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## Preface

The aim in writing this book has been to give a survey of the main applications of group and representation theory to particle physics. It provides the essential notions of relativistic invariance, space-time symmetries and internal symmetries employed in the standard University courses of Relativistic Quantum Field Theory and Particle Physics. However, we point out that this is neither a book on these subjects, nor it is a book on group theory.

Specifically, its main topics are, on one side, the analysis of the Lorentz and Poincaré groups and, on the other side, the internal symmetries based mainly on unitary groups, which are the essential tools for the understanding of the interactions among elementary particles and for the construction of the present theories. At the same time, these topics give important and enlightening examples of the essential role of group theory in particle physics. We have attempted to present a pedagogical survey of the matter, which should be useful to graduate students and researchers in particle physics; the only prerequisite is some knowledge of classical field theory and relativistic quantum mechanics. In the Bibliography, we give a list of relevant texts and monographs, in which the reader can find supplements and detailed discussions on the questions only partially treated in this book.

One of the most powerful tools in dealing with invariance properties and symmetries is *group theory*. Chapter 1 consists in a brief introduction to group and representation theory; after giving the basic definitions and discussing the main general concepts, we concentrate on the properties of Lie groups and Lie algebras. It should be clear that we do not claim that it gives a self-contained account of the subject, but rather it represents a sort of glossary, to which the reader can refer to recall specific statements. Therefore, in general, we limit ourselves to define the main concepts and to state the relevant theorems without presenting their proofs, but illustrating their applications with specific examples. In particular, we describe the root and weight diagrams, which provide a useful insight in the analysis of the classical Lie groups and their representations; moreover, making use of the Dynkin diagrams, we present a classification of the classical semi-simple Lie algebras and Lie groups.

The book is divided into two parts that, to large extent, are independent from one another. In the first part, we examine the invariance principles related to the symmetries of the physical space-time manifold. Disregarding gravitation, we consider that the geometry of space-time is described by the Minkowski metric and that the inertial frames of reference of special relativity are completely equivalent in the description of the physical phenomena. The co-ordinate transformations from one frame of reference to another form the so-called *inhomogeneous Lorentz group* or *Poincaré group*, which contains the space-time translations, besides the pure Lorentz transformations and the space rotations. The introductory and didactic nature of the book influenced the level of the treatment of the subject, for which we renounced to rigorousness and completeness, avoiding, whenever possible, unnecessary technicalities.

In Chapter 2 we give a short account of the three-dimensional rotation group, not only for its important role in different areas of physics, but also as a specific illustration of group theoretical methods. In Chapter 3 we consider the main properties of the homogeneous Lorentz group and its Lie algebra. First, we examine the restricted Lorentz group, which is nothing else than the non-compact version  $SO(3,1)$  of the rotation group in four dimensions. In particular we consider its finite dimensional irreducible representations: they are non unitary, since the group is non compact, but they are very useful, in particle physics, for the derivation of the relativistic equations. Chapter 4 is devoted to the Poincaré group, which is most suitable for a quantum mechanical description of particle states. Specifically, the transformation properties of one-particle and two-particle states are examined in detail in Chapter 5. In this connection, a covariant treatment of spin is presented and its physical meaning is discussed in both cases of massive and massless particles. In Chapter 6 we consider the transformation properties of the particle states under the discrete operations of parity and time reversal, which are contained in the homogeneous Lorentz group and which have important roles in particle physics. In Chapter 7, the relativistic wave functions are introduced in connection with one-particle states and the relative equations are examined for the lower spin cases, both for integer and half-integer values. In particular, we give a group-theoretical derivation of the Dirac equation and of the Maxwell equations.

The second part of the book is devoted to the various kinds of internal symmetries, which were introduced during the extraordinary development of particle physics in the second half of last century and which had a fundamental role in the construction of the present theories. A key ingredient was the use of the unitary groups, which is the subject of Chapter 8. In order to illustrate clearly this point, we give a historical overview of the different steps of the process that lead to the discovery of elementary particles and of the properties of fundamental interactions. The main part of this chapter is devoted to the analysis of *hadrons*, i.e. of particle states participating in strong interactions. First we consider the isospin invariance, based on the group  $SU(2)$  and on the assumption that the members of each family of hadrons, almost degenerate in

mass but with different electric charge, are assigned to the same irreducible representation. Further analysis of the different kinds of hadrons lead to the introduction of a larger symmetry, now called flavor  $SU(3)$  invariance, which allowed the inclusion of different isospin multiplets in the same irreducible representation of the  $SU(3)$  group and gave rise to a more complete classification of hadrons. Moreover, it provided a hint to the introduction of *quarks* as the fundamental constituents of matter. Finally, the analysis of the hadronic states in terms of quarks lead to the discovery of a new degree of freedom, called *color*, that gave a deeper understanding of the nature of strong interactions. It was clear from the very beginning that the flavor  $SU(3)$  symmetry was only approximate, but it represented an important step toward the more fundamental symmetry of color  $SU(3)$ .

Chapter 9 is a necessary complement of the previous chapter, since it describes a further successful step in the development of particle physics, which is the introduction of gauge symmetry. After reminding the well-known case of quantum electrodynamics, we briefly examine the field theory based on the gauge color  $SU(3)$  group, i.e. quantum chromodynamics, which provides a good description of the peculiar properties of the strong interactions of quarks. Then we consider the electroweak Standard Model, the field theory based on the gauge  $SU(2) \otimes U(1)$  group, which reproduces with great accuracy the properties of weak interactions of leptons and quarks, combined with the electromagnetic ones. An essential ingredient of the theory is the so-called spontaneous symmetry breaking, which we illustrate in the frame of a couple of simple models. Finally, we mention the higher gauge symmetries of Grand Unification Theories, which combine strong and electroweak interactions.

The book contains also three Appendices, which complete the subject of unitary groups. In Appendix A, we collect some useful formulas on the rotation matrices and the Clebsch-Gordan coefficients. In Appendix B, the symmetric group is briefly considered in connection with the problem of identical particles. In Appendix C, we describe the use of the Young tableaux for the study of the irreducible representations of the unitary groups, as a powerful alternative to the use of weight diagrams.

Each chapter, except the first, is supplied with a list of problems, which we consider useful to strengthen the understanding of the different topics discussed in the text. The solutions of all the problems are collected at the end of the book.

The book developed from a series of lectures that both of us have given in University courses and at international summer schools. We have benefited from discussions with students and colleagues and we are greatly indebted to all of them.

## Notation

The natural system of units, where  $\hbar = c = 1$ , is used throughout the book. In this system:  $[length] = [time] = [energy]^{-1} = [mass]^{-1}$ .

Our conventions for special relativity are the following. The metric tensor is given by

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad (0.1)$$

and the *contravariant* and *covariant* four-vectors are denoted, respectively, by

$$x^\mu = (x^0, \mathbf{x}), \quad x_\mu = g_{\mu\nu}x^\nu = (x^0, -\mathbf{x}). \quad (0.2)$$

Greek indices run over 0, 1, 2, 3 and Latin indices denote the spacial components. Repeated indices are summed, unless otherwise specified.

The derivative operator is given by

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left( \frac{\partial}{\partial x^0}, \nabla \right). \quad (0.3)$$

The Levi-Civita tensor  $\epsilon^{0123}$  is totally antisymmetric; we choose, as usual,  $\epsilon^{0123} = +1$  and consequently one gets  $\epsilon_{0123} = -1$ .

The complex conjugate, transpose and Hermitian adjoint of a matrix  $M$  are denoted by  $M^*$ ,  $\tilde{M}$  and  $M^\dagger = \tilde{M}^*$ , respectively.

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