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Editorial Introduction

The present volume, entitled "*Interpolation, Schur functions and moment problems II*", contains a selection of essays on various topics in Schur Analysis and addresses the goals set in 2006 in Volume 165 "*Interpolation, Schur functions and moment problems*" of the OT Series.

The origins of Schur Analysis lie in I. Schur's remarkable 1917/1918 two-part paper [33] which includes his insightful approach to constructing an algorithm for solving the coefficient problem for functions holomorphic on the unit disk $\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| < 1\}$ that are also bounded by 1. The class of all such functions is now referred to as the *Schur class.* The algorithm has come to be known as the *Schur algorithm.* The English translation of Schur's paper [33] and research papers on the Schur algorithm were published in volume 18 of the OT Series (see [19]).

Schur's parametrization ([33]) of a Schur function's Taylor coefficients using a sequence of contractive parameters was the starting point for many applications of Schur analytic methods to various types of block matrices. Schur showed that lower triangular Toeplitz matrices resulting from the Taylor coefficient sequence of a Schur function are contractive. Schur's results represent a particularly noteworthy special case, which opened the gateway to more general cases and led to the development of Schur analytic methods for other block matrices. The theory developed for non-negative Hermitian matrices is a particularly interesting example of these later developments. More specifically, it was shown that non-negative Hermitian matrices can be fully described using their diagonal blocks and a triangular configuration of contractive matrices, often referred to as a *choice triangle.* Every non-negative Hermitian matrix corresponds to a triangular configuration of matrix balls. A contractive parameter taken from the choice triangle then indicates the position of the relevant block within the corresponding matrix ball. Further information on Schur Analysis of block matrices and related topics can be found, for instance, in Bakonyi/Woerdeman [9], Constantinescu [11] as well as Dubovoj/Fritzsche/Kirstein [12] and references therein.

R. Nevanlinna deserves particular mention for being among the first to recognize, adapt and further develop Schur's ideas. In his dissertation (see [27]), R. Nevanlinna arrived at a modification of the Schur algorithm with which he was able to describe all Schur functions satisfying certain interpolation conditions. Nevanlinna was unaware of the fact (likely due to the tumultuous times of World

War I in which he obtained his results) that Pick [30] had already determined necessary and sufficient conditions for the aforementioned interpolation problem in terms of whether or not a particular matrix (later given the name *Pick matrix*) constructed from the given data was non-negative Hermitian. Interpolation problems remained a favourite subject of R. Nevanlinna. His most significant achievements in this field are found in his 1929 article [29] in which he discusses many new ideas on characterizing the connection between limit point and limit circle cases. It should be mentioned that reprints of the papers Schur [33], Nevanlinna [29] and Pick [30] as well as Herglotz [23] and Weyl [36] are all to be found in the collection [17].

R. Nevanlinna is also responsible for introducing a very interesting approach to power moment problems on the real axis (see [28]). Inspired by his work on interpolation problems in [27], R. Nevanlinna found a way to restate the Hamburger moment problem on the real axis as an equivalent problem. The equivalent problem involves finding functions holomorphic in the open upper half-plane $\Pi^+ := \{z \in \mathbb{C} \ : \ \operatorname{Im} z \in (0, +\infty)\}$ with a given asymptotic expansion and having non-negative imaginary parts. This reformulation of the problem makes it possible to apply Schur's method for power moment problems. The, up to this point, groundbreaking work of Stieltjes [34] and Hamburger [20] was based on continued fractions methods. A survey of the theory of classical power moment problems is found in Akhiezer [5].

