Chapter 2
The Spaces $W^{k,p}$

The theory of the spaces $W^{k,p}$ for $p = 2$, outlined in Chap. 1, has been substantially developed; now there exist plenty of spaces of analogous type. The fundamental sources are: S.L. Sobolev [1], S.M. Nikolskii [2], J. Deny, J.L. Lions [1], E. Gagliardo [1, 2]; see also N. Aronszajn [3], N. Aronszajn, K.T. Smith [1–3], N. Aronszajn, F. Mulla, P. Szeptycki [1], V.M. Babich [1], O.V. Besov [1, 2], S. Campanato [3–7], E. Gagliardo [3], V.P. Ilyin [1, 2], G.N. Jakovlev [1], W. Kondrashov [1], L.D. Kudriavcev [2], J.L. Lions [5], E. Magenes [4], K. Maurin [1], N.G. Meyers, J. Serrin [1], J. Nečas [11], S.M. Nikolskii [3–8], G. Prodi [1], L.N. Slobodetskii [1, 2], V.I. Smirnov [1], S.V. Uspenskii [1–4], L. De Vito [1].

2.1 Definitions and Auxiliary Theorems

2.1.1 Classification of Domains, Pseudotopology in $C_0^\infty(\Omega)$

In Sect. 1.1.3, we introduced domains with continuous or lipschitzian boundaries; it was a particular case of a more general definition:

A bounded domain Ω is of type $\mathfrak{N}^{k,\mu}$, where k is a non-negative integer or infinity, $0 \le \mu \le 1$, if there exist functions a_r as in 1.1.3 defined in the closures of cubes $\overline{\Delta}_r = \{x' \in \mathbb{R}^{N-1},\ |x_{ri}| \le \alpha,\ i = 1,2,\dots,N-1\}$, μ-hölderian together with their derivatives of order $\le k$, which means that for $x'_r, y'_r \in \overline{\Delta}_r$, there is $|D^i a_r(x'_r) - D^i a_r(y'_r)| \le c|x'_r - y'_r|^\mu$, $|i| \le k$.[1] If $\mu = 0$, the functions a_r and their derivatives of order $\le k$ are only continuous in $\overline{\Delta}_r$, and for simplicity we shall write $\mathfrak{N}^{k,0} = \mathfrak{N}^k$.

Let Ω be a domain in $\mathbb{R}^N$, k a non-negative integer or $k = \infty$, $0 \le \mu \le 1$. We denote by $C^{k,\mu}(\overline{\Omega})$ the space of complex-valued functions whose derivatives of

[1]Hereafter various constants will be mostly denoted by the same letter c. If necessary, we shall use indices or another appropriate notation.

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order $\leq k$ are μ-hölderian on the closure of Ω. If $\mu = 0$, the functions and their derivatives of order $\leq k$ are only continuous on $\overline{\Omega}$; we then write simply $C^k(\overline{\Omega})$ instead of $C^{k,0}(\overline{\Omega})$. If $k < \infty$, we endow $C^k(\overline{\Omega})$ with the following norm:

$$|u|_{C^k(\overline{\Omega})} = \sum_{|\alpha|\leq k} \max_{x\in\overline{\Omega}} |D^\alpha u(x)|, \tag{2.1}$$

and $C^{k,\mu}(\overline{\Omega})$ with the norm defined by:

$$|u|_{C^{k,\mu}(\overline{\Omega})} = |u|_{C^k(\overline{\Omega})} + \sum_{|\alpha|=k} \sup_{x,y\in\Omega, x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\mu}. \tag{2.2}$$

The spaces $C^k(\overline{\Omega})$ and $C^{k,\mu}(\overline{\Omega})$ are Banach spaces. We denote also by $C^k(\Omega)$ (resp. $C^{k,\mu}(\Omega)$) the spaces of functions continuous (resp. $\mu-$hölderian) together with derivatives of order $\leq k$ in Ω.

For $C_0^\infty(\Omega)$, see 1.1.1.

On $C_0^\infty(\Omega)$ we introduce a *pseudotopology* (cf. L. Schwartz [1]): Let φ_n be a sequence in $C_0^\infty(\Omega)$. Then $\lim_{n\to\infty} \varphi_n = \varphi$ in $C_0^\infty(\Omega)$, if there exists $\Omega' \subset \overline{\Omega'} \subset \Omega$, Ω' bounded, such that the support of φ_n (denoted by $\operatorname{supp} \varphi_n$) and the support of φ are included in $\overline{\Omega'}$, and for all $k \geq 0$, $\lim_{n\to\infty} \varphi_n = \varphi$ in $C^k(\overline{\Omega'})$.

Following L. Schwartz [1], we will for some time denote the space $C_0^\infty(\Omega)$ by $\mathscr{D}(\Omega)$. The *space of distributions* on Ω – the dual of $\mathscr{D}(\Omega)$ – will be denoted by $\mathscr{D}'(\Omega)$. For $f \in \mathscr{D}'(\Omega)$, we denote the value of f at the point $\varphi \in \mathscr{D}(\Omega)$ by $\langle \varphi, f \rangle$.

The derivative $D^i f$ of the distribution f is again a distribution, defined by the formula

$$\langle \varphi, D^i f \rangle = (-1)^{|i|} \langle D^i \varphi, f \rangle, \quad \varphi \in \mathscr{D}(\Omega). \tag{2.3}$$

If f_n is a sequence in $\mathscr{D}'(\Omega)$, we say that $\lim_{n\to\infty} f_n = f$ in $\mathscr{D}'(\Omega)$ if for all $\varphi \in \mathscr{D}(\Omega)$ we have:

$$\lim_{n\to\infty} \langle \varphi, f_n \rangle = \langle \varphi, f \rangle.$$

Let $L^1_{loc}(\Omega)$ be the space of locally integrable functions on Ω. We define the imbedding $L^1_{loc}(\Omega) \subset \mathscr{D}'(\Omega)$ by:

$$\langle \varphi, f \rangle = \int_\Omega \varphi f \, \mathrm{d}x, \quad \varphi \in \mathscr{D}(\Omega), \quad f \in L^1_{loc}.$$

Obviously we have:

Proposition 1.1. *If $f_1, f_2 \in L^1_{loc}(\Omega)$ and if for every $\varphi \in \mathscr{D}(\Omega)$*

$$\langle \varphi, f_1 \rangle = \langle \varphi, f_2 \rangle, \tag{2.4}$$

then $f_1 = f_2$ almost everywhere in Ω.

Indeed: It follows from (2.4) that for every interval $I \subset \Omega$, we have:

$$\int_I f_1 \, \mathrm{d}x = \int_I f_2 \, \mathrm{d}x.$$

□

Let B be a Banach space such that $\mathscr{D}(\Omega) \subset B$; B is called *normal* if $\mathscr{D}(\Omega)$ is dense in B.

If $\mathscr{D}(\Omega) \subset B$ algebraically and topologically, then $B' \subset \mathscr{D}'(\Omega)$, algebraically and topologically. B' is a subspace of distributions.

Exercise 1.1. If $f \in C^k(\Omega)$, then, for $|i| \leq k$, $D^i f$ in the classical sense and $D^i f$ in the sense of distributions coincide.

Hereafter if $f \in \mathscr{D}'(\Omega)$, $D^i f$ will denote the derivative in the distribution sense.

2.1.2 The Space $L^p(\Omega)$, Mean Continuity

Let $1 \leq p < \infty$. We denote by $L^p(\Omega)$ the space of $p-$integrable functions on Ω with the norm:

$$|f|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, \mathrm{d}x \right)^{1/p}; \tag{2.5}$$

this space is a Banach space, it has a countable basis, it is reflexive for $p > 1$. The following *mean continuity* property holds:

Theorem 1.1. *Let Ω be an open set in $\mathbb{R}^N$, $f \in L^p(\Omega)$, $f(x) = 0$ for $x \notin \Omega$. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|z| < \delta \Longrightarrow \left(\int_\Omega |f(x+z) - f(x)|^p \, \mathrm{d}x \right)^{1/p} < \varepsilon.$$

Proof. Let us assume Ω bounded and $\varepsilon > 0$. There exists $\nu <$ meas (Ω) such that

$$M \subset \Omega, \quad \text{meas}\,(M) < \nu \Longrightarrow \left(\int_M |f(x)|^p \, \mathrm{d}x \right)^{1/p} < \varepsilon/3.$$

According to Lusin's theorem, there exists a closed set $F \subset \Omega$, meas $(F) >$ meas $(\Omega) - (1/2)\nu$ such that f is continuous on F. Then there exists $\delta > 0$ such that

$$|z| < \delta \Longrightarrow x \in F, \quad x + z \in F, \quad |f(x+z) - f(x)| < \frac{\varepsilon}{3(\text{meas}\,(\Omega))^{1/p}}.$$

For z fixed, $|z| < \delta$, we denote

$$H_z = \{y \in \mathbb{R}^N,\ y = x + z,\ x \in F\}, \quad F_z = F \cap H_z = F - (F - H_z).$$

We can choose δ sufficiently small such that $H_z \subset \Omega$. Then we have:

$$\begin{aligned} \text{meas}\,(F_z) &> \text{meas}\,(\Omega) - \frac{\nu}{2} - \left[\text{meas}\,(\Omega) - \left(\text{meas}\,(\Omega) - \frac{\nu}{2} \right) \right] \\ &= \text{meas}\,(\Omega) - \nu, \end{aligned}$$

i.e.

$$\text{meas}\,(\Omega - F_z) < \nu.$$

We get:

$$\left(\int_\Omega |f(x+z)-f(x)|^p\,\mathrm{d}x\right)^{1/p} \le \left(\int_{F_z} |f(x+z)-f(x)|^p\,\mathrm{d}x\right)^{1/p}$$

$$+\left(\int_{\Omega-F_z} |f(x+z)-f(x)|^p\,\mathrm{d}x\right)^{1/p} < \text{meas}\,(F_z)^{1/p}\frac{\varepsilon}{3(\text{meas}\,(\Omega))^{1/p}}$$

$$+\left(\int_{\Omega-F_z} |f(x+z)|^p\,\mathrm{d}x\right)^{1/p} + \left(\int_{\Omega-F_z} |f(x)|^p\,\mathrm{d}x\right)^{1/p} < \varepsilon;$$

if Ω is unbounded, for any $\varepsilon > 0$, we can find a ball $K(r)$ with radius $r > 1$ such that

$$\left(\int_{\Omega\ -K(r-1)} |f(x)|^p\,\mathrm{d}x\right)^{1/p} < \varepsilon/3.$$

Then we can repeat the proof given previously for the bounded set $\Omega \cap K(r-1)$ with $\varepsilon/3$ and $\delta \le 1$. □

2.1.3 The Regularizing Operator

Let $h > 0$. We define the regularizing kernel by:

$$\omega(x,h) = \begin{cases} \exp(|x|^2/(|x|^2-h^2)) & \text{for } |x| < h, \\ 0 & \text{for } |x| \ge h. \end{cases}$$

The kernel is a function in $C^\infty(\mathbb{R}^N)$. The *regularizing operator* is the operator mapping $L^p(\Omega)$ into itself and defined by

$$f_h(x) = \frac{1}{\kappa h^N}\int_\Omega \omega(x-y,h)f(y)\,\mathrm{d}y, \tag{2.6}$$

where

$$\kappa = \int_{|x|<1} \omega(x,1)\,\mathrm{d}x = \frac{1}{h^N}\int_{|x|<h} \omega(x,h)\,\mathrm{d}x.$$

Of course, we can use also other regularizing kernels; Sect. 2.5.5.

By an immediate computation we get:

$$f_h(x) = \frac{1}{\kappa}\int_{|z|<1} \omega(z,1)f(x+hz)\,\mathrm{d}z$$

where $f(x) = 0$ for $x \notin \Omega$.

In what follows, we will use the following notation: let B_1, B_2 be two Banach spaces, and T a bounded linear mapping defined on B_1 with values in B_2; we shall write $T \in [B_1 \to B_2]$.

Theorem 1.2. *The operator which defines $f_h(x)$ by (2.6), has the following properties: it belongs to $[L^p(\Omega) \to L^p(\Omega)]$, $f_h \in C^\infty(\overline{\Omega})$ (and also $\in C^\infty(\mathbb{R}^N)$), and $\lim_{h\to 0} f_h = f$ in $L^p(\Omega)$.*

Proof. We have to prove:

$$|f_h|_{L^p(\Omega)} \leq C|f|_{L^p(\Omega)}, \quad \lim_{h\to 0} f_h = f \text{ in } L^p(\Omega); \tag{2.7}$$

the other properties are clear. We always set $f(x) = 0$ for $x \notin \Omega$; then we have:

$$\begin{aligned}
&\int_\Omega |f_h(x) - f(x)|^p \, \mathrm{d}x \\
&\qquad = \int_\Omega \left| \frac{1}{\kappa} \int_{|z|<1} \omega(z,1) f(x+hz) \, \mathrm{d}z - \frac{1}{\kappa} \int_{|z|<1} \omega(z,1) f(x) dz \right|^p \mathrm{d}x \\
&\qquad \leq \int_\Omega \left(\frac{1}{\kappa} \int_{|x|<1} \omega(z,1) |f(x+hz) - f(x)| \, \mathrm{d}z \right)^p \mathrm{d}x \equiv I(h).
\end{aligned}$$

If $p = 1$ then the Fubini theorem implies:

$$I(h) \leq c \int_{|z|<1} dz \int_\Omega |f(x+hz) - f(x)| \, \mathrm{d}x. \tag{2.8}$$

If $p > 1$ then using the Hölder inequality and the Fubini theorem we get:

$$I(h) \leq c \int_\Omega \mathrm{d}x \int_{|z|<1} |f(x+hz) - f(x)|^p \, \mathrm{d}z = c \int_{|z|<1} dz \int_\Omega |f(x+hz) - f(x)|^p \, \mathrm{d}x; \tag{2.9}$$

the result follows according to (2.8), (2.9) and Theorem 1.1. The inequality in (2.7) can be obtained by a trivial modification of the proof. □

2.1.4 Compactness Condition

The following theorem is due to Kolmogorov.

Theorem 1.3. *Let Ω be a bounded domain, $M \subset L^p(\Omega)$. M is precompact if and only if:*

$$M \text{ is a bounded set,} \tag{2.10}$$

$$\text{the functions } f \in M \text{ are mean-equicontinuous.} \tag{2.11}$$

Proof. First, let us prove that the conditions are sufficient; as previously we set $f(x) = 0$ for $x \notin \Omega$. According to (2.8), (2.9) we have:

$$|f_h - f|_{L^p(\Omega)} \leq c \sup_{|z|<h} \left(\int_\Omega |f(x+z) - f(x)|^p \,\mathrm{d}x \right)^{1/p}. \tag{2.12}$$

Let $\varepsilon > 0$; we can find $\delta > 0$ such that $h < \delta \Longrightarrow |f_h - f|_{L^p(\Omega)} < \varepsilon/2$. Let $M_h = \{f_h, f \in M, h \text{ fixed}\}$. The functions f_h are bounded by the same constant and they are equicontinuous, hence M_h is a relatively compact set in $C^0(\overline{\Omega})$, and there exists an $\varepsilon/2(\text{meas}\,(\Omega))^{1/p}$-net, say S_ε, in $C^0(\overline{\Omega})$. It follows from (2.12) that S_ε is an ε-net in $L^p(\Omega)$.

Now we prove the necessity: If M is relatively compact then (2.9) holds. Let $f_1, f_2, \ldots, f_k$ be an $(\varepsilon/3)$-net in M. According to Theorem 1.1, each f_i, $i = 1, 2, \ldots, k$ is mean continuous, and there exists $\delta > 0$ such that

$$|z| < \delta \Longrightarrow (\int_\Omega |f(x+z) - f(x)|^p \,\mathrm{d}x < (\varepsilon/3)^p, \quad i = 1, 2, \ldots, k.$$

Hence, if $f \in M$, there exists an index i such that $|f - f_i|_{L^p(\Omega)} < \varepsilon/3$, and

$$|z| < \delta \Longrightarrow \left(\int_\Omega |f(x+z) - f(x)|^p \,\mathrm{d}x \right)^{1/p} < \varepsilon.$$

□

Exercise 1.2. Prove Theorem 1.1 using the fact that for Ω bounded we have $L^p(\Omega) = \overline{C(\overline{\Omega})}$.

Exercise 1.3. Prove Theorem 1.3 for Ω unbounded with the following additional condition: for each $\varepsilon > 0$ there exists $r > 0$ such that

$$f \in M \Longrightarrow \int_{\Omega - \overline{K(r)}} |f(x)|^p \,\mathrm{d}x < \varepsilon,$$

where $K(r)$ is a ball with center at the origin and with radius r.

2.2 The Spaces $W^{k,p}(\Omega)$

2.2.1 A Property of the Regularizing Operator

Let us recall that $L^p_{loc}(\Omega), p \geq 1$ is the space of complex-valued functions defined on Ω which are locally $p-$integrable on Ω (i.e. on every compact set in Ω). The following proposition is obvious:

Proposition 2.1. *Let $f_i \in \mathscr{D}'(\Omega), i = 1,2$. Then*

$$D^\alpha(f_1 + f_2) = D^\alpha f_1 + D^\alpha f_2, \quad D^\alpha(\lambda f_i) = \lambda D^\alpha f_i, \quad D^\alpha(D^\beta f_i) = D^{\alpha+\beta} f_i.$$

Theorem 2.1. *Let $u \in \mathscr{D}'(\Omega)$, $\overline{\Omega}^* \subset \Omega$, Ω^* bounded. Suppose that $D^\alpha u \in L^p_{loc}(\Omega)$, $p \geq 1$. Then there exists $h_0 > 0$ such that for $h \leq h_0$, $x \in \Omega^*$ we have:*

$$D^\alpha u_h(x) = (D^\alpha u)_h(x), \tag{2.13}$$

$$\lim_{h\to 0} D^\alpha u_h = D^\alpha u \text{ in } L^p(\Omega^*). \tag{2.14}$$

Proof. Indeed: if $\varphi_h(y) = (1/\kappa h^N)\omega(x-y,h)$, then $u_h(x) = \langle \varphi_h, u\rangle$ for $h \leq h_0$, $x \in \Omega^*$, with $h_0 \leq \operatorname{dist}(\Omega^*, \partial\Omega)$. Using the definition from 2.1.1, it follows that

$$\begin{aligned} D^\alpha u_h(x) &= (-1)^{|\alpha|}\langle D^\alpha \varphi_h, u\rangle = \langle \varphi_h, D^\alpha u\rangle \\ &= \frac{1}{\kappa h^N}\int_\Omega \omega(x-y,h) D^\alpha u(y)\,\mathrm{d}y = (D^\alpha u)_h(x); \end{aligned}$$

(2.14) is a direct consequence of Theorem 1.2. □

2.2.2 The Absolute Continuity

Let Ω be a domain in $\mathbb{R}^N$, P a line verifying $P \cap \Omega \neq \emptyset$. A function defined almost everywhere in Ω is said *absolutely continuous on the line P* if it is continuous on each closed interval of $P \cap \Omega$.

Theorem 2.2. *Suppose $u \in L^1_{loc}(\Omega)$ and $\partial u/\partial x_i \in L^p(\Omega)$, $p \geq 1$. This function changed on a set of measure zero is absolutely continuous on almost all lines parallel to the axis x_i.*[2] *Let us denote by $[\partial u/\partial x_i]$ the usual derivative and by $\partial u/\partial x_i$ the distribution derivative. Then we have almost everywhere $[\partial u/\partial x_i] = \partial u/\partial x_i$.*

Conversely, if $u \in L^1_{loc}(\Omega)$ is absolutely continuous on almost all lines parallel to the axis x_i with $[\partial u/\partial x_i] \in L^p(\Omega)$, then we have $\partial u/\partial x_i = [\partial u/\partial x_i]$.

Proof. Let us prove the second part: If $\varphi \in \mathscr{D}(\Omega)$, by integration by parts we get $\langle \varphi, [\partial u/\partial x_i]\rangle = \langle \varphi, \partial u/\partial x_i\rangle$. For the first part, let $\Omega = \cup_{j=1}^\infty C_j$, where C_j are cubes; this cover is locally finite, which is always possible. Let C be one of these cubes, and $\psi \in \mathscr{D}(\Omega)$ such that $\psi(x) = 1$ for $x \in C$. Let us put $v = u\psi; v \in L^1(\Omega)$. Obviously $\partial v/\partial x_i = (\partial u/\partial x_i)\psi + u(\partial\psi/\partial x_i)$. Put $v = \partial v/\partial x_i = 0$ for $x \notin \Omega$. Let K be a cube big enough such that $\overline{\Omega} \subset K$.

[2] The set of all intersections of parallel hyperplanes where u is not absolutely continuous, with the hyperplane $x_i = 0$, is a set M such that $\operatorname{meas}_{(N-1)} M = 0$.

Let us define $v^*(x)$ *by*:

$$v^*(x) = \int_{-\infty}^{x_i} \frac{\partial v}{\partial x_i}(x_1,\dots,x_{i-1},\xi,x_{i+1},\dots,x_N)\,\mathrm{d}\xi \tag{2.15}$$

for the points $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_N)$, where

$$\int_{-\infty}^{\infty} \left| \frac{\partial v}{\partial x_i}(x_1,\dots,x_{i-1},\xi,x_{i+1},\dots,x_N) \right| \mathrm{d}\xi < \infty.$$

Let $\chi \in \mathscr{D}(\Omega)$ be a function with $\chi(x) = 1$ on $\operatorname{supp} v$.

For all $\varphi \in \mathscr{D}(\Omega)$, we have:

$$\begin{aligned}\int_\Omega \varphi v^*\,\mathrm{d}x &= \int_\Omega \Big(\int_{x_i}^{\infty} \varphi(x_1,\dots,x_{i-1},\xi,x_{i+1},\dots,x_N)\,\mathrm{d}\xi\Big)\frac{\partial v}{\partial x_i}(x)\,\mathrm{d}x \\ &= \int_\Omega \Big(\int_{x_i}^{\infty} \varphi\,\mathrm{d}\xi\Big)\chi\frac{\partial v}{\partial x_i}\,\mathrm{d}x = \int_\Omega \varphi v\,\mathrm{d}x,\end{aligned}$$

then almost everywhere $v(x) = v^*(x)$; it is clear that v^* is absolutely continuous on almost all lines parallel to the axis x_i, and almost everywhere $[\partial v^*/\partial x_i] = \partial v/\partial x_i$. But since almost everywhere on C $v^*(x) = v(x)$, the result follows. □

Remark 2.1. According to Theorem 2.2, a function f not absolutely continuous on $[0,1]$ (or continuous) which has almost everywhere a derivative such that $[\mathrm{d}f/\mathrm{d}x] \in L^1(0,1)$, satisfies $[\mathrm{d}f/\mathrm{d}x] \neq \mathrm{d}f/\mathrm{d}x$. The well known example is a monotone function continuous on $(0,1)$, $f(0) = 0$, $f(1) = 1$, $\mathrm{d}f/\mathrm{d}x = 0$ almost everywhere.

2.2.3 *The Spaces* $W^{k,p}(\Omega)$

For an integer $k \geq 0$, and $p \geq 1$, we denote by $W^{k,p}(\Omega)$[3] the subspace of functions $f \in L^p(\Omega)$ such that for $|\alpha| \leq k, D^\alpha u \in L^p(\Omega)$. On $W^{k,p}(\Omega)$ we define a norm by:

$$|u|_{W^{k,p}(\Omega)} = \Big(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u|^p dx\Big)^{1/p}, \tag{2.16}$$

and, if $p = 2, W^{k,2}(\Omega)$ is an Hilbert space for the scalar product defined by (1.1.2). The membership of u in $W^{k,p}(\Omega)$ is a local property, indeed.

[3] If $u \in W^{k,2}(\Omega)$ by the definition from 1.1.1, then in general it is the same as in the definition given here; the converse is not true. Later we shall see that for $\Omega \in \mathfrak{N}^0$ the two definitions coincide. In N.G. Meyers, J. Serrin [1], it is proved that $C^\infty(\overline{\Omega}) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$.

Proposition 2.2. *Let $\Omega_i, i = 1,2,\dots,l$, be domains satisfying $\cup_{i=1}^{l}\Omega_i \supset \Omega$. If $u \in W^{k,p}(\Omega_i)$, $i = 1,2,\dots,l$, then $u \in W^{k,p}(\Omega)$, and*

$$|u|_{W^{k,p}(\Omega)} \leq c\sum_{i=1}^{l} |u|_{W^{k,p}(\Omega_i)}.$$

Proof. Let $|\alpha| \leq k$. We denote by g_i the derivatives $D^\alpha u$ in Ω_i; it follows immediately from the definition of $D^\alpha u$ that if $\Omega_i \cap \Omega_j \neq \emptyset$, then $g_i = g_j$ almost everywhere in $\Omega_i \cap \Omega_j$. We define $g(x) = g_i(x)$ in Ω_i, and let $\varphi \in C_0^\infty(\Omega)$. According to Proposition 1.2.3, there exist functions $\psi_i \in C_0^\infty(\Omega_i)$ such that

$$x \in \text{supp}\ \varphi \Longrightarrow \sum_{i=1}^{l} \psi_i(x) = 1.$$

We have:

$$\langle D^\alpha \varphi, u\rangle = \langle D^\alpha (\sum_{i=1}^{l} \psi_i\varphi), u\rangle = (-1)^{|\alpha|}\sum_{i=1}^{l}\langle \psi_i\varphi, g_i\rangle$$
$$= (-1)^{|\alpha|}\sum_{i=1}^{l}\langle \psi_i\varphi, g\rangle = (-1)^{|\alpha|}\langle \varphi, g\rangle.$$

□

Remark 2.2. According to Theorem 2.2, there is a definition equivalent to the previous: $u \in W^{1,p}(\Omega)$, if $u \in L^p(\Omega)$ and if, after a modification of u on a set of zero measure, u remains absolutely continuous on almost all lines parallel to the x_1 axis and if $[\partial u/\partial x_1] \in L^p(\Omega)$ (cf. Theorem 2.2). By another modification $[\partial u/\partial x_2] \in L^p(\Omega)$, etc.

There exists another definition for $p = 2$, $k = 1$ due to B. Levi related to Remark 2.2: $W^{1,p}(\Omega)$ is the subspace in $L^p(\Omega)$ of functions u which, and after a modification on a set of zero measure, remain absolutely continuous on almost all lines parallel to the axes $x_1, x_2, \dots, x_N$; the derivatives $[\partial u/\partial x_i] \in L^p(\Omega)$.

