Chapter 22
More on Duality

This final chapter is devoted to duality, the dual variety and the conormal scheme of an embedded projective variety are given as applications. Reflexivity and biduality are studied, in particular duality of hyperplane sections and projections. An application we present here is a very nice theorem of Hefez and Kleiman. Finally we give a brief presentation of some further results on duality and reflexivity.

22.1 The Dual Variety and the Conormal Scheme

Let $\mathbb{P}^N = \mathbb{P}(V)$ denote projective N-space, V being an $N+1$-dimensional vector space over k. $\mathbb{P}^{N\vee} = \mathbb{P}(V^*)$ denotes the *dual* projective space, whose k-points are identified with the hyperplanes in $\mathbb{P}^N$:

$$\mathbb{P}^{N\vee} = \left\{ H \;\middle|\; H \subset \mathbb{P}^N \text{ hyperplane} \right\}.$$

Let $X \hookrightarrow \mathbb{P}^N$ be a projective, closed subscheme of $\mathbb{P}^N$. By definition the dual variety is the (reduced) closure of the set of hyperplanes which are tangent to X at some smooth point:

$$X^\vee = \overline{\left\{ H \in \mathbb{P}^{N\vee} \;\middle|\; \begin{array}{l} H \text{ tangent to } X \\ \text{at a smooth point } x \in X \end{array} \right\}}.$$

The condition in the above definition can be expressed as

$$H \supset T_{X,x} \quad \text{for some } x \in X_{sm}.$$

By definition $X^\vee$ is reduced, but it need neither be irreducible nor of pure dimension. While the definition does make sense for non reduced subvarieties X, the nilpotent components do not contribute to $X^\vee$, and in particular $X^\vee = \emptyset$ if and only if X has no reduced components.

A. Holme, *A Royal Road to Algebraic Geometry*,

DOI 10.1007/978-3-642-19225-8_22,

To understand $X^\vee$ better, we look at the following diagram:

$$\begin{tikzcd}[column sep=small]
 & & & Z(X) \subset \mathbb{P}^N \times \mathbb{P}^{N\vee} \arrow[dlll, "p"'] \arrow[dll, "\mathrm{pr}_1"] \arrow[drr, "\lambda"'] \arrow[drrr, "\mathrm{pr}_2"] & & & \\
X \subset & \mathbb{P}^N & & & & X^\vee \subset & \mathbb{P}^{N\vee}
\end{tikzcd}$$

Here

$$Z(X) = \overline{\left\{ (x, H) \in X_{sm} \times \mathbb{P}^{N\vee} \,\middle|\, \begin{array}{l} H \text{ tangent to } X \\ \text{at the smooth point } x \in X \end{array} \right\}}.$$

By definition $X^\vee$ is the image of $Z(X)$ under the projection pr_2, thus the morphism λ is induced. Also, pr_1 induces a morphism p.

Assume for the moment that X is smooth. Then we have the following key diagram, where

$$\mathcal{N}_{X/\mathbb{P}^N} = (\mathcal{I}_{X/\mathbb{P}^N}/\mathcal{I}^2_{X/\mathbb{P}^N})^\vee$$

and where $i : X \hookrightarrow \mathbb{P}^N$ is the embedding of X into $\mathbb{P}^N$. Moreover, $\mathcal{P}^1(X)$ denotes the locally free sheaf of *principal parts* of X:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{N}^\vee_{X/\mathbb{P}^N}(1) & \to & i^*\Omega^1_{\mathbb{P}^N}(1) & \to & \Omega^1_X(1) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{N}^\vee_{X/\mathbb{P}^N}(1) & \to & \mathcal{O}_X^{N+1} & \to & \mathcal{P}^1(X) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_X(1) & = & \mathcal{O}_X(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{22.1}$$

The left injective map in the lower exact sequence induces a surjective

$$\mathcal{O}_X^{N+1} \twoheadrightarrow \mathcal{N}_{X/\mathbb{P}^N}(-1)$$

which gives the closed embedding

$$Z(X) = \mathbb{P}(\mathcal{N}_{X/\mathbb{P}^N}(-1)) \hookrightarrow \mathbb{P}(\mathcal{O}_X^{N+1}) = X \times \mathbb{P}^N.$$

The twist by -1 ensures that the tautological line bundle (i.e., the invertible sheaf) $\mathcal{O}_{\mathbb{P}(\mathcal{N}_{X/\mathbb{P}^N}(-1))}(1)$ is the restriction of the invertible sheaf $\mathrm{pr}_2^*(\mathcal{O}_{\mathbb{P}^{N\vee}}(1))$.