Bolstered by the advances in operator theory and influenced by the needs of electrical engineering, signal transmission and processing, as well as prediction theory for stationary stochastic processes, Schur's method experienced a renaissance near the end of the 1960s. The publications of Adamjan–Arov–Krein [1–4] laid the groundwork for this resurgence of the Schur method and led to intensive research on matrix and operator versions of classical interpolation and moment problems. The flexibility and variety of Schur analytic methods led to the publishing of an exceptionally large number of books dedicated to this topic: Alpay [6], Alpay/Dijksma/Rovnyak/de Snoo [7], Bakonyi/Constantinescu [8], Bakonyi/-Woerdeman [9], Ball/Gohberg/Rodman [10], Constantinescu [11], Dubovoj/Fritzsche/Kirstein [12], Dym [13], Foias/Frazho [15], Foias/Frazho/Gohberg/Kaashoek [16], Helton [24], Katsnelson [25] (see also [26], which is the English translation of the first part of [25]), Rosenblum/Rovnyak [31], Sakhnovich [32]. These books share the common feature that they all deal with matrix and operator generalizations of Schur's method.

Schur Analysis' rapid growth and evolution is documented by the numerous articles, particularly the many recent articles, published on topics in Schur Analysis. It is to be expected that these developments will continue for quite a while to come. This volume addresses a number of current trends and developments in Schur Analysis. By and large, the focus is on matricial generalizations of Schur's and Nevanlinna's earlier described classical results.

The first central theme of this volume involves matrix versions of power moment problems on partial intervals of the real axis as well as how these problems might be approached. The solvability of such moment problems can be characterized by regarding a block Hankel matrix constructed from the initial data and considering its Hermitian non-negativeness. This leads to a need to know more about the inner structure of block Hankel matrices. Before going into further detail on this, it is worth recalling a similar topic, very much related to non-negative Hermitian block Toeplitz matrices. As part of developing approaches to matrix versions of interpolation problems for the Carathéodory and Schur function classes, much was done in the 1980s to describe the inner structure of non-negative Hermitian and contractive block matrices (see, for instance, Dubovoj/Fritzsche/Kirstein [12]). The focus was on block Toeplitz matrices and it was determined that a non-negative Hermitian block Toeplitz matrix could be uniquely described with a sequence of contractive parameters. This parameter sequence, often referred to as a *Schur parameter sequence*, contains considerable information on the inner structure of a given block Toeplitz matrix. The next task was to find a similarly useful inner parametrization for non-negative Hermitian block Hankel matrices. The canonical Hankel parametrization, which was introduced in [14] and [18], proved to be a very effective tool. A number of questions concerning the inner structure of special non-negative Hermitian block Hankel matrices could thus be resolved. Closely related to the search for these answers is the need for a corresponding Schur-type algorithm for finite and infinite sequences of complex matrices.

The second main theme of this volume relates to the theory of matricial Carathéodory functions and deals with various aspects of this theory. The close connection to the theory of orthogonal rational matrix functions on the unit circle is among the most important tools used in discussing these topics.

This volume contains six papers, and we now review their contents.

Reciprocal sequences (of finite and infinite sequences of matrices) and their applications: The paper "*On the concept of invertibility for sequences of complex $p \times q$ matrices and its applications to holomorphic $p \times q$ matrix-valued functions*" by **B. Fritzsche**, **B. Kirstein**, **C. Mädler** and **T. Schwarz** focuses on a type of invertibility for finite and infinite sequences of complex $p \times q$ matrices. The problem of characterizing all invertible sequences of complex $p \times q$ matrices naturally leads to what will be referred to as *first term dominant* sequences of complex $p \times q$ matrices. Given an invertible sequence of complex $p \times q$ matrices, the question of how to determine its inverse sequence (of complex $q \times p$ matrices) is answered using a recursively defined sequence of $q \times p$ matrices, called its *reciprocal sequence*. This definition opens up many possibilities for analyzing and solving a great number of matricial complex function theory problems. This includes, for example, the problem of characterizing the holomorphicity of a holomorphic $p \times q$ matrix-function F's Moore-Penrose inverse F^+.