Here an adaptation of a theorem of J. Deny, J.L. Lions [1]:

Theorem 2.3. *For $W^{1,p}(\Omega)$, the definition from 2.2.3 and the definition of B. Levi are equivalent.*

Proof. If $u \in W^{1,p}(\Omega)$ by the B. Levi definition, then it is the same as by our definition according to Theorem 2.2. Let $u \in W^{1,p}(\Omega)$; we use the steps used in the proof of Theorem 2.2: $u\psi = v \in W^{1,p}(K)$, where K is a cube $(-l,l)^N$ sufficiently large such that $\text{supp}\, v \subset K$. According to Theorem 2.1, $\lim_{h\to 0} v_h = v$ in $L^p(\Omega)$, $\lim_{h\to 0} \partial v_h \partial x_i = \partial v/\partial x_i$ in $L^p(K)$, $i = 1,2,\dots,N$. Let us set for $i = 1,2,\dots,N$:

$$g_i(x) = \int_{-\infty}^{x_i} \frac{\partial v}{\partial x_i}(x_1,\dots,x_{i-1}\xi, x_{i+1},\dots,x_N)\,\mathrm{d}\xi. \tag{2.17}$$

Then $g_i(x)$ is an absolutely continuous function on almost all lines parallel to the axis x_i; we have:

$$\lim_{h\to 0} \underbrace{\int_{-l}^{l}\cdots\int_{-l}^{l}}_{(N-1)\text{-times}} \left(\int_{-l}^{l}\left|\frac{\partial v_h}{\partial x_1}-\frac{\partial v}{\partial x_1}\right| \mathrm{d}x_1\right)\mathrm{d}x' = 0,$$

then we can find a sequence h_n, $\lim_{n\to\infty} h_n = 0$, such that for almost all lines parallel to the axis x_1

$$\lim_{n\to\infty}\int_{-l}^{l}\left|\frac{\partial v_{h_n}}{\partial x_1}-\frac{\partial v}{\partial x_1}\right| \mathrm{d}x_1 = 0.$$

We deduce that $\lim_{n\to\infty} v_{h_n}(x) = g_1(x)$ on almost all lines parallel to x_1. The same property holds for $i=2$: we can extract a subsequence h_{n_m} of the sequence h_n such that $\lim_{m\to\infty} v_{h_{n_m}}(x) = g_2(x)$ on almost all lines parallel to the axis x_2, etc. Step by step we construct a sequence v_{h_s}, such that $\lim_{s\to\infty} v_{h_s}(x) = g_i(x)$ on almost all lines parallel to $x_1, x_2, \ldots, x_N$; it is clear that $\lim_{s\to\infty} v_{h_s} = v^*(x) = v(x)$ almost everywhere in Ω. We conclude as in the proof of Theorem 2.2. $\square$

Exercise 2.1. If $u \in W^{1,p}(\Omega)$, prove that $|u| \in W^{1,p}(\Omega)$ and $||u||_{W^{1,p}(\Omega)} \leq |u|_{W^{1,p}(\Omega)}$.

Hint: use Theorem 2.3.

2.2.4 *The Spaces* $W^{k,p}(\Omega)$ *(Continuation)*

Proposition 2.3. *The space* $W^{k,p}(\Omega)$ *is a Banach space with a countable basis, reflexive for* $p > 1$.

Proof. The space $W^{k,p}(\Omega)$ is complete, this is a consequence of the definition from 2.2.3; $W^{k,p}(\Omega)$ is a closed subspace of the space $[L^p(\Omega)]^s$ which has a countable basis and which is reflexive if $p > 1$. Here, s is the number of indices $|\alpha| \leq k$. $\square$

Proposition 2.4. *Let* u_i *be a sequence in* $\mathscr{D}'(\Omega)$, *with* $|D^\alpha u_i|_{L^p(\Omega)} \leq c_1$, $p > 1$, *and* $\lim_{i\to\infty} u_i = u$ *in the sense of distributions. Then* $D^\alpha u \in L^p(\Omega)$, *and* $|D^\alpha u|_{L^p(\Omega)} \leq c_1$.

Proof. For $\varphi \in C_0^\infty(\Omega)$, we have:

$$\lim_{i\to\infty}\langle \varphi, D^\alpha u_i\rangle = \lim_{i\to\infty}(-1)^{|\alpha|}\langle D^\alpha\varphi, u_i\rangle = (-1)^{|\alpha|}\langle D^\alpha \varphi, u\rangle,$$

$$\langle \varphi, D^\alpha u_i\rangle = \int_\Omega \varphi D^\alpha u_i \,\mathrm{d}x.$$

But $\overline{\mathscr{D}(\Omega)} = L^q(\Omega)$ for $q \geq 1$, hence $\lim_{i\to\infty} D^\alpha u_i = g$ weakly in $L^p(\Omega)$, and $\langle\varphi, g\rangle = \lim_{i\to\infty}\langle\varphi, D^\alpha u_i\rangle = \lim_{i\to\infty}(-1)^{|\alpha|}\langle D^\alpha\varphi, u_i\rangle = (-1)^{|\alpha|}\langle D^\alpha \varphi, u\rangle$. $\square$

Remark 2.3. If $p = 1$, Proposition 2.4 is true if the sequence $D^\alpha u_i$ is weakly compact.

2.2.5 *The Spaces* $W_0^{k,p}(\Omega)$

We denote $W_0^{k,p}(\Omega) = \overline{\mathscr{D}(\Omega)}$, the closure of $\mathscr{D}(\Omega)$ with respect to the norm of $W^{k,p}(\Omega)$, and $W^{-k,q}(\Omega) = (W_0^{k,p}(\Omega))'$ the dual space of $W_0^{k,p}(\Omega)$.

Proposition 2.5. *Suppose $p > 1$; then every function $f \in W^{-k,q}(\Omega)$ can be written (not uniquely) in the following form:*

$$f = \sum_{|\alpha| \le k} D^\alpha f_\alpha, \tag{2.18}$$

where

$$f_\alpha \in L^q(\Omega) \quad \textit{and} \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Let s be as in the proof of Proposition 2.3. We have $W_0^{k,p}(\Omega) \subset [L^p(\Omega)]^s$. According to the Hahn-Banach theorem, f can be extended on $[L^p(\Omega)]^s$. But $([L^p(\Omega)]^s)' = [L^q(\Omega)]^s$, hence

$$v \in [L^p(\Omega)]^s \Longrightarrow fv = \sum_{i=1}^{s} \int_\Omega v_i g_i \,\mathrm{d}x$$

with $g_i \in L^q(\Omega)$. If $v \in W_0^{k,p}(\Omega)$, we get:

$$fv = \sum_{|\alpha| \le k} (-1)^{|\alpha|} \int_\Omega D^\alpha v f_\alpha \,\mathrm{d}x$$

with $f_\alpha \in L^q(\Omega)$; then

$$v \in \mathscr{D}(\Omega) \Longrightarrow fv = \langle v, \sum_{|\alpha| \le k} D^\alpha f_\alpha \rangle.$$

□

Exercise 2.2. If $p > 1$, prove that the closed unit ball in $W^{k,p}(\Omega)$ is weakly compact.

Remark 2.4. If Ω is a domain such that its complement $\complement\Omega$ has a positive measure, then for $k \ge 1$ we cannot have $W_0^{k,p}(\Omega) = W^{k,p}(\Omega)$; details can be found in J.L. Lions [5]. But if $\Omega = \mathbb{R}^N$ we have:

Proposition 2.6. $W_0^{k,p}(\mathbb{R}^N) = W^{k,p}(\mathbb{R}^N)$.

Proof. Let $\varphi \in \mathscr{D}(\mathbb{R}^N)$ satisfy $\varphi(x) = 1$ for $|x| \le 1$, $\varphi(x) = 0$ for $|x| \ge 2$. If $u \in W^{k,p}(\mathbb{R}^N)$, let us put $u_r(x) = u(x)\varphi(x/r)$. Clearly $\lim_{r\to\infty} u_r = u$ in $W^{k,p}(\mathbb{R}^N)$. Using Theorem 2.1, we get $\lim_{h\to 0} u_{rh} = u_r$ in $W^{k,p}(\mathbb{R}^N)$ (we use (2.6)); but $u_{rh} \in \mathscr{D}(\mathbb{R}^N)$. □

2.3 Imbedding Theorems

2.3.1 The Lipschitz Transform

Lemma 3.1. *Let Ω, O be two bounded open sets, T a one-to-one continuous mapping, $T : O \to \Omega$, with a Lipschitz inverse, i.e.*

$$|T^{-1}(x) - T^{-1}(y)| \leq c|x-y|. \tag{2.19}$$

Let $u \in L^p(\Omega), p \geq 1$. Then $v(y) = u(T(y)) \in L^p(O)$, and we have:

$$|v|_{L^p(O)} \leq c|u|_{L^p(\Omega)}. \tag{2.20}$$

Proof. Using the regularizing operator (we put $u(x) = 0$ for $x \notin \Omega$), we get $\lim_{h\to 0} u_h = u$ in $L^p(\Omega)$. Let S_d be a rectangular lattice in $\mathbb{R}^N$ formed by cubes with sidelength d; let us consider cubes $C_1, C_2, \ldots, C_{m_d}$ whose closures are contained in O; we have:

$$\int_O |u_h(T(y))|^p \,\mathrm{d}y = \lim_{d\to 0} \sum_{i=1}^{m_d} d^n \inf_{y\in C_i} |u_h(T(y))|^p. \tag{2.21}$$

According to (2.19) we have:

$$\text{meas } (C_i) \leq c_i \,\text{meas } (T(C_i)). \tag{2.22}$$

Indeed: if y_i is the center of C_i, and if $\partial T(C_i)$ is the boundary of $T(C_i)$, we get $T^{-1}(\partial T(C_i)) \subset \partial C_i$. Using (2.19) if $x \in \partial T(C_i)$, and denoting $x_i = T(y_i)$, we have $|x - x_i| \geq c_2 |T^{-1}(x) - y_i| \geq (1/2)dc_2$, hence $T(C_i)$ contains a ball with center x_i and radius $(1/2)dc_2$ and we have (2.22). Furthermore,

$$d^n \sum_{i=1}^{m_d} \inf_{y\in C_i} |u_h(T(y))|^p \leq c_1 \sum_{i=1}^{m_d} \text{meas } (T(C_i)) \inf_{y\in C_i} |u_h(T(y))|^p \leq c_1 \int_\Omega |u_h(x)|^p \,\mathrm{d}x, \tag{2.23}$$

and then

$$\int_O |u_h(T(y))|^p \,\mathrm{d}y \leq c_1 \int_\Omega |u_h(x)|^p \,\mathrm{d}x. \tag{2.24}$$

Now $\lim_{h\to 0} u_h = u$ in $L^p(\Omega)$, and we can extract a subsequence, say u_{h_i}, such that $\lim_{i\to\infty} u_{h_i} = u$ almost everywhere in Ω. It follows from (2.19) that $\lim_{i\to\infty} u_{h_i}(T(y)) = u(T(y))$ almost everywhere in O. The Fatou lemma gives (2.24). □

Lemma 3.2. *Let Ω, O be two bounded open sets, and T and T^{-1} one-to-one Lipschitz mappings, $T : O \to \Omega$. Let $u \in W^{1,p}(\Omega)$, $p \geq 1$. We have $u(T(y)) \in W^{1,p}(O)$, and if we set $v(y) = u(T(y))$ we get:*

$$|v|_{W^{1,p}(O)} \leq c|u|_{W^{1,p}(\Omega)}. \tag{2.25}$$

Proof. Let us put $u \equiv 0$ outside of Ω, and let u_h be the regularized function. The function $v^{(h)}(y) = u_h(T(y))$ is a Lipschitz function in $\overline{O}$, hence *a posteriori* it is Lipschitz on all lines parallel to $y_1, y_2, \dots, y_N$. Then we have in the usual sense:

$$\frac{\partial v^{(h)}}{\partial y_i} = \sum_{j=1}^{N} \frac{\partial u_h}{\partial x_j}\frac{\partial x_j}{\partial y_i}; \tag{2.26}$$

according to Theorem 2.2 it holds in the sense of distributions. We have $\lim_{h\to 0} \partial u_h/\partial x_j = \partial u/\partial x_j$ in $L^p(\Omega^*)$ for $\overline{\Omega^*} \subset \Omega, \lim_{h\to 0} u_h = u$ in $L^p(\Omega)$. Let us denote $O^* = T^{-1}(\Omega^*)$. According to the previous lemma, $\partial v^{(h)}/\partial y_i$ is a Cauchy sequence in $L^p(O^*)$. We get:

$$\frac{\partial v}{\partial y_i} = \sum_{j=1}^{N} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial y_i}, \tag{2.27}$$

and then, using again the previous lemma, we have $v \in W^{1,p}(O)$, and hence the inequality (2.25). $\square$

2.3.2 *Density of* $C^\infty(\overline{\Omega})$ *in* $W^{k,p}(\Omega)$

In Chap. 1, we introduced another definition of $W^{k,2}(\Omega)$ (cf. the definition in 1.1.1); we can generalize it for $p \geq 1$.

Problem 3.1. Characterize the domains such that $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$.

We know that these two definitions are equivalent under certain conditions (cf. E. Gagliardo [2]), V.P. Iljin [1]); let us denote by $C^\infty_{\overline{\Omega}}(\mathbb{R}^N)$ the space of restrictions to $\overline{\Omega}$ of functions in $C^\infty(\mathbb{R}^N)$ (clearly we have $C^\infty_{\overline{\Omega}}(\mathbb{R}^N) \subset C^\infty(\overline{\Omega})$). We have:

Theorem 3.1. *Let* $\Omega \in \mathfrak{N}^0$. *There is* $\overline{C^\infty_{\overline{\Omega}}(\mathbb{R}^N)} = W^{k,p}(\Omega)$.

Proof. Using the notations introduced in 1.2.4, we set $u_r = u\varphi_r$. We get immediately $u_r \in W^{k,p}(\Omega)$; using the local charts (x'_r, x_{rN}), let us define $u_{r\lambda}$, $r \leq m$ by $u_{r\lambda}(x'_r, x_{rN}) = u_r(x'_r, x_{rN} + \lambda)$. If λ is sufficient small, we have $u_{r\lambda} \in W^{k,p}(\Omega)$, and according to Theorem 1.1 $\lim_{\lambda\to 0} u_{r\lambda} = u_r$ in $W^{k,p}(\Omega)$. We have $u_{r\lambda} \in W^{k,p}(\Omega_\lambda)$, with $\Omega_\lambda \supset \overline{\Omega}$. It follows from Theorem 2.1 that $\lim_{h\to 0} u_{r\lambda h} = u_{r\lambda}$ in $W^{k,p}(\Omega)$. If $r = m+1$, we have $\lim_{h\to 0} u_{m+1,h} = u_{m+1}$ in $W^{k,p}(\Omega)$, so $u_{\lambda h} = \sum_{r=1}^{m+1} u_{r\lambda h}$, and we get $\lim_{h\to 0,\lambda\to 0} u_{\lambda h} = u$ in $W^{k,p}(\Omega)$. $\square$

Example 3.1. Let Ω be the disc in $\mathbb{R}^2$ with center at origin and radius 1, without the segment $0 \leq x_1 \leq 1, x_2 = 0$. Then $\overline{C^\infty(\overline{\Omega})} \neq W^{k,p}(\Omega)$, if $k \geq 1$, $p \geq 2$ (the result is true if $p \le 2$, but at the moment, we are not able to prove it). Indeed, according to Theorem 1.1.2 we can define traces "from top" and "from bottom" on the segment mentioned. If we have $\overline{C^\infty(\overline{\Omega})} = W^{k,p}(\Omega)$, these traces will coincide, but this is not possible in polar coordinates (r,Θ) if we consider the function $r^k\Theta \in W^{k,p}(\Omega)$.

A bounded domain is called *starshaped with respect to the origin* if there exists a positive continuous function on the unit sphere, say $h(x/|x|)$, such that $\Omega = \{x \in \mathbb{R}^N, |x| < h(x/|x|)\}$. We have (cf. V.I. Smirnov [1]):

Theorem 3.2. *Let Ω be a starshaped domain with respect to the origin. Then $W^{k,p}(\Omega) = \overline{C^\infty_{\overline{\Omega}}(\mathbb{R}^N)}$.*

Proof. Indeed: let $u \in W^{k,p}(\Omega)$ and put $u_\lambda(x) = u(\lambda x)$, $0 < \lambda < 1$. According to Theorem 1.1, $\lim_{\lambda\to 1} u_\lambda = u$ in $W^{k,p}(\Omega)$. Denoting $\Omega_\lambda = \{x \in \mathbb{R}^N,\ x = y/\lambda, y \in \Omega\}$, we have $u_\lambda \in W^{k,p}(\Omega_\lambda)$, but $\overline{\Omega} \subset \Omega_\lambda$. Obviously, for every $\varepsilon > 0$, there exist λ, $h > 0$ such that $|u_{\lambda h} - u|_{W^{k,p}(\Omega)} < \varepsilon$. □

In the statements of Theorems 3.1, 3.2, the properties of Ω are used; but it is possible to generalize these theorems in the following form:

Theorem 3.3. *Let Ω be a bounded domain such that there exists a sequence of domains Ω_n, $n = 1,2,\dots$ such that $\overline{\Omega} \subset \Omega_n$, $\Omega_n \supset \Omega_{n+1}$; $\cap_{n=1}^\infty \Omega_n = \Omega$, and let us assume that for every $u \in W^{k,p}(\Omega)$, there exists $u_n \in W^{k,p}(\Omega_n)$ such that $\lim_{n\to\infty} |u_n - u|_{W^{k,p}(\Omega)} = 0$. Then $\overline{C^\infty_{\overline{\Omega}}(\mathbb{R}^N)} = W^{k,p}(\Omega)$.*

Proof. Let $u_n = 0$ outside of Ω_n; according to Theorem 2.1 we can find a sequence h_n such that $\lim_{n\to\infty} |u_{nh_n} - u_n|_{W^{k,p}(\Omega)} = 0$. □

2.3.3 *The Gagliardo Lemma*

Let us prove a lemma due to E. Gagliardo [2]:

Lemma 3.3. *Let C be the cube $(-1,1)^N$, let C_i denote its faces $(-1,1)^{N-1}$ for $x_i = 0$. Let $f_i \in L^{N-1}(C_i)$, $i = 1,2,\dots,N$, and define f_i in C by $f_i(x) = f_i(x_1,\dots,x_{i-1},x_{i+1},\dots,x_N)$. Then*

$$\int_C \Big|\prod_{i=1}^N f_i\Big| dC \le \prod_{i=1}^N \Big(\int_{C_i} |f_i|^{N-1} dC_i\Big)^{1/(N-1)}. \tag{2.28}$$

Proof. We use a recurrence process over N. For $N = 2$ inequality (2.28) is a simple consequence of Fubini's theorem:

$$\int_C |f_1||f_2| dC = \int_{-1}^1 |f_1(x_2)|\,\mathrm{d}x_2 \int_{-1}^1 |f_2(x_1)|\,\mathrm{d}x_1.$$

Let be $N > 2$, and let us assume that (2.28) holds for $N-1$; we have:

$$I = \int_C |f_1 f_2 \ldots f_N| dC = \int_{C_1} |f_1| dC_1 \int_{-1}^{1} |f_2 f_3 \ldots f_N| \, \mathrm{d}x_1.$$

Using the Hölder inequality for $p_i = N-1$, $i = 1,2,\ldots,N-1$, we get:

$$I \le \int_{C_1} |f_i| dC_1 \prod_{i=2}^{N} \Big(\int_{-1}^{1} |f_i|^{N-1} \, \mathrm{d}x_1 \Big)^{1/(N-1)}. \tag{2.29}$$

Now using again the Hölder inequality for $p = N-1$, $q = (N-1)/(N-2)$ in (2.29), we get:

$$I \le \big(\int_{C_1} |f_1| dC_1 \big)^{1/(N-1)} \big(\int_{C_1} \prod_{i=2}^{N} \big(\int_{-1}^{1} |f_i|^{N-1} \, \mathrm{d}x_1 \big)^{1/(N-2)} dC_1 \big)^{(N-2)/(N-1)}. \tag{2.30}$$

Using now the recurrence hypothesis, denoting $C_{1i}, i = 2,\ldots,N$, the projections of C_1 on the hyperplane $x_i = 0$, we finally get:

$$\begin{aligned} &\int_{C_1} \prod_{i=2}^{N} \big(\int_{-1}^{1} |f_i|^{N-1} \, \mathrm{d}x_1 \big)^{1/(N-2)} dC_1 \\ &\quad \le \prod_{i=2}^{N} \big(\int_{C_{1i}} dC_{1i} \int_{-1}^{1} |f_i|^{N-1} \, \mathrm{d}x_1 \big)^{1/(N-2)} = \prod_{i=2}^{N} \big(\int_{C_i} |f_i|^{N-1} dC_i \big)^{1/(N-2)}. \end{aligned}$$

Then taking into account (2.30), we get (2.28). □

2.3.4 The Sobolev Imbedding Theorems

Now, let us start with the first of the *imbedding theorems*; basically these theorems are due to Sobolev (cf. S.L. Sobolev [1].) Here we use an adaptation of the method of Gagliardo (cf. E. Gagliardo [2]). Let us recall that if for two Banach spaces $B_1, B_2, B_1 \subset B_2$ is an imbedding algebraically and topologically, then this means that each element in B_1 is an element of B_2, and for every $x \in B_1$, $|x|_{B_2} \le c|x|_{B_1}$. We use the notation introduced in 1.2.4.

Theorem 3.4. *Let $\Omega \in \mathfrak{N}^{0,1}$, $1 \le p < N$. If $1/q = 1/p - 1/N$, then $W^{1,p}(\Omega) \subset L^q(\Omega)$ algebraically and topologically.*

Proof. It is sufficient to prove that $u_r = u\varphi_r \in L^q(V_r)$. We define a mapping T on the cube $C = \{|y_i| < 1, i = 1,2,\ldots,N\}$, $T : C \to V_r$ using $x = (x'_r, x_{rN})$, $y = (y'_r, y_{rN})$; for simplicity we omit the index r, and we set:

$$x' = \alpha y', \quad x_N = (\beta/2) y_N + a(\alpha y') + \beta/2. \tag{2.31}$$

The mapping T and its inverse are lipschitzian between $\overline{C}$ and $\overline{V}$. According to Lemma 3.2 it is sufficient to prove that $v \in L^q(C)$ where $v(y) = u(T(y))$, and the corresponding inequality. Let $v \in W^{1,p}(C)$, and equal zero in a neighborhood of the sides of the cube, except the side $y_N = 1$, $|y_i| < 1$, $i = 1,2,\ldots,N-1$, cf. Fig. 2.1

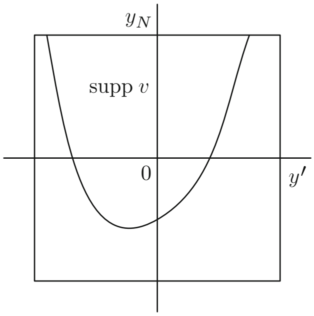

Fig. 2.1

Theorem 2.1 implies the existence of a sequence v_l, $\lim_{l\to\infty} v_l = v$ in $W^{1,p}(C), v_l \in C^\infty(\overline{C})$ with the support mentioned. We prove first the following inequality:

$$|v_l|_{L^q(C)} \le c|v_l|_{W^{1,p}(C)}, \tag{2.32}$$

then we can pass to the limit as $l \to \infty$ and we obtain:

$$|v|_{L^q(C)} \le c|v|_{W^{1,p}(C)}. \tag{2.33}$$

Let $v \in C^\infty(\overline{\Omega})$ with the support mentioned and let us consider the function:

$$y_i \to |v(y)|^{(Np-p)/(N-p)}$$

as a function of the variable y_i (all other local charts are fixed). On the interval $(-1,1)$ we have almost everywhere:

$$\frac{\partial}{\partial y_i}|v(y)|^{(Np-p)/(N-p)} \le \frac{Np-p}{N-p}|v|^{(Np-N)/(N-p)}\left|\frac{\partial v}{\partial y_i}\right|; \tag{2.34}$$

in fact, it is sufficient to use the inequality

$$\left|\frac{d|f|}{dt}\right| \le \left|\frac{df}{dt}\right| \tag{2.35}$$

which holds for an absolutely continuous function almost everywhere on $(-1,1)$. Using (2.34) we get:

$$\sup_{|y_i|\le 1} |v(y)|^{(Np-p)/(N-p)} \le \frac{Np-p}{N-p}\int_{-1}^{1} |v|^{(Np-N)/(N-p)} \left|\frac{\partial v}{\partial y_i}\right| \mathrm{d}y_i. \tag{2.36}$$

Let C_i be the projection of C on the hyperplane $y_i = 0$, $p > 1$. Using Hölder's inequality we get from (2.36),

$$\begin{aligned}
&\int_{C_i} \sup_{|y_i|\le 1} |v(y)|^{(Np-N)/(N-p)} dC_i \\
&\le \frac{Np-p}{N-p}\left(\int_C |v|^{Np/(N-p)} dC\right)^{(p-1)/p} \left(\int_C \left|\frac{\partial v^p}{\partial y_i}\right| dC\right)^{1/p} \\
&\le \frac{Np-p}{N-p}\left(\int_C |v|^{Np/(N-p)} dC\right)^{(p-1)/p} |v|_{W^{1,p}(C)}.
\end{aligned} \tag{2.37}$$

Taking into account Lemma 3.3 and (2.37), it follows that

$$\begin{aligned}
\int_C |v|^{Np/(N-p)} dC &\le \int_C \prod_{i=1}^{N} \sup_{|y_i|<1} |v|^{p/(N-p)} dC \\
&\le \prod_{i=1}^{N} \left(\int_{C_i} \sup_{|y_i|<1} |v|^{(Np-p)/(N-p)} dC_i\right)^{1/(N-1)} \\
&\le \left(\frac{Np-p}{N-p}\right)^{N/(N-1)} \left(\int_C |v|^{Np/(N-p)} dC\right)^{(Np-N)/(Np-p)} |v|^{N/(N-1)}_{W^{1,p}(C)},
\end{aligned} \tag{2.38}$$

and hence

$$\left(\int_C |v|^{Np/(N-p)} dC\right)^{(N-p)/Np} \le \frac{Np-p}{N-p} |v|_{W^{1,p}(C)}. \tag{2.39}$$

Going back to the index r, for $r = m+1$ we cover U_{m+1} by a finite number of cubes and using a partition of unity we again arrive at the inequality (2.39), and we have the result for $p > 1$. If $p = 1$ then (2.36) becomes:

$$\sup_{|y_i|<1} |v(y)| \le \int_{-1}^{1} \left|\frac{\partial v}{\partial y_i}\right| \mathrm{d}y,$$

and by Lemma 3.3, we obtain

$$\int_C |v|^{N/(N-1)} dC \le \int_C \prod_{i=1}^{N} \sup_{|y_i|<1} |v|^{1/(N-1)} dC$$

$$\le \prod_{i=1}^{N} \left(\int_{C_i} \sup_{|y_i|<1} |v| dC_i\right)^{1/(N-1)} \le \prod_{i=1}^{N} \left(\int_C \left|\frac{\partial v}{\partial y_i}\right| dC\right)^{1/(N-1)} \le |v|^{N/(N-1)}_{W^{1,1}(C)},$$

which completes the proof. □

Remark 3.1. Theorem 3.4 is obviously true for a finite union of domains of the type $\mathfrak{N}^{0,1}$.

A domain has the *interior cone property* if there exists a fixed cone such that each point of Ω is the vertex of this cone appropriately placed into Ω. We can easily prove that such a domain can be decomposed into a finite union of domains of the type $\mathfrak{N}^{0,1}$, cf. E. Gagliardo [2].

Remark 3.2. In J.L. Lions [5], Theorem 3.4 is proved for $\Omega = \mathbb{R}^N$. The proof follows the same ideas.

