Chapter 2
Spontaneous Symmetry Breaking and the Goldstone Theorem

2.1 Degenerate Ground States

Before discussing the case of a *continuous* symmetry, we will first have a look at a field theory with a *discrete* internal symmetry. This will allow us to distinguish between two possibilities: a dynamical system with a unique ground state or a system with a finite number of distinct degenerate ground states. In particular, we will see how, for the second case, an infinitesimal perturbation selects a particular vacuum state.

To that end we consider the Lagrangian of a real scalar field $\Phi(x)$ [8]

$$\mathscr{L}(\Phi, \partial_\mu \Phi) = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{2}\Phi^2 - \frac{\lambda}{4}\Phi^4, \tag{2.1}$$

which is invariant under the discrete transformation $R : \Phi \mapsto -\Phi$. The corresponding classical energy density reads

$$\mathscr{H} = \Pi\dot{\Phi} - \mathscr{L} = \frac{1}{2}\dot{\Phi}^2 + \frac{1}{2}(\vec{\nabla}\Phi)^2 + \underbrace{\frac{m^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4}_{\equiv \mathscr{V}(\Phi)}, \tag{2.2}$$

where one chooses $\lambda > 0$ so that $\mathscr{H}$ is bounded from below. The field Φ_0 which minimizes the Hamilton density $\mathscr{H}$ must be constant and uniform since in that case the first two terms take their minimum values of zero everywhere. It must also minimize the "potential" $\mathscr{V}$ since $\mathscr{V}(\Phi(x)) \geq \mathscr{V}(\Phi_0)$, from which we obtain the condition

$$\mathscr{V}'(\Phi) = \Phi(m^2 + \lambda\Phi^2) = 0.$$

We now distinguish two different cases:

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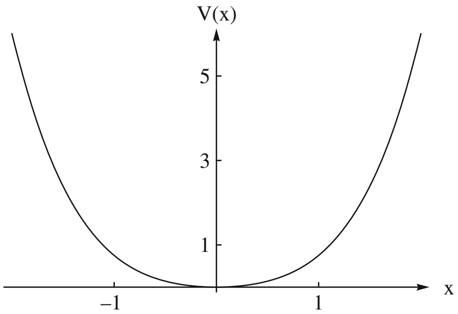

Fig. 2.1 $\mathscr{V}(x) = x^2/2 + x^4/4$ (Wigner-Weyl mode)

1. $m^2 > 0$ (see Fig. 2.1): In this case the potential $\mathscr{V}$ has its minimum for $\Phi = 0$. In the quantized theory we associate a unique ground state $|0\rangle$ with this minimum. Later on, in the case of a continuous symmetry, this situation will be referred to as the Wigner-Weyl realization of the symmetry.
2. $m^2 < 0$ (see Fig. 2.2): Now the potential exhibits two distinct minima. (In the continuous symmetry case this will be referred to as the Nambu-Goldstone realization of the symmetry.)

We will concentrate on the second situation, because this is the one which we would like to generalize to a continuous symmetry and which ultimately leads to the appearance of Goldstone bosons. In the present case, $\mathscr{V}(\Phi)$ has a local maximum for $\Phi = 0$ and *two* minima for

$$\Phi_{\pm} = \pm\sqrt{\frac{-m^2}{\lambda}} \equiv \pm\Phi_0. \tag{2.3}$$

As will be explained below, the quantized theory develops two degenerate vacua $|0,+\rangle$ and $|0,-\rangle$ which are distinguished through their vacuum expectation values of the field $\Phi(x)$:[1]

$$\begin{aligned}\langle 0,+|\Phi(x)|0,+\rangle &= \langle 0,+|e^{iP\cdot x}\Phi(0)e^{-iP\cdot x}|0,+\rangle = \langle 0,+|\Phi(0)|0,+\rangle \equiv \Phi_0,\\ \langle 0,-|\Phi(x)|0,-\rangle &= -\Phi_0.\end{aligned} \tag{2.4}$$

We made use of translational invariance, $\Phi(x) = e^{iP\cdot x}\Phi(0)e^{-iP\cdot x}$, and the fact that the ground state is an eigenstate of energy and momentum. We associate with the transformation $R : \Phi \mapsto \Phi' = -\Phi$ a unitary operator $\mathscr{R}$ acting on the Hilbert space of our model, with the properties

[1] The case of a quantum field theory with an infinite volume V has to be distinguished from, say, a nonrelativistic particle in a one-dimensional potential of a shape similar to the function of Fig. 2.2. For example, in the case of a symmetric double-well potential, the solutions with positive parity always have lower energy eigenvalues than those with negative parity (see, e.g., Ref. [11]).

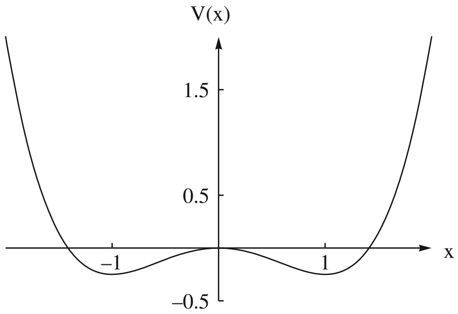

Fig. 2.2 $\mathscr{V}(x) = -x^2/2 + x^4/4$ (Nambu-Goldstone mode)

$$\mathscr{R}^2 = I, \quad \mathscr{R} = \mathscr{R}^{-1} = \mathscr{R}^\dagger.$$