We now assume only that X is a reduced, projective scheme of pure dimension n. Letting X_{sm} denote the open subset of X consisting of smooth points, we have

$$Z(X) = \overline{\mathbb{P}(\mathcal{N}_{X_{sm}/\mathbb{P}^N - \mathrm{Sing}(X)}(-1))}.$$

Much information about $X^\vee$ is contained in the *Chow cohomology class*

$$[Z(X)] \in A^*(\mathbb{P}^N \times \mathbb{P}^{N\vee}) = \mathbf{Z}[s,t]$$

where $A^*(S)$ denotes the Chow (cohomology) ring of the smooth, projective variety S, $s = \mathrm{pr}_1^*([H])$, $t = \mathrm{pr}_2^*([H'])$ and H, H' denote hyperplanes in $\mathbb{P}^N$ and $\mathbb{P}^{N\vee}$, respectively.

If $n = \dim(X)$, then clearly

$$\dim(Z(X)) = n + (N - n - 1) = N - 1$$

and hence, since $Z(X)$ maps onto $X^\vee$ by λ,

$$\dim(X^\vee) \leq N - 1.$$

We have the following expression, where the $\delta_j(X)$ are integers for all j:

$$\begin{aligned}[Z(X)] = \delta_0(X)s^N t + \delta_1(X)s^{N-1}t^2 + \cdots + \delta_n(X)s^{N-n}t^{n+1} \\ + \delta_{n+1}(X)s^{N-n-1}t^{n+2} + \cdots + \delta_{N-1}(X)st^N.\end{aligned} \tag{22.2}$$

It follows easily, and will be shown in the next sections, that

$$\Delta = \delta_{n+1}(X)s^{N-n-1}t^{n+2} + \cdots + \delta_{N-1}(X)st^N$$

is actually equal to *zero*.

22.2 Reflexivity and Biduality

Suppose that we construct the basic diagram for X, $Z(X)$ and $X^\vee$, but this time with $X^\vee$ instead of X:

$$\begin{tikzcd}[column sep=small]
 & & & Z(X^\vee) \subset \mathbb{P}^{N\vee} \times \mathbb{P}^{N\vee\vee} \arrow[dlll, "p^\vee"'] \arrow[dl, "\mathrm{pr}_1"] \arrow[dr, "\lambda^\vee"] \arrow[drrr, "\mathrm{pr}_2"] & & & \\
X^\vee & \subset & \mathbb{P}^{N\vee} & & X^{\vee\vee} & \subset & \mathbb{P}^{N\vee\vee}
\end{tikzcd}$$

If we make the canonical identification $\mathbb{P}^N = \mathbb{P}^{N\vee\vee}$, then we have $\mathbb{P}^{N\vee} \times \mathbb{P}^{N\vee\vee} \cong \mathbb{P}^N \times \mathbb{P}^{N\vee}$, where the isomorphism is the canonical one interchanging the two copies of projective space $\mathbb{P}^N$ and $\mathbb{P}^{N\vee}$. Thus we can compare the subschemes $Z(X^\vee)$ and $Z(X)$. In the "good" cases these two subschemes are equal:

Definition 22.1 The embedded, reduced projective variety $X \hookrightarrow \mathbb{P}^N$ is said to be *reflexive* if

$$Z(X^\vee) = Z(X).$$

If $X = X^{\vee\vee}$, then we say that *biduality* holds for X.

It is clear that reflexivity implies biduality. But the converse is false, for this there are counterexamples. The following classical result is an important fact, it implies in particular that reflexivity and hence biduality always holds in characteristic zero:

Theorem 22.1 *The embedded, reduced projective variety $X \hookrightarrow \mathbb{P}^N$ is reflexive if and only if the morphism $\lambda : Z(X) \longrightarrow X^\vee$ is generically smooth.*

For the history of this theorem, as well as modern proof using Lagrangian geometry, we refer to Kleiman's article [32]. As a corollary we obtain the following geometric criterion:

Corollary 22.2 *X is reflexive if and only if the contact locus* $\mathrm{Cont}(H,X)$ *with X of the generic tangent hyperplane H to X is a linear subspace of* $\mathbb{P}^N$.