Example 3.2. The number $q = Np/(N-p)$ in Theorem 3.4 is the best possible: the function $u(x) = |x|^{-(N/2)+1}\ln^{-1}|x|$ defined on the ball $\Omega = \{x \in \mathbb{R}^N, |x| < 1/2\}$, $N \geq 3$, is in $W^{1,2}(\Omega)$; on the other hand,

$$\int_\Omega |u|^{(2N/(N-2))+\varepsilon}\,\mathrm{d}x = \infty \quad \text{for} \quad \varepsilon > 0.$$

Theorem 3.5. *Let $\Omega \in \mathfrak{N}^{0,1}, p = N$. Obviously $W^{1,p}(\Omega) \subset L^q(\Omega)$ algebraically and topologically for any $q, 1 \leq q < \infty$.*

Theorem 3.6. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p \geq 1$, $kp < N$. Put $1/q = 1/p - k/N$. Then $W^{k,p}(\Omega) \subset L^q(\Omega)$ algebraically and topologically.*

Proof. We proceed by recurrence with respect to k: the theorem is true if $k = 1$; we assume that it is true for $k-1$. Then $D^\alpha u \in W^{1,p}(\Omega)$, $|\alpha| \leq k-1$, hence $D^\alpha u \in L^{q^*}(\Omega)$ with $1/q^* = 1/p - 1/N \Longrightarrow u \in W^{k-1,q^*}(\Omega) \Longrightarrow u \in L^q(\Omega)$. $\square$

Obviously, we have also

Theorem 3.7. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p \geq 1$, $kp = N$. Then $W^{k,p}(\Omega) \subset L^q(\Omega)$ algebraically and topologically for any q, $1 \leq q < \infty$.*

2.3.5 The Sobolev Imbedding Theorems (Continuation)

Theorem 1.1.11 is a particular case of an imbedding theorem if $kp > N$. Now we follow the ideas of C.B. Morrey [3]. Hereafter we shall say that for a *Banach space B, $B \subset C^0(\overline{\Omega})$ algebraically and topologically* if every function $f \in B$ (where B is a subspace of measurable functions on Ω) can be modified on a set of measure zero in such a way that this modified function is absolutely continuous on $\overline{\Omega}$; moreover we have:

$$\max_{x\in\overline{\Omega}} |f(x)| \leq c|f|_B.$$

Theorem 3.8. *Let $\Omega \in \mathfrak{N}^{0,1}, p \geq 1, kp > N$, and denote*

$$\mu \begin{cases} = k-(N/p) & \textit{if} \quad k-(N/p)<1, \\ <1 & \textit{if} \quad k-(N/p)=1, \\ =1 & \textit{if} \quad k-(N/p)>1. \end{cases}$$

Then $W^{k,p}(\Omega) \subset C^{0,\mu}(\overline{\Omega})$ *algebraically and topologically.*

Proof. We proceed as in the proof of Theorem 3.4: we consider u_r, $r \le m$; obviously $u_r \in W^{k,p}(V_r)$, and

$$|u_r|_{W^{k,p}(V_r)} \le c_1 |u|_{W^{k,p}(\Omega)}. \tag{2.40}$$

Let us consider the case $k-(N/p)<1$; we have $(k-1)p<N$, and thus, by Theorem 3.6,

$$|u_r|_{W^{1,q}(V_r)} \le c_2 |u|_{W^{k,p}(V_r)} \tag{2.41}$$

where $1/q = 1/p-(k-1)/N$. For simplicity we omit the index r. If we use the mapping (2.31) we have to consider $u \in W^{1,q}(C)$, and due to Theorem 3.1, it is sufficient to assume that $u \in C^\infty(\overline{C})$, and to prove the inequality

$$|u|_{C^{0,\mu}(\overline{C})} \le c_3 |u|_{W^{1,q}(C)}. \tag{2.42}$$

To do this, let $y_{[1]}, y_{[2]} \in \overline{C}$; it is always possible to find a cube C_ρ with faces parallel to the faces of C such that $y_{[1]}, y_{[2]} \in \overline{C_\rho} \subset \overline{C}$ and with sides of length equal to ρ, $\rho \le |y_{[1]} - y_{[2]}| \le \sqrt{N}\rho$, cf. Fig. 2.2 ($N=2$).

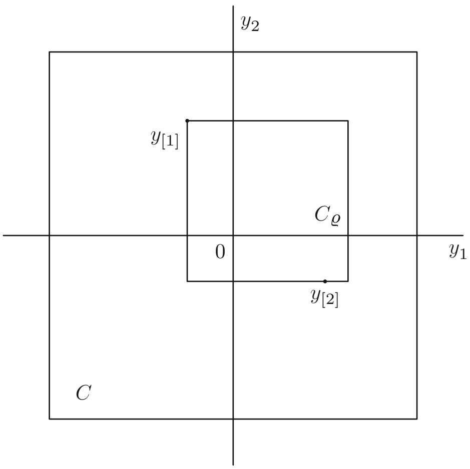

Fig. 2.2

Let $y \in C_\rho$. We have:

$$|u(y) - u(y_{[j]})| = \left| \int_0^1 \sum_{i=1}^N \frac{\partial u}{\partial y_i}(y_{[j]} + t(y - y_{[j]}))(y_i - y_{[j]i})\,\mathrm{d}t \right|$$
$$\leq c_4\rho \int_0^1 \sum_{i=1}^N \left| \frac{\partial u}{\partial y_i}(y_{[j]} + t(y - y_{[j]})) \right| \mathrm{d}t,$$

and then using the change of variables $z = y_{[j]} + t(y - y_{[j]})$, we get:

$$\left| \frac{1}{\rho^N} \int_{C_\rho} u(y)\,\mathrm{d}y - u(y_{[j]}) \right| \leq \frac{1}{\rho^N} \int_{C_\rho} |u(y) - u(y_{[j]})|\,\mathrm{d}y$$
$$\leq c_4 \rho^{-N+1} \int_0^1 t^{-N} dt \int_{C_{\rho t}} \sum_{i=1}^N \left| \frac{\partial u}{\partial y_i}(z) \right| dz, \tag{2.43}$$

where

$$C_{\rho t} = \{z \in C_\rho, z = y_{[j]} + t(y - y_{[j]}), y \in C_\rho\}.$$

Now from the Hölder inequality and from (2.43),

$$\left| \frac{1}{\rho^N} \int_{C_\rho} u(y)\,\mathrm{d}y - u(y_{[j]}) \right| \leq c_5 |u|_{W^{1,q}(C)} \rho^{k-(N/p)},$$

hence

$$|u(y_{[1]}) - u(y_{[2]})| \leq 2c_5 |u|_{W^{1,q}(C)} \rho^{k-(N/p)} \leq 2c_5 |u|_{W^{1,q}(C)} |y_{[1]} - y_{[2]}|^{k-(N/p)}. \tag{2.44}$$

Let $y_{[0]}, y \in C$; we have:

$$|u(y_{[0]})| \leq |u(y)| + \int_0^1 \sum_{i=1}^N \left| \frac{\partial u}{\partial y_i}(y_{[0]} + t(y - y_{[0]}))(y_i - y_{[0]i}) \right| \mathrm{d}t,$$

then by integration with respect to y over C we obtain as above:

$$|u(y_{[0]})| \leq c_6 |u|_{W^{1,q}(C)}, \tag{2.45}$$

and with (2.44) we have (2.42).

Concerning u_{m+1} it is sufficient to cover U_{m+1} by a finite number of cubes contained in Ω, and using a partition of unity, we obtain again (2.42). Hence, the case $k - (N/p) < 1$ is proved.

If $k - (N/p) = 1$, we use $p' < p$, and the result follows.

If $k - (N/p) > 1$, there exists a positive integer m such that $k - (N/p) - 1 \leq m < k - (N/p) \Longrightarrow 0 < k - m - (N/p) \leq 1$, then $D^\alpha u \in W^{k-m,p}(\Omega)$, $|\alpha| \leq m; D^\alpha u = v$

is continuous, the case $0 < k - (N/p) < 1$ having been proved, thus $u \in C^{0,1}(\overline{\Omega})$; the theorem is proved completely. □

Remark 3.3. As an immediate consequence of the previous theorems, we obtain imbedding theorems for the derivatives. For instance, if $u \in W^{k,p}(\Omega)$, $\Omega \in \mathfrak{N}^{0,1}$, $(k-m)p < N \Longrightarrow D^\alpha u \in L^q(\Omega)$ with $1/q = 1/p - (k-m)/N$ and $|\alpha| \leq m$.

Example 3.3. Theorem 3.4 does not hold if for instance Ω is a domain in $\mathbb{R}^2$ and its boundary contains a sharp cuspidal point: Assume

$$\Omega = \{x \in \mathbb{R}^2, 0 < x_1 < 1, |x_2| < x_1^{2p} \exp(-p/x_1)\}, \quad 1 \leq p < 2,$$

and $u(x) = \exp(1/x_1)$. We have:

$$\int_\Omega |u|^p \, \mathrm{d}x = 2\int_0^1 \exp(p/x_1)\exp(-p/x_1)x_1^{2p}\, \mathrm{d}x_1 < \infty,$$

$$\int_\Omega \left|\frac{\partial u}{\partial x_1}\right|^p \mathrm{d}x = 2\int_0^1 \frac{1}{x_1^{2p}} \exp(p/x_1)\exp(-p/x_1)x_1^{2p}\, \mathrm{d}x_1 < \infty,$$

on the other hand, if $q > p$,

$$\int_\Omega |u|^q \, \mathrm{d}x = 2\int_0^1 \exp(q/x_1)\exp(-p/x_1)x_1^{2p}\, \mathrm{d}x_1 = \infty,$$

Concerning the imbedding theorems if $\Omega = \mathbb{R}^N$ cf. J.L. Lions [5]; if Ω is unbounded, cf. J. Deny, J.L. Lions [1]; for Ω unbounded and the estimate of type (1.1.4); cf. also the previous paper.

2.3.6 Extension, the Nikolskii Method

Let $u \in W_0^{k,p}(\Omega)$. We define the *extension of* u *on* $\mathbb{R}^N$: let $u = \lim_{n\to\infty} \varphi_n$, $\varphi_n \in C_0^\infty(\Omega)$. φ_n is a Cauchy sequence in $W^{k,p}(\Omega)$, thus in $W^{k,p}(\mathbb{R}^N)$, the sequence converges to $v \in W^{k,p}(\mathbb{R}^N)$. Obviously, the restriction operator (denote it by R) which maps a function from $W^{k,p}(\mathbb{R}^N)$ onto its restriction on Ω satisfies $Rv = u$. We write $v = Pu$ where P is the extension operator $P \in [W_0^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^N)]$; P is linear and continuous.

Corollary 3.1. *Let* Ω *be a bounded domain. Then Theorems 3.4-3.7 hold if we replace* $W^{k,p}(\Omega)$ *by* $W_0^{k,p}(\Omega)$.

Proof. Indeed, take a cube $C \supset \overline{\Omega}$; for the extension P, we have $P \in [W_0^{k,p}(\Omega) \to W^{k,p}(C)]$. □

If $\Omega \subset \Omega_1$ and $u \in W^{k,p}(\Omega_1)$, we have $u \in W^{k,p}(\Omega)$ and we can formulate a question:

Problem 3.2. For what domains Ω does $R(W^{k,p}(\mathbb{R}^N)) = W^{k,p}(\Omega)$?

A.P. Calderon [2] proved that this equality holds for $\Omega \in \mathfrak{N}^{0,1}$, $p > 1$; we shall prove it in the case $\Omega \in \mathfrak{N}^{k-1,1}, p \geq 1$ and for $\Omega \in \mathfrak{N}^{0,1}$, $p = 2$.

Lemma 3.4. *Let Ω, O be two bounded open sets and T a one-to-one continuous mapping, $T : O \to \Omega$. We assume that T is $(k-1)$-times continuously differentiable in $\overline{O}$, the derivatives of order $k-1$ are lipschitzian in $\overline{O}$; we assume also T^{-1} lipschitzian in $\overline{\Omega}$. Let $u \in W^{k,p}(\Omega), v(y) = u(T(y))$. Then we have:*

$$|v|_{W^{k,p}(O)} \leq c|u|_{W^{k,p}(\Omega)}. \tag{2.46}$$

Proof. Let $u \in W^{k,p}(\Omega)$, we put $u = 0$ outside of Ω, and let u_h be the regularized function. Let $v_{[h]}(y) = u_h(T(y))$. We can compute the derivatives of order $\leq (k-1)$ of $v_{[h]}$ in the usual sense; for $|\alpha| = k-1, D^\alpha v_{[h]}$ is an absolutely continuous function on all lines parallel to the axes y_i. For $|\alpha| \leq k$, the derivatives $D^\alpha v_{[h]}$, taken in the distribution sense, are bounded on $\overline{\Omega}$. If $|\alpha| \leq k$, $D^\alpha v_{[h]}$ is a Cauchy sequence in $L^p(\Omega^*)$ due to Lemma 3.1. We deduce $v \in W^{k,p}(\Omega^*)$; according to formulae as (2.26) for $D^\alpha v, |\alpha| \leq k$, and according to Lemma 3.1 the result follows. □

The following theorem is based on an idea of S.M. Nikolskii, cf. V.M. Babich [1].

Theorem 3.9. *Let $\Omega \in \mathfrak{N}^{k-1,1}$, $1 \leq p < \infty$. The extension operator P exists and maps $W^{k,p}(\Omega)$ linearly and continuously into $W^{k,p}(\mathbb{R}^N)$.*

Proof. Let $u_r = u\varphi_r$, the function u_{m+1} belongs to $W_0^{k,p}(\Omega)$, hence we extend it by zero outside of Ω. Let us consider $u_r, r \leq m$, and omit the index r. Let T be the mapping of the prism

$$K = \{y \in \mathbb{R}^N, y = (y', y_N), |y_i| < \alpha, i = 1, 2, \ldots, N-1, 0 < y_N < \beta\}$$

to V (cf. 1.2.4, Chap. 1), defined by

$$x' = y', \quad x_N = y_N + a(y'). \tag{2.47}$$

All hypotheses in Lemma 3.4 are satisfied, $u \in W^{k,p}(K)$ and the inequality (2.46) holds. We denote:

$$K' = \{y \in \mathbb{R}^N, |y_i| < \alpha, i = 1, 2, \ldots, N-1, |y_N| < \beta\},$$

$$K_+ = \{y \in \mathbb{R}^N, |y_i| < \alpha, i = 1, 2, \ldots, N-1, 0 < y_N < k\beta\},$$

$$K_- = \{y \in \mathbb{R}^N, |y_i| < \alpha, i = 1, 2, \ldots, N-1, -\beta < y_N < 0\}.$$

Let us extend u by zero outside of K on K_+, and put for $y_N < 0$

$$u(y', y_N) = \sum_{i=1}^{k} \lambda_i u(y', -iy_N), \tag{2.48}$$

where the coefficients λ_i are defined by

$$1 = \sum_{i=1}^{k} (-i)^{\alpha} \lambda_i, \quad \alpha = 0, 1, \ldots, k-1. \tag{2.49}$$

The determinant of this linear system is not equal to zero, thus the λ_i's are uniquely determined. Clearly $u \in W^{k,p}(K_-)$, but we have also $u \in W^{k,p}(K')$. To see this let us consider $D^{\alpha}u$, $|\alpha| \leq k$. Assume that $u \in C^{\infty}(K)$; according to Theorem 3.1, if we pass to the limit we get $u \in W^{k,p}(K)$. Let $\varphi \in C_0^{\infty}(K')$, and let us consider:

$$\int_{K'} D^{\alpha}\varphi(y)u(y)\,\mathrm{d}y = \int_K D^{\alpha}\varphi(y)u(y)\,\mathrm{d}y + \sum_{i=1}^{k} \lambda_i \int_{K_-} D^{\alpha}\varphi(y)u(y', -iy_N)\,\mathrm{d}y. \tag{2.50}$$

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ and let us set $\alpha = \alpha' + \alpha''$, $\alpha'' = (0, \ldots, 0, \alpha_N)$. Now using integration by parts in (2.50) we get:

$$\begin{aligned}
\int_{K'} D^{\alpha}\varphi(y)u(y)\,\mathrm{d}y &= (-1)^{|\alpha'|} \int_K \frac{\partial^{\alpha_N}}{\partial y_N^{\alpha_N}} D^{\alpha'} u(y)\,\mathrm{d}y \\
&\quad + (-1)^{|\alpha'|} \sum_{i=1}^{k} \lambda_i \int_{K_-} \frac{\partial^{\alpha_N}\varphi(y)}{\partial y_N^{\alpha_N}} D^{\alpha'} u(y', -iy_N)\,\mathrm{d}y \\
&= (-1)^{|\alpha|} \int_K \varphi(y) D^{\alpha}u(y)\,\mathrm{d}y + (-1)^{|\alpha|} \int_{K_-} \varphi(y) \sum_{i=1}^{k} \lambda_i (-i)^{\alpha'} D^{\alpha} u(y', -iy_N)\,\mathrm{d}y \\
&\quad + \sum_{j=1}^{\alpha_N} (-1)^j \Big(\sum_{i=1}^{k} \lambda_i (-i)^{j-1} - 1\Big) \int_{\Delta} \frac{\partial^{\alpha_N - j}\varphi(y', 0)}{\partial y_N^{\alpha_N - j}} \frac{\partial^{j-1}}{\partial y_N^{j-1}} D^{\alpha'} u(y', 0)\,\mathrm{d}y.
\end{aligned} \tag{2.51}$$

According to (2.49), the last term of (2.51) is equal to zero, then

$$|u|_{W^{k,p}(K')} \leq c|u|_{W^{k,p}(K)}. \tag{2.52}$$

The mapping (2.47) and its inverse satisfy the hypotheses of Lemma 3.4 for $\Omega = U$, $O = K'$.

We obtain the extension of u on U such that $u \in W^{k,p}(U)$ and u is equal to zero in a neighborhood of ∂U. If we put $u = 0$ outside of U we have constructed an *extension operator*. Now we go back to the index r, and denote this extension operator by P_r; we have the result if we put $Pu = \sum_{r=1}^{m+1} P_r u_r$. $\square$

2.3.7 Extension, the Calderon Method

We describe the method of A.P. Calderon only for $p = 2$: we shall introduce some simplifications as in the work of J. Nečas [9]. We shall finish with remarks concerning the case $p \neq 2$.

Let $a(x')$ be a lipschitzian function in the cube $\Delta = \{|x_i| < \alpha, i = 1, 2, \ldots, N-1\}$. Let us denote by $C(y)$ the cone $x_N > y_N$, $|x' - y'| < \kappa(x_N - y_N)$. We can find κ sufficiently small such that the set $\Lambda = \{x \in \mathbb{R}^N, x' \in \Delta, x_N = a(x')\}$ has an empty intersection with $C((x', a(x')))$, $x' \in \Delta$. Let us denote $K_\infty = \cup_{x' \in \Delta} C((x', a(x')))$; we choose $\gamma > 0$ such that $x' \in \overline{\Delta} \Longrightarrow a(x') < \gamma$, and define $K = \{x \in K_\infty, x_N < \gamma\}$; cf. Fig. 2.3.

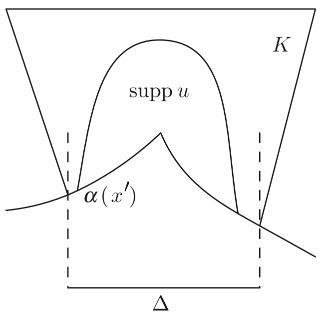

Fig. 2.3

Let $u \in C^\infty(\overline{K})$ be a function with support in $K \cup \Lambda$, cf. Fig. 2.3. We denote by S the unit sphere with center at the origin O and let $\sigma \in S \cap C(0)$. Now for $x \in K$, we have:

$$\int_0^\infty t^{k-1} \frac{d^k}{dt^k}(\exp(-t)u(x+t\sigma))\,\mathrm{d}t = (-1)^k(k-1)!u(x). \tag{2.53}$$

Let us denote $\alpha! = \alpha_1!\alpha_2!\ldots\alpha_N!$, $\sigma^\alpha = \sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\ldots\sigma_N^{\alpha_N}$; we get:

$$\frac{d^k}{dt^k}(\exp(-t)u(x+t\sigma)) = \exp(-t)(-1)^k \sum_{|\alpha|\le k} D^\alpha u(x+t\sigma)\sigma^\alpha \frac{|\alpha|!}{\alpha!}\binom{k}{|\alpha|}(-1)^{|\alpha|}. \tag{2.54}$$

Now let $\nu(\sigma)$ be a real infinitely differentiable function on S with support in $S \cap C(0)$, $\nu(\sigma) \geq 0$, $\int_S \nu(\sigma)\,\mathrm{d}S = 1$. From (2.53) and (2.54) we get:

$$\begin{aligned} u(x) =& \frac{1}{(k-1)!}\int_S \nu(\sigma)dS \int_0^\infty t^{k-1}\exp(-t)(-1)^k \\ &\times \sum_{|\alpha|\le k}(-1)^{|\alpha|}\frac{|\alpha|!}{\alpha!}\binom{k}{|\alpha|}\sigma^\alpha D^\alpha u(x+t\sigma)dt, \end{aligned}$$

and the substitution $y = x + t\sigma$ leads to

$$u(x) = \frac{1}{(k-1)!}\int_{R^N}\left[\sum_{|\alpha|\le k}\frac{|\alpha|!}{\alpha!}(\binom{k}{|\alpha|})\frac{(x-y)^\alpha}{|x-y|^{|\alpha|+N-k}}D^\alpha u(y)\right] \times \exp(-|x-y|)\mu\left(\frac{x-y}{|x-y|}\right)\mathrm{d}y, \tag{2.55}$$

where $\mu(x) = \nu(-x)$; formula (2.55) is an adaptation of the Sobolev identity, cf. S.L. Sobolev [1].

If $f \in L^1(\mathbb{R}^N)$, we denote by $\hat{f}(\xi)$ the Fourier transform of f defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^N}\exp(-\mathrm{i}(x,\xi))f(x)\,\mathrm{d}x;$$

if $f \in L^2(\mathbb{R}^N), \hat{f}$ is defined by the Plancherel method (cf. S. Bochner, K. Chandrasekharan [1]). We have

Lemma 3.5.

$$u \in W^{k,2}(\mathbb{R}^N) \Longleftrightarrow \int_{\mathbb{R}^N}|\hat{u}(\xi)|^2(1+|\xi|^2)^k\,\mathrm{d}\xi < \infty,$$

and

$$c_1|u|_{W^{k,2}(\mathbb{R}^N)} \le \left(\int_{\mathbb{R}^N}|\hat{u}(\xi)|^2(1+|\xi|^2)^k\,\mathrm{d}\xi\right)^{1/2} \le c_2|u|_{W^{k,2}(\mathbb{R}^N)}.$$

Indeed, it is sufficient to use the Parseval identity (cf. the book of S. Bochner, K. Chandrasekharan [1]); for $f, g \in L^2(\mathbb{R}^N)$, we have:

$$\int_{\mathbb{R}^N} f\overline{g}\,\mathrm{d}x = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}\hat{f}\overline{\hat{g}}\,\mathrm{d}\xi,$$

by the definition of $W^{k,2}(\mathbb{R}^N)$ in 2.2.3 and by the well known properties of the Fourier transform such as

$$\varphi \in C_0^\infty(\mathbb{R}^N) \Longrightarrow \widehat{\left(\frac{\partial\varphi}{\partial x_i}\right)}(\xi) = i\xi_i\hat{\varphi}(\xi), \text{ etc.}$$

Remark 3.4. If $\Omega = \mathbb{R}^N$, then from Lemma 3.5 follows easily a particular case of Theorem 3.8, i.e.: if $2k > N$, then $W^{k,2}(\mathbb{R}^N) \subset C^0(\mathbb{R}^N)$ algebraically and topologically. This follows from

$$\int_{\mathbb{R}^N}|\hat{f}(\xi)|\,\mathrm{d}\xi \le \left(\int_{\mathbb{R}^N}|\hat{f}(\xi)|^2(1+|\xi|^2)^k\,\mathrm{d}\xi\right)^{1/2}\left(\int_{\mathbb{R}^N}\frac{1}{(1+|\xi|^2)^k}\right)^{1/2},$$

and from the inverse transform:

$$f(x) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \hat{f}(\xi) \exp(\mathrm{i}(x,\xi))\, \mathrm{d}\xi.$$

With the notations of this section, $K, \dots$, we have:

Lemma 3.6. *Let $M \subset K \cup \Lambda$ be a closed set, and $u \in W^{k,2}(K)$,* $\operatorname{supp} u \subset M$. *Let us denote by $W \subset W^{k,2}(\Omega)$ the set of functions with support in M. Then there exists a mapping $P \in [W \to W^{k,2}(\mathbb{R}^N)]$ such that $RPu = u$.*

Proof. Let $u \in W \cap C^\infty(\overline{K})$. Let us put for $|\alpha| \le k$:

$$f_\alpha(x) = \begin{cases} D^\alpha u(x) & \text{for} \quad x \in K, \\ 0 \quad \text{for} & x \notin K. \end{cases}$$

For $x \in \mathbb{R}^N$, we define:

$$\begin{aligned} v(x) = \frac{1}{(k-1)!} \int_{\mathbb{R}^N} \left[\sum_{|\alpha| \le k} \frac{|\alpha|!}{\alpha!} \binom{k}{|\alpha|} \frac{(x-y)^\alpha}{|x-y|^{|\alpha|+N-k}} f_\alpha(y) \right] \times \\ \times \exp(-|x-y|) \mu \left(\frac{x-y}{|x-y|} \right) \mathrm{d}y = \sum_{|\alpha| \le k} \int_{\mathbb{R}^N} I_\alpha(x-y) f_\alpha(y)\, \mathrm{d}y. \end{aligned} \tag{2.56}$$

It is clear that $\int_{\mathbb{R}^N} |I_\alpha(x)|\, \mathrm{d}x < \infty$, $|\alpha| \le k$, thus according the to theorem about the Fourier transform of a convolution, we get:

$$\hat{v}(\xi) = \sum_{|\alpha| \le k} \hat{I}_\alpha(\xi) \hat{f}_\alpha(\xi). \tag{2.57}$$

For $|\alpha| \le k$, we have

$$|\hat{I_\alpha}(\xi)| \le C_1 (1 + |\xi|^2)^{-k/2}. \tag{2.58}$$

Indeed, it is sufficient to consider:

$$\begin{aligned} \hat{J}_\alpha(\xi) &= \frac{1}{(k-1)!} \int_{\mathbb{R}^N} \frac{x^\alpha}{|x|^{|\alpha|+N-k}} \exp(-|x|) \mu \left(\frac{x}{|x|} \right) \exp(-\mathrm{i}(x,\xi)\, \mathrm{d}x \\ &= \int_S \frac{\sigma^\alpha \mu(\sigma)}{(1 + \mathrm{i}(\sigma,\xi))^k}\, \mathrm{d}S \end{aligned}$$

Clearly we have:

$$\max_{|\xi| \le 1} |\hat{J_\alpha}(\xi)| \le c_2; \tag{2.59}$$

let $|\xi| \ge 1$. Let us consider the vector $\eta = \xi/|\xi|$. Without loss of generality we can assume the support of μ sufficiently small such that if $\mu(\sigma) \ne 0$, then $(\sigma, \sigma_0) > 1/2$, where $\sigma_0 = (0,0,\ldots,-1)$; then we can find $\varepsilon > 0$ such that if $|\eta_N| \le \varepsilon$, $\eta \notin \operatorname{supp}\mu$, and if $|\eta_N| > \varepsilon$, $\sigma \in \operatorname{supp}\mu, |(\sigma,\eta)| \ge c_3$. In the case $|\eta_N| \le \varepsilon$ let us introduce a new system of charts generated by vectors $\sigma^1, \sigma^2, \ldots, \sigma^N$, such that $-(\sigma^N, \sigma_0) = (1-\eta_N^2)^{1/2}$, $\sigma^1 = \eta$, where the coordinates are $\tau_1, \tau_2, \ldots, \tau_N$. We put $\tau = (\tau', \tau_N)$ and obtain:

$$\hat{f}_\alpha(\xi) = \int_{|\tau'|\le 1} \frac{\lambda(\tau')\,\mathrm{d}\tau'}{(1+\mathrm{i}\tau_1|\xi|)^k},$$

with λ indefinitely differentiable if $|\tau'| < 1$, and with $\operatorname{supp}\lambda$ in $|\tau'| < 1$.