In accord with Eq. 2.4 the action of the operator $\mathscr{R}$ on the ground states is given by

$$\mathscr{R}|0, \pm\rangle = |0, \mp\rangle. \tag{2.5}$$

For the moment we select one of the two expectation values and expand the Lagrangian about $\pm\Phi_0$:[2]

$$\begin{aligned} \Phi &= \pm\Phi_0 + \Phi', \\ \partial_\mu\Phi &= \partial_\mu\Phi'. \end{aligned} \tag{2.6}$$

Exercise 2.1 Show that

$$\mathscr{V}(\Phi) = \tilde{\mathscr{V}}(\Phi') = -\frac{\lambda}{4}\Phi_0^4 + \frac{1}{2}(-2m^2)\Phi'^2 \pm \lambda\Phi_0\Phi'^3 + \frac{\lambda}{4}\Phi'^4.$$

Thus, the Lagrangian in terms of the shifted dynamical variable reads

$$\mathscr{L}'(\Phi', \partial_\mu\Phi') = \frac{1}{2}\partial_\mu\Phi'\partial^\mu\Phi' - \frac{1}{2}(-2m^2)\Phi'^2 \mp \lambda\Phi_0\Phi'^3 - \frac{\lambda}{4}\Phi'^4 + \frac{\lambda}{4}\Phi_0^4. \tag{2.7}$$

In terms of the new dynamical variable Φ', the symmetry R is no longer manifest, i.e., it is hidden. Selecting one of the ground states has led to a spontaneous symmetry breaking which is always related to the existence of several degenerate vacua.

At this stage it is not clear why the ground state of the quantum system should be one or the other of $|0, \pm\rangle$ and not a superposition of both. For example, the linear combination

$$\frac{1}{\sqrt{2}}(|0, +\rangle + |0, -\rangle)$$

[2] The field Φ' instead of Φ is assumed to vanish at infinity.

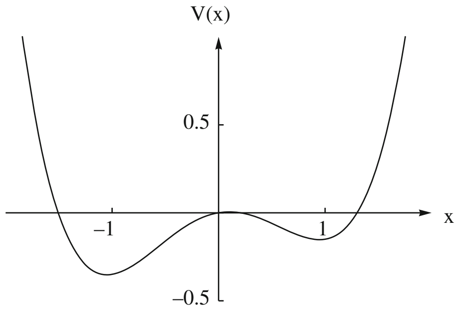

Fig. 2.3 Potential with a small odd component: $\mathscr{V}(x) = x/10 - x^2/2 + x^4/4$

is invariant under $\mathscr{R}$ as is the original Lagrangian of Eq. 2.1. However, this superposition is not stable against any infinitesimal external perturbation which is odd in Φ (see Fig. 2.3),

$$\mathscr{R}\varepsilon H'\mathscr{R}^\dagger = -\varepsilon H'.$$

Any such perturbation will drive the ground state into the vicinity of either $|0,+\rangle$ or $|0,-\rangle$ rather than $\frac{1}{\sqrt{2}}(|0,+\rangle \pm |0,-\rangle)$. This can easily be seen in the framework of perturbation theory for degenerate states. Consider

$$|1\rangle = \frac{1}{\sqrt{2}}(|0,+\rangle + |0,-\rangle), \quad |2\rangle = \frac{1}{\sqrt{2}}(|0,+\rangle - |0,-\rangle),$$

such that

$$\mathscr{R}|1\rangle = |1\rangle \quad \mathscr{R}|2\rangle = -|2\rangle.$$

The condition for the energy eigenvalues of the ground state, $E = E^{(0)} + \varepsilon E^{(1)} + \cdots$, to first order in ε results from

$$\det\begin{pmatrix} \langle 1|H'|1\rangle - E^{(1)} & \langle 1|H'|2\rangle \\ \langle 2|H'|1\rangle & \langle 2|H'|2\rangle - E^{(1)} \end{pmatrix} = 0.$$

Due to the symmetry properties of Eq. 2.5, we obtain

$$\langle 1|H'|1\rangle = \langle 1|\mathscr{R}^{-1}\mathscr{R}H'\mathscr{R}^{-1}\mathscr{R}|1\rangle = \langle 1| - H'|1\rangle = 0$$

and similarly $\langle 2|H'|2\rangle = 0$. Setting $\langle 1|H'|2\rangle = a > 0$, which can always be achieved by multiplication of one of the two states by an appropriate phase, one finds

$$\langle 2|H'|1\rangle \stackrel{H'=H'^\dagger}{=} \langle 1|H'|2\rangle^* = a^* = a = \langle 1|H'|2\rangle,$$

resulting in

$$\det\begin{pmatrix} -E^{(1)} & a \\ a & -E^{(1)} \end{pmatrix} = E^{(1)^2} - a^2 = 0. \quad \Rightarrow E^{(1)} = \pm a.$$

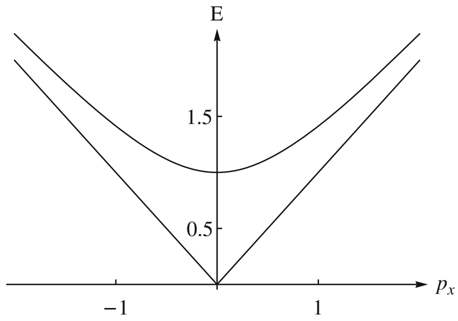

Fig. 2.4 Dispersion relation $E = \sqrt{1 + p_x^2}$ and asymptote $E = |p_x|$

In other words, the degeneracy has been lifted and we get for the energy eigenvalues

$$E = E^{(0)} \pm \varepsilon a + \cdots . \tag{2.8}$$

The corresponding eigenstates of zeroth order in ε are $|0,+\rangle$ and $|0,-\rangle$, respectively. We thus conclude that an arbitrarily small external perturbation which is odd with respect to R will push the ground state to either $|0,+\rangle$ or $|0,-\rangle$.