Proof of the Corollary Evidently $\mathrm{Cont}(H,X) = Z(X)_h = \lambda^{-1}(h)$, where h is the point of $X^\vee$ which corresponds to the tangent hyperplane H. Thus reflexivity implies that $\mathrm{Cont}(H,X)$ equals the fiber $\lambda^{-1}(h) =$

$\mathbb{P}(\mathcal{N}_{X^\vee{}_{sm}/\mathbb{P}^{N\vee}-\mathrm{Sing}(X^\vee)}(-1))_h$, provided that H corresponds to a smooth point h of $X^\vee$. Conversely, assume that $\mathrm{Cont}(H,X) = Z(X)_h = \lambda^{-1}(h)$ is linear, hence in particular equidimensional and geometrically regular, for all $h \in U$ where $U \subset X^\vee$ is an open dense subset. Then making U smaller if necessary we may assume that λ is flat over U. Hence λ is smooth over U by standard facts on smooth morphisms, say [18], Chap. III Theorem 10.2. □

In the next section we shall give a *numerical criterion* for reflexivity. At this point we note the following fact, which was noted in [39]:

Theorem 22.3 (T. Urabe) *If X is reflexive, then*

$$\delta_j(X^\vee) = \delta_{N-1-j}(X).$$

Proof By the identifications above and the assumption of reflexivity it is clear that

$$\begin{aligned} &\delta_0(X)s^N t + \cdots + \delta_i(X)s^{N-i}t^{i+1} + \cdots \\ &\quad = \delta_0(X^\vee)st^N + \cdots + \delta_i(X^\vee)s^{j+1}t^{N-j} + \cdots \end{aligned}$$

The claim is immediate from this. □

22.3 Duality of Projective Varieties

If X is smooth, then it is easily seen from diagram (22.1) that we have the following formulas for the numerical invariants occurring in formula (22.2):

$$\text{For all } i \geq 0, \quad \delta_i(X) = \sum_{j=i}^{n} (-1)^{n-j} \binom{j+1}{i+1} \deg(c_{n-j}(X)). \tag{22.3}$$

To prove this, we use a general fact which is referred to as *Scott's Formula*: Let

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

be an exact sequence of locally free sheaves of the finite ranks e, f and g, respectively, on the (smooth projective) scheme S. Then there is a canonical closed embedding

$$\mathbb{P}(\mathcal{G}) \hookrightarrow \mathbb{P}(\mathcal{F})$$

so that we get a class

$$[\mathbb{P}(\mathcal{G})] \in A(\mathbb{P}(\mathcal{F})) = A(S)[\xi_F]$$

where $\xi_F \in A(\mathbb{P}(\mathcal{F}))$ denotes $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1))$. Letting $p : \mathbb{P}(\mathcal{F}) \longrightarrow S$ denote the canonical projection, we then have Scott's Formula

$$[\mathbb{P}(\mathcal{G})] = \sum_{i \geq 0} p^*(c_{e-i}(\mathcal{E}^\vee))\xi_F^i.$$

A proof of this can be found in [12], p. 61 or in Sect. 2 of the Appendix to [22]. Now a straightforward computation applying this formula to the exact sequence on X

$$0 \longrightarrow \mathcal{P}^1(X)^\vee \longrightarrow \mathcal{O}_X^{N+1} \longrightarrow \mathcal{N}_{X/\mathbb{P}^N}(-1) \longrightarrow 0$$

where the Chern classes of $\mathcal{P}^1(X)$ are computed by means of the exact sequence

$$0 \longrightarrow \Omega_X(1) \longrightarrow \mathcal{P}^1(X) \longrightarrow \mathcal{O}_X(1) \longrightarrow 0$$

yields (22.3).

Formula (22.3) has been generalized to the *singular case* by R. Piene in [37]. In fact this paper was an important contribution to the theory of duality in the presence of singularities, making it possible to understand the relation between singular Chern- or Segre classes on one hand and the *Polar classes* which have the *delta–invariants* as their degrees, on the other.