Using integration by parts, we get:

$$\begin{aligned}\int_{|\tau'|\le 1} \frac{\lambda(\tau')\,\mathrm{d}\tau'}{(1+\mathrm{i}\tau_1|\xi|)^k} &= \frac{1}{(k-1)!}\frac{1}{(\mathrm{i}|\xi|)^{k-1}} \int_{|\tau'|<1} \frac{(\partial^{k-1}\lambda/\partial\tau_1^{k-1})\,\mathrm{d}\tau'}{1+\mathrm{i}\tau_1|\xi|} \\ &= \frac{1}{(k-1)!}\frac{1}{(\mathrm{i}|\xi|)^{k-1}} \int_{|\tau'|<1} \frac{(\partial^{k-1}\lambda/\partial\tau_1^{k-1})(1-\mathrm{i}\tau_1|\xi|)\,\mathrm{d}\tau'}{1+\tau_1^2|\xi|^2}.\end{aligned} \tag{2.60}$$

We have:

$$\left|\int_{|\tau'|<1} \frac{(\partial^{k-1}\lambda/\partial\tau^{k-1})\,\mathrm{d}\tau'}{1+\tau_1^2|\xi|^2}\right| \le c_4 \int_{-1}^{1} \frac{\mathrm{d}\tau_1}{1+\tau_1^2|\xi|^2} \le \frac{c_5}{|\xi|}. \tag{2.61}$$

To simplify the notation, let us set $\delta = \partial^{k-1}\lambda/\partial\tau_1^{k-1}$; we get:

$$\begin{aligned}\int_{|\tau'|<1} \frac{\delta(\tau')|\xi|\tau_1\,\mathrm{d}\tau'}{1+\tau_1^2|\xi|^2} &= \int_{|\tau'|<1} \frac{\delta(0,\tau_2,\ldots,\tau_{N-1})|\xi|\tau_1\,\mathrm{d}\tau'}{1+\tau_1^2|\xi|^2}\,\mathrm{d}\tau' \\ &\quad + \int_{|\tau'|<1} \frac{\delta(\tau')-\delta(0,\tau_2,\ldots,\tau_{N-1})}{1+\tau_1^2|\xi|^2}|\xi|\tau_1\,\mathrm{d}\tau'.\end{aligned} \tag{2.62}$$

The first integral in the right hand side is equal zero, and using in the second integral the inequality $|\delta(\tau') - \delta(0,\tau_2,\ldots,\tau_{N-1})| \le c_6|\tau_1|$ we get

$$\left|\int_{|\tau'|<1} \frac{\delta(\tau')-\delta(0,\tau_2,\ldots,\tau_{N-1})}{1+\tau_1^2|\xi|^2}|\xi|\tau_1 d\tau'\right| \le \frac{c_7}{|\xi|}. \tag{2.63}$$

Inequality (2.58) follows from (2.60) to (2.63).

If $|\eta_N| > \varepsilon$, then $|(\sigma,\eta)| \ge c_3$ for $\sigma \in \operatorname{supp}\mu$, and in this case (2.58) again holds.

Finally, according to (2.57), (2.58) and Lemma 3.5, we get:

$$|v|_{W^{k,2}(\mathbb{R}^N)} \le c_9|u|_{W^{k,2}(K)}, \tag{2.64}$$

and (2.56) defines the extension operator P. □

Now we can prove

Theorem 3.10. *Let $\Omega \in \mathfrak{N}^{0,1}$. Then there exists $P \in [W^{k,2}(\Omega) \to W^{k,2}(\mathbb{R}^N)]$, such that $RPu = u$.*

Proof. We use always φ_r, V_r, etc. as in 1.2.4; for $u\varphi_r$, $r \le m$, we use Lemma 3.6, with $K_r \subset V_r$, such that supp $\varphi_r \subset K_r \cup \Lambda_r$. Let P_r be as in Lemma 3.6, for $r = m+1$ let us define P_{m+1} by 2.3.6; and if we put: $Pu = \sum_{r=1}^{m+1} P_r u$, the result holds. □

Remark 3.5. Theorem 3.9 is true also in the general case $p > 1$, $p \neq 2$. For the proof we can use again (2.56), and in addition the theory of singular operators due to A.P. Calderon, A. Zygmund [1], cf. J.L. Lions [5], or (2.57) and in addition the multiplier theorems, cf. S.G. Mikhlin [4], B. Malgrange [2].

The Nikolskii and Calderon extensions are different at one point: the Nikolskii extension operator can be the same for $W^{k,p}(\Omega)$, $k = 1, 2, \ldots, \kappa$, the Calderon extension depends on k which can be in some situations inconvenient, for instance if we use interpolation, cf. J.L. Lions [5].

Let us formulate an unsolved problem:

Problem 3.3. Is Theorem 3.10 true for $p = 1$?

The questions concerning the existence of extension operators P are closely related with the whole theory of Sobolev spaces $W^{k,p}(\Omega)$; if P exists for one $W^{k,p}(\Omega)$ we can restrict our consideration solely to the case of $W^{k,p}(\mathbb{R}^N)$ as far as concerns density theorems (cf. Theorem 3.1), imbeddings, compactness of imbedding operators, and their consequences such as trace theorems etc. We can also formulate *converse* problems, such as, e.g.,

Problem 3.4. If Ω is bounded and if Theorem 3.4 holds, does P exist?

In some particular cases the answer is negative, for instance: if $\Omega \in \mathfrak{N}^0$, Theorem 3.1 is true, but in general, P does not exist, according to Example 3.3. If P exists, we can see that Ω must be *almost* in $\mathfrak{N}^{0,1}$, etc.

2.3.8 The Spaces $W^{k,p}(\Omega)$, k Non-integer

It is natural, in particular in problems of traces, cf. Chap. 1, Theorem 1.2 and the two following sections, to introduce the spaces $W^{k,p}(\Omega)$, $p \ge 1$, where k is a real number ≥ 0, in general not integer. The definition of these spaces is due to S. Aronszajn [3], L.N. Slobodetskii [1], see also N. Aronszajn, K.T. Smith [1]:

Let Ω be a domain in $\mathbb{R}^N$, $k \ge 0$, $p \ge 1$. If *k is not integer, the space* $W^{k,p}(\Omega)$ is a subspace of $W^{[k],p}(\Omega)$, where $[k]$ is the integer part of k, of functions such that for $|\alpha| = [k]$

$$\int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{N+p(k-[k])}} \, dx \, dy < \infty; \tag{2.65}$$

the norm in $W^{k,p}(\Omega)$ is defined as

$$|u|_{W^{k,p}(\Omega)} = \Big(|u|^p_{W^{[k],p}(\Omega)} + \sum_{|\alpha|=[k]} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{N+p(k-[k])}} \mathrm{d}x\,\mathrm{d}y\Big)^{1/p}. \quad (2.66)$$

For $k \geq 0$ we define also $W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}$.

If $k < 0$, we put $W^{k,p}(\Omega) = (W_0^{-k,q}(\Omega))'$, the dual space of $W_0^{-k,q}(\Omega)$, $1/p + 1/q = 1$.

Clearly we have:

Proposition 3.1. *The space $W^{k,p}(\Omega), k \geq 0$, integer, is a Banach space, separable, and reflexive for $p > 1$.*

Proof. $W^{k,p}(\Omega)$ is obviously complete. We can consider this space as a closed subspace of the product $[L^p(\Omega)]^s \times [L^p(\Omega \times \Omega)]^t$ {where s is as in Proposition 2.2 and t is the number of indices $\alpha, |\alpha| = k$} of elements of the following form:

$$\left(u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial^{[k]} u}{\partial x_N^{[k]}}, \frac{\frac{\partial^{[k]} u(x)}{\partial x_1^{[k]}} - \frac{\partial^{[k]} u(y)}{\partial x_1^{[k]}}}{|x-y|^{N/p+(k-[k])}}, \ldots, \frac{\frac{\partial^{[k]} u(x)}{\partial x_N^{[k]}} - \frac{\partial^{[k]} u(y)}{\partial x_N^{[k]}}}{|x-y|^{N/p+(k-[k])}} \right).$$

This space is separable, and reflexive if $p > 1$, thus the same is true for $W^{k,p}(\Omega)$. □

Actually, the theory of the spaces $W^{k,p}(\Omega)$ can be extended to other spaces whose definition is closely related to $W^{k,p}(\Omega)$; we have the Nikolskii spaces $H^{k,p}$, the Besov spaces $B^{k,p}$ (cf. O.V. Besov [1, 2]), the Lizorkin spaces $L^{k,p}$, the weighted spaces $W_\alpha^{k,p}$, the Morrey spaces $W^{k,p,\lambda}$, etc. For details and a survey cf. S.M. Nikolskii [2], E. Magenes [4], the following section and Chap. 7; concerning the spaces $W^{k,p,\lambda}$ cf. S. Campanato [6, 7].

2.4 The Problem of Traces

2.4.1 Lemmas

We have solved this problem partly in Theorem 1.1.2. Here in this section,we consider the problem in more detail. In fact, most of these theorems are due to S.L. Sobolev [1]; here we shall follow the ideas of E. Gagliardo [1]. We obtain slightly stronger results than S.L. Sobolev. It is important to observe that we are interested in traces on $(N-1)-$dimensional manifolds; for smaller dimension cf. the monograph of S.L. Sobolev [1]; if $p = 2$, it is possible to consult K. Maurin [1].

Let $\Omega \in \mathfrak{N}^{0,1}$ and f a function defined almost everywhere on $\partial\Omega$ which means that $f(x_r', a_r(x_r'))$ is defined almost everywhere in Δ_r, $r = 1, 2, \ldots, m$, cf. 1.2.4. If $f(x_r', a_r(x_r')) \in L^p(\Delta_r)$, $r = 1, 2, \ldots, m$, $p \geq 1$, we say that $f \in L^p(\partial\Omega)$ with the following norm:

$$|f|_{L^p(\partial\Omega)} = \left[\sum_{r=1}^{m} \int_{\Delta_r} |f(x'_r, a_r(x'_r))|^p \, \mathrm{d}x'_r\right]^{1/p}. \quad (2.67)$$

[4]

Theorem 4.1. *The space $L^p(\partial\Omega)$ is a separable Banach space.*

Proof. We must prove that $L^p(\partial\Omega)$ is complete. Indeed, let f_i be a Cauchy sequence. We have $\lim_{i\to\infty} f_i(x'_r, a_r(x'_r)) = f_{[r]}(x'_r, a_r(x'_r))$; let $U_r \cap U_s \cap \partial\Omega = M \neq \emptyset$. Let P_r (resp. P_s) be the projection of M onto the plane $x_{rN} = 0$ (resp. $x_{sN} = 0$), and T the mapping $T : \overline{P}_s \to \overline{P}_r$ defined for $x'_s \in P_s$ by $T(x'_s) = x'_r$ (e.g. x'_r and x'_s are projections of the same point on m to Δ_r and Δ_s, resp). If it is necessary we can rotate the coordinate system (x'_s, x_{sN}) (resp. (x'_r, x_{rN})) to get:

$$x_{ri} = x_{si} + \lambda_i, \quad i = 1, 2, \ldots, N-2,$$
$$x_{rN-1} = x_{sN-1}\cos\varphi - x_{sN}\sin\varphi + \lambda_{N-1}, x_{rN} = x_{sN-1}\sin\varphi + x_{sN}\cos\varphi + \lambda_N, \quad (2.68)$$

where φ is the angle between the axes x_{sN}, x_{rN}. The mapping $T : \overline{P_s} \to \overline{P_r}$ is one-to-one and lipschitzian as well as T^{-1}; indeed:

$$\begin{aligned}
|x'_r - y'_r|^2 &= \sum_{i=1}^{N-2} (x_{ri} - y_{ri})^2 + (x_{rN-1} - y_{rN-1})^2 \\
&= \sum_{i=1}^{N-2} (x_{si} - y_{si})^2 + [(x_{sN-1} - y_{sN-1})\cos\varphi - (a_s(x'_s) - a_s(y'_s))\sin\varphi]^2 \\
&\le 2|x'_s - y'_s|^2 + |(a_s(x'_s) - a_s(y'_s)|^2 \le (2 + c_1)|x'_s - y'_s|^2.
\end{aligned}$$

We prove by the same approach that T^{-1} is lipschitzian. Without loss of generality we can assume that $f_i(x'_s, a_s(x'_s))$ (resp. $f_i(x'_r, a_r(x'_r))$) converges almost everywhere in Δ_s (resp. in Δ_r), as $i \to \infty$; we get:

$$f_{[s]}(x'_s, a_s(x'_s)) = f_{[r]}(x'_r, a_r(x'_r))$$

almost everywhere on P_s (resp. P_r), where $x'_r = T(x'_s)$. □

We will use also a fundamental lemma:

Lemma 4.1. *Let f be a function defined almost everywhere on $\partial\Omega$, different from zero at most only for $x'_r \in \Delta_r$, r fixed. Let*

$$\int_{\Delta_r} |f(x'_r, a_r(x'_r))|^p \, \mathrm{d}x'_r < \infty.$$

Then $f \in L^p(\partial\Omega)$, and

[4] In Chap. 3 we shall consider surface integrals in more detail. We shall define $(\int_{\partial\Omega} |f|^p dS)^{1/p}$ precisely by a norm equivalent to (2.67): we shall deduce that $L^p(\partial\Omega)$ does not depend on the systems of local coordinates (x'_r, x_{rN}).

$$|f|_{L^p(\partial\Omega)} \le c\left(\int_{\Delta_r} |f(x'_r, a_r(x'_r))|^p \, \mathrm{d}x'_r\right)^{1/p}.$$

Proof. We proceed as in the proof of Theorem 1.1; we use $T : P_s \to P_r$, and Lemma 3.1. □

2.4.2 Imbedding Theorems

Theorem 4.2. *Let $\Omega \in \mathfrak{N}^{0,1}$, $1 \le p < N$, $1/q = 1/p - [1/(N-1)](p-1)/p$. Then there exists exactly one mapping $Z \in [W^{1,p}(\Omega \to L^q(\partial\Omega))]$ such that $u \in C^\infty(\overline{\Omega}) \Longrightarrow Zu = u$.*

Proof. It suffices to consider $u \in C^\infty(\overline{\Omega})$, use a partition of unity as in 1.2.4, and investigate the functions $u_r, r = 1, 2, \ldots, m$. To simplify the notation we omit the index r. Let us put

$$v(x', a(x')) = |u(x', a(x'))|^{(Np-p)/(N-p)}.$$

We have:

$$v(x', a(x')) = -\int_{a(x')}^{a(x')+\beta} \frac{\partial v}{\partial x_N}(x', \eta)\, \mathrm{d}\eta,$$

then

$$|u(x', a(x'))|^{(Np-p)/(N-p)} \le \frac{Np-p}{N-p} \int_{a(x')}^{a(x')+\beta} |u(x', \eta)|^{(Np-p)/(N-p)} \left|\frac{\partial u}{\partial x_N}(x', \eta)\right| \mathrm{d}\eta,$$

and for $p > 1$

$$\begin{aligned} &\int_\Delta |u(x', a(x'))|^{(Np-p)/(N-p)} \, \mathrm{d}x' \\ &\qquad \le \frac{Np-p}{N-p}\left(\int_V |u|^{Np/(N-p)} \, \mathrm{d}x\right)^{(p-1)/p} \left(\left|\int_V \frac{\partial u}{\partial x_N}\right|\right)^{1/p}; \end{aligned} \tag{2.69}$$

this inequality also holds for $p = 1$. According to Theorem 3.4, we obtain for $u \in C^\infty(\overline{\Omega})$

$$|u|_{L^q(\partial\Omega)} \le c|u|_{W^{1,p}(\Omega)} \tag{2.70}$$

and the result follows from Theorem 3.1. □

Using the terminology introduced in Chap. 1, we call *Zu the trace of u;* to simplify we shall write u instead of Zu.

Exercise 4.1. Use Theorem 3.8 and prove, with the hypotheses of Theorem 4.2, that $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega) \Longrightarrow Zu = u$.

Theorem 4.3. *Let $\Omega \in \mathfrak{N}^{0,1}$, $u \in W^{1,p}(\Omega)$, $1 \le p < N$. Then, after a modification on a set of measure zero, u is absolutely continuous for almost all $x'_r \in \Delta_r$ on the interval $a_r(x'_r) \le x_{rN} \le a_r(x'_r) + \beta$, and $u(x'_r, a_r(x'_r)) = (Zu)(x'_r, a_r(x'_r))$ almost everywhere in Δ_r.*

Proof. Using Theorem 3.9, we extend u from V_r to U_r; then we use Theorem 2.2. As in the previous proof, after a modification of u on a set of measure zero, we obtain the inequality (2.69). □

Remark 4.1. If Ω is a smooth domain (cf. 1.2.3) or if $\partial\Omega$ is sufficiently smooth, in Theorem 4.3 we can use G_r instead of U_r (cf. 1.2.4). Then we obtain that for almost all interior normals, the function $u(\sigma_r, t)$ is absolutely continuous for $0 \le t \le \delta$, and modulo a set of measure zero, its limit value coincides with the trace of u.

Theorem 4.4. *Let us consider $u \in W^{1,p}(V)$, $1 \le p < N, a(x')$ a lipschitzian function in $\overline{\Delta}$, cf. 1.2.4. Then if $1/q = 1/p - [1/(N-1)](p-1)/p$, we have in the sense of traces*

$$\lim_{\varepsilon \to 0} \int_\Delta |u(x', a(x') + \varepsilon) - u(x', a(x'))|^q \, \mathrm{d}x' = 0, \tag{2.71}$$

$$\int_\Delta |u(x', a(x') + \varepsilon) - u(x', a(x'))|^p \, \mathrm{d}x' \le c\varepsilon^{p-1} |u|^p_{W^{1,p}(V_\varepsilon)}, \tag{2.72}$$

where

$$V_\varepsilon = \{x \in \mathbb{R}^N, x = (x', x_N), x' \in \Delta, \ a(x') < x_N < a(x') + \varepsilon\},$$

Proof. Let $V' = \{x \in \mathbb{R}^N, x = (x', x_N), x' \in \Delta, a(x') < x_N < a(x') + \beta/2\}$, and $u(x', x_N + \varepsilon) - u(x', x_N)$ in V', $\varepsilon < \beta/2$. Using inequality (2.69), we get (2.71) according to Theorem 1.1.

To obtain (2.72) we restrict ourselves only to the case $u \in C^\infty(\overline{V})$; we have:

$$u(x', a(x') + \varepsilon) - u(x', a(x')) = \int_{a(x')}^{a(x')+\varepsilon} \frac{\partial u}{\partial x_N}(x', \xi) \, \mathrm{d}\xi,$$

then

$$|u(x', a(x') + \varepsilon) - u(x', a(x'))|^p \le \varepsilon^{p-1} \int_{a(x')}^{a(x')+\varepsilon} \left| \frac{\partial u}{\partial x_N}(x', \xi) \right|^p \mathrm{d}\xi,$$

and (2.72) follows by integration with respect to x'. □

Let $\Omega_n, \Omega \in \mathfrak{N}^k$. We shall say that $\lim_{n\to\infty} \Omega_n = \Omega$ in $\mathfrak{N}^k$, if $\Omega_n \subset \Omega$ and if $\partial\Omega_n$ and $\partial\Omega$ are described by the same system of charts; let a_{rn} be the corresponding functions defined on the charts. Let us assume that

$$\lim_{n\to\infty} |a_{rn} - a_r|_{C^{k,1}(\overline{\Delta_r})} = 0.$$

Let Ω_n, $\Omega \in \mathfrak{N}^{k,1}$. We shall say that $\lim_{n\to\infty} \Omega_n = \Omega$ in $\mathfrak{N}^{k,1}$ if $\lim_{n\to\infty} \Omega_n = \Omega$ in $\mathfrak{N}^k$, if

$$\lim_{n\to\infty} |a_{rn} - a_r|_{W^{k+1,2}(\Delta_r)} = 0$$

and if $|a_{rn}|_{C^{k,1}(\Delta r)} \leq$ const.

If $\lim_{n\to\infty} \Omega_n = \Omega$ in $\mathfrak{N}^{0,1}$, and if $g_n \in L^q(\partial\Omega_n)$, $g \in L^q(\partial\Omega)$, we shall say that $\lim_{n\to\infty} g_n = g$ in $L^q(\partial\Omega)$, if

$$\lim_{n\to\infty} \sum_{r=1}^{m} \int_{\Delta_r} |g_n(x'_r, a_{rn}(x'_r)) - g(x'_r, a_r(x'_r))|^q \, \mathrm{d}x'_r = 0.$$

We have:

Theorem 4.5. *Let Ω_n, $\Omega \in \mathfrak{N}^{0,1}$, $\lim_{n\to\infty} \Omega_n = \Omega$ in $\mathfrak{N}^{0,1}$.*[5] *Let $u \in W^{1,p}(\Omega)$, $1 \leq p < N$. Then if $Zu = g$ on $\partial\Omega$ (resp. $Zu = g_n$ on $\partial\Omega_n$), we have $\lim_{n\to\infty} g_n = g$ in L^{q_1}, $1/q_1 > 1/q = 1/p - [1/(N-1)](p-1)/p$.*

Proof. We consider again V_r as above, and prove as in (2.72):

$$\lim_{n\to\infty} \int_{\Delta_r} |u(x'_r, a_r(x'_r)) - u(x', a_{rn}(x'_r))|^p \, \mathrm{d}x'_r = 0.$$

If $p = 1$, everything is proved. Let $p > 1$; in this case according to (2.69), we get:

$$\int_{\Delta_r} |u(x', a_{rn}(x'_r))|^q \, \mathrm{d}x'_r \leq c_1;$$

then

$$\begin{aligned}
&\int_{\Delta_r} |u(x'_r, a(x'_r)) - u(x', a_{rn}(x'_r))|^{q_1} \, \mathrm{d}x'_r \\
&\leq \int_{\Delta_r} |u - u_n|^{[q/(q_1-q)]/(q-1)} |u - u_n|^{(q-q_1)/(q-1)} \, \mathrm{d}x'_r \qquad (2.72\,\text{bis}) \\
&\leq \left(\int_{\Delta_r} |u - u_n|^q \, \mathrm{d}x'_r \right)^{(q_1-1)/(q-1)} \left(\int_{\Delta_r} |u - u_n| \, \mathrm{d}x'_r \right)^{(q-q_1)/(q-1)},
\end{aligned}$$

where $u_n = u(x'_r, a_{rn}(x'_r))$. $\square$

Obviously we have:

Theorem 4.6. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p = N$. Then Theorems 4.2, 4.4 hold for arbitrary $q \geq 1$.*

Theorem 4.7. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p \geq 1$, $kp < N$. Let $1/q = 1/p - [1/(N-1)] \times (kp - 1)/p$. Then the mapping Z from Theorem 4.2 satifies $Z \in [W^{k,p}(\Omega) \to L^q(\partial\Omega)]$.*

[5] In this theorem it is sufficient that $\lim_{n\to\infty} \Omega_n = \Omega$ in the following sense:

$$\lim_{n\to\infty} |a_{rn} - a_r|_{C^0(\overline{\Delta_r})} = 0, \ |a_{rn}|_{C^{0,1}(\overline{\Delta_r})} \leq \text{const.}$$

Indeed, if $u \in W^{k,p}(\Omega)$, then $(\partial u/\partial x_i) \in W^{k-1,p}(\Omega)$, thus we can apply Theorem 3.6: $u \in W^{1,q^*}(\Omega)$ with $1/q^* = 1/p - (k-1)/N$. Then we use Theorem 4.2. $\square$

It is immediately clear that we have:

Theorem 4.8. *Let $\Omega \in \mathfrak{N}^{0,1}$, $kp = N$. Then the mapping Z defined in Theorem 4.2 satisfies: $Z \in [W^{k,p}(\Omega) \to L^q(\partial\Omega)]$ for every $q \geq 1$.*

2.4.3 Two Trace Theorems

Let $p < N$. In Sect. 2.5 we shall see that for $\Omega \in \mathfrak{N}^{0,1}$, the space $L^q(\partial\Omega)$ with

$$\frac{1}{q} = \frac{1}{p} - \frac{1}{N-1}\frac{p-1}{p}$$

is not a trace space but a larger space: the mapping $Z: W^{1,p}(\Omega) \to L^q(\partial\Omega)$ is not surjective.