In the above discussion, we have tacitly assumed that the Hamiltonian and the field $\Phi(x)$ can simultaneously be diagonalized in the vacuum sector, i.e., $\langle 0,+|0,-\rangle = 0$. Following Ref. [18], we will justify this assumption which will also be crucial for the continuous case to be discussed later.

For an infinite volume, a general vacuum state $|v\rangle$ is defined as a state with momentum eigenvalue $\vec{0}$,

$$\vec{P}|v\rangle = \vec{0},$$

where $\vec{0}$ is a *discrete* eigenvalue as opposed to an eigenvalue of single- or many-particle states for which $\vec{p} = \vec{0}$ is an element of a continuous spectrum (see Fig. 2.4). We deal with the situation of several degenerate ground states[3] which will be denoted by $|u\rangle$, $|v\rangle$, etc., and start from the identity

$$0 = \langle u|[H, \Phi(x)]|v\rangle \quad \forall \quad x, \tag{2.9}$$

from which we obtain for $t = 0$

$$\int d^3y \, \langle u|\mathscr{H}(\vec{y},0)\Phi(\vec{x},0)|v\rangle = \int d^3y \, \langle u|\Phi(\vec{x},0)\mathscr{H}(\vec{y},0)|v\rangle . \tag{2.10}$$

Let us consider the left-hand side,

[3] For continuous symmetry groups one may have a non-countably infinite number of ground states.

$$\int d^3y\,\langle u|\mathscr{H}(\vec{y},0)\Phi(\vec{x},0)|v\rangle = \sum_w \langle u|H|w\rangle\langle w|\Phi(0)|v\rangle$$
$$+\int d^3y \int d^3p \sum_n \langle u|\mathscr{H}(\vec{y},0)|n,\vec{p}\rangle\langle n,\vec{p}|\Phi(0)|v\rangle e^{-i\vec{p}\cdot\vec{x}},$$

where we inserted a complete set of states which we split into the vacuum contribution and the remainder, and made use of translational invariance. We now define

$$f_n(\vec{y},\vec{p}) = \langle u|\mathscr{H}(\vec{y},0)|n,\vec{p}\rangle\langle n,\vec{p}|\Phi(0)|v\rangle$$

and assume f_n to be reasonably behaved such that one can apply the lemma of Riemann and Lebesgue,

$$\lim_{|\vec{x}|\to\infty} \int d^3p\, f(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} = 0.$$

At this point the assumption of an infinite volume, $|\vec{x}\,| \to \infty$, is crucial. Repeating the argument for the right-hand side and taking the limit $|\vec{x}\,| \to \infty$, only the vacuum contributions survive in Eq. 2.10 and we obtain

$$\sum_w \langle u|H|w\rangle\langle w|\Phi(0)|v\rangle = \sum_w \langle u|\Phi(0)|w\rangle\langle w|H|v\rangle$$

for arbitrary ground states $|u\rangle$ and $|v\rangle$. In other words, the matrices $(H_{uv}) \equiv (\langle u|H|v\rangle)$ and $(\Phi_{uv}) \equiv (\langle u|\Phi(0)|v\rangle)$ commute and can be diagonalized simultaneously. Choosing an appropriate basis, one can write

$$\langle u|\Phi(0)|v\rangle = \delta_{uv} v, \quad v \in \mathbb{R},$$

where v denotes the expectation value of Φ in the state $|v\rangle$.

In the above example, the ground states $|0,+\rangle$ and $|0,-\rangle$ with vacuum expectation values $\pm\Phi_0$ are thus indeed orthogonal and satisfy

$$\langle 0,+|H|0,-\rangle = \langle 0,-|H|0,+\rangle = 0.$$

2.2 Spontaneous Breakdown of a Global, Continuous, Non-Abelian Symmetry

Using the example of the O(3) sigma model we recall a few aspects relevant to our subsequent discussion of spontaneous symmetry breaking [16].[4] To that end, we consider the Lagrangian

[4] The linear sigma model [6, 7, 17] is constructed in terms of the O(4) multiplet $(\sigma, \pi_1, \pi_2, \pi_3)$. Since the group O(4) is locally isomorphic to SU(2) × SU(2), the linear sigma model is a popular framework for illustrating the spontaneous symmetry breaking in two-flavor QCD.

$$\begin{aligned}\mathscr{L}(\vec{\Phi}, \partial_\mu \vec{\Phi}) &= \mathscr{L}(\Phi_1, \Phi_2, \Phi_3, \partial_\mu \Phi_1, \partial_\mu \Phi_2, \partial_\mu \Phi_3) \\ &= \frac{1}{2}\partial_\mu \Phi_i \partial^\mu \Phi_i - \frac{m^2}{2}\Phi_i \Phi_i - \frac{\lambda}{4}(\Phi_i \Phi_i)^2, \end{aligned} \tag{2.11}$$

where $m^2 < 0, \lambda > 0$, with Hermitian fields Φ_i. By choosing $m^2 < 0$, the symmetry is realized in the Nambu-Goldstone mode [9, 13].[5]

The Lagrangian of Eq. 2.11 is invariant under a global "isospin" rotation,[6]

$$g \in \mathrm{SO}(3) : \Phi_i \mapsto \Phi'_i = D_{ij}(g)\Phi_j = \left(e^{-i\alpha_k T_k}\right)_{ij}\Phi_j. \tag{2.12}$$

For the Φ'_i to also be Hermitian, the Hermitian T_k must be purely imaginary and thus antisymmetric (see Eqs. 1.69). The iT_k provide the basis of a representation of the so(3) Lie algebra and satisfy the commutation relations $[T_i, T_j] = i\varepsilon_{ijk}T_k$. We use the representation of Eqs. 1.69, i.e., the matrix elements are given by $t_{i,jk} = -i\varepsilon_{ijk}$. We now look for a minimum of the potential which does not depend on x.

