

Chapter 2

Principles of Analytical Mechanics

2.1 Constraints

As we have already mentioned in [Chap. 1](#), a particle (or a system of particles) is *subject to constraints* if its motion is restricted by a *constraint force* on a certain surface, or on some curve, etc. The notion of *constraint* is essential in understanding the analytical mechanics formalism, and we shall begin this chapter with a thorough analysis of this basic concept.

By definition, a constraint is a geometric or kinematic condition that limits the possibilities of motion of a mechanical system. For example, a body sliding on an inclined plane cannot leave the plane, or a pebble inside a soccer ball is compelled to move within a given volume, etc.

2.1.1 One-Particle Systems

Assuming Cartesian coordinates are used, let us begin our investigation with a single particle. If \mathbf{r} is the radius-vector of the particle and \mathbf{v} its velocity at time t , then a relation of the form

$$f(\mathbf{r}, \mathbf{v}, t) = 0 \quad (2.1.1)$$

is the mathematical expression for a *constraint*. One says that the particle is *subject to the constraint* (2.1.1).

We can classify the constraints according to three criteria:

- (a) A constraint can be expressed either by an equality

$$f(x, y, z, t) = 0, \quad (2.1.2)$$

or by an inequality

$$f(x, y, z) \leq 0, \quad f(x, y, z) \geq 0. \quad (2.1.3)$$

The first type of constraint is called *bilateral* and the second *unilateral*. For example, the relation

$$(x - at)^2 + (y - bt)^2 + (z - ct)^2 = R^2 \quad (2.1.4)$$

indicates that the particle is permanently on a moving sphere, with its centre at the point (at, bt, ct) , while the inequality

$$x^2 + y^2 + z^2 - R^2 \leq 0 \quad (2.1.5)$$

shows that the motion of the particle is restricted inside a fixed sphere of radius R .

- (b) If the time t does not explicitly appear in the equation of the constraint, this is called a *scleronomous* or *stationary* constraint. Such a constraint is, for instance, (2.1.5). If the constraint is time-dependent, like (2.1.4), it is named a *rheonomous* or *non-stationary* constraint. An example of rheonomous constraints is provided by the system

$$f_1(x, y, z, t) = 0, \quad f_2(x, y, z, t) = 0, \quad (2.1.6)$$

meaning that the particle is forced to slide on a moving curve.

- (c) A velocity-dependent constraint is called a *kinematic* or *differential* constraint, like

$$f(x, y, z, \dot{x}, \dot{y}, \dot{z}) = 0, \quad (2.1.7)$$

while a constraint in which the components of the velocity do not appear is named a *geometric* or *finite* constraint. For example, the constraints (2.1.2)–(2.1.6) are geometric, while (2.1.1) is kinematic. From now on, we shall consider only those differential constraints which are *linear* in the velocity components, as

$$a_i \dot{x}_i + b = 0, \quad (2.1.8)$$

where

$$a_i = a_i(\mathbf{r}, t), \quad b_i = b_i(\mathbf{r}, t) \quad (i = 1, 2, 3) \quad (2.1.9)$$

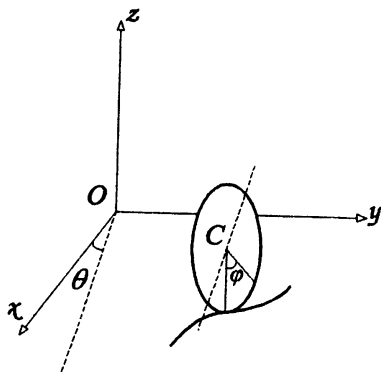
and the summation convention has been used. Taking the total time derivative of (2.1.2), we have:

$$\frac{\partial f}{\partial x_i} \dot{x}_i + \frac{\partial f}{\partial t} = 0, \quad (2.1.10)$$

meaning that a geometric constraint can be written as a linear differential constraint. Obviously, the reciprocal of this statement is not true.

Those differential constraints which can be put in a finite form are called *integrable* constraints. The geometric constraints, together with the integrable constraints, form the class of *holonomic* constraints. Such constraints are, for

Fig. 2.1 The motion of a coin of radius a , rolling on a horizontal plane, as an example of non-integrable constraint.



example, those given by (2.1.2), (2.1.4), (2.1.6). The non-integrable constraints, together with constraints expressed by inequalities, are said to be *non-holonomic* constraints.

As an example of non-integrable constraint, let us consider a coin of radius a , rolling on a horizontal plane and keeping always a vertical position (Fig. 2.1). If θ is the angle between the x -axis and the normal to the coin, and φ is the angle of rotation of the coin, the velocity of the point C is

$$\mathbf{v} = a\dot{\varphi}\mathbf{u}, \quad (2.1.11)$$

which is permanently orthogonal to the axis of the coin. The components of the velocity are:

$$v_x = a\dot{\varphi} \cos \theta, \quad v_y = a\dot{\varphi} \sin \theta, \quad v_z = 0,$$

or, in differential form,

$$dx - a \cos \theta d\varphi = 0, \quad dy - a \sin \theta d\varphi = 0. \quad (2.1.12)$$

These two equations cannot be integrated, because their left hand sides do not represent total differentials of some functions. Consequently, they provide an example of a non-holonomic (vector) constraint.

There are no general methods of solving problems involving non-holonomic constraints. Each case must be studied separately. Fortunately, most of the problems arising in mechanics are connected with holonomic constraints.

A constraint can be characterized simultaneously upon all possible criteria. For instance, the constraint expressed by (2.1.2) is bilateral, scleronomous and geometric, while the constraint (2.1.7) is bilateral, scleronomous and differential.

There is a close relation between the number of constraints and the number of degrees of freedom of a mechanical system. The minimal number of real independent parameters that determine the position of a particle defines the *number of degrees of freedom* of that particle. A free particle, i.e. a particle subject only to applied forces, has three degrees of freedom. If the coordinates of the particle are connected by a relation of type (2.1.2), the number of its degrees of freedom

reduces to two. In the same way, the existence of the two constraints (2.1.6) implies that the particle moves on a curve: the position of the particle is determined by a single parameter, corresponding to a single degree of freedom. In general, *each geometric bilateral constraint applied to a system reduces its number of degrees of freedom by one.*

Note that the coordinates of a particle cannot simultaneously obey more than two independent constraints; a third constraint would either keep the particle fixed, or make its motion known without considering the forces acting on it.

2.1.1.1 The Fundamental Equation of Motion

As we have already mentioned in Chap. 1, the existence of a constraint can be connected with a *reaction* or *constraint force*, which determines the particle to obey the constraint. If we denote by \mathbf{F} and \mathbf{L} the resultants of the applied and constraint forces, respectively, acting on a particle P of mass m , the differential equation of motion reads:

$$m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{L}. \quad (2.1.13)$$

The fundamental problem of mechanics of a system subject to both applied and constraint forces is: *given \mathbf{F} and the initial conditions, consistent with the constraints, find the motion of the system and determine the reaction force \mathbf{L} .* The constraint force \mathbf{L} is *a priori* unknown, therefore in order to use Eq. (2.1.13) one must make certain assumptions on it. The following two examples will familiarize the reader with the methods of solving problems involving constraint forces.

(1) *Motion on a curve.* First, assume that the curve, considered to be fixed, is given by its *parametric* equations:

$$x_i = x_i(q) \quad (i = 1, 2, 3), \quad (2.1.14)$$

where q is a real, time-dependent parameter. On the other hand, projecting (2.1.13) on the axes, we have:

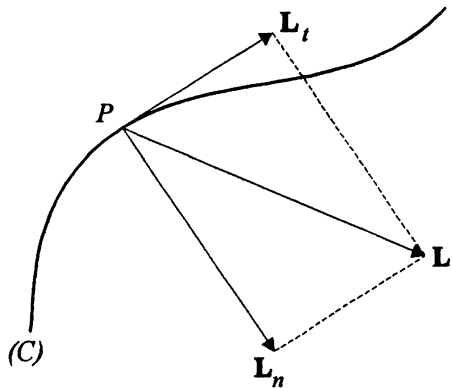
$$m\ddot{x}_i = F_i + L_i \quad (i = 1, 2, 3), \quad (2.1.15)$$

where F_x, F_y, F_z are given as functions of $\mathbf{r}, \dot{\mathbf{r}}, t$ or, in view of (2.1.14),

$$F_i = F_i(q, \dot{q}, t) \quad (i = 1, 2, 3). \quad (2.1.16)$$

We have arrived at a system of three second-order differential equations (2.1.15), with four unknowns, L_x, L_y, L_z, q . To solve the problem, one decomposes the constraint force \mathbf{L} into two vector components, \mathbf{L}_n and \mathbf{L}_t (Fig. 2.2). The component \mathbf{L}_n lies in the plane normal to the curve (C) at the point P , while the component \mathbf{L}_t is tangent to the curve and points in the direction of motion of the particle.

Fig. 2.2 Decomposition of the constraint force \mathbf{L} into two vector components, one normal and one tangent to the trajectory.



The component \mathbf{L}_n is called the *normal reaction* and \mathbf{L}_t – the *force of friction*. If $\mathbf{L}_t = 0$, the particle moves *without friction* and the curve is *perfectly smooth* or *ideal*. If $\mathbf{L}_n = 0$, the force \mathbf{L} is tangent to a *perfectly rough* curve.

Assuming $\mathbf{L}_t = 0$, since \mathbf{v} is always directed along the tangent to the trajectory (which in our case coincides with the constraint), we can write

$$\mathbf{v} \cdot \mathbf{L} = \dot{x}L_x + \dot{y}L_y + \dot{z}L_z = 0. \quad (2.1.17)$$

We are now in possession of four equations (2.1.15) and (2.1.17) for the unknowns L_x, L_y, L_z, q . Therefore, we are able to determine both $q = q(t)$, i.e. the motion of the particle on the curve, and the components of the constraint force.

Second, let us suppose that the fixed curve is given in the implicit form

$$f_1(x, y, z) = 0, \quad f_2(x, y, z) = 0. \quad (2.1.18)$$

In this case, the differential equations (2.1.15), together with the frictionlessness condition (2.1.17) and the constraint equations (2.1.18) form a system of six equations for six unknowns: x, y, z, L_x, L_y, L_z .

The problem can be solved somewhat differently by decomposing the force \mathbf{L} into two vector components, along the normals to the two surfaces whose intersection produces the curve (C) (Fig. 2.3). Then we may write

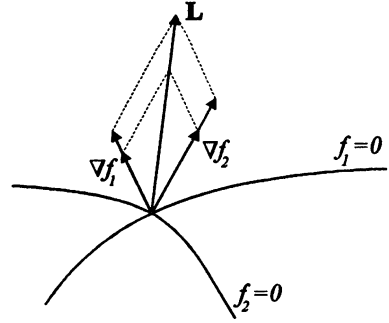
$$\mathbf{L} \equiv \mathbf{L}_n = \lambda \text{grad } f_1 + \mu \text{grad } f_2, \quad (2.1.19)$$

where λ and μ are two scalar multipliers. The equation of motion is then

$$m\ddot{x}_i = F_i + \lambda \frac{\partial f_1}{\partial x_i} + \mu \frac{\partial f_2}{\partial x_i} \quad (i = 1, 2, 3). \quad (2.1.20)$$

Thus, we are left with a system of five equations (2.1.18) and (2.1.20) for the unknowns x, y, z, λ, μ . In this way, both the motion of the particle and the constraint force are determined.

Fig. 2.3 Decomposition of the constraint force \mathbf{L} into two vector components, along the normals to the two surfaces whose intersection produces the curve (C).



Observations:

- (a) If the component \mathbf{L}_t is non-zero but known, it can be included in \mathbf{F} :

$$m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{L}_t + \mathbf{L}_n = \mathbf{F}' + \mathbf{L}_n \quad (2.1.21)$$

and one then follows the usual procedure.

- (b) An alternative form of the equations of motion for a stationary curve is obtained by projecting (2.1.13) on the axes of a natural system of coordinates (see Appendix B):

$$m\dot{v} = F_\tau, \quad \frac{mv^2}{\rho} = F_v + L_v, \quad 0 = F_\beta + L_\beta, \quad (2.1.22)$$

where the index τ shows the tangent to the curve, v – the principal normal and β – the bi-normal.

(2) *Motion on a surface.* Here the procedure is similar, though a little more complicated. Following the same order as in the previous case, let us first suppose that the surface is given in the parametric form

$$x_i = x_i(q^1, q^2) \quad (i = 1, 2, 3). \quad (2.1.23)$$

Since

$$\dot{\mathbf{r}} = \sum_{\alpha=1}^2 \frac{\partial \mathbf{r}}{\partial q^\alpha} \dot{q}^\alpha, \quad \ddot{\mathbf{r}} = \sum_{\alpha, \beta=1}^2 \frac{\partial^2 \mathbf{r}}{\partial q^\alpha \partial q^\beta} \dot{q}^\alpha \dot{q}^\beta + \sum_{\alpha=1}^2 \frac{\partial \mathbf{r}}{\partial q^\alpha} \ddot{q}^\alpha,$$

the equation of motion (2.1.13) becomes

$$m \left(\sum_{\alpha, \beta=1}^2 \frac{\partial^2 \mathbf{r}}{\partial q^\alpha \partial q^\beta} \dot{q}^\alpha \dot{q}^\beta + \sum_{\alpha=1}^2 \frac{\partial \mathbf{r}}{\partial q^\alpha} \ddot{q}^\alpha \right) = \mathbf{F} + \mathbf{L} \quad (2.1.24)$$

or, in components,

$$m \left(\sum_{\alpha, \beta=1}^2 \frac{\partial^2 x_i}{\partial q^\alpha \partial q^\beta} \dot{q}^\alpha \dot{q}^\beta + \sum_{\alpha=1}^2 \frac{\partial x_i}{\partial q^\alpha} \ddot{q}^\alpha \right) = F_i + L_i \quad (i = 1, 2, 3). \quad (2.1.25)$$

There are five unknowns occurring in (2.1.25): q^1, q^2, L_x, L_y, L_z . To solve this problem, one decomposes \mathbf{L} into two components, \mathbf{L}_t and \mathbf{L}_n , the first being tangent to the surface and showing the direction of motion, and the second along the normal to the surface. If $\mathbf{L}_t = 0$, then $\mathbf{L} = \mathbf{L}_n$. To express this property, we observe that \mathbf{L} is normal to two parametric curves on the surface, i.e.

$$L_t \frac{\partial x_i}{\partial q^\alpha} = 0 \quad (\alpha = 1, 2). \quad (2.1.26)$$

We therefore have five equations, (2.1.25) and (2.1.26), for the unknowns q^1, q^2, L_x, L_y, L_z .

If the surface is perfectly smooth, the constraint force \mathbf{L} can be eliminated by multiplying (2.1.24) by $\frac{\partial \mathbf{r}}{\partial q^\gamma}$ ($\gamma = 1, 2$). Then, in view of (2.1.26), we have:

$$m \left(\frac{\partial \mathbf{r}}{\partial q^\gamma} \cdot \sum_{\alpha, \beta=1}^2 \frac{\partial^2 \mathbf{r}}{\partial q^\alpha \partial q^\beta} \dot{q}^\alpha \dot{q}^\beta + \frac{\partial \mathbf{r}}{\partial q^\gamma} \cdot \sum_{\alpha=1}^2 \frac{\partial \mathbf{r}}{\partial q^\alpha} \ddot{q}^\alpha \right) = Q_\gamma \quad (\gamma = 1, 2), \quad (2.1.27)$$

where

$$Q_\gamma = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q^\gamma} \quad (\gamma = 1, 2) \quad (2.1.28)$$

are the covariant components of the applied force \mathbf{F} along the tangents of any two parametric lines of the surface.

The solution of (2.1.27) is

$$q^\alpha = q^\alpha(t, C_1, \dots, C_4) \quad (\alpha = 1, 2) \quad (2.1.29)$$

and, if the initial conditions are known, the finite equations of motion can be determined. The solution (2.1.29) is then introduced into (2.1.23) and thus we obtain the motion of the particle in the real (physical) space.

Equation (2.1.27) can be set in a more condensed form by using the *metric tensor* $g_{\alpha\beta}$ (see (2.6.35)), which is defined by

$$g_{\alpha\beta} = \frac{\partial \mathbf{r}}{\partial q^\alpha} \cdot \frac{\partial \mathbf{r}}{\partial q^\beta} \quad (\alpha, \beta = 1, 2). \quad (2.1.30)$$

We have:

$$\frac{\partial g_{\alpha\beta}}{\partial q^\gamma} = \frac{\partial^2 \mathbf{r}}{\partial q^\alpha \partial q^\gamma} \cdot \frac{\partial \mathbf{r}}{\partial q^\beta} + \frac{\partial \mathbf{r}}{\partial q^\alpha} \cdot \frac{\partial^2 \mathbf{r}}{\partial q^\beta \partial q^\gamma}$$

and, making a cyclic permutation of the indices α, β, γ and then combining the three obtained relations, we get

$$\frac{\partial \mathbf{r}}{\partial q^\gamma} \cdot \frac{\partial^2 \mathbf{r}}{\partial q^\alpha \partial q^\beta} = \Gamma_{\alpha\beta, \gamma} \quad (\alpha, \beta, \gamma = 1, 2), \quad (2.1.31)$$

where the quantities

$$\Gamma_{\alpha\beta,\gamma} = \frac{1}{2} \left(\frac{\partial g_{\beta\gamma}}{\partial q^\alpha} + \frac{\partial g_{\gamma\alpha}}{\partial q^\beta} - \frac{\partial g_{\alpha\beta}}{\partial q^\gamma} \right) \quad (2.1.32)$$

are called the *Christoffel symbols of the first kind*. By virtue of (2.1.31), Eq. (2.1.27) can be put in the form

$$m \left(\sum_{\alpha=1}^2 g_{\gamma\alpha} \ddot{q}^\alpha + \sum_{\alpha,\beta=1}^2 \Gamma_{\alpha\beta,\gamma} \dot{q}^\alpha \dot{q}^\beta \right) = Q_\gamma \quad (\gamma = 1, 2). \quad (2.1.33)$$

If we multiply this equation by $g^{\gamma\sigma}$ and perform the summation over γ , we find

$$m \left(\ddot{q}^\sigma + \sum_{\alpha,\beta=1}^2 \Gamma_{\alpha\beta}^{\sigma} \dot{q}^\alpha \dot{q}^\beta \right) = Q^\sigma \quad (\sigma = 1, 2), \quad (2.1.34)$$

where

$$\Gamma_{\alpha\beta}^{\sigma} = \sum_{\gamma=1}^2 g^{\gamma\sigma} \Gamma_{\alpha\beta,\gamma} \quad (2.1.35)$$

are the *Christoffel symbols of the second kind* and

$$Q^\sigma = \sum_{\gamma=1}^2 g^{\sigma\gamma} Q_\gamma \quad (2.1.36)$$

are the contravariant components of the quantities (2.1.28).