Nevertheless we have:

Theorem 4.9. *Let $\Omega \in \mathfrak{N}^{0,1}, p \geq 1$. Then $\overline{Z(W^{1,p}(\Omega))} = L^p(\partial\Omega)$.*

Proof. Let $f \in L^p(\partial\Omega)$ and set, for $x \in \partial\Omega, f_r(x) = f(x)\varphi_r(x)$. We have that

$$f_r \in L^p(\partial\Omega), \quad \sum_{r=1}^{m} f_r = f.$$

It is sufficient to prove our theorem for f_r, $r = 1, 2, \ldots, m$. For simplicity we omit the index r. Fix $\varepsilon > 0$. The function $f(x', a(x'))$ belongs to $L^p(\Delta)$, thus there exists $\varphi \in C_0^\infty(\Delta)$ such that $|f - \varphi|_{L^p(\Delta)} < \varepsilon$; let us set $v(x', x_N) = \varphi(x')$, and let $\psi \in C_0^\infty(U)$ be such that $x \in \partial\Omega \Longrightarrow v(x)\psi(x) = v(x)$. Then $Z(v\psi) = \varphi$. $\square$

In Chap. 1 we introduced the subspace:

$$V = \{v \in W^{k,2}(\Omega), v = \frac{\partial v}{\partial n} = \frac{\partial^2 v}{\partial n^2} \ldots = \frac{\partial^{k-1} v}{\partial n^{k-1}} = 0 \text{ on } \partial\Omega\}.$$

Let us recall that $W_0^{k,p}(\Omega) = \overline{C_0^\infty(\Omega)}$. We have:

Theorem 4.10. *Let $\Omega \in \mathfrak{N}^{0,1}, p \geq 1$. Then $W_0^{1,p}(\Omega) = W \equiv \{v \in W^{1,p}(\Omega),$ $v = 0$ on $\partial\Omega\}$.*

Proof. As $u = 0$ on $\partial\Omega$, we have $u_r = u\varphi_r = 0$ on $\partial\Omega, r = 1, 2, \ldots, m$. For u_{m+1}, it is clear that $u_{m+1} \in W_0^{1,p}(\Omega) : \lim_{h\to 0} u_{m+1,h} = u_{m+1}$ in $W^{1,p}(\Omega)$ and $u_{m+1,h} \in C_0^\infty(\Omega)$. Let now $r \leq m$; for simplicity we omit the index r. We have $u \in W^{1,p}(V)$, and set $u = 0$ in U outside of V. Then $u \in W^{1;p}(U)$: Indeed, using the transformation (2.47), denoting

$$K = \{y \in \mathbb{R}^N, |y|_i < \alpha, i = 1,2,\ldots,N-1, |y_N| < \beta\},$$
$$K_+ = \{y \in \mathbb{R}^N, |y|_i < \alpha, i = 1,2,\ldots,N-1, 0 < y_N < \beta\},$$
$$K_- = \{y \in \mathbb{R}^N, |y|_i < \alpha, i = 1,2,\ldots,N-1, -\beta < y_N < 0\},$$

and setting $v(y) = u(T(y))$ we have that $v \in W^{1,p}(K_+)$ and according to Lemma 3.2, we get:

$$v(y',0) = 0. \tag{2.73}$$

If $\psi \in C_0^\infty(K)$, we get:

$$\int_{K_+} \frac{\partial \psi}{\partial y_i} v \, \mathrm{d}y = -\int_{K_+} \psi \frac{\partial v}{\partial y_i} \, \mathrm{d}y, \quad i = 1,2,\ldots,N-1, \tag{2.74}$$

$$\int_{K_+} \frac{\partial \psi}{\partial y_N} v \, \mathrm{d}y = -\int_{\Delta} \psi(y',0) v(y',0) \, \mathrm{d}y' - \int_{K_+} \psi \frac{\partial v}{\partial y_N} \, \mathrm{d}y. \tag{2.75}$$

It follows from (2.73)–(2.75) that $v \in W^{1,p}(K) \Longrightarrow u \in W^{1,p}(U)$ by the transformation (2.46) extended to K. We denote $u_\lambda(x', x_N) = u(x', x_N - \lambda)$. For $\lambda > 0$ sufficiently small, $u_\lambda \in W^{1,p}(V)$, with $\operatorname{supp} u_\lambda \subset V$. Then we have $u_\lambda \in W_0^{1,p}(V)$, and since $\lim_{\lambda \to 0} u_\lambda = u$, we get: $W \subset W_0^{1,p}(\Omega)$. Obviously $W_0^{1,p}(\Omega) \subset W$, and the result follows. □

Let $\Omega \in \mathfrak{N}^{k-1,1}$. Denote by $W^{k,p}(\partial\Omega)$, $p \geq 1$, $k \geq 1$ integer, the subspace of $L^p(\partial\Omega)$ of functions for which $f(x'_r, a_r(x'_r)) = f_r \in W^{k,p}(\Delta_r)$, $r = 1,2,\ldots,m$. On $W^{k,p}(\partial\Omega)$ we introduce the norm:

$$|f|_{W^{k,p}(\partial\Omega)} = \left(\sum_{r=1}^m |f_r|^p_{W^{k,p}(\Delta_r)} \right)^{1/p},$$

where $f_r(x'_r) = f(x'_r, a_r(x'_r))$. $W^{k,p}(\partial\Omega)$ is a separable Banach space.

2.4.4 Some Other Properties of Traces

Lemma 4.2. *Let $\Omega \in \mathfrak{N}^{0,1}$. The exterior (or interior) normal exists almost everywhere on $\partial\Omega$.*

Proof. It is sufficient to prove that the function a_r, $r = 1,2,\ldots$, has almost everywhere in Δ_r a total differential. We again omit the index r. Let M be a countable set on the unit sphere $|x'| = 1$ dense on this sphere; we assume that M contains the points of intersection of all axes of coordinates with the sphere. Let $m \in M$. The function $a(x')$ has almost everywhere in Δ a derivative $\partial a/\partial m$; as M is countable, there exists $B \subset \Delta$, $\operatorname{meas} B = \operatorname{meas} \Delta$, such that the function $a(x')$ has for $x' \in B, m \in M$ a derivative $(\partial a/\partial m)$; we denote this derivative $a_m(x')$. According to Theorem 2.2, $a_m(x')$ is the distributional derivative and hence, if $m \in M, x \in B$,

$$a_m(x') = \sum_{i=1}^{N-1} \frac{\partial a}{\partial x_i}(x')m_i. \tag{2.76}$$

This equality holds for all $x \in B$ and all directions. Indeed: Let n be a normed vector and $m_{[j]} \in M$ with $\lim_{j\to\infty} m_{[j]} = n$ in $\mathbb{R}^{N-1}$. Fix $\varepsilon > 0$. We have:

$$\left| \frac{a(x'+tn) - a(x')}{t} - \sum_{i=1}^{N-1} \frac{\partial a}{\partial x_i}(x')n_i \right| < \varepsilon$$

for t sufficiently small; indeed, according to $a \in C^{0,1}(\overline{\Delta})$, there exists a sequence $m_{[j]}$, and a number $\delta > 0$ such that for $|t| < \delta$

$$\left| \frac{a(x'+tm_{[j]}) - a(x'+tn)}{t} - \sum_{i=1}^{N-1} \frac{\partial a}{\partial x_i}(x')m_{[j],i} + \sum_{i=1}^{N-1} \frac{\partial a}{\partial x_i}(x')n_i \right| < \varepsilon/2.$$

Let $m_{[j]}$ be fixed; using (2.76) for $m_{[j]} \in M$, the result follows. If moreover $x \in B$, then

$$\lim_{t\to 0} \left(\sup_{|n|=1} \left| \frac{a(x'+tn) - a(x')}{t} - \sum_{i=1}^{N-1} \frac{\partial a}{\partial x_i}(x')n_i \right| \right) = 0. \tag{2.77}$$

If (2.77) does not hold, there would exist an $\varepsilon > 0$, a sequence $t_k, \lim_{k\to\infty} t_k = 0$, and normed vectors $n_{[k]}$ with $\lim_{k\to\infty} n_{[k]} = n$ such that

$$\left| \frac{a(x'+t_k n_{[k]}) - a(x')}{t_k} - \sum_{i=1}^{N-1} \frac{\partial a}{\partial x_i}(x')n_{[k]_i} \right| \ge \varepsilon;$$

this would imply that for k sufficient large

$$\left| \frac{a(x'+t_k n) - a(x')}{t_k} - \sum_{i=1}^{N-1} \frac{\partial a}{\partial x_i}(x')n_i \right| \ge \varepsilon/2,$$

and this inequality is a contradiction to (2.76). But (2.77) implies the existence of the total differential at x. $\square$

Let $\Omega \in \mathfrak{N}^{0,1}$. If $u \in W^{k,p}(\Omega)$ then $D^\alpha u \in L^p(\partial\Omega)$, $|\alpha| \le k-1$, and we define the exterior normal derivative of order $l \le k-1$ by:

$$\frac{\partial^l u}{\partial n^l} = \sum_{|\alpha|=l} \frac{l!}{\alpha!} D^\alpha u n^\alpha, \tag{2.78}$$

where n is the exterior normal, $n = (n_1, n_2, \ldots, n_N)$ and $n^\alpha = n_1^{\alpha_1} n_2^{\alpha_2} \ldots n_N^{\alpha_N}$.

Theorem 4.11. *Let $\Omega \in \mathfrak{N}^{0,1}$; if $1 \le p < N$, put $1/q = 1/p - [1/(N-1)](p-1)/p$; if $p = N$, put $q \ge 1$. There exists a unique mapping $Z \in [W^{2,p}(\Omega) \to W^{1,q}(\partial\Omega)]$ such that $u \in C^\infty(\overline{\Omega}) \Longrightarrow Zu = u$.*

Proof. Fix $u \in C^\infty(\overline{\Omega})$. Then we have $u(x', a_r(x'_r)) \in W^{1,q}(\Delta_r)$, and according to Theorem 2.2, if we denote $v(x'_r) = u(x', a_r(x'_r))$, we have:

$$\frac{\partial v}{\partial x_{ri}} = \frac{\partial u}{\partial x_{ri}} + \frac{\partial u}{\partial x_{rN}} \frac{\partial a_r}{\partial x_{ri}}, \quad i = 1,2,\ldots,N-1. \tag{2.79}$$

We have $\partial u/\partial x_{ri} \in W^{1,p}(V_r), i = 1,2,\ldots,N$.

Using Theorems 4.2, 4.5 and (2.79) we get:

$$|u|_{W^{1,q}(\partial\Omega)} \le c|u|_{W^{2,p}(\Omega)}.$$

□

Remark 4.2. It is easy to see that for $u \in W^{2,p}(\Omega)$, (2.79) holds for the derivatives $\frac{\partial u}{\partial x_{ri}}, i = 1,2,\ldots,N$, considered in the sense of traces.

Remark 4.3. If $\Omega \in \mathfrak{N}^{k-2,1}$, $k \ge 2$, we have with the same notations as in Theorem 4.11 that $Z \in [W^{k,p}(\Omega) \to W^{k-1,q}(\partial\Omega)]$.

Theorem 4.12. *If $\Omega \in \mathfrak{N}^{0,1}$, $u \in W^{2,p}(\Omega)$, $p \ge 1$, $u = \partial u/\partial n = 0$ on $\partial\Omega$, then $u \in W_0^{2,p}(\Omega)$.*

Proof. For $r = 1,2,\ldots,m$, we have according to (2.79)

$$0 = \sum_{i=1}^{N-1} \frac{\partial u}{\partial x_{ri}} \frac{\partial a_r}{\partial x_{ri}} - \frac{\partial u}{\partial x_{rN}},\text{[6]} \tag{2.80}$$

$$0 = \frac{\partial u}{\partial x_{rN}} + \frac{\partial u}{\partial x_{rN}} \frac{\partial a_r}{\partial x_{ri}}, \quad 1,2,\ldots,N-1. \tag{2.81}$$

Starting from (2.80), (2.81) we can compute the derivatives $\partial u/\partial x_{ri}$, $i = 1,2,\ldots,N$, and we get a homogeneous linear system with nonzero determinant, thus $\partial u/\partial x_{ri} = 0$ on $\partial\Omega$. As in Theorem 4.10, the function u, extended by zero outside of Ω, is in $W^{2,p}(\mathbb{R}^N)$; the remaining part of the proof is the same as in Theorem 4.10. □

Theorem 4.13. *If $\Omega \in \mathfrak{N}^{k,1}$, $u \in W^{k,p}(\Omega)$ with $u = \partial u/\partial n = \ldots = \partial^{k-1}u/\partial n^{k-1} = 0$ on $\partial\Omega$, then $u \in W_0^{k,p}(\Omega)$.*

Proof. For $k \le 2$ this theorem was proved for $\Omega \in \mathfrak{N}^{0,1}$; hence we can assume $k \ge 3$. It is sufficient to prove that u extended by zero outside of Ω is in $W^{k,p}(\mathbb{R}^N)$.

---

[6] Clearly $n = \left(1 + \sum_{i=1}^{N-1} (\partial a_r/\partial x_{ri})^2\right)^{-1/2} \left(\partial a_r/\partial x_{r1}, \partial a_r/\partial x_{r2} \ldots, \partial a_r/\partial x_{r(N-1)}, -1\right)$.

To do this we use the domains $G_1, G_2, \dots, G_M$, ψ_i, $i = 1, 2, \dots, M$, given in 2.2.4, and the mapping (1.35) from Chap. 1; this mapping is smooth, one-to-one from $\overline{\Delta} \times (-\delta, \delta)$ to $\overline{G_r}$, the inverse has the same properties, they are generated by functions in $C^{k-1,1}(\overline{\Delta})$. Using Lemma 3.4 and $u_r = u\psi_r$ in G_r, after the mapping we get $u_r \in W^{k,p}(C_+)$ with $C_+ = \Delta \times (0, \delta)$. For $t = 0$ we have:

$$u_r = \frac{\partial u_r}{\partial t} = \dots = \frac{\partial^{k-1} u_r}{\partial t^{k-1}} = 0,$$

then $D^\alpha u_r = 0$ for $t = 0, |\alpha| \le k-1$. The function u extended by zero if $\sigma \in \Delta$, $t < 0$, is in $W^{k,p}(C)$ with $C = \Delta \times (-\delta, \delta)$. By the mapping inverse to (1.35), we get $u_r \in W^{k,p}(G_r)$, $r = 1, 2, \dots, M$. Obviously $u_{M+1} \in W_0^{k,p}(\Omega)$, so

$$u = \sum_{r=1}^{M+1} u_r \in W^{k,p}(\mathbb{R}^N).$$

□

The notion of smoothness almost everywhere for $\partial\Omega$ was introduced in 1.2.3.

Theorem 4.14. *Let $u \in W^{k,p}(\Omega)$, $\partial\Omega$ almost everywhere smooth, $u = \partial u/\partial n = \dots = \partial^{k-1}/\partial n^{k-1} = 0$ on $\partial\Omega$ in the sense of traces. Then $u \in W_0^{k,p}(\Omega)$.*

Proof. It is sufficient to prove that $D^\alpha u = 0$ on $\partial\Omega$ for $|\alpha| \le k-1$. Let $y \in \partial\Omega$ be a regular point. We use the set U from 1.2.4 which corresponds to that point, and proceed as the proof of the previous theorem. □

Problem 4.1. Is Theorem 4.14 true if $\Omega \in \mathfrak{N}^{0,1}$?

Remark 4.4. Let $\Omega \in \mathfrak{N}^{0,1}$, $u \in W^{2,p}(\Omega)$. We can prove that the subspace of $W^{1,p}(\partial\Omega) \times L^p(\partial\Omega)$, generated by $(u, \partial u/\partial n), u \in W^{2,p}(\Omega)$, is dense in this space. Cf. Chap. 5.

Exercise 4.2. Let $\Omega \in \mathfrak{N}^{1,1}$, $u \in W^{3,p}(\Omega)$, $u = \partial u/\partial n = \partial^2 u/\partial n^2 = 0$ on $\partial\Omega$ in the sense of traces. Prove that $u \in W_0^{3,p}(\Omega)$.

2.5 The Problem of Traces (Continuation)

2.5.1 Application of the Fourier Transform

In the previous section we pointed out that the spaces $L^p(\partial\Omega)$ in Theorem 4.2 are larger than the trace spaces. For instance, if $\Omega \in \mathfrak{N}^{0,1}$, the natural topology of the space of traces of $W^{1,p}(\Omega)$ is the topology of the quotient space $W^{1,p}(\Omega)/W_0^{1,p}(\Omega)$. On the other hand, this approach is rather formal and does not give a characterization of the trace space.

If $\Omega = \mathbb{R}^N_+ = \{x \in \mathbb{R}^N,\ x_N > 0\}$, $p = 2$, $k \geq 1$, we can easily characterize the trace space using the Fourier transform, cf. for instance J.L. Lions [4]. To do this we introduce for $\Omega = \mathbb{R}^N$ the notion of $W^{k,2}(\mathbb{R}^N)$ with k an arbitrary real number; later we shall see that the new definition coincides for $k \geq 0$ with the definition given in 2.3.8.

If $k \geq 0$, $W^{k,2}(\mathbb{R}^N)$ *is defined as the subset of* $L^2(\mathbb{R}^N)$ of functions $f(x)$ such that their Fourier transforms satisfy:

$$\left(\int_{\mathbb{R}^N} |\hat{f}(\xi)|^2 (1+|\xi|^2)^k \,\mathrm{d}\xi\right)^{1/2} < \infty. \tag{2.82}$$

The left hand side of (2.82) is a *norm*, moreover $W^{k,2}(\mathbb{R}^N)$ is an *Hilbert* space with the scalar product:

$$[f,g] = \int_{\mathbb{R}^N} \hat{f}(\xi)\overline{\hat{g}(\xi)}(1+|\xi|^2)^k \,\mathrm{d}\xi. \tag{2.83}$$

If $k < 0$, we define $W^{k,2}(\mathbb{R}^N) = (W^{-k,2}(\mathbb{R}^N))'$. It is possible to define $W^{k,2}(\mathbb{R}^N)$ directly for all real k in the setting of tempered distributions and their Fourier transforms, cf. L. Schwartz [2].

Let us denote as usual $x = (x', x_N), \xi = (\xi', \xi_N)$.

Theorem 5.1. *Let k be an integer, $k = 1, 2 \ldots$. Then*

$$u \in W^{k,2}(\mathbb{R}^N_+) \Longrightarrow u(x',0) \in W^{k-1/2,2}(\mathbb{R}^{N-1}),$$

and if $g(x') = u(x',0)$, we have:

$$|g|_{W^{k-1/2,2}(\mathbb{R}^{N-1})} \leq c|u|_{W^{k,2}(\mathbb{R}^N_+)} \tag{2.84}$$

Proof. By immediate application of (1.1.10) we obtain $g \in L^2(\mathbb{R}^{N-1})$; we extend u to $\mathbb{R}^N$ using (2.48) and get obviously $|u|_{W^{k,2}(\mathbb{R}^N)} \leq c_1|u|_{W^{k,2}(\mathbb{R}^N_+)}$. Then according to Lemma 3.5 it follows:

$$\left(\int_{\mathbb{R}^N} |\hat{u}(\xi)|^2 (1+|\xi|^2)^k \,\mathrm{d}\xi\right)^{1/2} \leq c_2|u|_{W^{k,2}(\mathbb{R}^N)}. \tag{2.85}$$

Due to Proposition 2.5, it is sufficient to prove (2.84) for $u \in C_0^\infty(\mathbb{R}^N)$; we have:

$$\hat{g}(\xi') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\xi)\,\mathrm{d}\xi_N,$$

and then

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} |\hat{g}(\xi)|^2 (1+|\xi'|^2)^{k-1/2} \,\mathrm{d}\xi' \\
&\quad = \frac{1}{4\pi^2} \int_{\mathbb{R}^{N-1}} \left| \int_{-\infty}^{\infty} \hat{u}(\xi)\,\mathrm{d}\xi_N \right|^2 (1+|\xi'|^2)^{k-1/2} \,\mathrm{d}\xi' \\
&\quad \leq \frac{1}{4\pi^2} \int_{\mathbb{R}^{N-1}} \Big[(1+|\xi'|^2)^{k-1/2} \int_{-\infty}^{\infty} |\hat{u}(\xi)|^2 (1+|\xi|^2)^k \,\mathrm{d}\xi_N \times \\
&\qquad\qquad \times \int_{-\infty}^{\infty} (1+|\xi'|^2)^{-k} \,\mathrm{d}\xi_N \Big] \,\mathrm{d}\xi'.
\end{aligned} \tag{2.86}$$

We have

$$\int_{-\infty}^{\infty} (1+|\xi'|^2)^{-k} \,\mathrm{d}\xi_N = \pi (1+|\xi'|^2)^{-k+1/2},$$

and the result follows from (2.86). □

Remark 5.1. Clearly we have: if $u \in W^{k,2}(\mathbb{R}^N_+)$ then we have for $l = 1,2,\dots,k-1$ that $(\partial^l u/\partial x_N^l)(x',0) \in W^{k-l-1/2,2}x'_N(\mathbb{R}^{N-1})$, with

$$\sum_{l=0}^{k-1} \left| \frac{\partial^i u}{\partial x_N^i} \right|_{W^{k-l-1/2,2}(\mathbb{R}^{N-1})} \leq c|u|_{W^{k,2}(\mathbb{R}^N_+)}.$$

In what follows, $B_1, B_2, \dots, B_k$ will be Banach spaces; we denote $B_1 \times B_2 \times \cdots \times B_k$ the Cartesian product of B_i, i.e. the set of elements $u = (u_1, u_2, \dots, u_k)$, where $u_i \in B_i$; and we equip $\prod_{i=1}^k B_i$ with the norm $\sum_{i=1}^k |u_i|_{B_i}$ or some equivalent norm.

We have a "converse" of Remark 5.1:

Theorem 5.2. *There exists a mapping:*

$$Z \in \Big[\prod_{l=0}^{k-1} W^{k-l-1/2,2}(\mathbb{R}^{N-1}) \to W^{k,2}(\mathbb{R}^N_+) \Big]$$

such that if

$$g = (g_1, g_2, \dots, g_{k-1}) \in \prod_{l=0}^{k-1} W^{k-l-1/2,2}(\mathbb{R}^{N-1}),$$

then for $u = Zg$

$$\frac{\partial^l u}{\partial x_N^l}(x',0) = g_l(x').$$

Proof. We define:

$$Z_l \in [W^{k-l-1/2,2}(\mathbb{R}^{N-1}) \to W^{k,2}(\mathbb{R}^N_+)], \quad l = 1,2,\dots,k-1,$$

taking

$$Z_l g_l = u_l,$$

where

$$\hat{u}_l(\xi', x_N) = x_N^l \exp(-(1+|\xi'|)x_N)\hat{g}_l(\xi'). \tag{2.87}$$

Z_l is of the type mentioned; indeed, let $|\alpha| \le k$, let us consider $w_\alpha = D^\alpha u_l$ in $\mathbb{R}^N_+$ and set $w_\alpha = 0$ for $x_N < 0$. Let us denote $\alpha' = (\alpha_1, \dots, \alpha_{N-1}, 0)$, $\alpha'' = (0, \dots, 0, \alpha_N)$. Hence $\hat{w_\alpha}(\xi)$ is a finite sum of expressions like:

$$\begin{aligned} aI(\xi) = a\int_0^\infty \exp(-\mathrm{i}x_N\xi_N)(\xi')^{\alpha'}(1+|\xi'|)x_N^{l-j}\times \\ \times \exp(-(1+|\xi'|)x_N)\hat{g}_l(\xi')\,\mathrm{d}x_N, \end{aligned}$$

where a is a constant, $j = 0, 1, \dots, \min(\alpha_N, l)$. We have:

$$I(\xi) = \frac{(l-j)!(\xi')^{\alpha'}(1+|\xi'|)^{\alpha_N-1}\hat{g}_l(\xi')}{(1+|\xi'|+\mathrm{i}\xi_N)^{l-j+1}},$$

and so

$$\begin{aligned} &\int_{\mathbb{R}^N} |I(\xi)|^2\,\mathrm{d}\xi \\ &\le c_1\int_{\mathbb{R}^{N-1}} \Bigg[|\xi|^{2|\alpha'|}(1+|\xi'|^{2\alpha_N-2j})|\hat{g}_l(\xi')|^2\times \\ &\qquad \times \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi_N}{((1+|\xi'|)^2+\xi_N^2)^{l-j+1}}\Bigg]\mathrm{d}\xi' \\ &= c_2\int_{\mathbb{R}^{N-1}} |\xi|^{2|\alpha'|}(1+|\xi'|)^{2\alpha_N-2l-1}|\hat{g}_l(\xi')|^2\,\mathrm{d}\xi' \\ &\le c_3\int_{\mathbb{R}^{N-1}} (1+|\xi'|)^{2\alpha-2l-1}|\hat{g}_l(\xi')|^2\,\mathrm{d}\xi' \\ &\le c_4\int_{\mathbb{R}^{N-1}} (1+|\xi'|)^{k-l-1/2}|\hat{g}_l(\xi')|^2\,\mathrm{d}\xi'. \end{aligned}$$

Now we have $\partial^j u_l/\partial x_N^j(x', 0) = 0$ for $j < l$. We construct Z as a linear combination of Z_l by setting

$$\begin{aligned} Zg = Z_0g_0 + Z_1\left(g_1 - \frac{\partial}{\partial x_N}Z_0g_0\right) \\ + \frac{1}{2!}Z_2\left(g_2 - \frac{\partial^2}{\partial x_N^2}Z_0g_0 - \frac{\partial^2}{\partial x_N^2}Z_1\left(g_1 - \frac{\partial}{\partial x_N}Z_0g_0\right)\right) + \dots \end{aligned} \tag{2.88}$$

□

2.5.2 Lemmas Based on the Hardy Inequality

The method used in 2.5.1 does not work if $p \neq 2$; here we generalize the Gagliardo approach, E. Gagliardo [1], assuming $p > 1$. Moreover we assume Ω bounded.

Let $k \geq 0$, $\Omega \in \mathfrak{N}^{[k]'-1,1}$, where $[k]'$ is the smallest integer such that $k \leq [k]'$. We define $W^{k,p}(\partial\Omega)$ as the subset of functions from $W^{[k],p}(\partial\Omega)$ such that $f_r(x'_r) = f(x'_r, a_r(x'_r)) \in W^{k,p}(\Delta_r), r = 1,2,\dots,m$; the norm is defined by:

$$|f|_{W^{k,p}(\partial\Omega)} = \left(\sum_{r=1}^{m} |f_r|^p_{W^{k,p}(\Delta_r)} \right)^{1/p} . \tag{2.89}$$

Remark 5.2. The space $W^{k,p}(\partial\Omega)$ is a separable Banach space and reflexive if $p > 1$; this is an immediate consequence of Proposition 3.1.

In this section we consider only the case $p > 1$. If $p = 1$ we have to use another approach, cf. E. Gagliardo [1].

Now we shall prove a *Hardy inequality*, cf. G.H. Hardy, J.L. Littlewood, G. Pólya [1].

