Chapter 2
Distribution Functions

2.1 Introduction

In the process of calculating uncertainty in measurements, we come across quite a few statistical terms such as random variable, independent event, distribution function and probability density function. In this chapter, we deal with random variable, distribution functions, and probability density functions, discrete and continuous functions. The normal (Gaussian) probability density function and its properties are discussed.

The word "density" used in probability density function is seldom used. In loose terms every cumulative distribution function or probability density function is called as distribution. It is only in context to other things that a distinction between cumulative and density function is made.

2.2 Random Variable

Random variable is a real number representing an outcome of any random experiment. For example, tossing of n unbiased coins and counting number of heads appearing therein, then any one of the possible outcomes of this experiment will represent a random variable. Any outcome ω will be a natural number like 0, 1, 2, $\ldots, n$.

If S is the sample space then

$$\omega \in S. \tag{2.1}$$

That is for each ω there exist $X(\omega)$ – the probability of happening belonging to a set of real numbers R.

For example, ω in the above example may be r number of heads, then $X(\omega)$ will be ${}^{n}C_{r}/2^{n}$ for n unbiased coins. The value of probability is obtained by using the binomial distribution.

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Similarly any observation indicated by an unbiased instrument is an outcome ω of a random experiment. Here, ω belongs to the set of observations S indicated by the instrument. Then $X(\omega)$ is also a random variable belonging to the set of real numbers. In case of measurement with unbiased instruments, $X(\omega)$ will belong to a normal distribution.

Formal definition of a random variable may be as follows:

For the sample space S associated with a random experiment, there exists a real-valued variable $X(\omega)$ belonging to a real space $(-\infty, +\infty)$.

This is an example of one-dimensional random variable. If the functional values are ordered pair then it is called as a two-dimensional random variable. In general for an n-dimensional random variable whose domain is S has $X(\omega)$ is the collection of n-tuples of real numbers (vectors in n-space).

2.3 Discrete and Continuous Variables

Discrete variable is that which takes only finite number of values in a given interval. Similarly continuous variable is that which takes infinite number of values in a given interval. The interval may be large or small. That is discrete variable will take only certain values in small steps.

2.4 Discrete Functions

2.4.1 Probability Distribution of a Random Variable

Probability distribution of a random variable is a function giving probability that a random variable takes any given value, which belongs to a given set of values.

2.4.2 Discrete Probability Function

In case of statistical data of discrete variables, each variable x_i will have a specific frequency f_i. It means that a particular variable x_i will occur f_i times.

If N is the total frequency then

$$\sum_{i=1}^{i=n} f_i = N. \tag{2.2}$$

Then f_i/N is the probability for the existence of the variable x_i.

2.5 Distribution Function

Distributions function is the sum of all probabilities of a random variable such that the random variable is less than the given value. In fact, it is the cumulative sum of all frequencies such that X is less than or equal to the given value.

2.5.1 Continuous Distribution Function

A function $F(x)$ giving, for every value of a random variable x, the probability that the random value of X be less than or equal to x is a continuous distribution function. It is expressed as

$$F(x) = \Pr(X \leq x). \tag{2.3}$$

The distribution function defined in this way is also called a cumulative distribution function. The word cumulative is seldom used before the cumulative continuous distribution function.

2.5.2 Discrete Distribution

In statistical data of discrete variable, the set of relative cumulative frequencies (cumulative frequency divided by total frequency) is a distribution function.

For example, if there are n independent variables $x_1, x_2, x_3, \ldots, x_n$ with $f_1, f_2, f_3, \ldots, f_n$ with

$$\sum_{i=1}^{i=n} f_i = N,$$

then f_i/N is the relative frequency and $\sum_{i=1}^{i=r} f_i/N$ is the relative cumulative frequency of x_i for all values of i. Such a set of relative cumulative frequencies is a discrete distribution function of $x_1, x_2, x_3, \ldots, x_n$. In this case also, the word cumulative is seldom used before the cumulative discrete distribution function.

2.6 Probability Density Function

For a continuous random variable X, the probability density function is the derivative (if it exists) of its distribution function $F(X)$ i.e.

$$f(x) = \mathrm{d}F(x)/\mathrm{d}x. \tag{2.4}$$

$f(x)\mathrm{d}x$ is the probability element such that the random variable X lies in between x and $x + \mathrm{d}x$.

Mathematically

$$f(x) = \Pr(x < X < x + \mathrm{d}x). \tag{2.5}$$

The integral or the sum of all the probabilities of a continuous variable taking every value in between $-\infty$ and $+\infty$ is a certainty; hence

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1. \tag{2.6}$$

2.6.1 Discrete Probability Function

If a discrete random variable X can take values $x_1, x_2, x_3, \ldots, x_n$, with probabilities $p_1, p_2, p_3, \ldots, p_n$, such that

$$p_1 + p_2 + p_3 + \cdots + p_n = 1 \tag{2.7}$$

and

$$p_i \geq 0 \quad \text{for all } i, \tag{2.8}$$

then these two sets constitute a discrete probability distribution.

A function P_r for each value of x_i of discrete random variable X, giving the probability p_i when the random variable takes the value x_i such that

$$p_i = \Pr(X = x_i) \tag{2.9}$$

is the probability function.

