Chapter 2
Basics of the Dimensional Analysis

2.1 Preliminary Remarks

In this introductory chapter some basic ideas of the dimensional analysis are outlined using a number of the instructive examples. They illustrate the applications of the Pi-theorem in the field of hydrodynamics and heat and mass transfer.

The systems of units and dimensional and dimensionless quantities, as well as the principle of dimensional homogeneity are discussed in Sect. 2.2. Section 2.3 deals with non-dimensionalization of the mass and momentum balance equations, as well as the energy and diffusion equations. In Sect. 2.4 the dimensionless groups characteristic of hydrodynamic and heat and mass transfer phenomena are presented. Here the physical meaning of several dimensionless groups and similarity criteria is discussed, In addition, similitude and modeling characteristic of the experimental investigations of thermohydrodynamic processes are considered. The Pi-theorem is formulated in Sect. 2.5.

2.2 Basic Definitions

2.2.1 Dimensional and Dimensionless Parameters

Momentum, heat and mass transfer in continuous media occur in processes characterized by the interaction and coupling of the effects of hydrodynamic and thermal nature. The intensity of these interactions and coupling is determined by the magnitudes of physical quantities involved which characterize the physical properties of the medium, its state, motion and interactions with the surrounding boundaries and penetrating fields. The magnitudes of these quantities are determined experimentally by comparing the readings of the measuring devices with some chosen scales, which are taken as units of the measured characteristics,

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e.g. length, mass, time, etc. For example, an actual pipe diameter, fluid velocity or temperature are expressed as

$$d = nL_*, \; v = mV_*, \; T = kT_* \tag{2.1}$$

where n, m and k are some numbers, whereas L_*, V_* and T_* are units of length, velocity and temperature, respectively.

The quantities which characterize flow and heat and mass transfer of fluids are related to each other by certain expressions based on the laws of nature. For example, the volumetric flow rate Q_v of viscous fluid through a round pipe of radius r, and the drag force F_d acting on a small spherical particle slowly moving with constant velocity in viscous fluid are expressed by the Poiseuille and Stokes laws

$$Q_v = \frac{\pi r^4 \Delta P}{8\mu l} \tag{2.2}$$

$$F_d = 6\pi\mu u r \tag{2.3}$$

In (2.2) and (2.3) ΔP is the pressure drop on a length l, μ is the fluid viscosity, and u is the particle velocity. Equations 2.2 and 2.3 show that units of the volumetric flow rate Q_v and drag force F_d can be expressed as some combinations of the units of length, velocity, viscosity and pressure drop. In particular, the unit of r coincides with the unit of length L, of u is expressed through the units of length and time as LT^{-1}, the unit of $[\mu] = L^{-1}MT^{-1}$ in addition involves the unit of mass, as well as the unit of the pressure drop $[\Delta P] = L^{-1}MT^{-2}$ (cf. Table 2.1). Here and hereinafter symbol $[A]$ denotes units of a dimensional quantity A.

It is emphasized that the units of numerous physical quantities can be expressed via a few fundamental units. For example, we have just seen that the units of volumetric flow rate and drag force are expressed via units of length, mass and time only, as $[Q_v] = L^3T^{-1}$, *and* $[F_d] = LMT^{-2}$. A detailed information the units of measurable quantities is available in the book by Ipsen (1960). The possibility to express units of any physical quantities as a combination of some fundamental units allows subdividing all physical quantities into two characteristic groups, namely (1) primary or fundamental quantities, and (2) derivative (secondary or dependent) ones. The set of the fundamental units of measurements that is sufficient for expressing the other measurement quantities of a certain class of phenomena is called the system of units. Historically, different systems of units were applied to physical phenomena (Table 2.2).

In the present book we will use mainly the International System of Units (Table 2.3).

In this system of units (hereinafter called SI Units) an amount of a substance is measured with a special unit- mole (mol). Also, two additional dimensionless units: one for a plane angle- radian (rad), and another one for a solid angle- steradian (sr), are used. A detailed description of the SI Units can be found in the books of Blackman (1969) and Ramaswamy and Rao (1971).

Table 2.1 Physical quantities

Quantity	Dimensions	Derived units
A. (Mechanical quantities)		
Acceleration	LT^{-2}	$m.s^{-2}$
Action	ML^2T^{-1}	$kg.m^2_.s^{-1}$
Angle (plane)	1	$rad.$
Angle (solid)	1	$sterad.$
Angular acceleration	T^{-2}	$rad.s^{-2}$
Angular momentum	ML^2T^{-1}	$kg.m^2_.s^{-1}$
Area	L^2	m^2
Curvature	L^{-1}	m^{-1}
Surface tension	MT^{-2}	$kg.s^{-2}$
Density	ML^{-3}	$kg{\bullet}m^{-3}$
Elastic modulus	$ML^{-1}T^{-2}$	$kg.m^{-1}_.s^{-2}$
Energy (work)	ML^2T^2	J
Force	MLT^{-2}	N
Frequency	T^{-1}	s^{-1}
Kinematic viscosity	L^2T^{-1}	$m^2_.s^{-1}$
Mass	M	kg
Momentum	MLT^{-1}	$kg.m.s^{-1}$
Power	ML^2T^{-3}	W
Pressure	$ML^{-1}T^{-2}$	$N.m^{-2}$
Time	T	s
Velocity	LT^{-1}	$m.s^{-1}$
Volume	L^3	m^3
B. (Thermal quantities)		
Enthalpy	ML^2T^2	J
Entropy	$ML^2T^2\theta^{-1}$	$J.K^{-1}$
Gas constant	$L^2T^{-1}\theta^{-1}$	$J.kg.^{-1}K^{-1}$
Heat capacity per unit mass	$L^2T^{-2}\theta^{-1}$	$J.kg^{-1}_.K^{-1}$
Heat capacity per unit volume	$ML^{-1}T^{-2}\theta^{-1}$	$J.m.^{-3}K^{-1}$
Internal energy	ML^2T^2	J
Latent heat of phase change	L^2T^{-2}	$J.kg^{-1}$
Quantity of heat	ML^2T^{-2}	J
Temperature	θ	K
Temperature gradient	$L^{-1}\theta$	$K.m^{-1}$
Thermal conductivity	$MT^{-3}L\theta^{-1}$	$W.m^{-1}_.K^{-1}$
Thermal diffusivity	L^2T^{-1}	$m^2_.s^{-1}$
Heat transfer coefficient	$MT^3\theta^{-1}$	$W.m^2_.K^{-1}$

The numerical values of the physical quantities expressed through fundamental units depend on the scales of arbitrarily chosen for the latter in any given system of units. For example, the velocity magnitude of a solid body moving in fluid, which is 1 m/s in SI units is 100 cm/s in the Gaussian CGS (centimeter, gram, second) System of Units. The physical quantities whose numerical values depend on the

Table 2.2 Systems of units

	Absolute			Technical		
Quantity	CGS	MKS	FPS	CGS	MKS	FPS
Mass	Gram	Kilogram	Pound	9.81 g	9.81 kg	Slug
Force	Dyne	Newton	Poundal	Gram-force	Kilogram-force	Pound-force
Length	Centimeter	Meter	Foot	Santimeter	Meter	Foot
Time	Second	Second	Second	Second	Second	Second

Table 2.3 International system of units-SI

Quantity	Units	Abbreviation
Mass	Kilogram	kg
Length	Meter	m
Time	Second	s
Temperature	Kelvin	K
Electric current	Ampere	A
Luminous intensity	Candela	cd

fundamental units are called dimensional. For such quantities, units are derivative and are expressed through the fundamental unites according to the physical expressions involved. For example, units of the gravity force $F_g = mg$ are expressed through the fundamental units bearing in mind the previous expression and the fact that $[m] = M$, and $[g] = LT^{-2}$ as

$$[F_g] = LMT^{-2} \tag{2.4}$$

In fact, units of any physical quantity can be expressed through a power law[1]

$$[A] = L^{\alpha_1} M^{\alpha_2} T^{\alpha_3} \tag{2.5}$$

where the exponents α_i are found by using the principle of dimensional homogeneity.

The quantities whose numerical values are independent of the chosen units of measurements are called dimensionless. For example, the relative length of a pipe $\bar{l} = \frac{l}{d}$ (where l and d are the length and diameter of the pipe, respectively) is dimensionless. Formally this means that $[\bar{l}] = 1$.

In the general case, physical quantities can be characterized by their magnitude and direction. Such quantities as, for example, temperature and concentration are scalar and are characterized only by their magnitudes, whereas such quantities as velocity and force are vectors and are characterized by their magnitudes and directions. Vectors can also be characterized by introducing a so-called vector length $\boldsymbol{L}$ (Williams 1892). Projections of the vector length $\boldsymbol{L}$ on, say, the axes of

[1] A demonstration of this statement can be found in Sedov (1993).

a Cartesian coordinate system x, y and z are denoted as $L_{x,}$ L_y and L_z, respectively. A number of instructive examples of application of vector length for studying different problems of applied mechanics are presented in the monographs by Huntley (1967) and Douglas (1969). The application of the idea of vector length in studying of drag and heat transfer at a flat plate subjected to a uniform flow of the incompressible fluid is discussed by Barenblatt (1996) and Madrid and Alhama (2005).

The expansion of a number of the fundamental units allows a significant improvement of the results of the dimensional analysis. For this aim it is useful to consider different properties the mass: (1) mass as the quantity of matter M_μ, and (2) mass as the quantity of the inertia M_i. Similarly, using projections of a vector $\boldsymbol{L}$ on the Cartesian coordinate axes as the fundamental units it is possible to express the units of such derivative (secondary) quantities as volume V and velocity vector $\mathbf{v}$ as $[V] = L_x L_y L_z$ and $[u] = L_x T^{-1}$, $[v] = L_y T^{-1}$, and $[\mathrm{w}] = \mathrm{L_z T^{-1}}$ where u,v and w denote the projections of $\boldsymbol{v}$ on the coordinate axes as is traditionally done in fluid mechanics. It is emphasized that using two different quantities of mass and projections of a vector allows one to reveal more clearly the physical meaning of the corresponding quantities. For example, the dimensions of work W in a rectilinear motion and torque T in rotation system of units LMT are the same L^2MT^{-2}, whereas in the system of units$L_x L_y L_z MT$ they are different, namely $[W] = L_x^2 MT^{-2}$, whereas $[\mathrm{T}] = \mathrm{L_x L_y MT^{-2}}$.

2.2.2 *The Principle of Dimensional Homogeneity*

Principle of dimensional homogeneity expresses the key requirements to a structure of any meaningful algebraic and differential equations describing physical phenomena, namely: all terms of these equations must to have the same dimensions. To illustrate this principle, we consider first the expression for the drag force acting on a spherical particle slowly moving in highly viscous fluid. The Stokes formula describing F_d reads

$$F_d = 6\pi\mu u r \tag{2.6}$$

Here $[F_d] = LMT^{-2}$ is the drag force, $[\mu] = L^{-1}MT^{-1}$ is the viscosity of the fluid, $[u] = LT^{-1}$ and $[r] = L$ are the particle velocity and its radius, respectively. It is easy to see that (2.6) satisfies the principle dimensional homogeneity. Indeed, substitution of the corresponding dimensions to the left hand side and the right hand side of (2.6) results in the following identity

$$LMT^{-2} = (L^{-1}MT^{-1})(LT^{-1})(L) = LMT^{-2} \tag{2.7}$$

As a second example, we consider the Navier–Stokes and continuity equations. For flows of incompressible fluids they read

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{v} \tag{2.8}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{2.9}$$

where $\mathbf{v} = [LT^{-1}]$ is the velocity vector, $[\rho] = L^{-3}M$,$[\nu] = L^2T^{-1}$ and $[P] = L^{-1}MT^{-2}$ are the density, kinematic viscosity ν and pressure, respectively.