Lemma 5.1. *Let be $f \in L^p(a,b)$, $-\infty < a < b < \infty$, $p > 1$. The following inequalities hold:*

$$\int_a^b \left[\frac{1}{x-a} \int_a^x |f(\xi)| \,\mathrm{d}\xi \right]^p \mathrm{d}x \leq \left(\frac{p}{p-1} \right)^p \int_a^b |f(x)|^p \,\mathrm{d}x, \tag{2.90}$$

$$\int_a^b \left[\frac{1}{b-x} \int_x^b |f(\xi)| \,\mathrm{d}\xi \right]^p \mathrm{d}x \leq \left(\frac{p}{p-1} \right)^p \int_a^b |f(x)|^p \,\mathrm{d}x. \tag{2.91}$$

Proof. Let us define:

$$f_\varepsilon(x) = \begin{cases} f(x) & \text{for} \quad x \geq a + \varepsilon, \\ 0 & \text{for} \quad a < x < a + \varepsilon. \end{cases}$$

We have

$$\begin{aligned} &\int_a^b \frac{1}{(x-a)^p} \left[\int_a^x |f_\varepsilon(\xi) \,\mathrm{d}\xi| \right]^p \mathrm{d}x \\ &\qquad = \frac{1}{(b-a)^{p-1}} \frac{1}{p-1} \left[\int_a^b |f_\varepsilon(\xi)| \,\mathrm{d}\xi \right]^p \\ &\qquad\quad + \frac{p}{p-1} \int_a^b \frac{1}{(x-a)^{p-1}} \left(\int_a^x |f_\varepsilon(\xi)| \,\mathrm{d}\xi \right)^{p-1} |f_\varepsilon(x)| \,\mathrm{d}x \end{aligned}$$

$$\leq \frac{p}{p-1}\int_a^b \frac{1}{(x-a)^{p-1}}\left(\int_a^x |f_\varepsilon(\xi)|\,\mathrm{d}\xi\right)^{p-1}|f_\varepsilon(x)|\,\mathrm{d}x$$
$$\leq \frac{p}{p-1}\left(\int_a^b \frac{1}{(x-a)^p}\left(\int_a^x |f_\varepsilon(\xi)|\,\mathrm{d}\xi\right)^p \mathrm{d}x\right)^{(p-1)/p}\left(\int_a^b |f_\varepsilon(x)|^p\,\mathrm{d}x\right)^{1/p}.$$

This implies inequality (2.90) for the function f_ε, and consequently

$$\int_a^b \left[\frac{1}{x-a}\int_a^x |f_\varepsilon(\xi)|\,\mathrm{d}\xi\right]^p \mathrm{d}x \leq \left(\frac{p}{p-1}\right)^p \int_a^b |f_\varepsilon(x)|^p\,\mathrm{d}x$$
$$\leq \left(\frac{p}{p-1}\right)^p \int_a^b |f(x)|\,\mathrm{d}x.$$

Now, Fatou's lemma gives (2.90) for f. Inequality (2.91) can be proved in the same way. □

Lemma 5.2. *Let $\Delta = \{x\in\mathbb{R}^2, 0<x_1<1, 0<x_2<x_1\}$, $u\in W^{1,p}(\Delta)$, $p>1$. Then we have the inequality*

$$\int_0^1\int_0^1 \left|\frac{u(t,t)-u(\tau,\tau)}{t-\tau}\right|^p dt\,\mathrm{d}\tau < c|u|^p_{W^{1,p}(\Delta)}. \tag{2.92}$$

Proof. According to Theorem 3.1, it is sufficient to consider $u\in C^\infty(\overline{\Delta})$. Let $0\leq \tau<t\leq 1$. If we denote $f(t)=u(t,t)$, then

$$\left|\frac{f(t)-f(\tau)}{t-\tau}\right| \leq \frac{1}{t-\tau}\int_\tau^t \left|\frac{\partial u}{\partial x_1}(x_1,\tau)\right|\mathrm{d}x_1 + \frac{1}{t-\tau}\int_\tau^t \left|\frac{\partial u}{\partial x_2}(t,x_2)\right|\mathrm{d}x_2,$$

and

$$\begin{aligned}\left|\frac{f(t)-f(\tau)}{t-\tau}\right|^p \leq 2^{p-1}\Bigg[&\frac{1}{(t-\tau)^p}\left(\int_\tau^t \left|\frac{\partial u(x_1,\tau)}{\partial x_1}(x_1,\tau)\right|\mathrm{d}x_1\right)^p \\ &+ \frac{1}{(t-\tau)^p}\left(\int_\tau^t \left|\frac{\partial u(t,x_2)}{\partial x_2}(t,\tau)\right|\mathrm{d}x_2\right)^p\Bigg].\end{aligned} \tag{2.93}$$

By integration with respect to t, $\tau<t<1$, and then to τ, $0<\tau<1$, we get according to Lemma 5.1

$$\begin{aligned}&\int_0^1 \mathrm{d}\tau\int_\tau^1 \left|\frac{f(t)-f(\tau)}{t-\tau}\right|^p \mathrm{d}t \\ &\leq 2^{p-1}\left(\frac{p}{p-1}\right)^p\Bigg[\int_0^1 \mathrm{d}\tau\int_\tau^1 \left|\frac{\partial u(x_1,\tau)}{\partial x_1}(x_1,\tau)\right|^p \mathrm{d}x_1 \\ &+ \int_0^1 \mathrm{d}t\int_0^t \left|\frac{\partial u(t,x_2)}{\partial x_2}(t,\tau)\right|^p \mathrm{d}x_2\Bigg] \leq 2^{p-1}\Big(\frac{p}{p-1}\Big)^p|u|^p_{W^{1,p}(\Delta)}.\end{aligned} \tag{2.94}$$

□

Lemma 5.3. *Let C be the cube $(-1,1)^{N-1}, p > 1, u \in L^p(C)$; moreover let us assume:*

$$c_i^p = \underbrace{\int_{-1}^{1} \dots \int_{-1}^{1}}_{(N-2)times} \mathrm{d}x_1 \dots \mathrm{d}x_{i-1}\,\mathrm{d}x_{i+1} \dots \mathrm{d}x_{N-1} \times \\ \times \left(\int_{-1}^{1} \int_{-1}^{1} \frac{|A_i(t) - A_i(\tau)|^p}{|t-\tau|^p}\,\mathrm{d}t\,\mathrm{d}\tau \right) < \infty, \tag{2.95}$$

where $A_i(t) = u(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_{N-1})$.

In this case we have the inequality:

$$|u|_{W^{1-1/p,p}(C)} \leq c[|u|_{L^p(C)} + \sum_{i=1}^{N-1} c_i^p]^{1/p}. \tag{2.96}$$

Proof. For $x, y \in C$. Denote $x_{[i]} = (y_1, \dots, y_i, x_{i+1}, \dots, x_{N-1})$, $i = 0, 1, \dots, N-1$. We have:

$$\int_C \int_C \frac{|u(x)-u(y)|^p}{|x-y|^{N-2+p}}\,\mathrm{d}x\,\mathrm{d}y \leq c \sum_{i=1}^{N-1} \int_C \int_C \frac{|u(x_{[i]}) - u(x_{[i-1]})|^p}{|x-y|^{N-2+p}}\,\mathrm{d}x\,\mathrm{d}y.$$

For instance, let us consider:

$$\begin{aligned} &\int_C \int_C \frac{|u(x_{[1]}) - u(y)|^p}{|x-y|^{N-2+p}}\,\mathrm{d}x_1 \dots \mathrm{d}x_{N-1}\,\mathrm{d}y_1 \dots \mathrm{d}y_{N-1} \\ &\quad = \int_C \int_{-1}^{1} |u(x_{[1]}) - u(x)|^p\,\mathrm{d}x\,\mathrm{d}y_1 \underbrace{\int_{-1}^{1} \dots \int_{-1}^{1}}_{(N-2)\text{times}} \frac{\mathrm{d}y_2 \dots \mathrm{d}y_{N-1}}{|x-y|^{N-2+p}} \\ &\quad \leq c \int_C \int_{-1}^{1} \frac{|u(x_{[1]}) - u(x)|^p}{|x_{[1]} - x|^p}\,\mathrm{d}x\,\mathrm{d}y_1 = c c_1^p. \end{aligned}$$

□

Exercise 5.1. For the cube C in the previous lemma, prove the converse of (2.96).

Let

$$[|u|_{L^p(C)} + \sum_{i=1}^{N-1} c_i^p]^{1/p}. \tag{2.96 bis}$$

Notice that (2.96 bis) defines an equivalent norm on $W^{k,p}(C)$ (this is a consequence of the previous exercise).

Exercise 5.2. Replace C by $\mathbb{R}^N$ and prove the equivalence of (2.96 bis) and (2.90).

2.5.3 Imbedding Theorems, Application of the Spaces $W^{1-1/p,p}(\partial\Omega)$

Theorem 5.3. *Let C be the cube $(-1,1)^N$, $p>1$. Let C_i be the faces $x_i=1$, $|x_j|<1$, $j\neq i$, and $u\in W^{1,p}(C)$. Then we have the inequality:*

$$|u|_{W^{(1-1/p),p}(C_i)} \leq c|u|_{W^{1,p}(C)}, \quad i=1,2,\ldots,N.$$

Proof. According to Theorem 3.1, it is sufficient to take $u\in C^\infty(\overline{C})$. Let i be fixed, $1\leq i\leq N$, $j\neq i$ (for instance $j<i$.) By Lemma 5.2, we have:

$$\underbrace{\int_{-1}^{1}\cdots\int_{-1}^{1}}_{(N-2)\text{times}} \mathrm{d}x_1\ldots\mathrm{d}x_{j-1}\,\mathrm{d}x_{j+1}\ldots\mathrm{d}x_{i-1}\,\mathrm{d}x_{i+1}\ldots\mathrm{d}x_N\times$$

$$\times\left(\int_{-1}^{1}\int_{-1}^{1}\frac{|B_{ji}(t)-B_{ji}(\tau)|^p}{|t-\tau|^p}\,\mathrm{d}t\,\mathrm{d}\tau\right)$$

$$\leq c\underbrace{\int_{-1}^{1}\cdots\int_{-1}^{1}}_{(N-2)\text{times}} \mathrm{d}x_1\ldots\mathrm{d}x_{j-1}\,\mathrm{d}x_{j+1}\ldots\mathrm{d}x_{i-1}\,\mathrm{d}x_{i+1}\ldots\mathrm{d}x_N\times$$

$$\times\int_{-1}^{1}\int_{-1}^{1}\left(\left|\frac{\partial u}{\partial x_j}\right|^p+\left|\frac{\partial u}{\partial x_i}\right|^p\right)\mathrm{d}x_j\,\mathrm{d}x_i,$$

where $B_{ji}(t)=u(x_1,\ldots x_{j-1},t,x_{j+1},\ldots,x_{i-1},1,x_{i+1}\ldots,x_N)$ and the result follows from Lemma 5.3. □

Theorem 5.4. *Let $u\in W^{k,p}(C)$; under the same hypotheses as that of Theorem 5.3, we have for $l\leq k-1$*

$$\left|\frac{\partial^l u}{\partial x_i^l}\right|_{W^{k-l-1/p,p}(C_i)} \leq c|u|_{W^{k,p}(C)} \quad i=1,2,\ldots,N.$$

Indeed, $D^\alpha u\in W^{1,p}(C)$ for $|\alpha|\leq k-1$, and the result follows from the previous theorem. □

2.5.4 Imbedding Theorems, Application of the Spaces $W^{1-1/p,p}(\partial\Omega)$ (Continuation)

Lemma 5.4. *Let $\Omega\in\mathfrak{N}^{0,1}$, $p\geq 1$, $k\geq 0$, $0\leq\lambda<k$. Then $W^{k,p}(\Omega)\subset W^{\lambda,p}(\Omega)$ algebraically and topologically.*

Proof. The only nontrivial case is the case of λ non integer, k integer; the other cases are either trivial or consequences of the case considered. It is sufficient to investigate the case $0<\lambda<1, k=1$. We set $u_r = u\varphi_r$, use the transformation (2.31), and then everything reduces to $\Omega = C = (-1,1)^N$. According to Theorem 3.1, we can assume $v \in C^\infty(\overline{C})$, and prove the inequality:

$$\int_C \int_C \frac{|v(x)-v(y)|^p}{|x-y|^{N+p\lambda}}\,\mathrm{d}x\,\mathrm{d}y \le c_1 |v|^p_{W^{1,p}(C)}. \tag{2.97}$$

We use the Hölder inequality and write:

$$\begin{aligned}
\int_C \int_C \frac{|u(x)-u(y)|^p}{|x-y|^{N+p\lambda}}\,\mathrm{d}x\,\mathrm{d}y &= \int_C \int_C \frac{\left|\int_0^1 (d/dt)v(y+t(x-y))\,\mathrm{d}t\right|^p}{|x-y|^{N+p\lambda}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le c_2 \int_C \int_C \frac{\left|\int_0^1 \sum_{i=1}^N (\partial v/\partial x_i)(y+t(x-y))\,\mathrm{d}t\right|^p}{|x-y|^{N+p\lambda-p}}\,\mathrm{d}x\,\mathrm{d}y \\
&\le c_3 \sum_{i=1}^N \int_C \int_C \int_0^1 \frac{|(\partial v/\partial x_i)(y+t(x-y))|^p}{|x-y|^{N+p\lambda-p}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned}$$

We transform the set of points $(x,y,t) \in C \times C \times (0,1)$ to G by:

$$\tau = t, \quad \eta = y, \quad \xi = y + t(x-y).$$

We have for $i = 1,2,\dots,N$:

$$\begin{aligned}
\int_C \int_C \int_0^1 \frac{|(\partial v/\partial x_i)(y+t(x-y))|^p}{|x-y|^{N+p\lambda-p}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t &= \int_G \frac{|(\partial v)/\partial x_i(\xi)|^p \tau^{p(\lambda-1)}}{|\xi-\eta|^{N+p\lambda-p}}\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}t \\
= \int_0^1 \tau^{p(\lambda-1)}\,\mathrm{d}\tau \int_C \left|\frac{\partial v}{\partial x_i}(\xi)\right|^p \mathrm{d}\xi \int_{G_{\tau,\xi}} \frac{\mathrm{d}\eta}{|\xi-\eta|^{N+p\lambda-p}}.
\end{aligned} \tag{2.98}$$

Since $|\xi(1-\tau)^{-1}-\eta| \le c_4 \tau(1-\tau)^{-1}$, we have $|\xi-\tau| \le c_5\tau(1-\tau)^{-1}$, and

$$\begin{aligned}
&\int_0^1 \tau^{p(\lambda-1)}\,\mathrm{d}\tau \int_C \left|\frac{\partial v}{\partial x_i}(\xi)\right|^p \mathrm{d}\xi \int_{G_{\tau,\xi}} \frac{\mathrm{d}\eta}{|\xi-\eta|^{N+p\lambda-p}} \\
&\quad \le c_6 \int_0^{1/2} \tau^{p(\lambda-1)}\,\mathrm{d}\tau \int_C \left|\frac{\partial v}{\partial x_i}(\xi)\right|^p \mathrm{d}\xi \int_{G_{\tau,\xi}} \frac{\mathrm{d}\eta}{|\xi-\eta|^{N+p\lambda-p}} \\
&\quad + c_6 \int_{1/2}^1 \tau^{p(\lambda-1)}\,\mathrm{d}\tau \int_C \left|\frac{\partial v}{\partial x_i}(\xi)\right|^p \int_{G_{\tau,\xi}} \frac{\mathrm{d}\eta}{|\xi-\eta|^{N+p\lambda-p}}\,\mathrm{d}\xi \le c_7 \int_C \left|\frac{\partial v}{\partial x_i}(\xi)\right|^p \mathrm{d}\xi,
\end{aligned}$$

and finally (2.97) follows from (2.98). □

Lemma 5.5. *Let $\Omega \in \mathfrak{N}^{0,1}$, $u \in W^{k,p}(\Omega)$, $p \geq 1$, $k \geq 0$; $h \in C^{[k]'-1,1}(\overline{\Omega})$. Then $hu \in W^{k,p}(\Omega)$, and*

$$|uh|_{W^{k,p}(\Omega)} \leq c|u|_{W^{k,p}(\Omega)}|h|_{C^{[k]'-1,1}(\overline{\Omega})}.$$

Proof. If k is an integer the result is trivial. Let k be a non-integer; we have for $|\alpha| < [k]$:

$$D^\alpha(uh) = \sum_{|\beta|\leq|\alpha|} h_\beta D^\beta u,\ h_\beta \in C^{0,1}(\overline{\Omega}).$$

If $|\beta| \leq [k]$, then

$$\begin{aligned}
&\int_\Omega\int_\Omega \frac{|h_\beta(x)D^\beta u(x) - h_\beta(y)D^\beta u(y)|^p}{|x-y|^{N+p(k-[k])}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad \leq 2^{p-1}\int_\Omega\int_\Omega \frac{|h_\beta(x)(D^\beta u(x) - D^\beta u(y))|^p}{|x-y|^{N+p(k-[k])}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 2^{p-1}\int_\Omega\int_\Omega \frac{|D^\beta u(y)|^p|h_\beta(x) - h_\beta(y))|^p}{|x-y|^{N+p(k-[k])}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

The first integral on the right hand side is less than or equal to:

$$|u|^p_{W^{k,p}(\Omega)}|h|^p_{C^{[k]'-1,1}(\overline{\Omega})},$$

and the second integral is less than or equal to:

$$c_1\int_\Omega |D^\beta u(y)|^p\,\mathrm{d}y\int_\Omega \frac{\mathrm{d}x}{|x-y|^{N-1+p([k-[k]])}} \leq c_2\int_\Omega |D^\beta u(y)|^p\,\mathrm{d}y.$$

□

Theorem 5.5. *Let $\Omega \in \mathfrak{N}^{k-1,1}$, $u \in W^{k,p}(\Omega)$, $p > 1$, k an integer. If $l \leq k-1$, then the following inequality holds:*

$$\left|\frac{\partial^l u}{\partial n^l}\right|_{W^{k-l-1/p,p}(\partial\Omega)} \leq c|u|_{W^{k,p}(\Omega)}.$$

Proof. It is sufficient to prove that $D^\alpha u \in W^{k-l-1/p,p}(\partial\Omega)$ and the corresponding inequality for $|\alpha| = l$; we have $D^\alpha u \in W^{k-l,p}(\Omega)$. Let us put $v = D^\alpha u, v_r(x'_r) = v(x'_r, a_r(x'_r))$. Differentiating v_r with respect to coordinates x'_r $|\beta|$-times, $|\beta| \leq k-l-1$, we obtain that $v \in W^{k-l-1,p}(\Delta_r)$; then, by the previous lemma, everything goes back to knowing whether $D^\beta v \in W^{1-1/p,p}(\Delta_r)$. But $D^\beta v$ is a linear combination of terms $aD^\gamma u, |\gamma| \leq |\beta|$, $a \in C^{0,1}(\overline{\Delta}_r)$, hence using once more the previous lemma it is sufficient to see that $D^\delta u \in W^{1-1/p,p}(\partial\Omega), |\delta| \leq k-1$, with the corresponding estimate.

Setting $w = D^\delta u$, we have $w \in W^{1,p}(\Omega)$. Using the transformation (2.31) and the inverse transformation which are at least lipschitzian, the result follows from Theorem 5.4. $\square$

2.5.5 A Lemma

We have to prove the "converse" of Theorem 5.5. We do this with more restrictive conditions concerning the domains.

A function $R(z), z \in \mathbb{R}^N$ will be called a *regularizing kernel* if $R \in C^\infty(\mathbb{R}^N)$, with support contained in the closed unit ball with center at the origin. We shall write $R \in \mathfrak{R}_N$. If $\int_{\mathbb{R}^N} R(z)dz = 1$, we define the *regularizing operator* by:

$$u_h(x) = \int_{|z|<1} R(z)u(hz+x)\,\mathrm{d}z = \frac{1}{h^N}\int_{|y-x|<h} R\left(\frac{y-x}{h}\right)u(y)\,\mathrm{d}y.$$

Lemma 5.6. *Let P be the pyramid defined by: $P = \{x \in \mathbb{R}^N,\ 0 < x_N < 1,\ |x_i| < 1 - x_N,\ i = 1,2,\ldots,N-1\}$, $\varphi_0 \in W^{k-1/p,p}(C)$, $p > 1$, where $C = \{x' \in \mathbb{R}^{N-1}, |x_i| < 1\}$. There exists a mapping $Z \in [W^{k-1/p,p}(C) \to W^{k,p}(P)]$ such that if $Z\varphi_0 = u$, then $u = \varphi_0$ on the basis C of P.*

Proof. Let $R \in \mathfrak{R}_{N-1}$, $\int_{\mathbb{R}^{N-1}} R(z')\,\mathrm{d}z' = 1$, let us set $x = (x', x_N)$, and if $x \in P$:

$$u(x', x_N) = \frac{1}{x_N^{N-1}}\int_{|y'-x'|<x_N} R\left(\frac{y'-x'}{x_N}\right)\varphi_0(y')\,\mathrm{d}y'.$$

Clearly we have $u \in C^\infty(P)$. Let$|\alpha| \le k-1$ and consider $D^\alpha u$; we have:

$$D^\alpha u(x) = \sum_{\substack{|\beta|=|\alpha|\\|\lambda|=(\alpha_N)}} c_{\beta\lambda}\int_{|z'|<1} R(z'){z'}^\lambda D^\beta\varphi_0(x_N z' + x)\,\mathrm{d}z'.$$

Now, if we proceed as in the proof of Theorem 1.2, we get the inequality:

$$\int_{|x'|<1-x_N} |D^\alpha u(x', x_N)|^p\,\mathrm{d}x' \le c\sum_{|\beta|=|\alpha|}\int_{|x'|<1} |D^\beta\varphi_0(x')|^p\,\mathrm{d}x',$$

which implies

$$|u|_{W^{k-1,p}(P)} \le c|\varphi_0|_{W^{k-1/p,p}(C)}. \tag{2.98 bis}$$

It remains to prove: if $R \in \mathfrak{R}_{N-1}$ and $f \in W^{k-1/p,p}(C)$, then for

$$v(x) = \int_{|z'|<1} R(z')f(x_N z' + x')\,\mathrm{d}z'$$

we have:

$$|v|_{W^{1,p}(P)} \leq c|f|_{W^{1-1/p,p}(C)}.$$

First we consider $\partial v/\partial x_i, 1 \leq i \leq N-1$; without loss of generality let us consider $\partial v/\partial x_1$; we have:

$$\frac{\partial v}{\partial x_1} = -\frac{1}{x_N^N}\int_{|y'-x'|<x_N} \frac{\partial R}{\partial z_1}\left(\frac{y'-x'}{x_N}\right) f(y')\,\mathrm{d}y'.$$

We have:

$$\frac{1}{x_N^N}\int_{|y'-x'|<x_N} \frac{\partial R}{\partial z_1}\left(\frac{y'-x'}{x_N}\right) \mathrm{d}y' = 0,$$

so

$$\begin{aligned}\frac{\partial v}{\partial x_1}(x) &= \frac{1}{x_N^N}\int_{|y'-x'|<x_N} \frac{\partial R}{\partial z_1}\left(\frac{y'-x'}{x_N}\right) (f(x')-f(y'))\,\mathrm{d}y' \\ &= \int_{|z'|<1} \frac{\partial R}{\partial z_1}(z')\frac{f(x')-f(x_N z'+x')}{x_N}\,\mathrm{d}z'.\end{aligned}$$

From this, we get:

$$\begin{aligned}&\int_P \left|\frac{\partial v}{\partial x_1}(x)\right|^p \mathrm{d}x \leq c_1 \int_P \mathrm{d}x \int_{|z'|<1}\left|\frac{f(x')-f(x_N z'+x')}{x_N}\right|^p dz' \\ &= c_1\int_P x_N^{(-p-N+1)}\,\mathrm{d}x\int_{|y'-x'|<x_N}\frac{|f(x')-f(y')|^p}{|x'-y'|^{N-2+p}}|x'-y'|^{N-2+p}\,\mathrm{d}y' \\ &= c_1\int_{(\max_{1\leq i\leq N-1}|x_i|)<1}\mathrm{d}x'\int_{|y'-x'|<1-(\max_{1\leq i\leq N-1}|x_i|)}\frac{|f(x')-f(y')|^p}{|x'-y'|^{N-2+p}}\,\mathrm{d}y'\times \\ &\quad\times\int_{|x'-y'|<1}^{|y'-x'|<1-(\max_{1\leq i\leq N-1}|x_i|)} |x'-y'|^{N-2+p}|x_N^{-p-N+1}\,\mathrm{d}x_N \\ &\leq \frac{2c_1}{N+p-2}\int_{(\max_{1\leq i\leq N-1}|x_i|)<1}\mathrm{d}x'\int_{|y'-x'|<1-(\max_{1\leq i\leq N-1}|x_i|)}\frac{|f(x')-f(y')|^p}{|x'-y'|^{N-2+p}}\,\mathrm{d}y',\end{aligned}$$

and then

$$\left|\frac{\partial v}{\partial x_1}\right|_{L^p(P)} \leq c|f|_{W^{1-1/p,p}(C)}.$$

Now, let us consider

$$\begin{aligned}\frac{\partial v}{\partial x_N} &= -\frac{N-1}{x_N^N}\int_{|y'-x'|<x_N} R\left(\frac{y'-x'}{x_N}\right) f(y')\,\mathrm{d}y' \\ &\quad -\frac{1}{x_N^N}\int_{|y'-x'|<x_N}\sum_{i=1}^{N-1}\frac{\partial R}{\partial z_i}\left(\frac{y'-x'}{x_N}\right)\frac{y_i-x_i}{x_N}f(y')\,\mathrm{d}y'.\end{aligned}$$

We have:

$$\frac{N-1}{x_N^N}\int_{|y'-x'|<x_N} R\left(\frac{y'-x'}{x_N}\right)\mathrm{d}y' + \frac{1}{x_N^N}\int_{|y'-x'|<x_N}\sum_{i=1}^{N-1}\frac{\partial R}{\partial z_i}\left(\frac{y'-x'}{x_N}\right)\frac{y_i-x_i}{x_N}\,\mathrm{d}y' = 0;$$

finally as above we get the inequality:

$$\left|\frac{\partial v}{\partial x_N}\right|_{L^p(P)} \le c|f|_{W^{1-1/p,p}(C)}.$$

Using Theorem 1.2, we see that $u = \varphi_0$ on the basis of P. □

2.5.6 The Converse Theorem

Lemma 5.7. *Let $R \in \mathfrak{R}_{N-1}$, $\int_{\mathbb{R}^{N-1}} R(z')\,\mathrm{d}z' = 1$, $l \ge 0$ an integer. We have:*

$$\frac{\partial^i}{\partial x_N^l}\left[x_N^{l-N+1}R\Big(\frac{y'-x'}{x_N}\Big)\right] = \frac{1}{x_N^{N-1}}H\Big(\frac{y'-x'}{x_N}\Big),$$

where $H \in \mathfrak{R}_{N-1}, \int_{\mathbb{R}^{N-1}} H(z')\,\mathrm{d}z' = l!$

Proof. We have to prove that $\int_{\mathbb{R}^{N-1}} H(z')\,\mathrm{d}z' = l!$. We have:

$$\int_{|x'-y'|<x_N}\frac{\partial^i}{\partial x_N^l}\left[x^{l-N+1}R\Big(\frac{y'-x'}{x_N}\Big)\right]\mathrm{d}y$$
$$= \frac{\partial^l}{\partial x_N^l}\int_{|x'-y'|<x_N} x_N^{l-N+1}R\Big(\frac{y'-x'}{x_N}\Big)\,\mathrm{d}y' = \frac{\partial^l}{\partial x_N^l}(x_N^l) = l!.$$

□

Lemma 5.8. *Let P be the pyramid as in Lemma 5.6, and C its basis, $\varphi_i \in W^{k-l-1/p,p}(C)$, l,k integers such that $0 \le l \le k-1$, $p > 1$. Then there exists a mapping*

$$Z_l \in [W^{k-l-1/p,p}(C) \to W^{k,p}(P)]$$

such that $Z_l\varphi_l = u_l$ satisfies:

$$u_l = \frac{\partial u_l}{\partial x_N} = \ldots \frac{\partial^{l-1}u_l}{\partial x_N^{l-1}} = 0 \;\; on\ C, \tag{2.99a}$$

$$\frac{\partial^l u_l}{\partial x_N^l} = \varphi_l \;\; on\ C. \tag{2.99b}$$

Proof. Let $R \in \mathscr{R}_{N-1}, \int_{\mathbb{R}^{N-1}} R(z')\,\mathrm{d}z' = 1/l!$, and let us set:

$$u_l(x', x_N) = x_N^{l-N+1} \int_{|x'-y'|<x_N} R(\frac{y'-x'}{x_N})\varphi_l(y')\,\mathrm{d}y'.$$

Obviously, $u_l \in C^\infty(P)$. Let $|\alpha| \leq k-l-1$; we get:

$$u_l(x', x_N) = x_N^l \int_{|z'|<1} R(z')\varphi_l(x_N z' + x')\,\mathrm{d}z',$$

thus

$$\left|D^\alpha u_l\right|_{L^p(P)} \leq c|\varphi_l|_{W^{k-l-1/p,p}(C)}, \tag{2.100}$$

for the same reasons as that used previously in the proof of Lemma 5.6, starting from the proof of (2.98bis). Let $k-l \leq |\alpha| \leq k-1$; let us set $\alpha = \alpha' + \alpha''$, $|\alpha''| = k-l-1$. We have for $M \in \mathfrak{R}_{N-1}$:

$$D^{\alpha'} u_l(x', x_n) = x_N^{|\alpha|'-1} \int_{|z'|<1} M(z')\varphi_l(x_N z' + x')\,\mathrm{d}z'. \tag{2.101}$$

If we apply the operator $D^{\alpha''}$ to (2.101), we get (2.100) with $|\alpha| \leq k-1$ for the same reasons as in the proof of (2.98bis). If $|\alpha| = k$, we use the ideas of the proof of Lemma 5.6 and we get:

$$Z_l \in [W^{k-l-1/p,p}(C) \to W^{k,p}(P)].$$

It is clear that (2.98) holds; concerning (2.99a, b), we use the previous lemma. □

Theorem 5.6. *Let P be the pyramid as in Lemma 5.6 and C its basis, k an integer, $k \geq 1$, $p > 1$, $\varphi_l \in W^{k-l-1/p,p}(C)$, $l = 0, 1, \ldots, k-1$. There exists a mapping:*

$$Z \in \left[\prod_{l=0}^{k-1} W^{k-l-1/p,p}(C) \to W^{k,p}(P)\right]$$

such that if $Z(\varphi_0, \varphi_1, \ldots, \varphi_{k-1}) = u$, then $\frac{\partial^l u}{\partial x_N^l} = \varphi_l$ on C.