Exercise 2.2 Determine the minimum of the potential

$$\mathscr{V}(\Phi_1, \Phi_2, \Phi_3) = \frac{m^2}{2}\Phi_i \Phi_i + \frac{\lambda}{4}(\Phi_i \Phi_i)^2.$$

We find

$$|\vec{\Phi}_{\text{min}}| = \sqrt{\frac{-m^2}{\lambda}} \equiv v, \quad |\vec{\Phi}| = \sqrt{\Phi_1^2 + \Phi_2^2 + \Phi_3^2}. \tag{2.13}$$

Since $\vec{\Phi}_{\text{min}}$ can point in any direction in isospin space we have a non-countably infinite number of degenerate vacua. Any infinitesimal external perturbation that is not invariant under SO(3) will select a particular direction which, by an appropriate orientation of the internal coordinate frame, we denote as the 3 direction in our convention,

$$\vec{\Phi}_{\text{min}} = v\hat{e}_3. \tag{2.14}$$

Clearly, $\vec{\Phi}_{\text{min}}$ of Eq. 2.14 is *not* invariant under the full group $G = \mathrm{SO}(3)$ since rotations about the 1 and 2 axes change $\vec{\Phi}_{\text{min}}$.[7] To be specific, if

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[5] In the beginning, the discussion of spontaneous symmetry breaking in field theories [9, 13–15] was driven by an analogy with the theory of superconductivity [1, 2, 4, 5].

[6] The Lagrangian is invariant under the full group O(3) which can be decomposed into its two components: the proper rotations connected to the identity, SO(3), and the rotation-reflections. For our purposes it is sufficient to discuss SO(3).

[7] We say, somewhat loosely, that T_1 and T_2 do not annihilate the ground state or, equivalently, finite group elements generated by T_1 and T_2 do not leave the ground state invariant. This should become clearer later on.

$$\vec{\Phi}_{\min} = v \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we obtain

$$T_1\vec{\Phi}_{\min} = v \begin{pmatrix} 0 \\ -i \\ 0 \end{pmatrix}, \quad T_2\vec{\Phi}_{\min} = v \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}, \quad T_3\vec{\Phi}_{\min} = 0. \tag{2.15}$$

Note that the set of transformations which do not leave $\vec{\Phi}_{\min}$ invariant does *not* form a group, because it does not contain the identity. On the other hand, $\vec{\Phi}_{\min}$ is invariant under a subgroup H of G, namely, the rotations about the 3 axis:

$$h \in H : \quad \vec{\Phi}' = D(h)\vec{\Phi} = e^{-i\alpha_3 T_3}\vec{\Phi}, \quad D(h)\vec{\Phi}_{\min} = \vec{\Phi}_{\min}. \tag{2.16}$$

Exercise 2.3 Write Φ_3 as

$$\Phi_3(x) = v + \eta(x), \tag{2.17}$$

where $\eta(x)$ is a new field replacing $\Phi_3(x)$, and express the Lagrangian in terms of the fields $\Phi_1, \Phi_2,$ and $\eta,$ where $v = \sqrt{-m^2/\lambda}$.

The new expression for the potential is given by

$$\tilde{\mathscr{V}} = \frac{1}{2}(-2m^2)\eta^2 + \lambda v\eta(\Phi_1^2 + \Phi_2^2 + \eta^2) + \frac{\lambda}{4}(\Phi_1^2 + \Phi_2^2 + \eta^2)^2 - \frac{\lambda}{4}v^4. \tag{2.18}$$

Upon inspection of the terms quadratic in the fields, one finds after spontaneous symmetry breaking two massless Goldstone bosons and one massive boson:

$$\begin{aligned} m^2_{\Phi_1} &= m^2_{\Phi_2} = 0, \\ m^2_\eta &= -2m^2. \end{aligned} \tag{2.19}$$

The model-independent feature of the above example is given by the fact that for each of the two generators T_1 and T_2 which do not annihilate the ground state one obtains a *massless* Goldstone boson. By means of a two-dimensional simplification (see the "Mexican hat" potential shown in Fig. 2.5) the mechanism at hand can easily be visualized. Infinitesimal variations orthogonal to the circle of the minimum of the potential generate quadratic terms, i.e., "restoring forces" linear in the displacement, whereas tangential variations experience restoring forces only of higher orders.