2.7 Discrete Probability Functions

2.7.1 Binomial Probability Distribution

Binomial distribution is one of the most important probability function used in practical applications. The applications range from sampling inspection to the failure of rocket engines.

Suppose that a series of n independent trials have been made, each of which can be a success with probability p or a failure with probability $(1 - p)$. The number of success, which is observed, will be any natural number between 0 and n.

An event with r successes necessarily means an event with r successes and $(n - r)$ failures. Such an event is denoted as $p^r(1-p)^{n-r}$, but r successes and $n-r$

failures may be arranged in ${}^nC_r$ ways, so the probability of the event $p^r(1-p)^{n-r}$ is ${}^nC_r p^r(1-p)^{n-r}$. If this probability is denoted as p_r then $p_1, p_2, p_3, \ldots, p_n$ are the respective probabilities of 1, 2, 3, …, n successes, giving

$$\begin{aligned} {}^nC_0(1-p)^n + {}^nC_1 p(1-p)^{n-1} &+ {}^nC_2 p^2(1-p)^{n-2} \\ &+ \cdots {}^nC_{n-1} p^{n-1}(1-p) \\ &+ {}^nC_n p^n \end{aligned} \tag{2.10}$$

Binomial probability distribution is applicable whenever a series of trials is made satisfying the following conditions:

Each trial has only two outcomes, which are mutually exclusive. One of the two outcomes is denoted as success then other is failure, for example head and tail in a coin, go and not go, and defective and non-defective in industrial production

1. Probability p of a success is constant in each trial. This also means that probability of failure $(1-p)$ is also constant.
2. The outcomes of successive trials are independent.

Larger is the sample size the outcomes will fit better to the binomial function.

2.7.1.1 Probability of the Binomial Distribution

It may be noticed that independent variable, in case of binomial distribution, is r with relative frequency f_r, which is same as the probability p_r for r success and $n-r$ failures

$$P_r = {}^nC_r p^r q^{n-r} = f_r \quad \text{for all from 1 to } n. \tag{2.11}$$

Here q, for the sake of brevity, is written for $(1-p)$.

2.7.1.2 Moments

In general, $\sum_{r=1}^{r=n} r^k f_r$ is called the kth moment of the random variable r. The arithmetic mean is the first moment. Second moment in conjunction of first moment will give variance. If arithmetic mean is zero then second moment is the variance.

2.7.1.3 Arithmetic Mean

Hence the mean μ of the binomial distribution is given as

$$\sum_{r=0}^{r=n} r \cdot {}^nC_r p^r q^{n-r}, \tag{2.12}$$

but

$$^{n}C_r = \frac{n!}{r!(n-r)!} = \frac{n(n-1)!}{r \times (r-1)! \times \{(n-1)-(r-1)\}!} = ^{n-1}C_{r-1}(n)/r.$$

Substituting this value of $^{n}C_r$ in (2.12) gives us

$$\begin{aligned}
\mu &= np \sum {}^{n-1}C_{r-1} p^{r-1} q^{n-1-(r-1)} \\
&= np\,(p+q)^{n-1} \qquad (2.13) \\
&= np\,(p+1-p)^{n-1} \\
&= np.
\end{aligned}$$

2.7.1.4 Standard Deviation

Similarly one can find the standard deviation of the binomial distribution.

$$\text{Second moment} = \sum r^2 \cdot {}^{n}C_r\, p^r q^{n-r}.$$

Following steps twice as we have done for arithmetic mean above, we get

$$\begin{aligned}
&\text{Second moment} = np(n-1)p + np, \\
&\text{Standard deviation} = \left[\text{second moment} - (\text{first moment})^2\right]^{1/2},
\end{aligned}$$

giving us

$$\left[np(n-1)p + np - (np)^2\right]^{1/2} = [np\{np - p + 1 - np\}]^{1/2}.$$

Standard deviation of a binomial distribution is

$$\{np(1-p)\}^{1/2}. \qquad (2.14)$$

2.7.2 Poisson's Distribution

Another important discrete distribution is the Poisson's distribution. When a probability of happening of an event is very small, i.e. p is small and n is quite large such that np the mean in binomial distribution is finite, then binomial distribution reduces to Poisson's distribution with np as the parameter. Examples are found in industrial production, for example defective blades in a blade-manufacturing factory. Overfilling of packages with an automatic filling machine in a packaging industry is another example.

Poisson's distribution with r as discrete random variable is given as

$$P_r = \frac{(np)^r}{r!}\exp(-np). \tag{2.15}$$

2.7.2.1 Mean of the Poisson's Distribution

$$\begin{aligned}
\text{Mean} &= \sum_{r=0}^{r=\infty} r\{(np)^r/r!\}\exp(-np) \\
&= np\exp(-np)\sum_{r=1}^{r=\infty}(np)^{r-1}/(r-1)! \qquad (2.16)\\
&= np\exp(-np)\exp(np) \\
&= np.
\end{aligned}$$

Arithmetic mean μ of Poisson's distribution is np.