Indeed: We use Lemmas 5.6 and 5.8 and proceed as in the construction of (2.88). □

2.5.7 The Converse Theorem (Continuation)

Let $k = 1$. We have:

Theorem 5.7. *Let $\Omega \in \mathfrak{N}^{0,1}, p > 1$. There exists a mapping $Z \in [W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)]$ such that for $h \in W^{1-1/p,p}(\partial\Omega), Zh = u$, we have $u = h$ on $\partial\Omega$.*

Proof. We use the partition of unity from 1.2.4 and put $h_r = h\varphi_r$, $1 \le r \le m$; to simplify we omit the index r. By projection onto the hyperplane $x_N = 0$, the function $h \in W^{(1-1/p),p}(\Delta)$. Without loss of generality we assume $\alpha = 1$ (in the definition of Δ in 1.2.4). We use Lemma 5.6 to construct $Z \in [W^{1-1/p,p}(\Delta) \to W^{1,p}(P)]$. Since $\operatorname{supp} h \subset \Delta$, with compact support, there exists $\psi \in C_0^\infty(\mathbb{R}^N)$, $\operatorname{supp} \psi \subset P \cup C$ such that $h\psi = h$ on C. We write $Zh = h$; we have $h\psi \in W^{1,p}(K_+)$, $K_+ = \{x \in \mathbb{R}^N, |x_i| < 1, i = 1,2,\ldots,N-1, 0 < x_N < 1\}$. We use the transformation (2.31), and going back to the index r starting from $h\psi$ we construct a function $v_r \in W^{1,p}(V_r)$, $\operatorname{supp} v_r \subset V_r \cup \Lambda_r$. Obviously, the function $v = \sum_{r=1}^m v_r$ gives the extension of h on Ω. □

Problem 5.1. Prove a theorem analogous to Theorem 5.7, but for $k \ge 2$, $\Omega \in \mathfrak{N}^{0,1}$ or for a nonsmooth boundary $\partial\Omega$. The problem can be posed in the following way: Let t be the number of indexes α, $|\alpha| = k-1$. We consider the closed subset $W \subset \left[W^{1-1/p,p}(\partial\Omega)\right]^t$ as the closure of the set M of elements $(D^{\alpha[1]}u, D^{\alpha[2]}u, \ldots, D^{\alpha[t]}u), u \in C^\infty(\overline{\Omega})$. Is the mapping $Z \in [W^{k,p}(\Omega) \to W]$, defined by $Zu = (D^{\alpha[1]}u, D^{\alpha[2]}u, \ldots, D^{\alpha[t]}u)$, surjective?

For $N = 2, \partial\Omega$ piecewise smooth, the solution with some modifications is given by G.N. Jakovlev [1]. Cf. also P. Grisvard [1].

If $\partial\Omega$ is sufficiently smooth, the problem is solved (cf. papers by L.N. Slobodetskii [1], S.V. Uspenskii [1]):

Theorem 5.8. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p > 1$, $h_l \in W^{k-l-1/p,p}(\partial\Omega)$, $l = 0, 1,2,\ldots,k-1$. There exists a mapping,*

$$Z \in \left[\prod_{l=0}^{k-1} W^{k-l-1/p,p}(\partial\Omega) \to W^{k,p}(\Omega)\right],$$

such that if $Z((h_0,h_1,\ldots,h_{k-1})) = u$, then on $\partial\Omega$ we have $\partial^l u/\partial n^l = h_l$, $l = 0,1,2,\ldots,k-1$.

Proof. Due to the previous theorem it is sufficient to consider the case $k \ge 2$. We use $G_r, \psi_r, \ldots$ as in 1.2.4, and the transformation (1.35) from Chap. 1; this transformation is one-to-one, with lipschitzian derivatives of order $\le k-1$ on $\overline{G}_r$, and the inverse transformation has the same properties on $\overline{K}_r$. Put $h_{lr} = h_l\psi_r$. By projection on the hyperplane $x_{rN} = 0$, we consider h_{lr} as a function of the variable σ; we have $h_{lr} \in W^{k-l-1/p,p}(\Delta_r)$.

Without loss of generality, we can assume in the definition of G_r that $\alpha = \delta = 1$. According to Theorem 5.6 we construct $u_r \in W^{k,p}(P)$ such that on C $\partial^l u_r/\partial x_N^l = (-1)^l h_{lr}$. Since $\operatorname{supp} h_{lr} \subset C$, there exists $\psi \in C_0^\infty(\mathbb{R}^N)$ with $\operatorname{supp} \psi \subset P \cup C$ such that $\partial^l(u_r\psi)/\partial x_N^l = (-1)^l h_{lr}$ on C. We have $u_r\psi \in W^{k,p}(K_+)$, $K_+ = \{y \in \mathbb{R}^N, y = (\sigma,t), |\sigma_i| < 1, 0 < t < 1\}$. Using the transformation 1.2.7, we get $v_r = u_r\psi \in W^{k,p}(G_r \cap \Omega)$; but according to the form of the support of ψ we have $v_r \in W^{k,p}(\Omega)$; on the other hand $\partial^l v_r/\partial n^l = h_{lr}$ on $\partial\Omega$, thus $v = \sum_{r=1}^M v_r$ gives the transformation such that $Z((h_0,h_1,\ldots,h_{k-1})) = v$. □

2.5.8 Remarks

It is possible to investigate the spaces $W^{k,p}(\Omega)$, where k is noninteger, in more detail. Concerning the questions about extension and density, cf. J.L. Lions, E. Magenes [4], for imbedding theorems cf. S.V. Uspenskii [1, 2, 4]. There is a strong link between interpolation in the sense of J.L. Lions and traces, cf. J.L. Lions [7–10]. From our considerations on these questions it is possible to obtain some consequences, for instance:

Corollary 5.1. *Let C be an $(N-1)-$dimensional cube, $p>1$. If $p<N$, $1/q = 1/p - [1/(N-1)][(p-1)/p]$, then $W^{1-1/p,p}(C) \subset L^q(C)$ algebraically and topologically; if $p=N$, q is an arbitrary real number ≥ 1; if $p>N$, then $W^{1-1/p,p}(C) \subset C^{0,\mu}(\overline{C})$ with $\mu = 1-N/p$, algebraically and topologically.*

Remark 5.3. Let us set $k = 1-1/p$. Then in Corollary 5.1 for $p<N$ we have $1/q = 1/p - k/(N-1)$, and we have the formula from Theorem 3.4. It holds in other cases, cf. S.V. Uspenskii [1].

Let C be an $(N-1)-$dimensional cube, $u \in W^{k-1/p,p}(C), k \geq 1$ an integer, $p>1$; according to Lemma 5.5 it is possible to extend u to the corresponding pyramid P so that $u \in W^{k,p}(P)$. $P \in \mathfrak{N}^{0,1} \Longrightarrow W^{k,p}(P) = \overline{C^\infty(\overline{P})} \Longrightarrow W^{k-1/p,p}(C) = \overline{C^\infty(\overline{C})}$, hence

Corollary 5.2. *Let C be an $(N-1)-$dimensional cube, $k \geq 1$ an integer, $p>1$. Then $W^{k-1/p,p}(C) = \overline{C^\infty(\overline{C})}$.*

Let C_{N-1} be an $(N-1)$-dimensional cube, $u \in W^{k-1/p,p}(C_{N-1})$, $k \geq 1$ an integer, $p>1$. According to Lemma 5.6 we can extend u to a cube C_N with C_{N-1} as its face. By the approach used in Theorem 3.8 it is possible to extend u to the whole $\mathbb{R}^N$ so that $|u|_{W^{k,p}(\mathbb{R}^N)} \leq c|u|_{W^{k,p}(C_N)}$. We get an extension of u to $\mathbb{R}^{N-1}, u \in W^{k-1/p,p}(\mathbb{R}^{N-1})$. We have also

Corollary 5.3. *Let C be an $(N-1)-$dimensional cube, $p>1$, $k \geq 1$ an integer. Then there exists $P \in [W^{k,p}(C) \to W^{k-1/p,p}(\mathbb{R}^{N-1})]$, such that $RP = I$, where R is the restriction operator, $R \in [W^{k-1/p,p}(\mathbb{R}^N) \to W^{k-1/p,p}(C)]$, and I the identity operator on $W^{k-1/p,p}(C)$.*

Remark 5.4. A well known simple counterexample due to J. Hadamard states the existence of a continuous function on the unit circle which is not the trace of a function from $W^{1,2}(K)$ where K is the unit disc, cf. for instance S.G. Mikhlin [2]. L. De Vito [1] has constructed an absolutely continuous function with the same property. It is easy to see that a piecewise smooth function cannot be the trace of a function $u \in W^{1,2}(\Omega)$. For sufficient conditions implying that f belongs to $W^{k-1/p,p}(\partial\Omega)$, cf. S.M. Nikolskii [2, 5–8], G. Prodi [1], J. Nečas [11], etc.

Remark 5.5. If $p=1$, $\Omega \in \mathfrak{N}^{0,1}$, we have obtained by Theorem 4.2 that the traces belong to $L^1(\partial\Omega)$. We have also the converse, i.e. $W^{1,1}(\Omega)/W_0^{1,1}(\Omega) = L^1(\partial\Omega)$; cf. E. Gagliardo [1].

Exercise 5.3. Prove that on the unit circle we have:

$$W^{1/2,2}(\partial\Omega) = \{u \in L^2(\partial\Omega), \quad \sum_{-\infty}^{\infty} n|a_n|^2 < \infty,\ a_n = \frac{1}{2\pi}\int_0^{2\pi} u(\vartheta)\exp(-in\vartheta)\,\mathrm{d}\vartheta\}.$$

Exercise 5.4. Prove directly that for $\Omega \in \mathfrak{N}^{0,1}$, $u \in W^{1,p}(\partial\Omega)$ we can extend u to Ω so that $u \in W^{1,p}(\Omega), p \geq 1$.

2.6 Compactness

2.6.1 The Kondrashov Theorem

In Chap. 1, we have proved a particular case of the compactness of the imbedding operator, cf. Chap. 1, Theorem 1.1.4. We shall give some generalizations; concerning the literature on the subject see W. Kondrashov [1], S.L. Sobolev [1], V.I. Smirnov [2], etc.

Theorem 6.1. *Let $\Omega \in \mathfrak{N}^{0,1}$, $1 \leq p < N$, $1 \geq 1/q > 1/p - 1/N$. The identity mapping $I : W^{1,p}(\Omega) \to L^q(\Omega)$ is compact.*

Proof. Let $u_n \in W^{1,p}(\Omega)$ be a bounded sequence, $|u_n|_{W^{1,p}(\Omega)} \leq 1$. It is possible to find elements $v_n \in C^\infty(\overline{\Omega})$ such that $|u_n - v_n|_{W^{1,p}(\Omega)} < 1/n$, $|v_n|_{W^{1,p}(\Omega)} \leq 1$; it is sufficient to prove the existence of a subsequence of v_n, for simplicity we denote this subsequence again v_n, which converges in $L^q(\Omega)$. Let us put $1/q^* = 1/p - 1/N$, let $\varepsilon > 0$, $\Omega^* \subset \overline{\Omega^*} \subset \Omega, \Omega^*$ a subdomain such that

$$\text{meas}\,(\Omega - \Omega^*) < \left(\frac{\varepsilon}{3c_1}\right)^{q^*/(q^*-1)},$$

where $|v_n|_{L^{q^*}(\Omega)} \leq c_1$ (cf. Theorem 3.4). Let $\delta > 0$ be sufficient small such that $x \in \Omega^*$, $|z| < \delta \Longrightarrow x + z \in \Omega$; we have:

$$|v_n(x+z) - v_n(x)| = \left|\int_0^{|z|} \sum_{i=1}^N \frac{\partial v_n}{\partial x_i}\left(x + \frac{z}{|z|}t\right)\frac{z_i}{|z|}\,\mathrm{d}t\right| \leq \int_0^{|z|} \sum_{i=1}^N \left|\frac{\partial v_n}{\partial x_i}\left(x + \frac{z}{|z|}t\right)\right| \mathrm{d}t,$$

then it follows that

$$\int_{\Omega^*} |v_n(x+z) - v_n(x)|\,\mathrm{d}x = \int_0^{|z|} dt \int_\Omega \sum_{i=1}^N \left|\frac{\partial v_n}{\partial x_i}(y)\right| \mathrm{d}y \leq |z| \cdot |v_n|_{W^{1,1}(\Omega)}.$$

Let $|v_n|_{W^{1,1}(\Omega)} \leq c_2$, and let us choose $\delta < \varepsilon/3c_2$. Let us extend v_n by zero outside of Ω; then we have:

$$\int_{\Omega} |v_n(x+z) - v_n(x)|\,\mathrm{d}x$$
$$\leq \int_{\Omega-\Omega^*} |v_n(x+z)|\,\mathrm{d}x + \int_{\Omega-\Omega^*} |v_n(x)|\,\mathrm{d}x + \int_{\Omega^*} |v_n(x+z) - v_n(x)|\,\mathrm{d}x \leq \varepsilon.$$

If $q = 1$ the sequence v_n has the same properties as that in Theorem 1.3. and the result is true in this case. If $1 < q < q^*$, we have:

$$\int_{\Omega} |v_m - v_n|^q\,\mathrm{d}x = \int_{\Omega} |v_m - v_n|^{[q^*(q-1)]/(q^*-1)} |v_m - v_n|^{(q^*-q)/(q^*-1)}\,\mathrm{d}x$$
$$\leq \left(\int_{\Omega} |v_m - v_n|^{q^*}\,\mathrm{d}x\right)^{(q-1)/(q^*-1)} \left(\int_{\Omega} |v_m - v_n|\,\mathrm{d}x\right)^{(q^*-q)/(q^*-1)};$$

this finishes the general case. □

Corollary 6.1. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p \geq 1$, $kp < N$, k an integer, $1 \geq 1/q > 1/p - k/N$. The identity mapping $I : W^{k,p}(\Omega) \to L^q(\Omega)$ is compact.*

Corollary 6.2. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p \geq 1$, $kp = N$, k an integer, $q \geq 1$ arbitrary. The identity mapping $I : W^{k,p}(\Omega) \to L^q(\Omega)$ is compact.*

Corollary 6.3. *Let $\Omega \in \mathfrak{N}^{0,1}$, $p \geq 1$, $kp > N$, k an integer. The identity mapping $I : W^{k,p}(\Omega) \to C(\overline{\Omega})$ is compact.*

2.6.2 Traces

As far as concern traces, we have

Theorem 6.2. *Let $\Omega \in \mathfrak{N}^{0,1}$, $1 < p < N$, $1 \geq 1/q > 1/p - [1/(N-1)](p-1)/p$. The mapping $Z \in [W^{1,p}(\Omega) \to L^q(\partial\Omega)]$, which defines the traces, is compact.*

Proof. As in proof of Theorem 6.1 it is sufficient to consider a sequence $v_n \in C^\infty(\overline{\Omega})$, $n = 1,2,\ldots$, bounded in $W^{1,p}(\Omega)$. With a partition of unity as in 1.2.4 we take v_r, $\operatorname{supp} v_r \subset V_r \cup \Lambda_r$. For $1 \leq p < \infty$ the function $p \to 1/p - [1/(N-1)](p-1)/p$ is decreasing, thus there exists exactly one value p^* such that $1/q = 1/p^* - [1/(N-1)](p^*-1)/p^*$. It is clear that it is sufficient to consider the case $q > 1$. According to (2.69) with $1/q^* = 1/p^* - 1/N$ (we omit the index r)

$$|v_n - v_m|^q_{L^q(\Lambda)} \leq c[|v_n - v_m|^q_{L^q(V)} + |v_n - v_m|^{(Np^*-N)/(N-p^*)}_{L^{q^*}(V)} |v_n - v_m|^q_{W^{1,p^*}(V)}]. \tag{2.102}$$

According to Theorem 6.1 we can extract a subsequence denoted also by v_n which converges in $L^{q^*}(V)$; but $q \leq q^*$, hence (2.102) implies the convergence of v_n in $L^q(V)$.

□

Exercise 6.1. Give a formulation of the previous theorem with the hypotheses $kp < N$ or $kp = N$ with $k \geq 1$ an integer.

Remark 6.1. Let $\Omega \in \mathfrak{N}^{0,1}$, $1 < p < N$. Then the mapping Z of $W^{1,p}(\Omega)$ into $L^q(\partial\Omega)$ with $1/q = 1/p - [1/(N-1)](p-1)/p$ is not surjective. Indeed: it is always possible to find a sequence v_n defined on $\partial\Omega$ and bounded by the same constant, with $|v_n|_{L^1(\partial\Omega)} = 1$, which converges weakly to zero in $L^1(\partial\Omega)$. If Z was surjective, we could extend v_n to Ω in such a way that v_n is bounded in $W^{1,p}(\Omega)$, and due to Theorem 6.2 there will exist a subsequence v_{n_k} which will converge in $L^1(\partial\Omega)$ to a limit equal to zero, and this is a contradiction to $|v_n|_{L^1(\partial\Omega)} = 1$.

2.6.3 The Lions Lemma, Another Theorem of Compactness

We can prove Lemma 5.1 from Chap. 1 under more general conditions (cf. J.L. Lions [4]):

Lemma 6.1. *Let B_i, $i = 1,2,3$, be three Banach spaces, $B_1 \subset B_2 \subset B_3$ algebraically and topologically. Assume that the identity mapping $I : B_1 \to B_2$ is compact. Then for every $\varepsilon > 0$ there exists $\lambda(\varepsilon)$ such that $u \in B_1 \Longrightarrow$*

$$|u|_{B_2} \leq \varepsilon |u|_{B_1} + \lambda(\varepsilon)|u|_{B_3}.$$

The proof is the same as that of Lemma 1.5.1.

Example 6.1. Let $B_1 = W^{1,p}(\Omega)$, $B_2 = L^q(\Omega)$, $B_3 = L^1(\Omega)$, $\Omega \in \mathfrak{N}^{0,1}$, $1/q > 1/p - 1/N$ if $p < N$, $q \geq 1$ if $p \geq N$. This satisfies the conditions of Lemma 6.1.

Theorem 6.3. *Let $\Omega \in \mathfrak{N}^{0,1}$, $1 \leq p$, $1 \leq q \leq p$. The identity mapping $I : W^{1,p}(\Omega) \to L^q(\Omega)$ is compact.*

Proof. It is sufficient to consider a sequence $v_n \in C^\infty(\overline{\Omega})$ bounded in $W^{1,p}(\Omega)$, and to extract a subsequence which converges in $L^p(\Omega)$. Fix $\varepsilon > 0$; we can find $\Omega' \subset \overline{\Omega}' \subset \Omega$ such that

$$\left(\int_{\Omega - \Omega'} |v_n(x)|^p \, \mathrm{d}x \right)^{1/p} < \frac{\varepsilon}{3}. \tag{2.102 bis}$$

To do this we consider the open sets V_r, $r = 1,2,\ldots,m$. If $a_r(x_r') < x_N < a_r(x_r') + \beta/2$, we have:

$$v_n(x_r', x_{rN}) = v_n(x_r', \tau) - \int_{x_{rN}}^{\tau} \frac{\partial v_n}{\partial x_{rN}}(x_r', \xi_N) \, \mathrm{d}\xi_N, \quad a_r(x_r') + \beta/2 < \tau < a_r(x_r') + \beta. \tag{2.103}$$

From this we get (we omit the index r):

$$
\begin{aligned}
|v_n(x',x_N)|^p &\le 2^{p-1}\left[|v_n(x',\tau)|^p + \left|\int_{x_N}^{\tau}\frac{\partial v_n}{\partial x_n}(x',\xi_N)\,\mathrm{d}\xi_N\right|^p\right] \\
&\le 2^{p-1}\left[|v_n(x',\tau)|^p + \beta^{p-1}\int_{a(x')}^{a(x')+\beta}\left|\frac{\partial v_n}{\partial x_n}(x',\xi_N)\right|^p\,\mathrm{d}\xi_N\right].
\end{aligned}
\tag{2.104}
$$

Now we integrate (2.104) with respect to $\tau \in (a(x')+\beta/2, a(x')+\beta)$ and obtain

$$
\frac{\beta}{2}|v_n(x',x_N)|^p \le 2^{p-1}\left[\int_{a(x')+\beta/2}^{a(x')+\beta}|v_n(x',\tau)|^p\,\mathrm{d}\tau + \frac{\beta^p}{2}\int_{a(x')}^{a(x')+\beta}\left|\frac{\partial v_n}{\partial x_n}(x',\xi_N)\right|^p\mathrm{d}\xi_N\right]; \tag{2.105}
$$

now by integration of (2.105) with respect to $x' \in \Delta$ and to $x_N \in (a(x'), a(x')+\gamma)$, γ sufficiently small, we get:

$$
\frac{\beta}{2}\int_{\Delta}\mathrm{d}x'\int_{a(x')}^{a(x')+\gamma}|v_n(x',x_N)|^p\,\mathrm{d}x_N \le 2^{p-1}\left(1+\frac{\beta^p}{2}\right)\gamma|v_n|^p_{W^{1,p}(\Omega)};
$$

this gives (2.102 bis) if γ is sufficiently small. Now let Ω'' be a subdomain satisfying $\Omega' \subset \Omega'' \subset \overline{\Omega''} \subset \Omega$ and $\delta > 0$ sufficiently small such that $x \in \Omega'', |z| < \delta \Longrightarrow x+z \in \Omega$. We have:

$$
\begin{aligned}
|v_n(x+z)-v_n(x)| &\le \left|\int_0^{|z|}\sum_{i=1}^{N}\frac{\partial v_n}{\partial x_i}\left(x+\frac{z}{|z|}t\right)\frac{z_i}{|z|}\,\mathrm{d}t\right| \\
&\le |z|^{1-1/p}\sum_{i=1}^{N}\left(\int_0^{|z|}\left|\frac{\partial v_n}{\partial x_i}\left(x+\frac{z}{|z|}t\right)\right|^p dt\right)^{1/p},
\end{aligned}
$$

hence

$$
|v_n(x+z)-v_n(x)|^p \le c|z|^{p-1}\sum_{i=1}^{N}\int_0^{|z|}\left|\frac{\partial v_n}{\partial x_i}\left(x+\frac{z}{|z|}t\right)\right|^p dt,
$$

and

$$
\int_{\Omega''}|v_n(x+z)-v_n(x)|^p\,\mathrm{d}x \le c|z|^p\sum_{i=1}^{N}\int_{\Omega}\left|\frac{\partial v_n}{\partial x_i}(y)\right|^p\mathrm{d}y \le c|z|^p|v_n|^p_{W^{1,p}(\Omega)},
$$

and thus

$$
\left(\int_{\Omega''}|v_n(x+z)-v_n(x)|^p\,\mathrm{d}x\right)^{1/p} \le c|z||v_n|_{W^{1,p}(\Omega)}.
$$

Finally, let δ be sufficiently small such that $x \in \Omega - \Omega''$, $|z| < \delta \Longrightarrow x+z \in \Omega - \Omega'$, and

$$
\left(\int_{\Omega''}|v_n(x+z)-v_n(x)|^p\,\mathrm{d}x\right)^{1/p} < \frac{\varepsilon}{3}.
$$

Let us set $v_n(x) = 0$ for $x \notin \Omega$. We get:

$$\left(\int_\Omega |v_n(x+z) - v_n(x)|^p \,\mathrm{d}x\right)^{1/p} \le \left(\int_{\Omega''} |v_n(x+z) - v_n(x)|^p \,\mathrm{d}x\right)^{1/p} + \left(\int_{\Omega-\Omega''} |v_n(x+z)|^p \,\mathrm{d}x\right)^{1/p} + \left(\int_{\Omega-\Omega''} |v_n(x)|^p \,\mathrm{d}x\right)^{1/p} \le \varepsilon.$$

Now Theorem 1.3 gives the assertion. □

Exercise 6.2. Prove Theorem 6.3 for a domain Ω starshapped with respect to the origin.

Remark 6.2. For an arbitrary domain Ω, bounded or unbounded, the restriction operator $R : W^{1,p}(\Omega) \to L^q(\Omega')$ with Ω' bounded, $\overline{\Omega}' \subset \Omega$, and with q as in Theorem 6.1, is compact.

Exercise 6.3. Let Ω be a bounded domain such that for any $\varepsilon > 0$, we can find $\Omega' \subset \overline{\Omega'} \subset \Omega$ such that $|u|_{W^{1,p}(\Omega)} \le 1 \Longrightarrow |u|_{L^p(\Omega-\Omega')} < \varepsilon$. Then the identity mapping $I : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact.

Remark 6.3. If there exists an extension operator $P \in [W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)]$, then Theorem 6.3 is true for Ω bounded.

Problem 6.1. Characterize bounded domains for which the imbedding theorem $W^{1,2}(\Omega) \subset L^2(\Omega)$ is compact. We find only equivalent statements as for instance the existence of a spectrum having the form given in Theorem 1.5.1.