The paper "*On reciprocal sequences of matricial Carathéodory sequences and associated matrix functions*" by **B. Fritzsche**, **B. Kirstein**, **A. Lasarow** and **A. Rahn** discusses, among other topics, applications of reciprocal sequences to a particular subclass from matricial complex function theory. The focus is, in particular, on the case $p = q$ and the class $\mathcal{C}_q(\mathbb{D})$ of all $q \times q$ Carathéodory functions on the open unit disk $\mathbb{D} := \{ w \in \mathbb{C} : |w| < 1 \}$. These are all $q \times q$ matrix-functions Ω that are holomorphic in $\mathbb{D}$ and for which $\frac{1}{2}[\Omega(w) + \Omega^*(w)]$ is non-negative Hermitian for all $w \in \mathbb{D}$. As an important result, it is shown that for any function $\Omega \in \mathcal{C}_q(\mathbb{D})$, the Moore-Penrose inverse Ω^+ of this function again belongs to $\mathcal{C}_q(\mathbb{D})$. It is also determined that the Taylor coefficient sequence of Ω^+ is the reciprocal sequence to Ω's Taylor coefficient sequence.

In the paper "*On a Schur-type algorithm for sequences of complex $p \times q$ matrices and its interrelations with the canonical Hankel parametrization*" by **B. Fritzsche**, **B. Kirstein**, **C. Mädler** and **T. Schwarz**, a further application of reciprocal sequences is discussed. The central focus is on special classes of sequences of complex $q \times q$ matrices that are closely related to a number of different matricial versions of the Hamburger Moment Problem on the real axis $\mathbb{R}$. Given a sequence $(s_j)_{j=0}^{\infty}$ of complex $p \times q$ matrices, the reciprocal sequence of complex $q \times p$ matrices $(s_j^{\sharp})_{j=0}^{\infty}$ is used to develop a Schur-type algorithm for sequences of $p \times q$ matrices. This makes it possible for an algebraic approach to matricial versions of the Hamburger Moment Problem to be modelled on the basis of this algorithm.

Matricial power moment problems: With "*The multiplicative structure of the resolvent matrix for the truncated Hausdorff matrix moment problem*", **Abdon E. Choque Rivero** offers a continued look into topics from his earlier collaborations with Yu.M. Dyukarev, B. Fritzsche and B. Kirstein. These collaborations yielded a constructive approach to obtaining a special polynomial resolvent matrix. Choque Rivero derives a factorization of this polynomial as a product of linear factors.

The paper "*On a special parametrization of matricial α-Stieltjes one-sided non-negative definite sequences*", by **B. Fritzsche**, **B. Kirstein** and **C. Mädler**, deals with topics concerning the matrix version of the Stieltjes moment problem for a half-infinite closed interval. The authors obtained characterizations of the solvability for the relevant moments in earlier collaborations with Yu.M. Dyukarev. These solvability criteria say that the initial data sequence must be one-sided α-Stieltjes non-negative definite. The class of all such matrix sequences serves as a starting point for this paper. The objective here is to derive an inner parametrization for sequences of this class. This parametrization should, in a particular manner, reflect the one-sided α-Stieltjes structure of the aforementioned sequences. The authors used a similar approach in their earlier work regarding Hankel non-negative definite sequences. There, they obtained that the structure of these sequences was clearly recognizable in their canonical Hankel parametrizations. The authors, furthermore, constructed a Schur-type algorithm which yielded the Hankel parametrization as part of its result. The approach used here for the α-Stieltjes case is quite similar. The first step involves a detailed look at the constructed α-Stieltjes parametrization. In particular, a relationship between this parametrization and the Hankel parametrization is established.

Orthogonal rational matrix-valued functions: In a series of papers, over the past decade, **B. Fritzsche**, **B. Kirstein** and **A. Lasarow** worked out essential features of a Szegő theory for orthogonal rational matrix-valued functions on the unit circle and applied this theory to corresponding interpolation and moment problems. The paper "*On maximal weight solutions of a moment problem for rational matrix-valued functions*" continues these studies and focuses on special canonical solutions of the matricial moment problem under consideration. These canonical solutions turn out to be molecular non-negative Hermitian measures on the unit circle, which means that they are concentrated on a finite set of points of the unit circle. It is shown that these canonical solutions have a particular extremal property which is connected to a maximal mass condition in a prescribed point of the unit circle.