It is seen that all the terms in (2.8) have dimensions LT^{-2} and in (2.9) have dimensions T^{-1}.

There are a number of important applications of the principle of the dimensional homogeneity. For example, it can be used for correcting errors in formulas or equations, which is advisable to students. Take the expression for the volumetric rate of incompressible fluid through a round pipe of radius r as

$$Q_v = \frac{\pi r^2}{8\mu}\left(\frac{\Delta P}{l}\right) \tag{2.10}$$

where Q_v is the volumetric flow rate, ΔP is the pressure drop over an arbitrary section of the pipe length of length l.

The dimension of the term on the left hand side in (2.10) is L^3T^{-1}, whereas of the one on the right hand side of this equation is LT^{-1}. Thus, (2.10) does not satisfy the principle of dimensional homogeneity. In order to find the correct form of the dependence of the volumetric flow rate on the governing parameters, we present (2.10) as follows

$$Q_v = \frac{\pi}{8} r^{\alpha_1}\mu^{\alpha_2}\left(\frac{\Delta P}{l}\right)^{\alpha_3} \tag{2.11}$$

where α_i are unknown exponents.

Bearing in mind the dimensions of Q_v, r, μ and $\left(\frac{\Delta P}{l}\right)$, we arrive at the following system of algebraical equations for the exponents α_i

$$\begin{aligned} \alpha_1 - \alpha_2 - 2\alpha_3 &= 3 \\ \alpha_2 + \alpha_3 &= 0 \\ -\alpha_2 - 2\alpha_3 &= -1 \end{aligned} \tag{2.12}$$

From (2.12) it follows that the exponents α_i are equal $\alpha_1 = 4$, $\alpha_2 = -1$, *and* $\alpha_3 = 1$. Then, the correct form of (2.10) reads as

$$Q_v = \frac{\pi r^4}{8\mu}\left(\frac{\Delta P}{l}\right) \tag{2.13}$$

The third example concerns the application the principle of dimensional homogeneity to determine the dimensionless groups from a set of dimensional parameters. Consider a set of dimensional parameters

$$a_1, a_2 \cdots a_k, a_{k+1} \cdots a_n \tag{2.14}$$

Assume that k parameters have independent dimensions. Accordingly, the dimensions of the other $n - k$ parameters can be expressed as

$$\begin{aligned}
[a_{k+1}] &= [a_1]^{\alpha_1'} \cdot \cdots \cdot [a_k]^{\alpha_k'} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
[a_n] &= [a_1]^{\alpha_1^{n-k}} \cdots [a_k]^{\alpha_k^{n-k}}
\end{aligned} \tag{2.15}$$

Therefore, the ratios

$$\begin{aligned}
\frac{a_{k+1}}{a_1^{\alpha_1'} \cdots a_k^{\alpha_k'}} &= \Pi_1 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\frac{a_n}{a_1^{n-k} \cdots a_k^{n-k}} &= \Pi_{n-k}
\end{aligned} \tag{2.16}$$

are dimensionless. Requiring that the dimensions of the numerator and denominator in the ratios (2.16) will be the same, we arrive at the system of algebraical equations for the unknown exponents.

In conclusion, we give one more instructive example of the application of the principle of dimensional homogeneity for the description of the equation of state of perfect gas. The general form of the equation of state reads (Kestin v.1 (1966) and v.2 (1968)):

$$F(P, v_s, T) = 0 \tag{2.17}$$

where P, $v_s$and T are the pressure, specific volume and temperature, respectively.

Equation 2.17 can be solved (at least in principle), with respect to any one of the three variables involved. In particular, it can be written as

$$P = f(v_s, T) \tag{2.18}$$

The set of the governing parameters involved in (2.18) is incomplete since the dimension of pressure $[P] = L^{-1}MT^{-2}$ cannot be expressed in the form of any combination of dimensions of specific volume $[v_s] = L^3M^{-1}$ and temperature $[T] = \theta$. Therefore, the function f on right hand side in (2.18) must include some dimensional constant c

$$P = f(c, v_s, T) \tag{2.19}$$

It is reasonable to choose as such a constant the gas constant R that account for the physical nature of the gas, but does not depend on its specific volume, pressure and temperature. Assuming that $c = R/\gamma$ (γ is a dimensionless constant), we write the dimension of this constant as $[c] = L^2T^{-2}\theta^{-1}$. All the parameters in (2.19) have independent dimensions. Then, according to the Pi-theorem (see Sect. 2.5), (2.19) takes the form

$$P = \gamma_1 c^{\alpha_1} v_s^{\alpha_2} T^{\alpha_3} \tag{2.20}$$

where γ_1 is a dimensionless constant.

Using the principle of the dimensional homogeneity, we find the values of the exponents α_i as $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 1$. Assuming $\gamma = \gamma_1$, we arrive at the Clapeyron equation

$$P = R\rho T \tag{2.21}$$

The equation of state of perfect gas can be also derived directly by applying the Pi-theorem to solve the problems of the kinetic theory and accounting for the fact pressure of perfect gas results from atom (molecule) impacts onto a solid wall.[2] Considering perfect gas as an ensemble of rigid spherical atoms (or molecules) moving chaotically in the space, we can assume that pressure of such gas is determined by atom (or molecule) mass m, their number per unit volume N and the average velocity squared $<v^2>$

$$P = f(m, N, <v^2>) \tag{2.22}$$

The dimensions of P and the governing parameters m, N and $<v^2>$ are

$$[P] = L^{-1}MT^{-2},\ [m] = M,\ [N] = L^{-3},\ \left[<v^2>\right] = L^2T^{-2} \tag{2.23}$$

All the governing parameters have independent dimensions. Therefore, the difference between the number of the governing parameters n and the number of the parameters with independent dimensions k equals zero. In this case the pressure can be expressed as Sedov (1993);

$$P = \gamma m^{\alpha_1} N^{\alpha_2} <v^2>^{\alpha_3} \tag{2.24}$$

where γ is a dimensionless constant.

[2] This idea was expressed first by D. Bernoulli in 1727 who wrote that pressure of perfect gas is related to molecule velocities squared.

Using the principle of dimensional homogeneity, we find the values of the exponents in (2.24) as $\alpha_1 = \alpha_2 = \alpha_3 = 1$. Then, (2.24) takes the form

$$P = \gamma m N \langle v^2 \rangle \tag{2.25}$$

Bearing in mind that $m\langle v^2\rangle$ is directly proportional $k_B T$ ($m\langle v^2\rangle = \gamma_1 k_B T$, *where* γ_1 is a dimensionless constant), we arrive at the following equation

$$P = \varepsilon k_B T N \tag{2.26}$$

Here $\varepsilon = \gamma\gamma_1$ is a dimensionless constant, $[k_B] = L^2 M T^{-2} \theta^{-1}$ is Boltzmann's constant, $[T] = \theta$ is the absolute temperature.

Applying (2.26) to a unit mole of a perfect gas, we can write the known thermodynamic relations as

$$N = N_\mu,\ k_B = \frac{\mu R}{N_\mu},\ \mu v_s = \text{constant} \tag{2.27}$$

Here N_μ is the Avogadro number, μ is the molecular mass, v_s is the specific volume, and $[R] = L^2 T^{-2} \theta^{-1}$ is the gas constant. Then, (2.27) takes the form

$$P = \rho R T \tag{2.28}$$

Summarizing, we see that the pressure of perfect gas is directly proportional to the product of the gas density, gas constant and the absolute temperature and does not depend on the mass of individual atoms (molecules). Note that (2.28) can be obtained directly from the functional equation $P = f(m, N, T, k_B)$(Bridgman 1922).

2.3 Non-Dimensionalization of the Governing Equations

It is beneficial in the analysis complex thermohydrodynamic phenomena to transform the system of mass, momentum, energy and species balance equations into a dimensionless form. The motivation for such transformation comes from two reasons. The first reason is related with the generalization of the results of theoretical and experimental investigations of hydrodynamics and heat and mass transfer in laminar and turbulent flows by presentation the data of numerical calculation and measurements in the form of dependences between dimensionless parameters. The second reason is related to the problem of modeling thermohydrodynamic processes by using similarity criteria that determine the actual conditions of the problem. The procedure of non-dimensionalization of the continuity (mass balance), momentum, energy and species balance equations is illustrated below by transforming the following model equation

$$\sum_{j=1}^{n} A_j^{(i)} = 0 \tag{2.29}$$

where $A_j^{(i)}$ includes differential operators, some independent variables, as well as constants; superscript i refers to the momentum $(i = 1)$, energy $(i = 2)$, species $(i = 3)$ and continuity $(i = 4)$ equations, n is the total number of terms in a given equation.

The terms in (2.29) account for different factors that affect the velocity, temperature and species fields: the inertia features of fluid, viscous friction, conductive and convective heat transfer, etc. These terms are dimensional. The dimension of $A_j^{(i)}$ in the system of units $LMT\theta$ is

$$\left[A_j^{(i)}\right] = L^{\alpha_j^{(i)}} M^{\beta_j^{(i)}} T^{\gamma_j^{(i)}} \theta^{\varepsilon_j^{(i)}} \tag{2.30}$$

where the values of the exponents $\alpha,\ \beta,\ \gamma$ and ε are determined by the magnitude of i and $j;$ all the terms that correspond to a given i have the same dimension:

$$\left[A_1^{(i)}\right] = \left[A_2^{(i)}\right] = \cdots \left[A_j^{(i)}\right] = \cdots \left[A_n^{(i)}\right] \tag{2.31}$$

The variables and constants included in (2.29) may be rendered dimensionless by using some characteristic scales of the density $[\rho_*] = L^{-3}M,$ velocity $[v_*] = LT^{-1},$ length $[l_*] = L,$ time $[t_*] = T$, etc. Then, the dimensionless variables and constants of the problem are expressed as

$$\overline{\rho} = \frac{\rho}{\rho_*}, \overline{v} = \frac{v}{v_*}, \overline{T} = \frac{T}{T_*}, \overline{c} = \frac{c}{c_*}, \overline{t} = \frac{t}{t_*}, \overline{P} = \frac{P}{P_*}, \overline{\mu} = \frac{\mu}{\mu_*},\ \overline{k} = \frac{k}{k_*}, \overline{D} = \frac{D}{D_*}, \overline{g} = \frac{g}{g_*} \tag{2.32}$$

where the asterisks denote the characteristic scales, and the dimensionless parameters are denoted by bars. In addition, $k_* = \left[LMT^{-3}\theta^{-1}\right], D_* = [L^2T^{-1}],$ *and* $g_* = [LT^{-2}]$ are the characteristic scales of thermal conductivity, diffusivity and gravity acceleration, respectively.

Taking into account (2.32), we can present all terms of (2.29) as follows

$$A_j^{(i)} = A_{j*}^{(i)} \overline{A}_j^{(i)} \tag{2.33}$$

where $A_{j*}^{(i)}$ is the corresponding dimensional multiplier comprised of the characteristic scales, $\overline{A}_j^{(i)} = A_j^{(i)}/A_{j*}^{(i)}$ is the dimensionless form of the jth term in (2.29). The exact form of the multipliers $A_{j*}^{(i)}$ is determined by the actual structure of the terms $A_j^{(i)}$. For example, the multiplier of the first term of the momentum balance equation is found from

$$A_1^{(i)} = \rho \frac{\partial v}{\partial t} = \frac{\rho_* v_*}{t_*} \frac{\partial (v/v_*)}{\partial (t/t_*)} = A_{1*}^{(i)} \overline{A}_1^{(i)} \tag{2.34}$$

where $A_{1*}^{(i)} = \dfrac{\rho_* v_*}{t_*}$, $\overline{A}_1^{(i)} = \dfrac{\partial \overline{v}}{\partial \overline{t}}$.