If no applied force \mathbf{F} acts on the particle, the kinetic energy theorem implies that the particle moves on the surface (2.1.23) with constant speed. In this case, the acceleration vector \mathbf{a} is oriented along the principal normal to the trajectory which, for $\mathbf{L}_t = 0$, coincides with the normal to the surface. The equations of equilibrium of the particle, written in a *geodetic form*, are then

$$\ddot{q}^\sigma + \sum_{\alpha,\beta=1}^2 \Gamma_{\alpha\beta}^{\sigma} \dot{q}^\alpha \dot{q}^\beta = 0 \quad (\sigma = 1, 2). \quad (2.1.37)$$

As an example, let us take $q^1 = \theta$, $q^2 = \varphi$. Then the metric (see Appendix B)

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \quad (r = \text{const.})$$

yields

$$g_{11} = r^2, \quad g_{22} = r^2 \sin^2 \theta, \quad g_{12} = 0$$

and, together with the condition

$$\sum_{\alpha=1}^2 g_{\alpha\beta} g^{\alpha\gamma} = \delta_{\beta}^{\gamma},$$

we find

$$g^{11} = \frac{1}{r^2}, \quad g^{22} = \frac{1}{r^2 \sin^2 \theta}, \quad g^{12} = 0.$$

We are now able to calculate the Christoffel symbols of the second kind, $\Gamma_{\alpha\beta}^{\gamma}$. The only non-zero symbols are

$$\Gamma_{22}^1 = -\sin \theta \cos \varphi, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \cot \theta$$

and Eqs. (2.1.37) read

$$\begin{aligned} \ddot{\theta} - \sin \theta \cos \varphi \dot{\varphi}^2 &= 0, \\ \ddot{\varphi} + \cot \theta \dot{\theta} \dot{\varphi} &= 0. \end{aligned}$$

We easily recognize the components of the acceleration vector (1.1.17) along the parametric lines θ, φ , taken for $r = \text{const}$.

If, as a second example, we choose $q^1 = x, q^2 = y$, we have $g_{\alpha\beta} = \delta_{\alpha\beta}$, which leads to the equation of motion

$$\ddot{x} = 0, \quad \ddot{y} = 0, \quad (2.1.38)$$

as expected.

If the fixed, ideal surface is given under the implicit form

$$f(x, y, z) = 0, \quad (2.1.39)$$

the components of the fundamental equation of motion are

$$m\ddot{x}_i = F_i + \lambda \frac{\partial f}{\partial x_i} \quad (i = 1, 2, 3). \quad (2.1.40)$$

The three equations (2.1.40), together with the equation of constraint (2.1.39), form a system of four equations in the unknowns x, y, z, λ . Both the motion and the constraint forces can then be determined.

2.1.1.2 Static Equilibrium of a Particle

(1) **Free particle.** A point mass m is in *equilibrium* relative to a certain frame if the resultant of the forces acting on it is zero

$$m\ddot{\mathbf{r}} = \mathbf{F} = 0. \quad (2.1.41)$$

The equilibrium positions are determined by solving the system of three equations with three unknowns

$$m\ddot{x}_i = F_i = 0. \quad (2.1.42)$$

If, in particular, the solution of the system (2.1.42) is unique, we have only one position of equilibrium.

Assume that the particle is subject to a conservative force field

$$\mathbf{F} = -\text{grad } V = \text{grad } U, \quad (2.1.43)$$

where we denote $U(\mathbf{r}) = -V(\mathbf{r})$. We also assume that $U(\mathbf{r})$ is a function of class C^2 . The last two relations give

$$\frac{\partial U}{\partial x_i} = 0 \quad (i = 1, 2, 3). \quad (2.1.44)$$

Consequently, in order for the position P_0 of the particle to be a position of equilibrium, it is necessary to have

$$\left(\frac{\partial U}{\partial x_i} \right)_{P_0} = 0 \quad (i = 1, 2, 3), \quad (2.1.45)$$

meaning that in P_0 the function $U(\mathbf{r})$ has either an extremum, or an inflection point.

A position P_0 of the particle P is a position of *stable equilibrium* if, setting the particle in a position P_1 close to P_0 , and giving it a sufficiently small initial velocity \mathbf{v}_0 , the trajectory of the particle remains in an infinitely small sphere. In other words, the displacement of the particle from the equilibrium position is infinitely small. More rigorously, for any $\varepsilon > 0$ there correspond the functions $\eta_1(\varepsilon) > 0, \eta_2(\varepsilon) > 0$, such that, if $|P_0 P_1| < \eta_1(\varepsilon)$ and $|\mathbf{v}_0| < \eta_2(\varepsilon)$, then $|P_0 P| < \varepsilon$ for any t .

The position P_0 is a position of maximum (or minimum) for $U(P)$, if there is a vicinity Q_{P_0} of P_0 in which $U(P) \leq U(P_0)$ (or $U(P) \geq U(P_0)$), for any $P \in Q_{P_0}$. When these conditions are fulfilled without the “equal” sign, we have a *strict maximum (minimum)*.

Using these definitions, we shall now demonstrate the *Lagrange–Dirichlet theorem*: *If in the position P_0 the function U has a strict maximum, then P_0 is a position of stable equilibrium.*

The proof begins with the observation that, since $U(P_0) = \max.$, then $V(P_0) = \min.$ But, as we know, the origin of the potential energy V can be arbitrarily chosen, so that we can take $V(P_0) = 0$. Consequently, there exists a vicinity Q_{P_0} of P_0 (except for P_0) for which $V(P) > 0$. Let L be the maximum value of $V(P)$ on the boundary of the domain Q_{P_0} and let us choose a vicinity $Q'_{P_0} \subset Q_{P_0}$ of P_0 , so that $V(P) < \frac{L}{2}$ for any $P \in Q'_{P_0}$. Suppose that at the initial time the particle is in $P_1 \in Q'_{P_0}$ and has the velocity \mathbf{v}_1 , chosen so as to have $\frac{1}{2}m|\mathbf{v}_1|^2 < \frac{L}{2}$. Applying the kinetic energy theorem (1.3.21), we have:

$$\frac{1}{2}m|\mathbf{v}|^2 - \frac{1}{2}m|\mathbf{v}_1|^2 = U(P) - U(P_1)$$

or, since the system is conservative,

$$\frac{1}{2}m|\mathbf{v}|^2 + V(P) = \frac{1}{2}m|\mathbf{v}_1|^2 + V(P_1) = \text{const.}$$

Thus, we may write

$$\frac{1}{2}m|\mathbf{v}|^2 + V(P) < \frac{L}{2} + \frac{L}{2} = L.$$

The quantities $T(P) = \frac{1}{2}m|\mathbf{v}|^2$ and $V(P)$ are positive, therefore $T(P) < L$, $V(P) < L$, showing that the velocity of the particle in position P cannot be greater than a certain value. Consequently the particle, starting from the position P_1 , will never touch the boundary of the domain Q_{P_0} . Recalling the definition of stable equilibrium, the proof is completed.

(2) *Particle subject to constraints*

- (a) *Equilibrium on a surface.* Assuming again a perfectly smooth (ideal) surface, in order for a particle to be in equilibrium it must obey the equation

$$m\ddot{\mathbf{r}} = \mathbf{F} + \mathbf{L}_n = 0. \quad (2.1.46)$$

In other words, the particle is in equilibrium relative to the surface if the resultant of the applied forces is directed along the normal to the surface. If the surface is given in the parametric form (2.1.23), the equations of equilibrium are:

$$F_x + \lambda \frac{\partial(y, z)}{\partial(q^1, q^2)} = 0, \quad F_y + \lambda \frac{\partial(z, x)}{\partial(q^1, q^2)} = 0, \quad F_z + \lambda \frac{\partial(x, y)}{\partial(q^1, q^2)} = 0, \quad (2.1.47)$$

where the functional determinants $\frac{\partial(y, z)}{\partial(q^1, q^2)}$, $\frac{\partial(z, x)}{\partial(q^1, q^2)}$, $\frac{\partial(x, y)}{\partial(q^1, q^2)}$ are the direction parameters of the normal to the surface. The determinant $\frac{\partial(y, z)}{\partial(q^1, q^2)}$, for example, is calculated by

$$\frac{\partial(y, z)}{\partial(q^1, q^2)} = \begin{vmatrix} \frac{\partial y}{\partial q^1} & \frac{\partial y}{\partial q^2} \\ \frac{\partial z}{\partial q^1} & \frac{\partial z}{\partial q^2} \end{vmatrix}.$$

If the surface is given under the implicit form (2.1.39), the equations of equilibrium are

$$F_i + \lambda \frac{\partial f}{\partial x_i} = 0 \quad (i = 1, 2, 3). \quad (2.1.48)$$

- (b) *Equilibrium on a curve.* Following the same procedure, we first consider the case where the curve is given under the parametric form as in (2.1.14). Let x' , y' , z' be the direction parameters of the tangent to the curve in the point where the particle is. Then, the constraint force \mathbf{L} is normal to the curve if

$$x'_i L_i = 0. \quad (2.1.49)$$

This condition is identically satisfied by the choice:

$$L_x = \lambda z' - \mu y', \quad L_y = \mu x' - \nu z', \quad L_z = \nu y' - \lambda x', \quad (2.1.50)$$

where λ , μ , ν are three arbitrary parameters.

Finally, if the curve is expressed by its implicit equations (2.1.18), in view of (2.1.20), the equilibrium condition reads:

$$F_i + \lambda \frac{\partial f_1}{\partial x_i} + \mu \frac{\partial f_2}{\partial x_i} = 0 \quad (i = 1, 2, 3). \quad (2.1.51)$$

Example. Let us find the equilibrium position of a heavy particle, sliding without friction on a fixed circle of radius R , situated in a vertical plane. The circle can be conceived as given by the intersection of a sphere of radius R and a plane passing through its centre. Choosing the origin of the coordinate system in the centre of the sphere and the x -axis along the descendent vertical, the equations of the circle are

$$f_1(x, y, z) \equiv x^2 + y^2 + z^2 - R^2 = 0, \quad f_2(x, y, z) \equiv z = 0. \quad (2.1.52)$$

The equilibrium positions are obtained by eliminating λ and μ from (2.1.51). Multiplying this equation by $\epsilon_{ijk} \frac{\partial f_1}{\partial x_j} \frac{\partial f_2}{\partial x_k}$, we have:

$$\begin{vmatrix} F_x & F_y & F_z \\ \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{vmatrix} = 0. \quad (2.1.53)$$

But $F_x = G = mg$, $F_y = F_z = 0$, therefore (2.1.53) gives $y = 0$. These results, when introduced into the first equation of (2.1.52), produce the following two conditions of equilibrium:

$$x = \pm R. \quad (2.1.54)$$

On the other hand, projecting Eq. (2.1.43) on axes, we have:

$$U = mgx + \text{const.} \quad (2.1.55)$$

One observes that U has a maximum for $x = R$ and a minimum for $x = -R$. According to the Lagrange–Dirichlet theorem, the position $P_0(x = +R, y = 0)$ is a position of *stable* equilibrium, while $P_1(x = -R, y = 0)$ is a position of *unstable* equilibrium.

2.1.2 Many-Particle Systems

Let P_1, \dots, P_N be a system of N particles. At any moment t , the radius-vectors of the particles $\mathbf{r}_1, \dots, \mathbf{r}_N$ and their velocities $\dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N$ can take arbitrary values. A relation of the form

$$f(\mathbf{r}_1, \dots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N, t) = 0 \quad (2.1.56)$$

is a *constraint* which restricts the motion of the particles. The criteria of classification of constraints for many-particle systems are similar to those encountered in the case of a single particle. For example, the relations

$$f_k(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = 0 \quad (k = \overline{1, s}; s \leq 3N) \quad (2.1.57)$$

express *s* bilateral, rheonomous, geometric constraints. They are also *holonomic* constraints. The number of constraints cannot exceed $3N$; in the case $s = 3N$, the N vectors $\mathbf{r}_1, \dots, \mathbf{r}_N$ would be completely determined by the constraints.

As for a single particle, we consider only those differential constraints which are linear in the velocities:

$$\sum_{i=1}^N \mathbf{g}_i^k(\mathbf{r}_1, \dots, \mathbf{r}_N, t) \cdot \dot{\mathbf{r}}_i + g_0^k(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = 0 \quad (k = \overline{1, s}). \quad (2.1.58)$$

It is seen that (2.1.57) can be written in a form similar to (2.1.58). Indeed, taking its total time derivative, we arrive at

$$\sum_{i=1}^N (\text{grad}_i f_k) \cdot \dot{\mathbf{r}}_i + \frac{\partial f_k}{\partial t} = 0 \quad (k = \overline{1, s}). \quad (2.1.59)$$

The constraints (2.1.58) can be integrable (holonomic) or non-integrable (non-holonomic). The non-integrable constraints are also called *Pfaffian*.

The fundamental equation of motion, written for the particle P_i of mass m_i of the system, is

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i + \mathbf{L}_i \quad (i = \overline{1, N}), \quad (2.1.60)$$

where \mathbf{L}_i is the resultant of the constraint forces acting on the particle.

Assuming that the holonomic constraints (2.1.57) are ideal, we can generalize the relation (2.1.40) by multiplying (2.1.57) by $\lambda_k(t)$, performing the summation over k , and introducing the result into (2.1.60):

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i + \sum_{k=1}^s \lambda_k \text{grad}_i f_k \quad (i = \overline{1, N}). \quad (2.1.61)$$

Equations (2.1.61), together with the constraints (2.1.57), represent $3N + s$ equations in the unknowns $\mathbf{r}_1, \dots, \mathbf{r}_N$ ($3N$ coordinates) and $\lambda_1, \dots, \lambda_s$. Equations (2.1.61) are called the *Lagrange equations of the first kind*. They are due to the

Italian-French mathematician *Joseph-Louis Lagrange*, as are many other results, concepts and formalisms which we shall encounter further in this book.

Observation: The problems involving static equilibrium of mechanical systems of particles are discussed in a way similar to that used for a single particle. Notice, nevertheless, that special care must be taken in the case of interacting particles.

2.2 Elementary Displacements

To determine the equilibrium conditions of a system of N particles subject to constraints using the method developed in the previous section, one must separately study the equilibrium of each particle, taking into account that the constraint forces are *a priori* unknown. If the number of particles is large, we have many equations with many unknowns. In this case the aforementioned procedure becomes complicated.

We shall now give a more general and more useful method for solving both dynamic and static problems of mechanics. The main difference from the already known formalism is that the effect of constraints is expressed not by constraint forces, but rather by *elementary displacements* associated with these forces.

Assume that our system is subject to s holonomic, scleronomic constraints

$$f_k(\mathbf{r}_1, \dots, \mathbf{r}_N) = 0 \quad (k = \overline{1, s}). \quad (2.2.1)$$

Being under the action of applied forces, the particles perform certain displacements which must be *consistent* with the constraints. Let $d\mathbf{r}_i$ be the infinitesimal displacement of the particle P_i during the time interval dt , subject to the applied forces and the initial conditions, and consistent with the constraints. Such a displacement takes place *effectively*, during the time interval dt , being *unique*. It is a *real displacement*. But, if we only fix the position of the particle at time t , we can have an infinite number of velocities $\dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N$, consistent with the constraints (2.2.1). The displacements performed by particles under these conditions are called *possible*. The real displacements belong to the multitude of possible displacements, being the subset that satisfies both the equations of motion and the initial conditions.

Now, let us consider a system of displacements $\delta\mathbf{r}_i$ ($i = \overline{1, N}$) that obey only one condition: they are consistent with the constraints. These purely geometric displacements are *synchronic*, i.e. they are taken at an instant t ($\delta t = 0$). These are usually called *virtual displacements*.

By differentiating (2.2.1), we get

$$\sum_{i=1}^N (\text{grad}_i f_k) \cdot d\mathbf{r}_i = 0 \quad (k = \overline{1, s}), \quad (2.2.2)$$

meaning that all the real (or possible) displacements $d\mathbf{r}_i$ lie in the planes tangent to the surfaces $f_1 = 0, \dots, f_s = 0$. Using the definition of virtual displacements $\delta\mathbf{r}_i$, we infer also:

$$\sum_{i=1}^N (\text{grad}_i f_k) \cdot \delta \mathbf{r}_i = 0 \quad (k = \overline{1, s}), \quad (2.2.3)$$

showing that any virtual displacement can become a possible one.

Passing now to the rheonomous constraints, given, for example, by Eq. (2.1.56), we realize that the possible displacements $d\mathbf{r}_i$ ($i = \overline{1, N}$) must obey the relation

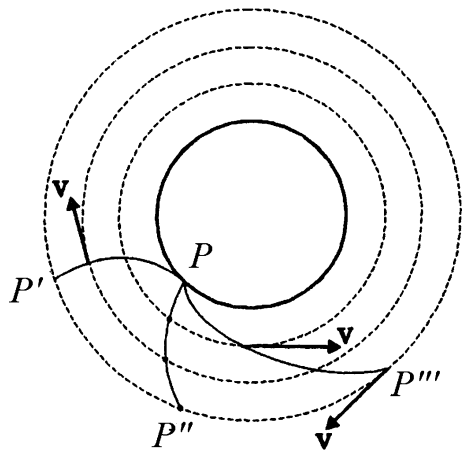
$$\sum_{i=1}^N (\text{grad}_i f_k) \cdot d\mathbf{r}_i + \frac{\partial f_k}{\partial t} dt = 0 \quad (k = \overline{1, s}), \quad (2.2.4)$$

while the virtual displacements $\delta \mathbf{r}_i$ satisfy an equation similar to Eq. (2.2.3). Writing (2.2.4) for two sets of possible displacements $d\mathbf{r}'_i$ and $d\mathbf{r}''_i$, and then subtracting the obtained relations, we arrive precisely at Eq. (2.2.3), where

$$\delta \mathbf{r}_i = d\mathbf{r}'_i - d\mathbf{r}''_i. \quad (2.2.5)$$

Therefore, *any virtual displacement can be considered as the difference between two possible displacements*. For example, consider a spherical balloon with a fixed centre, taken in the process of inflation (Fig. 2.4). At the moments $t_1 < t_2 < t_3 \dots$, the radii of the balloon will be $R_1 < R_2 < R_3 \dots$. An ant moving on the balloon, and being at time t_1 in the position P , could be at time $t_2 > t_1$ in any position P' , P'' , P''' etc., on the sphere of radius $R_2 > R_1$. The displacements $d\mathbf{r}'$, $d\mathbf{r}''$, $d\mathbf{r}'''$, etc. are *possible* displacements. Depending on the initial conditions (the ant is considered a mechanical system), only one of these displacements is *real*. Any *virtual* displacement $\delta \mathbf{r}$ at time t_2 lies in the plane tangent to the sphere of radius R_2 , and obeys the rule (2.2.5). The virtual displacements are *atemporal*, in our example being any displacement on the balloon surface, taken at an instant, while R is fixed.

Fig. 2.4 Intuitive examples of real, possible and virtual displacements.



2.3 Principle of Virtual Work

A method providing a very efficient way of eliminating the constraint forces appearing in a mechanical problem is the *principle of virtual work*. Assume again that a system of N particles P_1, \dots, P_N is in static equilibrium and subject to ideal constraints. If $\delta \mathbf{r}_i$ is a virtual displacement of the particle P_i , consistent with the constraints, then, by definition,

$$\delta W = \mathbf{F}_i \cdot \delta \mathbf{r}_i \quad (2.3.1)$$

is the *virtual work* of the force \mathbf{F}_i relative to the displacement $\delta \mathbf{r}_i$. In the case of static equilibrium ($\ddot{\mathbf{r}}_i = 0$), multiplying (2.1.60) by $\delta \mathbf{r}_i$ and summing for all the particles of the system, yields:

$$\sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0, \quad (2.3.2)$$

where we have used the property of ideal constraints

$$\sum_{i=1}^N \mathbf{L}_i \cdot \delta \mathbf{r}_i = 0. \quad (2.3.3)$$

Relation (2.3.2) expresses the *principle of virtual work*: *The necessary and sufficient condition for static equilibrium of a sclerononomous system subject to ideal constraints is that the virtual work of the applied forces, for virtual displacements consistent with the constraints, be zero*. If the particles were free, the displacements $\delta \mathbf{r}_i$ would be arbitrary.

Let us now show that from the principle of virtual work all the conditions of equilibrium discussed in Sect. 2.1 can be derived.