Here we confine ourselves to some simple observations, which in certain situations can be quite useful as we shall see later.

Before we state the result, we recall some background:

Nevertheless, we have some information in the singular case as well, and in certain situations this can be quite useful, as we shall see later. Before we state the result, we recall some background:

Let Z denote a *smooth, quasi-projective scheme* of pure dimension n, embedded as a (locally closed) subscheme of a projective space $i: \ Z \hookrightarrow \mathbb{P}^N$. Let $\overline{Z} \subset \mathbb{P}^N$ denote the closure of Z in $\mathbb{P}^N$, so $Z \subset \overline{Z}$ is an open subset, and denote the codimension of $S = \overline{Z} - Z$ in $\overline{Z}$ by r. Finally let $A^*(Y)$ and $A_*(Y)$ be the Chow cohomology ring, respectively the Chow homology module, of the singular quasi-projective scheme Y, in the sense of Chap. 18 or Fulton [12]. Recall that $A^*(Y)$ is a commutative, graded ring with 1, and $A_*(Y)$ is a graded module over $A^*(Y)$. Both multiplications will be denoted by $\cdot$ or by $\wedge$. We then have the graded homomorphism

$$\Psi_Y : A^*(Y) \longrightarrow A_*(Y)$$

which sends the element $\alpha \in A^*(Y)$ to the element $\alpha \wedge [Y] \in A_*(Y)$. Since $A^*(Y)$ is graded by codimension and $A_*(Y)$ by dimension, we have

$$\Psi_{Y\,j} : A^j(Y) \longrightarrow A_{\dim(Y)-j}(Y).$$

This homomorphism is compatible with the standard properties of the *covariant pair*

$$(A^*(\ ), A_*(\ )),$$

in particular it is functorial, and whenever Y is smooth, it is an isomorphism which only introduces a shift in the grading. Finally, the open embedding i above induces a *Gysin map* $i^* : A_*(Z) \longrightarrow A_*(\overline{Z})$, which fits into the exact sequence below, where $j : S \hookrightarrow \overline{Z}$ is the canonical closed embedding:

$$A_*(S) \xrightarrow{j_*} A_*(\overline{Z}) \xrightarrow{i^*} A_*(Z) \longrightarrow 0.$$

It follows that for all integers $\alpha \leq n - r - 1$ the above maps will induce isomorphisms

$$A^\alpha(\overline{Z}) \xrightarrow{\approx} A^\alpha(Z) \xrightarrow{\approx} A_{n-\alpha}(Z)).$$

We now take $X = X - S$, where $S = \mathrm{Sing}(X)$, thus $X = \overline{Z}$. We make the following

Definition 22.2 Let X be a projective scheme, of pure dimension n, and with singular locus $S = \mathrm{Sing}(X)$ which is of dimension m. Then for all $\alpha \leq n - m - 1$, the Chern class $c_\alpha(X) \in A^\alpha(X)$ is defined as the Chern class $c_\alpha(X - S) \in A^\alpha(X - S)$ where $A^\alpha(X - S)$ is identified with $A^\alpha(X)$ by the isomorphism given above.

We are now ready for a partial extension of (22.3) to the singular case:

Theorem 22.4 *Assume that the singular locus of X is of dimension m. Then the formula* (22.3) *above holds for all $i \geq m + 1$.*

Remark 22.5 It follows from this that the formula (22.3) holds for all $i \geq m+1$ for any functorial theory of singular Chern-classes which coincides with the usual Chern-classes in the smooth case.

Proof Let $U = X - \mathrm{Sing}(X)$, and let $\mathbf{V} = \mathbb{P}^N - \mathrm{Sing}(X)$. Then U is a closed subscheme of $\mathbf{V}$, and

$$p^{-1}(U) = \mathbb{P}(\mathcal{N}_{U/\mathbf{V}}(-1)).$$

Moreover, we have the exact sequence

$$0 \longrightarrow \mathcal{P}^1(X)^\vee|_U \longrightarrow \mathcal{O}_U^{N+1} \longrightarrow \mathcal{N}_{U/\mathbf{V}}(-1) \longrightarrow 0.$$