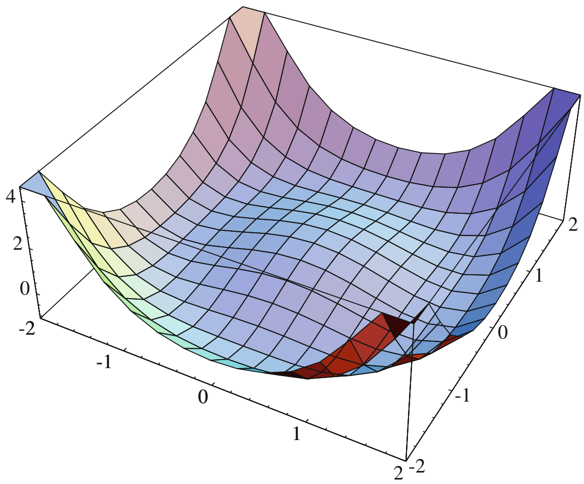

Fig. 2.5 Two-dimensional rotationally invariant potential: $\mathscr{V}(x,y) = -(x^2+y^2) + \frac{(x^2+y^2)^2}{4}$

Now let us generalize the model to the case of an arbitrary compact Lie group G of order n_G resulting in n_G infinitesimal generators.[8] Once again, we start from a Lagrangian of the form [10]

$$\mathscr{L}\left(\vec{\Phi}, \partial_\mu \vec{\Phi}\right) = \frac{1}{2}\partial_\mu \vec{\Phi} \cdot \partial^\mu \vec{\Phi} - \mathscr{V}\left(\vec{\Phi}\right), \tag{2.20}$$

where $\vec{\Phi}$ is a multiplet of scalar (or pseudoscalar) Hermitian fields. The Lagrangian $\mathscr{L}$ and thus also $\mathscr{V}$ are supposed to be globally invariant under G, where the infinitesimal transformations of the fields are given by

$$g \in G: \quad \Phi_i \mapsto \Phi_i + \delta\Phi_i, \quad \delta\Phi_i = -i\varepsilon_a t_{a,ij}\Phi_j. \tag{2.21}$$

The Hermitian representation matrices $T_a = (t_{a,ij})$ are again antisymmetric and purely imaginary. We now assume that, by choosing an appropriate form of $\mathscr{V}$, the Lagrangian generates a spontaneous symmetry breaking resulting in a ground state with a vacuum expectation value $\vec{\Phi}_{\text{min}} = \langle \vec{\Phi} \rangle$ which is invariant under a continuous subgroup H of G. We expand $\mathscr{V}$ about $\vec{\Phi}_{\text{min}}$, $|\vec{\Phi}_{\text{min}}| = v$, i.e., $\vec{\Phi} = \vec{\Phi}_{\text{min}} + \vec{\chi}$,

$$\mathscr{V}\left(\vec{\Phi}\right) = \mathscr{V}\left(\vec{\Phi}_{\text{min}}\right) + \underbrace{\frac{\partial \mathscr{V}\left(\vec{\Phi}_{\text{min}}\right)}{\partial \Phi_i}}_{=0} \chi_i + \frac{1}{2}\underbrace{\frac{\partial^2 \mathscr{V}\left(\vec{\Phi}_{\text{min}}\right)}{\partial \Phi_i \partial \Phi_j}}_{\equiv m_{ij}^2} \chi_i \chi_j + \cdots. \tag{2.22}$$

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[8] The restriction to compact groups allows for a complete decomposition into finite-dimensional irreducible unitary representations.

The matrix $M^2 = (m_{ij}^2)$ must be symmetric and, since one is expanding about a minimum, positive semidefinite, i.e.,

$$\sum_{i,j} m_{ij}^2 x_i x_j \geq 0 \quad \forall \quad \vec{x}. \tag{2.23}$$

In that case, all eigenvalues of M^2 are nonnegative. Making use of the invariance of $\mathscr{V}$ under the symmetry group G,

$$\begin{aligned}\mathscr{V}\left(\vec{\Phi}_{\min}\right) &= \mathscr{V}\left(D(g)\vec{\Phi}_{\min}\right) = \mathscr{V}\left(\vec{\Phi}_{\min} + \delta\vec{\Phi}_{\min}\right)\\ &\overset{(2.22)}{=} \mathscr{V}\left(\vec{\Phi}_{\min}\right) + \frac{1}{2} m_{ij}^2 \delta\Phi_{\min,i} \delta\Phi_{\min,j} + \cdots,\end{aligned} \tag{2.24}$$

one obtains, by comparing coefficients,

$$m_{ij}^2 \delta\Phi_{\min,i} \delta\Phi_{\min,j} = 0. \tag{2.25}$$

Differentiating Eq. 2.25 with respect to $\delta\Phi_{\min,k}$ and using $m_{ij}^2 = m_{ji}^2$ results in the matrix equation

$$M^2 \delta\vec{\Phi}_{\min} = \vec{0}. \tag{2.26}$$

Inserting the variations of Eq. 2.21 for arbitrary ε_a, $\delta\vec{\Phi}_{\min} = -i\varepsilon_a T_a \vec{\Phi}_{\min}$, we conclude

$$M^2 T_a \vec{\Phi}_{\min} = \vec{0}. \tag{2.27}$$

Recall that the T_a represent generators of the symmetry transformations of the Lagrangian of Eq. 2.20. The solutions of Eq. 2.27 can be classified into two categories:

1. T_a, $a = 1, \ldots, n_H$, is a representation of an element of the Lie algebra belonging to the subgroup H of G, leaving the selected ground state invariant. Therefore, invariance under the subgroup H corresponds to

$$T_a \vec{\Phi}_{\min} = \vec{0}, \quad a = 1, \ldots, n_H,$$

such that Eq. 2.27 is automatically satisfied without any knowledge of M^2.