2.3.1 Free Particle

The principle (2.3.2) for one particle is written as

$$\mathbf{F} \cdot \delta \mathbf{r} = 0. \quad (2.3.4)$$

Since $\delta \mathbf{r}$ is completely arbitrary, it follows that $\mathbf{F} = 0$, in agreement with (2.1.41).

2.3.2 Particle Subject to Constraints

If the constraint is an ideal surface $f(x, y, z) = 0$, then the condition

$$\text{grad } f \cdot \delta \mathbf{r} = 0 \quad (2.3.5)$$

expresses the fact that the particle lies on the surface. Multiplying (2.3.5) by some scalar λ and adding the result to (2.3.4), we fall back on the relation (2.1.48). In the case of an ideal curve, the condition (2.3.4) must be completed with

$$\text{grad } f_1 \cdot \delta \mathbf{r} = 0, \quad \text{grad } f_2 \cdot \delta \mathbf{r} = 0, \quad (2.3.6)$$

leading together to (2.1.51).

2.3.3 System of Free Particles

For arbitrary virtual displacements $\delta \mathbf{r}_i$, we have

$$\sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0, \quad (2.3.7)$$

yielding the conditions of equilibrium

$$\mathbf{F}_i = 0 \quad (i = \overline{1, N}), \quad (2.3.8)$$

which are also obtained from (2.1.60) for $\ddot{\mathbf{r}}_i = 0$, $\mathbf{L}_i = 0$. Notice that, by using the principle of virtual work, the N relations (2.3.8) are replaced by a single relation (2.3.7).

2.3.4 System of Particles Subject to Constraints

Assuming that the constraints are given by (2.1.57), we may write:

$$\sum_{i=1}^N \left(\frac{\partial f_k}{\partial x_i} \delta x_i + \frac{\partial f_k}{\partial y_i} \delta y_i + \frac{\partial f_k}{\partial z_i} \delta z_i \right) = 0 \quad (k = \overline{1, s}). \quad (2.3.9)$$

From (2.3.2) we have also

$$\sum_{i=1}^N (X_i \delta x_i + Y_i \delta y_i + Z_i \delta z_i) = 0, \quad (2.3.10)$$

where X_i, Y_i, Z_i are the components of the force \mathbf{F}_i . The displacements $\delta x_i, \delta y_i, \delta z_i$ are not arbitrary anymore, but they must obey the s relations (2.3.9).

The $(s + 1)$ equations (2.3.9) and (2.3.10) can be written as a single relation by using the method of *Lagrange multipliers*. Let us amplify each of the equations (2.3.9) by λ_k , then perform the summation over the index k and add the result to (2.3.10). We obtain:

$$\sum_{i=1}^N \left[\left(X_i + \sum_{k=1}^s \lambda_k \frac{\partial f_k}{\partial x_i} \right) \delta x_i + \left(Y_i + \sum_{k=1}^s \lambda_k \frac{\partial f_k}{\partial y_i} \right) \delta y_i + \left(Z_i + \sum_{k=1}^s \lambda_k \frac{\partial f_k}{\partial z_i} \right) \delta z_i \right] = 0. \quad (2.3.11)$$

This relation must be satisfied by *any* $\delta x_i, \delta y_i, \delta z_i$ ($i = \overline{1, N}$). Since these variations must obey the s linear homogeneous equations (2.3.10), it follows that $3N - s$ of these displacements can be taken as being independent. Then, in (2.3.11) are determined λ_k so that the parentheses which multiply the s dependent displacements are zero, leading to a number of s equations. Next, we make vanish the parentheses multiplying the $3N - s$ independent displacements, and get more $3N - s$ equations. Finally, we are left with $3N$ equations:

$$X_i + \sum_{k=1}^s \lambda_k \frac{\partial f_k}{\partial x_i} = 0, \quad Y_i + \sum_{k=1}^s \lambda_k \frac{\partial f_k}{\partial y_i} = 0, \quad Z_i + \sum_{k=1}^s \lambda_k \frac{\partial f_k}{\partial z_i} = 0 \quad (i = \overline{1, N}). \quad (2.3.12)$$

The $3N$ equations (2.3.12), together with the s equations (2.1.57), form a system of $3N + s$ equations for $3N + s$ unknowns: the equilibrium coordinates x_i, y_i, z_i ($i = \overline{1, N}$) and the multipliers $\lambda_1, \dots, \lambda_s$.

Observation: The principle of virtual work applies to the study of the equilibrium conditions of a *rigid body* as well. Anticipating, we shall use (4.3.10) to write the velocity \mathbf{v}_i of a particle P_i of the rigid body, relative to a fixed frame $Oxyz$:

$$\mathbf{v}_i = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}'_i. \quad (2.3.13)$$

Here, \mathbf{v}_0 is the velocity of some particle O' of the body and \mathbf{r}'_i the radius-vector of P_i relative to O' . If $\delta \mathbf{r}'_i$ is a virtual displacement of P_i , consistent with the rigidity constraints, we can write:

$$\mathbf{v}_i = \frac{\delta \mathbf{r}_i}{\delta t}, \quad \mathbf{v}_0 = \frac{\delta \mathbf{r}_0}{\delta t} \quad (2.3.14)$$

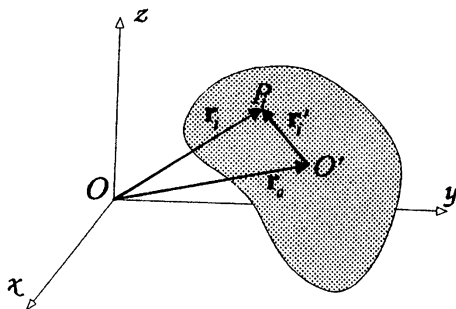
and (2.3.13) becomes

$$\delta \mathbf{r}_i = \delta \mathbf{r}_0 + (\boldsymbol{\omega} \times \mathbf{r}'_i) \delta t. \quad (2.3.15)$$

The principle of virtual work (2.3.2) reads then:

$$\sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i = \delta \mathbf{r}_0 \cdot \sum_{i=1}^N \mathbf{F}_i + \delta t \boldsymbol{\omega} \cdot \sum_{i=1}^N \mathbf{r}'_i \times \mathbf{F}_i = 0, \quad (2.3.16)$$

Fig. 2.5 Choice of the systems of coordinates to find the equilibrium conditions of a free rigid body (2.3.20).



being true for completely arbitrary variations $\delta \mathbf{r}_0$ and $\omega \delta t$. Therefore, we obtain:

$$\sum_{i=1}^N \mathbf{F}_i = \mathbf{F} = 0, \quad \sum \mathbf{r}'_i \times \mathbf{F}_i = \mathcal{M}' = 0, \quad (2.3.17)$$

where the torque \mathcal{M}' is taken relative to O' . But $\mathbf{r}'_i = \mathbf{r}_i - \mathbf{r}_0$ (see Fig. 2.5), so that

$$\sum_{i=1}^N \mathbf{r}'_i \times \mathbf{F}_i = \sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_i - \mathbf{r}_0 \times \sum_{i=1}^N \mathbf{F}_i = 0 \quad (2.3.18)$$

and, using (2.3.17),

$$\mathcal{M} = \sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_i = 0. \quad (2.3.19)$$

Since O is arbitrary, we conclude that \mathcal{M} can be taken with respect to *any* point. Therefore, the equilibrium conditions of a free rigid body are:

$$\mathbf{F} = 0, \quad \mathcal{M} = 0. \quad (2.3.20)$$

Note that any point of a free rigid body has two independent virtual vector displacements, $\delta \mathbf{r}_0$ and $\omega \delta t$, equivalent to six components. Consequently, a free rigid body possesses six degrees of freedom.

The equilibrium conditions for a rigid body subject to constraints are obtained in a similar way. For instance, if the body has a fixed point, say O' , the reaction force of the point O' can be considered as an applied force, and so the body can be regarded as being free. Hence, in view of (2.3.17),

$$\mathbf{L} + \sum_{i=1}^N \mathbf{F}_i = 0, \quad \sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_i = 0. \quad (2.3.21)$$

Since O' is fixed relative to O , we have $\delta \mathbf{r}_0 = 0$, and so

$$\delta \mathbf{r}_i = (\omega \times \mathbf{r}'_i) \delta t, \quad (2.3.22)$$

i.e. a rigid body with a fixed point has three degrees of freedom.

2.3.5 Application

Using the principle of virtual work, let us find the equilibrium positions of a particle A of mass m , which can slide without friction on an ellipse of semi-axes a and b , rotating with constant angular velocity ω about its minor axis, directed along the vertical as shown in Fig. 2.6.

Our particle is subject to two applied forces:

$$\text{force of gravity: } \mathbf{F}_g = m\mathbf{g} = -mg\mathbf{j}, \quad (2.3.23)$$

$$\text{centrifugal force: } \mathbf{F}_{cf} = m\omega^2\mathbf{r} = m\omega^2x\mathbf{i}, \quad (2.3.24)$$

and a constraint force, due to the restriction of moving on the ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (2.3.25)$$

The principle of virtual work (2.3.2) yields:

$$\mathbf{F} \cdot \delta\mathbf{r} = m\omega^2x\delta x - mg\delta y = 0. \quad (2.3.26)$$

On the other hand, by differentiating (2.3.25), we have:

$$\frac{x\delta x}{a^2} + \frac{y\delta y}{b^2} = 0. \quad (2.3.27)$$

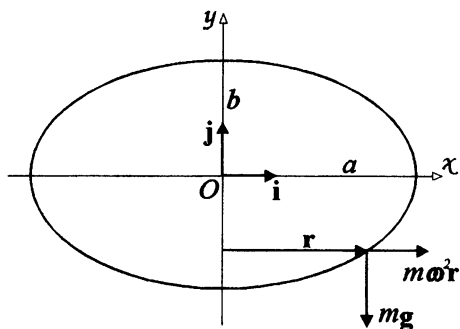
Eliminating δy between the last two equations, we obtain

$$x\left(\frac{1}{a^2} + \frac{\omega^2 y}{b^2 g}\right) = 0.$$

This means that either

$$(a) \quad x = 0, \quad \frac{1}{a^2} + \frac{\omega^2 y}{b^2 g} \neq 0,$$

Fig. 2.6 A particle sliding without friction on a rotating ellipse.



or

$$(b) \quad x \neq 0, \quad \frac{1}{a^2} + \frac{\omega^2 y}{b^2 g} = 0.$$

Consequently, these two cases lead to the following possible equilibrium conditions:

$$(a) \quad x = 0, \quad y = \pm b, \quad (2.3.28)$$

$$(b) \quad x = \pm a \sqrt{1 - \frac{b^2 g^2}{a^4 \omega^4}}, \quad y = -\frac{b^2 g}{a^2 \omega^2}. \quad (2.3.29)$$

Obviously, if $\omega \rightarrow \infty$, we have $x \rightarrow \pm a$, $y \rightarrow 0$.

2.4 Generalized Coordinates

Consider again a system of N particles P_1, \dots, P_N , of radius-vectors $\mathbf{r}_1, \dots, \mathbf{r}_N$ relative to a Cartesian orthogonal frame $Oxyz$, subject to s holonomic independent constraints

$$f_k(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = 0 \quad (k = \overline{1, s}). \quad (2.4.1)$$

Due to the existence of the constraints, the $3N$ coordinates of particles are not independent, therefore the number of independent coordinates will be

$$3N - s = n, \quad (2.4.2)$$

meaning that our system has $3N - s = n$ degrees of freedom. For instance, a system of two particles, at a fixed distance one from the other, has $6 - 1 = 5$ degrees of freedom.

If the number of particles is large, the presence of constraints makes the determination of the coordinates x_i, y_i, z_i a difficult task. We shall attach to the n degrees of freedom a number of n independent variables q_1, \dots, q_n , called *generalized coordinates* or *Lagrangian variables*. The $3N$ Cartesian coordinates \mathbf{r}_i are then expressed in terms of q_1, \dots, q_n by

$$\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_n, t) \equiv \mathbf{r}_i(q, t) \quad (i = \overline{1, N}). \quad (2.4.3)$$

The generalized coordinates q_j ($j = \overline{1, n}$) satisfy the following properties:

(a) Any independent variation of q_1, \dots, q_n yields

$$f_k[\mathbf{r}_1(q, t), \dots, \mathbf{r}_N(q, t), t] \equiv 0. \quad (2.4.4)$$

(b) Any $\mathbf{r}_1, \dots, \mathbf{r}_N$, consistent with the constraints (2.4.1), can be obtained from (2.4.3).

(c) There also exists the inverse transformation of (2.4.3), namely

$$q_j = q_j(\mathbf{r}_1, \dots, \mathbf{r}_N, t) \quad (j = \overline{1, n}), \quad (2.4.5)$$

for $\mathbf{r}_1, \dots, \mathbf{r}_N$ satisfying (2.4.1).

Similarly to the Cartesian coordinates, the generalized coordinates are assumed to be continuous functions of time, at least twice differentiable. On the other hand, in contrast to the Cartesian coordinates, the generalized coordinates do not necessarily have the dimension of length. We can choose as Lagrangian coordinates any suitable assembly of geometrical objects, such as: segments of straight lines, arcs, angles, surfaces, components of angular velocities, etc.

The choice of generalized coordinates is somewhat arbitrary. It is always possible to find a point transformation

$$q_j \rightarrow q'_j = q'_j(q_1, \dots, q_n, t) \quad (j = \overline{1, n}), \quad (2.4.6)$$

such that q'_1, \dots, q'_n are a new set of Lagrangian variables.

If the system is not subject to constraints, we can choose as generalized coordinates the $3N$ Cartesian coordinates of the particles, but there are also other possible choices. For instance, the position of a free particle can be defined either by its Cartesian coordinates x, y, z , its spherical coordinates r, θ, φ , or its cylindrical coordinates ρ, φ, z , etc.

Example. A particle P is constrained to remain on the moving sphere

$$(x - at)^2 + (y - bt)^2 + (z - ct)^2 = R^2. \quad (2.4.7)$$

Since $n = 2$, we can choose $q_1 = \theta$, $q_2 = \varphi$. At time t , the centre of the sphere is at the point (at, bt, ct) , therefore we can write

$$x = at + R \sin \theta \cos \varphi, \quad y = bt + R \sin \theta \sin \varphi, \quad z = ct + R \cos \theta, \quad (2.4.8)$$

representing the transition from Cartesian to spherical coordinates.

2.4.1 Configuration Space

The set of radius-vectors $\mathbf{r}_1, \dots, \mathbf{r}_N$ define the so-called *configuration* of the system of particles, in the real space. If we choose q_1, \dots, q_n as coordinates of a n -dimensional space R_n , then to each set of values of the variables q_1, \dots, q_n will correspond a *representative point* in this space, known as the *configuration space*. In other words, any configuration of a mechanical system can be represented by a single point in the configuration space R_n . Note that the configuration space does not generally have an intuitive meaning, as does the Euclidean space used in Newtonian mechanics; but, as we shall prove, the abstract notions of *generalized coordinates* and *configuration space* are very useful not only in mechanics, but in other physical disciplines as well.

As the mechanical system changes its configuration with time, the configuration point traces a curve in configuration space, called *generalized trajectory*. This is by no means any of the real trajectories of particles, but describes the motion of the *whole* system. The generalized trajectory can be conceived as a succession of representative points, each of them corresponding to a certain configuration of the system. To know the *law of motion* in the configuration space means to know

$$q_j = q_j(t) \quad (j = \overline{1, n}). \quad (2.4.9)$$

These are also the *parametric equations* of the generalized trajectory. Once (2.4.9) are known, by means of (2.4.3), the motion of the particles in real space can also be determined.

2.4.2 Generalized Forces

In view of (2.4.3), a real infinitesimal displacement $d\mathbf{r}_i$ of particle P_i , during the time interval dt , is

$$d\mathbf{r}_i = \sum_{j=1}^n \frac{\partial \mathbf{r}_i}{\partial q_j} dq_j + \frac{\partial \mathbf{r}_i}{\partial t} dt \quad (i = \overline{1, N}), \quad (2.4.10)$$

while a virtual displacement $\delta \mathbf{r}_i$ satisfies the relation

$$\delta \mathbf{r}_i = \sum_{j=1}^n \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \quad (i = \overline{1, N}). \quad (2.4.11)$$

The displacements dq_j and δq_j in the configuration space are similar to the displacements $d\mathbf{r}_i$ and $\delta \mathbf{r}_i$ defined in the real space. Thus, by dq_j we mean *real* (or *possible*) displacements of the representative point during time dt , while δq_j are *virtual displacements*, taken at $t = \text{const.}$ (i.e. $\delta t = 0$). If q_1, \dots, q_n are independent, $\delta q_1, \dots, \delta q_n$ are also independent and can be considered as a set of n completely arbitrary displacements at an instant.

Let us now write the virtual work δW , done by applied forces $\mathbf{F}_1, \dots, \mathbf{F}_N$ on the particles, in terms of virtual displacements in the configuration space. In view of (2.4.3), we have:

$$\delta W = \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_{j=1}^n \left(\sum_{i=1}^N \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) \delta q_j.$$

If we define the *generalized forces* by

$$Q_j = \sum_{i=1}^N \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \quad (j = \overline{1, n}), \quad (2.4.12)$$

the work can be written as

$$\delta W = \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_{j=1}^n Q_j \delta q_j. \quad (2.4.13)$$

Since, in general, the forces \mathbf{F}_i are functions of the form

$$\mathbf{F}_i = \mathbf{F}_i(\mathbf{r}_1, \dots, \mathbf{r}_N, \dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N, t) \quad (i = \overline{1, N}), \quad (2.4.14)$$

we conclude that the generalized forces Q_j have the following functional dependence:

$$Q_j = Q_j(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) \equiv Q_j(q, \dot{q}, t) \quad (j = \overline{1, n}). \quad (2.4.15)$$

The quantities

$$\dot{q}_j = \frac{dq_j}{dt}$$

are called *generalized velocities* and are related to the real velocities $\mathbf{v}_1, \dots, \mathbf{v}_N$ by

$$\dot{\mathbf{r}}_i = \mathbf{v}_i = \sum_{j=1}^n \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \quad (i = \overline{1, N}). \quad (2.4.16)$$

The physical meaning of the generalized forces Q_j emerges from the significance of their associated generalized coordinates. For example, if the transition from Cartesian coordinates x, y, z to orthogonal curvilinear coordinates q_1, q_2, q_3 is defined by

$$x_i = x_i(q_1, q_2, q_3) \quad (i = \overline{1, N}), \quad (2.4.17)$$

then $\frac{\partial \mathbf{r}}{\partial q_k}$ is a vector tangent to the curve $q_k = \text{variable}$, while $Q_k = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_k}$ is the component of \mathbf{F} on this direction. In particular, the choice $q_1 = r, q_2 = \theta, q_3 = \varphi$ yields (see Appendix B):

$$\begin{aligned} Q_1 &= Q_r = \mathbf{F} \cdot \mathbf{u}_r = F_r, \\ Q_2 &= Q_\theta = \mathbf{F} \cdot (r \mathbf{u}_\theta) = r F_\theta, \\ Q_3 &= Q_\varphi = \mathbf{F} \cdot (r \sin \theta \mathbf{u}_\varphi) = r \sin \theta F_\varphi. \end{aligned} \quad (2.4.18)$$

The generalized forces do not generally have the dimension of force, but the product $[qQ]$ has always the dimension of work.