References

[1] V.M. Adamjan, D.Z. Arov, and M.G. Kreĭn, *Infinite Hankel matrices and generalized problems of Carathéodory-Fejér and F. Riesz*, Funkcional. Anal. i Priložen. **2** (1968), no. 1, 1–19 (Russian). MR0234274 (38 #2591)

[2] ———, *Infinite Hankel matrices and generalized Carathéodory-Fejér and I. Schur problems*, Funkcional. Anal. i Priložen. **2** (1968), no. 4, 1–17 (Russian). MR0636333 (58 #30446)

[3] ———, *Analytic properties of the Schmidt pairs of a Hankel operator and the generalized Schur-Takagi problem*, Mat. Sb. (N.S.) **86(128)** (1971), 34–75 (Russian). MR0298453 (45 #7505)

[4] ———, *Infinite Hankel block matrices and related problems of extension*, Izv. Akad. Nauk Armjan. SSR Ser. Mat. **6** (1971), no. 2-3, 87–112 (Russian, with Armenian and English summaries). MR0298454 (45 #7506)

[5] N.I. Akhiezer, *The classical moment problem and some related questions in analysis*, Translated by N. Kemmer, Hafner Publishing Co., New York, 1965. MR0184042 (32 #1518)

[6] D. Alpay, *The Schur algorithm, reproducing kernel spaces and system theory*, SMF/AMS Texts and Monographs, vol. 5, American Mathematical Society, Providence, RI, 2001. Translated from the 1998 French original by Stephen S. Wilson. MR1839648 (2002b:47144)

[7] D. Alpay, A. Dijksma, J. Rovnyak, and H. de Snoo, *Schur functions, operator colligations, and reproducing kernel Pontryagin spaces*, Operator Theory: Advances and Applications, vol. 96, Birkhäuser Verlag, Basel, 1997. MR1465432 (2000a:47024)

[8] M. Bakonyi and T. Constantinescu, *Schur's algorithm and several applications*, Pitman Research Notes in Mathematics Series, vol. 261, Longman Scientific & Technical, Harlow, 1992. MR1192547 (94b:47022)

[9] M. Bakonyi and H.J. Woerdeman, *Matrix completions, moments, and sums of Hermitian squares*, Princeton University Press, Princeton, NJ, 2011. MR2807419

[10] J.A. Ball, I. Gohberg, and L. Rodman, *Interpolation of rational matrix functions*, Operator Theory: Advances and Applications, vol. 45, Birkhäuser Verlag, Basel, 1990. MR1083145 (92m:47027)

[11] T. Constantinescu, *Schur parameters, factorization and dilation problems*, Operator Theory: Advances and Applications, vol. 82, Birkhäuser Verlag, Basel, 1996. MR1399080 (97g:47012)

[12] V.K. Dubovoj, B. Fritzsche, and B. Kirstein, *Matricial version of the classical Schur problem*, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 129, B.G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1992. With German, French and Russian summaries. MR1152328 (93e:47021)

[13] H. Dym, *J contractive matrix functions, reproducing kernel Hilbert spaces and interpolation*, CBMS Regional Conference Series in Mathematics, vol. 71, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1989. MR1004239 (90g:47003)

[14] Yu.M. Dyukarev, B. Fritzsche, B. Kirstein, C. Mädler, and H.C. Thiele, *On distinguished solutions of truncated matricial Hamburger moment problems*, Complex Anal. Oper. Theory **3** (2009), no. 4, 759–834, DOI 10.1007/s11785-008-0061-2. MR2570113

[15] C. Foias and A.E. Frazho, *The commutant lifting approach to interpolation problems*, Operator Theory: Advances and Applications, vol. 44, Birkhäuser Verlag, Basel, 1990. MR1120546 (92k:47033)

[16] C. Foias, A.E. Frazho, I. Gohberg, and M.A. Kaashoek, *Metric constrained interpolation, commutant lifting and systems*, Operator Theory: Advances and Applications, vol. 100, Birkhäuser Verlag, Basel, 1998. MR1635831 (99i:47027)