2. T_a, $a = n_H + 1, \ldots, n_G$, is *not* a representation of an element of the Lie algebra belonging to the subgroup H. In that case $T_a \vec{\Phi}_{\min} \neq \vec{0}$, and $T_a \vec{\Phi}_{\min}$ is an eigenvector of M^2 with eigenvalue 0. To each such eigenvector corresponds a massless Goldstone boson. In particular, the different $T_a \vec{\Phi}_{\min} \neq \vec{0}$ are linearly independent, resulting in $n_G - n_H$ independent Goldstone bosons. (If they were not linearly independent, there would exist a nontrivial linear combination

$$\vec{0} = \sum_{a=n_H+1}^{n_G} c_a\left(T_a\vec{\Phi}_{\rm min}\right) = \underbrace{\left(\sum_{a=n_H+1}^{n_G} c_a T_a\right)}_{\equiv T}\vec{\Phi}_{\rm min},$$

such that T is an element of the Lie algebra of H in contradiction to our assumption.)

Remark It may be necessary to perform a similarity transformation on the fields in order to diagonalize the mass matrix.

Let us check these results by reconsidering the example of Eq. 2.11. In that case $n_G = 3$ and $n_H = 1$, generating two Goldstone bosons (see Eq. 2.19).

We conclude this section with two remarks.

1. The number of Goldstone bosons is determined by the structure of the symmetry groups. Let G denote the symmetry group of the Lagrangian with n_G generators, and H the subgroup with n_H generators which leaves the ground state invariant after spontaneous symmetry breaking. For each generator which does not annihilate the vacuum one obtains a massless Goldstone boson, i.e., the total number of Goldstone bosons equals $n_G - n_H$.
2. The Lagrangians used in *motivating* the phenomenon of a spontaneous symmetry breakdown are typically constructed in such a fashion that the degeneracy of the ground states is built into the potential at the classical level (the prototype being the "Mexican hat" potential of Fig. 2.5). As in the above case, it is then argued that an *elementary* Hermitian field of a multiplet transforming nontrivially under the symmetry group G acquires a vacuum expectation value signaling a spontaneous symmetry breakdown. However, there also exist theories such as QCD where one cannot infer from inspection of the Lagrangian whether the theory exhibits spontaneous symmetry breaking. Rather, the criterion for spontaneous symmetry breaking is a nonvanishing vacuum expectation value of some Hermitian operator, not an elementary field, which is generated through the dynamics of the underlying theory. In particular, we will see that the quantities developing a vacuum expectation value may also be local Hermitian operators composed of more fundamental degrees of freedom of the theory. Such a possibility was already emphasized in the derivation of Goldstone's theorem in Ref. [10].

2.3 Goldstone Theorem

By means of the above example, we motivate another approach to Goldstone's theorem without delving into all the subtleties of a quantum field-theoretical approach (for further reading, see Sect. 2 of Ref. [3]). Given a Hamilton operator with a global symmetry group $G = \mathrm{SO}(3)$, let $\vec{\Phi}(x) = (\Phi_1(x), \Phi_2(x), \Phi_3(x))$ denote a triplet of local Hermitian operators transforming as a vector under G,

$$g \in G: \vec{\Phi}(x) \mapsto \vec{\Phi}'(x) = e^{i\sum_{k=1}^{3}\alpha_k Q_k}\vec{\Phi}(x)e^{-i\sum_{l=1}^{3}\alpha_l Q_l} = e^{-i\sum_{k=1}^{3}\alpha_k T_k}\vec{\Phi}(x), \quad (2.28)$$

where the Q_i are the generators of the SO(3) transformations on the Hilbert space satisfying $[Q_i, Q_j] = i\varepsilon_{ijk}Q_k$ and the $T_i = (t_{i,jk})$ are the matrices of the three-dimensional representation satisfying $t_{i,jk} = -i\varepsilon_{ijk}$. We assume that one component of the multiplet acquires a nonvanishing vacuum expectation value:

$$\langle 0|\Phi_1(x)|0\rangle = \langle 0|\Phi_2(x)|0\rangle = 0, \quad \langle 0|\Phi_3(x)|0\rangle = v \neq 0. \quad (2.29)$$

Then the two generators Q_1 and Q_2 do not annihilate the ground state, and to each such generator corresponds a massless Goldstone boson.

In order to prove these two statements, let us expand Eq. 2.28 to first order in the α_k :

$$\vec{\Phi}' = \vec{\Phi} + i\sum_{k=1}^{3}\alpha_k[Q_k, \vec{\Phi}] = \left(1 - i\sum_{k=1}^{3}\alpha_k T_k\right)\vec{\Phi} = \vec{\Phi} + \vec{\alpha}\times\vec{\Phi}.$$

Comparing the terms linear in the α_k,

$$i[\alpha_k Q_k, \Phi_l] = \varepsilon_{lkm}\alpha_k\Phi_m,$$

and noting that all three α_k can be chosen independently, we obtain

$$i[Q_k, \Phi_l] = -\varepsilon_{klm}\Phi_m,$$

which expresses the fact that the field operators Φ_i transform as a vector.[9] Using $\varepsilon_{klm}\varepsilon_{kln} = 2\delta_{mn}$, we find

$$-\frac{i}{2}\varepsilon_{kln}[Q_k, \Phi_l] = \delta_{mn}\Phi_m = \Phi_n.$$

In particular,

$$\Phi_3 = -\frac{i}{2}([Q_1, \Phi_2] - [Q_2, \Phi_1]), \quad (2.30)$$

with cyclic permutations for the other two cases.