If the forces \mathbf{F}_i ($i = \overline{1, N}$) derive from a potential (see (1.3.22)):

$$\mathbf{F}_i = -\text{grad}_i V \quad (i = \overline{1, N}), \quad (2.4.19)$$

then the generalized forces Q_j obey a similar equation:

$$Q_j = - \sum_{i=1}^N \frac{\partial V}{\partial \mathbf{r}_i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = - \frac{\partial V}{\partial q_j} \quad (j = \overline{1, n}), \quad (2.4.20)$$

where $V = V(q_1, \dots, q_n, t)$ is the potential in terms of the new variables.

From (2.4.13) follows that a position of the representative point in the configuration space, at time t , is a position of equilibrium, if

$$\sum_{j=1}^n Q_j \delta q_j = 0, \quad (2.4.21)$$

which expresses the *principle of virtual work in R_n* . If the virtual displacements δq_j are arbitrary and independent, it results in

$$Q_j = 0 \quad (j = \overline{1, n}), \quad (2.4.22)$$

meaning that: a certain position of a system of particles, subject to holonomic constraints, is a position of equilibrium, if all the generalized forces corresponding to that position are zero.

2.4.3 Kinetic Energy in Generalized Coordinates

It is most useful in the development of our formalism to express the kinetic energy T of the system in terms of the generalized coordinates q_1, \dots, q_n , and the generalized velocities $\dot{q}_1, \dots, \dot{q}_n$. In view of (2.4.5), we have:

$$T = \frac{1}{2} \sum_{i=1}^N m_i \sum_{j=1}^n \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_i}{\partial t} \right) \cdot \sum_{k=1}^n \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \right).$$

Set

$$a = \frac{1}{2} \sum_{i=1}^N m_i \left| \frac{\partial \mathbf{r}_i}{\partial t} \right|^2, \quad a_j = \sum_{i=1}^N m_i \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial t}, \quad a_{jk} = \frac{1}{2} \sum_{i=1}^N \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}, \quad (2.4.23)$$

where a, a_j, a_{jk} are continuous and differentiable functions of q_1, \dots, q_n, t . Thus,

$$T = a + \sum_{j=1}^n a_j \dot{q}_j + \sum_{j=1}^n \sum_{k=1}^n a_{jk} \dot{q}_j \dot{q}_k = T_0 + T_1 + T_2, \quad (2.4.24)$$

where the meaning of T_0, T_1, T_2 is obvious.

If the constraints are scleronomous, the terms T_0 and T_1 in (2.4.24) vanish and the kinetic energy $T = T_2$ becomes a homogeneous quadratic form of the generalized velocities \dot{q}_j :

$$T = \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n a_{jk} \dot{q}_j \dot{q}_k. \quad (2.4.25)$$

Keeping in mind the definition of a_{jk} , we see that the quadratic form T_2 is positively defined, $T_2 \geq 0$ (the equality sign is valid only if all $\dot{q}_1, \dots, \dot{q}_n$ are zero). For example, the kinetic energy of a particle of mass m in spherical coordinates is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2).$$

We can therefore conclude that, in general, the kinetic energy has the following functional dependence:

$$T = T(q, \dot{q}, t). \quad (2.4.26)$$

2.5 Differential and Integral Principles in Analytical Mechanics

As Newtonian mechanics is based on the well-known principles of inertia, of force and of reciprocal interactions, so another formulation of mechanics is constructed on some fundamental axioms, called *principles of analytical mechanics*. These postulates serve to deduce the differential equations of motion in the configuration space. The principles of analytical mechanics are more general than those of Newtonian mechanics; they allow not only to obtain the results of Newtonian mechanics, but also to approach a large variety of non-mechanical problems. As a matter of fact, the methods provided by analytical mechanics play an important role in other physical disciplines, such as: theory of elasticity, quantum field theory, electrodynamics, theory of relativity, etc.

The principles of analytical mechanics can be grouped in two categories:

- (a) *Differential principles*, which give us information about the state of a system, at different times, and take care of the behaviour of the system under *infinitesimal variations* of general coordinates and velocities in the configuration space. In general, the differential equations of motion (in both real and configuration spaces) can be considered as mathematical forms of certain differential principles. Such a principle is, for example, *D'Alembert's principle*.
- (b) *Integral principles*, which consider the motion of a system during a *finite* time interval. These principles operate with *global* variations in configuration space. In this category fall *variational principles*, that use the methods of variational calculus, for global displacements along the generalized trajectories. The *Hamilton* and the *Maupertuis principles* belong to this category.

The distinction between these two groups is not absolute. As we shall see later on, there is an intimate relation between all the principles of analytical mechanics.

2.5.1 D'Alembert's Principle

Consider again a system of N particles P_1, \dots, P_N , subject to applied and (holonomic) constraint forces. The Newtonian equation of motion of the particle P_i is (see (2.1.60)):

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i + \mathbf{L}_i \quad (i = \overline{1, N}). \quad (2.5.1)$$

Let us denote

$$\mathbf{J}_i = -m_i \ddot{\mathbf{r}}_i \quad (i = \overline{1, N}) \quad (2.5.2)$$

and call it the *inertial force* acting on particle P_i . Then

$$\mathbf{F}_i + \mathbf{L}_i + \mathbf{J}_i = 0 \quad (i = \overline{1, N}). \quad (2.5.3)$$

This vector equation expresses one of the forms of D'Alembert's principle: *there is an equilibrium, at any moment, between the applied, the constraint and the inertial forces acting on a particle*. This is the initial form of the principle, discovered by *Jean-Baptiste le Rond D'Alembert*.

In the case of ideal constraints, D'Alembert's principle can be written in an alternative form, which is very useful in some applications. To find this expression, we multiply (2.5.3) by the virtual displacement $\delta \mathbf{r}_i$, and then we take the sum over all particles of the system. Since the virtual work associated with the ideal constraints is zero, we arrive at

$$\sum_{i=1}^N (\mathbf{J}_i + \mathbf{F}_i) \cdot \delta \mathbf{r}_i = 0 \quad (2.5.4)$$

or, in a slightly different form,

$$\sum_{i=1}^N (\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0, \quad (2.5.5)$$

meaning that: *The sum of the virtual works of applied and inertial forces, acting on a system subject to ideal constraints, is zero*. This form of D'Alembert's principle is most useful, because it does not contain the constraint forces anymore. It was given by *Lagrange* and is used to deduce the differential equations of motion in configuration space.

2.5.2 Lagrange Equations for Holonomic Systems

We are now prepared to derive the differential equations of motion of a system of N particles, subject to ideal and independent constraints, in terms of generalized coordinates q_1, \dots, q_n . To this end, we shall express both the variations $\delta \mathbf{r}_i$ and the

derivatives $\ddot{\mathbf{r}}_i$ occurring in (2.5.5) in the configuration space. The virtual displacements $\delta \mathbf{r}_i$ can be written as (see (2.4.11)):

$$\delta \mathbf{r}_i = \sum_{j=1}^n \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \quad (i = \overline{1, N}), \quad (2.5.6)$$

hence

$$\begin{aligned} \sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i &= \sum_{j=1}^n \left(\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) \delta q_j \\ &= \sum_{j=1}^n \left[\frac{d}{dt} \left(\sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - \sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \right] \delta q_j. \end{aligned} \quad (2.5.7)$$

But

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) &= \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) + \sum_{k=1}^n \frac{\partial}{\partial q_k} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \dot{q}_k = \frac{\partial}{\partial q_j} \left(\frac{\partial \mathbf{r}_i}{\partial t} + \sum_{k=1}^n \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k \right) \\ &= \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j}. \end{aligned} \quad (2.5.8)$$

On the other hand, (2.4.16) yields

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}. \quad (2.5.9)$$

The substitution of (2.5.8) and (2.5.9) into (2.5.7) gives

$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_{j=1}^n \left[\frac{d}{dt} \left(\sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial \dot{q}_j} \right) - \sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right] \delta q_j. \quad (2.5.10)$$

Recalling that

$$T = \frac{1}{2} \sum_{i=1}^N m_i |\dot{\mathbf{r}}_i|^2$$

is the kinetic energy of the system of particles, it is easy to observe that (2.5.10) becomes

$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_{j=1}^n \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right] \delta q_j. \quad (2.5.11)$$

The last step is now to introduce (2.4.13) and (2.5.11) into the expression for D'Alembert's principle (2.5.5). The result is:

$$\sum_{j=1}^n \left[\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right] \delta q_j = 0. \quad (2.5.12)$$

Since the constraints are independent, the virtual displacements δq_j are completely arbitrary. Therefore (2.5.12) holds true only if all the square brackets are zero, i.e.

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q_j \quad (j = \overline{1, n}), \quad (2.5.13)$$

which are called *Lagrange equations of the second kind*. From now on, we shall use these equations under the shorter name of *Lagrange equations*. They represent a system of n second-order differential equations in the variables q_j . The general integral of (2.5.13),

$$q_j = q_j(t, C_1, \dots, C_{2n}) \quad (j = \overline{1, n}), \quad (2.5.14)$$

expresses the *law of motion* in the configuration space R_n . The $2n$ arbitrary constants C_1, \dots, C_{2n} are determined by $2n$ initial conditions: at time $t = t_0$, we must know both the generalized coordinates and the generalized velocities,

$$q_j^0 = q_j(t_0, C_1, \dots, C_{2n}), \quad \dot{q}_j^0 = \dot{q}_j(t_0, C_1, \dots, C_{2n}). \quad (2.5.15)$$

Once the motion in configuration space is determined, the solution (2.5.14) is introduced into (2.4.3), giving the motion in real space.

If, in particular, there are no constraints acting on the particles, we can choose as generalized coordinates the Cartesian coordinates, thus falling on Newton's second law discussed in Chap. 1.

Assume now that the applied forces \mathbf{F}_i are *potential*. Then, according to (2.4.20), the generalized forces Q_j are also potentials and we obtain:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} = 0 \quad (j = \overline{1, n}),$$

where $V = V(q, t)$. Introducing the *Lagrangian function* or, simply, the *Lagrangian* L by

$$L(q, \dot{q}, t) = T(q, \dot{q}, t) - V(q, t), \quad (2.5.16)$$

we finally arrive at

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0 \quad (j = \overline{1, n}). \quad (2.5.17)$$

These equations are remarkably useful for several reasons. First, as we have already mentioned, they do not contain constraint forces. Second, all the

information regarding the behaviour of the system is contained in a single scalar function, the Lagrangian. These equations are widely applied in many branches of physics, as we shall show in our further development of this formalism.

To solve a problem using the Lagrangian technique, one should proceed as follows:

1. Identify the n degrees of freedom of the system and choose suitable generalized coordinates q_j ;
2. Construct either the functions T , Q_j , or the Lagrangian L ;
3. Impose initial conditions;
4. Integrate the Lagrange equations and then, if necessary, determine the trajectories of the particles;
5. Obtain the constraint forces by means of (2.1.60):

$$\mathbf{L}_i = m_i \ddot{\mathbf{r}}_i - \mathbf{F}_i \quad (i = \overline{1, N}). \quad (2.5.18)$$

In particular, if there is no applied force acting on the particles, the Lagrange equations determine the geodesics of the configuration space R_n . The already known form of the equation of geodesics (see (2.1.37)) is obtained recalling that, for scleronomous constraints, the kinetic energy can be written as (see (2.4.25)):

$$T = \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n a_{jk}(q) \dot{q}_j \dot{q}_k. \quad (2.5.19)$$

The Lagrange equations (2.5.13) then yield:

$$\sum_{k=1}^n a_{jk} \ddot{q}_k + \sum_{k=1}^n \sum_{l=1}^n \frac{\partial a_{jk}}{\partial q_l} \dot{q}_k \dot{q}_l - \frac{1}{2} \sum_{k=1}^n \sum_{l=1}^n \frac{\partial a_{kl}}{\partial q_j} \dot{q}_k \dot{q}_l = Q_j \quad (j = \overline{1, n}).$$

Introducing the Christoffel symbols of the first kind,

$$\Gamma_{kl,j} = \frac{1}{2} \left(\frac{\partial a_{jk}}{\partial q_l} + \frac{\partial a_{lj}}{\partial q_k} - \frac{\partial a_{kl}}{\partial q_j} \right), \quad (2.5.20)$$

we have:

$$\sum_{k=1}^n a_{jk} \ddot{q}_k + \sum_{k=1}^n \sum_{l=1}^n \Gamma_{kl,j} \dot{q}_k \dot{q}_l = Q_j. \quad (2.5.21)$$

If $Q_j = 0$, we arrive at the geodetic form of the equilibrium equations in configuration space R_n (see (2.1.37)).

2.5.3 Velocity-Dependent Potential

Let us show that the Lagrange equations (2.5.17) keep their form in the case of a *generalized* or *velocity-dependent* potential $V(q, \dot{q}, t)$, linear in \dot{q}_j , if Q_j can be taken as

$$Q_j = \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{q}_j} \right) - \frac{\partial V}{\partial q_j} \quad (j = \overline{1, n}). \quad (2.5.22)$$

Consider the potential

$$V(q, \dot{q}, t) = \sum_{j=1}^n \alpha_j \dot{q}_j + V_0 = V_1 + V_0, \quad (2.5.23)$$

where α_j ($j = \overline{1, n}$) and V_0 are functions of q_j and t , and add the quantity

$$\frac{\partial V}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial V}{\partial \dot{q}_j} \right)$$

to both sides of (2.5.13). Then it is obvious that, if (2.5.23) is true, we arrive at the Lagrange equations (2.5.17), where

$$L(q, \dot{q}, t) = T(q, \dot{q}, t) - V(q, \dot{q}, t). \quad (2.5.24)$$

A classic example of generalized potential is offered by the motion of an electrically charged particle in an external electromagnetic field. It is well known that the electromagnetic force acting on a particle of mass m and charge e , moving with velocity \mathbf{v} in the field \mathbf{E} , \mathbf{B} , is:

$$\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (2.5.25)$$

The fields \mathbf{E} and \mathbf{B} are usually given in terms of the electromagnetic potentials $\mathbf{A}(\mathbf{r}, t)$ and $\phi(\mathbf{r}, t)$ as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (2.5.26)$$

Since the particle is free, it has three degrees of freedom. We choose $q_i = x_i$, $\dot{q}_i = \dot{x}_i = v_i$ ($i = 1, 2, 3$). Recalling that q_i and \dot{q}_i are independent with respect to each other, we can write (see Appendix B):

$$\nabla(\mathbf{v} \cdot \mathbf{A}) = \mathbf{v} \times (\nabla \times \mathbf{A}) + (\mathbf{v} \cdot \nabla)\mathbf{A}. \quad (2.5.27)$$

We also have:

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A}. \quad (2.5.28)$$

Using (2.5.26)–(2.5.28), we get from (2.5.25):

$$\begin{aligned} F_i &= e \left\{ -\frac{\partial \phi}{\partial x_i} - \frac{\partial A_i}{\partial t} + [\mathbf{v} \times (\nabla \times \mathbf{A})]_i \right\} \\ &= e \left\{ -\frac{\partial}{\partial x_i} (\phi - \mathbf{v} \cdot \mathbf{A}) - \frac{d}{dt} \left[\frac{\partial}{\partial v_i} (\mathbf{v} \cdot \mathbf{A}) \right] \right\}. \end{aligned}$$

If we define the velocity-dependent potential by

$$V = e(\phi - \mathbf{v} \cdot \mathbf{A}), \quad (2.5.29)$$

which is of the form (2.5.23), we see that F_i can indeed be derived from (2.5.29). Therefore, the Lagrangian of our system is

$$L = \frac{1}{2} m |\mathbf{v}|^2 - e\phi + e\mathbf{v} \cdot \mathbf{A}. \quad (2.5.30)$$

In this example we started from the equation of motion (2.5.25) and arrived at the Lagrangian (2.5.30), but usually the problem is posed in an inverse way: given the Lagrangian, we are supposed to find the differential equations of motion.

Observations:

- (a) Systems admitting a simple or a generalized potential are called *natural*. In view of the definition (2.5.16), we can write:

$$L = L_0 + L_1 + L_2, \quad (2.5.31)$$

where

$$L_0 = b, \quad L_1 = \sum_{j=1}^n b_j \dot{q}_j, \quad L_2 = \sum_{j=1}^n \sum_{k=1}^n b_{jk} \dot{q}_j \dot{q}_k.$$

The coefficients b, b_j, b_{jk} are functions of q_1, \dots, q_n, t . Taking into account (2.4.24), in the case of a simple potential $V(q, t)$,

$$L_0 = T_0 - V, \quad L_1 = T_1, \quad L_2 = T_2, \quad (2.5.32)$$

while for a generalized potential $V(q, \dot{q}, t)$ (see (2.5.22))

$$L_0 = T_0 - V, \quad L_1 = T_1 - V_1, \quad L_2 = T_2. \quad (2.5.32')$$

- (b) Conservative forces represent a particular case of potential forces, therefore the Lagrange equations are used in the form (2.5.17), observing that now the function V is the *potential energy* of the system.
- (c) If the Lagrangian L does not depend on one of the generalized coordinates q_1, \dots, q_n , say q_k (k fixed), the Lagrange equations (2.5.17) yield:

$$\frac{\partial L}{\partial \dot{q}_k} = \text{const.} \quad (2.5.33)$$

Such a generalized coordinate is called *cyclic* or *ignorable*, and (2.5.33) is a first integral of (2.5.17).

- (d) Let us add to the Lagrangian a term which is the total time derivative of some function $F(q, t)$:

$$L' = L(q, \dot{q}, t) + \frac{d}{dt}F(q, t). \quad (2.5.34)$$

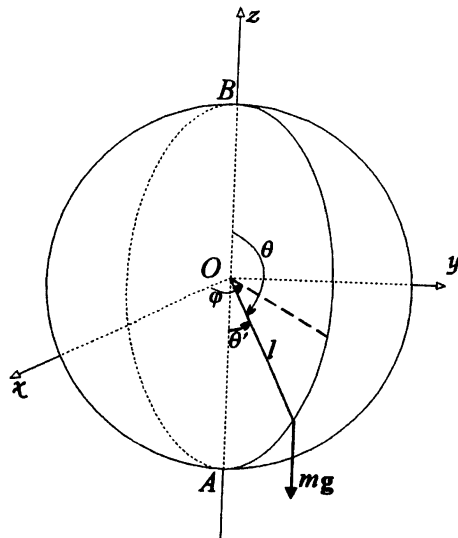
Introducing (2.5.34) into the Lagrange equations (2.5.17), the terms containing F give zero, and we obtain the same system of equations for L' . This simple exercise is left to the reader. In conclusion, the terms having the form of a total time derivative can be omitted from a Lagrangian. In other words, two Lagrangian functions which differ from one another by terms being total time derivatives give the same description of the motion and therefore the two Lagrangians L and L' are *equivalent*.

A Heavy Particle Moving on a Spherical Surface

Let us find the differential equations of motion of a particle of mass m , moving without friction under the influence of gravity on a fixed spherical surface of radius l (a spherical pendulum) (Fig. 2.7).

This system has two degrees of freedom. Using spherical coordinates and choosing $q_1 = \theta$, $q_2 = \varphi$, we have:

Fig. 2.7 A particle moving without friction on a fixed sphere (spherical pendulum).



$$T = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2), \quad V = mgl \cos \theta,$$

hence

$$L = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2) - mgl \cos \theta. \quad (2.5.35)$$

Performing the calculations in (2.5.17), we obtain the equations of motion:

$$\ddot{\theta} - \sin \theta \cos \theta \dot{\varphi}^2 - \frac{g}{l} \sin \theta = 0, \quad (2.5.36)$$

$$\ddot{\varphi} + 2 \cot \theta \dot{\theta} \dot{\varphi} = 0. \quad (2.5.37)$$

These two second-order differential equations are non-linear. If $\varphi = \text{const.}$, i.e. if the motion is performed on a vertical circle of radius l , we are left with a single equation:

$$\ddot{\theta} - \frac{g}{l} \sin \theta = \ddot{\theta}' + \frac{g}{l} \sin \theta' = 0, \quad (2.5.38)$$

which is the differential equation of a *plane* (or *simple*) pendulum. The problem of the mathematical pendulum will be thoroughly discussed in [Chap. 3](#).