2.7.2.2 Standard Deviation of the Poisson's Distribution

Variance of Poisson's distribution V is written as

$$\begin{aligned}
V &= \text{Second moment} - (\text{first moment})^2 \\
&= \{\exp(-np)\sum_{r=0}^{r=\infty} r^2(np)^r/r!\} - (np)^2 \\
&= \exp(-np)\sum_{r=1}^{r=\infty}\{r(r-1)+r\}(np)^r/r!\} - (np)^2 \\
&= \exp(-np)\left[(np)^2\sum_{r=2}^{r=\infty}(np)^{r-2}/(r-2)!\right. \\
&\quad \left. + np\sum_{r=1}^{r=\infty}(np)/(r-1)\right] - (np)^2 \qquad (2.17)\\
&= (np)^2 + (np) - (np)^2 \\
&= np.
\end{aligned}$$

Hence standard deviation of the Poisson's Distribution is $\sqrt{np}$ and mean is np.

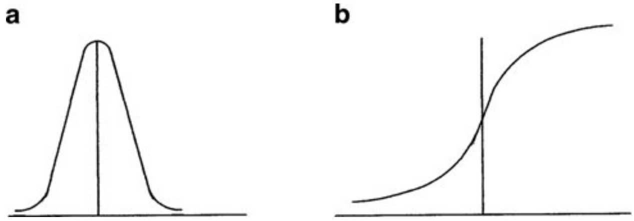

Fig. 2.1 (**a**) Normal Probability Function, (**b**) Normal Cumulative Distribution

2.8 Continuous Probability Distributions

2.8.1 Normal Probability Function

A binomial distribution, in which non of p or $(1-p)$ is small and n approaches to ∞, reduces to normal or Gaussian distribution. This bell-shaped distribution is most well known and is most widely used.

Figure 2.1a represents the normal probability function. The curve is symmetrical about the mean μ. Taking σ as standard deviation, the Gaussian probability density function can mathematically be expressed as

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-(x-\mu)^2}{2\sigma^2}\right\}. \tag{2.18}$$

The cumulative distribution curve for the Gaussian probability function is shown in Fig. 2.1b and is mathematically expressed as

$$P(X \le x_1) = \int_{-\infty}^{x_1} P(x)\mathrm{d}x, \tag{2.19}$$

$P(x)$ being the probability function of a random variable x; hence by definition,

$$\int_{-\infty}^{\infty} P(x)\mathrm{d}x = 1.$$

The Gaussian function has the following properties:

The curve for the normal (Gaussian) probability distribution is bell shaped and is symmetrical about the line $x = \mu$.

- Mean, mode and median of a normal distribution are the same.
- $P(x)$ decreases rapidly as the numerical values of x increases.
- $P(x)$ is maximum at $x = \mu$ and is equal to $1/\sigma\sqrt{2\pi}$.
- The x-axis is an asymptote to the curve.
- The points of inflexion are at $x = \mu \pm \sigma$ and the ordinates of the points are $\frac{1}{\sigma\sqrt{2\pi}} \exp(-1/2)$.

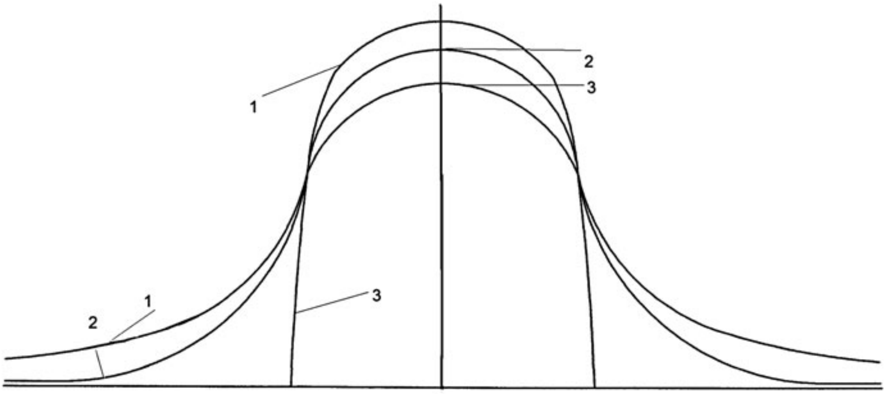

Fig. 2.2 Normal and similar curves

- As the curve represents a probability, which cannot be negative so no portion of the curve will lie below the x-axis.
- The semi-range (from mean to extreme value on either side of mean) is almost three times the standard deviation σ.
- The range of $\mu \pm 3\sigma$ covers 99.73% of area covered by the curve and x-axis; i.e. the probability of the random variable lying between $\mu \pm 3\sigma$ is 0.9973. Hence 99.73% of all normal variates will lie in this interval.
- The range of $\mu \pm 2\sigma$ covers 95.45% of all normal variates; i.e. the area covered in between $\mu \pm 2\sigma$ is 95.45% of the total area.
- The range of $\mu \pm \sigma$ covers 68.27% of normal variates.
- The range of $\mu \pm 0.6749\sigma$ covers 50% of the normal variates; 0.6749σ is called as probable error.

From the property of the area covered for different values of x helps us in deciding as to which curve is normal and which is not. In Fig. 2.2, though all the three curves have same mean and standard deviation, but only one of them represents the normal distribution. Making use of the aforesaid properties about the area covered between various ordinates, we can say that only curve 2 represents the normal curve because it has very small area (about 0.03%) covered beyond $x = 3\sigma$.