The substitution of the expression (2.33) into (2.29) yields

$$\sum_{j=1}^{n} A_{j*}^{(i)} \overline{A}_j^{(i)} = 0 \tag{2.35}$$

Dividing the left and right hand sides of (2.35) by a multiplier $A_{k*}^{(i)}$ $(1 \le k \le n)$, we arrive at the dimensionless form of the conservation equations

$$\overline{A}_k^{(i)} + \left\{ \sum_{j=1}^{k-1} \prod_{j*}^{(i)} \overline{A}_j^{(i)} + \sum_{j=k+1}^{n} \prod_{j*}^{(i)} \overline{A}_j^{(i)} \right\} = 0 \tag{2.36}$$

where $\prod_{j*}^{(i)} = A_{j*}/A_{k*}$ are the dimensionless groups.

To illustrate the general approach described above, we render dimensionless the Navier–Stokes equations, the energy and species balance equations, as well as the continuity equation. For incompressible fluids these equations read

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \tag{2.37}$$

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_P (\mathbf{v} \cdot \nabla) T = k \nabla^2 T + \phi \tag{2.38}$$

$$\rho \frac{\partial c_\xi}{\partial t} + \rho (\mathbf{v} \cdot \nabla) c_\xi = \rho D \nabla^2 c_\xi \tag{2.39}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{2.40}$$

where ρ, $\mathbf{v}$ T, P and c_ξ are the density, velocity vector, the temperature, pressure and the concentration of the species ξ. In particular, let us use the Cartesian coordinate system where vector $\mathbf{v}$ has components u, v and w in projections to the x, y and z axes. In addition, μ, k and D are the viscosity, thermal conductivity and diffusivity which are assumed to be constant, g the magnitude of the gravity acceleration $\mathbf{g}$, ϕ is the dissipation function $\phi = 2\mu\left[(\partial u/\partial x)^2 + (\partial v/\partial y)^2 + (\partial w/\partial z)^2\right] + \mu(\partial u/\partial y + \partial v/\partial x)^2 + \mu(\partial v/\partial z + \partial w/\partial y)^2 + \mu(\partial w/\partial x + \partial u/\partial z)^2$.

The multipliers $A_{j*}^{(i)}$ in (2.37)–(2.40) are listed below

$$A_{1*}^{(1)} = \frac{\rho_* v_*}{t_*}, A_{2*}^{(1)} = \frac{\rho_* v_*^2}{l_*}, A_{3*}^{(1)} = \frac{P_*}{l_*}, A_{4*}^{(1)} = \rho_* g_* \tag{2.41}$$

$$A_{1*}^{(2)} = \frac{\rho_* c_{P*} T_*}{t_*},\ A_{2*}^{(2)} = \frac{\rho_* c_{P*} v_* T_*}{l_*}, A_{3*}^{(2)} = \frac{k_* T_*}{l_*}, A_{4*}^{(2)} = \frac{\mu_* v_*^2}{l_*}$$

$$A_{1*}^{(3)} = \frac{\rho_* c_*}{t_*}, A_{2*}^{(3)} = \frac{\rho_* c_*}{l_*}, A_{3*}^{(3)} = \frac{\rho_* D_* c_*}{l_*^2}$$

$$A_{1*}^{(4)} = \frac{v_*}{l_*}, A_{2*}^{(4)} = \frac{v_*}{l_*}$$

Dividing the multipliers $A_{j*}^{(1)}$ by $A_{2*}^{(1)}$, $A_{j*}^{(2)}$ by $A_{2*}^{(2)}$, $A_{j*}^{(3)}$ by $A_{2*}^{(3)}$ and $A_{j*}^{(4)}$ by $A_{2*}^{(4)}$, we arrive at the following system of dimensionless equations

$$St\frac{\partial \overline{\mathbf{v}}}{\partial \overline{t}} + (\overline{\mathbf{v}} \cdot \nabla)\overline{\mathbf{v}} = -Eu\nabla\overline{P} + \frac{1}{\mathrm{Re}}\nabla^2\overline{\mathbf{v}} + \frac{1}{Fr} \tag{2.42}$$

$$St\frac{\partial \overline{T}}{\partial \overline{t}} + (\overline{\mathbf{v}} \cdot \nabla)\overline{T} = \frac{1}{Pe}\nabla^2\overline{T} + \frac{Br}{\mathrm{Re}}\overline{\phi} \tag{2.43}$$

$$St\frac{\partial \overline{c_\xi}}{\partial \overline{t}} + (\overline{\mathbf{v}} \cdot \nabla)\overline{c_\xi} = \frac{1}{Pe_d}\nabla^2\overline{c_\xi} \tag{2.44}$$

$$\nabla \cdot \overline{\mathbf{v}} = 0 \tag{2.45}$$

where $St = l_*/v_* t_*$, $Eu = P_*/\rho_* v_*^2$, $\mathrm{Re} = v_* l_*/\nu_*$, $Pe = v_* l_*/\alpha_*$, $Pe_d = v_* l_*/D_*$, $Fr = v_*^2/g_* l_*$, $Br = \mu_* v_*^2/k_* T_*$ are the Strouhal, Euler and Reynolds numbers, as well as the thermal and diffusion Peclet numbers, and the Froude and Brinkman numbers, respectively, ν and α are the kinematic viscosity and thermal diffusivity, and the dimensionless dissipation function $\overline{\phi} = \phi/\left[\mu(v_*/l_*)^2\right], \overline{v} = v/v_*, \overline{P} = P/\rho v_*^2, \overline{T} = T/T_*$ and $\overline{c}_\xi = c/c_*$ are the dimensionless variables.

The non-dimensionalization of the initial and boundary conditions is similar to the one described above. In that case each of the independent variables $x,\ y,\ z$ and t, as well as the flow characteristics $u,\ v,\ T$ and c_ξ are also rendered dimensionless by using some scales that have the same dimensions as the corresponding parameters. For example, consider the non-dimensionalization of the initial and boundary conditions for the following three problems of the theory of viscous fluid flows: (1) steady flow in laminar boundary layer over a flat plate, (2) laminar flow about a flat plate which instantaneous started to move in parallel to itself, and (3) submerged laminar jet issued from a round nozzle.

In case (1), let the velocity and temperature of the undisturbed fluid far enough from the plate be u_∞, T_∞, and the wall temperature be $T_w = const$. Then, the boundary conditions read

$$x = 0,\ 0 \le y \le \infty,\ u = u_\infty,\ T = T_\infty \tag{2.46}$$

$$x > 0,\ y = 0,\ u = v = 0,\ T = T_w;\ y \to \infty,\ u \to u_\infty,\ T \to T_\infty$$

Introducing as the scales of length some L, velocity u_∞ and temperature $T_w - T_\infty$, we rearrange (2.46) to the following dimensionless form[3]

$$\bar{x} = 0,\ 0 \le \bar{y} \le \infty \quad \bar{u} = 1,\ \Delta\overline{T} = 1 \tag{2.47}$$

$$\bar{x} > 0,\ \bar{y} = 0\ \ \bar{u} = \bar{v} = 0,\ \Delta\overline{T} = 0;\ \bar{y} \to \infty\ \ \bar{u} \to 1,\ \Delta\overline{T} \to 1$$

where $\bar{x} = x/L,\ \bar{y} = y/L,\ \bar{u} = u/u_\infty, \bar{v} = v/u_\infty,\ \Delta\overline{T} = (T_w - T)/(T_w - T_\infty)$.

The equation for the heat flux at the wall is used to introduce the heat transfer coefficient h:

$$h(T_w - T_\infty) = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{2.48}$$

Being rendered dimensionless, the heat transfer coefficient is expressed in the following form

$$Nu = \left(\frac{\partial \Delta\overline{T}}{\partial \bar{y}}\right)_{\bar{y}=0} \tag{2.49}$$

where $Nu = hL/k$ is the dimensionless heat transfer coefficient is called the Nusselt number.

In case (2), the initial and boundary conditions of the problem on a plate starting to move from rest with velocity U in the x-direction in contact with the viscous fluid read

$$t = 0,\ \ 0 \le y \le \infty\ \ u = 0 \tag{2.50}$$

$$t > 0, \quad y = 0 \qquad u = U;\ y = \infty,\ u = 0$$

Since no time or length scales are given, we use as the characteristic time scale $t_* = \nu/U^2$ and as the characteristic length scale ν/U. Then, (2.50) take the following dimensionless form

$$\bar{t} = 0,\ 0 \le \bar{y} \le \infty\ \ \bar{u} = 0;\ \bar{t} > 0,\ \bar{y} = 0\ \bar{u} = 1,\ \bar{y} \to \infty\ \bar{u} \to 0 \tag{2.51}$$

In case (3), the boundary conditions for a submerged laminar jet are

[3] It is emphasized that in the problem on flow in the boundary layer over a semi-infinite plate, a given characteristic scale L is absent. According to the self-similar Blasius solution of this problem, the dimensionless coordinate $\bar{y} = y/(\nu x/u_\infty)^{1/2}$ with $(\nu x/u_\infty)^{1/2}$ playing the role of the length scale (Sedov 1993).

$$x = 0,\ 0 \le y \le r_0,\ u = u_0,\ T = T_0;\ \ y > r_0\ u = 0,\ T = T_\infty \tag{2.52}$$

$$x > 0,\ y = 0,\ \frac{\partial u}{\partial y} = 0,\ \frac{\partial T}{\partial y} = 0;\ y \to \infty,\ u \to 0,\ T \to T_\infty$$

where $r_0$is the nozzle radius.

The dimensionless form of the conditions (2.52) is

$$\bar{x} = 0,\ 0 \le \bar{y} \le 1,\ \bar{u} = 1\ \Delta\overline{T} = 1;\ \bar{y} > \infty,\ \bar{u} \to 0,\ \Delta\overline{T} \to 0 \tag{2.53}$$

$$\bar{x} > 0,\ \bar{y} = 0,\ \frac{\partial \bar{u}}{\partial \bar{y}} = 0,\ \frac{\partial \Delta\overline{T}}{\partial \bar{y}} = 0;\ \bar{y} = \infty,\ \bar{u} \to 0,\ \Delta\overline{T} \to 0$$

where $\bar{x} = x/r_0,\ \bar{y} = y/r_0,\ \bar{u} = u/u_0,\ \Delta\overline{T} = (T_\infty - T)/(T_\infty - T_0)$.

At large enough distance from the jet origin at $x/r_0 >> 1$, it is possible to use the integral condition $\int_0^\infty \bar{u}^2 \bar{y} d\bar{y} = const$, instead of the condition (2.52) at $x = 0$. Note that there is another way of rendering the system of fundamental equations of hydrodynamics and heat and mass transfer theory dimensionless. It consists in rendering dimensionless each quantity in these equations using for this aim the scales of the density, velocity, temperature, etc. Requiring that the convective terms of these equations do not contain any dimensional multipliers, it is not easy to arrive at the equations identical to (2.42)–(2.45). To illustrate this approach to non-dimensionalization of the mass, momentum, energy and species conservation equations, consider, for example, the system of equations describing flows of reactive gases

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2.54}$$

$$\rho \frac{\partial v}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P + \nabla \cdot (\mu \nabla \mathbf{v}) + \rho \mathbf{g} \tag{2.55}$$

$$\rho \frac{\partial h}{\partial t} + \rho(\mathbf{v} \cdot \nabla)h - \nabla \cdot (k \nabla T) = q W_k \tag{2.56}$$

$$\rho \frac{\partial c_k}{\partial t} + \rho(\mathbf{v} \cdot \nabla)c_k - \nabla \cdot (\rho D \nabla c_k) = -W_k \tag{2.57}$$

$$P = \frac{\gamma - 1}{\gamma} \rho h \tag{2.58}$$

where $\mathbf{v}$ is the velocity vector, $\rho,\ P,\ h$ and T are the density, pressure, enthalpy and temperature, $c_k = \rho_k/\rho$ is the relative concentration of the k^{th} species, $\rho = \Sigma \rho_k,\ with\ \rho_k$ being density of the k^{th} species, $W_k(c_k, T)\ and\ W$ are the chemical reaction rates, q is the heat of the overall reaction, and $\gamma = c_p/c_v$is the ratio of specific heat at constant pressure to the one at constant volume (the adiabatic index). Note that in the energy balance equation (2.56) the dissipation term is neglected.