In order to prove that Q_1 and Q_2 do not annihilate the ground state, let us consider Eq. 2.28 for $\vec{\alpha} = (0, \pi/2, 0)$,

$$e^{-i\frac{\pi}{2}T_2}\vec{\Phi} = \begin{pmatrix} \cos\left(\frac{\pi}{2}\right) & 0 & \sin\left(\frac{\pi}{2}\right) \\ 0 & 1 & 0 \\ -\sin\left(\frac{\pi}{2}\right) & 0 & \cos\left(\frac{\pi}{2}\right) \end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = \begin{pmatrix} \Phi_3 \\ \Phi_2 \\ -\Phi_1 \end{pmatrix} = e^{i\frac{\pi}{2}Q_2}\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix}e^{-i\frac{\pi}{2}Q_2}.$$

---

[9] Using the replacements $Q_k \to \hat{l}_k$ and $\Phi_l \to \hat{x}_l$, note the analogy with $i[\hat{l}_k, \hat{x}_l] = -\varepsilon_{klm}\hat{x}_m$.

From the first row we obtain

$$\Phi_3 = e^{i\frac{\pi}{2}Q_2}\Phi_1 e^{-i\frac{\pi}{2}Q_2}.$$

Taking the vacuum expectation value

$$v = \left\langle 0 \middle| e^{i\frac{\pi}{2}Q_2}\Phi_1 e^{-i\frac{\pi}{2}Q_2} \middle| 0 \right\rangle$$

and using Eq. 2.29, clearly $Q_2|0\rangle \neq 0$, since otherwise the exponential operator could be replaced by unity and the right-hand side would vanish. A similar argument shows $Q_1|0\rangle \neq 0$.

At this point let us make two remarks.

1. The "states" $Q_{1(2)}|0\rangle$ cannot be normalized. In a more rigorous derivation one makes use of integrals of the form

$$\int d^3x\, \langle 0|[J_k^0(t,\vec{x}), \Phi_l(0)]|0\rangle,$$

and first determines the commutator before evaluating the integral [3].

2. Some derivations of Goldstone's theorem right away start by assuming $Q_{1(2)}|0\rangle \neq 0$. However, for the discussion of spontaneous symmetry breaking in the framework of QCD it is advantageous to establish the connection between the existence of Goldstone bosons and a nonvanishing expectation value (see Sect. 3.2).

Let us now turn to the existence of Goldstone bosons, taking the vacuum expectation value of Eq. 2.30:

$$0 \neq v = \langle 0|\Phi_3(0)|0\rangle = -\frac{i}{2}\langle 0|([Q_1,\Phi_2(0)] - [Q_2,\Phi_1(0)])|0\rangle \equiv -\frac{i}{2}(A-B).$$

We will first show $A = -B$. To that end we perform a rotation of the fields as well as the generators by $\pi/2$ about the 3 axis [see Eq. 2.28 with $\vec{\alpha} = (0,0,\pi/2)$]:

$$e^{-i\frac{\pi}{2}T_3}\vec{\Phi} = \begin{pmatrix} -\Phi_2 \\ \Phi_1 \\ \Phi_3 \end{pmatrix} = e^{i\frac{\pi}{2}Q_3}\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} e^{-i\frac{\pi}{2}Q_3},$$

and analogously for the charge operators

$$\begin{pmatrix} -Q_2 \\ Q_1 \\ Q_3 \end{pmatrix} = e^{i\frac{\pi}{2}Q_3}\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} e^{-i\frac{\pi}{2}Q_3}.$$

We thus obtain

$$
\begin{aligned}
B = \langle 0|[Q_2, \Phi_1(0)]|0\rangle = \langle 0|\Big(& e^{i\frac{\pi}{2}Q_3}(-Q_1)\underbrace{e^{-i\frac{\pi}{2}Q_3}e^{i\frac{\pi}{2}Q_3}}_{=1}\Phi_2(0)e^{-i\frac{\pi}{2}Q_3} \\
& - e^{i\frac{\pi}{2}Q_3}\Phi_2(0)e^{-i\frac{\pi}{2}Q_3}e^{i\frac{\pi}{2}Q_3}(-Q_1)e^{-i\frac{\pi}{2}Q_3}\Big)|0\rangle \\
&= -\langle 0|[Q_1, \Phi_2(0)]|0\rangle = -A,
\end{aligned}
$$

where we made use of $Q_3|0\rangle = 0$, i.e., the vacuum is invariant under rotations about the 3 axis. In other words, the nonvanishing vacuum expectation value v can also be written as

$$
0 \neq v = \langle 0|\Phi_3(0)|0\rangle = -i\langle 0|[Q_1, \Phi_2(0)]|0\rangle = -i\int d^3x\, \langle 0|[J_1^0(t,\vec{x}), \Phi_2(0)]|0\rangle. \tag{2.31}
$$

We insert a complete set of states $1 = \sum\!\!\!\!\!\!\int_n |n\rangle\langle n|$ into the commutator[10]

$$
v = -i\sum\!\!\!\!\!\!\int_n \int d^3x\, \left(\langle 0|J_1^0(t,\vec{x})|n\rangle\langle n|\Phi_2(0)|0\rangle - \langle 0|\Phi_2(0)|n\rangle\langle n|J_1^0(t,\vec{x})|0\rangle\right),
$$

and make use of translational invariance

$$
\begin{aligned}
&= -i\sum\!\!\!\!\!\!\int_n \int d^3x\big(e^{-iP_n\cdot x}\langle 0|J_1^0(0)|n\rangle\langle n|\Phi_2(0)|0\rangle - \cdots\big) \\
&= -i\sum\!\!\!\!\!\!\int_n (2\pi)^3\delta^3(\vec{P}_n)\big(e^{-iE_nt}\langle 0|J_1^0(0)|n\rangle\langle n|\Phi_2(0)|0\rangle - e^{iE_nt}\langle 0|\Phi_2(0)|n\rangle\langle n|J_1^0(0)|0\rangle\big).
\end{aligned}
$$