As in the case when X is smooth, we now proceed by Scott's Formula, where as before the Chern classes of $\mathcal{P}^1(X)|_U = \mathcal{P}^1(U)$ are computed by means of the exact sequence

$$0 \longrightarrow \Omega_U^1(1) \longrightarrow \mathcal{P}^1(U) \longrightarrow \mathcal{O}_U(1) \longrightarrow 0.$$

The claim follows from this. $\square$

Corollary 22.6 *$\delta_n(X)$ is equal to the degree of X, and for $j \geq n+1$, $\delta_j(X) = 0$.*

Proof Immediate from (22.3). □

The invariants δ_j determine the dimension of the dual scheme $X^\vee$, and if X is reflexive, then the degree of $X^\vee$ is also given by these invariants: This was proved by *R. Piene* in [37].

In the following theorem we give a stronger version of this result, which gives a *numerical criterion* for reflexivity.

Theorem 22.7 (i) $X^\vee$ *is of dimension* $N-1-r$ *if*

$$\delta_0(X) = \cdots = \delta_{r-1}(X) = 0, \quad \delta_r(X) \neq 0.$$

(ii) *With* r *as in* (i), X *is reflexive if and only if*

$$\delta_r(X) = \deg(X^\vee).$$

Proof (i): We have the following sequence of biimplications, where P^r_{generic} denotes a linear r-space in $\mathbb{P}^{N\vee}$ in general position:

$$\begin{gathered}\dim(X^\vee) \leq N-1-r \\ \Updownarrow \\ X^\vee \cap P^r_{\text{generic}} = \emptyset \\ \Updownarrow \\ Z(X) \cap \mathrm{pr}_2^{-1}(P^r_{\text{generic}}) = \emptyset \\ \Updownarrow \\ [Z(X)] \cdot t^{N-r} = \delta_0 s^N t^{N-r+1} + \cdots + \delta_{r-1} s^{N-r} t^N = 0 \\ \Updownarrow \\ \delta_0 = \cdots = \delta_{r-1} = 0.\end{gathered}$$

(ii): Let P^{r+1}_{generic} be a linear subspace of $\mathbb{P}^{N\vee}$ in general position, as above. It intersects $X^\vee$ in exactly $\deg(X^\vee)$ smooth points. Letting ϵ denote the degree of the general fiber of λ, we then have the formula

$$\epsilon \cdot \deg(X^\vee) = \delta_r(X).$$

See the definition of $f_*([V])$ in Sect. 18.2. In the proof of Theorem 22.9 we show $\varepsilon = 1$. The claim follows from this, together with Corollary 22.2. □

22.4 Duality of Hyperplane Sections and Projections

A basic observation in the study of projective duality is that the operation of embedding a projective space as a linear subspace of another, is dual to the operation of projecting a larger space onto a smaller projective space with a linear center: In order to make this correspondence precise, we have to resolve a conflict of notation: Namely, if $L \hookrightarrow \mathbb{P}^N$ is a linear subspace, then the notation $L^\vee$ could mean either the dual variety of L considered as a subvariety of $\mathbb{P}^N$, or the dual space of the linear space L itself. Normally there is no need to distinguish between these two concepts in the notation, since the situation will be clear from the context. But whenever there is a possibility of confusion, *we shall denote the dual of the projective space L itself by L^**. We then have the following elementary observation, the proof of which is immediate:

Proposition 22.8 (i) *Let $L \hookrightarrow \mathbb{P}^N$ be a linear subspace of dimension r. Then $L^\vee \hookrightarrow \mathbb{P}^{N\vee}$ consists of those hyperplanes in $\mathbb{P}^N$ which are tangent to L, i.e. they contain L.*

(ii) *Let $\mathrm{pr}_L : \mathbb{P}^N - - \rightarrow \mathbb{P}^{N-r-1}$ be the projection with center in the linear r-dimensional subspace L. The pullback of a hyperplane in $\mathbb{P}^{N-r+1}$ yields a hyperplane in $\mathbb{P}^N$ which contains L, thus a point in $L^\vee$. This correspondence is bijective, and establishes an embedding*

$$\mathbb{P}^{N-r-1\vee} \hookrightarrow \mathbb{P}^{N\vee}$$

which identifies $\mathbb{P}^{N-r-1}$ with $L^\vee$.

We next give a simple proof of the theorem below, which is shown in [37].