The curve 1 is not a normal curve as the area covered by it beyond $x = 3\sigma$ is much more than 0.03%. Similarly the curve 3 is also not a normal curve as all the area is covered between $x = -2\sigma$ and $x = 2\sigma$.

2.8.2 Cumulative Distribution of the Normal Probability Function

Distribution or cumulative function $F(X \le x)$ means relative cumulative frequency or the total area of the normal curve covered by it with the x-axis from x equal to

$-\infty$ to the ordinate at x_1. In any experiment with an unbiased instrument this also represents the percentage of observations likely to fall within the limit when x varies from $-\infty$ to x.

$$F(X \leq x_1) = (1/\sigma\sqrt{2\pi}) \int_{-\infty}^{x_1} \exp(-(x-\mu)^2/2\sigma^2\,\mathrm{d}x. \tag{2.20}$$

And

$$F(-\infty < x < \infty) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-(x-\mu)^2/2\sigma^2\mathrm{d}x = 1. \tag{2.21}$$

Putting $z = (x-\mu)/\sigma$ in (2.21) gives us

$$F(-\infty < z < \infty) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-z^2/2)\mathrm{d}z. \tag{2.22}$$

As the integrand is an even function, (2.22) can be written as

$$\begin{aligned} F(0 < z < \infty) &= 2(1/\sqrt{2\pi})S \int_{0}^{\infty} \exp(-z^2/2)\,\mathrm{d}z = 1 \\ \text{or} &= (1/\sqrt{2\pi}) \int_{0}^{\infty} \exp(-z^2/2)\,\mathrm{d}z = 1/2 \\ &= (1/\sqrt{2\pi}) \int_{-\infty}^{0} \exp(-z^2/2)\,\mathrm{d}z. \end{aligned} \tag{2.23}$$

Limits in above integrals are z equal to zero to z equal to ∞ and $z = -\infty$ to $z = 0$, but $z = (x-\mu)/\sigma$; hence corresponding limits of x in the second integral will be $x = -\infty$ to $x = \mu$. So the last integral in (2.23) can be written as

$$F(-\infty < X < \mu) = \int_{-\infty}^{x=\mu} \exp\left\{\frac{(x-\mu)^2}{2\sigma^2}\right\} = 1/2. \tag{2.24}$$

2.8.3 Normal Distribution and Probability Tables

Table A.1 gives the probability of happening for the given value of variable z. The values of z in steps of 0.01 have been taken from 0 to 3.49.

$z = (x-\mu)/\sigma$, $z = 1$ corresponds one standard deviation.

Table A.2 gives the cumulative frequency (area covered) from $-\infty$ to different value of z $\left\{\frac{1}{\sqrt{2\pi}} \int_0^z \exp(-z^2/2)\mathrm{d}z\right\}$.

In fact, the table gives the cumulative normal distribution against deviation of the variate from the mean expressed in terms of standard deviation.

Table A.3 gives the area covered by the variable from 0 to z. In fact, Table A.3 can be derived from Table A.2 by subtracting 0.5 from each entry.

Table A.4 gives the probability interval for the given value of z. It is the area covered by the variables from $-z$ to $+z$. For given value of z, every entry in Table A.4 is twice the entry in Table A.3.

Table A.5 gives the values of z for the given probability interval.

2.8.4 Mean and Variance of a Linear Combination of Normal Variates

Let z be a linear combinations of two normal variates and is given by

$$z = ax + by. \tag{2.25}$$

Then the probability distribution of z will also be a normal distribution giving

$$\text{Mean of } z = \mu_z = a\mu_x + b\mu_y \tag{2.26}$$

and

$$\text{Variance of } z = \sigma_z^2 = a^2\sigma_x^2 + b^2\sigma_y^2. \tag{2.27}$$

Generalizing the above statements, if z is a linear combination of n normal variates given as

$$z = \sum_{p=1}^{p=n} a_p x_p. \tag{2.28}$$

Then probability distribution of z will also be a normal distribution with mean and standard deviation given by

$$\text{Mean} = \mu_z = \sum_{p=1}^{p=n} a_p \mu_p, \tag{2.29}$$

$$\sigma_z = \left(\sum_{p=1}^{p=n} a_p^2 \sigma_p^2 \right)^{1/2}. \tag{2.30}$$

2.8.5 Standard Deviation of Mean

Let there be n normal variates $x_1, x_2, x_3, \ldots, x_n$, then mean $\bar{x}$ of these variates is given by

$$\bar{x} = (1/n) \sum_{p=1}^{p=n} x_p. \tag{2.31}$$

Following (2.30) and taking $1/n = a_p$ for all values of p, then standard deviation from (2.30) is given by

$$\sigma_{\bar{x}} = \left(\sum_{p=1}^{p=n} \sigma_p^2 / n^2 \right)^{1/2} . \tag{2.32}$$

If these n normal variates belong to the same population for example observations of an unbiased measuring instrument, then each variate will have the same σ, giving us

$$\sigma_{\bar{x}} = (n\sigma^2/n^2)^{1/2} = \sigma/\sqrt{n}. \tag{2.33}$$

2.8.6 Deviation from the Mean

From (2.16), the mean deviation is given by

$$|x_r - \mu| = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |(x-\mu)| \exp\{-(x-\mu)^2/2\sigma^2\}\, \mathrm{d}x.$$