Introducing dimensionless parameters as follows $\overline{a} = \frac{a}{a_*}$ (the asterisk denotes the scale of a parameter a), we arrive at the following equations

$$\frac{\rho_*}{t_*}\frac{\partial \overline{\rho}}{\partial \overline{t}} + \frac{\rho_* v_*}{L_*}\nabla \cdot (\overline{\rho \mathbf{v}}) = 0 \tag{2.59}$$

$$\frac{\rho_* v_*}{t_*}\frac{\partial \overline{v}}{\partial \overline{t}} + \frac{\rho_* v_*}{L_*}\overline{\rho}(\overline{\mathbf{v}} \cdot \nabla)\overline{\mathbf{v}} = -\frac{P_*}{L_*}\nabla \overline{P} + \frac{\mu_* v_*}{L_*^2}\nabla \cdot (\overline{\mu}\nabla \overline{\mathbf{v}}) + \rho_* g_* \overline{\rho}\overline{\mathbf{g}} \tag{2.60}$$

$$\frac{\rho_* h_*}{t_*}\overline{\rho}\frac{\partial \overline{h}}{\partial \overline{t}} + \frac{\rho_* v_* h_*}{L_*}\overline{\rho}(\overline{\mathbf{v}} \cdot \nabla)\overline{h} - \frac{k_* T_*}{L_*^2}\nabla \cdot (\overline{k}\nabla \overline{T}) = qW_{k.*}\overline{W}_k \tag{2.61}$$

$$\frac{\rho_*}{t_*}\overline{\rho}\frac{\partial c_k}{\partial \overline{t}} + \frac{\rho_* v_*}{L_*}\overline{\rho}(\overline{\mathbf{v}} \cdot \nabla)c_k - \frac{\rho_* D_*}{L_*^2}\nabla \cdot (\overline{\rho}\overline{D}\nabla c_k) = -W_{k.\cdot *}\overline{W}_k \tag{2.62}$$

$$\overline{P} = \frac{\gamma - 1}{\gamma}\frac{\rho_* h_*}{P_*}\overline{\rho}\overline{h} \tag{2.63}$$

where ρ_*, v_*, P_*, T_*, h_* and L_* are the scales of density, velocity, pressure, temperature, enthalpy and length, respectively.

Requiring that the second terms on left hand sides in (2.59)–(2.62) do not contain any dimensionless multipliers and also accounting for the fact that for perfect gas $\rho_* h_* / P_* = \gamma/(\gamma - 1)$, we obtain

$$St\frac{\partial \overline{\rho}}{\partial \overline{t}} + \nabla \cdot (\overline{\rho \mathbf{v}}) = 0 \tag{2.64}$$

$$St\frac{\partial \overline{v}}{\partial \overline{t}} + \overline{\rho}(\overline{\mathbf{v}} \cdot \nabla)\overline{\mathbf{v}} = -Eu\nabla \overline{P} + \frac{1}{\mathrm{Re}}\nabla \cdot (\overline{\mu}\nabla \overline{\mathbf{v}}) + \frac{1}{Fr}\overline{\rho}\overline{\mathbf{g}} \tag{2.65}$$

$$St\frac{\partial \overline{T}}{\partial \overline{t}} + \overline{\rho}(\overline{\mathbf{v}} \cdot \nabla)\overline{T} - \frac{1}{Pe}\nabla \cdot (\overline{k}\nabla \overline{T}) = Da_3 \overline{W}_k \tag{2.66}$$

$$St\frac{\partial c_k}{\partial \overline{t}} + \overline{\rho}(\overline{\mathbf{v}} \cdot \nabla)c_k - \frac{1}{Pe_d}\nabla \cdot (\overline{\rho}\overline{D}\nabla c_k) = Da_1 \overline{W}_k \tag{2.67}$$

$$\overline{P} = \overline{\rho}\overline{h} \tag{2.68}$$

where in addition to previously introduced Strouhal, Reynolds, Euler, the thermal and diffusion Peclet numbers, and the Froude number, two Damkohler numbers $Da_1 = W_{k.*}L_*/\rho_* v_*,\ and\ Da_3 = qW_{k.*}L_*/\rho_* v_* h_*$ (defined according to the Handbook of Chemistry and Physics,1968) appear.

2.4 Dimensionless Groups

2.4.1 Characteristics of Dimensionless Groups

As was shown in Sect. 2.3, the dimensionless momentum, energy and diffusion equations contain a number of dimensionless groups, which represent themselves some combinations of the physical properties of fluid, acting forces, heat fluxes, etc. The physical meaning and number of these groups is determined by a specific situation, as well as by a particular model used for description of the physical phenomena characteristic of that situation (Table 2.4).[4]

Consider in detail some particular dimensionless groups. The Prandtl, Schmidt and Lewis numbers belong to a subgroup of dimensional groups that incorporate only quantities that account for the physical properties of fluid. They are expressed as the following ratios (cf. Table 2.4)

$$\Pr = \frac{v}{a}, \quad Sc = \frac{v}{D}, \quad Le = \frac{a}{D} \tag{2.69}$$

where v, a and D are the kinematic viscosity, thermal diffusivity and diffusivity, respectively.

Consider, for example the Prandtl number. It represents itself the ratio of kinematic viscosity to thermal diffusivity, i.e. of the characteristics of fluid responsible for the intensity of momentum and heat transfer. Accordingly, the Prandtl number can be considered as a parameter that characterizes the ratio of the extent of propagation of the dynamic and thermal perturbations. Therefore, at very low Prandtl numbers (for example, in flows of liquid metals), the thickness of the thermal boundary layer δ_T is much larger than the thickness of the dynamical one, δ. In contrast, at $\Pr >> 1$ (in flows of oils) the equality $\delta >> \delta_T$ is valid. The Schmidt number is the diffusion analog of the Prandtl number. It determines the ratio of the thicknesses of the dynamical and diffusion boundary layers.

The Reynolds number belongs to the subgroup of the dimensionless groups which are ratios of the acting forces. It can be considered as the ratio of the inertia force $F_i$to the friction force F_f

[4] Dimensionless groups can be also found directly by transformation of the functional equations of a specific problem using the Pi-theorem (see Sect. 2.5). A detailed list of dimensionless groups related to flows of incompressible and compressible fluids in adiabatic and diabatic conditions, flows of non-Newtonian fluids and reactive mixtures can be found in Handbook of Chemistry and Physics, 68th Edition, 1987–1988, CBC Inc. Boca Roton, Florida, and in Chart of Dimensionless Numbers, OMEGA Technology Company. See also Lykov and Mikhailov (1963) and Kutateladze (1986).

Table 2.4 Dimensionless groups

Name	Symbol	Definition	Comparison ratio	Field of use
Archimedes number	Ar	$\frac{gL^3\rho}{\mu^2}(\rho - \rho_f)$	Gravity force to viscous force	Motion of fluid due to density differences (buoyancy)
Biot number	Bi	$\frac{hL}{k_s}$	Convection heat transfer to conduction heat transfer	Heat transfer
Bond number	Bo	$\frac{\rho g L^2}{\sigma}$	Gravitaty force to surface tension	Motion of drops and bubbles. Atomization
Brinkman number	Br	$\frac{\mu v^2}{k\Delta T}$	Heat dissipation to heat transferred	Viscous flows
Capillary number	Ca	$\frac{\mu v}{\sigma}$	Viscous force to surface tension force	Two-phase flow. Atomization. Moving contact lines
Damkohler number	Da_1 Da_3	$\frac{WL}{V_m}$ $\frac{qWL}{\rho v c_P \Delta T}$	Chemical reaction rate to bulk mass flow rate. Heat released to convected heat	Chemical reactions, momentum, and heat transfer
Darcy number	Da_2	$\frac{vL}{D_*}$	Inertia force to permeation force	Flow in porous media
Dean number	De	$\frac{vR\rho}{\mu}\sqrt{\frac{R}{r}}$	Centrifugal force to inertial force	Flow in curved channels and pipes
Deborah number	De	$\frac{\tau_r}{\tau_0}$	Relaxation time to the characteristic hydrodynamic time	Non-Newtonian hydrodynamics. Rheology
Eckert number	Ec	$\frac{v_\infty^2}{c_P \Delta T}$	Kinetic energy to thermal energy	Compressible flows
Ekman number	Ek	$\left(\frac{\mu}{2\rho\omega L^2}\right)^{1/2}$	(Viscous force to Coriolis force)$^{1/2}$	Rotating flows
Euler number	Eu	$\frac{\rho v^2}{\Delta P}$	Pressure drop to dynamic pressure	Fluid friction in conduits
Grashof number	Gr	$\frac{\rho^2 g\beta L^3 \Delta T}{\mu^2}$	Buoyancy force to viscous force	Natural convection
Jacob number	Ja	$\frac{c_P \rho_f \Delta T}{r\rho_V}$	Heat transfer to heat of evaporation	Boiling
Knudsen number	Kn	$\frac{\lambda}{L}$	Mean free path to characteristic dimension	Rarefied gas flows and flows in micro- and nano-capillaries
Kutateladze number	K	$\frac{r_v}{c_P \Delta T}$	Latent heat of phase change to convective heat transfer	Combined heat and mass transfer in evaporation
Lewis number	Le	$\frac{k}{\rho c_P D}$	Thermal diffusivity to diffusivity	Combined heat and mass transfer
Mach number	M	$\frac{v}{C}$	Flow speed to local speed of sound	Compressible flows
	Nu	$\frac{hL}{k}$		Forced convection

(continued)

Table 2.4 (continued)

Name	Symbol	Definition	Comparison ratio	Field of use
Nusselt number			Total heat transfer to conductive heat transfer	
Peclet number	Pe	$\frac{L\rho v c_p}{k}$	Bulk heat transfer to conductive heat transfer	Forced convection
Prandtl number	Pr	$\frac{\mu c_P}{k}$	Momentum diffusivity to thermal diffusivity	Heat transfer in fluid flows
Rayleigh number	Ra	$\frac{g\beta L^3\rho^2 c_P}{\mu k}$	Thermal expansion to thermal diffusivity and viscosity	Natural convection
Richardson number	Ri	$-\left(\frac{g}{\rho}\frac{\partial P}{\partial L_h}\right)\Big/\left(\frac{\partial v}{\partial L_h}\right)_w$	Gravity force to the inertia force	Stratified flow of multilayer systems
Rossby number	Ro	$\frac{v}{\omega L\sin\Lambda}$	The inertia force to Coriolis force	Geophysical flows. Effect of earth's rotation on flow in pipes
Schmidt number	Sc	$\frac{\mu}{\rho D}$	Kinematic viscosity to molecular diffusivity	Diffusion in flow
Senenov number	Se	$\frac{h_m}{K}$	Intensity of heat transfer to intensity of chemical reaction	Reaction kinetics. Convective heat transfer.
Sherwood number	Sh	$\frac{h_m L}{D}$	Mass diffusivity to molecular diffusitivy	Mass transfer
Stenton number	St	$\frac{h}{\rho v c_P}$	Heat transferred to thermal capacity of fluid	Forced convection
Strouhal number	St	$\frac{fL}{v}$	Time scale of flow to oscillation period	Unsteady flow. Vortex shedding
Taylor number	Ta	$\left(\frac{2\omega L^2\rho}{\mu}\right)^2$	(Coriolis force to viscous force)^2	Effect of rotation on natural convection
Weber number	We	$\frac{v^2\rho L}{\sigma}$	The dynamic pressure to capillary pressure	Bubble formation, drop impact

$$\mathrm{Re} = \frac{vL}{\nu} = \frac{\rho v^2}{\mu(v/L)} = \frac{\rho v^2/L}{\mu(v/L^2)} \tag{2.70}$$

where ρ, μ and L are the density, viscosity and the characteristic length.