2.7 Quotient Spaces, Equivalent Norms

2.7.1 Equivalent Norms

The methods used in this paragraph are strongly related to these introduced in Chap. 1, Sect. 1.1 concerning the same questions, and also with the work of J. Deny, J.L. Lions [1], J.L. Lions [2]. As in 1.1.7 we denote by $P_{(k-1)}$ *the space of polynomials of degree* $\le k-1$. We shall consider only domains such that

$$v \in P_{(k-1)} \Longrightarrow |v|_{L^p(\Omega)} < \infty. \tag{2.106}$$

Lemma 7.1. *Let Ω be a domain satisfying (2.106), $p \ge 1$, $k \ge 1$ an integer. Then there exist functionals f_i, $i = 1, 2, \dots, l$, on $W^{k,p}(\Omega)$ such that $v \in P_{(k-1)}$ implies the equivalence:*

$$\sum_{i=1}^{l} |f_i v|^p = 0 \Longleftrightarrow v \equiv 0. \tag{2.106bis}$$

Proof. There are many ways to construct f_i: for instance we take Ω^*, a bounded nonempty subdomain of Ω, and define:

$$f_\alpha v = \int_{\Omega^*} x^\alpha v(x)\,\mathrm{d}x, \quad |\alpha| \le k-1, \tag{2.107a}$$

or

$$f_\alpha v = \int_{\Omega^*} D^\alpha v(x)\,\mathrm{d}x, \quad |\alpha| \le k-1. \tag{2.107b}$$

$\square$

Let us formulate the first theorem on equivalent norms in $W^{k,p}(\Omega)$:

Theorem 7.1. *Let $\Omega \in \mathfrak{N}^0$, f_i functionals satisfying (2.106 bis), $p \ge 1$, $k \ge 1$ an integer. We have the inequality:*

$$c_1 |u|_{W^{k,p}(\Omega)} \le \left[\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)} + \sum_{i=1}^{l} |f_i u|^p \right]^{1/p} \le c_2 |u|_{W^{k,p}(\Omega)}.$$

Proof. We have to prove the inequality:

$$c_1 |u|_{W^{k,p}(\Omega)} \le \left[\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)} + \sum_{i=1}^{l} |f_i u|^p \right]^{1/p}. \tag{2.108}$$

We proceed by contradiction: suppose that (2.108) does not hold for any constant c_1, for instance for $c_1 = 1/n$. Then there exists a function $u_n \in W^{k,p}(\Omega)$, $|u_n|_{W^{k,p}(\Omega)} = 1$ such that

$$\frac{1}{n} > \left[\sum_{|\alpha|=k} |D^\alpha u_n|^p + \sum_{i=1}^{l} |f_i u_n|^p \right]^{1/p}. \tag{2.109}$$

From this we get, for $|\alpha| = k$:

$$\lim_{n\to\infty} D^\alpha u_n = 0 \text{ in } L^p(\Omega). \tag{2.110}$$

According to Theorem 6.3, the identity mapping $I : W^{k,p}(\Omega) \to W^{k-1,p}(\Omega)$ is compact, hence there exists a subsequence u_{n_m} of u_n which converges in $W^{k-1,p}(\Omega)$ and by (2.110) in $W^{k,p}(\Omega)$. Let $u = \lim_{m\to\infty} u_{n_m}$. We have $D^\alpha u = 0 \Longrightarrow u \in P_{(k-1)}$, but $P_{(k-1)}$ is of finite dimension and hence closed in $W^{k,p}(\Omega)$. Then, due to (2.108), we have:

$$\sum_{i=1}^{l} |f_i u|^p = 0 \Longrightarrow u = 0,$$

which is a contradiction to the fact that

$$\lim_{m\to\infty} |u_{n_m}|_{W^{k,p}(\Omega)} = |u|_{W^{k,p}(\Omega)} = 1.$$

$\square$

Remark 7.1. Clearly Theorem 7.1 holds if the identity mapping $I : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact.

2.7.2 Quotient Spaces

Let $P \subset P_{(k-1)}$, and denote by $W^{k,p}(\Omega)/P$ the quotient space (cf. 1.1.7) with the topology associated with the usual norm:

$$\text{For } \tilde{u} \in W^{k,p}(\Omega)/P, \quad |\tilde{u}|_{W^{k,p}(\Omega)/P} = \inf_{u \in \tilde{u}} |u|_{W^{k,p}(\Omega)}. \tag{2.111}$$

Theorem 7.2. *Let $\Omega \in \mathfrak{N}^0$, and let us assume that the identity mapping $I : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact. Then we have:*

$$c_1 |\tilde{u}|_{W^{k,p}(\Omega)/P_{(k-1)}} \leq \Big[\sum_{|\alpha|=k} \big[D^\alpha u|^p_{L^p(\Omega)}\big]^{1/p} \leq c_2 |\tilde{u}|_{W^{k,p}(\Omega)/P_{(k-1)}}. \tag{2.111 bis}$$

If $p = 2$, $W^{k,2}(\Omega)/P_{(k-1)}$ is a Hilbert space with the scalar product

$$(\tilde{v}, \tilde{u}) = \sum_{|\alpha|=k} \int_\Omega D^\alpha v D^\alpha \overline{u} \,\mathrm{d}x. \tag{2.112}$$

Proof. First, $W^{k,p}(\Omega)/P_{(k-1)}$ is complete with respect to the norm

$$\Big[\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)} \Big]^{1/p}. \tag{2.113}$$

Indeed: let $\tilde{u}_n$ be a Cauchy sequence for (2.113). We can choose $u_n \in \tilde{u}_n$ such that $f_i u_n = 0$, $i = 1, 2, \ldots, l, f_i$ satisfying (2.106 bis); this is always possible by Lemma 7.1. We apply Theorem 7.1, hence u_n is a Cauchy sequence in $W^{k,p}(\Omega)$; let $\lim_{n\to\infty} u_n = u$, which implies $\lim_{n\to\infty} \tilde{u}_n = \tilde{u}$. Denote by B_1 the quotient space $W^{k,p}(\Omega)/P_{(k-1)}$ with the norm (2.111), and by B_2 the same space but with the norm (2.113). The identity mapping $I : B_1 \to B_2$ is continuous, due to the Banach theorem, cf. Chap. 1, Sect. 1.1, and the same property is true for inverse transformation. The result follows. □

Hereafter, when we shall use the mentioned Banach theorem we will not specify the spaces B_1, B_2, etc.

Theorem 7.3. *Let $\Omega \in \mathfrak{N}^0$, and assume that the identity mapping $I : W^{1,p}(\Omega) \to L^p(\Omega)$ is compact, $P \subset P_{(k-1)}$, f_i, $i = 1, 2, \ldots, l$, functionals on $W^{k,p}(\Omega), k \geq 1$ an integer, $p \geq 1$ such that*

$$\textit{for } v \in P_{(k-1)}, \quad \sum_{i=1}^{l} |f_i v|^p = 0 \Longleftrightarrow v \in P.$$

Then

$$c_1 |\tilde{u}|_{W^{k,p}(\Omega)/P} \leq \left([\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)}]^p + \sum_{i=1}^{l} |f_i u|^p \right)^{1/p} \leq c_2 |\tilde{u}|_{W^{k,p}(\Omega)/P}.$$

If $p = 2$, $W^{k,p}(\Omega)/P$ is a Hilbert space with the scalar product:

$$(\tilde{v}, \tilde{u}) = \sum_{|\alpha|=k} \int_\Omega D^\alpha v D^\alpha \overline{u} \, \mathrm{d}x + \sum_{i=1}^{l} f_i v \overline{f_i u}.$$

Proof. It is sufficient to prove that $W^{k,p}(\Omega)/P$ is complete with the norm

$$\Big[\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)} + \sum_{i=1}^{l} |f_i u|^p \Big]^{1/p}. \tag{2.114}$$

Let $\tilde{u}_n$ be a Cauchy sequence. According to Theorem 7.2, we can find polynomials $p_n \in P_{(k-1)}$ such that $\lim_{n\to\infty}(u_n + p_n) = u$ in $W^{k,p}(\Omega)$. Since $\bar{u}_n$ is a Cauchy sequence with respect to the norm (2.114), the same holds for the sequence $\tilde{p}_n$. Clearly $(\sum_{i=1}^{l} |f_i u|^p)^{1/p}$ is a norm in $P_{(k-1)}/P$, hence $\lim_{n\to\infty} \tilde{p}_n = \tilde{p}$ in $W^{k,p}(\Omega)/P$, and $\lim_{n\to\infty} \tilde{u}_n = \tilde{u} - \tilde{p}$. □

Remark 7.2. Let $\Omega^* \subset \Omega$, Ω^* non empty, L the orthogonal complement of P in $P_{(k-1)}$ for the space $L^2(\Omega^*)$; $p_1, p_2, \ldots, p_l$ a basis of L. Then the functionals $f_i u = \int_{\Omega^*} u p_i \, \mathrm{d}x$ satisfy the hypotheses of Theorem 7.3.

2.7.3 The Spaces $V^{k,p}(\Omega)$

Let us formulate a theorem on equivalent norms:

Theorem 7.4. *Let $\Omega \in \mathfrak{N}^0$, Ω^* a nonempty and open subset of Ω. Then we have the inequality*

$$|u|_{W^{k,p}(\Omega)} \leq c \left[\sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^p \, \mathrm{d}x + \int_{\Omega^*} |u|^p \, \mathrm{d}x \right]^{1/p}. \tag{2.115}$$

Proof. It is sufficient to prove that $W^{k,p}(\Omega)$ with the norm (2.115) is complete. Let u_n be a Cauchy sequence. According to Theorem 7.2 there exist $p_n \in P_{(k-1)}$

such that $\lim_{n\to\infty}(u_n+p_n)=u$ in $W^{k,p}(\Omega)$. Since u_n+p_n is a Cauchy sequence for the norm (2.115), $\lim_{n\to\infty} p_n = p$ in $L^p(\Omega^*)$ and in $W^{k,p}(\Omega)$, which implies that $\lim_{n\to\infty} u_n = u-p$ in $W^{k,p}(\Omega)$. □

Fix $\Omega\subset\mathbb{R}^N$. Denote by $V^{k,p}(\Omega)$ the space of functions in $L^p_{loc}(\Omega)$ whose distributional derivatives of order k belong to $L^p(\Omega)$. Let $\Omega^*\subset\overline{\Omega}^*\subset\Omega$; Ω^* bounded; *on* $V^{k,p}(\Omega)$ *we define the norm by*

$$\left[\sum_{|\alpha|=k}|D^\alpha u|^p_{L^p(\Omega)}+|u|^p_{L^p(\Omega^*)}\right]^{1/p}. \tag{2.116}$$

Theorem 7.5. *The space $V^{k,p}(\Omega)$ is a Banach space. If we change Ω^* in such a way that $\overline{\Omega^*}\subset\Omega$, we obtain equivalent norms. Let $\Omega'\subset\overline{\Omega}'\subset\Omega$, Ω' bounded. Then the restriction operator R satisfies $R\in[V^{k,p}(\Omega)\to W^{k,p}(\Omega')]$.*

Proof. Let $u\in V^{k,p}(\Omega)$, and extend u by zero outside of Ω. We have $u_h\in C^\infty(\overline{\Omega})$ and $u_h\in W^{k,p}(\Omega')$. On the other hand $\lim_{h\to 0}u_h=u$ in $L^p(\Omega')$, and $\lim_{h\to 0}D^\alpha u_h = D^\alpha u$ in $L^p(\Omega')$ for $|\alpha|=k$. Without loss of generality we can assume $\Omega'\in\mathfrak{N}^0$. According to Theorem 7.4, $u\in W^{k,p}(\Omega')$, and we have:

$$|u|_{W^{k,p}(\Omega')}\le c_1|u|_{V^{k,p}(\Omega)}. \tag{2.117}$$

Using (2.117) we see that the topology of $V^{k,p}(\Omega)$ does not depend on Ω^*. Let u_n be a Cauchy sequence in $V^{k,p}(\Omega)$. Due to (2.117) there exists $u\in L^p_{loc}(\Omega)$, $u\in W^{k,p}(\Omega')$ for all bounded Ω', $\Omega'\subset\overline{\Omega}'\subset\Omega$, and $\lim_{n\to\infty}u_n=u$ in $L^p(\Omega')$, $\lim_{n\to\infty}D^\alpha u_n = D^\alpha u$ in $L^p(\Omega)$ for $|\alpha|=k$. □

Remark 7.3. It is a priori clear that we could assume in the definition of $V^{k,p}(\Omega)$ that $u\in L^1_{loc}(\Omega)$ and $D^\alpha u\in L^p(\Omega), |\alpha|=k$. It is sufficient to consider $u\in\mathscr{D}'(\Omega)$ with $D^\alpha u\in L^p(\Omega)$ and we obtain the same space, cf. J. Deny, J.L. Lions [1].

Theorem 7.6. *Let $\Omega\in\mathfrak{N}^0$. Then $W^{k,p}(\Omega)=V^{k,p}(\Omega)$ algebraically and topologically.*

Proof. It is sufficient to prove that $W^{k,p}(\Omega)=V^{k,p}(\Omega)$ algebraically. Let $|\alpha| = k-1$, and let us consider $v=D^\alpha u$. We have $v\in V^{1,p}(\Omega)$ due to Theorem 7.5. Let us consider v in V_r, $r=1,2,\dots,m$. For simplicity we omit the index r. Now using Theorem 2.2 change v on a set of measure zero in such a way that the function obtained is absolutely continuous in V on almost all lines parallel to the axis x_N. We have:

$$v(x',x_N)=\int_{y_N}^{x_N}\frac{\partial v}{\partial x_N}(x',\eta)\,\mathrm{d}\eta+v(x',y_N),\quad x_N,y_N\in(a(x'),a(x')+\beta).$$

From this relation we get:

$$|v(x', x_N)|^p \le 2^{p-1} \left[\beta^{p-1} \int_{a(x')}^{a(x')+\beta} \left| \frac{\partial v}{\partial x_N}(x', \eta) \right|^p \mathrm{d}\eta + |v(x', y_N)|^p \right].$$

Integrating this inequality with respect to y_N on the interval $(a(x') + \beta/2, a(x') + \beta)$, we get:

$$\frac{\beta}{2}|v(x', x_N)|^p \le 2^{p-2}\beta^p \int_{a(x')}^{a(x')+\beta} \left| \frac{\partial v}{\partial x_N}(x', \eta) \right|^p \mathrm{d}\eta + 2^{p-1} \int_{a(x')+\beta/2}^{a(x')+\beta} |v(x', y_N)|^p \,\mathrm{d}y_N. \tag{2.118}$$

Finally we integrate (2.118) with respect to x_N on $(a(x'), a(x') + \beta)$, and then with respect to x' on Δ, and get:

$$\begin{aligned} &\frac{\beta}{2} \int_\Delta \mathrm{d}x' \int_{a(x')}^{a(x')+\beta} |v(x', x_N)|^p \,\mathrm{d}x_N \\ &\qquad \le 2^{p-2}\beta^{p+1} \int_\Delta \mathrm{d}x' \int_{a(x')}^{a(x')+\beta} \left| \frac{\partial v}{\partial x_N}(x', \eta) \right|^p \mathrm{d}\eta \\ &\qquad + 2^{p-1}\beta \int_\Delta \mathrm{d}x' \int_{a(x')+\beta/2}^{a(x')+\beta} |v(x', y_N)|^p \,\mathrm{d}y_N. \end{aligned} \tag{2.119}$$

The estimates (2.119), (2.117) give $v \in W^{1,p}(\Omega)$; the result follows by recurrence. □

Remark 7.4. In R. Courant, D. Hilbert [1] we can find an example of a bounded domain such that $W^{k,p}(\Omega) \subset V^{k,p}(\Omega)$ holds strictly.

Remark 7.5. For Ω bounded, Theorem 7.1 holds with f_α as in (2.107a) or (2.107b) for $V^{k,p}(\Omega)$.

For other examples of equivalent norms on $W^{k,p}(\Omega)$ cf. Chap. 1, Sect. 1.1.

2.7.4 Nikodym Domains

Theorem 7.7. *Let $k \ge 1$ be an integer, $p \ge 1, \Omega$ a domain such that $v \in P_{(k-1)} \Longrightarrow \int_\Omega |v|^p \,\mathrm{d}x < \infty$. We have the inequality:*

$$c_1 |\tilde{u}|_{V^{k,p}(\Omega)/P_{(k-1)}} \le \Big[\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)} \Big]^{1/p} \le c_2 |\tilde{u}|_{V^{k,p}(\Omega)/P_{(k-1)}}. \tag{2.120}$$

Proof. Let Ω_i be an increasing sequence of domains in $\mathfrak{N}^0$, $i = 1, 2, \dots$, such that $\lim_{i\to\infty} \Omega_i = \Omega$, and let $\tilde{u}_n$ be a Cauchy sequence for the norm (2.113).

According to Theorems 7.6 and 7.2, (2.120) holds for Ω_i, $i=1,2,\ldots$. There exists $\tilde{u}_{[i]} \in W^{k,p}(\Omega_i)/P_{(k-1)}$ such that $\lim_{n\to\infty} \tilde{u}_n = \tilde{u}_{[i]}$ in $W^{k,p}(\Omega_i)/P_{(k-1)}$. It is clear that the restriction of $u_{[i+1]}$ to Ω_i is $u_{[i]}$, and hence there exists $u \in L^p_{loc}(\Omega)$ such that $\lim_{n\to\infty} \tilde{u}_n = \tilde{u}$ in $W^{k,p}(\Omega_i)/P_{(k-1)}$, $i=1,2,\ldots$, and $\lim_{n\to\infty} D^\alpha u_n = D^\alpha u$ in $L^p(\Omega)$ for $|\alpha|=k$. Thus $V^{k,p}(\Omega)/P_{(k-1)}$ with the norm (2.113) is complete and the result follows from the Banach theorem. □

Remark 7.6. Taking into account Remark 7.5, we can proceed in the proof of Theorem 7.7 as in the proof of Theorem 7.2.

Exercise 7.1. Using Theorem 7.7 and the regularizing operator, prove that $u \in \mathscr{D}'(\Omega)$, $D^\alpha u \in L^p(\Omega)$, $|\alpha|=k$, imply $u \in V^{k,p}(\Omega)$.

A bounded open set Ω is called a *Nikodym domain* if for all $p \geq 1$, $V^{1,p}(\Omega) = W^{1,p}(\Omega)$.[7]

If we have to be more precise, we say that the bounded domain Ω is (k,p)-*Nikodym* if $V^{k,p}(\Omega) = W^{k,p}(\Omega)$.

According to Theorem 7.5 we get obviously

Proposition 7.1. *If Ω is a Nikodym domain, it is $(k,p)-$Nikodym for $k \geq 1$, $p \geq 1$.*

We have another characterization of Nikodym domains (for $p=2$, cf. J. Deny, J.L. Lions [1]):

Theorem 7.8. *The domain Ω is a Nikodym domain if and only if for $\tilde{u} \in W^{1,p}(\Omega)/P_{(0)}$ the following inequality holds:*

$$c_1 |\tilde{u}|_{W^{1,p}(\Omega)/P_{(0)}} \leq \left[\sum_{i=1}^{N} \left| \frac{\partial u}{\partial x_i} \right|^p_{L^p(\Omega)} \right]^{1/p}. \tag{2.121}$$

Proof. If Ω is a Nikodym domain, the identity mapping

$$I : W^{1,p}(\Omega)/P_{(0)} \to V^{1,p}(\Omega)/P_{(0)}$$

is surjective. The norms $|\tilde{u}|_{W^{1,p}(\Omega)/P_{(0)}}$ and $|\tilde{u}|_{V^{1,p}(\Omega)/P_{(0)}}$ are equivalent, and we have (2.120) and then (2.121).

The relation (2.121) being satisfied, due to (2.120) it is sufficient to prove the density of $W^{1,p}(\Omega)/P_{(0)}$ in $V^{1,p}(\Omega)/P_{(0)}$. To do this, let $\tilde{u} \in V^{1,p}(\Omega)/P_{(0)}, u \in \tilde{u}$; without loss of generality we can assume u real. We define:

$$u_n = \begin{cases} u & \text{for} \quad |u| < n, \\ n & \text{for} \quad u \geq n, \\ -n & \text{for} \quad u \leq -n. \end{cases}$$

[7] The original definition was less restrictive: Ω is a Nikodym domain if $V^{1,2}(\Omega) = W^{1,2}(\Omega)$.

It follows from Theorem 2.3 that $u_n \in W^{1,p}(\Omega)$. For $F_n = \{x \in \Omega, |u(x)| \geq n\}$, we have obviously $\lim_{n\to\infty}$ meas $F_n = 0$, hence $\lim_{n\to\infty} \tilde{u}_n = \tilde{u}$ in $V^{1,p}(\Omega)/P_{(0)}$. □

Problem 7.1. Is in general $W^{k,p}(\Omega)/P_{(k-1)}$ dense in $V^{k,p}(\Omega)/P_{(k-1)}$ for instance for Ω bounded?

Exercise 7.2. For Ω a Nikodym domain prove the *Poincaré inequality:*

$$u \in W^{1,2}(\Omega) \Longrightarrow |u|^2_{L^2(\Omega)} - \frac{1}{\text{meas}\,\Omega}\left|\int_\Omega u(x)\,\mathrm{d}x\right|^2 \leq c\sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^2_{L^2(\Omega)}.$$

Remark 7.7. If Ω is $(k,p)-$Nikodym, then we have the inequality:

$$c_1|\tilde{u}|_{W^{k,p}(\Omega)/P_{(k-1)}} \leq \big[\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)}\big]^{1/p}.$$

We proceed as in the first part of the proof of Theorem 7.8.

Now we formulate a more general theorem concerning equivalent norms, which generalizes Theorems 7.1, 7.3, 7.4:

Theorem 7.9. *Let Ω be a Nikodym domain, $|u|_1$ a seminorm on $W^{k,p}(\Omega)$; we assume that $|u|_1 \leq c|u|_{W^{k,p}(\Omega)}$. Let*

$$|\tilde{u}|_2 = \left[|u|_1^p + \sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)}\right]^{1/p},$$

and let $P \subset P_{(k-1)}$ be the set all polynomials u such that $|u|_2 = 0$. Then we get the following inequalities:

$$c_1|\tilde{u}|_{W^{k,p}(\Omega)/P} \leq |\tilde{u}|_2 \leq c_2|\tilde{u}|_{W^{k,p}(\Omega)/P}.$$

Proof. It is sufficient to prove that $W^{k,p}(\Omega)/P$ with the norm $|\tilde{u}|_2$ is complete. Let $\tilde{u}_n$ be a Cauchy sequence. According to Theorem 7.7, and since Ω is a Nikodym domain, there exists a sequence $p_n \in P_{(k-1)}$ such that $\lim_{n\to\infty}(u_n + p_n) = u$ in $W^{k,p}(\Omega)$. But $|\tilde{p}|_2$ is a norm on $P_{(k-1)}/P$, therefore $\lim_{n\to\infty} \tilde{p}_n = \tilde{p}$ in $P_{(k-1)}/P$, and consequently also in $W^{k,p}(\Omega)/P$. Thus $\lim_{n\to\infty} \tilde{u}_n = \tilde{u} - \tilde{p}$. □

Remark 7.8. If Ω is a Nikodym domain, we deduce from Theorem 7.9 the *generalized Poincaré inequality*, i.e. for $u \in W^{k,p}(\Omega)$ we have

$$|u|_{W^{k,p}(\Omega)} \leq c\big[\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)} + \sum_{|\alpha|\leq k-1} \big|\int_\Omega D^\alpha u\,\mathrm{d}x\big|^p\big]^{1/p}.$$

Exercise 7.3. Let C be the unit disc without the interval $y=0$, $0 \le x \le 1$. Prove that C is a Nikodym domain.

Exercise 7.4. Let Ω be a starshapped domain with respect to the origin. Prove that Ω is a Nikodym domain.

Theorem 7.10. *Let Ω be a Nikodym domain, $V \subset W^{k,p}(\Omega)$ a closed subset, and $P_V = V \cap P_{(k-1)}$. Then $(\sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)})^{1/p}$ is an equivalent norm on V/P_V.*

Proof. Let $P_{(k-1)} = P_V \dotplus L$ be the direct sum with $L \cap P_V = 0$, $q_1, q_2, \ldots, q_\kappa$ a basis of L (the functions q_i are linearly independent). It is sufficient to prove that V/P_V with the norm given above is complete. Let $\tilde{u}_n$ be a Cauchy sequence. According to Theorem 7.7 and due to the Nikodym property of Ω, there exist $p_n \in P_{(k-1)}$ such that $u_n + p_n$ is a Cauchy sequence in $W^{k,p}(\Omega)$, hence $\tilde{u}_n + \tilde{p}_n$ is also a Cauchy sequence in $W^{k,p}(\Omega)/P_V$. We have:

$$p_n = a_n + b_n, \quad a_n \in P_V,\ b_n \in L,\ \tilde{b}_n = \sum_{i=1}^{\kappa} \lambda_{ni} \tilde{q}_i.$$

Let us consider:

$$\tilde{u}_n + \sum_{i=1}^{\kappa-1} \lambda_{ni} \tilde{q}_i. \tag{2.122}$$

The space $V/P_V \dotplus \cup_{i=1}^{\kappa-1} \tilde{q}_i = \tilde{W}$ is closed in $W^{k,p}(\Omega)/P_V$ and by the Hahn-Banach theorem, there exists a functional f on $W^{k,p}(\Omega)/P_V$ such that $f\tilde{v} = 0$ for $\tilde{v} \in \tilde{W}$ and $f\tilde{q}_\kappa = 1$. The sequence $\tilde{u}_n + \tilde{b}_n$ is a Cauchy sequence, therefore we have $f(\tilde{u}_n + \tilde{b}_n) = \lambda_{n\kappa}$ and $\lim_{n\to\infty} \lambda_{n\kappa} = \lambda_\kappa$. Using a recurrence process we get $\lim_{n\to\infty} \lambda_{ni} = \lambda_i$, $i = 1, 2, \ldots, \kappa$, and $\tilde{u}_n$ is a Cauchy sequence in $W^{k,p}(\Omega)/P_V$. □

Theorem 7.11. *Let Ω be a Nikodym domain, $V \subset W^{k,p}(\Omega)$ a closed subspace, and $|u|_1$ a seminorm on V; we assume $u \in V \Longrightarrow |u|_1 \le c|u|_{W^{k,p}(\Omega)}$. Let*

$$|u|_2 = \Big[|u|_1^p + \sum_{|\alpha|=k} |D^\alpha u|^p_{L^p(\Omega)} \Big]^{1/p}$$

and $P \subset P_{(k-1)} \cap V$ the space of polynomials in $P_{(k-1)} \cap V$ for which $|u|_1 = 0$. Then we have:

$$c_1 |\tilde{u}|_{V/P} \le |u|_2 \le c_2 |\tilde{u}|_{V/P}.$$

Proof. As in Theorem 7.9, it is sufficient to prove that a Cauchy sequence $\tilde{u}_n$ for the norm $|\tilde{u}|_2$ is a Cauchy sequence in V/P. With the same notations as in the previous theorem $u_n + p_n$ is a Cauchy sequence in $W^{k,p}(\Omega)$, and moreover $\tilde{u}_n + \tilde{p}_n$ is a Cauchy sequence in $W^{k,p}(\Omega)/P$; $p_n \in P_V$. But since $\tilde{p}_n$ is a Cauchy sequence in the norm $|\tilde{u}|_1$, it is also a Cauchy sequence in P_V/P. Hence $\tilde{p}_n$ is a Cauchy sequence in $W^{k,p}(\Omega)/P$. □

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