Putting

$$y = (x-\mu)/\sigma\sqrt{2},$$
$$\text{we get } \mathrm{d}y = \mathrm{d}x/\sigma\sqrt{2} \text{ or}$$
$$\sigma\sqrt{2}\,\mathrm{d}y = \mathrm{d}x,$$

giving

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma\sqrt{2}\,|y| \exp(-y^2) \left\{(\sigma\sqrt{2})\mathrm{d}y\right\} \tag{2.34}$$
$$= 2\sigma\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} |y| \exp(-y^2)\, \mathrm{d}y$$
$$= 2\sigma\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{0} -y \exp(-y^2)\mathrm{d}y + 2\sigma\sqrt{\frac{1}{2\pi}} \int_{0}^{\infty} y \exp(-y^2)\mathrm{d}y. \tag{2.35}$$

In the first integral, putting

$$y = -z,$$
$$\mathrm{d}y = -\mathrm{d}z$$

lower limit of $z = -\infty$ and upper limit of $z = 0$, the first integral becomes

$$= 2\sigma\sqrt{\frac{1}{2\pi}} \int_{0}^{\infty} z \exp(-z^2)\mathrm{d}z$$

$$= 2\sigma\sqrt{\frac{1}{2\pi}}\int_0^\infty z\exp(-z^2)\mathrm{d}z.$$

Hence (2.35) becomes

$$= 4\sigma\sqrt{\frac{1}{2\pi}}\int_0^\infty z\exp(-z^2)\mathrm{d}z = 4\sigma\frac{1}{\sqrt{2\pi}}\left[\frac{\exp(-z^2)}{-2}\right]_0^\infty$$

$$= -\sigma\sqrt{\frac{2}{\pi}}[0-1] = \sigma\sqrt{\frac{2}{\pi}}. \tag{2.36}$$

2.8.7 Standard Deviation of Standard Deviation

We will first find out the probability density function for the standard deviation from the first principle. In deriving the expression, we will apply the fact that any linear combination of normal random variables has a normal distribution. We will then compare it with normalized normal probability function and get the corresponding value of the standard deviation of the standard deviation.

We know σ the standard deviation is calculated by the formula

$$\sigma^2 = \sum_{p=1}^{p=\infty}\frac{(x_p-\bar{x})^2}{n}.$$

This formula is valid if n is very large say more than 200. For smaller values of n the best estimate of σ is s and is given by

$$s^2 = \sigma_v^2 = \sum_{p=1}^{p=n}\frac{(x_p-\bar{x})^2}{n-1}.$$

We know that $\bar{x}$ is sum of n number of normal variable; hence $\varepsilon_p = (x_p - \bar{x})$ is also a normal variable and therefore will follow a normal probability distribution. If there are n such deviations each will follow the normal distribution; hence, if σ_1 is the standard deviation, the probability P of occurrence of all the n deviation is given by

$$P = \frac{\exp\left(-\sum\limits_{p=1}^{p=n}\varepsilon_p^2/2\sigma_1^2\right)}{\sigma_1^n(2\pi)^{n/2}}. \tag{2.37}$$

Similarly if standard deviation is $\sigma_1 + \delta$, where δ is a small quantity, then probability P_1 is given by

$$P_1 = \frac{\exp\left(-\sum\limits_{p=1}^{p=n} \varepsilon_p^2/2(\sigma_1+\delta)^2\right)}{(\sigma_1+\delta)^n(2\pi)^{n/2}}.$$

Thus, the ratio of $P/P_1 = Q$ is given by

$$\begin{aligned}
Q &= \left(1+\frac{\delta}{\sigma_1}\right)^{-n} \exp\left[\frac{1}{2}\sum_{p=1}^{p=n} \varepsilon_p^2 \left\{\frac{1}{\sigma_1^2} - \frac{1}{(\sigma_1+\delta)^2}\right\}\right] \\
&= \left(1+\frac{\delta}{\sigma_1}\right)^{-n} \exp\left\{\frac{1}{2}\sum_{p=1}^{p=n} \varepsilon_p^2 \frac{(2\delta\sigma_1+\delta^2)}{\sigma_1^2(\sigma_1+\delta)^2}\right\} \\
&= \exp\left\{\frac{1}{2}\sum_{p=1}^{p=n} \varepsilon_p^2 \frac{(2\delta\sigma_1+\delta^2)}{\sigma_1^2(\sigma_1+\delta)^2} - n\log\left(1+\frac{\delta}{\sigma_1}\right)\right\}
\end{aligned} \tag{2.38}$$

Next if σ_1 is the value, which makes P to be maximum, then partial derivative of P with respect σ_1 must be zero.