Integration with respect to the momentum of the inserted intermediate states yields an expression of the form

$$
= -i(2\pi)^3\sum_n{}' \left(e^{-iE_nt}\cdots - e^{iE_nt}\cdots\right),
$$

where the prime indicates that only states with $\vec{p} = \vec{0}$ need to be considered. Due to the Hermiticity of the symmetry current operators J_k^μ as well as the Φ_l, we have

$$
c_n \equiv \langle 0|J_1^0(0)|n\rangle\langle n|\Phi_2(0)|0\rangle = \langle n|J_1^0(0)|0\rangle^*\langle 0|\Phi_2(0)|n\rangle^*,
$$

such that

$$
v = -i(2\pi)^3\sum_n{}' \left(c_n e^{-iE_nt} - c_n^* e^{iE_nt}\right). \tag{2.32}
$$

---

[10] The abbreviation $\sum\!\!\!\!\!\!\int_n |n\rangle\langle n|$ includes an integral over the total momentum $\vec{p}$ as well as all other quantum numbers necessary to fully specify the states.

From Eq. 2.32 we draw the following conclusions.

1. Due to our assumption of a nonvanishing vacuum expectation value v, there must exist states $|n\rangle$ for which both $\langle 0|J^0_{1(2)}(0)|n\rangle$ and $\langle n|\Phi_{1(2)}(0)|0\rangle$ do not vanish. The vacuum itself cannot contribute to Eq. 2.32 because $\langle 0|\Phi_{1(2)}(0)|0\rangle = 0$.
2. States with $E_n > 0$ contribute (φ_n is the phase of c_n)
$$\frac{1}{i}\left(c_n e^{-iE_n t} - c_n^* e^{iE_n t}\right) = \frac{1}{i}|c_n|\left(e^{i\varphi_n} e^{-iE_n t} - e^{-i\varphi_n} e^{iE_n t}\right) = 2|c_n| \sin(\varphi_n - E_n t)$$
to the sum. However, v is time independent and therefore the sum over states with $(E, \vec{p}) = (E_n > 0, \vec{0})$ must vanish.
3. The right-hand side of Eq. 2.32 must therefore contain the contribution from states with zero energy as well as zero momentum thus zero mass. These zero-mass states are the Goldstone bosons.

2.4 Explicit Symmetry Breaking: A First Look

Finally, let us illustrate the consequences of adding a small perturbation to our Lagrangian of Eq. 2.11 which *explicitly* breaks the symmetry. To that end, we modify the potential of Eq. 2.11 by adding a term $a\Phi_3$,

$$\mathscr{V}(\Phi_1, \Phi_2, \Phi_3) = \frac{m^2}{2}\Phi_i\Phi_i + \frac{\lambda}{4}(\Phi_i\Phi_i)^2 + a\Phi_3, \tag{2.33}$$

where $m^2 < 0$, $\lambda > 0$, and $a > 0$, with Hermitian fields Φ_i. Clearly, the potential no longer has the original O(3) symmetry but is only invariant under O(2). The conditions for the new minimum, obtained from $\vec{\nabla}_\Phi \mathscr{V} = 0$, read

$$\Phi_1 = \Phi_2 = 0, \quad \lambda\Phi_3^3 + m^2\Phi_3 + a = 0.$$

Exercise 2.4 Solve the cubic equation for Φ_3 using the perturbative ansatz

$$\langle\Phi_3\rangle = \Phi_3^{(0)} + a\Phi_3^{(1)} + O(a^2). \tag{2.34}$$

The solution reads

$$\Phi_3^{(0)} = \pm\sqrt{-\frac{m^2}{\lambda}}, \quad \Phi_3^{(1)} = \frac{1}{2m^2}.$$

As expected, $\Phi_3^{(0)}$ corresponds to our result without explicit perturbation. The condition for a *minimum* (see Eq. 2.23) excludes $\Phi_3^{(0)} = +\sqrt{-\frac{m^2}{\lambda}}$. Expanding the potential with $\Phi_3 = \langle\Phi_3\rangle + \eta$ we obtain, after a short calculation, for the masses

$$
\begin{aligned}
m^2_{\Phi_1} = m^2_{\Phi_2} = a\sqrt{\frac{\lambda}{-m^2}},\\
m^2_\eta = -2m^2 + 3a\sqrt{\frac{\lambda}{-m^2}}.
\end{aligned}
\tag{2.35}
$$

The important feature here is that the original Goldstone bosons of Eq. 2.19 are now massive. The squared masses are proportional to the symmetry breaking parameter a. Calculating *quantum* corrections to observables in terms of Goldstone-boson loop diagrams will generate corrections which are nonanalytic in the symmetry breaking parameter such as $a\ln(a)$ [12]. Such so-called chiral logarithms originate from the mass terms in the Goldstone-boson propagators entering the calculation of loop integrals. We will come back to this point in the next chapter when we discuss the masses of the pseudoscalar octet in terms of the quark masses which, in QCD, represent the analogue to the parameter a in the above example.

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