2.5.4 Non-potential Forces

Assume that on a system of particles act both potential and non-potential forces. If we denote by \tilde{Q}_j ($j = \overline{1, n}$) the generalized non-potential forces, then the Lagrange equations (2.5.17) take the form

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = \tilde{Q}_j \quad (j = \overline{1, n}), \quad (2.5.39)$$

where, obviously, the Lagrangian $L = T - V$ includes only the potential forces. The infinitesimal virtual work done by the non-potential forces is

$$\delta \tilde{W} = \sum_{i=1}^N \tilde{\mathbf{F}}_i \cdot \delta \mathbf{r}_i = \sum_{j=1}^n \tilde{Q}_j \delta q_j. \quad (2.5.40)$$

Let us define the *power* $\tilde{\mathcal{P}}$ of non-potential forces:

$$\tilde{\mathcal{P}} = \frac{\delta \tilde{W}}{\delta t} = \sum_{i=1}^N \tilde{\mathbf{F}}_i \cdot \mathbf{v}_i = \sum_{j=1}^n \tilde{Q}_j \dot{q}_j, \quad (2.5.41)$$

and consider two remarkable cases:

(1) The non-potential forces of negative power ($\tilde{\mathcal{P}} < 0$) are called *dissipative forces*. Such a force is, for example, the friction force. If this can be written as

$$\mathbf{F}_i^f = -k\mathbf{v}_i \quad (i = \overline{1, N}, k > 0), \quad (2.5.42)$$

then there exists a scalar function \mathcal{T} ,

$$\mathcal{T} = \frac{1}{2}k \sum_{i=1}^N |\mathbf{v}_i|^2, \quad (2.5.43)$$

so that

$$\mathbf{F}_i^f = -\frac{\partial \mathcal{T}}{\partial \mathbf{v}_i} = -\nabla_{\mathbf{v}_i} \mathcal{T} \quad (i = \overline{1, N}), \quad (2.5.44)$$

where $\nabla_{\mathbf{v}_i}$ stands for the partial derivative with respect to \mathbf{v}_i . The function \mathcal{T} is called the *Rayleigh dissipation function*. It is obvious that Rayleigh's function for a scleronomous system is quadratic and homogeneous in the generalized velocities \dot{q}_j :

$$\mathcal{T} = \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n C_{jk} \dot{q}_j \dot{q}_k. \quad (2.5.45)$$

The physical significance of \mathcal{T} is found by writing the power of the friction forces:

$$\tilde{\mathcal{P}} = \sum_{i=1}^N \mathbf{F}_i^f \cdot \mathbf{v}_i = -k \sum_{i=1}^N |\mathbf{v}_i|^2 = -2\mathcal{T}, \quad (2.5.46)$$

i.e. the function \mathcal{T} is equal to half of the power dissipated by friction.

The generalized forces \tilde{Q}_j^f , associated with the friction forces \mathbf{F}_i^f , are:

$$\tilde{Q}_j^f = \sum_{i=1}^N \mathbf{F}_i^f \cdot \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}_j} = -\frac{\partial \mathcal{T}}{\partial \dot{q}_j} \quad (j = \overline{1, n}). \quad (2.5.47)$$

Therefore, in our case, the Lagrange equations (2.5.39) become

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} - \frac{\partial \mathcal{T}}{\partial \dot{q}_j} = 0 \quad (j = \overline{1, n}). \quad (2.5.48)$$

(2) If the power of non-potential forces is zero,

$$\sum_{i=1}^N \tilde{\mathbf{F}}_i \cdot \mathbf{v}_i = \sum_{j=1}^n \tilde{Q}_j \dot{q}_j = 0, \quad (2.5.49)$$

we deal with *gyroscopic forces*. Remark that, in order for (2.5.49) to be valid, $\tilde{\mathbf{F}}_i$ must be written as a cross product of two vectors, where one of them is \mathbf{v}_i or, equivalently, \tilde{Q}_j must have the form

$$\tilde{Q}_j = \sum_{k=1}^n h_{jk} \dot{q}_k \quad (j = \overline{1, n}), \quad (2.5.50)$$

the coefficients h_{jk} being antisymmetric:

$$h_{jk} = -h_{kj}. \quad (2.5.51)$$

As two examples of gyroscopic forces, we give the *Lorentz force* acting on a particle of charge e :

$$\mathbf{F}_L = e(\mathbf{v} \times \mathbf{B}), \quad (2.5.52)$$

and the *Coriolis force* (see (4.3.7)):

$$\mathbf{F}_i = -2m_i(\boldsymbol{\omega} \times \mathbf{v}_i), \quad (2.5.53)$$

where \mathbf{v}_i is the relative velocity of the particle m_i and $\boldsymbol{\omega}$ is the instantaneous vector of rotation.

The definition (2.5.49) shows that the instantaneous rate of the work done by a scleronomous system subject to gyroscopic forces is zero:

$$\frac{d}{dt}(\delta \tilde{W}) = \sum_{j=1}^n \sum_{k=1}^n h_{jk} \dot{q}_j \dot{q}_k = 0, \quad (2.5.54)$$

and therefore there exists the energy first integral.

In the next two chapters we shall give special attention to the application of this formalism on concrete models of dissipative and gyroscopic systems.

Observation: The Lagrange equations for non-holonomic systems are derived and applied in Chap. 4 (see (4.6.52)).

2.6 Elements of Calculus of Variations

Hamilton's principle (see Sect. 2.7), which is one of the most important principles of theoretical physics, belongs to the category of *variational principles*. For a better understanding of the formalism implied by the use of this principle, let us briefly review some elements of variational calculus.

The calculus of variations deals with the study of extremum values of functions depending on a curve, or on another function, rather than a real number. For the beginning, let us consider a function $f(x)$ of class at least C^2 (i.e. continuous,

together with its second partial derivatives), and expand it according to Taylor's formula, about a fixed value x_0 :

$$f(x) = f(x_0) + \frac{x - x_0}{1!} f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \dots \quad (2.6.1)$$

The quantity

$$\delta f = (x - x_0) f'(x_0) = f'(x_0) \delta x$$

is called the *first variation* of f at the point x_0 . The *necessary* and *sufficient* condition that the function f has a *stationary value* at x_0 is that $\delta f = 0$, for any arbitrary variation δx . This yields:

$$f'(x_0) = \left(\frac{\partial f}{\partial x} \right)_{x=x_0} \equiv \left(\frac{\partial f}{\partial x} \right)_0 = 0, \quad (2.6.2)$$

which reminds us of the condition of static equilibrium (2.1.45). Going further, we can define the *second variation* of f as

$$\delta^2 f = \frac{1}{2} (x - x_0)^2 f''(x_0). \quad (2.6.3)$$

If $f''(x_0) \geq 0$, we have a local minimum at x_0 , while if $f''(x_0) \leq 0$, x_0 is a local maximum.

Assume now that f is of the form $f(x_1, \dots, x_n)$. Then its first variation at (x_1^0, \dots, x_n^0) is

$$\delta f = \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} \right)_0 \delta x_j, \quad \delta x_j = x_j - x_j^0, \quad (2.6.4)$$

while the condition that f has a *stationary value* at (x_1^0, \dots, x_n^0) , for independent and arbitrary δx_j , reads:

$$\left(\frac{\partial f}{\partial x_j} \right)_0 = 0 \quad (j = \overline{1, n}). \quad (2.6.5)$$

If the variables x_1, \dots, x_n must obey s independent constraint equations

$$g_k(x_1, \dots, x_n) = 0 \quad (k = \overline{1, s}), \quad (2.6.6)$$

where g_1, \dots, g_s are functions of class C^2 , the variations δx_j are no longer independent, but must satisfy the system of s equations

$$\delta g_k = \sum_{j=1}^n \left(\frac{\partial g_k}{\partial x_j} \right)_0 \delta x_j = 0 \quad (k = \overline{1, s}). \quad (2.6.7)$$

To find the stationary conditions in the presence of constraints, one multiplies (2.6.7) by some arbitrary Lagrange multipliers λ_k ($k = \overline{1, s}$) and add the result to (2.6.4). Thus, we have:

$$\sum_{j=1}^n \left[\left(\frac{\partial f}{\partial x_j} \right)_0 + \sum_{k=1}^s \lambda_k \left(\frac{\partial g_k}{\partial x_j} \right)_0 \right] \delta x_j = 0. \quad (2.6.8)$$

Here the variations δx_j are independent, therefore the stationarity condition reads:

$$\left(\frac{\partial f}{\partial x_j} \right)_0 + \sum_{k=1}^s \lambda_k \left(\frac{\partial g_k}{\partial x_j} \right)_0 = 0 \quad (j = \overline{1, n}). \quad (2.6.9)$$

Let us now consider the definite integral

$$I[y(x)] = \int_{x_1}^{x_2} f(x, y, y') dx, \quad (2.6.10)$$

where $y = y(x)$ is a curve in the xy -plane and $y' = \frac{dy}{dx}$. The function $f(x, y, y')$ is of class C^2 in each of its arguments. The integral (2.6.10) is a *functional* of $y(x)$, giving the correspondence between the function f and the number I , associated to the curve $y = y(x)$.

One of the central problems of the calculus of variations is to find the curve $y = y(x)$ for which the associated integral (2.6.10) is an extremum in the given interval $x_1 \leq x \leq x_2$.

Denote by (C) the path $y = y(x)$ that makes the integral (2.6.10) an extremum and consider a neighbouring curve (C^*) , given by

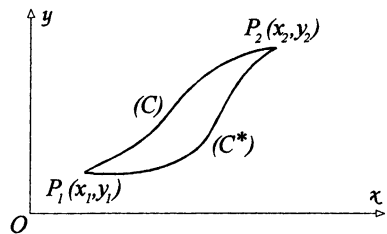
$$y^*(x) = y(x) + \epsilon \eta(x), \quad (2.6.11)$$

where ϵ is a small parameter independent of x , while $\eta(x)$ is a function of class C^1 which satisfies the conditions

$$\eta(x_1) = \eta(x_2) = 0. \quad (2.6.12)$$

Therefore, the two paths (C) and (C^*) have the same initial and final points, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ (Fig. 2.8). Varying the parameter ϵ , we obtain a family of curves C_1^*, C_2^*, \dots , all of them passing through P_1 and P_2 . The functional associated to the curve (C^*) is

Fig. 2.8 The two paths (C) and (C^*) , having the same initial and final points.



$$I[y^*(x)] = \int_{x_1}^{x_2} f(x, y^*, y'^*) dx. \quad (2.6.13)$$

Since

$$f(x, y^*, y'^*) = f(x, y + \epsilon \eta, y' + \epsilon \eta') = f(x, y, y') + \epsilon \eta \frac{\partial f}{\partial y} + \epsilon \eta' \frac{\partial f}{\partial y'} + \dots,$$

the first variation of the integral I is

$$\delta I = \epsilon \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \right) dx, \quad (2.6.14)$$

or, upon integration by parts of the second term and using (2.6.12):

$$\delta I = \epsilon \int_{x_1}^{x_2} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) \right] dx. \quad (2.6.15)$$

Recalling that $\eta(x)$ is arbitrary, except for the condition (2.6.12), the necessary and sufficient condition for a stationary value of I is

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0. \quad (2.6.16)$$

Consequently, among all curves passing through the fixed points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, the curve which makes the integral I stationary satisfies Eq. (2.6.16).

These considerations can be generalized for functionals of the type

$$I[y_1(x), \dots, y_n(x)] = \int_{x_1}^{x_2} f(x, y_1, \dots, y_n, y'_1, \dots, y'_n) dx, \quad (2.6.17)$$

where $y_i = y_i(x)$ ($i = \overline{1, n}$) are n functions of class C^2 and $y'_i = \frac{dy_i}{dx}$ ($i = \overline{1, n}$). Let x, y_1, \dots, y_n be the coordinates of a point in a $(n + 1)$ -dimensional space Q_{n+1} and let $P_1(x^1, y_k^1), P_2(x^2, y_k^2)$ be two fixed points in Q_{n+1} . Then, if

$$y_i = y_i(x) \quad (i = \overline{1, n}) \quad (2.6.18)$$

are the equations of the curve which makes (2.6.17) an extremum, and

$$y_i^* = y_i(x) + \epsilon \eta_i(x) \quad (2.6.19)$$

are the equations of a neighbouring curve passing through the same initial and final points, then, following a similar procedure, we obtain the condition which $y_i(x)$ has to obey so that the integral (2.6.17) be an extremum, in the form

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'_i} \right) = 0 \quad (i = \overline{1, n}). \quad (2.6.20)$$

These equations were first obtained by *Leonhard Euler* in 1744 and later used by *Lagrange* in mechanics. They are usually called the *Euler–Lagrange equations*.

Before going any further, we shall apply this formalism to some classical problems of variational calculus.

2.6.1 Shortest Distance Between Two Points in a Plane

Our aim is to minimize the integral

$$I = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2} dx, \quad (2.6.21)$$

where $s = \sqrt{x^2 + y^2}$ is the arc length in the xy -plane. Comparing (2.6.21) with (2.6.10), we get $f = \sqrt{1 + y'^2}$, and the Euler–Lagrange equation (2.6.16) yields:

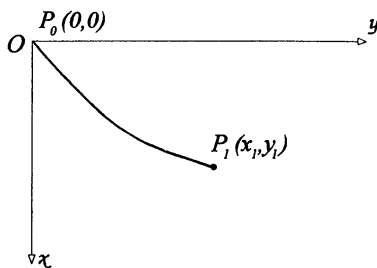
$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}, \quad \frac{\partial f}{\partial y} = 0,$$

hence

$$\frac{y'}{\sqrt{1 + y'^2}} = C, \quad y' = \frac{C}{\sqrt{1 - C^2}} = C_1,$$

leading by integration to the equation of a *straight line*, $y = C_1 x + C_2$. The constants C_1 and C_2 are determined by the condition that the curve must pass through the points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$. Note that our solution produces an *extremum* for (2.6.21) and we cannot know the nature of this extremum at the beginning. But the investigation of the problem, together with our common sense, tells us that the extremum is a *minimum*.

Fig. 2.9 Choice of coordinates for the brachistochrone problem. The points $P_0(0, 0)$ and $P_1(x_1, y_1)$ are fixed.



2.6.2 Brachistochrone Problem

Among all curves lying in a vertical plane and passing through two fixed points, find the one for which a heavy particle would slide down the curve without friction in minimum (extremum) time.

This problem was formulated in 1696 by the Swiss mathematician *Johann Bernoulli*, being the problem which lead to the calculus of variations. The word *brachistochrone* derives from the Greek *brachistos* (shortest) and *chronos* (time).

Choosing the coordinates as in Fig. 2.9, with the fixed points $P_0(0, 0)$ and $P_1(x_1, y_1)$, the time of descent from P_0 to P_1 , on any curve, can be written as

$$t = \int_{P_0}^{P_1} \frac{ds}{v}, \quad (2.6.22)$$

where v is the speed of the particle along the curve. Using the kinetic energy theorem (1.3.21),

$$mgx = \frac{1}{2}mv^2, \quad (2.6.23)$$

and the relation $ds = \sqrt{1 + y'^2}dx$ for the path element, we obtain:

$$t = \frac{1}{\sqrt{2g}} \int_0^{x_1} \sqrt{\frac{1 + y'^2}{x}} dx. \quad (2.6.24)$$

Next, we apply the Euler–Lagrange equation (2.6.16), where $f = \sqrt{\frac{1 + y'^2}{x}}$. Performing simple calculations, we have:

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x(1 + y'^2)}} = \frac{1}{\sqrt{2a}}, \quad (2.6.25)$$

where a is a constant. Separating the variables and integrating, we arrive at

$$y = \int_0^x \sqrt{\frac{x}{2a - x}} dx. \quad (2.6.26)$$

Fig. 2.10 Generation of the cycloid $x = a(1 - \cos \theta)$.

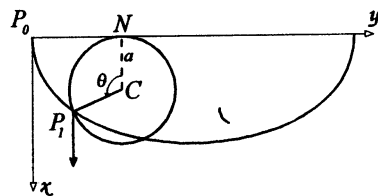
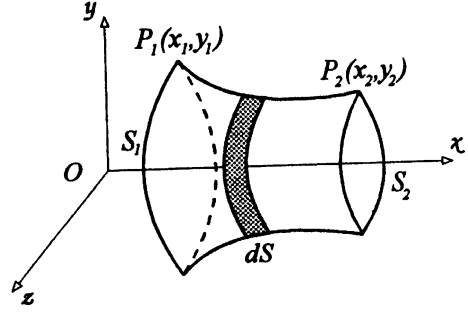


Fig. 2.11 Surface of revolution of minimum area. The points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are fixed.



To perform the integration, one makes the change of variable

$$\frac{x}{2a - x} = u^2, \quad (2.6.27)$$

hence

$$y = -2a \int_0^u u d\left(\frac{1}{1+u^2}\right) = -2a \left(\left[\frac{u}{1+u^2} \right]_0^u - [\arctan u]_0^u \right).$$

A new substitution

$$u = \tan \frac{\theta}{2} \quad (2.6.28)$$

yields finally

$$y = a(\theta - \sin \theta), \quad (2.6.29)$$

while x is found from (2.6.27) and (2.6.28):

$$x = a(1 - \cos \theta). \quad (2.6.30)$$

Equations (2.6.29) and (2.6.30) are the parametric equations of a cycloid, having the y -axis as basis and the concavity upwards. The constant a is the radius of the circle that generates the cycloid (Fig. 2.10). In fact, we have shown that the path of the cycloid insures a stationary value of t , but it is obvious that the extremum must be a minimum.

2.6.3 Surface of Revolution of Minimum Area

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be two fixed points in the xy -plane. Find the curve $y = y(x)$ passing through P_1 and P_2 which would generate by revolution about an axis (say, x) a surface of minimum area.

Examining Fig. 2.11, one observes that upon a revolution about the x -axis, a geometric volume has appeared, with fixed basis areas S_1 and S_2 . With two planes orthogonal to the x -axis, we delimit an elementary cylinder, of lateral area

$$dS = 2\pi y ds = 2\pi y \sqrt{1 + y'^2} dx.$$

The area generated by the curve passing through P_1 and P_2 is

$$S = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + y'^2} dx. \quad (2.6.31)$$

To make (2.6.31) a minimum (a maximum would not make any sense), the integrand $f = y\sqrt{1 + y'^2}$ must satisfy the Euler–Lagrange equation (2.6.16). The easiest way to get the result is to observe that, since f does not explicitly depend on x , Eq. (2.6.16) admits the first integral

$$y' \frac{\partial f}{\partial y'} - f = \text{const.} \quad (2.6.32)$$

Since

$$\frac{\partial f}{\partial y'} = \frac{yy'}{\sqrt{1 + y'^2}},$$

from (2.6.32) we obtain:

$$\frac{yy'^2}{\sqrt{1 + y'^2}} - y\sqrt{1 + y'^2} = C_1,$$

and, separating the variables and integrating,

$$x = C_1 \int \frac{dy}{\sqrt{y^2 - C_1^2}} + C_2 = C_1 \operatorname{arccosh} \frac{y}{C_1} + C_2,$$

yielding finally:

$$y = C_1 \cosh \frac{x - C_2}{C_1}, \quad (2.6.33)$$

which is the equation of a *catenary* (from the Latin *catena*, meaning *chain*). This is the shape, for instance, of a uniform, flexible heavy chain under gravity, when it is held fix at two points. The constants C_1 and C_2 are determined from the boundary conditions.