The dimensions of the numerator and denominator in right hand side ratio in (2.70) are $[\rho v^2/L] = \left[\mu\left(v/L^2\right)\right] = L^{-2}MT^{-2}$, i.e. the same as the dimensions of the terms $\rho[\partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{v}]$ and $\mu\nabla^2\mathbf{v}$ accounting for the inertia and viscous forces in the momentum balance equation. The terms $\rho v^2/L$ and $\mu v/L^2$ can be treated as the specific inertia and viscous forces $f_i = F_i/V$ and $f_f = F_f/V$, respectively, with the dimensions $[F_i] = LMT^{-2}$, $\left[F_f\right] = LMT^{-2}$, and $[V] = L^3$.

At small Reynolds numbers when the influence of viscosity is dominant, any chance perturbations of the flow field decay very quickly. At large Re such perturbations increase and result in laminar-turbulent transition. Therefore, the

Reynolds number is sensitive indicator of flow regimes. For example, in flows of an incompressible fluid in a smooth pipe, three kinds of flow regime can be realized depending on the value of the Reynolds number: (1) laminar ($\mathrm{Re} \leq 2300$), transitional ($2300 \leq \mathrm{Re} \leq 3500$), and developed turbulent ($\mathrm{Re} > 3500$).

The Peclet number is an example of a dimensionless group that is a ratio of heat fluxes of different nature. It reads

$$Pe = \frac{vL}{a} = \frac{\rho v c_P \Delta T}{k\left(\frac{\Delta T}{L}\right)} \tag{2.71}$$

where k and c_P are the thermal conductivity and specific heat at constant pressure, ΔT is the characteristic temperature difference.

The Peclet number is the ratio of the heat flux due to convection to the heat flux due to conduction. It can be considered as a measure of the intensity of molar to molecular mechanisms of heat transfer.

We mention also the Damkohler number that characterize the conditions of chemical reaction which proceeds in a reactive mixture, i.e. in the process accompanied by consumption of the initial reactants, formation of the combustion products, as well as an intensive heat release. Under these conditions the evolution of the temperature and concentration fields is determined by two factors: (1) hydrodynamics of the flow of reacting mixture, and (2) the rate of chemical reaction. The contribution of each of these factors can be estimated by the ratio of the characteristic hydrodynamic time $\tau_h \sim W^{-1}$ to the chemical reaction time $\tau_r \sim V_v^{-1}$ i.e. by the Damkohler number

$$Da_1 = \frac{\tau_h}{\tau_r} \tag{2.72}$$

If the Damkohler number is much less than unity, the influence of the chemical reaction on the temperature (concentration) field is negligible. At large values of Da_1 the effect of the chemical reaction and its heat release is dominant.

2.4.2 Similarity

Before closing the brief comments on the dimensionless groups, we outline how such groups are used in modeling of hydrodynamic and thermal phenomena. For this aim, we turn back to (2.64)–(2.68) that describe the mass, momentum, heat and species transfer in flows of incompressible fluids with constant physical properties. These equations contain eight dimensionless groups, namely, St, Re, Pe, Pe_d, Eu, Fr, Da_1 and Da_3. If the initial and boundary conditions of a particular problem do not contain any additional dimensionless groups (as, for example, the conditions $y = 0 \quad \overline{v} = 0,\ \overline{T} = 0,\ c_k = 0, \quad y \to \infty \quad \overline{v} = 1,\ \overline{T} = 1,\ c_k = 1$), the velocity, temperature and concentration fields determined by (2.64)–(2.68) can be expressed as follows

$$\overline{\mathbf{v}} = f_v(\overline{x}, \overline{y}, \overline{z}, St, Re, Eu, Fr) \tag{2.73}$$

$$\overline{T} = f_t(\overline{x}, \overline{y}, \overline{z}, St, Pe, Da_1) \tag{2.74}$$

$$c_k = f_c(\overline{x}, \overline{y}, \overline{z}, St, Pe_d, Da_3) \tag{2.75}$$

In (2.73) and (2.75) $\overline{T} = (T - T_w)/(T_\infty - T_w)$, and $\mathrm{c_k} = (\mathrm{c_k} - \mathrm{c_{k,w}})/(\mathrm{c_{k,\infty}} - \mathrm{c_{k,w}})$; subscripts w, and ∞ correspond to the values at the wall and in undisturbed fluid.

The expressions (2.73)–(2.75) are universal in a sense that the fields of dimensionless velocity, temperature and concentration determined by these expressions do not depend on the absolute values of the characteristic scales. That means that in geometrically similar systems (for example, cylindrical pipes of different diameter) values of dimensionless velocity, temperature and concentration at any similar point (with $\overline{x}_1 = \overline{x}_2 = \cdots = \overline{x}_i$; $\overline{y}_1 = \overline{y}_2 = \cdots = \overline{y}_i$; $\overline{z}_1 = \overline{z}_2 = \cdots = \overline{z}_i$) are the same if the values of the corresponding dimensionless groups are the same. Thus, the necessary conditions of the dynamic and thermal similarity in geometrically similar systems consist in equality of dimensionless groups (similarity numbers) relevant for the compared systems, i.e.

$$\begin{aligned} &St = idem,\ Re = idem,\ Eu = idem,\ Fr = idem,\ Pe = idem, \\ &Pe_d = idem,\ Da_1 = idem,\ Da_3 = idem \end{aligned} \tag{2.76}$$

for a considered class of flows. It is emphasized that in geometrically similar systems the boundary conditions should also be identical in such comparisons.

The conditions (2.76) allow modeling the momentum, heat and mass transfer processes in nature and technical applications by using the results of the experiments with miniature geometrically similar models. Note that among the totality of similarity numbers it is possible to select a family of dimensionless groups that contain combinations of only scales of the considered flow family and the physical parameters of a medium involved in a situation under consideration. Such similarity numbers are called similarity criteria (Loitsyanskii 1966). A number of similarity criteria can be less than the number of similarity numbers. For example, hydraulic resistance of cylindrical pipes with fully developed incompressible viscous fluid flow with a given throughput is characterized by two similarly numbers, namely, the Reynolds and Euler numbers. The first of them $Re = v_0 d/\nu$ is the similarity criterion, since it contains known parameters: the average velocity of fluid v_0, its viscosity ν and pipe diameter d. In contrast, the Euler number is not a similarity criterion, since it contains an unknown pressure drop which has to be found by solving the problem or measured experimentally (Loitsyanskii 1966).

2.5 The Pi-Theorem

2.5.1 General Remarks

This whole book is devoted to the Buckingham Pi-theorem (1914), which is widely used in a number of important problems of modern physics and, in particular, mechanics. The proof of this theorem, as well as numerous instructive examples of its applications for the analysis of various scientific and technical problems are contained in the monographs by Bridgman (1922), Sedov (1993), Spurk (1992) and Barenblatt (1987). Referring the readers to these works, we restrict our consideration by applications of the Pi-theorem to problems of hydrodynamics and the heat and mass transfer only.

The study of thermohydrodynamical processes in continuous media consists in establishing the relations between some characteristic quantities corresponding to a particular phenomenon and different parameters accounting for the physical properties of the matter, its motion and interaction with the surrounding medium. Such relations can be expressed by the following functional equation

$$a = f(a_1, a_2 \cdots a_n) \tag{2.77}$$

where a is the unknown quantities (for example, velocity, temperature, heat or mass fluxes, etc.), $a_1,\ a_2, \cdots a_n$ are the governing parameters (the characteristics of an undisturbed fluid, physical constants, time and coordinates of a considered point).

Equation 2.77 indicates only the existence of some relation between the unknown quantities and the governing parameters. However, it does not express any particular form of such relation. There are two approaches to determine an exact form of a relation of the type of (2.77): one is experimental, and the other one theoretical. The first approach is based on generalization of the results of measurements of unknown quantities a while varying the values of the governing parameters $a_1,\ a_2, \cdots a_n$. The second, theoretical, approach relies on the analytical or numerical solutions of the mass, momentum, energy and species balance equations. In both cases the establishment of a particular exact form of (2.77) does not entail significant difficulties while studying the simplest one-dimensional problems when (2.77) takes the form $a = f(a_1)$. On the contrary, a comprehensive experimental and theoretical analysis of a multiparametric equation $a = f(a_1, a_2 \cdots a_n)$ is extremely complicated and often represents itself an insoluble problem. The latter can be illustrated by the problem on a drag force acting on a body moving with a constant velocity in an infinite bulk of incompressible viscous fluid. In this case the drag force F_d acting from the fluid to the body depends on four dimensional parameters, namely, the fluid density ρ and viscosity μ, a characteristic size of the body d, and its velocity v. Then, the functional equation (2.77) takes the form

$$F_d = f(\rho, \mu, d, v) \tag{2.78}$$

In order to find experimentally the drag force, it is necessary to put the body into a wind tunnel and measure the drag force at a given velocity by an aerodynamic scale. That is the experimental way of solving the problem under consideration but only for one point on the parametric plane drag force-velocity. To determine the dependence of the drag force on velocity within a certain range of velocity v, it is necessary to reiterate the measurement of F_d at N values of v to determine the dependence $F_d = f(v)$ within a range $[v_1,\ v_2]$ at fixed values of $\rho,\ \mu$ and d. If we want to find the dependence F_d on all four governing parameters, we have to perform N^4 measurement.[5] Therefore, if the number of data points forF_d at varying one governing parameter is $N = 10^2,$ the total number of measurements that one needs will be equal to 10^8! It is evident that such number of measurements is practically impossible to perform. Moreover, even if we have an experimental data bank with 10^8 measurement points, we cannot say anything about the behavior of the function $F_d = f(\rho, \mu, v, d)$ outside the studied range of the governing parameters. An analytical or numerical calculation of the dependence of drag force on density, viscosity, velocity and size of the body is also an extremely complicated problem in the general case (at the arbitrary values of $\rho,\ \mu,\ v,$ and d) due to the difficulties involved in integrating the system of nonlinear partial differential equations of hydrodynamics.

Essentially both approaches to study the dependence of drag force on density, viscosity, velocity and size of the body allow a significant simplification of the problem by using the Pi-theorem. The latter points at the way of transformation of the function of n dimensional variables into a function of m $(with\ m < n)$ dimensionless variables. As a matter of fact, the Pi-theorem suggests how many dimensionless variables are needed for describing a given problem containing n dimensional parameters.