Theorem 22.9 (R. Piene) *Let X be a reduced scheme of pure dimension n.*

(1) *Assume that X is not a hypersurface in $\mathbb{P}^N$. If $\mathrm{pr}_P : \mathbb{P}^N - - \rightarrow \mathbb{P}^{N-1}$ is a generic projection with center in the point P, then*

$$\delta_i(X) = \delta_i(\mathrm{pr}_P(X))$$

for all $i = 0, \dots, n$.

(2) *Let $H \subset \mathbb{P}^N$ be a generic hyperplane. Then*

$$\delta_i(X \cap H) = \delta_{i+1}(X)$$

for all $i = 0, \dots, n-1$.

Proof We first show (1). By the proposition we have that in the set up

$$\mathrm{pr}_P : \mathbb{P}^N - - \rightarrow \mathbb{P}^{N-1}$$

$P^\vee$ is identified with $\mathbb{P}^{N-1\vee}$. Under this identification, we find

$$Z(\mathrm{pr}_P(X)) = (\mathrm{pr}_P \times \mathrm{id}_{\mathbb{P}^{N\vee}})(Z(X)) \cap (\mathbb{P}^{N-1} \times \mathbb{P}^{N-1\vee}).$$

A proof of this observation is given in Sect. 6 of [26] as Proposition 6.3, (2).

Letting

$$j : \mathbb{P}^{N-1} \times \mathbb{P}^{N-1\vee} \hookrightarrow \mathbb{P}^{N-1} \times \mathbb{P}^{N\vee}$$

be the canonical embedding, we thus find

$$\varepsilon[Z(\mathrm{pr}_P(X))] = j^*((\mathrm{pr}_P \times \mathrm{id}_{\mathbb{P}^{N\vee}})_*([Z(X)]))$$

where ε is an integer, we will show shortly that $\varepsilon = 1$. The symbol $(\mathrm{pr}_P \times \mathrm{id}_{\mathbb{P}^{N\vee}})_*$ does of course represent an abuse of notation, since pr_P is not defined at P. The pushdown is defined by the standard diagram extended to $\mathbb{P}^{N\vee}$:

$$\begin{tikzcd}
& \widetilde{\mathbb{P}^N} \times \mathbb{P}^{N\vee} \arrow[dl, "\pi_P \times \mathrm{id}_{\mathbb{P}^{N\vee}}"'] \arrow[dr, "\lambda \times \mathrm{id}_{\mathbb{P}^{N\vee}}"] & \\
\mathbb{P}^N \times \mathbb{P}^{N\vee} \arrow[rr, dashed, "\mathrm{pr}_P \times \mathrm{id}_{\mathbb{P}^{N\vee}}"] & & \mathbb{P}^{N-1} \times \mathbb{P}^{N\vee}
\end{tikzcd}$$

Here π_P denotes the blowing up with center P, and λ is the corresponding bundle map on to $\mathbb{P}^{N-1}$.

As usual we write

$$A(\mathbb{P}^N \times \mathbb{P}^{N\vee}) = \mathbf{Z}[s,t]$$

where s and t are the pullbacks of the hyperplane classes from $\mathbb{P}^N$ and $\mathbb{P}^{N\vee}$, respectively. Similarly

$$A(\mathbb{P}^N \times \mathbb{P}^{N-1\vee}) = \mathbf{Z}[s,\overline{t}]$$
$$A(\mathbb{P}^{N-1} \times \mathbb{P}^{N\vee}) = \mathbf{Z}[\overline{s},t]$$

and

$$A(\mathbb{P}^{N-1} \times \mathbb{P}^{N-1\vee}) = \mathbf{Z}[\overline{s},\overline{t}].$$

For $i \geq 1$ we have $\mathrm{pr}_{P*}(s^i) = \overline{s}^{i-1}$ (where pr_{P*} is defined in the obvious way via the blowing up with center P), and for all i and j

$$j^*(\overline{s}^i t^j) = \overline{s}^i \overline{t}^j.$$

Thus the expressions

$$[Z(X)] = \delta_0(X)s^N t + \delta_1(X)s^{N-1}t^2 + \cdots + \delta_n(X)s^{N-n}t^{n+1}$$

and

$$\begin{aligned}[Z(\mathrm{pr}_P(X))] = \delta_0(\mathrm{pr}_P(X))\overline{s}^{N-1}\overline{t} + \delta_1(\mathrm{pr}_P(X))\overline{s}^{N-2}\overline{t}^2 \\ + \cdots + \delta_n(\mathrm{pr}_P(X))\overline{s}^{N-1-n}\overline{t}^{n+1}\end{aligned}$$

immediately yield the claim, since $\delta_n(X) = \delta_n(\mathrm{pr}_P(X)) = \deg(X)$ so that $\varepsilon = 1$.