2.6.4 Geodesics

A *geodesic* is defined as the shortest distance between two points in a given space. We have already encountered this notion earlier in this chapter, but only in two particular cases. We wish now to give a general theory of geodesics, useful not only in classical mechanics, but also in the general theory of relativity.

First, we must define the *metric tensor*. Let E_m be an Euclidean space with the Cartesian coordinates y_1, \dots, y_m , and write the line element in the form

$$ds^2 = \sum_{j=1}^m dy_j dy_j. \quad (2.6.34)$$

Let now \mathcal{R}_n ($n < m$) be a n -dimensional manifold in E_m and let x^1, \dots, x^n be the coordinates of a point in \mathcal{R}_n . Since $y_j = y_j(x^1, \dots, x^n)$, we have:

$$ds^2 = \sum_{j=1}^m \sum_{i,k=1}^n \frac{\partial y_j}{\partial x^i} \frac{\partial y_j}{\partial x^k} dx^i dx^k$$

and, with the notation

$$g_{ik}(x^1, \dots, x^n) = g_{ki} = \sum_{j=1}^m \frac{\partial y_j}{\partial x^i} \frac{\partial y_j}{\partial x^k} \quad (2.6.35)$$

for the *metric tensor*, we arrive at the following form of the *metric* (squared line element):

$$ds^2 = \sum_{i,k=1}^n g_{ik} dx^i dx^k. \quad (2.6.36)$$

If $g_{ik} = \delta_{ik}$, i.e. if the manifold \mathcal{R}_n is Euclidean, we fall back on the metric (2.6.34).

If the metric (2.6.36) is invariant under a general coordinate transformation

$$x'^i = x'^i(x^1, \dots, x^n) \quad (i = \overline{1, n}), \quad (2.6.37)$$

the manifold \mathcal{R}_n is called *Riemannian*. In our case, the metric tensor g_{ik} is associated with transition from Cartesian to general coordinates, but the transformation can be performed between two manifolds of the same dimension. We can also write

$$ds^2 = \sum_{i=1}^n dx_i dx^i, \quad (2.6.38)$$

hence

$$dx_i = \sum_{k=1}^n g_{ik} dx^k \quad (i = \overline{1, n}). \quad (2.6.39)$$

This can be considered as a system of n linear algebraic equations in the unknown quantities dx^1, \dots, dx^n . Solving the system by Cramer's rule, we get:

$$dx^k = \sum_{i=1}^n g^{ki} dx_i \quad (k = \overline{1, n}), \quad (2.6.40)$$

where

$$g^{ki} = g^{ik} = \frac{G_{ki}}{g} \quad (2.6.41)$$

are the contravariant components of the metric tensor, while G_{ki} is the algebraic complement of the element g_{ki} in the determinant

$$g = \det(g_{ki}). \quad (2.6.42)$$

Since

$$dx_i = \sum_{k=1}^n g_{ik} dx^k = \sum_{k,l=1}^n g_{ik} g^{kl} dx_l,$$

we must have

$$\sum_{k=1}^n g_{ik} g^{kl} = g_i^l = \delta_i^l \quad (i, l = \overline{1, n}). \quad (2.6.43)$$

These elements of tensor calculus are useful in the derivation of the differential equation of geodesics.

Let $x^i (i = \overline{1, n})$ be the coordinates of a particle in \mathcal{R}_n and let

$$x^i = x^i(s) \quad (i = \overline{1, n}) \quad (2.6.44)$$

be the parametric equations of a curve passing through the given points 1 and 2. The arc length between the two points is

$$L = \int_1^2 ds = \int_1^2 \sqrt{\sum_{i,k=1}^n g_{ik} \dot{x}^i \dot{x}^k} ds, \quad \dot{x}^i = \frac{dx^i}{ds}. \quad (2.6.45)$$

In order for the curve (2.6.44) to be a geodesic, the function

$$f(x, \dot{x}, s) = \sqrt{\sum_{i,k=1}^n g_{ik} \dot{x}^i \dot{x}^k} \quad (2.6.46)$$

must satisfy the Euler–Lagrange equations (2.6.20):

$$\frac{d}{ds} \left(\frac{\partial f}{\partial \dot{x}^m} \right) - \frac{\partial f}{\partial x^m} = 0 \quad (m = \overline{1, n}). \quad (2.6.47)$$

Evaluating the derivatives, we have successively:

$$\begin{aligned}\frac{\partial f}{\partial \dot{x}^m} &= \sum_{k=1}^n g_{mk} \dot{x}^k, \\ \frac{d}{ds} \left(\frac{\partial f}{\partial \dot{x}^m} \right) &= \sum_{k=1}^n g_{mk} \ddot{x}^k + \sum_{i,k=1}^n \frac{\partial g_{mk}}{\partial x^i} \dot{x}^i \dot{x}^k, \\ \frac{\partial f}{\partial x^m} &= \frac{1}{2} \sum_{i,k=1}^n \frac{\partial g_{ik}}{\partial x^m} \dot{x}^i \dot{x}^k,\end{aligned}$$

and (2.6.47) yields:

$$\sum_{k=1}^n g_{mk} \ddot{x}^k + \frac{1}{2} \sum_{i,k=1}^n \left(\frac{\partial g_{mk}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^m} \right) \dot{x}^i \dot{x}^k = 0.$$

If we denote

$$\Gamma_{ik,m} = \frac{1}{2} \left(\frac{\partial g_{mk}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^m} \right), \quad (2.6.48)$$

the last equation becomes

$$\sum_{k=1}^n g_{mk} \ddot{x}^k + \sum_{i,k=1}^n \Gamma_{ik,m} \dot{x}^i \dot{x}^k = 0. \quad (2.6.49)$$

The quantities (2.6.48) are the *Christoffel symbols of the first kind*. Multiplying (2.6.49) by g^{ml} , then summing over m and using (2.6.43), we finally arrive at the differential equation of *geodesics* in \mathcal{R}_n :

$$\ddot{x}^l + \sum_{i,k=1}^n \Gamma_{ik}^l \dot{x}^i \dot{x}^k = 0 \quad (l = \overline{1, n}), \quad (2.6.50)$$

where

$$\Gamma_{ik}^l = \sum_{m=1}^n g^{ml} \Gamma_{ik,m} \quad (2.6.51)$$

are the *Christoffel symbols of the second kind*. It can be shown that the Christoffel symbols are not tensors (except for linear transformations).

If we take four dimensions, then (2.6.50) are the equations of geodesics in the Riemannian manifold \mathcal{R}_4 , used in the relativistic theory of gravitation. Here, Γ_{ik}^l determine the *intensity* of the field, while the components of the metric tensor g_{ik} play the role of *potentials* of the gravitational field.

Observation: If we denote

$$\phi = \frac{1}{2} \sum_{i,k=1}^n g_{ik} \dot{x}^i \dot{x}^k, \quad (2.6.52)$$

then the equations

$$\frac{d}{ds} \left(\frac{\partial \phi}{\partial \dot{x}^m} \right) - \frac{\partial \phi}{\partial x^m} = 0 \quad (m = \overline{1, n}) \quad (2.6.53)$$

yield the same result (2.6.50). Consequently, the following two variational equations:

$$\delta \int_1^2 \sqrt{\sum_{i,k=1}^n g_{ik} \dot{x}^i \dot{x}^k} ds = 0$$

and

$$\delta \int_1^2 \sum_{i,k=1}^n g_{ik} \dot{x}^i \dot{x}^k ds = 0, \quad (2.6.54)$$

are *equivalent*.

Geodesics of a Sphere

Let us find the geodesics of a sphere of constant radius $R = 1$. The sphere can be imagined as a two-dimensional Riemannian manifold embedded in the three-dimensional Euclidean space E_3 . The arc element on the sphere of unit radius is

$$ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2, \quad (2.6.55)$$

and thus our variational principle can be put in the form

$$\delta \int ds = \delta \int \frac{ds^2}{ds^2} ds = \delta \int (\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2) ds = 0, \quad (2.6.56)$$

where $\dot{\theta} = \frac{d\theta}{ds}$ and $\dot{\varphi} = \frac{d\varphi}{ds}$. Obviously, in our case

$$f = \dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2 = 1. \quad (2.6.57)$$

The geodesic for the variable φ is obtained by using the Euler–Lagrange equations (2.6.20). Performing elementary derivatives, we arrive at:

$$\ddot{\varphi} + 2 \cot \theta \dot{\theta} \dot{\varphi} = 0. \quad (2.6.58)$$

To obtain the *explicit* equation of the geodesic $\varphi = \varphi(\theta)$, we must eliminate the parameter s between the last two equations. First, we observe that (2.6.58) can be written as

$$d\dot{\varphi} + 2\dot{\varphi} \cot \theta d\theta = 0,$$

giving by integration

$$\dot{\varphi} = \frac{C}{\sin^2 \theta}, \quad (2.6.59)$$

where C is a constant. Then, we can write:

$$\dot{\theta} = \frac{d\theta}{ds} = \frac{d\theta}{d\varphi} \dot{\varphi}$$

and, in view of (2.6.57) and (2.6.59),

$$\sqrt{1 - \frac{C^2}{\sin^2 \theta}} = \frac{d\theta}{d\varphi} \frac{C}{\sin^2 \theta}.$$

Separating the variables, we have:

$$d\varphi = \frac{C}{\sin \theta \sqrt{\sin^2 \theta - C^2}} d\theta,$$

therefore

$$\cos(\varphi - \varphi_0) = \frac{C}{\sqrt{1 - C^2}} \cot \theta, \quad (2.6.60)$$

where φ_0 is a constant of integration. This is the equation of a plane through the origin of the coordinate system, which is also the centre of the sphere. Being at the intersection of this plane with the sphere, the geodesics of our problem are *great circles*. To make our result more obvious, let us write (2.6.60) in Cartesian coordinates. Using the well-known formula for $\cos(\varphi - \varphi_0)$ and the relations of transformation

$$x = \sin \theta \cos \varphi, \quad y = \sin \theta \sin \varphi, \quad z = \cos \theta,$$

we find the equation of our plane in the *normal* form:

$$\sqrt{1 - C^2} (x \cos \varphi_0 + y \sin \varphi_0) - Cz = 0. \quad (2.6.61)$$

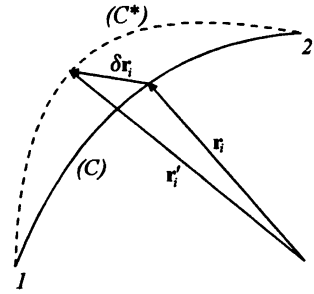
The constants φ_0 and C are determined by the choice of the fixed points.

2.7 Hamilton's Principle

The purpose of the previous section was to prepare the reader with regard to the characteristics of the variational principles of mechanics. For a better understanding of the importance and usefulness of these principles, we shall begin our study in real, physical space.

Let us consider a system of N particles, subject to ideal holonomic constraints of the form (2.1.57), and suppose we know the *real* motion of the particles during the time interval (t_1, t_2) , i.e. we know the functions

Fig. 2.12 Virtual variation of the radius-vector of a particle from one point of the real trajectory (C) to the corresponding point of the varied path (C^*).



$$\mathbf{r}_i = \mathbf{r}_i(t), \quad t_1 \leq t \leq t_2 \quad (i = \overline{1, N}). \quad (2.7.1)$$

Let us also consider another law of motion, given by

$$\mathbf{r}_i^* = \mathbf{r}_i^*(t), \quad (2.7.2)$$

which is consistent with the constraints (2.1.57), but does not obey the equations of motion, expressing, as we already know, a *virtual* motion of the system. We also assume that

$$\mathbf{r}_i(t_\alpha) = \mathbf{r}_i^*(t_\alpha) \quad (\alpha = 1, 2, i = \overline{1, N}), \quad (2.7.3)$$

meaning that both the real and the virtual trajectories pass, at times t_1 and t_2 , through the same fixed points of the real three-dimensional space. Hence, the virtual displacements

$$\delta \mathbf{r}_i = \mathbf{r}_i(t) - \mathbf{r}_i^*(t) \quad (i = \overline{1, N}) \quad (2.7.4)$$

represent the variation of the radius-vector of the particle P_i from one point of the real trajectory (C) to the corresponding point of the varied path (C^*) (Fig. 2.12). It follows that

$$\delta \mathbf{r}_i(t_1) = \delta \mathbf{r}_i(t_2) = 0 \quad (i = \overline{1, N}), \quad (2.7.5)$$

as well as

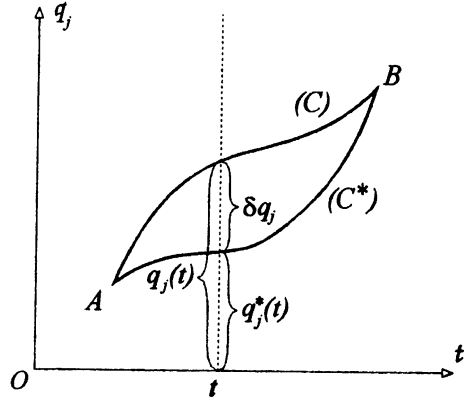
$$\frac{d}{dt}(\delta \mathbf{r}_i) = \frac{d}{dt}(\mathbf{r}_i - \mathbf{r}_i^*) = \mathbf{v}_i - \mathbf{v}_i^* = \delta \mathbf{v}_i \quad (i = \overline{1, N}). \quad (2.7.6)$$

Let us now direct our attention to the kinetic energy T^* , associated with the virtual motion. Supposing the trajectory (C^*) is infinitely close to (C), we may write:

$$T^* \simeq \frac{1}{2} \sum_{i=1}^N m_i (|\dot{\mathbf{r}}_i|^2 - 2\dot{\mathbf{r}}_i \cdot \delta \dot{\mathbf{r}}_i) = T - \sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \delta \dot{\mathbf{r}}_i. \quad (2.7.7)$$

Using this result, we shall make some transformation in D'Alembert's principle (2.5.5), as follows:

Fig. 2.13 Virtual displacement in the configuration space.



$$\sum_{i=1}^N m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \frac{d}{dt} \left(\sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \right) - \sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \delta \dot{\mathbf{r}}_i = \frac{d}{dt} \left(\sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \right) - \delta T, \quad (2.7.8)$$

where $\delta T = T - T^*$ is given by (2.7.7). In view of (2.4.13), we can write:

$$\frac{d}{dt} \left(\sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i \right) = \delta(T + W). \quad (2.7.9)$$

Integrating with respect to time between the fixed limits t_1 and t_2 and using (2.7.5), we obtain:

$$\int \delta(T + W) dt = 0. \quad (2.7.10)$$

In the case of potential forces, the virtual potential V^* can be calculated in terms of the real potential V in a similar way:

$$\begin{aligned} V^* &= V(\mathbf{r}_1^*, \dots, \mathbf{r}_N^*, t) = V(\mathbf{r}_1 - \delta \mathbf{r}_1, \dots, \mathbf{r}_N - \delta \mathbf{r}_N, t) \\ &\simeq V(\mathbf{r}_1, \dots, \mathbf{r}_N, t) - \sum_{i=1}^N (\text{grad}_i V) \cdot \delta \mathbf{r}_i, \end{aligned} \quad (2.7.11)$$

therefore the virtual work done by the system is:

$$\delta W = \sum_{i=1}^N \mathbf{F}_i \cdot \delta \mathbf{r}_i = V^* - V = -\delta V, \quad (2.7.12)$$

and relation (2.7.10), with notation (2.5.16), becomes:

$$\int_{t_1}^{t_2} \delta L dt = \delta \int_{t_1}^{t_2} L dt = 0. \quad (2.7.13)$$

This relation expresses *Hamilton's principle*. Since \mathbf{r}_i and \mathbf{r}_i^* are not independent variables, but must satisfy the same constraint relations, this principle is usually used in the configuration space. Let $q_j = q_j(t)$ and $q_j^* = q_j^*(t)$ ($j = \overline{1, n}$) be the parametric equations of generalized trajectories corresponding to the real and (one of the) virtual motions, respectively, and define the virtual displacements δq_j by

$$\delta q_j(t) = q_j(t) - q_j^*(t) \quad (j = \overline{1, n}). \quad (2.7.14)$$

According to the condition (2.7.5), the generalized trajectories corresponding to the real and virtual displacements pass through the same points in configuration space, i.e.

$$\delta q_j(t_1) = \delta q_j(t_2) = 0 \quad (j = \overline{1, n}). \quad (2.7.15)$$

Except for (2.7.15), the virtual variations δq_j are *independent* and, as pointed out in Fig. 2.13, they are orthogonal to the t -axis. Therefore, we can write (2.7.13) as

$$\delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0. \quad (2.7.16)$$

The integral

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \quad (2.7.17)$$

is called the *action integral*. We are now able to formulate Hamilton's principle: *Out of all the possible generalized paths passing through two fixed points, corresponding to the times t_1 and t_2 , the real motion is performed on that path for which the action is stationary.* Hamilton's principle is also called the *principle of stationary action*. Since, in general, the stationary extremum is a minimum, sometimes it is named the *principle of least action*.

The principle was published in 1834 by *William Rowan Hamilton*. Its discovery played an important role in the development of various aspects of theoretical physics and we shall support this statement by many examples.

2.7.1 Euler–Lagrange Equations for the Action Integral

Let us prove that the Lagrange differential equations of motion, for both potential and non-potential forces, can be derived from Hamilton's principle. Taking the first (virtual) variation of the action, we have:

$$\delta S = \int_{t_1}^{t_2} \sum_{j=1}^n \left(\frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right) dt,$$

or, if we integrate by parts the second term,

$$\delta S = \left[\sum_{j=1}^n \frac{\partial L}{\partial \dot{q}_j} \delta q_j \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_{j=1}^n \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \right] \delta q_j dt.$$

Using (2.7.15) and the arbitrariness of the variations δq_j , we arrive at the Lagrange equations (2.5.17).

In a similar way, starting from (2.7.10), we can obtain the Lagrange equations for non-potential systems (2.5.13). Indeed, we may write

$$\int_{t_1}^{t_2} \sum_{j=1}^n \left(\frac{\partial T}{\partial q_j} \delta q_j + \frac{\partial T}{\partial \dot{q}_j} \delta \dot{q}_j + Q_j \delta q_j \right) dt = 0$$

and, after an integration by parts,

$$\left[\sum_{j=1}^n \frac{\partial T}{\partial \dot{q}_j} \delta q_j \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) + Q_j \right] \delta q_j dt = 0.$$

Since $\delta q_j(t_1) = \delta q_j(t_2) = 0$, for independent and arbitrary δq_j we arrive at (2.5.13), as expected.

The condition (2.7.10), sometimes called *generalized Hamilton's principle*, can also be used to derive the Lagrange equations in case of existence of a velocity-dependent potential $V(q, \dot{q}, t)$. To show this, we must prove that there exists a function $L = T - V(q, \dot{q}, t)$, such that (2.7.10) is equivalent with (2.7.16) if the condition (2.5.23) is satisfied. Indeed, we have:

$$\int_{t_1}^{t_2} (\delta T + \delta W) dt = \int_{t_1}^{t_2} \left\{ \delta T + \sum_{j=1}^n \left[\frac{d}{dt} \left(\frac{\partial V}{\partial \dot{q}_j} \right) - \frac{\partial V}{\partial q_j} \right] \delta q_j \right\} dt = 0$$

and, after integration by parts,

$$\int_{t_1}^{t_2} \left[\delta T - \sum_{j=1}^n \left(\frac{\partial V}{\partial q_j} \delta q_j + \frac{\partial V}{\partial \dot{q}_j} \delta \dot{q}_j \right) \right] dt = \int_{t_1}^{t_2} \delta (T - V) dt = \delta \int_{t_1}^{t_2} L dt = 0,$$

which completes the proof.