Taking log of P in (2.37), we get

$$\log P = -n\log\sigma_1 - \sum_{p=1}^{p=n} (\varepsilon_p^2/2\sigma_1^2) - \frac{n}{2}\log(2\pi). \tag{2.39}$$

Differentiating and putting it to zero, we get

$$\frac{1}{P}\frac{\mathrm{d}p}{\mathrm{d}\sigma_1} = -\frac{n}{\sigma_1} - \frac{\sum\limits_{p=1}^{p=n} -2\varepsilon_p^2}{\sigma_1^3} = 0,$$

giving us

$$\sigma_1^2 = \frac{\sum\limits_{p=1}^{p=n} \varepsilon_p^2}{n}. \tag{2.40}$$

Substituting for $\sum\limits_{p=1}^{p=m} \varepsilon_p^2$ from (2.40) in (2.38), we get

$$Q = \exp\left\{\frac{1}{2}n\frac{(2\delta\sigma_1+\delta^2)}{(\sigma_1+\delta)^2} - n\log\left(1+\frac{\delta}{\sigma_1}\right)\right\}. \tag{2.41}$$

Expanding the exponent in terms of δ/σ_1 and neglecting terms containing δ^3 and higher powers, we get Q in simplified form as

$$Q = \exp(-n\delta^2/\sigma^2).$$

Thus, the probability that the value of σ_1 lies between $\sigma_1 +\delta$ and $\sigma_1 +\delta + d\delta$ is given by

$$Q_1 = K \exp(-n\delta^2/\sigma_1^2)\, d\delta. \tag{2.42}$$

K is to be such that total probability of Q_1 is unity when δ varies from $-\infty$ to $+\infty$ is 1, giving us

$$\int_{-\infty}^{\infty} K \exp(-n\delta^2/\sigma_1)\, d\delta = 1. \tag{2.43}$$

Put

$$\delta\sqrt{n}/\sigma_1 = y,$$

giving

$$d\delta = \frac{\sigma_1}{\sqrt{n}} dy.$$

Hence (2.41) becomes

$$K\frac{\sigma_1}{\sqrt{n}}\int_{-\infty}^{\infty} \exp(-y^2)\, dy = 1,$$

but $\int_{-\infty}^{\infty} \exp(-y^2)\, dy = \sqrt{\pi}$.

Hence giving us

$$K = \frac{\sqrt{n}}{\sigma_1\sqrt{\pi}}.$$

Substituting the value of K in (2.40), we get

$$\begin{aligned} Q_1 &= \frac{\sqrt{n}}{\sigma_{1\sqrt{\pi}}} \exp(-n\delta^2/\sigma_1^2)\, d\delta \\ &= \frac{1}{\left(\sigma_1/\sqrt{2n}\right)\sqrt{2\pi}} \exp\left[-\left\{\frac{\delta^2}{2\left(\sigma_1/\sqrt{2n}\right)^2}\right\}\right]. \end{aligned} \tag{2.44}$$

Comparing (2.42) with the standard form of normal distribution namely

$$\frac{\exp(-x^2/2\sigma^2)}{\sigma\sqrt{2\pi}}.$$

we note that standard deviation of the density function in (2.40) is

$$\sigma_1/\sqrt{2n}.$$

Hence standard deviation of standard deviation σ is

$$\frac{\sigma 1}{\sqrt{2n}}. \tag{2.45}$$

2.8.8 Nomenclature for Normal Distribution

Normal distribution is characterized by its mean μ and variance σ^2; hence quite often it is denominated as $N(\mu, \sigma^2)$. A normal distribution designated as $N(12, 5)$ is mathematically equivalent to

$$f(x) = \frac{1}{\sqrt{10\pi}} \exp\left\{(x-12)^2/10\right\}.$$

We know that the sum of two normal variates is also a normal variate. Hence normal distribution of the sum of two normal variates having designations $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ will be $N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$ and will be mathematically expressed as

$$f(x_1+x_2) = \frac{1}{(\sigma_1+\sigma_2)\sqrt{2\pi}} \exp -\{(x_1+x_2)-(\mu_1+\mu_2)\}/2(\sigma_1^2+\sigma_2^2). \tag{2.46}$$

2.8.9 Probability Function of the Ratio of Two Normal Variates [1]

Let x and y be two normal variables with means μ_1 and μ_2 and standard deviations σ_1 and σ_2, respectively. We wish to derive a probability distribution of z where z is given as

$$z = \frac{(x-\mu_1)}{(y-\mu_2)}. \tag{2.47}$$

Equation (2.47) may be written as

$$\frac{\sigma_2^2}{\sigma_1^2} z^2 = \frac{(x-\mu_1)^2/\sigma_1^2}{(y-\mu_2)^2/\sigma_2^2}, \tag{2.48}$$

but $(x-\mu_1)^2/\sigma_1^2$ and $(y-\mu_2)^2/\sigma_2^2$ are the squares of independent standardized normal variables and σ_2^2/σ_1^2 is an independent χ^2 variable with 1 degree of freedom. But ratio of the squares of two independent variables is also a χ^2 variable of degree 1. Thus, $z^2/(\sigma_1^2/\sigma_2^2)$ is the quotient of two independent χ^2 variables each with 1 degree of freedom.