(2) is shown in an analogous way: Let

$$i_H : \mathbb{P}^{N-1} \hookrightarrow \mathbb{P}^N$$
$$\mathrm{pr}_H : \mathbb{P}^{N\vee} - - \rightarrow \mathbb{P}^{N-1\vee}$$

denote the embedding with image H, and dually the projection with center H regarded as a point of $\mathbb{P}^{N\vee}$. We then have the relation

$$Z(X \cap H) = (\mathrm{id}_{\mathbb{P}^{N-1}} \times \mathrm{pr}_H)(Z(X) \cap (\mathbb{P}^{N-1} \times \mathbb{P}^{N\vee}))$$

which again is shown in Sect. 6. Here $\mathbb{P}^{N-1}$ is identified with H via i_H. Thus

$$[Z(X \cap H)] = (\mathrm{id}_{\mathbb{P}^{N-1}} \times \mathrm{pr}_H)_*((i_H \times \mathrm{id}_{\mathbb{P}^{N\vee}})^*([Z(X)]).$$

Now we have

$$\begin{aligned}[Z(X \cap H)] = \delta_0(X \cap H)\overline{s}^{N-1}\overline{t} + \delta_1(X \cap H)\overline{s}^{N-2}\overline{t}^2 \\ + \cdots + \delta_{n-1}(X \cap H)\overline{s}^{N-1-(n-1)}\overline{t}^n\end{aligned}$$

and since

$$(\mathrm{id}_{\mathbb{P}N-1} \times \mathrm{pr}_H)_*((i_H \times \mathrm{id}_{\mathbb{P}^{N\vee}})^*(s^u t^v) = \overline{s}^u \overline{t}^{v-1}$$

we also have

$$\begin{aligned}&(\mathrm{id}_{\mathbb{P}N-1} \times \mathrm{pr}_H)_*((i_H \times \mathrm{id}_{\mathbb{P}^{N\vee}})^*([Z(X)]) \\ &= \delta_1(X)\overline{s}^{N-1}\overline{t} + \delta_2(X)\overline{s}^{N-2}\overline{t}^2 + \cdots + \delta_n(X)\overline{s}^{N-1-(n-1)}\overline{t}^n\end{aligned}$$

where again a multiplicity ε turns out to be 1 since $\delta_n(X) = \delta_{n-1}(X \cap H) = \deg(X)$. This gives the claim. $\square$

22.5 A Theorem of Hefez-Kleiman

In this section we prove a theorem by Hefez and Kleiman [32] in the following form:

Theorem 22.10 *Let X be a reduced, projective scheme of pure dimension n, and let r be the integer such that*

$$\delta_0(X) = \cdots = \delta_{r-1}(X) = 0, \quad \delta_r(X) \neq 0.$$

Then for all $i \in [r, n]$,

$$\delta_i(X) \geq 1.$$

In characteristic zero, this can be strengthened to

$$\delta_i(X) \geq 2$$

provided that X *is not a linear subspace of* $\mathbb{P}^N$.

Proof Replacing X by the intersection with an appropriate generic linear subspace if necessary, we may assume that $r = 0$, i.e., that $X^\vee$ is a hypersurface. The claim then amounts to showing the theorem below, which is classical in spirit, but was first discovered by *A. Wallace* in [40] (Theorem 1 on p. 7, as well as the lemmas d and e):

Theorem 22.11 (Wallace) *If* $X^\vee$ *is a hypersurface in* $\mathbb{P}^N$, *then* $(X \cap H)^\vee$ *is a hypersurface in* $\mathbb{P}^{N-1\vee}$, *provided* H *is a hyperplane in general position. More generally, if* X *is cut by an* r*-dimensional linear subspace* L *in general position, then the dual of the linear section is a hypersurface in* $L^* \cong \mathbb{P}^{r\vee}$.