Finally, let us show that Hamilton's principle (2.7.16) is a *variational principle*. Comparing (2.6.10) with (2.7.17), it appears obvious that the action S is a *functional* of q_1, \dots, q_n , while the correspondence

$$x \rightarrow t, \quad y_i(x) \rightarrow q_j(t), \quad f(y, y', x) \rightarrow L(q, \dot{q}, t), \\ I(y_1, \dots, y_n) \rightarrow S(q_1, \dots, q_n)$$

shows that the Euler–Lagrange equations (2.6.20) are formally identical with the Lagrange equations for natural systems (2.5.17).

Before closing this section, we wish to emphasize the importance of the variational principles, Hamilton's principle being one of them. (Another variational principle will be discussed in Sect. 2.9; its understanding needs some extra background, which by then will be given.) Compared with several other formulations of mechanics, this principle offers certain advantages. First, since we deal with quantities defined with respect to any frame, the principle does not depend on the choice of coordinates. Second, once a single scalar function, the Lagrangian, is known, one can obtain both the differential equations of motion and the associated laws of conservation, in a direct and simple way. Third, the variational principles can be used for a unitary description of some other systems, like fields. This extension is possible because the Lagrangian has the dimension of energy and this quantity can be defined for any type of motion, while not all interactions can be described by forces. As we shall see later on, the fundamental equations of electrodynamics (Maxwell's equations), of the theory of linear elasticity (Lamé's equations), of quantum mechanics (Schrödinger equation), etc. can be derived from Hamilton's principle.

2.7.2 Criteria for the Construction of Lagrangians

There are several criteria which must be obeyed in constructing the Lagrangian function used in our formalism. They are:

- (a) *Superposition principle*. If the physical system consists of two (or more) interacting particles, the Lagrangian is composed of three groups of terms: (i) The Lagrangians of each particle, when the others are absent; (ii) Terms expressing the interaction between particles; (iii) Terms describing the interaction between the system and the exterior fields (if there are any).
- (b) *Invariance principle*. The action must be invariant with respect to the appropriate group of transformations (e.g. Galilei group in Newtonian mechanics, Lorentz group in relativistic mechanics).
- (c) *Correspondence principle*. The Lagrangian must be constructed in such a way, that all results of Newtonian mechanics be obtained by Hamilton's principle.
- (d) *Principle of physical symmetry*. The choice of the generalized coordinates must provide a Lagrangian function not only simple, but also useful, i.e. suitable to the symmetry properties of the system.

2.8 Symmetry Properties and Conservation Theorems

We already know that the motion of a mechanical system can be determined, in principle, by integrating the Lagrange equations (2.5.17). We say 'in principle', because there are circumstances when this operation is neither useful nor even

possible. Nevertheless, there exist some cases when it is possible to obtain information about our system without a full integration of the equations of motion. This is done by using the *first integrals*.

2.8.1 First Integrals as Constants of Motion

Consider a system of N particles with n degrees of freedom, subject to holonomic constraints, and assume that we found a relation of the type:

$$f(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) = C(\text{const.}), \quad (2.8.1)$$

which is identically satisfied by *any* solution of the Lagrange equations and for *any* initial conditions. Then (2.8.1) is called a *first integral* of (2.5.17) or a *constant of motion* (see Chap. 1, Sect. 1.2)

Suppose we know h distinct first integrals

$$f_s(q, \dot{q}, t) = C_s \quad (s = \overline{1, h}). \quad (2.8.2)$$

Then any function

$$F(f_1, \dots, f_h) = \text{const.} \quad (2.8.3)$$

is also a constant of motion, but not independent of (2.8.2). Since the general integral of the Lagrange equations (2.5.17),

$$q_j = q_j(t, C_1, \dots, C_{2n}) \quad (j = \overline{1, n}), \quad (2.8.4)$$

depends on $2n$ arbitrary independent constants, it follows that the maximum number of *distinct* first integrals is $2n$. The constants C_1, \dots, C_{2n} are determined from the initial conditions:

$$q_j^0 = q_j(t_0, C_1, \dots, C_{2n}), \quad \dot{q}_j^0 = \dot{q}_j(t_0, C_1, \dots, C_{2n}) \quad (j = \overline{1, n}). \quad (2.8.5)$$

The integration of the Lagrange equations is considerably facilitated by the application of first integrals, because:

- (i) Finding a solution of a *first-order* differential equation is an easier task;
- (ii) A first integral offers information on the physical nature of the system, as well as its symmetry properties;
- (iii) In some cases, first integrals express the conservation of fundamental physical quantities, such as linear and angular momenta, energy, etc.

Consequently, finding the first integrals (if there are any) is a necessary step in solving a problem by the Lagrangian technique.

As we have already mentioned in Sect. 2.5, if q_k is a cyclic coordinate, then the quantity $\frac{\partial L}{\partial q_k}$ is a constant of motion. Let us define the quantities

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad (j = \overline{1, n}) \quad (2.8.6)$$

and call them the *generalized momenta* associated (or conjugated) to the generalized coordinates q_j . The dimensions of p_j are given by those of q_j : if q is a distance, then p is a linear momentum; if q is an angle, then p is an angular momentum, etc. But in any case, we must have:

$$[p_j \dot{q}_j] = [\text{ENERGY}] = ML^2T^{-2}. \quad (2.8.7)$$

Introducing (2.8.6) into the Lagrange equations (2.5.17), we obtain:

$$\dot{p}_j = \frac{\partial L}{\partial q_j} \quad (j = \overline{1, n}). \quad (2.8.8)$$

If the coordinate q_k is cyclic, then (2.8.8) yields

$$p_k = \text{const.}, \quad (2.8.9)$$

expressing the *conservation of the generalized momentum associated with a cyclic coordinate*.

Equation (2.8.9) gives either the conservation of linear momentum, or that of angular momentum. It is valid not only in mechanics, but also for other physical systems. For example, since the coordinate x_k does not appear in the Lagrangian (2.5.30) describing the behaviour of a charged particle in an external electromagnetic field, the conjugated momentum is conserved:

$$p_k = mv_k + eA_k = \text{const.} \quad (2.8.10)$$

Therefore, in solving a concrete problem we should follow the rule: look for cyclic coordinates, each of them being associated with a first integral. Then, if there are not any, search for another set of generalized coordinates q'_j ($j = \overline{1, n}$), of which at least one being cyclic.

A useful example is offered by a particle moving in a central field. When expressed in Cartesian coordinates, the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x, y)$$

does not display any cyclic coordinate, but if it is written in terms of polar coordinates,

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r),$$

it shows the ignorable coordinate φ , leading to the first integral

$$p_\varphi = mr^2\dot{\varphi} = \text{const.} \quad (2.8.11)$$

2.8.2 Symmetry Transformations

As we have mentioned, the first integrals are related with the conservation of the fundamental physical quantities. This fact emerges from the intrinsic properties of the space–time continuum and expresses the connection between different types of motion and the conservation of associated quantities.

The study of a large amount of experimental data has led to the conclusion that the real, physical space is *homogeneous* (there are no privileged reference frames) and *isotropic* (there are no privileged directions), while time passes *uniformly* (there are no privileged moments of time). The homogeneity of space results in the fact that the properties of an isolated mechanical system do not change if all particles of the system perform infinitesimal translations with the same velocity \mathbf{v} , while the isotropy yields the conservation of the properties of the system if all particles execute infinitesimal rotations of the same angle, about the same direction. Finally, the uniformity of time shows that the origin of the time interval can be arbitrarily chosen, meaning that the properties of the system remain unchanged at an infinitesimal displacement of the time origin.

An important role in the study of physical systems is played by those transformations which leave the form of the differential equations of motion unchanged. These are called *symmetry transformations*. For example, we can cite space–time transformations, gauge transformations, etc. In the first category are included the *space displacements* (translations, rotations) and *time transformations*. The *gauge transformations* appear when one (or more) physical quantity is not completely determined by its equation of definition, and we shall familiarize the readers with them later on in this book.

2.8.3 Noether's Theorem

As we mentioned earlier in this section, the cyclic coordinates lead to constants of motion, expressing the symmetry of the Lagrangian with respect to certain space–time transformations. But not all constants of motion come from evident symmetry properties of the system, or have a simple form. That is why there appears the necessity of giving a general method to obtain the first integrals. Such formalism was provided in 1918 by the German Jewish mathematician *Emmy Noether*.¹

Let us consider a physical system, described by the generalized coordinates q_j and velocities \dot{q}_j , and assume that the Lagrangian $L(q, \dot{q}, t)$ of the system is known. A transformation of coordinates and time

$$q'_j = q'_j(q_1, \dots, q_n, t), \quad t' = t'(q_1, \dots, q_n, t) \quad (2.8.12)$$

¹ Noether, E.: Invariante Variationsprobleme. Nachr. Kgl. Ges. Wiss. Göttingen **2**, 235 (1918).

is a *symmetry transformation* if it leaves Hamilton's principle invariant or, in view of (2.5.34), if

$$L\left(q', \frac{dq'}{dt'}, t'\right) dt' = \left[L(q, \dot{q}, t) + \frac{dF(q, t)}{dt} \right] dt. \quad (2.8.13)$$

Let us now specify the transformation (2.8.12), namely that it is an *infinitesimal* transformation of the form:

$$q'_j = q_j + \epsilon \eta_j(q, t) \quad (j = \overline{1, n}), \quad t' = t + \epsilon \tau(q, t), \quad (2.8.14)$$

where η_j and τ are arbitrary functions, while the parameter ϵ is small enough as to keep only terms linear in it. Since

$$\frac{dt'}{dt} = 1 + \epsilon \frac{d\tau}{dt}, \quad \frac{dt}{dt'} \simeq 1 - \epsilon \frac{d\tau}{dt}, \quad (2.8.15)$$

we have:

$$\frac{dq'}{dt'} = \frac{dq'}{dt} \frac{dt}{dt'} \simeq \left(\dot{q}_j + \epsilon \frac{d\eta_j}{dt} \right) \left(1 - \epsilon \frac{d\tau}{dt} \right) \simeq \dot{q}_j + \epsilon \left(\frac{d\eta_j}{dt} - \dot{q}_j \frac{d\tau}{dt} \right). \quad (2.8.16)$$

Introducing (2.8.15) and (2.8.16) into (2.8.13), we arrive at:

$$L\left[q + \epsilon \eta, \dot{q} + \epsilon \left(\frac{d\eta}{dt} - \dot{q} \frac{d\tau}{dt} \right), t + \epsilon \tau \right] \left(1 + \epsilon \frac{d\tau}{dt} \right) = L(q, \dot{q}, t) + \epsilon \frac{d\phi}{dt},$$

where we took $F(q, t) = \epsilon \phi(q, t)$ because, obviously, F must be infinitesimal and linear in ϵ . Using Taylor's formula for series expansion in the l.h.s. and keeping only terms linear in ϵ , after some reduction and rearranging of terms we are left with

$$\epsilon \left[L \frac{d\tau}{dt} + \sum_{j=1}^n \eta_j \frac{\partial L}{\partial q_j} + \sum_{j=1}^n \left(\frac{d\eta_j}{dt} - \dot{q}_j \frac{d\tau}{dt} \right) \frac{\partial L}{\partial \dot{q}_j} + \tau \frac{\partial L}{\partial t} - \frac{d\phi}{dt} \right] = 0. \quad (2.8.17)$$

The transformation (2.8.14) is a symmetry transformation if, for a given L , there exist some functions η_j ($j = \overline{1, n}$) and τ , so that the l.h.s. of (2.8.17) is a total time derivative of a function of q_j, t . The first integrals of motion are obtained from the condition of stationary action S , on any path where the equations of motion (2.5.17) are satisfied. Before applying the action principle, we should mention that the variations $\delta q_j = q'_j - q_j$ ($j = \overline{1, n}$) and $\delta t = t' - t$ differ from those previously used by the fact that they perform a transition between two possible trajectories. Therefore, the first variation of the action S reads:

$$\delta S = \epsilon \int_{t_1}^{t_2} \left[\sum_{j=1}^n \eta_j \frac{\partial L}{\partial q_j} + \sum_{j=1}^n \left(\frac{d\eta_j}{dt} - \dot{q}_j \frac{d\tau}{dt} \right) \frac{\partial L}{\partial \dot{q}_j} + L \frac{d\tau}{dt} + \tau \frac{\partial L}{\partial t} - \frac{d\phi}{dt} \right] dt,$$

or, recalling that L is a function of q_j, \dot{q}_j, t :

$$\begin{aligned} \delta S = \epsilon \int_{t_1}^{t_2} & \left\{ \frac{d}{dt} \left[\sum_{j=1}^n \eta_j \frac{\partial L}{\partial \dot{q}_j} - \tau \left(\sum_{j=1}^n \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \right) - \phi \right] \right. \\ & \left. + \sum_{j=1}^n (\tau \dot{q}_j - \eta_j) \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \right] \right\} dt. \end{aligned} \quad (2.8.18)$$

The invariance of Hamilton's principle under the symmetry transformation (2.8.14) means $\delta S = 0$ for any time interval within which the Lagrange equations are valid. This implies

$$\sum_{j=1}^n \eta_j \frac{\partial L}{\partial \dot{q}_j} - \tau \left(\sum_{j=1}^n \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \right) - \phi = C, \quad (2.8.19)$$

where C is a constant. This equation expresses *Noether's theorem* for discrete systems of particles: *To any continuous symmetry transformation (2.8.14), one can associate a first integral (2.8.19).* Noether's theorem can also be written for infinitesimal quantities $\epsilon \eta_j, \epsilon \tau, \epsilon \phi, \epsilon C$, which is useful in some applications.

We shall now consider some particular cases, which will show the connection between Noether's theorem and the general theorems of mechanics discussed in Chap. 1.

1. Let us consider an isolated system of N particles ($V^{(e)} = 0$) with n degrees of freedom and assume that the particle P_i performs an infinitesimal space displacement of the form

$$\mathbf{r}'_i = \mathbf{r}_i + \delta \mathbf{r}_i \quad (i = \overline{1, N}), \quad t' = t. \quad (2.8.20)$$

Since

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_j} = \sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j},$$

we obtain:

$$\epsilon \sum_{j=1}^n \eta_j \frac{\partial L}{\partial \dot{q}_j} = \epsilon \sum_{i=1}^N \left(\sum_{j=1}^n m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \eta_j \right) = \sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i. \quad (2.8.21)$$

Suppose now that our space displacement is a *translation*, i.e. all particles of the system perform a straight motion in the same direction with the same velocity. Then we have

$$\delta \mathbf{r}_i \equiv \delta \mathbf{r} = \mathbf{n} \delta r, \quad (2.8.22)$$

\mathbf{n} being the unit vector along the direction of translation. Taking $\phi = 0$ in (2.8.19), we get:

$$\delta \mathbf{r} \cdot \mathbf{n} \cdot \sum_{i=1}^N m_i \dot{\mathbf{r}}_i = \text{const.}, \quad (2.8.23)$$

showing the conservation of total linear momentum in the direction of translation (Chap. 1, Sect. 1.3)

2. If the infinitesimal space transformation (2.8.20) is a *rotation* of all the particles, about the same axis and in the same direction:

$$\delta \mathbf{r}_i = \delta \boldsymbol{\theta} \times \mathbf{r}_i = \delta \theta \mathbf{s} \times \mathbf{r}_i, \quad (2.8.24)$$

where \mathbf{s} is the unit vector along the axis of rotation and $\delta \theta$ is the constant angle of rotation, then:

$$\epsilon \sum_{j=1}^n \eta_j \frac{\partial L}{\partial \dot{q}_j} = \sum_{i=1}^N m_i \dot{\mathbf{r}}_i \cdot (\delta \boldsymbol{\theta} \times \mathbf{r}_i) = \delta \theta \mathbf{s} \cdot \sum_{i=1}^N m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i$$

and, according to Noether's theorem (2.8.19),

$$\delta \theta \mathbf{s} \cdot \sum_{i=1}^N m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = \text{const.}, \quad (2.8.25)$$

which is nothing else but the conservation of the total angular momentum (Chap. 1, Sect. 1.3)

3. A special type of space transformation is that associated with the Newtonian mechanics principle which states that two inertial frames are equivalent in describing the motion of a mechanical system. The transition from one frame to another is given by an *infinitesimal Galilean transformation*:

$$\mathbf{r}'_i = \mathbf{r}_i + (\delta \mathbf{v}_0) t, \quad (2.8.26)$$

where the infinitesimal constant vector $\delta \mathbf{v}_0$ is the relative velocity of the frames. Indeed, taking the time derivative of (2.8.26), we have:

$$\mathbf{v}'_i = \mathbf{v}_i + \delta \mathbf{v}_0. \quad (2.8.27)$$

Choosing

$$\epsilon \phi = -\delta \mathbf{v}_0 \cdot \sum_{i=1}^N m_i \mathbf{r}_i, \quad \epsilon C = -\left(\sum_{i=1}^N m_i \right) \mathbf{r}_G^0 \cdot \delta \mathbf{v}_0 \quad (2.8.28)$$

in (2.8.19) and using (2.8.26), we obtain:

$$\delta \mathbf{v}_0 \cdot \left(\frac{\sum_{i=1}^N m_i \mathbf{r}_i}{\sum_{i=1}^N m_i} - \mathbf{r}_G^0 - \frac{\sum_{i=1}^N m_i \mathbf{v}_i}{\sum_{i=1}^N m_i} t \right) = 0,$$

or

$$\mathbf{n} \cdot (\mathbf{r}_G - \mathbf{r}_G^0 - \mathbf{v}_G t) = 0, \quad (2.8.29)$$

which is the centre of mass theorem for isolated systems (Chap. 1, Sect. 1.3).

Here, \mathbf{n} is the unit vector of $\delta\mathbf{v}_0$, while the meaning of \mathbf{r}_G and \mathbf{v}_G is obvious.

4. Let us now consider a pure time transformation and take

$$\tau = 1, \quad \delta q_j = 0, \quad \phi = 0$$

in (2.8.19). Hence:

$$\sum_{j=1}^n \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L = C. \quad (2.8.30)$$

To understand the physical significance of this equation of conservation, we shall first make some comments on the function

$$H = \sum_{j=1}^n p_j \dot{q}_j - L, \quad (2.8.31)$$

where p_j is given by (2.8.6). This function is called the *Hamiltonian* of the system. Recalling Euler's theorem for homogeneous functions:

$$\sum_{i=1}^n \frac{\partial f}{\partial x_i} x_i = m f, \quad (2.8.32)$$

where $f(x_1, \dots, x_n)$ is a homogeneous function of grade m , and using (2.5.31), we may write:

$$\begin{aligned} H &= \sum_{j=1}^n \frac{\partial L_1}{\partial \dot{q}_j} \dot{q}_j + \sum_{j=1}^n \frac{\partial L_2}{\partial \dot{q}_j} \dot{q}_j - L = L_1 + 2L_2 - (L_0 + L_1 + L_2) = L_2 - L_0 \\ &= T_2 - T_0 + V. \end{aligned} \quad (2.8.33)$$

If the constraints are scleronomous, then $T_0 = 0$, $T = T_2$, and we arrive at

$$H = T + V = \text{const.}, \quad (2.8.34)$$

which shows that *the Hamiltonian of a conservative system represents the total energy*, being a constant of motion.

The function H is of great importance in analytical mechanics. We shall encounter it again in Chap. 5 and discuss there its properties more thoroughly.