The Pi-theorem can be stated as follows. Let some dimension physical quantities a depend on n dimensional parameters $a_1,\ a_2 \cdots a_n,$ where k of them have an independent dimension. Then the functional equation for the quantities a

$$a = f(a_1, a_2 \cdots a_k, a_{k+1} \cdots a_n) \tag{2.79}$$

can be reorganized to the form of the dimensionless equation

$$\Pi = \varphi(\Pi_1, \Pi_2 \cdots \Pi_{n-k}) \tag{2.80}$$

that contain $n - k$ dimensionless variables. The latter are expressed as

$$\Pi_1 = \frac{a_1}{a_1^{\alpha_1'} a_2^{\alpha_2'} \cdots a_k^{\alpha_k'}},\ \Pi_2 = \frac{a_2}{a_1^{\alpha_1''} a_2^{\alpha_2''} \cdots a_k^{\alpha_k''}} \cdots\ \Pi_{n-k} = \frac{a_n}{a_1^{\alpha_1^{n-k}} a_2^{\alpha_2^{n-k}} \cdots a_k^{\alpha_k^{n-k}}} \tag{2.81}$$

The dimensionless form of the unknown quantities a is

[5] With an equal number of data points for each one of the four governing parameters.

$$\Pi = \frac{a}{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k}} \tag{2.82}$$

To illustrate the application of the Pi-theorem to hydrodynamic problems, return to the drag force acting on a body moving in viscous fluid. The unknown quantities and governing parameters of the corresponding problem have the following dimensions

$$[F_d] = LMT^{-2}, \quad [\rho] = L^{-3}M, \quad [\mu] = L^{-1}MT^{-1}, \quad [d] = L, \quad [v] = LT^{-1} \tag{2.83}$$

Three from the four governing parameters of this problem have independent dimensions. That means that a dimension of any governing parameters in this case can be expressed as a combination of dimensions of the three others. The dimension of the unknown quantity is also expressed as a combination of the governing parameters having independent dimensions $[F_d] = LMT^{-2} = [\rho v^2 d^2] = [\mu^2/\rho] = [\mu v d]$.

In accordance with the Pi-theorem, (2.78) takes the form

$$\Pi = \varphi(\Pi_1) \tag{2.84}$$

where $\Pi = \frac{F_d}{\rho^{\alpha_1} v^{\alpha_2} d^{\alpha_3}}$, and $\Pi_1 = \frac{\mu}{\rho^{\alpha'_1} v^{\alpha'_2} d^{\alpha'_3}}$.

Taking into account the dimension of the drag forceF_d and governing parameters with independent dimension ρ, v and d and using the principle of the dimensional homogeneity, we find the values of the exponents α_i and α'_i

$$\alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = 2; \quad \alpha'_1 = 1, \quad \alpha'_2 = 1, \quad \alpha'_3 = 1 \tag{2.85}$$

Then (2.84) reads

$$C_d = \varphi(\mathrm{Re}) \tag{2.86}$$

where $C_d = F_d/\rho v^2 d^2$ is the drag coefficient, and $Re{=}\rho v d/\mu$ is the Reynolds number.

The exact form of the function $\varphi(Re)$ cannot be determined by means of the dimensional analysis. However, this fact does not diminish the importance of the obtained result. Indeed, the dependence of the drag coefficient on only one dimensionless group (the Reynolds number) allows generalization of the experimental data on drag related to motions of bodies of different sizes moving with different velocities in fluids with different densities and viscosities. All this data can be presented in a collapsed form of a single curve $C_d(Re)$. Moreover, in some limiting cases corresponding to motion with low velocities (the so-called, creeping flows with Re $<<$ 1) or high speeds when Re $>>$ 1, it is possible to determine the exact forms of the dependence of the drag coefficient on Re.

In particular, at $Re << 1$ the inertia effects become negligible. Returning to (2.78), we can assume that the drag force depends on fluid viscosity, body size and its velocity

$$F_d = f(\mu, v, d) \tag{2.87}$$

All the governing parameters in (2.87) have independent dimensions ($n - k = 0$). Therefore, in this case (2.87) reduces to

$$F_d = c\mu^{\alpha_1} v^{\alpha_2} d^{\alpha_3} \tag{2.88}$$

where c is a dimensionless constant and $\alpha_1 = 1, \;\; \alpha_2 = 1, \;\; \alpha_3 = 1.$

Substituting the values of the exponents α_1, α_2 and α_3 into (2.88) leads to the following expression for the drag coefficient

$$C_d = \frac{c}{\mathrm{Re}} \tag{2.89}$$

It is evident that to determine the dependence $C_d(Re)$ at $\mathrm{Re} << 1$ it is sufficient to perform only one measurement in order to establish the value of the constant c.

It is emphasized that the efficiency of using the Pi-theorem in studies of physical phenomena is determined by the value of the difference $n - k$, i.e. by the number of the governing dimensionless groups. In all cases (excluding $k = 0$) the transformation of the functional equation by the Pi-theorem allows one to decrease number of variables. The most interesting two cases correspond to the difference $n - k$ being either 0 or 1. In the first case the functional equation takes the form

$$a = ca_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \tag{2.90}$$

In the second one it becomes

$$\Pi = \varphi(\Pi_1) \tag{2.91}$$

where Π represents itself the dimensionless group corresponding to the unknown parameter. Decreasing the number of dimensionless variables in (2.91) to only one is equivalent to the transformation of partial differential equations into the ordinary ones. A number of examples of transformation of the functional equations similar to (2.77) to a dimensionless form, as well as transformations of partial differential equations into the ordinary ones is given in the following sections.

2.5.2 Choice of the Governing Parameters

The theoretical study of hydrodynamic and heat and mass transfer processes is based on the system of partial differential equations that include the mass, momentum, energy and species conservation balances. This system of equations is supplemented by an equation of state and correlations determining the physical properties of the medium. The exact and approximate solutions of hydrodynamic and heat transfer problems in the framework of the continuum approach yield comprehensive answers to different problems of the theory. In distinction the dimensional analysis of hydrodynamic and heat and mass transfer problems meets some difficulties that arise already at the first step of the investigation when choosing the governing parameter of the problem. They stem from certain vagueness in choosing the governing parameters beginning from a pure intuitive evaluation of the features of a phenomenon under consideration. In addition, such approach to choosing the governing parameters often involves a number of parameters whose influence will appear to be negligible at the end. The latter makes it difficult to foresee the results of the dimensional analysis from scratch in generalizing hydrodynamic and heat and mass transfer. In order to improve the procedure of choosing the governing parameters and simplify and the following analysis, it is possible to use the system of the mass, momentum, energy and species balance equations.

Let us illustrate such an approach by the following examples.[6] We begin with the drag force acting on a spherical particle moving with a constant velocity in an infinite bulk of viscous incompressible fluid. It is reasonable to assume that the force that acts on the particle depends on its size d, velocity v and physical properties of the fluid, namely its density ρ and viscosity μ. In this case the functional equation for the drag force F_d reads

$$F_d = f(\rho, \mu, d, v) \tag{2.92}$$

The dimensional analysis of (2.92) leads to the following transformation in the form of the drag coefficient

$$C_d = \varphi_*(\mathrm{Re}) \tag{2.93}$$

where $C_d = F_d/\rho v^2 d^2$ is the drag coefficient, and $\mathrm{Re} = vd/\nu$ is the Reynolds number.

It is emphasized that the function $\varphi_*(\mathrm{Re})$on the right hand side of (2.93) can be presented as $c\varphi(\mathrm{Re})$, where c is a dimensionless constant with its value being chosen according to the experimental data. For example, for creeple motion of a small spherical particle when the drag force is given by the Stokes law $F_d = 3\pi\mu vd$, constant $c = 8/\pi$. Then the expression for the drag coefficient takes the form $C_d = 24/\mathrm{Re}$.

To determine the exact form of the dependence (2.93), one needs to integrate the continuity and the Navier–Stokes equations subjected to the no-slip condition at the particle surface. In some limiting cases corresponding to special conditions of

[6] A detailed analysis of these problems see in Chaps. 4 and 7

particle motion (say, very slow or fast), it is possible to find the exact form of the function $\varphi_*(\mathrm{Re})$ in (2.93) using these equations only for determining the set of the governing parameters. For example, in the case of slow motion (creeping flows) the inertia terms on the left hand side of the Navier–Stokes equations $\rho(\mathbf{v}\cdot\nabla)\mathbf{v}$ is much less than the terms in on the right hand side of these equations. That allows one to omit the inertial term and thereby exclude density from the governing parameters. As a result, the functional equation for the drag force reduces to the form of (2.6). Such simplification of the problem formulation is a key element which allows establishing an exact form of the dependence of the drag force on viscosity, velocity and diameter of the particle as $F_d \approx \mu v d$ that coincide (up to a numerical factor) with the exact result (the Stokes force) derived from the Navier–Stokes equations.

In the second case corresponding to a rapid body motion (the case of a large Reynolds number) the dominant role belongs to the turbulent transfer. The average characteristics of fully developed turbulent flows are governed by the Reynolds equations (Hinze 1975; Loitsyanskii 1966)

$$\frac{\partial \overline{v}_i}{\partial t} + \overline{v}_j \frac{\partial \overline{v}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} + \nu\nabla^2\overline{v}_i + \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(-\rho\overline{v_i' v_j'}\right) \tag{2.94}$$

(bars over parameters denote the average values).

In high Reynolds number flows the term $\nu\nabla^2\overline{v}_i$ associated with the effect of the molecular momentum transfer through molecular viscosity mechanism can be omitted. Then, assuming steady state average turbulent flow, the drag force which does not depend on molecular viscosity and time is given by

$$F_d = f(\rho, \overline{v}, d) \tag{2.95}$$

Applying the Pi-theorem to (2.95), we can rearrange it to the following form

$$F_d \approx \rho\overline{v}^2 d^2 \tag{2.96}$$

which agrees with the Newton law for drag.

Another example of employing the conservation equations to facilitate the dimensional analysis of complicated hydrodynamic and heat transfer problems is related to mass transfer to a vertical reactive plate in contact with a liquid solution of a reactive species (a reagent) which is initially at rest. When the rate of a heterogeneous reaction at the plate surface is much larger than the rate of diffusion transport of the reagent toward the surface, its concentration there equals zero, whereas far from the surface it is equal c_∞. The gradient of the reagent concentration across the thickness of the diffusion boundary layer results in a non-uniform density field. That, in turn, triggers buoyancy force which results in liquid motion near the wall. It is reasonable to assume that the velocity and concentration of the reactive species in the dynamic and diffusion boundary layers are determined by four parameters ρ_∞, c_∞, ν, g and two independent variables x and y

$$u = f_u(\rho_\infty, c_\infty, \nu, g, x, y) \tag{2.97}$$

$$c = f_c(\rho_\infty, c_\infty, \nu, g, x, y) \tag{2.98}$$

where we consider for brevity only one component of the velocity vector u; ν and D are the kinematic viscosity and diffusivity, and $g$is the acceleration due to gravity.

The functional equations (2.97) and (2.98) contain six governing parameters. Three of them have independent dimensions. Choosing $\rho_\infty,\ \nu$ and g or $\rho_\infty,\ D$ and g as parameters with the independent dimensions, we transform (2.97) and (2.98) to the following form

$$\Pi_u = \varphi_u(\Pi_1, \Pi_2, \Pi_3) \tag{2.99}$$

$$\Pi_c = \varphi_c(\Pi_{1*}, \Pi_{2*}, \Pi_{3*}) \tag{2.100}$$

where $\Pi_u = u/(g\nu)^{1/3}$, $\Pi_1 = c_\infty/\rho_\infty$, $\Pi_2 = x/(\nu^2/g)^{1/3}$, $\Pi_3 = y/(\nu^2/g)^{1/3}$, and $\Pi_c = c/\rho_\infty$, $\Pi_{1*} = c_\infty/\rho_\infty$, $\Pi_{2*} = x/(D^2/g)^{1/3}$, $\Pi_{3*} = y/(D^2/g)^{1/3}$.

Equations (2.99) and (2.100) show that the dimensionless velocity and concentration of the reactive species are the function of three dimensionless groups, which makes the analysis of the problem under consideration difficult. Therefore, employ also the conservation equations. The momentum and species balance equations that describe flow in the boundary layer and mass transfer to the vertical reactive wall read (Levich 1962) (see Sect. 3.10)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g_* c_* \tag{2.101}$$

$$u\frac{\partial c_*}{\partial x} + v\frac{\partial c_*}{\partial y} = D\frac{\partial^2 c_*}{\partial y^2} \tag{2.102}$$

where $g_* = g(c_\infty/\rho)(\partial\rho/\partial c)_{c=c_\infty}$, $c_* = (c_\infty - c)/c_\infty$, and $\rho = \rho(\mathrm{c})$.