[17] B. Fritzsche and B. Kirstein (eds.), *Ausgewählte Arbeiten zu den Ursprüngen der Schur-Analysis*, Teubner – Archiv zur Mathematik, vol. 16, B.G. Teubner Verlagsgesellschaft, Stuttgart, 1991. MR1162976 (93d:01095)

[18] B. Fritzsche, B. Kirstein, and C. Mädler, *On Hankel nonnegative definite sequences, the canonical Hankel parametrization, and orthogonal matrix polynomials*, Complex Anal. Oper. Theory **5** (2011), no. 2, 447–511, DOI 10.1007/s11785-010-0054-9. MR2805417

[19] I. Gohberg (ed.), *I. Schur methods in operator theory and signal processing*, Operator Theory: Advances and Applications, vol. 18, Birkhäuser Verlag, Basel, 1986. MR902600 (88d:00006)

[20] H. Hamburger, *Über eine Erweiterung des Stieltjesschen Momentenproblems*, Math. Ann. **81** (1920), no. 2-4, 235–319, DOI 10.1007/BF01564869 (German). MR1511966

[21] ———, *Über eine Erweiterung des Stieltjesschen Momentenproblems*, Math. Ann. **82** (1920), no. 1-2, 120–164, DOI 10.1007/BF01457982 (German). MR1511978

[22] ———, *Über eine Erweiterung des Stieltjesschen Momentenproblems*, Math. Ann. **82** (1921), no. 3-4, 168–187, DOI 10.1007/BF01498663 (German). MR1511981

[23] G. Herglotz, *Über Potenzreihen mit positivem, reellen Teil im Einheitskreis*, Berichte über die Verhandlungen der Königlich Sächsischen Gesellschaft der Wissenschaften zu Leipzig. Mathematisch-physische Klasse **63** (1911), 501–511 (German).

[24] J.W. Helton, *Operator theory, analytic functions, matrices, and electrical engineering*, CBMS Regional Conference Series in Mathematics, vol. 68, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1987. MR896034 (89f:47001)

[25] V.È. Katsnel′son, *Methods of J-theory in continuous interpolation problems of analysis*, 1983. Deposited in VINITI. Part 1 translated from the Russian in [26].

[26] ———, *Methods of J-theory in continuous interpolation problems of analysis. Part I*, translated by T. Ando, T. Ando Hokkaido University, Sapporo, 1985. MR777324 (86i:47048)

[27] R. Nevanlinna, *Über beschränkte Funktionen, die in gegebenen Punkten vorgeschriebene Werte annehmen*, Ann. Acad. Sci. Fenn. **A 13** (1919), no. 1, 1–71.

[28] ———, *Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjessche Momentenproblem*, Ann. Acad. Sci. Fenn. **A 18** (1922), no. 5, 1–53.

[29] ———, *Über beschränkte analytische Funktionen*, Ann. Acad. Sci. Fenn. **A 32** (1929), no. 7, 1–75.

[30] G. Pick, *Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden*, Math. Ann. **77** (1915), no. 1, 7–23, DOI 10.1007/BF01456817 (German). MR1511844

[31] M. Rosenblum and J. Rovnyak, *Hardy classes and operator theory*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1985. Oxford Science Publications. MR822228 (87e:47001)

[32] L.A. Sakhnovich, *Interpolation theory and its applications*, Mathematics and its Applications, vol. 428, Kluwer Academic Publishers, Dordrecht, 1997. MR1631843 (99j:47016)

[33] I. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind*. Teil I **147** (1917), 205–232; Teil II **148** (1918), 122–145, J. reine u. angew. Math.

[34] T.-J. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. **8** (1894), no. 4, J1–J122 (French). MR1508159

[35] ———, *Recherches sur les fractions continues* [*Suite et fin*], Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. **9** (1895), no. 1, A5–A47 (French). MR1508160

[36] H. Weyl, *Über das Pick–Nevanlinna'sche Interpolationsproblem und sein infinitesmales Analogon*, Annals of Mathematics **36** (1935), no. 2, 230–254 (German). MR1503221

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