We know that if χ_1^2 and χ_2^2 are two independent χ^2 variables with n_1 and n_2 degrees of freedom, respectively, then

χ_1^2/χ_2^2 is a $\beta_2(\mu/2,\, \nu/2)$ variate, whose probability density function $f(x)$ is given by definition

$$\begin{aligned} f(x)\mathrm{d}x &= \frac{1}{B(\mu,\nu)}\frac{x^{\mu-1}}{(1+x)^{\mu+\nu}}\mathrm{d}x \quad \text{for positive values of } \mu,\ \nu \text{ and } x \\ &= 0 \text{ otherwise.} \end{aligned} \tag{2.49}$$

Here $B(\mu,\nu) = \frac{\Gamma\nu\Gamma\mu}{T(\nu+\mu)}$, and Γ stands for Gamma function.

Hence, the probability function of $z^2(\sigma_1^2/\sigma_2^2)$, which is the ratio of two χ^2 variables each having 1 degree of freedom; hence its probability density function is given by

$$f\left(\frac{z^2}{\sigma_1^2/\sigma_2^2}\right) = \frac{\Gamma(1/2+1/2)}{\Gamma(1/2)\Gamma(1/2)} \cdot \frac{\left(\frac{\sigma_2^2 z^2}{\sigma_1^2}\right)^{1/2-1}}{\left(1+\sigma_2^2 z^2/\sigma_1^2\right)} \cdot \sigma_2^2 d z^2/\sigma_1^2. \tag{2.50}$$

Now

$\Gamma 1 = 1$ and $\Gamma(1/2) = \sqrt{\pi}$; substituting these values in (2.50), we get

$$f(z) = \frac{\sigma_1\sigma_2}{\pi(\sigma_1^2+\sigma_2^2 z^2)z}\mathrm{d}z^2 \text{ for } 0 \le z^2 \le \infty,$$

giving

$$f(z) = \frac{2\sigma_1\sigma_2}{\pi(\sigma_1^2+\sigma_2^2 z^2)}\mathrm{d}z \quad \text{for } 0 \le z \le \infty$$

or

$$f(z) = \frac{\sigma_1\sigma_2}{\pi(\sigma_1^2+\sigma_2^2 z^2)}\mathrm{d}z \quad \text{for} -\infty \le z \le \infty. \tag{2.51}$$

This probability function is important for the experiment in which the output quantity is the ratio of two independent normal variables. For example, measurement of resistance of a resistor by measuring current A passing through it and potential difference V across it. V and A are normal variables and resistance R is related to A and V as

$$R = \frac{V}{A}.$$

Hence

$$f(R)\,\mathrm{d}R = \frac{\sigma_1\sigma_2}{\pi(\sigma_1^2+\sigma_2^2 R^2)}\mathrm{d}R. \tag{2.52}$$

Here σ_1^2 and σ_2^2 are the variances of voltage and current measurements, respectively.

2.8.10 Importance of Normal Distribution

Most of the discrete distributions such as binomial, Poisson, and hypergeometric approach to the normal distribution.

Many of the sampling distributions such as Student's t, Snedecor's F, Chi square etc. tend to be a normal distribution for larger samples (of size greater than 10).

Quite often even if the variable is not normally distributed, it can be made to follow normal distribution by simple transformation. For example if the distribution of a variable X is skewed, the distribution of $\sqrt{X}$ might become a normal distribution.

Distributions of sample mean and sample variance tend to follow normal distribution.

The entire theory of small sample tests, for example, Student's t, Snedecor's F and Chi square, is based on the assumption that parent population from which samples are drawn follows normal distribution.

All readings indicated by an unbiased measuring instrument belong to the normal distribution. Random errors of every unbiased instrument follow the normal distribution with zero mean. All industrial products manufactured by automatic devices tend to follow normal distribution. Hence, normal distribution finds largest applications in statistical quality control in industry.

2.8.11 Collation of Data from Various Laboratories [2]

2.8.11.1 Most Probable Mean of the Data

All measuring unbiased instruments indicate the value of the measurand, which follows the normal distribution. The problem of collating the data given by different laboratories is quite common. Each laboratory gives the value of the measurand along with the uncertainty. The problem is to find the best estimate of the mean value and the uncertainty associated with it. For example, if there are n independent normal variates $x_1, x_2, x_3, \ldots, x_n$, then $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n$ respectively are their standard deviations. Assuming that X is the most probable value of x, then deviations from the probable value are $X - x_1, X - x_2, X - x_3, \ldots, X - x_n$. As each variate follows normal distribution, the probability of precisely this set of deviations, therefore, is the product of n normal probability functions for the aforesaid values of deviations, giving us

$$\frac{\exp(-(x_1 - X)/2\sigma_1^2}{\sigma_1\sqrt{2\pi}}\,\frac{\exp-(x_2 - X)/2\sigma_2^2}{\sigma_2\sqrt{2\pi}}\,\frac{\exp-(x_3 - X)/2\sigma_3^2}{\sigma_3\sqrt{2\pi}}\cdots\frac{\exp-(x_n - X)/2\sigma_n^2}{\sigma_n\sqrt{2\pi}}.$$

This expression may be written as

$$\frac{\exp\left\{-\sum\limits_{p=1}^{p=n}(x_p - X)^2/2\sigma_p^2\right\}}{(2\pi)^{n/2}\prod\limits_1^n \sigma_p}, \tag{2.53}$$

where $\prod\limits_1^n \sigma_p = \sigma_1 \times \sigma_2 \times \sigma_3 \times \cdots \times \sigma_n$.