2.9 Principle of Least Action

This principle was discovered in 1745 by *Pierre-Louis Moreau de Maupertuis* and it is the first integral principle of mechanics. Its initial formulation was nebulous and it was the merit of Euler, Lagrange and Jacobi that the principle acquired its current form.

In the discussion of Hamilton's principle, we used the notion of virtual displacements consistent with the constraints δq_j , being performed by the representative point in the configuration space, when passing from a point P of the real generalized trajectory (C), to some point P^* of an infinitely close trajectory (C^*) at the same time t (synchronic variations):

$$P(q, t) \rightarrow P^*(q + \delta q, t). \quad (2.9.1)$$

Since the boundary points were supposed to be fixed, we also had:

$$\delta q_j(t_1) = \delta q_j(t_2) = 0. \quad (2.9.2)$$

Summarizing, we can state that δ is a linear operator that satisfies the following conditions:

- (i) δq_j are arbitrary except for the end points, where $\delta q_j = 0$;
- (ii) $\delta t = 0$.

Let us now introduce a new operator Δ , including the variation of both space and time variables, defined by:

$$\Delta = \delta + \Delta t \frac{d}{dt}, \quad (2.9.3)$$

with the properties:

- (i) Δq_j are arbitrary, except for end points, where $\Delta q_j = 0$;
- (ii) Δt is arbitrary.

As we can see, the *asynchronous* variations given by Δ are less restrictive than those produced by δ . Applying Δ to q_j , we have:

$$\Delta q_j = \delta q_j + \dot{q}_j \Delta t, \quad (2.9.4)$$

expressing the correspondence between two points, one on the real and the other on the neighbouring path (Fig. 2.14). In the end points $\delta q_j(t_1) \neq 0$, $\delta q_j(t_2) \neq 0$, but

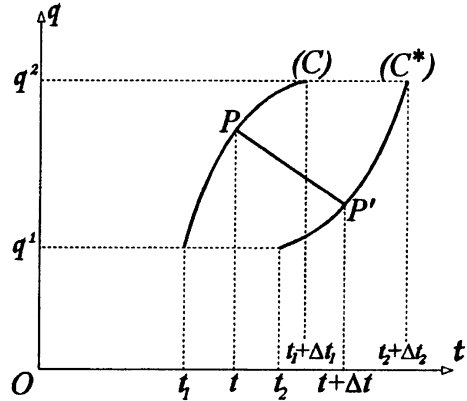
$$\Delta q_j(t_1) = \Delta q_j(t_2) = 0. \quad (2.9.5)$$

Let us now apply the operator Δ to some function $f(q, \dot{q}, t)$:

$$\begin{aligned} \Delta f &= \delta f + \frac{df}{dt} \Delta t = \sum_{j=1}^n \left(\frac{\partial f}{\partial q_j} \delta q_j + \frac{\partial f}{\partial \dot{q}_j} \delta \dot{q}_j \right) + \sum_{j=1}^n \left(\frac{\partial f}{\partial q_j} \dot{q}_j + \frac{\partial f}{\partial \dot{q}_j} \ddot{q}_j \right) \Delta t + \frac{\partial f}{\partial t} \Delta t \\ &= \sum_{j=1}^n \left(\frac{\partial f}{\partial q_j} \Delta q_j + \frac{\partial f}{\partial \dot{q}_j} \Delta \dot{q}_j \right) + \frac{\partial f}{\partial t} \Delta t, \end{aligned} \quad (2.9.6)$$

which is the usual differential of $f(q, \dot{q}, t)$. Next, we apply Δ to the action integral

Fig. 2.14 Correspondence between two points, one on the real and the other on a neighbouring path, when both space and time variations are considered.



$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt, \quad (2.9.7)$$

where the limits t_1 and t_2 are now variable. Let $A(t)$ be the primitive function of the Lagrangian $L[q(t), \dot{q}(t), t]$. It then follows that

$$\Delta S = \Delta \int_{t_1}^{t_2} L dt = \Delta A(t_2) - \Delta A(t_1), \quad (2.9.8)$$

or, in view of (2.9.3),

$$\Delta S = \delta A(t_2) - \delta A(t_1) + \dot{A}(t_2)\Delta t_2 - \dot{A}(t_1)\Delta t_1 = \delta \int_{t_1}^{t_2} L dt + [L\Delta t]_{t_1}^{t_2}. \quad (2.9.9)$$

On the other hand, the Lagrange equations (2.5.17) allow us to write

$$\begin{aligned} \delta L &= \sum_{j=1}^n \left(\frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right) = \sum_{j=1}^n (\dot{p}_j \delta q_j + p_j \delta \dot{q}_j) \\ &= \frac{d}{dt} \left(\sum_{j=1}^n p_j \delta q_j \right) = \frac{d}{dt} \sum_{j=1}^n (p_j \Delta q_j - p_j \dot{q}_j \Delta t). \end{aligned} \quad (2.9.10)$$

Therefore, using (2.9.5) and (2.8.31), we obtain:

$$\Delta S = \Delta \int_{t_1}^{t_2} L dt = \left[\left(\sum_{j=1}^n -p_j \dot{q}_j + L \right) \Delta t \right]_{t_1}^{t_2} = -[H\Delta t]_{t_1}^{t_2}. \quad (2.9.11)$$

If the system is conservative, then

$$H = E = T + V = \text{const.}, \quad (2.9.12)$$

meaning that on any varied path (C^*), the energy has the same value as on the real path (C). Therefore,

$$[H\Delta t]_{t_1}^{t_2} = H(\Delta t_2 - \Delta t_1) = H\Delta \int_{t_1}^{t_2} dt = \Delta \int_{t_1}^{t_2} H dt. \quad (2.9.13)$$

Introducing this result into (2.9.11), we arrive at

$$\Delta \int_{t_1}^{t_2} \sum_{j=1}^n p_j \dot{q}_j dt = 0, \quad (2.9.14)$$

which is one of the forms of the *principle of least action*: *The action taken for a real generalized trajectory is stationary with respect to any neighbouring isoenergetic path.* The quantity

$$W = \int_{t_1}^{t_2} \sum_{j=1}^n p_j \dot{q}_j dt \quad (2.9.15)$$

is called *Maupertuisian action*.

The principle of least action can be written in different equivalent forms. For example, recalling that the system is conservative, the kinetic energy T is a quadratic homogeneous form of generalized velocities:

$$\sum_{j=1}^n p_j \dot{q}_j = \sum_{j=1}^n \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j = 2T$$

and (2.9.14) yields:

$$\Delta \int_{t_1}^{t_2} 2T dt = 0. \quad (2.9.16)$$

Another form of this principle was given by *Carl Jacobi*. To write it, let us extract dt from the kinetic energy formula,

$$T = \frac{1}{2} \sum_{i=1}^N m_i \left| \frac{d\mathbf{r}_i}{dt} \right|^2 = \frac{1}{2} \sum_{i=1}^N m_i \left(\frac{ds_i}{dt} \right)^2,$$

and then introduce it into (2.9.16):

$$\Delta \int_{P_1}^{P_2} \sqrt{2(E - V) \sum_{i=1}^N m_i ds_i^2} = 0, \quad (2.9.17)$$

where P_1 and P_2 are the positions of the system at the times t_1 and t_2 , in the real space. If ds_i^2 is expressed in terms of q_j , then P_1 and P_2 are end points in configuration space. For a single particle, (2.9.17) reduces to

$$\Delta \int_{P_1}^{P_2} \sqrt{2m(E - V)} ds = 0. \quad (2.9.18)$$

Let us define the n -dimensional manifold \mathcal{R}_n by the metric

$$d\sigma^2 = 2(E - V) \sum_{i=1}^N m_i ds_i^2. \quad (2.9.19)$$

Then, as we know,

$$\Delta \int_{P_1}^{P_2} d\sigma = 0$$

defines the *geodesic line* in \mathcal{R}_n , between P_1 and P_2 . If, in particular, \mathcal{R}_n is the configuration space R_n , then the metric is (see (2.4.24)):

$$d\sigma^2 = 2(E - V) \sum_{j=1}^n \sum_{k=1}^n a_{jk} dq_j dq_k \quad (2.9.20)$$

and the principle of least action finally acquires the form:

$$\Delta \int_{P_1}^{P_2} \sqrt{2(E - V) \sum_{j=1}^n \sum_{k=1}^n a_{jk} dq_j dq_k} = 0. \quad (2.9.21)$$

Equal-Action Wave Front

Let us first determine the connection between the Hamiltonian (S) and Maupertuisian (W) actions. Assuming again that the system is conservative, we have:

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} (T - V) dt = \int_{t_1}^{t_2} (2T - E) dt = W - E(t_2 - t_1).$$

Taking $t_2 = t$, $t_1 = 0$, we arrive at:

$$S(q, t) = -Et + W(q). \quad (2.9.22)$$

On the other hand, Hamilton's principle (2.7.16) and the definition of the Hamiltonian (2.8.31) yield:

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_{j=1}^n \frac{\partial S}{\partial q_j} \dot{q}_j = \sum_{j=1}^n p_j \dot{q}_j - H, \quad (2.9.23)$$

therefore, in view of (2.9.22),

$$p_j = \frac{\partial S}{\partial q_j} \quad (j = \overline{1, n}). \quad (2.9.24)$$

(This is only a rough deduction of generalized momenta in terms of S , needed in this application; for more details, see Chap. 5).

Let our conservative system be a single particle and choose $q_j = x_j$ ($j = 1, 2, 3$). Then

$$p_j = \frac{\partial S}{\partial q_j} = \frac{\partial W}{\partial q_j} = (\text{grad } W)_j \quad (j = 1, 2, 3) \quad (2.9.25)$$

which, together with the formula of the Hamiltonian,

$$\frac{1}{2m} |\text{grad } W|^2 + V = E,$$

yields

$$|\text{grad } W| = \sqrt{2m(E - V)}. \quad (2.9.26)$$

With our choice of coordinates, the configuration space coincides with the real, three-dimensional space, while the generalized trajectory is just the real path of the particle. Then the equation

$$W(x, y, z) = \text{const.} \quad (2.9.27)$$

represents a family of fixed surfaces, while

$$S(x, y, z, t) = \text{const.} \quad (2.9.28)$$

stands for a family of moving surfaces. For example, if at $t = 0$ the surfaces S_1 and W_1 coincide, after a time interval dt , the surface S_1 has passed from W_1 to $W_1 + E dt$ (Fig. 2.15) and so on, similarly to the propagation of a wave front. If ds is the elementary displacement of the wave front in normal direction, we can write

$$dW = |\text{grad } W| ds = E dt,$$

which helps to write the phase velocity of the wave front:

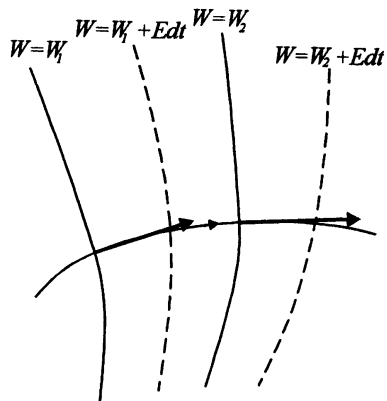
$$u = \frac{E}{|\text{grad } W|} = \frac{E}{\sqrt{2m(E - V)}} = \frac{E}{\sqrt{2mT}} = \frac{E}{p}, \quad (2.9.29)$$

where p is the momentum of the particle. In other words, (2.9.29) gives the phase velocity of propagation of the *equal-action wave front*.

To study the nature of these waves, one must find certain characteristic quantities, such as frequency and wavelength. These can be obtained by making an analogy with the propagation of light waves, whose equation is:

$$\Delta\psi - \frac{n^2}{c^2} \frac{\partial^2 \psi}{\partial t^2} = 0. \quad (2.9.30)$$

Fig. 2.15 Geometry of fixed ($W = \text{const.}$) and moving ($W + E dt = \text{const.}$) surfaces, propagating as a wave front.



Here, n is the index of refraction of the medium. If $n = \text{const.}$, the solution of (2.9.30) is

$$\psi = \psi_0(x, y, z) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}, \quad (2.9.31)$$

where $k = \frac{n\omega}{c} = nk_0$ is the wave number. Taking \mathbf{k} in the positive direction of the x -axis, we have:

$$\psi = \psi_0(x, y, z) e^{i(k_0 n x - \omega t)}. \quad (2.9.32)$$

If n is no longer a constant, but its variation is smooth, the solution of (2.9.30) is close to the form

$$\psi = \psi_0(x, y, z) e^{i[k_0 L(x, y, z) - \omega t]}. \quad (2.9.33)$$

The quantity L is called *eikonal*. If $n = \text{const.}$, then $L = nx$ (optical path length). Introducing (2.9.33) into (2.9.30), evaluating the derivatives and then separating the real and imaginary parts, we obtain:

$$\psi_0 k_0^2 [n^2 - |\text{grad } L|^2] + \Delta \psi_0 = 0, \quad (2.9.34)$$

$$\psi_0 \Delta L + 2(\text{grad } \psi_0) \cdot (\text{grad } L) = 0. \quad (2.9.35)$$

Suppose that the wavelength is small compared to the distance on which the medium displays its non-homogeneity. Then the presence of k_0^2 makes the first term in (2.9.34) much greater than the second, which results in

$$|\text{grad } L| = n. \quad (2.9.36)$$

This is the *eikonal* equation, fundamental in geometrical optics. The equal-phase surfaces are given by

$$f(x, y, z, t) = k_0 L(x, y, z) - \omega t = \text{const.} \quad (2.9.37)$$

Comparing (2.9.26) with the eikonal equation (2.9.36), we realize that they are similar: the quantity $\sqrt{2m(E - V)}$ plays the role of the refraction index and the

function W that of the eikonal, while the surfaces $S = \text{const.}$ are analogous to the surfaces $f = \text{const.}$ Therefore, we can set:

$$S = \alpha f, \quad W = \alpha k_0 L(x, y, z), \quad E = \alpha \omega, \quad (2.9.38)$$

where α is an arbitrary constant. Hence:

$$n = |\text{grad } L| = \frac{1}{\alpha k_0} |\text{grad } W| = \frac{c}{\alpha \omega} \sqrt{2m(E - V)}. \quad (2.9.39)$$

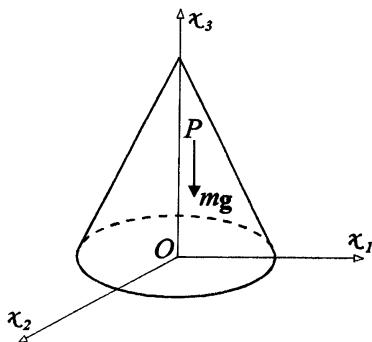
This analogy shows that the propagation of equal-action waves and light waves are similar phenomena. In their remarkable papers, *Erwin Schrödinger* and *Louis de Broglie* showed that the relation between wave and geometrical optics is similar to that between quantum and classical mechanics. If we apply the principle of least action in the form (2.9.18) and observe that according to (2.9.39) the integrand is proportional to n , we have:

$$\Delta \int n \, ds = 0, \quad (2.9.40)$$

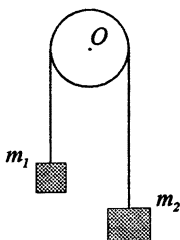
which is nothing else but the well-known *Fermat principle* of geometrical optics. Concluding our discussion, we can state that: *If applied forces are absent, the trajectory described by a particle of light (photon) is a geodesic (minimal optical path).*

2.10 Problems

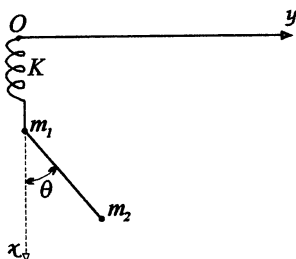
1. Determine the covariant and contravariant components of the metric tensor g_{ik} in spherical coordinates r, θ, φ .
2. Study the tensor properties of the Christoffel symbols $\Gamma_{ik,l}$ and Γ_{ik}^l .
3. Determine the shape of the curve traced by a catenary of mass m and length l , whose fixed ends are at the same height. The distance between ends is a .
4. Find the plane closed curve of given perimeter, which encloses the maximum area (isoperimetric problem).
5. Study the motion of a heavy particle, constrained to move without friction on the surface of a cone.



6. A particle of mass m and velocity v_1 passes from a semi-space in which its potential energy U_1 is constant, to a semi-space in which its potential energy U_2 is also constant. Determine the change in the direction of the particle.
7. A particle P of mass m moves without friction on the curve $y = f(x)$ passing through the origin. Assuming that the curve rotates about the vertical axis Oy with constant angular velocity ω_0 , find the shape of the curve so that the particle remains at rest with respect to the curve.
8. Construct the Lagrangian for a system of N charged particles interacting via Coulomb law, and placed in an external variable electromagnetic field \mathbf{E} , \mathbf{B} .
9. Find the Lagrangian of a double coplanar pendulum and write the differential equations of motion. Linearize these equations for small motions.
10. Two masses m_1 and m_2 are fastened at the ends of an inextensible, flexible rope, running over a massless pulley (Atwood machine). Determine the law of motion and the force of constraint.



11. Investigate the motion of a plane pendulum of mass m_1 whose point of support of mass m_2 is able to perform one of the following motions:
 - (a) A displacement on a horizontal straight line;
 - (b) A displacement on a vertical circle with constant angular velocity ω ;
 - (c) Oscillations along a horizontal line according to the law $a \cos \omega t$;
 - (d) Oscillations along a vertical line according to the law $a \sin \omega t$.
12. Determine the equations of motion and the period of small oscillations of the system shown in the figure.



13. The point of support of a simple pendulum of mass m moves uniformly on a vertical circle of radius R , with the constant angular velocity ω . Construct the Lagrangian and write the equation of motion.

14. A system is composed of a particle of mass M and n particles of mass m . Separate the motion of the centre of mass and reduce the problem to the motion of n particles.
15. Discuss the one-dimensional motion of a particle of mass m in the field with potential energy $U(x) = U_0(e^{-2\alpha x} - e^{-\alpha x})$ (the Morse potential). Here, $U_0 > 0$, $\alpha > 0$.
16. Assume that a particle of mass m moves in a field whose potential is either (a) $U = U(\rho)$, or (b) $U = U(z)$, where ρ , φ , z are cylindric coordinates. Find the first integrals of motion in both cases.
17. A particle moves on a helix of equations $\rho = a$, $z = b\varphi$, where a and b are constants. If the potential energy is

$$V = \frac{1}{2}k(\rho^2 + z^2),$$

where k is another constant, find the law of motion of the particle and the force of constraint.

18. Two particles m_1 and m_2 are connected by a cord passing through a hole in a horizontal table. The mass m_1 moves like a simple pendulum, while the mass m_2 slides without friction on the table. Identify the constraints and write the equations of motion.
19. Show that the transformation

$$x'_i = x_i + \eta_i a_i \sin \omega t, \quad t' = t \quad (i = 1, 2, 3),$$

where $\mathbf{a}(a_1, a_2, a_3)$ is an arbitrary constant vector and $\omega^2 = k/m$, is a symmetry transformation for the Lagrangian

$$L = \frac{m}{2} \dot{\mathbf{r}}^2 - \frac{1}{2} k r^2$$

(space oscillator) and find the first integral of motion associated with this transformation.

20. Using the first integral found in the previous problem, as well as some other first integrals corresponding to the motion described by this Lagrangian, determine the law of motion and the trajectory associated with the following initial conditions:

$$\mathbf{r}(0) = (x_0, 0, 0), \quad \dot{\mathbf{r}}(0) = (0, v_0, 0).$$

Mechanics

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