The boundary conditions for (2.101) and (2.102) are

$$u = v = 0 \quad c_* = 1 \text{ at } y = 0; \quad u = v = 0 \quad c_* = 0 \text{ at } y \to \infty \tag{2.103}$$

Equations (2.101) and (2.102) and the boundary conditions (2.103) contain four parameters that determine the local velocity and concentration fields

$$u = f_u(x, y, \nu, g_*) \tag{2.104}$$

$$c_* = f_c(x, y, D, g_*) \tag{2.105}$$

Applying the Pi-theorem to transform (2.104) and (2.105) to the dimensionless form, we obtain

$$\Pi_u = \psi_u(\Pi_1) \tag{2.106}$$

$$\Pi_c = \psi_c(\Pi_{1*}) \tag{2.107}$$

or equivalently,

$$u = (xg_*)^{1/2}\psi_u(\eta) \tag{2.108}$$

$$c_* = \psi_c(\eta\sqrt{Sc}) \tag{2.109}$$

where $\Pi_u = u/(xg_*)^{1/2}$, $\Pi_1 = y(g_*/x\nu^2)^{1/4}$, $\Pi_c = c_*$, $\Pi_{1*} = y(g_*/xD^2)^{1/4}$, $\eta = y(g_*/x\nu^2)^{1/4}$, and $Sc = \nu/D$ is the Schmidt number.

A number of instructive examples of application of the mass, momentum, energy and species conservation equations for dimensional analysis of the hydrodynamic and heat and mass transfer problems can be found in Chap. 7.

Problems

P.2.1. Transform the van der Waals equation (Kestin 1966; Jones and Hawkis 1986) to the dimensionless form. Show that such form is universal for any van der Waals gas if one uses the critical values of the pressure, volume and temperature as the characteristic scales.

In order to transform equation the van der Waals equation $(P + a/V^2)(V - b) = RT$(where a and b are constants) to dimensionless form, we present this equation as

$$(A_1 + A_2)(A_3 + A_4) = A_5 \tag{P.2.1}$$

where $A_1 = P$, $A_2 = a/V^2$, $A_3 = V$, $A_4 = -b$, $A_5 = RT$ with P, V and T being pressure, molar volume and temperature, respectively.

We can introduce some still undefined scales of pressure P_0, volume V_0 and temperature T_0 and write the expressions for scales of A_j as

$$A_{1*} = P_*,\ \ A_{2*} = \frac{a}{V_*^2},\ \ A_{3*} = V_*,\ \ A_{4*} = -b,\ \ A_{5*} = RT_* \tag{P.2.2}$$

Then (P.2.1) reduces to the form

$$(\overline{A_1} + \alpha\overline{A_2})(\beta\overline{A_3} + \gamma\overline{A}) = \varepsilon\overline{A}_5 \tag{P.2.3}$$

where $\overline{A_1} = A_1/A_{1*} = P/P$, $\overline{A_2} = A_2 \big/ A_{2*,} = (V/V_*)^{-2}$, $\overline{A_3} = A_3/A_{3*} = (V/V_*)$, $\overline{A_4} = A_4/A_{4*} = 1$ $\overline{A_5} = A_5/A_{5*} = T/T_*$ are the dimensionless variables, and $\alpha = A_{2*}/A_{1*} = a/\left(V_*^2 P_*\right)$, $\beta = A_{3*}/A_{1*} = V_*/P_*$, $\gamma = A_{4*}/A_{1*} = -b/P_*$, and $\varepsilon = A_{5*}/A_{1*} = RT_*/P_*$ are the dimensionless constants.

Equation (P.2.3) is the dimensionless van der Waals equation. For its further transformation one should define the characteristic scales of pressure, volume and temperature. For that purpose, take as the scales P_*, V_* and T_* the critical values of pressure, volume and temperature P_{cr}, V_{cr} and T_{cr}, respectively. Bearing in mind that the critical point is the inflection point where $(\partial P/\partial V)_T = 0$, and $(\partial^2 \mathrm{P}/\partial \mathrm{V}^2)_\mathrm{T} = 0$, we find

$$a = \frac{27}{64}\frac{R^2 T_{cr}^2}{P_{cr}}, \quad b = \frac{V_{cr}}{3} \tag{P.2.4}$$

$$P_{cr} = \frac{1}{27}\frac{a}{b^2}, \quad V_{cr} = 3b, \quad T_{cr} = \frac{8a}{27bR} \tag{P.2.5}$$

Using as the characteristic scales the critical values of pressure, temperature and specific volume, we transform (P.2.3) to the following final form

$$(\pi + \frac{3}{\omega^2})(3\omega - 1) = 8\tau \tag{P.2.6}$$

where $\pi = P/P_{cr}$, $\omega = V/V_{cr}$, and $\tau = T/T_{cr}$.

Equation (P.2.6) does not contain any constants accounting for the physical properties of any particular gas and, thus, is universal. It holds for any van der Waals gas.

P.2.2. (i) Transform the momentum and continuity equations for laminar flow of incompressible fluid over a plane plate in the boundary layer approximation to the dimensionless form using the *LMT* and $L_xL_yL_zMT$ systems of units. (ii) Show that the $L_xL_yL_zMT$ system of units cannot be used for transformation of the Navier–Stokes equations to the dimensionless form.

(i)-A: The *LMT*system of units. The boundary layer and continuity equations read

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{P.2.7}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + 0 \tag{P.2.8}$$

where the dimensions of u, v, ν, x and $y$are as follows

$$[u] = LT^{-1}, [v] = LT^{-1}, [\nu] = L^2T^{-1}, [x] = L, [y] = L \tag{P.2.9}$$

All the terms in (P.2.7) have the dimension LT^{-2}, whereas the dimension of the terms in (P.2.8) is T^{-1}. That shows that (P.2.7) and (P.2.8) can be transformed to the dimensionless form by using the multipliers $[N_1] = (LT^{-2})^{-1}$and $[N_2] = (T^{-1})^{-1}$, respectively. Introducing the scales of length L_*, velocity $V_*$and kinematic viscosity $\nu_* = \nu$, we write the expressions for the coefficients $A_{j*}^{(i)}$as follows

$$A_{1*}^{(1)} = A_{2*}^{(1)} = \frac{V_*^2}{L_*}, A_{3*}^{(1)} = \nu_* \frac{V_*}{L_*^2}; A_{1*}^{(2)} = A_{2*}^{(2)} = \frac{V_*}{L_*} \tag{P.2.10}$$

Bearing in mind the dimensions of $A_{j*}^{(i)}$, we express the multipliers N_1 and N_2 as

$$N_1 = \frac{1}{A_{1*}^{(1)}}, N_2 = \frac{1}{A_{1*}^{(2)}} \tag{P.2.11}$$

Then (P.2.7) and (P.2.8) reduce to the following form

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \frac{1}{\mathrm{Re}}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \tag{P.2.12}$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{P.2.13}$$

where $\bar{u} = u/V_*, \bar{v} = v/V_*, \bar{x} = x/L_*, \bar{y} = y/L_*,$ and $\mathrm{Re} = \mathrm{V_* L_*}/\nu_*$.

The coefficients $A_{j*}^{(i)}$ for the Navier–Stokes and continuity equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{P.2.14}$$

$$v\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{P.2.15}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{P.2.16}$$

are defined as follows

$$A_{1*}^{(1)} = A_{2*}^{(1)} = \frac{V_*^2}{L_*}, A_{3*}^{(1)} = \nu_* \frac{V_*}{L_*}; A_{1*}^{(2)} = A_{2*}^{(2)} = \frac{V_*^2}{L_*}, A_{3*}^{(2)} = \nu_* \frac{V_*}{L_*}; A_{1*}^{(3)} = A_{2*}^{(3)} = \frac{V_*}{L_*} \tag{P.2.17}$$

Using the multipliers $N_1 = 1/A_{1*}^{(1)}, N_2 = 1/A_{1*}^{(2)},$ and $\mathrm{N_3} = 1/\mathrm{A}_{1*}^{(3)}$, we reduce (P.2.14)–(P.2.16) to the following dimensionless form

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) \tag{P.2.18}$$

$$\bar{v}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} = \frac{1}{\mathrm{Re}}\left(\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right) \tag{P.2.19}$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{P.2.20}$$

(i)-B: The $L_xL_yL_zMT$ system of units. The dimensions of u, v, x and y are

$$[u] = L_xT^{-1}, [v] = L_yT^{-1}, [x] = L_x, [y] = L_y \tag{P.2.21}$$

where L_x and L_y are the scales of length in the x and $y$directions.

Introducing the characteristic scales of u, v, v, x and y as $[U_*] = L_xT^{-1}, [V_*] = L_yT^{-1}$, and $[v_*] = [v]$, we transform first of all the boundary layer and continuity equations (P.2.7) and (P.2.8). To this aim, we write the expressions for the coefficients $A_{j*}^{(i)}$ as

$$A_{1*}^{(1)} = \frac{U_*^2}{L_x},\; A_{2*}^{(1)} = \frac{U_*V_*}{L_y},\; A_{3*}^{(1)} = \frac{v_*U_*}{L_y^2},\; A_{1*}^{(2)} = \frac{U_*}{L_x},\; A_{2*}^{(2)} = \frac{V_*}{L_y} \tag{P.2.22}$$

Then (P.2.7) and (P.2.8) are transformed to

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \left(\frac{V_*L_x}{U_*L_y}\right)\bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \left(\frac{v_*L_x}{L_y^2U_*}\right)\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \tag{P.2.23}$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \left(\frac{V_*L_x}{U_*L_y}\right)\frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{P.2.24}$$

where $\bar{u} = u/U_*, \bar{v} = v/V_*, \bar{x} = x/L_x, \bar{y} = y/L_y$, and the multipliers before the second terms on left hand side of the boundary layer and continuity equations are dimensionless, i.e. $\left[V_*L_x/U_*L_y\right] = 1$.

In planar viscous flows in the x-direction with shear in the y-direction an important role is played by the shear component τ_{yx} of the stress tensor. The shear stress τ_{yx} can be presented as the ratio of the force F_{yx} to the surface area S_{zx} which have the following dimensions: $\left[F_{yx}\right] = ML_xT^{-2}$, and $[\mathrm{S_{zx}}] = \mathrm{L_zL_x}$. Then the dimension of the shear stress is $\left[\tau_{yx}\right] = ML_z^{-1}T^{-2}$. For viscous Newtonian fluids $\tau_{yx} = \mu du/dy$, where μ is the viscosity. Then, we find the dimension of the viscosity in the $L_xL_yL_zMT$ system of units as

$$[\mu] = \left[\tau_{yx}/(du/dy)\right] = L_x^{-1}L_yL_z^{-1}MT^{-1} \tag{P.2.25}$$

Bearing in mind that the dimension of density in the $L_xL_yL_zMT$ system of units is $[\rho] = L_x^{-1}L_y^{-1}L_z^{-1}M$, we determine the dimension of the kinematic viscosity ν as

$$[\nu] = [\mu/\rho] = L_y^2T^{-1} \tag{P.2.26}$$

Thus, the multiplier $\left(\nu_*L_x\big/L_y^2U_*\right)$ on the right hand side of (P.2.23) is dimensionless. It can be presented as Re_*^{-1}, where $\mathrm{Re}_* = \left(U_*L_y^2/\nu_*L_x\right)$ is a modified Reynolds number. Taking into account that the characteristic scales L_x, L_y, U_* and V_x are arbitrary, it is possible to assume that the ratio $\left(U_*L_y/V_*L_x\right) = 1$. Then, (P.2.23) and (P.2.24) take the following form

$$\overline{u}\frac{\partial\overline{u}}{\partial\overline{x}} + \overline{v}\frac{\partial\overline{u}}{\partial\overline{y}} = \frac{1}{\mathrm{Re}_*}\frac{\partial^2\overline{u}}{\partial\overline{y}^2} \tag{P.2.27}$$

$$\frac{\partial\overline{u}}{\partial\overline{x}} + \frac{\partial\overline{v}}{\partial\overline{y}}0 \tag{P.2.28}$$