Most probable value of X will be such that the above expression becomes maximum, which means that the exponent expression becomes a minimum. That is

$$\sum_{p=1}^{p=n}(x_p - X)^2/2\sigma_p^2 \quad \text{is a minimum.} \tag{2.54}$$

The expression is a minimum if its first differential with respect to X is zero. Differentiating it with respect to X and putting it equal to zero, we get

$$\sum -\frac{2(x_p - X)}{2\sigma_p^2} = 0, \tag{2.55}$$

giving us

$$X\sum_{p=1}^{p=n}\frac{1}{\sigma_p^2} = \sum_{r=1}^{r=n} x_p/\sigma_p^2. \tag{2.56}$$

Differentiating (2.56) again with respect to X, we get

$$\sum_{p=1}^{p=n}\frac{1}{\sigma_p^2}. \tag{2.57}$$

This is always positive. Hence the expression in (2.54) is a minimum. Hence giving the most probable value of X from (2.56) as

$$X = \frac{\sum\limits_{p=1}^{p=n} x_p/\sigma_p^2}{\sum\limits_{r=1}^{r=n}\frac{1}{\sigma_p^2}}. \tag{2.58}$$

If x_p is replaced by $\bar{x}_p$ the mean of the pth sample of n_p observations, then standard deviation of single observation should be replaced by standard deviation of the mean, which is equal to

$$\sigma_p/\sqrt{n_p}. \tag{2.59}$$

Hence most probable mean of results of several laboratories is given by

$$\bar{X} = \frac{\sum_{p=1}^{p=n} \left\{\bar{x}_p n_p/\sigma_p^2\right\}}{\sum_{p=1}^{p=n} (n_p/\sigma_p^2)}. \tag{2.60}$$

Weighted mean of $x_1, x_2, x_3, \ldots, x_n$ with respective weights of $w_1, w_2, w_3, \ldots, w_n$ is given as

$$\bar{X} = \sum_{p-1}^{p=n} w_p x_p \bigg/ \sum_{p=1}^{p=n} w_p. \tag{2.61}$$

Comparing (2.60) and (2.61), we get

$$\text{The weight factor } w_p \text{ of } \bar{x}_p = \left\{n_p/\sigma_p^2\right\} \bigg/ \sum_{p=1}^{p=n} (n_p/\sigma_p^2). \tag{2.62}$$

Hence in the above equation n_p/σ_p^2 is the weight factor of $\bar{x}_p$. Hence the collating laboratory must know about the number of observations taken for calculating the mean value by each laboratory.

We have seen that most probable value of the mean is not simple arithmetic means of the estimated values but a weight mean. The weight factor given in (2.62) is proportional to the number of observations and inversely proportional to the variance of each laboratory. It may be noticed that variance here is to be calculated by normal statistical means (Type A evaluation of standard uncertainty).

2.8.11.2 Standard Deviation of the Most Probable Mean

Here we have seen that weight factor of the mean is $\frac{n_p/\sigma_p^2}{\sum_{p=1}^{p=n} n_p/\sigma_p^2}$; we further know that if

$$\bar{X} = \sum_{p=1}^{p=n} a_p \bar{x}_p, \tag{2.63}$$

then variance of $\bar{X} = \sum_{p=1}^{p=n} a_p^2 \sigma_p^2/n_p$.

We know that σ_p^2/n_p is the variance of the mean $\bar{x}_p$ and $a_p = \frac{n_p/\sigma_p^2}{\sum_{p=1}^{p=n} n_p/\sigma_p^2}$, giving

$$\text{Variance of } \bar{X} = \frac{\sum_{p=1}^{p=n} \left(n_p/\sigma_p^2\right)^2 \sigma_p^2/n_p}{\left(\sum_{p=1}^{p=n} n_p/\sigma_p^2\right)^2} \tag{2.64}$$

and

$$\begin{aligned}\text{Variance of } \bar{X} &= \sum_{p=1}^{p=n} \left[\frac{\{(n_p/\sigma_p^2)^2\}}{n_p/\sigma_p^2}\right] \Bigg/ \left(\sum_{p-1}^{p=n} n_p/\sigma_p^2\right)^2 \\ &= \frac{\sum_{p=1}^{p=n} n_p/\sigma_p^2}{\left[\sum_{p=1}^{p=n} n_p/\sigma_p^2\right]^2} \\ &= \frac{1}{\sum_{p=1}^{p=n} n_p/\sigma_p^2}.\end{aligned} \tag{2.65}$$

$$\text{Standard deviation of the mean } \bar{X} = \left[\frac{1}{\sum_{p=1}^{p=n} n_p/\sigma_p^2}\right]^{1/2}. \tag{2.66}$$

The data sent for such collation not only contain the estimated value of the parameter and standard deviation but should also contain the number of observations taken by each laboratory.

References

1. S.C. Gupta, V.K. Kapoor, *Fundamentals of Mathematical Statistics*, 9th edn. (Sultan Chand, New Delhi, 1994), p. 13.27
2. C.F. Dietrich, *Uncertainty, Calibration and Probability* (Adam Hilger, New York, 1991), pp. 39–41
3. R.B. Frenkel, Fiducial inference applied to uncertainty estimation when identical readings are obtained under low instrument resolution. Metrologia **46**, 661–667 (2009)

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