proof of (b) is similar to the proof of (a), but using parts (a) and (c) of Lemma A.10. □

We next establish a few additional connections between $\mathbb{K}_{q\times q}$ and $\mathcal{R}_{q,\geq}$ as well as between $\mathbb{D}_{q\times q}$ and $\mathcal{R}_{q,>}$.

Lemma A.13. *Suppose* $A \in \mathbb{K}_{q\times q}$. *Then*

$$\operatorname{Re} A \in \mathbb{K}_{q\times q}, \qquad I_q + A \in \mathcal{R}_{q,\geq} \qquad \text{and} \qquad I_q + \operatorname{Re} A \in \mathcal{R}_{q,\geq}$$

Proof. We see that $\|\operatorname{Re} A\|_{\mathrm{S}} \leq \|A\|_{\mathrm{S}} \leq 1$. Hence, $\operatorname{Re} A \in \mathbb{K}_{q\times q}$. We have

$$\operatorname{Re}(I_q + A) = I_q + \operatorname{Re} A = [1 - \|\operatorname{Re} A\|_{\mathrm{S}}] \cdot I_q + [\operatorname{Re} A + \|\operatorname{Re} A\|_{\mathrm{S}} \cdot I_q]. \tag{A.2}$$

From $\|\operatorname{Re} A\|_{\mathrm{S}} \leq 1$, we obtain

$$[1 - \|\operatorname{Re} A\|_{\mathrm{S}}] \cdot I_q \in \mathbb{C}^{q\times q}_{\geq}. \tag{A.3}$$

We recall that $\operatorname{Re} A \in \mathbb{C}^{q\times q}_{\mathrm{H}}$. Thus, using the Bunjakowski-Cauchy-Schwarz Inequality, we get

$$\operatorname{Re} A + \|\operatorname{Re} A\|_{\mathrm{S}} \cdot I_q \in \mathbb{C}^{q\times q}_{\geq}. \tag{A.4}$$

From (A.2) - (A.4) it follows that $\operatorname{Re}(I_q + A) \in \mathbb{C}^{q\times q}_{\geq}$. Therefore, $I_q + A \in \mathcal{R}_{q,\geq}$. Since $\operatorname{Re} A \in \mathbb{K}_{q\times q}$, it follows that $I_q + \operatorname{Re} A \in \mathcal{R}_{q,\geq}$. □

Similar to Lemma A.13, we can show that the following lemma is true.

Lemma A.14. *Suppose* $A \in \mathbb{D}_{q\times q}$. *Then*

$$\operatorname{Re} A \in \mathbb{D}_{q\times q}, \qquad I_q + A \in \mathcal{R}_{q,>} \qquad \text{and} \qquad I_q + \operatorname{Re} A \in \mathcal{R}_{q,>}.$$

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