The Navier–Stokes and continuity equations (P.2.14) and (P.2.16) can be presented as

$$A_{1*}^{(1)}\overline{A}_1^{(1)} + A_{2*}^{(1)}\overline{A}_2^{(1)} = A_{3*}^{(1)}\overline{A}_3^{(1)} + A_{4*}^{(1)}\overline{A}_4^{(1)} \tag{P.2.29}$$

$$A_{1*}^{(2)}\overline{A}_1^{(2)} + A_{2*}^{(2)}\overline{A}_2^{(2)} = A_{3*}^{(2)}\overline{A}_3^{(2)} + A_{4*}^{(2)}\overline{A}_4^{(2)} \tag{P.2.30}$$

$$A_{1*}^{(3)}\overline{A}_1^{(3)} + A_{2*}^{(3)}\overline{A}_2^{(3)} = 0 \tag{P.2.31}$$

where $A_{1*}^{(1)} = U_*^2/L_x$, $A_{2*}^{(1)} = U_*V_*/L_y$, $A_{3*}^{(1)} = \nu_*U_*/L_x^2$, $A_{4*}^{(1)} = \nu_*U_*/L_y^2$, $A_{1*}^{(2)} = U_*V/L_x$, $A_{2*}^{(2)} = V_*^2/L_y$, $A_{3*}^{(2)} = \nu_*V_*/L_x^2$, $A_{4*}^{(2)} = \nu_*V_*/L_y^2$, $A_{1*}^{(3)} = U_*/L_x, A_{2*}^{(3)} = V_*/L_y$, $\overline{A}_1^{(1)} = \overline{u}\partial\overline{u}/\partial\overline{x}$, $\overline{A}_2^{(1)} = \overline{v}\partial\overline{u}/\partial\overline{y}$, $\overline{A}_3^{(1)} = \partial^2\overline{u}/\partial\overline{x}^2$, $\overline{A}_4^{(1)} = \partial^2\overline{u}/\partial\overline{y}^2$, $\overline{A}_1^{(2)} = \overline{v}\partial\overline{u}/\partial\overline{x}$, $\overline{A}_2^{(2)} = \overline{v}\partial\overline{v}/\partial\overline{y}$, $\overline{A}_3^{(2)} = \partial^2\overline{v}/\partial\overline{x}^2, \overline{A}_4^{(2)} = \partial^2\overline{v}/\partial\overline{y}^2$, $\overline{A}_1^{(3)} = \partial\overline{u}/\partial\overline{x}$, *and* $\overline{A}_2^{(3)} = \partial\overline{v}/\partial\overline{y}$.

Then (P.2.18)–(P.2.20) take the form

$$\overline{u}\frac{\partial\overline{u}}{\partial\overline{x}} + \left(\frac{V_*L_x}{U_*L_y}\right)\overline{v}\frac{\partial\overline{u}}{\partial\overline{y}} = \left(\frac{\nu_*L_x}{U_*L_y^2}\right)\left\{\left(\frac{L_y}{L_x}\right)^2\frac{\partial^2\overline{u}}{\partial\overline{x}^2} + \frac{\partial^2\overline{u}}{\partial\overline{y}^2}\right\} \tag{P.2.32}$$

$$\overline{v}\frac{\partial\overline{u}}{\partial\overline{x}} + \left(\frac{V_*L_x}{U_*L_y}\right)\overline{v}\frac{\partial\overline{v}}{\partial\overline{y}} = \left(\frac{\nu_*L_x}{U_*L_y^2}\right)\left\{\left(\frac{L_y}{L_x}\right)^2\frac{\partial^2\overline{v}}{\partial\overline{x}^2} + \frac{\partial^2\overline{v}}{\partial\overline{y}^2}\right\} \tag{P.2.33}$$

$$\frac{\partial\overline{u}}{\partial\overline{x}} + \left(\frac{V_*L_x}{U_*L_y}\right)\frac{\partial\overline{v}}{\partial\overline{y}} = 0 \tag{P.2.34}$$

The system of Eqs. (P.2.32)–(P.2.34) can be written as

$$\overline{u}\frac{\partial\overline{u}}{\partial\overline{x}}+\overline{v}\frac{\partial\overline{u}}{\partial\overline{y}}=\frac{1}{\mathrm{Re}}\left\{\left(\frac{L_y}{L_x}\right)^2\frac{\partial^2\overline{u}}{\partial\overline{x}^2}+\frac{\partial^2\overline{u}}{\partial\overline{y}^2}\right\} \tag{P.2.35}$$

$$\overline{v}\frac{\partial\overline{u}}{\partial\overline{x}}+\overline{v}\frac{\partial\overline{v}}{\partial\overline{y}}=\frac{1}{\mathrm{Re}_*}\left\{\left(\frac{L_y}{L_x}\right)^2\frac{\partial^2\overline{v}}{\partial\overline{x}^2}+\frac{\partial^2\overline{u}}{\partial\overline{y}^2}\right\} \tag{P.2.36}$$

$$\frac{\partial\overline{u}}{\partial\overline{x}}+\frac{\partial\overline{v}}{\partial\overline{y}}=0 \tag{P.2.37}$$

if we account for the fact that the dimension of the kinematic viscosity $[v_*]=L_y^2T^{-1}$. However, even in this case (P.2.35) and (P.2.36) are not dimensionless, since the dimension of the ratio L_y/L_x is not 1. Moreover, (P.2.35) and (P.2.36) do not satisfy the principle of the dimensional homogeneity under any assumption on the dimension of the kinematic viscosity. The latter shows that applying the $L_xL_yL_zMT$ system of units to transformation of the Navier–Stokes is incorrect.

P.2.3. (Reynolds 1886) Determine the resistance force acting on each of two circular disks of radii R which approach each other along the joint axis of symmetry with a constant velocity u, while the gap between the disks and the surrounding space are filled with incompressible viscous fluid. The pressure in the surrounding fluid far from the disks is equal P_*.

The liquid flow in the gap is axisymmetric. Therefore, we use cylindrical coordinates z, r, φ with the origin at the center of the lower disk which is assumed to be motionless (z and r correspond to the vertical and radial directions, respectively). Consider the low velocity case when the inertial effects are negligible. The effect of the gravity force we also will neglected. Then, it is possible to assume that the pressure gradient $\Delta P/r$ ($\Delta P = P - P_*$) is determined by the speed of the upper disk u, liquid viscosity μ, the instantaneous height of the gap h, and the radial position r

$$\frac{\Delta P}{r}=f(u,\mu,h,r) \tag{P.2.38}$$

For analyzing the problem, we use two different systems of units with a single (L) and two (L_z, L_r) length scales. In the first case the dimensions of the pressure gradient and the governing parameters can be expressed as

$$\left[\frac{\Delta P}{r}\right]=L^{-2}MT^{-2},[u]=LT^{-1},[\mu]=L^{-1}MT^{-1},[h]=L,[r]=L \tag{P.2.39}$$

Three of the four governing parameters in (P.2.38) have independent dimensions. Choosing $u, \mu,$ and r as the parameters with the independent dimensions, we reduce (P.2.38) according to the Pi-theorem to the following form

$$\Pi = \varphi(\Pi_1) \tag{P.2.40}$$

where $\Pi = (\Delta P/r)/u^{\alpha_1}\mu^{\alpha_2}r^{\alpha_3}$ and $\Pi_1 = \mathrm{h}/\mathrm{u}^{\alpha_1'}\mu^{\alpha_2'}\mathrm{r}^{\alpha_3'}$.

Using the principle of the dimensional homogeneity, we find the values of the exponents α_i and α_i': $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = -2; \alpha_1' = 0, \alpha_2' = 0,$ and $\alpha_3' = 1$. Accordingly, we arrive at the following expression

$$\frac{\Delta P}{r} = u\mu r^{-2}\varphi\left(\frac{h}{r}\right) \tag{P.2.41}$$

The force acting at the disk is found as

$$F_d = 2\pi \int^R \Delta P r dr \tag{P.2.42}$$

Substituting the expression (P.2.41) into (P.2.42), we obtain

$$F_d = 2\pi u\mu R \int_0^1 \varphi\left(\frac{\varepsilon}{\xi}\right) d\xi \tag{P.2.43}$$

where $\varepsilon = h/R, \xi = r/R,$ and $\int_0^1 \varphi(\varepsilon/\xi)\mathrm{d}\xi = \psi(\varepsilon)$.

Equation P.2.43 shows that the resistance force acting on a disk is directly proportional to its velocity, the radius of the disk, viscosity of the liquid, as well as a function of the ratio of the gap to the disk radius.

Additionally we transform (P.2.38) using the system of units with the two length scales L_z and L_r in the z and r directions, respectively. First, we determine the dimensions of the governing parameters and pressure gradient. The dimensions of the velocity u, gap thickness h and r are

$$[u] = L_z T^{-1}, [h] = L_z, [r] = L_r \tag{P.2.44}$$

To determine the dimensions of viscosity μ and pressure gradient $\Delta P/r$, we take into account the fact that in flows of viscous fluids in a narrow gap the dominant role is played by the radial velocity component, since the axial one is typically much smaller, $v_z << v_r$. In this case the force acting in the r-direction is much larger than in the z-direction, so that its dimension is $[F_r] = ML_rT^{-2}$. Accordingly, the dimension of the shear stress $\tau_{zr} = F_r/S_{rr}$ ($[S_{rr}] = L_r^2$) is $[\tau_{zr}] = ML_r^{-1}T^{-2}$. For Newtonian viscous fluids $\tau_{zr} = \mu(dv_r/dz)$. As a result, we find the dimension of viscosity $[\mu] = ML_r^{-2}L_zT^{-1}$. The dimensions of pressure and its gradient are

$$[\Delta P] = \frac{F_r}{S_{rz}} = \frac{ML_rT^{-2}}{L_rL_z} = ML_z^{-1}T^{-2} \tag{P.2.45}$$

$$\left[\frac{\Delta P}{r}\right] = \frac{ML_z^{-1}T^{-2}}{L_r} = ML_z^{-1}L_r^{-1}T^{-2} \tag{P.2.46}$$

Thus, the dimensions of all the governing parameters are expressed in the system of units with two length scales are independent. Then, according to the Pi-theorem, (P.2.38) takes the form

$$\frac{\Delta P}{r} = cu^{\alpha_1}\mu^{\alpha_2}h^{\alpha_3}r^{\alpha_4} \tag{P.2.47}$$

where c is a dimensionless constant.

Determining the values of the exponents α_i using the principle of the dimensional homogeneity as $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = -3$ and $\alpha_4 = 1$, we obtain

$$\frac{\Delta P}{r} = cu\mu\frac{r}{h^3} \tag{P.2.48}$$

Then, the substitution of (P.2.48) into (P.2.42) yields

$$F_d = \frac{c}{2}\pi u\mu R\left(\frac{R}{h}\right)^3 \tag{P.2.49}$$

The exact solution of this problem reads (Landau and Lifshitz 1987)

$$F_d = \frac{3}{2}\pi\mu u R\left(\frac{R}{h}\right)^3 \tag{P.2.50}$$

The comparison of (P.2.49) and (P.2.50) shows that the exact solution and the result of the dimensional analysis agree up to a dimensionless numerical factor. At the same time, the dimensional analysis of the problem using the system of units with a single length scale yields a less informative result, since (P.2.43) contains an unknown function $\psi(h/R)$.

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