Proof We give Wallace's proof, slightly reformulated: The dual of $X \cap H$ in $\mathbb{P}^{N\vee}$ is the closure of all points which correspond to hyperplanes H' which contain $T_{X,x} \cap H$ for some smooth point $x \in X$. Thus letting $T(X, X \cap H)$ denote the subvariety of $\mathbb{P}^{N\vee}$ which corresponds to (the closure of the set of) hyperplanes tangent to X at some (smooth) point of $X \cap H$, we find that the dual of the hyperplane section in $\mathbb{P}^{N\vee}$ is the *cone* over $T(X, X \cap H)$. Since $X^\vee$ is a hypersurface by assumption, the map λ is generically finite, thus $T(X, X \cap H)$ is of codimension 1 in $X^\vee$ whenever H is sufficiently general, hence $(X \cap H)^\vee$ in $\mathbb{P}^{N\vee}$ is a hypersurface as well, and a cone with vertex H. But since the correspondence between points in $\mathbb{P}^{N\vee}$ and points in $H^* = \mathbb{P}^{N-1\vee}$ is given by projection with center $H \in \mathbb{P}^{N\vee}$, this means that the dual of $(X \cap H)^\vee$ is a hypersurface in $\mathbb{P}^{N-1\vee}$.

Next, assume that the characteristic is zero. Replacing X by its intersection with a generic linear subspace of $\mathbb{P}^N$ if necessary, we may assume that $X^\vee$ is a hypersurface. Then cutting $X^\vee$ with a generic linear 2-space $\mathbb{P}2 \subset \mathbb{P}^N$ and using the duality of the delta-invariants, we have from Piene's theorem in the previous section

$$[Z(X^\vee \cap \mathbb{P}^2)] = \delta_0(X)\overline{s}^2\overline{t} + \delta_1(X)\overline{s}\overline{t}^2$$

where $\overline{s}$ and $\overline{t} \in A(\mathbb{P}^2 \times \mathbb{P}^{2\vee})$ are the pullbacks of line-classes from $\mathbb{P}^2$ and $\mathbb{P}^{2\vee}$, respectively. In particular, if $X^\vee$ is a hypersurface, then $X^\vee \cap \mathbb{P}^2$ must be a planar curve, of degree $\delta_0(X) \geq 2$: If it were of degree 1, then it would be a line, thus $X^\vee$ would be linear, hence X would be linear by biduality. Moreover, $\delta_1(X)$ is the degree of the curve $(X^\vee \cap \mathbb{P}^2)^\vee$. Hence we also have $\delta_1(X) \geq 2$. Thus we have shown the claim for $i = 0$ and 1. To proceed, we cut X with one hyperplane more and repeat the argument. □

A further sharpening is given by the

Proposition 22.12 *Assume that the characteristic of the ground field is zero, and that $X \subset \mathbb{P}^N$ is irreducible and is not a hypersurface in some linear subspace. Then the inequality in the theorem can be strengthened to*

$$\delta_i(X) \geq 3.$$

Proof By the same argument as in the last part of the proof of the theorem: Assume that $\delta_i(X) = 2$ for some i. Then we may assume that $i = r = 0$, if necessary after cutting X with an appropriate generic linear subspace of $\mathbb{P}^N$. Thus if $\mathbb{P}^2$ is a generic linear subspace of dimension 2, then $X^\vee \cap \mathbb{P}^2$ is an irreducible curve of degree 2, hence a smooth conic, thus $X^\vee$ is either smooth or a cone of degree 2. But it can not be a cone, as biduality holds and X is not contained in a hypersurface. Hence $X^\vee$ is smooth of degree 2, so X is also of degree 2. We are thus finished by observing that a variety of degree 2 is necessarily a hypersurface in some linear subspace of the ambient space. □

The number r referred to in the theorem above is called the *duality defect* of the embedded variety X. The concept is important for the classification of embedded projective varieties.

But here, with a view into a vast and very interesting field of research, is where our Royal Road ends.

Springer

http://www.springer.com/978-3-642-19224-1

A Royal Road to Algebraic Geometry
Holme, A.
2012, XIV, 366 p., Hardcover
ISBN: 978-3-642-19224-1